Symmetries of Distributions and Quadrature of Ordinary Differential Equations

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Abstract. We present a geometric exposition of S.Lie's and E.Cartan's theory of explicit integration of finite type (in particular, ordinary) differential equations. Numerous examples of how this theory works are given. In one of these, we propose a method of hunting for particular solutions of partial differential equations via symmetry preserving overdetermination.

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Introduction

It is widely known that the notion of a solvable Lie algebra is closely associated with differential equations solvable by quadratures and that this theory goes back to Sophus Lie. However, S. Lie himself never stated the general theorem underlying this fact. His main book on the subject [Lie] is a collection of geometrical algorithms for solving various classes of differential equations, unified by the idea of symmetry group. S. Lie managed to explain the whole variety of existing integration tricks from a uniform stand-point and also to create new powerful methods of quadrature and reduction.

Symmetries of differential equations can be used to reduce the dimension of the problem under study in either of the two ways, by passing over to quotient manifolds (first integration strategy—reduction of order) or by restricting the problem to a submanifold (second integration strategy—first integrals). One of the examples of the second integration method is provided by Theorem 49 of [Lie], which says that a linear first order partial differential equation in 4 variables (equivalent to a non-autonomous system of 3 o.d.e.'s) is reduced to quadratures, if it has a solvable 3-dimensional symmetry group satisfying certain non-degeneracy conditions. Propositions of this kind, apparently general in nature, do not appear in [Lie] as a universal theorem. As writes E. M. Polishchuk in his biographical book [Pol], 'this theorem seems to be dissolved in the totality of other Lie's results about groups admitted by equations'.

The second integration method of Lie was further developed by Elie Cartan who thought about the symmetries of arbitrary Pfaff systems (=distributions) in terms of vector fields and differential forms. Although E. Cartan, no doubt, understood the general fact equivalent to Theorem 3 below, he never stated it in [Ca]. Our aim is to clarify the results of Lie and Cartan placing them in the general differential-geometric context of manifolds and distributions [KN, St]. We give a precise statement of this theorem in a form that equally applies to single ordinary differential equations, systems of o.d.e.'s and to arbitrary systems of partial differential equations of finite type.

During the last 30 years, beginning with the pioneering book by L. V. Ovsiannikov [Ovs] whose Russian edition was published in 1962, an extensive literature on the symmetry groups of differential equations has appeared (cf. [Ib], [KLM], [Ol], [BK], [Sph]). We especially recommend the book by H. Stephani [Sph] which contains precise recipes of integration procedures, numerous suggestive examples, informal motivations and is very close to the original S. Lie's papers both in contents and style. In particular, it is as informal as possible to be read by people anxious just to solve their favourite ordinary differential equations.

Although many authors have nicely explained and developed S. Lie's approach, especially as regards his first integration method, there does not seem to exist any modern exposition of the relevant work of Elie Cartan who put the theorem of Lie into a more general setting. The books listed above do not unveil the differential-geometric background that unifies the second integration method as applied to ordinary differential equations, systems of such and finite type systems of partial differential equations. It goes without saying that specialists in group applications to differential equations know the Lie and Cartan's theorem in full generality, yet nobody will give you an exact reference to a written mathematical text that contains its precise statement. Our primary aim is to fill this gap.*

In sections 2 to 6, our style follows that of [BK], [Sph] (and also the classical literature of the beginning of the century) in that we do not state the exact notion of *integration by quadratures*, although it is crucial in the formulation of the main theorem. By doing so, we just adhere to the state of the art in this area. For the case of *linear algebraic* differential equations, the notion of quadratures is readily formalized in the framework of differential algebra (see [Ko], [Po2]). But, as far as the authors know, until now there have not been any successful formulations of nonlinear differential algebra that could meet our purposes. See [Cas] for one of the recent attempts to clarify the interrelation between differential geometry and differential algebra.

The paper proceeds as follows. In section 1, we recall basic definitions pertaining to distributions on manifolds and introduce the notion of symmetry. In section 2, we explain how to find first integrals of distributions using their symmetries. This theory is specialized for ordinary differential equations and finite type differential systems in sections 3 and 4, respectively. Section 5 contains some illustrative examples of integration of ordinary differential equations by means of the developed theory. Finally, in Section 6 we propose a trick which allows to find finite-parametric families of solutions of partial differential equations with ample symmetry algebras by reducing the problem to that of integration

* While revising the manuscript, the authors came across the research report by J. Sherring and G. Prince [SP] where Cartan's ideas are discussed in a spirit close to our own, but with applications only to systems of ordinary differential equations.

of a completely integrable distribution.

1. Symmetries of distributions

We begin by recalling some basic facts about distributions (see [KN], [St] for details).

Let P be a distribution of dimension m and codimension k on a smooth manifold M. This means that $\dim M = m + k$ and at every point x of M an m-dimensional subspace P_x of the tangent space T_xM is specified in such a way thet P_x smoothly depends on x. Let

$$D(P) = \{\, V \in \mathrm{Vect}(M) \mid V_x \in P_x \ \forall x \in M\}$$

be the set of all smooth vector fields lying in the distribution P and

$$\Lambda(P) = \{\, \omega \in \Omega^1(M) \mid \omega(V) = 0 \ \forall V \in D(P)\,\}$$

the set of all differential 1-forms vanishing on vectors from P.

Both $D(P)$ and $\Lambda(P)$ are modules over the ring of smooth functions $C^\infty(M)$ and a distribution is fixed whenever one specifies a system of generators of either of these modules. Both modules are usually free (this is always the case with their localizations to an appropriate open everywhere dense subset of M), so all one has to specify is either a set $V_1, \ldots, V_m$ of vector fields whose values at every point of the manifold are linearly independent or a set $\omega_1, \ldots, \omega_k$ of 1-forms subject to the same requirement.

Example 1. The 1-form $\omega = dy - zdx$ does not vanish in any point of the 3-space $\mathbf{R}^3$, so it defines a distribution of dimension 2 and codimension 1. For reasons explained below, it is called *Cartan's distribution*. A basis of the module $D(P)$ is formed by vector fields $V_1 = \partial_x + z\partial_y$ and $V_2 = \partial_z$.

Example 2. Let $M = \mathbf{R} \times \mathbf{R}^+ \times S^1$ with coordinates $x \in \mathbf{R}$, $y \in \mathbf{R}^+$, $\varphi \in S^1 = \mathbf{R} \bmod 2\pi$. The 1-form

$$\omega = 2\sin^2\frac{\varphi}{2}\, dx + \sin\varphi\, dy - y\, d\varphi$$

defines the so called *oricycle distribution* (explanation below). In this example both modules $D(P)$ and $\Lambda(P)$ are again free.

Example 3. We will describe this one in plain English, without formulas. Imagine a Möbius band and at every point of its surface the line, lying in the tangent plane and perpendicular to the central circle of the band. Both modules $D(P)$ and $\Lambda(P)$ associated with this distribution are not free, because the fibering of the Möbius band over its central circle is non-trivial. But both modules become free as soon as we cut the band in such a way that it becomes topologically trivial.

A distribution is by definition an infinitesimal object. It is connected with the world of finite things via the notion of *integral manifold*. A submanifold $N \subset M$ is said to be integral for the distribution P if $T_xN \subset P_x$ for all $x \in M$. An integral manifold is *maximal* if it is not contained in an integral manifold of greater dimension. For example, lines on the surface of the Möbius band which are perpendicular to the base circle, are maximal integral manifolds for the above mentioned distribution.

