Rationals and the Modular Group

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The modular group $\mathscr{M}$ is the quotient group $PSL_2(\mathbf{Z}) = SL_2(\mathbf{Z})/\{\pm I\}$ of $SL_2(\mathbf{Z})$, the group of 2×2 integer matrices of determinant 1. In [**1**] we gave an elementary proof that $\mathscr{M}$ has the structure of a free product of a cyclic group of order 2 generated by the image of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and a cyclic group of order 3 generated by the image of $B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

The free product structure provides a description of the non-trivial elements of $\mathscr{M}$ as unique strings of A's and B's with the property that there are no two consecutive A's and no three consecutive B's; we refer to these as *reduced strings*. We explained this free product structure in terms of the action of the modular group on the irrationals. In this note we describe the action on the rationals; this can be viewed as a way of describing the inverse of the Euclidean algorithm.

The group $SL_2(\mathbf{Z})$ acts via linear transformations on $\mathbf{R}^2$ as column vectors and this gives an action of $\mathscr{M}$ via linear fractional transformations on the projective line $P^1(\mathbf{R})$, the real numbers together with ∞. We may also view $P^1(\mathbf{R})$ as the slopes of non-zero vectors, that is, the equivalence classes of $\mathbf{R}^2 - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ induced by non-zero scalar multiplication; the equivalence class of the vector $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is denoted ∞, the equivalence class of the vector $\begin{pmatrix} p \\ q \end{pmatrix}$, $q \neq 0$ is the same as that of $\begin{pmatrix} p/q \\ 1 \end{pmatrix}$ and corresponds to the real number $z = p/q$. For the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbf{Z})$ the induced action on $P^1(\mathbf{R})$ is given by

$$z \longrightarrow \frac{az+b}{cz+d}.$$

The induced action for the generating elements is given as

$$A: z \longrightarrow \frac{-1}{z}, \quad B: z \longrightarrow \frac{-1}{z+1}, \quad B^2: z \longrightarrow -1 - \frac{1}{z}.$$

The orbit of e is easily seen to be in correspondence with the set of all first columns of matrices from $SL_2(\mathbf{Z})$. Thus the orbit of ∞ is in 1-1 correspondence with the projective line $P^1(\mathbf{Q})$, consisting of the set $\mathbf{Q}$ of all reduced fractions together with ∞; from elementary group theory this is in 1-1 correspondence with the set of left cosets of the stabilizer $\mathscr{N}$ of e, which is the image of the subgroup generated by AB in $\mathscr{M}$.

Using the free product description of $\mathscr{M}$ we can also describe the set of coset representatives as reduced strings of A's and B's . First, a non-trivial coset representative cannot end in AB or its inverse B^2A; therefore if it ends in A it is either A or of the form ZBA with Z ending in A or trivial; if it ends in B it is either B or ZB^2 with Z ending in A or trivial. Thus, as a first pass, the set of coset representatives is the set $\mathscr{R} = \{I\} \cup \{A\} \cup \{B\} \cup \{BA\} \cup \{B^2\} \cup \{ZBA | Z$ any string ending in $A\} \cup \{ZB^2 | Z$ any string ending in $A\}$. Next, to determine the distinct coset representatives we just observe that the free product description gives a unique expression for the elements. The coset equivalence relation $X \cong Y$ on reduced strings $X, Y \in \mathscr{M}$ is $Y = X(AB)^n$ or $Y = X(B^2A)^n$ for some non-negative integer n. We see easily that $A \cong B \bmod \mathscr{N}$ and hence also $XA \cong XB \bmod \mathscr{N}$ for any string X, and hence if this is reduced, $X \neq I$ must end in B. Thus we can simplify the description of the distinct coset representatives to $\mathscr{R} = \{I\} \cup \{B\} \cup \{B^2\} \cup \{ZB^2 | Z$ any string ending in $A\}$. It is easy to see that no two of these reduced strings are equivalent; for example, for $Z \neq W$, both ending in A, then $ZB^2 = WB^2(B^2A)^n$ and $ZB^2 = WB^2(AB)^n$ are impossible. Thus the coset representatives of $\mathscr{M}/\mathscr{N}$ are the distinct strings $\mathscr{R} = \{I\} \cup \{B\} \cup \{B^2\} \cup \{ZB^2 | Z$ any string ending in $A\}$.

