10766. *Proposed by Szilárd András, Babeş-Bolyai University, Cluj-Napoca, Romania.* Let x, y, and z be nonnegative real numbers. Prove that

(a) $(x+y+z)^{x+y+z}x^x y^y z^z \le (x+y)^{x+y}(y+z)^{y+z}(z+x)^{z+x}$.

(b) $(x+y+z)^{(x+y+z)^2}x^{x^2}y^{y^2}z^{z^2} \ge (x+y)^{(x+y)^2}(y+z)^{(y+z)^2}(z+x)^{(z+x)^2}$.

SOLUTIONS

Cramer's Rule for Non-Square Matrices

10618 [1997, 768]. *Proposed by S. Lakshminarayanan, S. L. Shah, and K. Nandakumar, University of Alberta, Edmonton, Canada.* Let A be a real $m \times n$ matrix of full rank with $m < n$ and let b be a real $m \times 1$ matrix. For $1 \le i \le n$, define

$$x_i = \frac{\det(A_i^* A^T) - \det(A_i A_i^T)}{\det(AA^T)},$$

where A_i^* is obtained by replacing the ith column of A by b, and A_i is obtained by deleting the ith column of A. Show that $x = [x_1, \ldots, x_n]^T$ is a solution to the linear system $Ax = b$.

Solution by the GCHQ Problems Group, Cheltenham, U. K. We write $A^i\langle b\rangle$ instead of A_i^* to emphasize the role of the vector b; thus $A^i\langle 0\rangle$ indicates A with its ith column zeroed out. Observe that $A_i A_i^T = A^i\langle 0\rangle A^T$, by comparing corresponding entries.

Extend A to a nonsingular $n \times n$ matrix $\binom{A}{C}$, where C is an $(n-m) \times n$ matrix whose rows form an orthonormal basis for the orthogonal complement of the row space of A. That is, each row of C has norm 1 and is orthogonal to all other rows of $\binom{A}{C}$. We have

$$\begin{pmatrix} A \\ C \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} AA^T & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^i\langle b\rangle \\ C \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} A^i\langle b\rangle A^T & M \\ 0 & I \end{pmatrix},$$

where I is the $(n-m) \times (n-m)$ identity matrix and M is some $n \times (n-m)$ matrix. By substituting these computations into the definition of x_i, canceling the nonzero factor $\det\binom{A}{C}^T$, and using the linearity of the determinant in its ith column, we obtain

$$x_i = \frac{\det\left(\binom{A^i\langle b\rangle}{C}\binom{A}{C}^T\right) - \det\left(\binom{A^i\langle 0\rangle}{C}\binom{A}{C}^T\right)}{\det\left(\binom{A}{C}\binom{A}{C}^T\right)} = \frac{\det\binom{A^i\langle b\rangle}{C} - \det\binom{A^i\langle 0\rangle}{C}}{\det\binom{A}{C}} = \frac{\det\binom{A}{C}^i\left\langle\binom{b}{0}\right\rangle}{\det\binom{A}{C}},$$

By Cramer's rule, x is the solution to the linear system $\binom{A}{C}x = \binom{b}{0}$, and hence x is a solution to $Ax = b$.

Solved also by J. Fuelberth & A. Gunawardena, J. H. Lindsey II, M. Sharma & P. G. Poonacha (India), WMC Problems Group, and the proposers.

An Identity for Strongly Connected Digraphs

10620 [1997, 870]. *Proposed by James Propp, Massachusetts Institute of Technology, Cambridge, MA.* A *digraph* on a vertex set V is a subset $A \subseteq \{(v, w)\colon v, w \in V, v \ne w\}$ and is *strongly connected* if it is possible to get from any vertex a to every other vertex e by a finite succession of arcs $(a, b), (b, c), \ldots, (d, e)$ in A. For $n \ge 1$, let E_n (respectively, O_n) denote the number of strongly connected digraphs on the vertex set $V = \{1, 2, \ldots, n\}$ with an even (respectively odd) number of arcs. Show that $E_n - O_n = (n-1)!$ for all $n \ge 1$.