A distribution is said to be *completely integrable* if the dimension of every maximal integral manifold is exactly m, the dimension of the distribution itself. In this case the entire manifold is the disjoint union of maximal integral manifolds of the distribution, which are the *leaves* of a *foliation*, so that the notion of a completely integrable distribution is equivalent to that of a foliation.

The striped Möbius band of Example 3 above is obviously a foliation (see Fig. 1). It is somewhat more difficult to decide whether the distributions of Examples 1 and 2 are completely integrable. The classical Frobenius' theorem gives two equivalent criteria for checking the complete integrability of a distribution. They are: 1) $D(P)$ should be a Lie algebra, and 2) the differential of any 1-form from $\Lambda(P)$ should belong to the ideal generated by $\Lambda(P)$. In the case of a distribution defined by a single 1-form ω the second condition simply means that $\omega \wedge d\omega = 0$.

The reader can readily verify that this condition holds for Example 2 and fails for Example 1. Maximal integral manifolds for the Cartan distribution are 1-dimensional; they will be described later in section 3. Maximal integral manifolds of the oricycle distribution are 2-dimensional; they can be easily found in a straightforward manner, but we will do that in a more elegant way — using symmetries — just to illustrate the meaning and the scope of applications of the latter.

By a (finite) *symmetry* of the distribution P we understand a (possibly local) diffeomorphism $f : M \to M$ which takes P into itself, i.e. such that $f_*(P_x) \subset P_{f(x)}$ for all $x \in M$. A vector field X is said to be an (infinitesimal) symmetry of the distribution if the flow generated by X consists of finite symmetries. The infinitesimal approach turns out to be much more constructive than its finite counterpart, so in what follows the word *symmetry* will always mean *infinitesimal symmetry* unless otherwise explicitely specified.

Example 4. In Examples 1 and 2 given above, the vector field ∂_x is a symmetry, because in both examples the coefficients of the basic 1-form ω do not depend on x and hence the corresponding finite transformations, i.e. translations in x, preserve these forms. In example 3, the infinitesimal rotation of the Möbius band along its center circle is also a symmetry.

The set of all symmetries $\mathrm{Sym}(P)$ forms a Lie algebra with respect to vector fields commutator, because finite symmetries obviously make a group. The following theorem allows one to deduce this fact by a simple manipulation with formulas. It gives two constructive characterizations of the symmetry algebra, one in terms of the associated module of vector fields $D(P)$, and another in terms of the dual module of 1-forms $\Lambda(P)$.

Theorem 1. *The following conditions are equivalent*:

(1) $X \in \mathrm{Sym}(P)$,
(2) $[X, D(P)] \subset D(P)$,
(3) $L_X(\Lambda(P)) \subset \Lambda(P)$,

where L_X is the Lie derivative operator along the vector field X.

Proof. Choose some (possibly local) bases $V_1, \ldots, V_m$ and $\omega_1, \ldots, \omega_k$ in the $C^\infty(M)$-modules $D(P)$ and $\Lambda(P)$, respectively. Conditions (2) and (3) can be rewritten as follows:

(2) there exist such functions α_{ij} that for all i one has

$$[X, V_i] = \sum_{j=1}^{m} \alpha_{ij} V_j,$$

(3) there exist such functions β_{ij} that for all i one has

$$L_X(\omega_i) = \sum_{j=1}^{k} \beta_{ij}\omega_j,$$

The theorem follows from the three implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (1), and we are going to prove them succesively.

(1) $\Rightarrow$ (2). Let $\{G_t\}$ be the 1-parametric transformation group corresponding to the vector field X which is a symmetry of P. Then for every time moment $t \in \mathbf{R}$ and all $i = 1, \ldots, m$ we have

$$(G_t)_*(V_i) = \sum_{j=1}^{m} \lambda_{ij}(t)V_j,$$

where λ_{ij} is a family of smooth functions on the manifold M smoothly depending on the parameter t.

Differentiating this over t at $t = 0$ and using the formula

$$L_X(V) = -\frac{d}{dt}\Big|_{t=0}(G_t)_*(V),$$

one gets

$$[X, V_i] = \sum_{j=1}^{m} \alpha_{ij}V_j,$$

with

$$\alpha_{ij} = -\frac{d}{dt}\Big|_{t=0}\lambda_{ij}(t).$$

(2) $\Rightarrow$ (3). Suppose that a vector field X satisfies (2). Now take a 1-form ω which vanishes on all the vector fields $V_1, \ldots, V_m$ and prove that $L_X(\omega)$ has the same property (and hence can be represented as a linear combination of $\omega_1, \ldots, \omega_k$). Indeed, for all $i = 1, \ldots, m$ we have

$$\begin{aligned} L_X\big(\omega(V_i)\big) &= \big(X \lrcorner\, d\omega + d(X \lrcorner\, d\omega)\big)(V_i) \\ &= d\omega(X, V_i) + V_i(\omega(X)) \\ &= X\big(\omega(V_i)\big) - \omega\big([X, V_i]\big) = 0 \end{aligned}$$

(3) $\Rightarrow$ (1). Consider the following differential $(k+1)$-forms, dependent upon the parameter t:

$$\Omega_i(t) = G_t^*(\omega_i) \wedge \omega_1 \wedge \ldots \wedge \omega_k.$$

Since $G_0^*(\omega_i) = \omega_i$, we have $\Omega_i(0) = 0$. We are going to prove that $\Omega_i(t) \equiv 0$; this will imply that $G_t^*(\omega_i)$ is a linear combination of $\omega_1, \ldots, \omega_k$ for all t and that X is indeed a symmetry of the distribution P. We have:

$$\frac{d}{dt}\Omega_i(t) = G_t^*(L_X\omega_i) \wedge \omega_1 \wedge \ldots \wedge \omega_k = \sum_{j=1}^{k} G_t^*(\beta_{ij})\Omega_j(t),$$

which means that the vector consisting of $(k+1)$-forms $\big(\Omega_1(t), \dots, \Omega_k(t)\big)$, is a solution to a linear homogeneous system of ordinary differential equations with zero initial conditions. Hence, it must vanish identically.

This completes the proof of the theorem.

Example 5. A simple computation using either of conditions 2 or 3 of the theorem, shows that vector field $X = y\,\partial_y + \sin\varphi\,\partial_\varphi$ is a symmetry of the oricycle distribution (see Example 2 above).

2. Symmetries and integration

The problem of *integration* of a distribution consists in describing its maximal integral manifolds. For a completely integrable distribution, this is equivalent to finding a *complete set of first integrals*. A function $h \in C^\infty(M)$ is called a first integral of the distribution P if every integral manifold of P lies entirely in some level surface $\{x \in M \mid h(x) = \text{const}\}$ of this function, or, equivalently, if $V(h) = 0$ for any $V \in D(P)$. A complete set of first integrals of the distribution P is a set of functions whose mutual level surfaces

$$\{x \mid h_1(x) = c_1, \dots, h_k(x) = c_k\}$$

represent the set of all maximal integral manifolds of P.

If X is a symmetry with the flow $\{G_t\}$ and Q a maximal integral manifold of the distribution P, then $G_t(Q)$ is also a maximal integral manifold for any t. That is, the 1-parameter transformation group generated by a symmetry preserves the set of maximal integral manifolds (but possibly rearranges them in some other order). There is, however, a distinguished class of symmetries which leave invariant every particular maximal integral manifold. These are called *characteristic symmetries*. By definition, a symmetry of the distribution P is said to be *characteristic* if it lies in P (or, more exactly, belongs to $D(P)$). Transformations corresponding to a characteristic symmetry move every maximal integral manifold along itself.

