Infinitesimal Classification of First Order Systems of Two Partial Differential Equations in Two Variables

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1 Introduction

We give a classification of symbols of partial differential systems that consist of two equations imposed on two functions in two independent variables, with respect to the natural action of the group of linear changes of dependent and independent variables.

The classification turns out to be finite, consisting of 5 types in the real case and of 4 types in the complex case. Apart from the elliptic, parabolic and hyperbolic types [1] constituting the well-known classification of second order equations for a function $u(x, y)$, in the present case we also have two non-equivalent 'degenerate' types.

Bifurcation diagrams for the singular orbits in the space of symbols imply certain properties of the singularities of linear and quasilinear systems of the considered type. I am indebted to S. V. Chmutov who explained me the meaning of versal deformations and bifurcation diagrams in the general context of an arbitrary Lie group action on a manifold.

In the last section of the paper we find, for every type of the symbol, the topological type of the corresponding integral Grassmanian — an ob-

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ject important in the study of singularities of solutions of partial differential equations [5].

All theorems stated in this text are quite straightforward, so we confine ourselves just to short comments, omitting the complete proofs.

2 Classification of Symbols

A system of k-th order partial differential equations imposed on m functions of n independent variables determines a submanifold $\mathcal{E}$ in the jet space $J_k(X,Y)$ where $\dim X = n$, $\dim Y = m$. The *symbol* of the system at a point $\theta \in \mathcal{E}$ is defined as the intersection of the tangent space to $\mathcal{E}$ and the tangent space to the fiber F_θ of the natural projection $J_k \to J_{k-1}$ containing θ. If $a \in X$ and $b \in Y$ are the source and the target of the jet θ, respectively, then the tangent space $T_\theta(F_\theta)$ is naturally isomorphic to $S^k V^* \otimes W$ where $V = T_a(X)$, $W = T_b(Y)$ and S^k denotes the k-th symmetric power (see [2]). In this sense, a symbol can be thought of as a linear subspace of the space $S^k(\mathbf{R}^n)^* \otimes \mathbf{R}^m$.

Denote by G_θ be the group of those local diffeomorphisms of the pair $(X \times Y, (a,b))$ which, after being lifted to the jet space $J_k(X,Y)$, do not move the point θ.

Theorem 1 *Let $\theta, \theta' \in \pi_{k,0}^{-1}(a,b)$ be two points lying in the same fiber of the natural projection $\pi_{k,0} : J_k(X,Y) \to J_0(X,Y) = X \times Y$. Then:*

1. *There exists a local diffeomorphism ϕ in a neighbourhood of (a,b) such that $\phi^{(k)}(\theta) = \theta'$.*

2. *The groups G_θ and $G_{\theta'}$ are conjugate via ϕ.*

3. *The action of G_θ in $T_\theta(F_\theta)$ agrees with the action of $G_{\theta'}$ in $T_{\theta'}(F_{\theta'})$ under the identification $T_\theta(F_\theta) \cong T_{\theta'}(F_{\theta'}) \cong S^k V^* \otimes W$.*

4. *In the case $X = \mathbf{R}^n$, $Y = \mathbf{R}^m$ and $\theta = (0,0,\ldots,0)$ in the standard jet coordinates the group G_θ consists of those local diffeomorphisms $(x,y) \mapsto (\tilde{x},\tilde{y})$ which satisfy $\frac{\partial \tilde{y}}{\partial x}|_{(0,0)} = 0$.*

5. *The action of this group in $T_\theta(F_\theta)$ depends only on the linear part of the diffeomorphism and decomposes as $G \to \mathrm{GL}(V) \times \mathrm{GL}(W) \to$*

$\mathrm{GL}(T_\theta(F_\theta))$, where the last arrow is nothing but the natural left-right action of the direct product $\mathrm{GL}(\mathbf{R}^n) \times \mathrm{GL}(\mathbf{R}^m)$ in $S^k(\mathbf{R}^n)^* \otimes \mathbf{R}^m$.

Thus, in a purely algebraic setting, the problem of symbol classification can be stated as follows. *Fix a quadruple of natural numbers n, m, k, r, and let V and W be linear spaces of dimensions n and m over a certain field. Describe the orbits of the group $\mathrm{GL}(V) \times \mathrm{GL}(W)$ in the Grassmanian of r-dimensional planes in $S^kV^* \otimes W$ with respect to the action that comes from the natural left-right action of this group in the underlying space $S^kV^* \otimes W$.* The problem makes sense over an arbitrary field; we will be only concerned with the reals and the complexes.

In the special case $m = 1$, $k = 2$, $r = 1$ (corresponding to one second order equation) we obtain the problem of classification of (real or complex) quadratic forms in n variables. Its well-known solution gives rise to the usual notion of type (hyperbolic, parabolic, elliptic) for the second order equations.

