Curves Whose Curvature Depends on Distance From the Origin

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1. INTRODUCTION. The fundamental existence and uniqueness theorem for curves in Euclidean space $\mathbb{R}^3$ states that a curve is uniquely determined up to rigid motion by its curvature and torsion, given as functions of its arc-length. Furthermore, given continuous functions $\kappa(s)$ and $\tau(s)$, with $\kappa(s)$ positive and continuously differentiable, there is a differentiable curve (of class at least C^3) with curvature κ and torsion τ.

In practice, such curves are often impossible to find explicitly, due to the difficulty in solving the *Frenet Equations*, the linear differential equations governing the curve; but see [**2**] for an example where the equations can be solved. In general, a result of Lie and Darboux shows that solving these equations is equivalent to solving a certain complex Riccati equation; see [**3**, p. 36] for details. Happily, in the planar case the Frenet equations can always be integrated by quadratures.

We consider a different sort of problem. Suppose the curvature of a proposed curve in the plane is given in terms of its *position*. Can the curve be determined, and if so, how? The general form of this problem requires one to solve a *nonlinear* differential equation:

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{\frac{3}{2}}} = \kappa(x(t), y(t)) \tag{1.1}$$

An interesting solved example of this problem occurs when the curvature is proportional to one of the coordinate functions, say $\kappa(x, y) = cy$. This is a remarkable property of the *Euler elasticae*, curves that minimize $\int \kappa(s)^2\, ds$ among curves of fixed length with fixed first-order boundary conditions. These are the "natural splines," formed by taking a thin inextensible wire of uniform thickness and pinning and "welding" the two ends in fixed positions; see [**4**].

Among the Euler elasticae there is a unique *closed* curve, which is in the shape of a figure-eight. The curvature of this curve is given by the elliptic function $\mathrm{sn}(u, p)$, where u is proportional to the arc-length parameter and $p = 0.9089086\ldots$, the elliptic modulus, satisfies the transcendental equation $2E(p) = K(p)$. Here K and E are the complete elliptic integrals of the first and second kind. The curvature vanishes at the crossing point, and the x-axis bisects the figure, with one loop above and the other below the axis.

2. A FIRST ATTEMPT. The curvature of a planar curve is most simply defined using a coordinate system that moves with the curve. Let T be the unit tangent vector to the curve $X(s)$, and let N be the vector orthogonal to T such that the

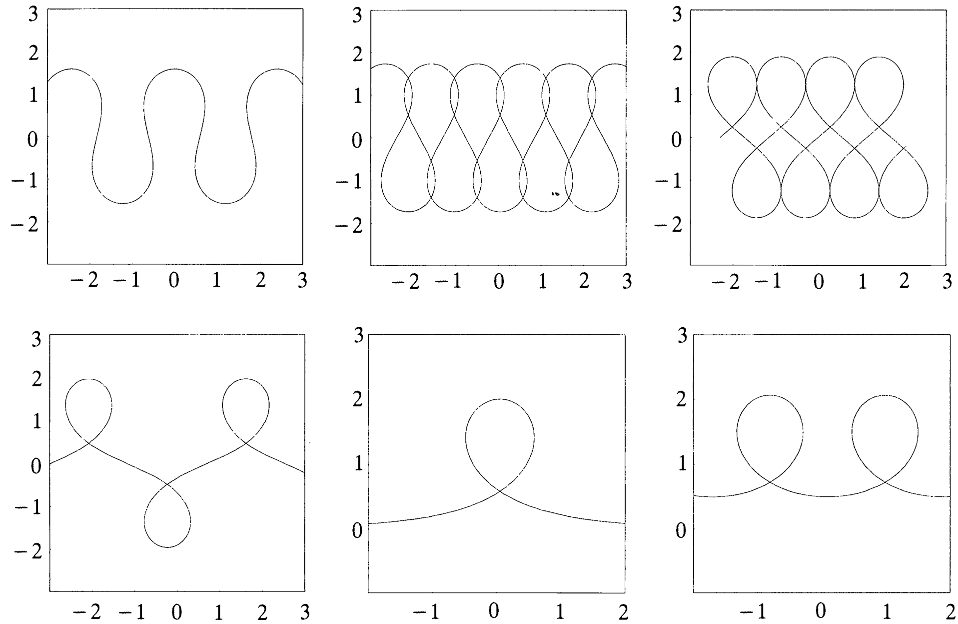

Figure 1. Some Euler elasticae.