Example. The symmetry mentioned above in Example 5 is characteristic while those given in Example 4 are not.

The classical theorem of Cartan [St] implies that the set of all characteristic symmetries

$$\operatorname{Char}(P) = \operatorname{Sym}(P) \cap D(P)$$

is an ideal of the Lie algebra $\operatorname{Sym}(P)$. Elements of the quotient algebra

$$\operatorname{Shuf}(P) = \operatorname{Sym}(P)/\operatorname{Char}(P)$$

will be referred to as *shuffling symmetries* of the distribution P. Flows corresponding to different representatives of a class

$$\bar{X} = X \bmod \operatorname{Char}(P)$$

rearrange (*shuffle*) the set of maximal integral manifolds of P in the same way.

Now suppose that $k = \operatorname{codim} P$ and $\mathcal{G} \subset \mathrm{Shuf}(P)$ is a k-dimensional Lie subalgebra which is *transversal* to the distribution in the sense that the natural mapping $\pi_x : \mathrm{Shuf}(P) \to T_x(M)/P_x$ is bijective at every point $x \in M$. Let $\bar{X}_1, \dots, \bar{X}_k$, where

$$\bar{X}_i = X_i \bmod \mathrm{Char}(P),\ X_i \in \mathrm{Sym}(P),$$

be a basis of $\mathcal{G}$ while $\omega_1, \dots, \omega_k$ be a basis of the $C^\infty(M)$-module $\Lambda(P)$. The transversality condition for the algebra $\mathcal{G}$ is equivalent to the requirement that the matrix

$$\Xi = \|\omega_i(X_j)\| \tag{1}$$

be nondegenerate at any point of the manifold. Hence one may choose another basis $\omega_1', \dots, \omega_k'$ of the module $\Lambda(P)$ in such a way that the corresponding matrix $\|\omega_i'(X_j)\|$ would be the unit matrix. This can be achieved by setting

$$\begin{pmatrix} \omega_1' \\ \vdots \\ \omega_k' \end{pmatrix} = \Xi^{-1} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_k \end{pmatrix}$$

Note that the values $\omega_i(X_j)$ do not depend on the choice of representatives $X_j \in \bar{X}_j \in \mathrm{Shuf}(P)$.

Theorem 2. *Let P be a completely integrable distribution defined by the set of 1-forms $\omega_1, \dots, \omega_k$. Let $\bar{X}_1, \dots, \bar{X}_k$ be the basis of an algebra of shuffling symmetries $\mathcal{G} \subset \mathrm{Shuf}(P)$. Suppose that $\omega_i(X_j) = \delta_{ij}$ and*

$$[\bar{X}_i, \bar{X}_j] = \sum_{s=1}^{k} c_{ij}^s \bar{X}_s,$$

where $c_{ij}^s \in \mathbf{R}$. Then

$$d\omega_s = -\frac{1}{2} \sum_{i,j} c_{ij}^s \omega_i \wedge \omega_j.$$

Proof. By Frobenius' theorem, $d\omega_s = \sum_j \gamma_{sj} \wedge \omega_j$, where γ_{ij} are appropriate 1-forms. Under the conditions specified, these 1-forms actually belong to $\Lambda(P)$, i.e. vanish on the vectors from the distribution. To see this, one should make use of the formula

$$L_{X_i}(\omega_s) = d(X_i \lrcorner\, \omega_s) + X_i \lrcorner\, d\omega_s.$$

According to Theorem 1, $L_{X_i}(\omega_s) \in \Lambda(P)$. Besides, by the premises of the present theorem, $d(X_i \lrcorner\, \omega_s) = 0$ for all i, s. Hence $X_i \lrcorner\, d\omega_s = \sum_j \gamma_{sj}(X_i)\omega_j - \gamma_{ij} \in \Lambda(P)$, which implies that $\gamma_{ij} \in \Lambda(P)$ and therefore

$$d\omega_s = \sum_{i<j} \alpha_{ij}^s \omega_i \wedge \omega_j$$

for some appropriate functions $\alpha_{ij} \in C^\infty(M)$. It follows that

$$d\omega_s(X_i, X_j) = \alpha_{ij}^s,$$

for $i < j$. On the other hand,

$$d\omega_s(X_i, X_j) = X_i(\omega_s(X_j)) - X_j(\omega_s(X_i)) - \omega_s([X_i, X_j]) = -\omega_s(\sum_{r=1}^{k} c_{ij}^r X_r) = -c_{ij}^s.$$

Therefore $\alpha_{ij}^s = -c_{ij}^s$ and

$$d\omega_s = -\sum_{i<j} c_{ij}^s \omega_i \wedge \omega_j = -\frac{1}{2}\sum_{i,j} c_{ij}^s \omega_i \wedge \omega_j.$$

Corollary. *If the algebra $\mathcal{G}$ is commutative, then in the conditions of the theorem a complete set of first integrals of the distribution can be found by quadratures.*

Indeed, in this case all the forms ω_i are closed and thus locally exact: $\omega_i = dh_i$ for some smooth functions $h_1, \ldots, h_k$. These functions can be recovered by computing the integrals

$$h_i(a) = \int_{a_0}^{a} \omega_i,$$

where a_0 is a fixed point of the manifold M.

Remark. As we mentioned in the introduction, we use the words ‘integration by quadratures’ in a somewhat vague sense. When we say that something can be found by quadratures, we mean that this something belongs to a suitable Liouville type extension of the basic field and there is an algorithmic procedure to recover it using known data. However, a notion of Liouville type differential extensions which is wide enough to incorporate finding implicit functions in the nonlinear case, is still awaiting its precise definition.

Example 6. Let us find a first integral of the oricycle distribution of Example 2 using its symmetry written down in Example 4 and in the last end explain the origin of this title to an inquisitive reader.

We have $\omega = 2\sin^2\frac{\varphi}{2}\,dx + \sin\varphi\,dy - y\,d\varphi$ and $X = \partial_x$. The pairing $\omega(X) = 2\sin^2\frac{\varphi}{2}$ is non-zero, so we can take another basic 1-form

$$\omega' = \frac{1}{2\sin^2\frac{\varphi}{2}}\omega = dx + \cot\frac{\varphi}{2}dy - \frac{1}{2\sin^2\frac{\varphi}{2}}d\varphi,$$

which must be exact, according to the theorem. And it really is. In fact,

$$\omega' = d\big(x + y\cot\frac{\varphi}{2}\big).$$

We see that the meaning of symmetries in this context is that they allow one to find the integrating factor for a Pfaffian equation.

Now about the sense of the term 'oricycle'. Consider the upper half-plane

$$H = \{x, y \mid y > 0\}$$

as a model of the Lobachevsky geometry. The role of straight lines in this model is played by semi-circles perpendicular to the x-axis (the 'absolute'). Let M be the unitary tangent manifold of H, i.e. the set of all unit tangent vectors. As the third coordinate φ on the 3-dimensional manifold M, we will take the angle between the upward vertical direction and the given vector (see Fig. 2). A simple calculation shows that a Lobachevsky line issuing from the point (x, y) at angle φ, arrives at the point of the 'absolute' with coordinate $x + y \cot \frac{\varphi}{2}$. This expression is the above found first integral of the distribution under study. It remains to note that the set of all vectors tending to the same point of the 'absolute', is called 'oricycle' in hyperbolic geometry.

Now suppose that we know a transversal algebra $\mathcal{G}$ of shuffling symmetries of a distribution P, which is not commutative. We will show that if this algebra is *solvable* then it is possible to decompose the problem of finding a complete set of first integrals for P into a finite number of steps, every one of which matches the assumptions of the Corollary above.

