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The purpose of this paper is to present two elementary (and perhaps somewhat novel) solutions of the Dirichlet problem for ellipsoids in $\mathbb{R}^n$. One of these is based on an elegant result of Ernst Fischer—of Riesz–Fischer fame.

By the *Dirichlet problem* (for the Laplacian) we mean the following: Given a bounded region (nonempty, open, connected set) Ω in $\mathbb{R}^n, n \geq 2$, and given a continuous function $f: \partial\Omega \to \mathbb{R}$ (called the *boundary data*), find a continuous function $u: \overline{\Omega} \to \mathbb{R}$ such that u is C^2 on Ω,

$$\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} u(x) = 0 \quad \text{for} \quad x \in \Omega \quad \text{and}$$

$$u(x) = f(x) \quad \text{for} \quad x \in \partial\Omega\text{—the boundary of } \Omega.$$

This is surely one of the most influential problems for the development of mathematics in the last two centuries; see, for example, [**4**], [**6**], and [**8**]. The case in which Ω is a disk in $\mathbb{R}^2$ is standard fare for writers of texts on the theory of analytic functions of one complex variable. In this paper we are concerned with the case in which Ω is an ellipsoid in $\mathbb{R}^n$, for arbitrary $n \geq 2$, and especially when f is (the restriction to $\partial\Omega$ of) a polynomial function.

1. BACKGROUND AND NOTATION. Let $2 \leq n \in \mathbb{Z}$. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let $x \cdot y = \sum_{k=1}^{n} x_k y_k$ and $|x| = (x \cdot x)^{1/2}$. For $1 \leq k \leq n$ we write ∂_k instead of $\partial/\partial x_k$ and define the *Laplacian* $\Delta = \sum_{k=1}^{n} \partial_k^2$. If Ω is a region in $\mathbb{R}^n$ and $v: \Omega \to \mathbb{R}$ then we say that v is *harmonic* on Ω provided v is C^2 (twice continuously differentiable) on Ω and v satisfies the *Laplace equation*

$$\Delta v(x) = 0 \quad \text{for all } x \in \Omega.$$

For completeness we include a well known proof [**9**, p. 103] of the weak form of

The Maximum Principle. *If Ω is a bounded region in $\mathbb{R}^n$, $v: \overline{\Omega} \to \mathbb{R}$, v is continuous on $\overline{\Omega}$, v is C^2 on Ω, and $\Delta v(x) \geq 0$ for all $x \in \Omega$, then v attains its maximum on $\partial\Omega$.*

Proof: For $0 < \epsilon \in \mathbb{R}$ let $v_\epsilon(x) = v(x) + \epsilon|x|^2$ for $x \in \overline{\Omega}$. Then v_ϵ is continuous on $\overline{\Omega}$, C^2 on Ω and, for all $x \in \Omega$,

$$\sum_{k=1}^{n} \partial_k^2 v_\epsilon(x) = \sum_{k=1}^{n} \partial_k^2 v(x) + 2n\epsilon \geq 2n\epsilon > 0.$$

Hence, for each $x \in \Omega$ there is a $k (1 \leq k \leq n)$ such that $\partial_k^2 v_\epsilon(x) > 0$; single variable calculus ensures that v_ϵ does *not* have a relative maximum at x. It follows that, for each $\epsilon > 0, v_\epsilon$ attains its maximum on $\partial\Omega$. That is, for each $\epsilon > 0$ there exists $x_\epsilon \in \partial\Omega$ such that $v_\epsilon(x) \leq v_\epsilon(x_\epsilon)$ for all $x \in \overline{\Omega}$. Hence, for $\epsilon > 0$ and $x \in \overline{\Omega}$ $v(x) \leq v_\epsilon(x) \leq v(x_\epsilon) + \epsilon|x_\epsilon|^2$, so that $v(x) \leq \max\{v(y) : y \in \partial\Omega\} + \epsilon R^2$, where $R = \max\{|x| : x \in \overline{\Omega}\}$. Since this is so for every $\epsilon > 0$, $v(x) \leq \max\{v(y) : y \in \partial\Omega\}$ for all $x \in \overline{\Omega}$. ∎

Corollary 1. *If Ω is a bounded region in $\mathbb{R}^n$, $u : \overline{\Omega} \to \mathbb{R}$, and u is continuous on $\overline{\Omega}$ and harmonic on Ω, then u attains its maximum and its minimum on $\partial\Omega$.*

For proof it suffices to note that since u is harmonic on Ω, so is $-u$.

