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The Fifty-Ninth William Lowell Putnam Mathematical Competition

Leonard F. Klosinski, Gerald L. Alexanderson, and Loren C. Larson

The results of the Fifty-ninth William Lowell Putnam Mathematical Competition held December 5, 1998, follow. They have been determined in accordance with the regulations governing the competition, a contest supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, a fund endowed by Mrs. Putnam in memory of her husband. The annual Competition is held under the auspices of the Mathematical Association of America.

The first prize, $25,000, was awarded to the Department of Mathematics at Harvard University. The members of the winning team were Michael L. Develin, Ciprian Manolescu, and Dragos N. Oprea; each was awarded a prize of $1000.

The second prize, $20,000, was awarded to the Department of Mathematics at the Massachusetts Institute of Technology. The members of the winning team were Amit Khetan, Eric H. Kuo, and Edward D. Lee; each was awarded a prize of $800.

The third prize, $15,000, was awarded to the Department of Mathematics at Princeton University. The members of the winning team were Craig R. Helfgott, Michael R. Korn, and Yuliy V. Sannikov; each was awarded a prize of $600.

The fourth prize, $10,000, was awarded to the Department of Mathematics at the California Institute of Technology. The members of the winning team were Christopher C. Chang, Christopher M. Hirata, and Hanhui Yuan; each was awarded a prize of $400.

The fifth prize, $5,000, was awarded to the Department of Mathematics at the University of Waterloo. The members of the winning team were Sabin Cautis, Derek I. E. Kisman, and Soroosh Yazdani; each was awarded a prize of $200.

The five highest ranking individual contestants, in alphabetical order, were Nathan G. Curtis, Duke University; Michael L. Develin, Harvard University; Kevin D. Lacker, Duke University; Ciprian Manolescu, Harvard University; and Ari M. Turner, Princeton University. Each of these has been designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of $2,500 by the Putnam Prize Fund.

The next ten highest ranking contestants, in alphabetical order, were: Sabin Cautis, University of Waterloo; Christopher C. Chang, California Institute of Technology; Donny C. Cheung, University of Waterloo; Dragos F. Ghioca, University of Pittsburgh; Andrei C. Gnepp, Harvard University; Michael Ostrovsky, Stanford University; Colin A. Percival, Simon Fraser University; Alexander H. Saltman, Harvard University; Daniel A. Stronger, Harvard University; and Liang Yang, Yale University; each was awarded a prize of $1,000.

The next eleven highest ranking contestants, in alphabetical order, were: Ian W. Caines, Dalhousie University; Scott H. Carnahan, California Institute of Technology; Adrian D. Corduneanu, University of Toronto; Matthew T. Gealy, University of Chicago; Larry D. Guth, Yale University; Amit Khetan, Massachusetts Institute of Technology; Michael R. Korn, Princeton University; Abhinav Kumar, Massachusetts Institute of Technology; Davesh Maulik, Harvard University;

Dragos Nicolae Oprea, Harvard University; and Ronfeng Sun, Clark University; each was awarded a prize of $250.

The following teams, named in alphabetical order, received honorable mention: University of Chicago, with team members Benjamin M. Cowan, Matthew T. Gealy, and Christopher D. Malon; Duke University, with team members Nathan G. Curtis, Andrew O. Dittmer, and Carl A. Miller; The Johns Hopkins University, with team members Alexander J. Diesl, Rakesh M. Lal, and Nehal S. Munshi; Stanford University, with team members Eugene G. Davydov, Alexander S. Dugas, and Michael Ostrovsky; and University of Toronto, with team members Cyrus Chen Hsia, Bhaskara M. Marthi, and Ryan O'Donnell.