Denote the commutator subalgebra of $\mathcal{G}$ by $\mathcal{G}^{(1)}$ and suppose that $\mathcal{G}^{(1)} \neq \mathcal{G}$. Then one can choose a basis $\bar{X}_1, \ldots, \bar{X}_k$ of $\mathcal{G}$ in such a way that $\bar{X}_1, \ldots, \bar{X}_r \notin \mathcal{G}^{(1)}$ while $\bar{X}_{r+1}, \ldots, \bar{X}_k \in \mathcal{G}^{(1)}$. In this case $c_{ij}^s = 0$ for all i, j, if $s \leq r$. Choose the basis $\omega_1, \ldots, \omega_k$ of $\Lambda(P)$ in such a way that $\omega_i(X_j) = \delta_{ij}$. Theorem 2 implies that 1-forms $\omega_1, \ldots, \omega_r$ are closed and hence in some open domain $\omega_1 = dh_1, \ldots, \omega_r = dh_r$. For an arbitrary choice of constants $c = (c_1, \ldots, c_r)$ the level surface

$$H_c = \{h_1 = c_1, \ldots, h_r = c_r\}$$

is invariant under the commutator subalgebra $\mathcal{G}^{(1)}$, since $X_j(h_i) = \omega_i(X_j) = 0$ if $j \geq r+1$, $i \leq r$.

Let P_c be the restriction of distribution P to the surface H_c. Distribution P_c is completely integrable — the foliation of M whose leaves are maximal integral manifolds of P cuts a foliation on the surface H_c. The dimension of P_c is equal to the dimension of P while its codimension is $k - r = \dim \mathcal{G}^{(1)}$.

Observe that the restriction $\mathcal{G}^{(1)}|_{H_c}$ constitutes a transversal algebra of shuffling symmetries of the distribution P_c. Indeed, a shift along the trajectories of any field X such that $\bar{X} \in \mathcal{G}^{(1)}$ shuffles the leaves of distribution P and preserves the manifold H_c; hence it must also shuffle the leaves of distribution P_c. Transversality of $\mathcal{G}^{(1)}$ follows from the fact that $\|\omega_i(X_j)\|$, $r < i, j \leq k$, is the unit matrix, due to the above assumptions.

We can subject the pair $(P_c, \mathcal{G}^{(1)}|_{H_c})$ to the same procedure which was formerly applied to the pair $(P, \mathcal{G})$. More precisely: let $\mathcal{G}^{(2)} = [\mathcal{G}^{(1)}, \mathcal{G}^{(1)}]$ be the commutant of $\mathcal{G}^{(1)}$. If $\mathcal{G}^{(2)} \neq \mathcal{G}^{(1)}$, then some of the 1-forms $\omega_{r+1}, \ldots, \omega_k$ are closed and give rise to local first integrals of the distribution P_c. Distribution P_c can be restricted to the mutual level surface of these integrals, etc.

Now suppose that the algebra $\mathcal{G}$ is *solvable*, i.e. the sequence $\mathcal{G} \supset \mathcal{G}^{(1)} \supset \mathcal{G}^{(2)} \supset \ldots$, where $\mathcal{G}^{(i+1)} = [\mathcal{G}^{(i)}, \mathcal{G}^{(i)}]$, becomes zero after a finite number of steps. Then the above described procedure sooner or later will pose one into the conditions of the Corollary of Theorem 2. Whence:

Theorem 3. *Let P be a distribution of codimension k. Suppose that a solvable k-dimensional algebra of shuffling symmetries of P, transversal to P, is known explicitely. Then P is integrable by quadratures, i.e. one can find a complete set of first integrals for P by integrating closed 1-forms and solving functional equations.*

By solution of *functional equations* (called so to distinguish them from differential equations) we mean finding functions defined by implicit formulas

$$F(z_1, z_2, \ldots, z_n) = 0 \Rightarrow z_1 = f(z_2, \ldots, z_n).$$

We will show how this theorem works on examples from the area of differential equations. But first we will explain the connection between equations and distributions.

3. Equations as distributions

In this section we will show how the above described algorithm can be applied to ordinary differential equations, giving the 'second integration procedure' of S. Lie (see [Sph] for a more traditional exposition).

The study of an ordinary differential equation

$$F(x, y, y', \ldots, y^{(k)}) = 0 \tag{2}$$

can be reduced to the study of a distribution in the following way.

Let $J^k\mathbf{R}$ be the manifold of k-jets of smooth functions on the line $\mathbf{R}$. By definition, this is a space whose points correspond to all conceivable sets of values taken by the independent variable x (coordinate in $\mathbf{R}$), dependent variable y and the derivatives of the latter with respect to the former up to order k. Hence, $J^k\mathbf{R}$ is a $(k+2)$-dimensional space $\mathbf{R}^{k+2}$ whose coordinates can be designated by x, y, p_1, …, p_k.

Let $\mathcal{C}$ be the distribution in $J^k\mathbf{R}$ defined by the set of 1-forms

$$\begin{aligned} \omega_0 &= dy - p_1 dx, \\ \omega_1 &= dp_1 - p_2 dx, \\ &\ldots \\ \omega_{k-2} &= dp_{k-2} - p_{k-1} dx, \\ \omega_{k-1} &= dp_{k-1} - p_k dx. \end{aligned} \tag{3}$$

Following [KLV], we will call $\mathcal{C}$ the *Cartan's distribution*. A particular case of this object appeared earlier as Example 1.

The characteristic property of Cartan's distribution, which shows its importance in the theory of differential equations, consists in the following. A curve in $J^k\mathbf{R}$ that projects on the x-axis without degeneration, is integral for $\mathcal{C}$ if and only if it has the form

$$y = y(x),\ p_1 = y'(x), \ldots,\ p_k = y^{(k)}(x), \tag{4}$$

where $y(x)$ is a smooth function of x. Apart from these, there are maximal integral manifolds of the second kind, namely straight lines parallel to the p_k axis. A general maximal integral manifold may include portions of both types, fitted together to make a smooth curve (see [KLV] for the proof).

Equation (2) corresponds to a hypersurface $\mathcal{E} \subset J^k\mathbf{R}$, defined by

$$F(x, y, p_1, \ldots, p_k) = 0,$$

The 2-dimensional Cartan's distribution $\mathcal{C}$, when restricted to $\mathcal{E}$, produces a 1-dimensional distribution $\mathcal{C}_\mathcal{E}$, which will be referred to as the *Cartan's distribution of the equation* $\mathcal{E}$. Maximal integral manifolds of this distribution can contain segments parallel to the p_k axis as well as portions of curves (4), where $y(x)$ must be a solution to equation (2) in order that the curve should lie on $\mathcal{E}$. These pieces are put together in the points where the projection to the x-axis degenerates, and such points may constitute only a discrete set on the curve.

Example 7. Consider the equation

$$yy' + x = 0.$$

We can take the variables y and $p = y'$ for coordinates on the corresponding surface $\mathcal{E} \subset J^1\mathbf{R}$. In these coordinate system the distribution $\mathcal{C}_\mathcal{E}$ is given by $(1+p^2)dy + yp\,dp = 0$. Its integral curves $y = C(1+p^2)^{-1/2}$ are depicted in Fig. 3a. Every pair of symmetrical curves produces one of the circles in the (x, y)-plane which correspond to two-valued solutions of the equation and are shown in Fig. 3b. Besides, there is a singular integral curve $y = 0$ on $\mathcal{E}$ which corresponds to the origin point in the (x, y)-plane and gives rise to no solution.