Corollary 2. *If Ω is a bounded region in $\mathbb{R}^n$ and $f : \partial\Omega \to \mathbb{R}$ is continuous, then there is at most one solution to the Dirichlet problem satisfying $u(x) = f(x)$ for all $x \in \partial\Omega$.*

Proof: Suppose that u_1 and u_2 were such solutions and let $v(x) = u_1(x) - u_2(x)$ for $x \in \overline{\Omega}$. Then v is continuous on $\overline{\Omega}$, harmonic on Ω, and vanishes on $\partial\Omega$. Hence, by Corollary 1, for every $x \in \overline{\Omega}$ we have

$$0 = \min\{v(y) : y \in \partial\Omega\} \le v(x) \le \max\{v(y) : y \in \partial\Omega\} = 0,$$

i.e., $0 = v(x) = u_1(x) - u_2(x)$ for all $x \in \overline{\Omega}$. ■

Let's fix $n \ge 2$ in $\mathbb{Z}$, let $r_1, \dots, r_n > 0$, let

$$b(x) = 1 - \sum_{k=1}^{n} x_k^2/r_k^2 \qquad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

and let $\mathscr{E}$ denote the ellipsoid $\{x \in \mathbb{R}^n : b(x) > 0\}$ so that $\partial\mathscr{E} = \{x \in \mathbb{R}^n : b(x) = 0\}$. We aim to solve the Dirichlet problem for $\mathscr{E}$ by first showing that it can be handled in the case of polynomial boundary data with the aid of an elementary consequence of some work of E. Fischer.

Let $\mathscr{P}$ denote the real algebra of all polynomial functions from $\mathbb{R}^n$ into $\mathbb{R}$. For $0 \le m \in \mathbb{Z}$, $\mathscr{P}_m$ denotes the *finite dimensional* linear subspace of $\mathscr{P}$ consisting of those members of $\mathscr{P}$ having degree at most m.

The following elegant result surely deserves to be better known. It has its origins in the 1917 paper [**7**] of Ernst Fischer; see the discussion of (2.8) on page 459 of [**10**]. Let's call it

Fischer's Lemma. *For $f \in \mathscr{P}$ define $L(f) = \Delta(fb)$. Then L is a linear, degree-preserving, bijection of $\mathscr{P}$ onto itself.*

Proof: Clearly L is a linear operator on $\mathscr{P}$. Suppose that $L(f) = 0$ for some $f \in \mathscr{P}$. Let $u = fb$ so that $\Delta u(x) = 0$ for all $x \in \mathbb{R}^n$ and $u(x) = 0$ for all $x \in \partial\mathscr{E}$. By Corollary 2, $u(x) = 0$ for all $x \in \overline{\mathscr{E}}$ so $f(x) = 0$ for all $x \in \mathscr{E}$ since $b(x) > 0$ for all $x \in \mathscr{E}$. But f is a polynomial; hence $f(x) = 0$ for all $x \in \mathbb{R}^n$. We have shown that L is one-to-one.

Now suppose that $0 \le m \in \mathbb{Z}$. If $f \in \mathscr{P}_m$ then $fb \in \mathscr{P}_{m+2}$ and hence $\Delta(fb) \in \mathscr{P}_m$. That is, L maps $\mathscr{P}_m$ *into* itself. Since $\mathscr{P}_m$ is finite dimensional and L is linear and one-to-one, L maps $\mathscr{P}_m$ *onto* itself. ■

2. THE DIRICHLET PROBLEM FOR ELLIPSOIDS AND POLYNOMIAL BOUNDARY DATA.

Fischer's Lemma and the Maximum Principle, yield a simple proof of

Theorem 1. *Suppose that $f \in \mathscr{P}_m$ for some $m \ge 0$. Then there exists a unique u in $\mathscr{P}_m$ such that*

$$\Delta u(x) = 0 \quad \textit{for all } x \in \mathbb{R}^n \quad \textit{and} \quad u(x) = f(x) \quad \textit{for all } x \in \partial\mathscr{E}. \qquad (\#)$$