Honorable mention was achieved by the following thirty-three individuals named in alphabetical order: Chetan Tukaram Balwe, University of Michigan, Ann Arbor; Adrian Birka, Massachusetts Institute of Technology; Dmitriy S. Boyarchenko, University of Pennsylvania; Li-Chung Chen, Harvard University; Constantin S. Chiscanu, Massachusetts Institute of Technology; John J. Clyde, Duke University; Shai M. Cohen, University of Toronto; Benjamin M. Cowan, University of Chicago; Samit Dasgupta, Harvard University; Kenneth K. Easwaran, Stanford University; Frederik H. Eaton, California Institute of Technology; Brad A. Friedman, University of Illinois, Urbana-Champaign; Craig R. Helfgott, Princeton University; Christopher M. Hirata, California Institute of Technology; Cyrus Chen Hsia, University of Toronto; Kai Huang, Massachusetts Institute of Technology; Liviu Ignat, University of Pittsburgh; Miro Jurisic, Massachusetts Institute of Technology; Scott P. Kempen, Marquette University; Derik I. E. Kisman, University of Waterloo; Eric H. Kuo, Massachusetts Institute of Technology; Chin-Aik Lee, Lebanon Valley College; Edward D. Lee, Massachusetts Institute of Technology; Christopher C. Mihelich, Harvard University; Carl A. Miller, Duke University; Mintcho P. Petkov, Dartmouth College; Dmitry L. Sagalovskiy, Harvard University; Yuliy V. Sannikov, Princeton University; Jan K. Siwanowicz, City College of New York; Mark J. Tilford, California Institute of Technology; Ian W. T. Vander Burgh, University of Waterloo; Cristian Voicu, St. Lawrence University; and Hoeteck Wee, Massachusetts Institute of Technology.

The other individuals who achieved ranks among the top 103, in alphabetical, order of their schools were: University of British Columbia, Lawrence Tang; California Institute of Technology, Ryan L. McCorvie, Michael A. Shulman, Hanhui Yuan, and Kaiwen Xu; Case Western Reserve University, Andrew D. Frohmader; University of Central Florida, Daniel E. Moraseski; University of Chicago, Charles D. Cadmon, Christopher D. Malon, and Sergey Vassiliev; University of Colorado, Boulder, Tao He; Duke University, Andrew O. Dittmer; Georgia Institute of Technology, Jeffrey M. Fowler; University of Georgia, Charles Rollin Mathis; Harvard University, Lukasz Fidkowski, Robert Ribciuc, and Joshua S. Vonkorff; Harvey Mudd College, Ranjithkumar Rajagopalan; The Johns Hopkins University, Alexander J. Diesl; Massachusetts Institute of Technology, Michael L. Brasher, Pokman Cheung, Shamik Das, Ashish Mishra, Brent J. Yen, and Boris Zbarsky; McGill University, Alexandru E. Ghitza; University of Michigan, Ann Arbor, Kurt A. Steinkraus; University of Nebraska, Lincoln, Travis W. Fisher; New York University, Ioana Dumitriu; Northeastern University, Ivo K. Nikolov; University of Pennsylvania, David Futer; Princeton University, Todd W. Geldon; Rice University, Brian David Rothbach; Simon's Rock College, Robert J. Young; University of South Carolina, Jason M. Burns; Stanford University, Eugene V. Davydov, Alex S. Dugas, and Siutaur Pang; University of Toronto, Bhaskara M.

Marthi, and Ryan W. O'Donnell; University of Vermont, Laura P. Riccio; Washington University, St. Louis, Dan B. Johnston; and University of Waterloo, Joel Kamnitzer, and Soroosh Yazdani.

There were 2,581 individual contestants from 419 colleges and universities in Canada and the United States in the Competition of December 5, 1998. Teams were entered by 319 institutions. The Questions Committee for the fifty-ninth Competition consisted of Michael J. Larsen (Chair), University of Pennsylvania; Steven G. Krantz, Washington University, St. Louis; and David J. Wright, Oklahoma State University; they composed the problems and were most prominent among those suggesting solutions. Alternate solutions to some problems have been published in *Mathematics Magazine*.

THE 59th ANNUAL WILLIAM LOWELL PUTNAM EXAMINATION

A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

A3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that $f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0$.

A4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2 A_1 = 10$, $A_4 = A_3 A_2 = 101$, $A_5 = A_4 A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

A5. Let $\mathcal{F}$ be a finite collection of open discs in $\mathbf{R}^2$ whose union contains a set $E \subseteq \mathbf{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supseteq E.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

A6. Let A, B, C denote distinct points with integer coordinates in $\mathbf{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

B1. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

B2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

B3. Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, and β are real numbers.

B4. Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

B5. Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = \underbrace{1111 \cdots 11}_{1998 \text{ digits}}.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

B6. Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

SOLUTIONS. In the 12-tuples $(n_{10}, n_9, n_8, n_7, n_6, n_5, n_4, n_3, n_2, n_1, n_0, n_{-1})$ following each problem number, n_i for $10 \geq i \geq 0$ is the number of students among the top 199 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A1. (156, 23, 4, 0, 0, 0, 0, 0, 0, 0, 16, 0)

Solution. Consider a plane cross-section through a vertex of the cube and the axis of the cone shown in Figure 1. Let s be the side of the cube. Segment DE is a