A vigilant reader might have noticed a flaw in our previous argument: there can exist such points $x \in \mathcal{E}$ where the plane of the Cartan's distribution is contained in $T_x\mathcal{E}$ and thus $\dim \mathcal{C}_\mathcal{E} = 2$ at x. In this case $\mathcal{C}_\mathcal{E}$ is no distribution at all. The study of such singular points requires special analysis. We will avoid it by confining ourselves from now on to equations resolved with respect to the highest derivative

$$y^{(k)} = f(x, y, y', \ldots, y^{(k-1)}). \tag{5}$$

It should be noted, however, that a major part of our considerations can be, *mutatis mutandis*, transferred to the general case.

For an equation of type (5) the integral manifolds of the Cartan's distribution can only consist of portions of type (4), so that the problem of finding the (multivalued) solutions of such an equation is equivalent to that of finding integral manifolds of the distribution $\mathcal{C}_\mathcal{E}$. This distribution is 1-dimensional and hence completely integrable. Its codimension is equal to the order k of the equation under study. Theorem 3 applied in this environment gives rise to the following

Theorem 4. *If one knows explicitely a solvable k-dimensional transversal Lie algebra of symmetries of an ordinary differential equation of order k, then one can find the general solution of this equation by quadratures.*

By *symmetries of the differential equation* we understand *shuffling* symmetries of its Cartan's distribution, i.e. elements of the quotient space

$$\mathrm{Sym}(\mathcal{E}) = \mathrm{Shuf}(\mathcal{C}_{\mathcal{E}}) = \mathrm{Sym}(\mathcal{C}_{\mathcal{E}})/\,\mathrm{Char}(\mathcal{C}_{\mathcal{E}}).$$

Elements of this space have a convenient description in terms of the so called *generating functions* ([KLV],[Ib]). After introducing these, we will explain what does the notion of transversality mean in terms of generating functions.

Let us view x, $y = p_0$, p_1, …, p_{k-1} as coordinates on the hypersurface $\mathcal{E}$. In this coordinate system any vector field is written as

$$X = \alpha\frac{\partial}{\partial x} + \beta_0\frac{\partial}{\partial p_0} + \ldots + \beta_{k-1}\frac{\partial}{\partial p_{k-1}}.$$

Note that the $C^\infty(M)$-module of characteristic symmetries of $\mathcal{C}_{\mathcal{E}}$ is generated by the field

$$D = \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial p_0} + \ldots + p_{k-1}\frac{\partial}{\partial p_{k-2}} + f\frac{\partial}{\partial p_{k-1}}, \tag{6}$$

the *total derivative operator* with respect to x on the equation $\mathcal{E}$. Hence, in the quotient algebra $\mathrm{Sym}(\mathcal{E})$ the following relation holds:

$$\frac{\partial}{\partial x} \equiv -p_1\frac{\partial}{\partial p_0} - \ldots - p_{k-1}\frac{\partial}{\partial p_{k-2}} - f\frac{\partial}{\partial p_{k-1}} \mod \mathrm{Char}(\mathcal{C}_{\mathcal{E}}),$$

and it is sufficient to search symmetries only among vector fields of the form

$$X = \beta_0\frac{\partial}{\partial p_0} + \ldots + \beta_{k-1}\frac{\partial}{\partial p_{k-1}}.$$

Suppose that such a field X is a symmetry of the Cartan's distribution. In virtue of theorem 1, this means that the 1-forms $L_X(\omega_i)$, $0 \le i < k$, belong to $\Lambda(\mathcal{C}_{\mathcal{E}})$, i.e. they are linear combinations of 1-forms ω_0, ω_1, …, ω_{k-1} with functional coefficients. Computing these Lie derivatives for $i < k-1$, we obtain

$$L_X(\omega_i) = (D(\beta_i) - \beta_{i+1})\,dx \mod \Lambda(\mathcal{C}_{\mathcal{E}}).$$

The latter 1-form is in $\Lambda(\mathcal{C}_{\mathcal{E}})$ if and only if $\beta_{i+1} = D(\beta_i)$. Denoting β_0 by φ, we arrive at the following expression for X:

$$X = X_\varphi = \sum_{i=0}^{k-1} D^i(\varphi)\frac{\partial}{\partial p_i}. \tag{7}$$

In particular, we see that the field X is entirely defined by one function φ, which is equal to the value of X on y. This function φ will be referred to as the *generating function* of the vector field X_φ.

The space of Lie vector fields is thus isomorphic to the space of smooth functions in $x, p_0, \dots, p_{k-1}$ and we only have to transfer the commutator operation from the Lie algebra of vector fields to the space of functions. The resulting operation is given by the Poisson type formula

$$\{\varphi, \psi\} = \sum_{i=0}^{k-1} D^i(\varphi)\frac{\partial \psi}{\partial p_i} - \sum_{i=0}^{k-1} D^i(\psi)\frac{\partial \varphi}{\partial p_i}$$

and will be exploited later in section 5.

The generating function of a symmetry of a given differential equation cannot be arbitrary. The condition $L_X(\omega_{k-1}) \in \Lambda(\mathcal{C}_\mathcal{E})$ which can be rewritten as

$$\big(D^k(\varphi) - \sum_{i=0}^{k-1} D^i(\varphi)\frac{\partial f}{\partial p_i}\big)dx \equiv 0 \bmod \Lambda(\mathcal{C}_\mathcal{E})$$

implies that φ must satisfy differential equation $\Delta(\varphi) = 0$, where the operator Δ ranging in the space of smooth functions in x, p_0, $\dots$, p_{k-1} is defined by

$$\Delta = D^k - \sum_{i=0}^{k-1} \frac{\partial f}{\partial p_i} D^i$$

(in the terminology of [KLV], operator Δ is the *universal linearization* of the function $F = p_k - f$ restricted to the equation $\mathcal{E}$).

The discussion we have just finished can be summarized as follows.

Theorem 5. $\mathrm{Sym}(\mathcal{E}) \cong \mathrm{Ker}\,\Delta$; *this isomorphism is effectuated by the correspondence* $X_\varphi \leftrightarrow \varphi$, *where* X_φ *is defined by* (6).

Since evidently $\omega_i(X_\varphi) = D^i(\varphi)$ for the basic 1-forms (3) of the Cartan's distribution, the pairing matrix (1) becomes $\Xi = D^i(\varphi_j)$, so that the algebra generated by $\varphi_1, \dots, \varphi_k$ is transversal if and only if this matrix is non-degenerate.

Our approach to symmetries of differential equations is related to the classical S.Lie's approach in the following way. Consider a vector field Y which is a symmetry of the Cartan's distribution $\mathcal{C}$ in $J^k\mathbf{R}$. If Y is tangent to the equation $\mathcal{E} \in J^k\mathbf{R}$, then its restriction to $\mathcal{E}$ defines a symmetry of the distribution $\mathcal{C}_\mathcal{E}$. In this case Y is said to be an *outer symmetry* of the equation $\mathcal{E}$. Symmetries of the equation $\mathcal{E}$ in the sense adopted above, i.e. symmetries of the distribution $\mathcal{C}_\mathcal{E}$, can be, correspondingly, called *inner symmetries*. Restrictions of outer symmetries to $\mathcal{E}$ are distinguished from the set of all inner symmetries by the property that their generating function depends only on x, y, p_1. This follows from the theorem of Lie and Bäcklund: *every symmetry of the distribution* $\mathcal{C}$ *coincides with the natural lift to* $J^k\mathbf{R}$ *of some contact vector field in* $J^1\mathbf{R}$. The lifting procedure is described in [IB, KLV, OL]; we shall not need its full description — for our aims it suffices to note that the contact field with generating function φ is

$$K_\varphi = -\varphi_p\frac{\partial}{\partial x} + (\varphi - p\varphi_p)\frac{\partial}{\partial y} + (\varphi_x + p\varphi_y)\frac{\partial}{\partial p},$$

where $p = p_1$, and its lift to $J^k\mathbf{R}$ differs from the right-hand part of this expression only in terms containing $\partial/\partial p_2, \ldots, \partial/\partial p_k$. Vector field K_φ coincides with the restriction of the operator $X_\varphi - \varphi_p D$ to the space of functions in x, y, p. Therefore, if X_φ is a symmetry of $\mathcal{E}$, then both X_φ and the lift of K_φ specify the same element of the quotient space $\mathrm{Sym}(\mathcal{E})$.