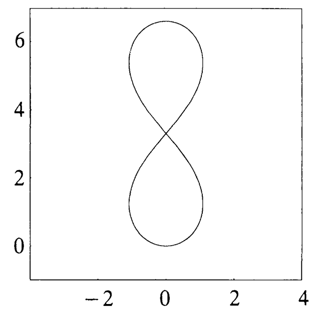

Figure 2. The figure-eight Euler elastica.

frame T, N is positively oriented. If s is the arc-length parameter, then $dX/ds = T$, and the Frenet equations are

$$\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T. \tag{2.1}$$

If we define the angle θ by writing the Euclidean coordinates of T as $(\cos\theta, \sin\theta)$, then we can rewrite (2.1) as

$$\frac{d\theta}{ds} = \kappa(s), \quad \frac{dx}{ds} = \cos\theta, \quad \text{and} \quad \frac{dy}{ds} = \sin\theta. \tag{2.2}$$

Equations (2.2) show that once κ has been determined, the curve can be found by three quadratures. They are also very useful in computing numerical and graphical solutions to the Frenet equations in the plane.

The condition we seek to have the curve satisfy is $\|X\| = \kappa$. A direct assault on (2.2) after substituting $\kappa = \sqrt{x^2 + y^2}$ is unpromising. A more straightforward approach might be to write $X(s) = (\kappa(s)\cos\phi, \kappa(s)\sin\phi)$ in polar coordinates. This leads to a pair of coupled second-order equations, namely

$$\frac{d^2\kappa}{ds^2} = \kappa\left(\frac{d\phi}{ds}\right)^2 - \kappa^2\frac{d\phi}{ds}, \quad 2\frac{d\kappa}{ds}\frac{d\phi}{ds} + \kappa\frac{d^2\phi}{ds^2} = \kappa\frac{d\kappa}{ds}, \tag{2.3}$$

which can be reduced by use of the first integral

$$\left(\frac{d\kappa}{ds}\right)^2 + \kappa^2\left(\frac{d\phi}{ds}\right)^2 = 1 \tag{2.4}$$

to a (somewhat unpleasant) second-order differential equation for κ. Once this is solved the curve can be determined by quadratures.

A slightly more devious approach is to define functions $\alpha(s)$ and $\beta(s)$ by the formula

$$X = \alpha T + \beta N \tag{2.5}$$

in order to parametrize the curve by coordinates that move with the curve. Differentiating (2.5) gives

$$\frac{d\alpha}{ds} = \beta\kappa + 1, \quad \frac{d\beta}{ds} = -\alpha\kappa. \tag{2.6}$$

The condition $\|X\| = \kappa$ in these coordinates is

$$\alpha^2 + \beta^2 = \kappa^2, \tag{2.7}$$

so we may define an angle ψ by $\alpha = \kappa\cos\psi$, $\beta = \kappa\sin\psi$. The equations (2.6) now lead to a pair of first order equations:

$$\frac{d\kappa}{ds} = \cos\psi, \quad \kappa\frac{d\psi}{ds} + \kappa^2 = -\sin\psi. \tag{2.8}$$

Unfortunately, it is again not clear how to solve the differential equations and determine κ.

3. MOTION UNDER GRAVITATIONAL FORCE. Although it is not always possible to find explicit solutions to a system of second-order differential equations, this can be done by quadratures for completely integrable Hamiltonian systems. Thus, we now attempt to reformulate our problem as a Hamiltonian system with sufficient symmetry to be integrable.