In this note we will solve, and derive the consequences of, the symbol classification problem in the case $n = 2$, $m = 2$, $k = 1$, $r = 2$, which, simple as it is, has never been studied in the literature, to the best of our knowledge. The algebraic problem to study is that of classifying two-dimensional linear spaces of linear operators in the plane with respect to the independent linear changes both in the source and in the target.

A complete list of all quadruples (n, m, k, r) for which the classification of symbols is finite, will be given in a separate paper [4].

3 Planes of linear operators in the plane

The direct product $\mathrm{GL}(\mathbf{R}^2) \times \mathrm{GL}(\mathbf{R}^2)$ acts in $\mathrm{End}(\mathbf{R}^2)$ according to the rule

$$(X, Y) \cdot A = XAY.$$

We are to find the orbits of the induced action on the set of 2-planes in $\mathrm{End}(\mathbf{R}^2)$.

In the four-dimensional space $\mathrm{End}(\mathbf{R}^2)$ we single out a three-dimensional cone K that consists of all degenerate operators. The trace left by this cone on a two-plane L in $\mathrm{End}(\mathbf{R}^2)$ and considered up to linear transformations of the plane, is an important invariant of L which is sufficient to distinguish all the orbits but two. The latter are made up of planes that are entirely contained in K and differ in that, for one of them, all the operators of the plane possess a common 1-dimensional kernel, while for the other, they have one and the same 1-dimensional image.

Theorem 2 *Two-dimensional planes of linear operators in $\mathbf{R}^2$ under the linear changes of variables in the source and in the target split into five orbits. Therefore, the symbol of any system of partial differential equations with $n = m = r = 2$ belongs to one of the five possible types.*

The orbits are listed in this table:

Type	$\dim(\mathcal{O})$	$L \cap K$	*Normal form of L*	*System of d.e.*
Hyperb.	*4*	*two lines*	$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$	$\begin{cases} u_y = 0 \\ v_x = 0 \end{cases}$
Ellipt.	*4*	*one point*	$\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$	$\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$
Parab.	*3*	*one line*	$\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle$	$\begin{cases} v_x = 0 \\ v_y = u_x \end{cases}$
Deg. I	*1*	*all plane*	$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$	$\begin{cases} u_y = 0 \\ v_y = 0 \end{cases}$
Deg. II	*1*	*all plane*	$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle$	$\begin{cases} v_x = 0 \\ v_y = 0 \end{cases}$

These five orbits can be conveniently vizualized in terms of elementary projective geometry. The Grassmanian of two-planes in the four-dimensional space $\mathrm{End}(\mathbf{R}^2)$ is isomorphic to the variety of projective lines in the three-dimensional projective space $P(\mathrm{End}\,\mathbf{R}^2)) \cong \mathbf{R}P^3$. The cone of degenerate operators shows up as a non-degenerate quadric $Q \subset \mathbf{R}P^3$. The action of the group of left and right linear changes carries over into the action of the group of projective transformations of $\mathbf{R}P^3$ that preserve the quadric Q. A line l belongs to the ellptic, parabolic or hyperbolic orbit if it intersects Q in 0, 1 or 2 points, respectively. The two families of straightline generators of the quadric correspond to the two degenerate types.

4 Versal Deformations

Let G be a Lie group smoothly acting on a manifold M. A *versal deformation* [3] of a point $p \in M$ is a local mapping $f : (\Lambda, 0) \to (M, p)$ of a certain parameter space $\Lambda \cong \mathbf{R}^s$ whose image is transverse to the orbit $G(p)$ and has dimension equal to the codimension of $G(p)$.

The partition of the space of parameters Λ into subsets corresponding to different orbits is called *bifurcation diagram*.

If the orbit of the point p has dimension equal to the dimension of M, then its bifurcation diagram is trivial and it has no versal deformations with $s > 0$. The type of such point p is said to be *stable*.

Theorem 3 *The versal deformations of the normal forms listed in Theorem 2 can be chosen in the following way:*

1. *The hyperbolic type is stable.*
2. *The elliptic type is stable.*
3. *For the parabolic type* $\{v_x = \lambda u_y,\ v_y = u_x\}$.
4. *For the first degenerate type* $\{u_y = \lambda_1 v_x,\ v_y = \lambda_2 u_x + \lambda_3 v_x\}$.
5. *For the second degenerate type* $\{v_x = \lambda_1 u_y,\ v_y = \lambda_2 u_x + \lambda_3 u_y\}$.

The bifurcation diagram for a parabolic point is one-dimensional and consists of an elliptic region on one side of the point considered and of a hyperbolic region on the other side.