A particular class of contact fields is composed of lifts of the so called infinitesimal *point transformations*, i.e. infinitesimal changes of independent and dependent variables or, in other words, vector fields in the space $J^0\mathbf{R}$ with coordinates x, y. Point transformations may be characterized by *linear* dependence of their generating functions on p. Indeed, according to (7), the generating function of a symmetry X is recovered by means of the formula $\varphi = \omega_0(X)$ and

$$\omega_0(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \ldots) = \beta - \alpha p.$$

Examples. Here are point transformations which are most frequently encountered in applications.

1. Translation in x is the lift of the field $\partial/\partial x$ with generating function $-p$.
2. Translation in y is the lift of the field $\partial/\partial y$ with generating function 1.
3. Scale transformation is the lift of $ax\partial/\partial x + by\partial/\partial y$, $a, b \in \mathbf{R}$. Its generating function is $by - apx$.

Remark. By commuting translations and scale transformations, one always obtains translations, so that any Lie algebra consisting of vector fields of these two kinds, is solvable.

As the reader may see, the 'inner' approach to symmetries is wider than the 'outer' approach which is confined only to contact transformations. From the 'outer' viewpoint, symmetries of the Cartan's distribution of an equation $\mathcal{E}$ should be considered as 'higher symmetries', since they are restrictions to $\mathcal{E}$ of symmetries of the Cartan's distribution on the space of jets of infinite order [KLV]. H. Stephani [Sph] calls them 'dynamic symmetries'.

However, all these 'higher' symmetries of ordinary differential equations can be reduced to mere point symmetries if one studies, instead of a single equation, the corresponding first order system. We will explain the relevant theory in the general context of finite type systems.

4. Finite type systems

Let us be given a system of differential equations

$$
\begin{aligned}
&F_1(x, u, \frac{\partial u}{\partial x}) = 0, \\
&\quad \dots \\
&F_r(x, u, \frac{\partial u}{\partial x}) = 0,
\end{aligned}
\tag{8}
$$

where u and x stand for the sets of dependent and independent variables $u^1, \dots, u^m$ and $x^1, \dots, x^n$, respectively, while $\partial u/\partial x$ designates the set of all derivatives of the former over the latter up to a fixed order k.

The study of such system is geometrized like before, by considering the submanifold $\mathcal{E}$ in the jet space $J^k(n, m)$ equipped with the Cartan's distribution. A natural system of coordinates in $J^k(n, m)$ is composed of all x^i for $i = 1, \dots, n$, all u^j for $j = 1, \dots, m$ and variables u^j_σ where $1 \le j \le m$ while $\sigma = i_1 i_2 \dots i_n$ is a multiindex with $|\sigma| = i_1 + i_2 + \dots + i_n \le k$. The Cartan's distribution is defined by the set of 1-forms $\omega^j_\sigma = du^j_\sigma - \sum_{i=1}^n u^j_{\sigma+(i)} dx^i$ for all $j = 1, \dots, m$ and $|\sigma| \le k - 1$, where $\sigma + (i)$ is obtained from σ by augmenting its i-th component by 1. Solutions of equation* (8) correspond to n-dimensional integral manifolds of the restricted distribution $\mathcal{C}_\mathcal{E}$ whose projection to the $(x^1, \dots, x^n)$-plane is non-degenerate.

The above described general notation for jet coordinates is rather cumbersome, so in the situation when we have one unknown function u of two independent variables x, y, we will make use of Monge's notation

$$x, y, u, p, q, r, s, t$$

instead of $x^1, x^2, u^1, u^1_{10}, u^1_{01}, u^1_{20}, u^1_{11}, u^1_{02}$, respectively.

Example 8. Consider the system

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1 - \cos u}{y}, \\
\frac{\partial u}{\partial y} &= \frac{\sin u}{y}
\end{aligned}
$$

in one unknown function u of two independent variables x, y. The relevant jet space $J^1(2, 1)$ is 5-dimensional with coordinates x, y, u, p, q. The equation under study specifies a 3-dimensional submanifold

$$\{p = \frac{1 - \cos u}{y},\ q = \frac{\sin u}{y}\}$$

in this space, and the form of Cartan

$$du - pdx - qdy,$$

* Here and below the word *equation* (singular) will be used as synonym to *system of equations*.

when restricted to this submanifold, becomes

$$du - \frac{1 - \cos u}{y} dx - \frac{\sin u}{y} dy,$$

which coincides, up to notations and a non-zero factor, with the 1-form ω of Example 2, provided that u is considered as a function defined in the upper half of the (x, y)-plane and taking values modulo 2π. It follows that our system has a 1-parametric family of solutions given by maximal integral manifolds of the oricycle distribution.

A system of differential equations is said to have *finite type* if its solution space is *finite-dimensional*. Any system of ordinary differential equations is obviously of finite type. In order that a system of partial differential equations might have finite type, it normally needs to be *overdetermined* in the sense that the number of equations in the system should exceed the number of unknown functions.*

If the Cartan's distribution of an equation is n-dimensional and completely integrable, as in Example 8, this equation is sure to have finite type. However, this is not the only class of finite type systems.

Example 9. The system

$$\begin{aligned} u_{xx} &= 0 \\ u_{yy} &= 0 \end{aligned}$$

is evidently of finite type, since its solutions are exausted by the 4-parametric family $u = axy + bx + cy + d,\ a, b, c, d \in \mathbf{R}$. However, the corresponding geometric image is a 6-dimensional plane $\{r = 0,\ t = 0\}$ in the 8-dimensional manifold $J^2(2, 1)$ whose Cartan distribution given by

$$\begin{aligned} du - pdx - qdy &= 0 \\ dp - sdy &= 0 \\ dq - sdx &= 0, \end{aligned}$$

is 3-dimensional and not completely integrable.

If we consider the *first prolongation* of this system

$$\begin{aligned} u_{xxx} &= 0 \\ u_{xxy} &= 0 \\ u_{xyy} &= 0 \\ u_{yyy} &= 0, \end{aligned}$$

obtained by taking total derivatives of all the equations of the initial system, we will arrive at a 8-dimensional submanifold (a plane, to be exact) in the jet space $J^3(2, 1) \cong \mathbf{R}^{12}$ equipped with a 2-dimensional completely integrable distribution. Thus, this example also can be studied in terms of completely integrable distributions, but to effectuate this reduction, one first has to prolongate the equation.

---

* This is always true, e.g., when the system is *involutive* (see [Ts], [Po1]).