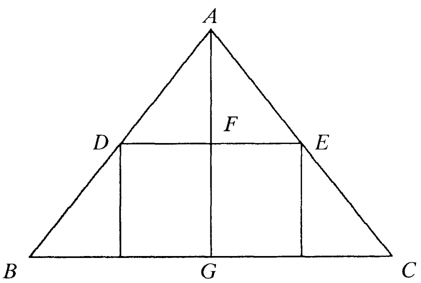

Figure 1.

diagonal of the top of the cube and has length $\sqrt{2}s$. By similarity of triangles ADE and ABC, we have $DE/BC = AF/AG$, or $\sqrt{2}s/2 = (3 - s)/3$. Hence

$$s = \frac{6}{3\sqrt{2} + 2} = \frac{3\sqrt{2}}{\sqrt{2} + 3} = \frac{9\sqrt{2} - 6}{7}.$$

A2. (103, 35, 26, 0, 0, 0, 0, 0, 9, 8, 12, 6)

Solution. Suppose the beginning and ending angles of s are α and β, respectively, where angles are measured counterclockwise from the positive horizontal axis.

$$A = \int_{\cos \beta}^{\cos \alpha} \sqrt{1 - x^2}\, dx = \int_{\beta}^{\alpha} \sqrt{1 - \cos^2 u}\, (-\sin u)\, du = \int_{\alpha}^{\beta} \sin^2 u\, du.$$

Similarly

$$B = \int_{\sin\alpha}^{\sin\beta} \sqrt{1-y^2}\, dy = \int_{\alpha}^{\beta} \cos^2 u\, du.$$

Then

$$A + B = \int_{\alpha}^{\beta} (\sin^2 u + \cos^2 u)\, du = \int_{\alpha}^{\beta} du = \beta - \alpha.$$

A3. (82, 34, 2, 0, 0, 0, 0, 0, 5, 0, 39, 37)

Solution. Assume $f(x)f'(x)f''(x)f'''(x) < 0$ for all x. By continuity, each of f, f', f'', f''' has constant sign. By replacing $f(x)$ by $f(-x)$, $-f(x)$, or $-f(-x)$, if necessary, we may assume that $f(x) > 0$ and $f'(x) > 0$ for all x. In either case, choosing $g = f$ or f' as necessary, we have a function g such that $g(x) > 0$, $g'(x) > 0$, and $g''(x) < 0$ for all x. Since g is strictly increasing and bounded below, there is a constant $C > 0$ such that $g(x) \to C^+$ as $x \to -\infty$. This horizontal asymptote forces g to be concave upwards at some point x, in contradiction to our assumption $g''(x) > 0$.

To be precise, fix x_1, and let $m = g'(x_1) > 0$. Since $\lim_{x_2 \to -\infty}(g(x_1) - g(x_2))/(x_1 - x_2) = 0$, there is an x_2, $x_2 < x_1$, such that $0 < (g(x_1) - g(x_2))/(x_1 - x_2) < m/2$. By the Mean Value Theorem, there is an x_3 strictly between x_2 and x_1 with $g'(x_3) = (g(x_1) - g(x_2))/(x_1 - x_2) < m/2$. Then there is an x_4 between x_3 and x_1 for which

$$g''(x_4) = \frac{g'(x_1) - g'(x_3)}{x_1 - x_3} > \frac{m - \frac{m}{2}}{x_1 - x_3} > 0.$$

A4. (39, 27, 52, 0, 0, 0, 0, 0, 49, 7, 14, 11)

Solution. A_n is divisible by 11 precisely when n is one more than a multiple of 6.

We know that a number is divisible by 11 if and only if the alternating sum of its digits is zero. So, let d_n be the alternating sum of the digits in A_n starting from the left. Then $d_n = d_{n-1} + (-1)^{r_{n-1}} d_{n-2}$, where r_n is the number of digits in A_n. Clearly, $r_n = F_n$ the nth Fibonacci number. The Fibonacci numbers have parity pattern $((-1)^{F_n})$: $-1, -1, 1, -1, -1, 1, \ldots$. Thus, the rules for calculating d_n are $d_1 = 0$, $d_2 = 1$, $d_3 = d_2 - d_1 = 1$, $d_4 = d_3 + d_2 = 2$, $d_5 = d_4 - d_3 = 1$, $d_6 = d_5 - d_4 = -1$, $d_7 = d_6 + d_5 = 0$, $d_8 = d_7 - d_6 = 1$, and so forth in this repeating pattern.