Proof: If $m \le 1$ the conclusion holds with $u = f$. Suppose that $m \ge 2$ and the degree of f is at least 2. We look for a u of the form $f + vb$ with $v \in \mathscr{P}_{m-2}$. For any such u, $u(x) = f(x)$ for all $x \in \partial\mathscr{E}$ and $\Delta u = \Delta f + \Delta(vb)$. By Fischer's Lemma, there exists a unique g in $\mathscr{P}_{m-2}$ such that $\Delta(gb) = -\Delta f$. Thus, if we define $u := f + gb$, then $u \in \mathscr{P}_m$ and (#) holds. Uniqueness follows from Corollary 2. ■

3. THE MEAN–VALUE PROPERTY AND THE WEIERSTRASS APPROXIMATION THEOREM. Let $B = \{x \in \mathbb{R}^n : |x| \le 1\}$, $S = \partial B = \{x \in \mathbb{R}^n : |x| = 1\}$, and $B_r(a) = \{x \in \mathbb{R}^n : |x - a| \le r\} = \{a + ry : y \in B\}$ for $a \in \mathbb{R}^n$ and $r > 0$. Denote by σ the normalized, $n - 1$ dimensional surface measure on S. The following result is a combination of [**1**, Theorems 1.2 and 1.20], which depend mainly upon the Divergence Theorem for B, a fairly self-contained exposition of which can be found in [**3**].

The Mean–Value Property. *Suppose that Ω is a region in $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$, and u is continuous. Then u is harmonic on Ω if and only if*

$$u(a) = \int_S u(a + rs)\, d\sigma(s) \quad \text{whenever } B_r(a) \subset \Omega.$$

According to [**6**, p. 35], this theorem can be traced back to an 1840 paper of Gauss.

Corollary. *If Ω is a region in $\mathbb{R}^n$, $u_k : \Omega \to \mathbb{R}$ is harmonic for each $k \in \mathbb{N}$, and $\{u_k\}_{k=1}^{\infty}$ converges uniformly on Ω to $u : \Omega \to \mathbb{R}$, then u is harmonic on Ω.*

Sketch of Proof. Suppose $B_r(a) \subset \Omega$. Then

$$u_k(a) = \int_S u_k(a + rs)\, d\sigma(s) \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad u(a) = \lim_{k \to \infty} u_k(a).$$

Moreover, since each u_k is continuous and the convergence is uniform on Ω, u is continuous on Ω and

$$\lim_{k \to \infty} \int_S u_k(a + rs)\, d\sigma(s) = \int_S u(a + rs)\, d\sigma(s).$$

Hence

$$u(a) = \int_S u(a + rs)\, d\sigma(s).$$

By the Mean Value Property, u is harmonic on Ω. ■

It does not appear to be well known that, in [**13**], Weierstrass proved his famous approximation theorem not only in the case of a single real variable but also in higher dimensions. That "approximate identity" proof, in the one dimensional case, is the subject of Chapter 59 of the beautiful book of Körner [**11**]. The same proof extends to higher dimensions without serious difficulty, see [**9**, p. 209 and Problem 1 on p. 213]. Because of its geometric appeal, its intimate relationship with the heat equation, and the fact that it affords C^m approximation, Weierstrass's own proof, in the author's opinion, has not been bettered.

The Weierstrass Approximation Theorem. *Given a rectangle I in $\mathbb{R}^n$, a continuous function $f : I \to \mathbb{R}$, and $\epsilon > 0$, there exists a p in $\mathscr{P}$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in I$.*

By a *rectangle* in $\mathbb{R}^n$ we mean a product of n closed bounded intervals.

4. THE DIRICHLET PROBLEM FOR ELLIPSOIDS; A SOLUTION

Theorem 2. *Given a continuous function* $f : \partial\mathscr{E} \to \mathbb{R}$, *there is a unique continuous function* $u : \overline{\mathscr{E}} \to \mathbb{R}$ *such that* u *is* C^2 *on* $\mathscr{E}$, $\Delta u(x) = 0$ *for all* $x \in \mathscr{E}$, *and* $u(x) = f(x)$ *for all* $x \in \partial\mathscr{E}$.

Proof: For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ let $\nu(x) = \Sigma_{k=1}^{n} x_k^2 / r_k^2$ and

$$\tilde{f}(x) = \begin{cases} \nu(x) f\big(\nu(x)^{-1} x\big) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\tilde{f}$ is continuous on $\mathbb{R}^n$ and therefore, by Weierstrass's Theorem, can be uniformly approximated on each rectangle by a polynomial. But $\tilde{f}(x) = f(x)$ for all $x \in \partial\mathscr{E}$. Hence there is a sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathscr{P}$ that converges to f uniformly on $\partial\mathscr{E}$.