Solution I by the proposer, currently at University of Wisconsin, Madison, WI. The terminology of the problem statement is somewhat nonstandard. In common usage, a digraph is a pair (V, A), in which A is the set of *arcs* in the digraph. Edges from a vertex to itself, and even multiple edges from one vertex to another, are ordinarily allowed; the digraphs considered here with no such edges are usually called *simple*. The "succession of arcs" described above is a *path* from a to e.

Let the *sign* of a digraph with arc set A be $(-1)^{|A|}$; the problem is to show that the sum s_n of signs of the strongly connected digraphs with vertex set $[n] = \{1, \ldots, n\}$ is $(n-1)!$. Let r_n be the sum of the signs of all the digraphs on the vertex set $[n]$ containing paths from n to all other vertices.

We prove first that $r_n = s_n - (n-1)s_{n-1}$ for $n > 1$. For $I \subseteq [n-1]$, let R_I be the set of digraphs on $[n]$ that contain paths from n to all other vertices and such that I is the set of vertices other than n that begin paths to vertex n. Let r_I be the sum of the signs of the digraphs in R_I, so $r_n = \sum r_I$.

When $|I| = n-1$, R_I is the set of strongly connected digraphs on $[n]$, and thus $r_I = s_n$.

When $|I| = n-2$, let j be the sole vertex of $[n-1] - I$. The signs of digraphs in R_I that have an arc from I to j sum to 0, since adding or deleting the arc (n, j) changes the sign without changing membership in this set. Thus we need to sum signs only for those digraphs in R_I that have no arc from I to j. Such digraphs consist of a strongly connected digraph on $[n] - \{j\}$ plus the arc (n, j), so the sum of their signs is $-s_{n-1}$. With $n-1$ choices for j, the sum of the contributions when $|I| = n-2$ is $-(n-1)s_{n-1}$.

When $|I| < n-2$, let i and j be the largest and second-largest elements of $[n-1] - I$. Adding or deleting the arc (i, j) changes the sign without changing membership in R_I. Hence these contributions cancel, and $r_I = 0$.

We have proved that $r_n = s_n - (n-1)s_{n-1}$. Since $s_1 = 1$, the proof is completed by proving that $r_n = 0$ for $n > 1$. For $I \subseteq [n-1]$, let Q_I denote the set of digraphs on $[n]$ in which I is the set of vertices other than n that are reachable by paths from vertex n. Let q_I be the sum of the signs of the digraphs in Q_I, so $r_n = q_{[n-1]}$.

When $|I| < n-1$, let i be the largest element of $[n-1] - I$. Adding or deleting the arc (i, n) changes the sign without changing membership in Q_I. Thus $q_I = 0$.

We conclude that $\sum_I q_I = r_n$. On the other hand, the sum of q_I over all I is the sum of the signs of all digraphs on $[n]$. This sum is 0, since adding or deleting the arc $(1, 2)$ changes the sign. Thus $r_n = 0$, as desired.

Composite solution II by Nikhil Bansal, Bombay, India, and the editors. Let $e(D)$ denote the number of arcs in a digraph D. The claim follows from comparing two expressions for $g(x)$, the exponential generating function for $a_n = \sum(-1)^{e(D)}$, where the sum is over *all* digraphs on $[n]$.

First, $g(x) = \sum_{n\geq 1} a_n x^n/n! = x$, since the number of digraphs with even size equals the number with odd size when $n > 1$, while $a_1 = (-1)^0 = 1$.

Alternatively, we group the contributions according to certain subdigraphs, counting those of even size minus those of odd size. The *strong components* of a digraph are the maximal strongly connected subdigraphs. The *condensation* D^* of a digraph D is the digraph obtained by contracting strong components to single points. The condensation of a digraph with k strong components is an acyclic digraph with k vertices.

To form a digraph on $[n]$ having k fixed strong components with vertex sets of sizes $n_1, \ldots, n_k$ respectively, we assign vertices to components, place a strongly connected digraph on each component, form an acyclic digraph on $[k]$ as the condensation, and expand edges of the condensation into sets of edges between the corresponding strong components. We divide by $k!$, because strong components are distinguished by the names of their vertices, but the names of the components are arbitrary.