A fundamental example of such a system arises from the problem of an orbiting object. Newton's equations of motion are $\vec{F}(X) = mX''$, where X is the position of the object. Suppose the object is moving through space under the influence of a central force $\vec{F}(X) = -f(r)X$, where $r = \|X\|$ is the distance from the origin and $f(r)$ is some continuous function. The motion satisfies the second order differential equations

$$X'' = -\frac{1}{m}f(r)X, \tag{3.1}$$

and the motion satisfies Kepler's Second Law (conservation of angular momentum): $X \times X' = C$, a constant vector, as can easily be seen by differentiating. The motion lies in a plane, which we assume is the (x, y)-plane, and if we put the curve in polar coordinates $X = (r\cos\phi, r\sin\phi)$, then

$$r^2\frac{d\phi}{dt} = \|X \times X'\| = c, \tag{3.2}$$

where $c = \|C\|$ is a constant (the angular momentum).

Now define a function $\Phi(r)$ by

$$\frac{d\Phi}{dr} = rf(r).$$

Then $\vec{F}(X) = -\nabla V$, where $V(x, y, z) = \Phi(\sqrt{x^2 + y^2 + z^2})$. The motion satisfies conservation of energy:

$$\frac{1}{2}m\|X'\|^2 + V = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) + \Phi(r) = E_0, \tag{3.3}$$

where E_0 is a constant (the total energy).

Using (3.2) and (3.3), it is possible to solve the equations of motion by quadrature. In the special case where $\Phi(r) = -1/r$, the force satisfies the inverse square law and we have the Kepler problem. The solutions in that case are ellipses, parabolas, and hyperbolas.

Now let us compute the curvature of solutions. Since the parametrization is no longer by arc length, we need the general formula for the curvature of a curve:

$$\kappa(t) = \frac{\|X' \times X''\|}{\|X'\|^3}. \tag{3.4}$$

Plugging in (3.1) and (3.2), this equation becomes

$$\kappa(t) = \frac{cf(r)}{m\|X'\|^3}. \tag{3.5}$$

Now using (3.3) to compute the denominator, we arrive at the formula

$$\kappa(t) = \frac{c\sqrt{m}\Phi'(r)}{\sqrt{8}r(E_0 - \Phi(r))^{\frac{3}{2}}}. \tag{3.6}$$

Note that the value of κ depends only on r. Define $\mu(r) = (E_0 - \Phi(r))^{-\frac{1}{2}}$. Then (3.6) reduces to

$$\frac{d\mu}{dr} = \frac{\sqrt{2}}{c\sqrt{m}}r\kappa(r). \tag{3.7}$$

If $\kappa(r)$ is any function such that $r\kappa(r)$ is continuous, we can solve (3.7) for μ. Define $\Phi(r) = -1/\mu(r)^2$, and consider solutions to (3.1) with energy $E_0 = 0$. Note that a specific choice of μ also specifies a value of the momentum c_0. That is, a solution to (3.1) has curvature *proportional* to the given function provided it has energy 0; its curvature is equal to the given function if its angular momentum $c = c_0$. This is part of the proof of our main result:

Theorem 3.1. *Let $\kappa(r)$ be a function such that $r\kappa(r)$ is continuous. Then the problem of determining a curve whose curvature is $\kappa(r)$, where r is the distance from the origin, is solvable by quadratures.*

4. THE CASE $\kappa(r) = r$. Using (2.2), it is not difficult to produce pictures of curves whose curvature is equal to the distance from the origin. Some of them are illustrated in Figure 3. We want to find analytic representations for these curves.

Theorem 3.1 is slightly deceptive. The fact that the differential equation is integrable by quadratures does not mean that it is easy to perform the integrations, as we now illustrate with the case that inspired this paper: $\kappa(r) = r$. The first step,