Bifurcation diagrams for both degenerate types are similar; they can be represented as a neighbourhood of zero in a three-space split by the cone $\lambda_3^2 + 4\lambda_1\lambda_2 = 0$ into an elliptic region (inside the cone) and a hyperbolic region (outside the cone); the points of the cone itself belong to the parabolic type.

Thus, the diagram of orbit adjacency looks like this:

$$\begin{tikzcd}
\boxed{\text{Degenerate I}} \arrow[dr] & & \boxed{\text{Elliptic}} \\
& \boxed{\text{Parabolic}} \arrow[ur] \arrow[dr] & \\
\boxed{\text{Degenerate II}} \arrow[ur] & & \boxed{\text{Hyperbolic}}
\end{tikzcd}$$

In the complex case all these results remain valid with one exception: the elliptic and hyperbolic orbits merge into one.

5 Degeneration of Symbols for Linear and Quasilinear Systems

So far, we have been concerned with the symbol of the system $\mathcal{E} \subset J_1(\mathbf{R}^2, \mathbf{R}^2)$ at one fixed point $\theta \in \mathcal{E}$. Let us now see what can be said about the dependence of the symbol $\sigma(\mathcal{E}, \theta)$ on this point θ.

For a general non-linear system the set $\{\sigma(\mathcal{E}, \theta)\}$ constitutes a six-parameter family, because the manifold $\mathcal{E}$ is six-dimensional.

For a quasilinear system

$$\begin{cases} a_1(x,y,u,v)u_x + b_1(x,y,u,v)u_y + c_1(x,y,u,v)v_x + d_1(x,y,u,v)v_y = 0 \\ a_2(x,y,u,v)u_x + b_2(x,y,u,v)u_y + c_2(x,y,u,v)v_x + d_2(x,y,u,v)v_y = 0 \end{cases} \tag{1}$$

this family is constant along the fibers of the projection $J_1 \to J_0$ and thus in fact depends only on four parameters.

For a linear system

$$\begin{cases} a_1(x,y)u_x + b_1(x,y)u_y + c_1(x,y)v_x + d_1(x,y)v_y = 0 \\ a_2(x,y)u_x + b_2(x,y)u_y + c_2(x,y)v_x + d_2(x,y)v_y = 0 \end{cases} \tag{2}$$

this family only depends on two parameters.

The results obtained in the previous section lead to the following conclusions about the behaviour of the regions in the parameter spaces where the symbol of the system belongs to one or another type.

Theorem 4 *For a linear system (2), the degenerations of the symbol of types I and II can be eliminated by a small movement. The parabolic points of a typical system form a curve in the plane (x, y), on one side of which the symbol of the system is elliptic while on the other side it is hyperbolic. This picture is stable with respect to small deformations.*

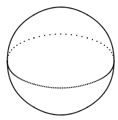
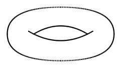
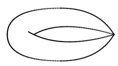
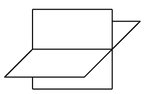
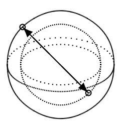
For a quasilinear systems (1) all degenerations of the symbol are stable. In a neighbourhood of every point (x, y, u, v) that belongs to either of the two degenerate types I or II, there is an entire curve made up of points of the same type; in a transversal to this curve one can always find parabolic, elliptic and hyperbolic points which, up to a diffeomorphism, fill up a quadratic cone, its inner part and its outer part.

6 Integral Grassmanians

According to [5], with every point θ of a submanifold $\mathcal{E} \subset J_k(X, Y)$, one can associate an algebraic variety $\mathrm{IG}(\mathcal{E}, \theta)$ which consists of all n-planes tangent to $\mathcal{E}$ and integral for the metasymplectic structure naturally defined on the jet bundle. This variety is called the *integral Grassmanian* of $\mathcal{E}$ at θ; its topology yields important information about the singularities that may appear in the solutions of the given system.

It is readily verified that the algebraic (and hence the topological) type of the integral Grassmanian $\mathrm{IG}(\mathcal{E}, \theta)$ is completely determined by the type of the symbol $\sigma(\mathcal{E}, \theta)$ in the sense of the previous classification and thus Theorem 2 allows to describe all possible types of integral Grassmanians for the first order systems 2×2.

Theorem 5 *The integral Grassmanians for the first order systems with two dependent and two independent variables may belong to one of the following five topological types defined by the type of the symbol:*

Symbol type	Integral Grassmanian	
Elliptic		Sphere S^2
Hyperbolic		Torus $S^1 \times S^1$
Parabolic		Torus with a meridian collapsed to a point
Degenerate I		Two intersecting planes in $\mathbf{R}P^3$
Degenerate II		Three-manifold $S^2 \times [0, 1]/(x, 0) \sim (-x, 1)$

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