According to Theorem 1, for each $k \in \mathbb{N}$ there is a unique $u_k \in \mathscr{P}$ such that $\Delta u_k(x) = 0$ for all $x \in \mathbb{R}^n$ and $u_k(x) = f_k(x)$ for all $x \in \partial\mathscr{E}$. By the Maximum Principle, for $j, k \in \mathbb{N}$ and $x \in \overline{\mathscr{E}}$,

$$|u_j(x) - u_k(x)| \le \max\{|f_j(y) - f_k(y)| : y \in \partial\mathscr{E}\} \to 0 \quad \text{as } j, k \to \infty.$$

Thus $\{u_k\}_{k=1}^{\infty}$ converges uniformly on $\overline{\mathscr{E}}$ to a continuous function $u : \overline{\mathscr{E}} \to \mathbb{R}$.

By the Corollary to the Mean Value Property, u is harmonic on $\mathscr{E}$. Moreover, for $x \in \partial\mathscr{E}$, $u(x) = \lim_{k\to\infty} u_k(x) = \lim_{k\to\infty} f_k(x) = f(x)$. ∎

5. A FINITE DIMENSIONAL DIRICHLET PRINCIPLE. Inspired by Gårding's discussion of the Dirichlet principle, [**8**, pp. 96–98], we present

Another Proof of Theorem 1. Assume $m \ge 2$ and $f \in \mathscr{P}_m$. For $\psi \in \mathscr{P}_m$ and $x \in \mathbb{R}^n$ let $\nabla\psi(x) = (\partial_1\psi(x), \dots, \partial_n\psi(x))$, the *gradient* of ψ at x, and let

$$D(\psi) = \int_{\mathscr{E}} |\nabla\psi(x)|^2 \, dx \text{—the } \textit{Dirichlet integral}.$$

For $\psi, \chi \in \mathscr{P}_m$ define $B(\psi, \chi) = \int_{\mathscr{E}} \nabla\psi(x) \cdot \nabla\chi(x)\, dx$. Notice that B is a symmetric bilinear form on $\mathscr{P}_m$ and $B(\psi, \psi) = D(\psi) \ge 0$ for all $\psi \in \mathscr{P}_m$. Moreover, for $\psi \in \mathscr{P}_m$, $D(\psi) = 0$ if and only if ψ is constant (i.e., $\psi \in \mathscr{P}_0$).

Let $V = \{wb : w \in \mathscr{P}_{m-2}\}$ and $A = \{f + v : v \in V\}$. Note that V is a linear subspace of $\mathscr{P}_m$ and every member of V vanishes on $\partial\mathscr{E}$. Hence, if we let $\langle \psi, \chi \rangle = B(\psi, \chi)$ for $\psi, \chi \in V$, then $\langle \cdot, \cdot \rangle$ is an inner product for V and its associated norm satisfies $\|v\|^2 = B(v, v) = D(v)$ for $v \in$ V.

Observe that A is an affine subspace of $\mathscr{P}_m$ and if $u \in A$ then $u(x) = f(x)$ for all $x \in \partial\mathscr{E}$.

We aim to prove that D has a unique minimizer on A, say u, and this u is a solution to our problem. For any $v \in V$ we have

$$D(f + v) = \int_{\mathscr{E}} |\nabla f(x) + \nabla v(x)|^2 \, dx = D(f) + 2B(f, v) + \|v\|^2.$$

Now the map $v \mapsto B(f, v)$, for $v \in V$, is a linear functional on the finite dimensional vector space V. Hence there is a unique g in V such that $B(f, v) = \langle g, v \rangle$ for all $v \in V$. For $v \in V$ we therefore have

$$D(f + v) = D(f) + 2\langle g, v \rangle + \|v\|^2 = D(f) + \|v + g\|^2 - \|g\|^2$$

and this is clearly least exactly when $v = -g$. Let $u = f - g \in A$ and conclude that $D(u) \le D(\psi)$ for all $\psi \in A$.