The above definition of finite type systems is not constructive in the sense that one should know the size of the space of solutions in order to decide whether the system is of finite type. There is a class of systems for which one can easily say that they are of finite type. We are speaking about systems whose *infinite prolongation* (see [KLV], [Ts]) is finite-dimensional. This implies that after some number of prolongations the system can be resolved with respect to all derivatives of a fixed highest order k. In this case all variables u^i_σ with $|\sigma| \geq k$ can be expressed through x^j, u^s and u^s_τ with $|\tau| < k$, so that the space $\mathcal{A} = C^\infty(J^{k-1})$ of smooth functions on the manifold of $(k-1)$-jets serves as a closed universe for all calculations related to the given system—much in the same way as the space of functions in $x, y, y_1, \ldots, y_{k-1}$ did in the case of ordinary differential equations.

In this space $\mathcal{A}$ the restrictions of total derivative operators

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma+(i)} \frac{\partial}{\partial u^j_\sigma}$$

are defined. A generating function of a symmetry is represented by a row $(\varphi^1, \ldots, \varphi^m) \in \mathcal{A}^m$, where m is the number of dependent variables $u^1, \ldots, u^m$. The vector field in J^{k-1} corresponding to this function is given by the formula

$$X_{(\varphi^1,\ldots,\varphi^m)} = \sum_{\sigma,j} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma},$$

where $D_{i_1,\ldots,i_n} = D_1^{i_1} \circ \cdots \circ D_n^{i_n}$. The generating function of a symmetry for a system like (8) should satisfy r equations

$$\sum_{\sigma,j} \frac{\partial F_i}{\partial u^j_\sigma} D_\sigma(\varphi^j) = 0, \; i = 1, \ldots, r.$$

This constitutes a counterpart of Theorem 5 in the case of finite type systems. All these formulas are valid for general systems of p.d.e.'s, but they have to be considered in infinite-dimensional spaces (see [KLV], [Ts] for proofs and details).

Example. The finite type system

$$\begin{aligned} u_x &= yv, \\ u_y &= xv, \\ v_x &= yu, \\ v_y &= xu \end{aligned}$$

is described by a 4-dimensional surface with coordinates $x = x^1$, $y = x^2$, $u = u^1$, $v = u^2$ in 8-dimensional jet space $J^1(2, 2)$. Below, we give the step-by-step records of the integration procedure of Section 2 applied in this case (however, not explaining how the symmetries S_1 and S_2 were found).

Basic forms of Cartan's distribution

$$\omega_1 = du - yvdx - xvdy,$$
$$\omega_2 = dv - yudx - xudy.$$

Symmetries:

$$S_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$
$$S_2 = e^{xy}(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}).$$

Their generating functions:

$$\varphi_1 = \begin{pmatrix} u \\ v \end{pmatrix},\ \varphi_2 = \begin{pmatrix} e^{xy} \\ e^{xy} \end{pmatrix}.$$

Pairing matrix $S_i(\varphi_j)$:

$$\Xi = \begin{pmatrix} u & e^{xy} \\ v & e^{xy} \end{pmatrix}.$$

New basic 1-forms:

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \Xi^{-1} \begin{pmatrix} du - yvdx - xvdy \\ dv - yudx - xudy \end{pmatrix} = \begin{pmatrix} \frac{d(u-v)}{u-v} + d(xy) \\ \frac{-vdu+udv}{e^{xy}(u-v)} - e^{-xy}(u+v)d(xy) \end{pmatrix}$$

First integrals:

$$\omega_1' = d[\log(u-v) + xy] \sim dh_1,\ h_1 = e^{xy}(u-v),$$
$$\omega_2'\big|_{h_1=\text{const}} = d[e^{-xy}(u+v)],\ h_2 = e^{-xy}(u+v).$$

General solution:

$$u = C_1e^{xy} + C_2e^{-xy},$$
$$v = C_1e^{xy} - C_2e^{-xy}.$$

5. More examples for o.d.e.'s

Here we give several examples of how the Theorem 4 can yield explicit solutions for o. d. e. 's. Further examples can be found in an extensive literature that comprises, e.g., the books [Ol], [Ovs], [Sph], [BK]. Note that the fundamental problem related to our method is to *find* enough symmetries of a given differential equation in order that the integration procedure could work. In Example 12, we show how one does this by equating coefficients in the case of point symmetries. This technique is already quite traditional and is described here for the sake of the readers who are not familiar with the literature. One should note, however, that in recent years many computer programs have appeared that enable one to search the symmetries interactively. Among these, we would like to mention the well-known REDUCE package by F. Schwarz and the DELiA program referred to below in the Acknowledgements.

Example 10. Let $\mathcal{E}$ be the linear equation

$$\alpha_k y^{(k)} + \cdots + \alpha_1 y' + \alpha_0 y = g,$$

where $\alpha_k, \ldots, \alpha_1, \alpha_0, g$ are given functions of x. An arbitrary solution of the corresponding homogeneous equation

$$\alpha_k y^{(k)} + \cdots + \alpha_1 y' + \alpha_0 y = 0$$

is a symmetry (generating function of a symmetry, to be precise) of the initial equation. The space of all solutions makes a k-dimensional commutative algebra, and if one knows its basis, i.e. a fundamental system of solutions of the homogeneous equation, then the inhomogeneous equation can be integrated by means of quadratures.

The computations needed to actually carry out the integration in this case, are precisely the same which one comes accross when applying the usual trick of 'variation of constants'. In particular, the matrix Ξ here consists of functions of x and hence is nothing but the usual Wronski matrix of the fundamental system involved.

Example 11. In the well-known reference book on ordinary differential equations [Ka] one can find the following equations:

$$4y^2 y''' - 18yy'y'' + 15y'^3 = 0,$$
$$9y^2 y''' - 45yy'y'' + 40y'^3 = 0,$$

which are treated as separate examples (No. 7.8 and 7.9). For each of them, a separate solution procedure is recommended. However, it is readily seen that both equations as well as the arbitrary equation of the form

$$ay^2 y''' + byy'y'' + cy'^3 = 0, \ a, b, c \in \mathbf{R}, \tag{9}$$

possess a 3-dimensional solvable Lie algebra of symmetries consisting of the translation in x and two independent scale transformations. It is convenient to take the functions $\varphi_1 = y$, $\varphi_2 = p$, $\varphi_3 = y + xp$ for a basis of the symmetry algebra and follow the above described scheme to accomplish the integration.

Remark. This equation can be integrated in three steps by succesively lowering its order, using first the translation invariance and then invariance with respect to the two scale transformations. Thus, the procedure of Lie and Cartan is a generalisation of standard methods of order reduction. Note that the sequence in which the symmetries are used in this procedure should conform to the Lie algebra structure of the given set of symmetries. For example, if one tries to use scale invariance of equation (9) first, then the translation symmetry will be lost. In general, on the first step the symmetries belonging to the last (smallest) commutator subalgebra should be used.

Example 12. This one is the longest example in all the paper— it even incorporates a theorem. In compensation, an industrious reader will have an opportunity to see in full detail how all the machinery works and what kind of results one is supposed to obtain. We will find all equations of the form

$$y'' = y' + f(y), \tag{10}$$

which possess a two-dimensional Lie algebra of point symmetries and then find the explicit expression for solutions of these equations in terms of quadratures.

The independent variable x does not enter explicitely into the equation, hence the x-translation is a symmetry. The problem is to determine when this equation has a second symmetry with generating function of the form

$$\varphi = \alpha p + \beta, \tag{11}$$

where $p = p_1$ and α and β are functions of x and y such that $\{p, \varphi\}$ is a linear combination of p and φ. In what follows, we exclude the trivial particular case when the function $f(y)$ is linear.