The contributions to parity within each strong component and within the expansion of each edge of the acyclic digraph are independent; the final digraph has even size if and only if the number of odd contributions is even. Thus we take the product, over each set of edges to be included, of the number of ways to include it with even size minus the number of ways to include it with odd size. Let $b_m = E_m - O_m$; this is the contribution for a strong component of order m.

At this point we have

$$\sum(-1)^{e(D)} = \sum_{k=1}^{n} \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1!\ldots n_k!}\frac{1}{k!} b_{n_1}\ldots b_{n_k} \sum_{C} \prod_{i,j\in\binom{[k]}{2}} (E_{ij} - O_{ij}),$$

where the inner sum is over acyclic digraphs C on $[k]$, the notation $\binom{[k]}{2}$ stands for the set of all 2-element subsets of $[k]$, and E_{ij} and O_{ij}, respectively, denote the number of ways to have an even or odd number of edges between components i and j in the expansion of C.

When i and j are not adjacent in C, there is no edge in the expansion, and $E_{ij} - O_{ij} = 1$. When i and j are adjacent, there must be at least one edge in the expansion, and all such edges agree in direction with the edge in C. Eliminating the empty set yields $E_{ij} - O_{ij} = -1$. With the factor -1 for each edge of C, the product is $(-1)^{e(C)}$.

To further simplify the formula, we claim that $\sum_C(-1)^{e(C)} = (-1)^{k-1}$. We define an involution on the acyclic digraphs that pairs up digraphs with sizes differing by 1, and we show that the only unpaired digraph is the digraph C_k with arc set $\{(k, j)\colon 1 \le j \le k-1\}$. A *source* is a vertex with no incoming arc; a *predecessor* of j is a vertex i such that (i, j) is an arc.

Every acyclic digraph C has at least one source. Let i be the least source vertex. When $i \neq k$, we add or delete the arc (i, k); it remains true that i is the least source. In the remaining digraphs, the only source vertex is k. For these, let j be the highest vertex having a predecessor other than k. Add or delete the arc (k, j). It remains true that k is the only source and that j is the highest vertex with a predecessor other than k. The only digraph in which j does not exist is C_k (with $k-1$ edges), which completes the claim.

The coefficient of $x^n/n!$ in our generating function is now

$$\sum_{k=1}^{n} \sum_{n_1+\cdots+n_k=n} \binom{n}{\{n_i\}} \prod b_{n_i} \frac{(-1)^{k-1}}{k!}.$$

Given the exponential generating function $f(x) = b_n x^n/n!$, this yields $g(x) = 1 - e^{-f(x)}$. Since $g(x) = x$, we obtain $f(x) = -\ln(1-x)$, and hence $b_k = (k-1)!$.

Editorial comment. The proposer's proof is adapted from an analysis of the unsigned case due to V. A. Liskovec (On a recurrence method of counting graphs with labelled vertices, *Soviet Math. Dokl.* 10 (1969) 242–256) and explicated by E. M. Wright (The number of strong digraphs, *Bull. London Math. Soc.* 3 (1971) 348–350). Problem 6673 [1991, 965; 1994, 686] in this MONTHLY is the analogous problem for undirected graphs. The proposer notes the following consequence. If the sign of a disjoint union of strongly connected (simple) digraphs is $(-1)^{e+k}$, where e is the total number of arcs and k is the number of components, then the sum of the signs of all n-vertex disjoint unions of strongly connected digraphs is 0. He asks whether there is a simple direct proof of this corollary. One might also ask for a simple signed involution to prove the original claim directly.

Solved also by R. J. Chapman (U. K.).

Simultaneous Squares from Arithmetic Progressions

10622 [1997, 870]. *Proposed by M. N. Deshpande, Nagpur, India.* Find infinitely many triples (a, b, c) of positive integers such that a, b, c are in arithmetic progression and such that $ab+1$, $bc+1$, and $ca+1$ are perfect squares.

Solution I by Hansruedi Widmer, Nussbaumen, Switzerland. Let $a_0 = 1$, $a_1 = 4$, and $a_{n+2} = 4a_{n+1} - a_n$ for $n \geq 0$, and set $b_n = 2a_{n+1}$ and $c_n = a_{n+2}$. We claim that