We can also describe this set $\mathscr{R}$ as the union of $\mathscr{R}_m$ defined inductively as

$$\begin{aligned} \mathscr{R}_0 &= \{I, B\}, \quad \mathscr{N}_0 = \{B^2\} \\ \mathscr{P}_m &= \{A\}\mathscr{N}_m, \quad \mathscr{N}_{m+1} = \{B^2, B\}\mathscr{P}_m \qquad (1) \\ \mathscr{R}_{m+1} &= \mathscr{R}_m \cup \mathscr{P}_m \cup \mathscr{N}_m \end{aligned}$$

It is easy to see that $\mathscr{R}_m$ has 2^{m+1} elements and $\mathscr{P}_m$, $\mathscr{N}_m$ each have 2^m elements. We can rewrite (1) as

$$\mathscr{P}_{m+1} = \{AB, AB^2\}\mathscr{P}_m, \quad \mathscr{N}_{m+1} = \{B^2A, BA\}\mathscr{N}_m. \qquad (2)$$

Simplifying (2) we obtain the following result. $\mathscr{FS}(x, y)$ denotes the free semigroup with the generators, x, y.

Proposition. *$P^1(\mathbf{Q})$ is in 1-1 correspondence with $\mathscr{R} = \{I\} \cup \{B\} \cup \mathscr{FS}(AB, AB^2) \cdot AB^2 \cup \mathscr{FS}(B^2A, BA) \cdot B^2$.*

Observing, that $I\infty = \infty$, $B\infty = 0$, $B^2\infty = -1$ and $AB^2\infty = 1$, and $P^1(\mathbf{Q}) - \{\infty, 0\}$ is the positive and negative rationals, we have the following

Corollary. *The set of positive rationals is the orbit of the free semigroup generated by AB and AB^2 on $z = 1$. The set of negative rationals is the orbit of the free semigroup generated by B^2A and BA on $z = -1$.*

These upper $U = AB$, $U^- = B^2A$ and lower $L = AB^2$, $L^- = BA$ triangular matrix actions corresponding to these semigroup generators are

$$U: z \longrightarrow z + 1, \quad L: z \longrightarrow \frac{z}{z+1},$$

$$U^-: z \longrightarrow z - 1, \; L^-: z \longrightarrow \frac{z}{1-z}.$$

Every positive rational is uniquely expressible in terms of semigroup generators as an element of the orbit of 1. Alternatively, starting from a reduced positive rational $z = \frac{p}{q}$ we can apply a greedy or Euclidean recursion to obtain a finite sequence that stabilizes at 1;

$$\frac{p}{q} \to \begin{cases} \dfrac{(p-q)}{q} & \text{if } p > q \\ \dfrac{p}{(q-p)} & \text{if } p < q \\ 1 & \text{if } p = q = 1 \end{cases}$$

Here we apply U^- or L^- depending on whether or not $z > 1$ or $z < 1$. For example, the sequence

$$\frac{34}{55}, \frac{34}{21}, \frac{13}{21}, \frac{13}{8}, \frac{5}{8}, \frac{5}{3}, \frac{2}{3}, \frac{2}{1}, 1$$

corresponds to $(U^- L^-)^4(\frac{34}{55}) = 1$. Since we have shown that the set of positive rationals can be described by a free semigroup, this means that $(LU)^4 L$ is *the* coset representative in $\mathscr{R}_9$ corresponding to 34/55 as described in the Corollary.

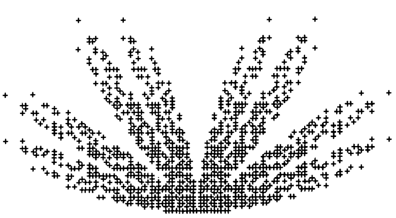

Figure 1. $\mathscr{R}_9$.

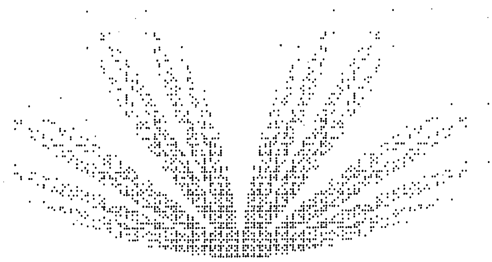

Figure 2. $\mathscr{R}_{10}$.

Finally, we consider the matrix action of the distinct non-trivial coset representatives in $\mathscr{R}$ on the column $\begin{pmatrix}1\\0\end{pmatrix}$ and the plots of these images in $\mathbf{R}^2$. These images are just the points with relatively prime coordinates in the upper half-plane. We obtain the fascinating plant-like structures in Figures 1 and 2. For example, the distant points from the root $\begin{pmatrix}0\\1\end{pmatrix}$ are the 'Fibonacci points' $\begin{pmatrix}\pm 34\\55\end{pmatrix}$, $\begin{pmatrix}\pm 55\\34\end{pmatrix}$ in $\mathscr{R}_9$.

REFERENCE

1. Roger C. Alperin, $PSL_2(\mathbf{Z}) = \mathbf{Z}_2 * \mathbf{Z}_3$ *Amer. Math. Monthly* **100** (1993) 385–386.

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