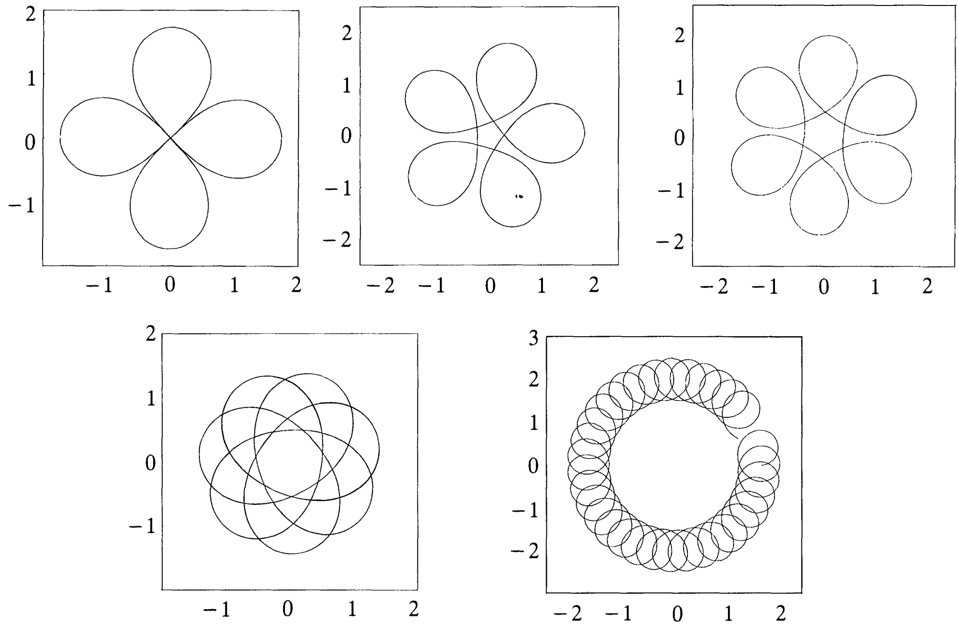

Figure 3. Curves with curvature equal to radial distance.

solving (3.7) is easy and gives

$$\mu(r) = \frac{\sqrt{2}}{3c_0\sqrt{m}}(r^3 + a), \quad \Phi(r) = -\frac{9c_0^2 m}{2(r^3 + a)^2}, \tag{4.1}$$

where a is a constant of integration. This corresponds to a force law in which the magnitude is proportional to $r^2/(r^3 + a)^3$.

Applying (3.3) and assuming that $r^2\phi' = c_0$, we obtain

$$(r')^2 + r^2(\phi')^2 = \frac{9c_0^2}{(r^3 + a)^2} = \frac{9r^4(\phi')^2}{(r^3 + a)^2}. \tag{4.2}$$

This has among its solutions the circular orbit $X(t) = (r_0 \cos(\alpha t), r_0 \sin(\alpha t))$, where $\alpha = 3c_0/r_0(r_0^3 + a)$ and $r_0^2\alpha = c_0$. This implies $3r_0 = r_0^3 + a$. However, X is a solution to (3.1) only when $\alpha^2 = 27c_0^2 r_0/(r_0^3 + a)^3$. This implies $r_0 = 1$, which of course we knew! Other solutions can have $r' = 0$ only at isolated points.

We can eliminate t and find $\phi = \phi(r)$ from

$$\phi = \pm \int \frac{r^3 + a}{r\sqrt{9r^2 - (r^3 + a)^2}}\, dr. \tag{4.3}$$

This is not always an elementary integral. One special case, however, is very pleasant, namely the case where $a = 0$. Then the integral becomes elementary, and the resulting curve is given by

$$\phi - \phi_0 = \frac{1}{2}\arcsin\left(\frac{r^2}{3}\right), \quad r^2 = 3\sin 2(\phi - \phi_0). \tag{4.4}$$

This curve is none other than (one leaf of) the *Bernoulli lemniscate* shown in Figure 4.

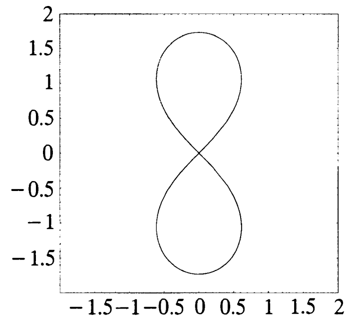

Figure 4. The Bernoulli lemniscate.