Suppose that $0 \neq v \in V$. If $t \in \mathbb{R}$ then $u + tv \in A$ so that $D(u) \leq D(u + tv) = D(u) + 2tB(u, v) + t^2\|v\|^2$, i.e., $0 \leq 2tB(u, v) + t^2\|v\|^2$ for all $t \in \mathbb{R}$. It follows that, for all $v \in V$, $0 = B(u, v) = \sum_{k=1}^{n} \int_{\mathscr{E}} \partial_k u(x) \partial_k v(x)\, dx$. But recall that every member of V vanishes on $\partial\mathscr{E}$. Integration by parts therefore leads us to conclude that $0 = \sum_{k=1}^{n} \int_{\mathscr{E}} (\partial_k^2 u(x)) v(x) dx = \int_{\mathscr{E}} \Delta u(x) v(x)\, dx$ for all $v \in V$. That is,

$$0 = \int_{\mathscr{E}} \Delta u(x) w(x) b(x)\, dx \quad \text{for every } w \in \mathscr{P}_{m-2}. \qquad (*)$$

Now $\Delta u \in \mathscr{P}_{m-2}$ and the map $(\psi, \chi) \mapsto \int_{\mathscr{E}} \psi(x)\chi(x)b(x)\, dx$ $(\psi, \chi \in \mathscr{P}_{m-2})$ is clearly an inner product for $\mathscr{P}_{m-2}$. Hence, $(*)$ ensures that $\Delta u = 0$. Since $u \in A$, $u(x) = f(x)$ for $x \in \partial\mathscr{E}$ and uniqueness is guaranteed by Corollary 2 of the Maximum Principle. ■

6. REMARKS

(i) Theorem 1 was proved in yet another nonstandard way in [**5**, Théorme 6, p. 60].

(ii) Suppose that $g \in \mathscr{P}$ and we are interested in solving the Dirichlet problem for *Poisson's equation*: $\Delta u(x) = g(x)$, $x \in \mathscr{E}$, and $u(x) = f(x)$, $x \in \partial\mathscr{E}$ where $f: \partial\mathscr{E} \to \mathbb{R}$ is a given continuous function. By Fischer's Lemma there exists a v in $\mathscr{P}$ such that $g = \Delta(vb)$. By Theorem 2, there exists a continuous $w: \overline{\mathscr{E}} \to \mathbb{R}$ such that w is C^2 on $\mathscr{E}$, $\Delta w(x) = 0$, for $x \in \mathscr{E}$, and $w(x) = f(x)$ for $x \in \partial\mathscr{E}$. Let $u(x) = w(x) + v(x)b(x)$ for $x \in \overline{\mathscr{E}}$. Then u is continuous on $\overline{\mathscr{E}}$ and C^2 on $\mathscr{E}$, $\Delta u(x) = \Delta w(x) + \Delta(vb)(x) = g(x)$ for $x \in \mathscr{E}$, and $u(x) = w(x) = f(x)$ for $x \in \partial\mathscr{E}$.

(iii) Theorems 1 and 2 apply to arbitrary ellipsoids and not just the "canonical" types considered so far. To see this it suffices to note that every ellipsoid is isometric to one of the kind we've considered and to check that the class of harmonic function is invariant under isometries; [**1**, pp. 2–3].

(iv) In place of the Laplacian one could substitute an operator of the form $\sum_{k=1}^{n} \lambda_k \partial_k^2$ with positive real λ_k.

(v) Extensions of Fischer's ideas and applications thereof to differential problems have been given by several authors; see [**10**], [**12**], and the references included therein.

(vi) Additional intriguing properties of harmonic polynomials, together with applications thereof to boundary value problems for B, can be found in the charming paper of Axler and Ramey [**2**].

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JOHN BAKER, since taking early retirement in 1996 and becoming (undistinguished) Professor Emeritus, has continued to engage in research and expository writing. He continues to occupy the bucket seats of classical analysis and functional equations at the University of Waterloo. A major change in his nonacademic activities involves a refocusing of his curmudgeonly writings from university politics to the local variety. In all likelihood, as you read this he'll still be involved in a battle to keep a box store development from desecrating the rural community in which he lives.
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Said a mathematician named Haar,
"Von Neumann can't see very far.
He missed a great treasure:
They call it Haar measure.
Poor Johnny's just not up to par."

Contributed by Paul R. Chernoff, UC Berkeley, who provides the following background: John von Neumann had proved the existence of invariant measures on compact topological groups, but tried to discourage Alfred Haar from working on the locally compact case on the grounds that it seemed unlikely to be true in that generality. Fortunately, Haar persisted.