# Descartes' Rule of Signs: Another Construction

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Descartes' Rule of Signs is a simple, classical bound on the number of positive roots of a polynomial, and analogously on the number of negative roots. In Descartes' own words, the rule is stated as follows: [**3**]

> An equation can have as many true [positive] roots as it has changes of sign, from $+$ to $-$ or from $-$ to $+$.

This statement is written in terms of sign changes of the coefficients, but the wording is very similar to the Intermediate Value Theorem, which says that a continuous function must have at least one root in an interval if it changes sign in that interval. This suggests a natural construction of the polynomial which achieves the bound by creating a correspondence between sign changes of its coefficients and sign changes of its value at designated points.

Other constructions that achieve the maximum number of roots are known [**2**]. In this MONTHLY, Anderson, Jackson, and Sitharam [**1**] give a different natural construction, which works as long as all signs are nonzero. They construct a polynomial by choosing the roots according to the desired signs; they then show that the coefficients of the polynomial have the correct signs. They also show that this polynomial can be modified to obtain a polynomial with the same sign sequence and any number of positive roots that is a positive even integer less than the maximum.

For our construction, we need only a single polynomial without signs attached to the coefficients; however we specify the signs, the value of the polynomial has the correct signs at the proper places.

**Theorem 1.** *Let $\sigma_0, \dots, \sigma_n$ be any sequence of $-1$, $0$, and $+1$. Then for any $k > n$, the polynomial*

$$p(x) = \sum_{j=0}^{n} \sigma_j k^{-j^2} x^j$$

*has the maximum number of positive and negative roots allowed by Descartes' Rule of Signs.*

*Proof:* If $\sigma_j \neq 0$, then at $x = k^{2i}$, the absolute value of the term of $x^j$ is $k^{2ij-j^2}$. Thus, at $x = k^{2j}$, the absolute value of the term of $x^{2j}$ is $k^{j^2}$ and the absolute values of the other terms are all at most $k^{j^2-1}$. Since there are only $n$ such terms and $k > n$, the term of $x^{2j}$ is larger in absolute value than all the others combined, and thus the sign of $p(k^{2j})$ is the same as $\sigma_j$. Thus, if $\sigma_i$ and $\sigma_j$ are two consecutive opposite signs, the polynomial must have a root between $x = k^{2i}$ and $x = k^{2j}$.

Analogously, for negative roots, the term of $x^{2j}$ is larger in absolute value than all the other terms at $x = -k^{2j}$, and thus if $(-1)^i\sigma_i$ and $(-1)^j\sigma_j$ are two consecutive opposite signs in $p(-x)$, the polynomial must have a root between $x = -k^{2j}$ and $x = -k^{2i}$. ■

Since this theorem is valid for polynomials with zero coefficients, it is also easy to construct a polynomial with fewer roots than the maximum by making some of the coefficients very small and using the remaining signed coefficients to force the roots. In particular, we can make the number of positive roots short of the maximum by any even number.

**Theorem 2.** *Let $\sigma_0, \dots, \sigma_n$ be any sequence of $-1$, $0$, and $+1$; we may assume $\sigma_0 \neq 0$ and $\sigma_n \neq 0$ since the nonzero roots are not affected by eliminating zeros at the ends of the sequence. Let $\tau_0, \dots, \tau_n$ be obtained by changing some of the internal $\sigma_i$ to zero, but keeping $\tau_0$ and $\tau_n$ nonzero. Then there is a polynomial $p(x)$ with sign sequence $\sigma$ that has as many positive roots as there are sign changes in the $\tau$ sequence and as many negative roots as there are sign changes in the $(-1)^i\tau_i$ sequence.*

**Corollary.** *If the $\sigma$ sequence has at least $2r$ sign changes, we can take the $\tau$ sequence by changing to zeros those signs that are opposite to $\sigma_0$ and precede the $2r$-th sign change. This gives a polynomial with $2r$ fewer positive roots than the number of sign changes in the $\sigma$ sequence.*

*Proof:* Let $q(x)$ be the polynomial for the $\tau$ sequence; let its roots be $x_1, \dots, x_m$. The polynomial $q(x)$ has zero coefficients where certain signed coefficients are needed. We replace these zeros in $q(x)$ by sufficiently small terms $\sigma_i \delta x^i$ in $p(x)$ without affecting the number or signs of roots.