A5. (85, 24, 15, 0, 0, 0, 0, 0, 5, 2, 8, 60)

Solution. Let D_1 be a disc of greatest radius. Select D_2 to be a disc of greatest radius that is disjoint from D_1. Proceeding inductively, suppose that $D_1, \ldots, D_{j-1}$ have been chosen. Select D_j to be a disc of greatest radius that is disjoint from $D_1, \ldots, D_{j-1}$.

The process eventually stops, producing a pairwise disjoint collection of discs, $D_1, \ldots, D_n$. We claim that $\bigcup_{j=1}^{n} 3D_j \supseteq E$. To prove this assertion, it suffices to see that $\bigcup_{j=1}^{n} 3D_j \supseteq \bigcup_{D \in \mathscr{F}} D$.

Let $D \in \mathscr{F}$. If D is one of the selected discs D_j then the assertion is trivial. If D is not one of the selected discs, then let D_m be the first disc in the sequence that intersects D. Then, by the way we chose the D_j, the radius of D_m is at least as great as the radius of D. But the triangle inequality implies that $3D_m \supseteq D$.

A6. (12, 1, 3, 0, 0, 0, 0, 0, 3, 9, 92, 79)

Solution. The set $\mathbf{Z}^2$ is preserved by 90° rotation about any of its points. Therefore, there exists a point C' in $\mathbf{Z}^2$ with $|BC'| = |BA|$, BC' perpendicular to BA such that C and C' belong to the same half-plane with respect to the line through A and B. If $C \neq C'$, then $|CC'| = s \geq 1$. Let $r = |AB|$. By rotating and translating the coordinate system, we may assume that $B = (0,0)$, $A = (r,0)$, $C' = (0,r)$, and $C = (0,r) + s(\cos\theta, \sin\theta)$, where θ is the angle to the line $C'C$ measured from the positive horizontal axis. Thus

$$\begin{aligned}(|AB| + |BC|)^2 &= \left(r + \sqrt{s^2\cos^2\theta + (r + s\sin\theta)^2}\right)^2 \\ &= 2r^2 + s^2 + 2rs\sin\theta + 2r\sqrt{s^2\cos^2\theta + (r+s\sin\theta)^2} \\ &\geq 2r^2 + 1 + 2rs\sin\theta + 2r(r + s\sin\theta) \\ &= 8\frac{r(r+s\sin\theta)}{2} + 1 = 8[ABC] + 1.\end{aligned}$$

B1. (112, 30, 30, 0, 0, 0, 0, 0, 3, 3, 12, 9)

Solution 1. Let $z = x + 1/x$. Then

$$\begin{aligned}\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)} &= \frac{z^6 - z^2(z^2-3)}{z^3 + z(z^2-3)} \\ &= z^3 - z(z^2 - 3) = 3z = 3(x + 1/x).\end{aligned}$$

Observe that $x + 1/x - 2 = (\sqrt{x} - 1/\sqrt{x})^2 \geq 0$, with equality if and only if $x = 1$, so that the minimum of our function is 6.

Solution 2. By direct expansion followed by long-division of polynomials,

$$\begin{aligned}\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)} &= \frac{3}{x}\left(\frac{2x^8 + 5x^6 + 6x^4 + 5x^2 + 2}{2x^6 + 3x^4 + 3x^2 + 2}\right) \\ &= \frac{3}{x}(x^2 + 1) = 3(x + 1/x).\end{aligned}$$

By the arithmetic-geometric mean inequality, $x + 1/x \geq 2\sqrt{x(1/x)} = 2$, with equality if and only if $x = 1$. It follows that the minimum value is 6 and occurs when $x = 1$.

B2. (82, 10, 9, 0, 0, 0, 0, 0, 3, 11, 38, 46)

Solution. With two reflections, one about the x-axis and the other about $y = x$, the shortest perimeter for such a triangle is seen to equal the distance between the points $(a, -b)$ and (b, a). This distance is $\sqrt{2(a^2 + b^2)}$.

B3. (44, 4, 3, 0, 0, 0, 0, 0, 7, 2, 62, 77)

Solution. Observe that twice the desired surface area equals the surface area of the whole sphere minus 5 spherical caps subtended by a "chord" with central angle

$2\pi/5$. The surface area of a spherical cap of angle α can be computed in spherical coordinates by

$$\int_0^{\alpha/2}\int_0^{2\pi}\sin\phi\, d\theta\, d\phi = 2\pi(1-\cos(\alpha/2)).$$

Thus, the desired area is

$$\tfrac{1}{2}(4\pi - 5(2\pi(1-\cos(\pi/5)))) = 5\pi\cos(\pi/5) - 3\pi.$$

That is, $A = -3\pi$, $B = 5\pi$, $\alpha = \pi/2$, $\beta = \pi/5$.