From this example it becomes clear why the previous, more straightforward, attempts led to such complicated equations. The solution to the original equations would have given the lemniscate parametrized by arc length. However, the arc length function for the lemniscate is an elliptic integral. Thus even this special case would be much more complicated to find. In fact, we have not yet solved (3.1) explicitly, even in the special case; we found instead a non-parametric representation of the trajectory. This is not difficult to do, however, as we observe in the next section.

5. SUB-AFFINE ARC LENGTH. Solutions to equations of motion for particles moving under the influence of a central force have a natural parametrization. From the equation $X'' = g(x, y)X$ we have observed that $\|X \times X'\|$ is constant. Now let $X(t)$ be a curve satisfying $X \times X' \neq \vec{0}$. Then there is a re-parametrization of the curve such that with respect to the new parameter $\sigma = \sigma(t)$ the curve satisfies

$$\left\| X(\sigma) \times \frac{dX}{d\sigma} \right\| \equiv 1. \tag{5.1}$$

This parameter is computed by

$$\sigma(t) = \int_0^t \|X \times X'\| \, dt. \tag{5.2}$$

In the case of the lemniscate $r^2 = 3\cos(2\theta)$, evaluation of (5.2) is an elementary calculation and yields $\sigma = \frac{3}{2}\sin(2\theta)$.

The parameter σ may be called the *sub-affine arc length parameter*. The motivation for this name is the following: If $Y(t)$ is a curve in the plane such that the vectors $Y'(t)$ and $Y''(t)$ are linearly independent, then we may re-parametrize the curve by a parameter σ such that with respect to this new parametrization $\|Y' \times Y''\| \equiv 1$. This parametrization is known as the *affine arc length*; see [**1**, p. 149]. It plays a role in affine geometry exactly analogous to the usual arc length parameter in Euclidean geometry. If a curve is parametrized by affine arc-length, then $Y''' + \rho Y' = 0$, where $\rho(\sigma)$ is the *affine curvature* of the curve Y; it is a geometric invariant of the curve under unimodular affine transformations of the plane.

If Y is a curve parametrized by affine arc length, then its derivative $X(\sigma) = dY/d\sigma$ is parametrized by subaffine arc length. We call $\rho(\sigma) = -d^2X/d\sigma^2$ the *subaffine curvature* of X. This is a geometric invariant of X under unimodular linear transformations of R^2.

Our problem was to find curves whose curvature is equal to the distance from the origin. Let γ be such a curve. If $\gamma(t) \times \gamma'(t) \neq 0$, then we can locally parametrize γ by subaffine arc length. Then

$$\frac{d^2\gamma}{d\sigma^2} = -\rho(\sigma)\gamma(\sigma). \tag{5.3}$$

The Euclidean curvature is given by the formula $\kappa(\gamma(\sigma)) = \rho(\sigma)/\|\gamma'(\sigma)\|^3$.

If we now assume that the radial distance from the origin varies monotonically on some interval $\sigma_0 \leq \sigma \leq \sigma_1$, we can solve for σ in terms of r and define $f(r) = \rho(\sigma(r))$. Thus, on any part of a curve along which $X \times X'$ does not vanish and $X \cdot X'$ does not vanish, the curve arises as a solution to an equation of the form (3.1).

Local extrema of r do not present any difficulty, since it is evident that the curve has a symmetry at each such point. Places where $X \times X' = 0$, however, represent limits of trajectories. For example, in the example of the lemniscate the origin is a singularity of the orbit. Note, however, that these are of necessity isolated points on the curve. This observation completes the proof of the theorem.

It is interesting to note that the graphical solution corresponding to the lemniscate actually produces *two* lemniscates at right angles to each other. Indeed, the solution to (2.2) with initial conditions $x = 0, y = 0$, θ arbitrary, sweeps out alternate halves of the two lemniscates, producing a flower with four loops. Note that the equation we have solved is for the *signed* curvature equaling the radial distance. Thus κ is required to remain non-negative. The curvature of the lemniscate changes sign as it passes through the origin.

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