Since $q(x)$ is known to have its roots all in distinct intervals, it cannot have any double roots. Thus there is some $\epsilon_1$ such that no root of $q'(x)$ is within $\epsilon_1$ of a root of $q(x)$; we also require $\epsilon_1 < |x_i|$ for all $i$, which we can require since $q(0) = \tau_0 \neq 0$. Since $|q'(x)|$ is continuous, it is bounded on each interval $[x_i - \epsilon_1, x_i + \epsilon_1]$; let $\epsilon_2$ be its minimum value on the union of all of these intervals. Finally, let $\epsilon_3$ be the minimum of $\epsilon_1\epsilon_2$ and the value of $|q(x)|$ on the intervals $[-k^{2n}, x_1 - \epsilon_1]$, $[x_{i-1} + \epsilon_1, x_i - \epsilon_1]$, and $[x_m + \epsilon_1, k^{2n}]$.

Then if we add to $q(x)$ any differentiable function $f(x)$ that has $|f(x)| < \epsilon_3$ and $|f'(x)| < \epsilon_2$ on $[-k^{2n}, k^{2n}]$, then $f(x) + q(x)$ must still have one simple root in each interval $[x_i - \epsilon_1, x_i + \epsilon_1]$, since it changes sign in each such interval and its derivative does not change sign there. Also, $f(x) + q(x)$ cannot have any other root in $[-k^{2n}, k^{2n}]$, since $|f(x)| < \epsilon_3 < |q(x)|$ outside these intervals. Thus $f(x) + q(x)$ has the same number of roots as $q(x)$ in $[-k^{2n}, k^{2n}]$, with the same signs.

By the construction of $q(x)$ with leading term $\pm k^{-n^2}x^n$ and other terms whose absolute values are all at most $k^{-n^2-1}x^n$ for $x > k^{2n}$, we have $|q(x)| \geq (k - n)(k^{-n^2-1})x^n$ for all $x \geq k^{2n}$. Thus if $|f(x)| < (k - n)(k^{-n^2-1})x^n$ for all $x \geq k^{2n}$, then $f(x) + q(x)$ has no roots with absolute value greater than $k^{2n}$, and thus has the same number of positive and negative roots as $q$.

We can thus let $p(x) = f(x) + q(x)$, where

$$f(x) = \sum_{j=1}^{n-1} (\sigma_j - \tau_j)\delta x^j,$$

with $\delta$ sufficiently small to meet the conditions on $f$. This polynomial has the correct signs and the correct number of positive and negative roots. ■

Note that this technique does not allow us to obtain simultaneously all possible numbers of positive and negative roots. In fact, this turns out to be impossible; some combinations of positive and negative roots cannot be obtained at all. The simplest impossible case is the sequence $+, 0, -, 0, +$, corresponding to the polynomial $ax^4 - bx^2 + c = 0$, which has $-x$ as a root if it has $x$ as a root and thus cannot simultaneously have no positive and two negative roots.

Even if no signs are zero, it may not be possible to obtain simultaneously all admissible numbers of positive and negative roots. For example, the sequence $+, -, -, -, +$ has two positive and two negative sign changes. It is possible for a polynomial with this sign sequence to have two negative or zero positive real roots, but not both simultaneously. A fourth-degree polynomial with only two negative real roots for which the sum of the roots was positive could be factored as $a(x^2 + bx + c)(x^2 - sx + t)$ with $a, b, c, s, t > 0$, $s^2 < 4t$, and $b^2 \geq 4c$. The product of these factors is $a(x^4 + (b - s)x^3 + (t + c - bs)x^2 + (bt - cs)x + st)$. To get the correct sign sequence, we need $b < s$ and $bt < cs$, which gives $b^2 t < s^2 c$ and thus $b^2/c < s^2/t$. But we have $b^2/c \geq 4 > s^2/t$.

This counterexample provides a negative answer to the question raised in **[1]** whether it is possible to get a polynomial with an arbitrary sign sequence and any simultaneous numbers of positive and negative roots allowed by Descartes' Rule of Signs. This suggests a new conjecture: the only possible numbers of positive and negative roots are the maximum values permitted by Descartes' Rule of Signs in a sequence obtained by changing some of the internal signs to zeros as in Theorem 2. The above cases and the analogous $+, +, -, +, +$ confirm the conjecture for degree 4.

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# Integrating Polynomials in Secant and Tangent


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This note provides a relatively painless way to integrate arbitrary polynomials in secant and tangent without invoking integration by parts or anything beyond elementary polynomial and trigonometric identities. The techniques involved also introduce some of the ideas behind the construction of Laurent polynomials, although the manner in which they do so is rather indirect. We begin with a theorem that covers almost all possibilities.

**Theorem 1.** *For each polynomial $P(s, t)$ in two variables, there are polynomials $F$ and $G$ in one variable and a constant $c$ such that*

$$\int P(\sec x, \tan x) \sec x \, dx = F(u) - G(v) + c \ln(u) + C$$

*where $u = \sec x + \tan x$ and $v = \sec x - \tan x$.*