B4. (42, 9, 22, 0, 0, 0, 0, 0, 21, 28, 24, 53)

Solution. It is necessary and sufficient that $mn/(m,n)^2$ is even, where (m,n) denotes the greatest common divisor. Indeed, if $m = ac$ and $n = bc$, then $\lfloor i/ac \rfloor + \lfloor i/bc \rfloor$ depends only on $\lfloor i/c \rfloor$, so

$$\sum_{i=0}^{abc^2-1}(-1)^{\left\lfloor \frac{i}{ac}\right\rfloor + \left\lfloor \frac{i}{bc}\right\rfloor} = c\sum_{i=0}^{abc-1}(-1)^{\left\lfloor \frac{i}{a}\right\rfloor + \left\lfloor \frac{i}{b}\right\rfloor}$$

$$= c\sum_{k=0}^{c-1}(-1)^{k(a+b)}\sum_{i=0}^{ab-1}(-1)^{\left\lfloor \frac{i}{a}\right\rfloor + \left\lfloor \frac{i}{b}\right\rfloor}.$$

If ab is odd then $a+b$ is even, so

$$\sum_{k=0}^{c-1}(-1)^{k(a+b)} = c$$

and the sum of ab terms, each equal to ± 1, is nonzero. Conversely, if $a+b$ is odd, then

$$\left\lfloor \frac{i}{a}\right\rfloor + \left\lfloor \frac{ab-1-i}{a}\right\rfloor + \left\lfloor \frac{i}{b}\right\rfloor + \left\lfloor \frac{ab-1-i}{b}\right\rfloor = b - 1 + a - 1 \equiv 1 \pmod 2.$$

Thus,

$$\sum_{i=0}^{ab-1}(-1)^{\left\lfloor \frac{i}{a}\right\rfloor + \left\lfloor \frac{i}{b}\right\rfloor} = \sum_{i=0}^{ab/2-1}(-1)^{\left\lfloor \frac{i}{a}\right\rfloor + \left\lfloor \frac{i}{b}\right\rfloor} + (-1)^{\left\lfloor \frac{ab-1-i}{a}\right\rfloor + \left\lfloor \frac{ab-1-i}{b}\right\rfloor} = 0.$$

B5. (55, 2, 29, 0, 0, 0, 0, 0, 8, 0, 29, 76)

Solution. We have

$$\sqrt{N} = \sqrt{\frac{10^{1998}-1}{9}} = \frac{10^{999}}{3}(1-10^{-1998})^{1/2}.$$

Taylor's theorem with remainder implies that

$$(1-10^{-1998})^{1/2} = 1 - \frac{10^{-1998}}{2} + \epsilon, \quad \epsilon < \frac{10^{-3996}}{8}.$$

Thus, the first digit after the decimal, point in

$$10^{999}\sqrt{N} = \frac{10^{1998}-1}{3} + \frac{1}{6} + \frac{10^{1998}\epsilon}{3}$$

is 1.

B6. (25, 8, 8, 0, 0, 0, 0, 0, 4, 5, 31, 118)

Solution. Let $n = 4m^2$. Then

$$\begin{aligned} n^3 + an^2 + bn + c &= 64m^6 + 16am^4 + 4bm^2 + c \\ &= (8m^3 + am)^2 + (4b - a^2)m^2 + c. \end{aligned}$$

If $m \gg 0$, $|(4b - a^2)m^2 + c| < 8m^3 + am$, so $n^3 + an^2 + bn + c$ can be a square only if $(4b - a^2)m + c = 0$. This can happen for only one value of m unless $4b - a^2 = c = 0$, and in this case $n^3 + an^2 + bn + c$ is a square if and only if n is square.

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UNDERSTANDING THE METRIC SYSTEM

1 million microphones = 1 megaphone
1 million bicycles = 2 megacycles
2000 mockingbirds = 2 kilomockingbirds
10 cards = 1 decacards
1/2 lavatory = 1 demijohn
1 millionth of a fish = 1 microfiche
453.6 graham crackers = 1 pound cake
10 rations = 1 decoration
10 millipedes = 1 centipede
3-1/3 tridents = 1 decadent
10 monologues = 5 dialogues
2 monograms = 1 diagram
8 nickels = 2 paradigms

Plucked from the Internet by Russ Hood, Rio Linda, CA