We will use variables x, y and p as a system of coordinates on the surface $\mathcal{E} \in J^2(1,1)$ corresponding to the given equation. In these coordinates the total derivative operator has the form

$$D = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + (p+f)\frac{\partial}{\partial p}. \tag{12}$$

The generating function of a symmetry $\varphi(x, y, p)$ has to satisfy the equation

$$D^2(\varphi) - D(\varphi) - f'\varphi = 0.$$

Taking into account relations (11) and (12), we can rewrite the last one as

$$\alpha_{yy}p^2 + (2\alpha_{xy} + 2\alpha_y + \beta_{yy})p^2 + (3\alpha_y f + \alpha_x + \alpha_{xx} + 2\beta_{xy})p + (2\alpha_x f + \beta_y f + \beta_{xx} - \beta_x - f'\beta) = 0,$$

which is equivalent to the system of equations

$$\begin{aligned} \alpha_{yy} &= 0, \\ 2\alpha_{xy} + 2\alpha_y + \beta_{yy} &= 0, \\ 3\alpha_y f + \alpha_x + \alpha_{xx} + 2\beta_{xy} &= 0, \\ 2\alpha_x f + \beta_y f + \beta_{xx} - \beta_x - f'\beta &= 0. \end{aligned} \tag{13}$$

The first equation yields $\alpha = \gamma y + \delta$, where γ and δ are some functions of x. Substituting this into the second equation, we obtain $\beta = -(\gamma + \gamma')y^2 + \varepsilon y + \zeta$, where ε and ζ are again functions of x. Then the third equation is reduced to

$$3\gamma f = 3(\gamma + \gamma')y - \delta' - \delta'' - 2\varepsilon'.$$

Since the function $f(y)$ was supposed to be non-linear, it follows that $\gamma = 0$ and $\varepsilon = (d - \delta - \delta')/2$, where $d = \text{const}$.

Now the last equation of the system (13) takes the form

$$(\varepsilon y + \zeta)f' - \eta f = \theta y + \lambda \tag{14}$$

with $\eta = 2\delta' + \varepsilon$, $\theta = \varepsilon'' - \varepsilon'$, which is an ordinary differential equation with respect to the function $f(y)$, where the variable x enters as parameter. Its general solution, under the assumptions $\varepsilon \neq 0$, $\eta \neq 0$, $\varepsilon \neq \eta$, is given by the formula

$$f = \mu\big(y + \frac{\zeta}{\varepsilon}\big)^{\eta/\varepsilon} + \frac{\theta}{\varepsilon - \eta}\big(y + \frac{\zeta}{\varepsilon}\big) + \frac{\theta\zeta - \varepsilon\lambda}{\varepsilon\eta}, \tag{15}$$

where μ is an arbitrary function of x.

Among all functions (15) we have to choose those which depend only on y and are nonlinear in y. The former requirement holds if and only if all of the functions μ, ζ/ε, η/ε, $\theta/(\varepsilon - \eta)$, $(\theta\zeta - \varepsilon\lambda)/(\varepsilon\eta)$ are constants. Denoting the constants μ, ζ/ε, η/ε by a, b and c respectively and taking into account all the relations among functions under consideration, we arrive at the following expressions:

$$\begin{aligned}
&\varepsilon = -\frac{k+1}{2}ae^{kx}, \\
&\zeta = b\varepsilon, \\
&\eta = c\varepsilon, \\
&\theta = (k^2 - k)\varepsilon, \\
&\lambda = (k^2 - k)b\varepsilon,
\end{aligned}$$

where $k = \frac{1-c}{c+3}$ (note that $c \neq -3$ if $\varepsilon \neq 0$). Hence

$$f(y) = a(y + b)^c - \frac{2c + 2}{(c + 3)^2}y.$$

Now consider the possibilities previuosly excluded. Either of assumptions $\eta = 0$ and $\eta = \varepsilon$ results in linearity of the function $f(y)$. In the case $\varepsilon = 0$ we obtain a new series of solutions to equation (14),

$$f(y) = ae^{by} - \frac{2}{b}, \;\; a, b \in \mathbf{R}.$$

The computations accomplished can be summarized as follows.

Theorem 6. *Among all nonlinear equations of the form $y'' = y' + f(y)$ only equations of the two following series:*

$$\text{(A)} \qquad y'' = y' + a(y+b)^c - \frac{2c+2}{(c+3)^2}y, \ a,b,c \in \mathbf{R}, \ c \neq -3,$$

$$\text{(B)} \qquad y'' = y' + ae^{by} - \frac{2}{b}, \ a,b \in \mathbf{R}, \ b \neq 0$$

possess a two-dimensional algebra of point symmetries. In case (A) *a basis of this algebra may be chosen to consist of functions*

$$\varphi_1 = p, \ \varphi_2 = e^{kx}(p - \frac{k+1}{2}y)$$

with $k = \frac{1-c}{c+3}$, in case (B) *the two functions*

$$\varphi_1 = p, \ \varphi_2 = e^{-x}(p - \frac{2}{b})$$

can be taken as a basis.

To complete the example, we will carry out the integration procedure for the equation of series (A) with $a = 1$, $b = 0$, $c = -2$,

$$y'' = y' + y^{-2} + 2y. \tag{16}$$

The corresponding algebra of point symmetries is generated by $\varphi_1 = p$ and $\varphi_2 = e^{3x}(p-2y)$. We have $\{\varphi_1, \varphi_2\} = -3\varphi_2$. The commutator subalgebra $\mathcal{G}^{(1)}$ is generated by φ_2, hence the basis φ_1, φ_2 is written exactly in the order prescribed by the algorithm. The pairing matrix of symmetries and basic Cartan's 1-forms is

$$\Xi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ D\varphi_1 & D\varphi_2 \end{pmatrix} = \begin{pmatrix} p & e^{3x}(p-2y) \\ q & e^{3x}(q+p-6y) \end{pmatrix},$$

where $q = p + y^{-2} + 2y$. Its determinant is equal to $e^{3x}T$, where $T = p^2 - 6yp + 2yq = p^2 - 4yp + 2y^{-1} + 4y^2$. The new basic 1-forms are computed as follows:

$$\begin{pmatrix} \omega_0' \\ \omega_1' \end{pmatrix} = \Xi^{-1} \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} = \frac{e^{-3x}}{T} \begin{pmatrix} e^{3x}(q+p-6y) & e^{3x}(2y-p) \\ -q & p \end{pmatrix} \begin{pmatrix} dy - pdx \\ dp - qdx \end{pmatrix},$$

whence

$$\omega_0' = \frac{1}{T}[(2p + y^{-2} - 4y)dy + (2y-p)dp] - dx,$$
$$\omega_1' = \frac{1}{T}[-e^{-3x}(p + y^{-2} + 2y)dy + e^{-3x}pdp].$$

The form ω_0' is closed; its integral is $-\frac{1}{2}\log|T| - x$. Instead of this function it is more convenient to take

$$h_1 = \exp(-2\int \omega_0') = e^{2x}(p^2 - 4yp + 2y^{-1} + 4y^2)$$

which is also a first integral of the equation.

Now we are to integrate the 1-form ω_1' on the surface H_{c_1}, defined by $h_1 = c_1 = \text{const}$. On this surface

$$p = 2y \pm \sqrt{c_1 e^{-2x} - 2y^{-1}}$$

and hence

$$\omega_1' = \big(-e^{-3x} \mp \frac{2ye^{-3x}}{\sqrt{c_1 e^{-2x} - 2y^{-1}}}\big)dx \pm \frac{e^{-3x}}{\sqrt{c_1 e^{-2x} - 2y^{-1}}}dy.$$

Computation of the integral of this 1-form depends on the sign of the constant c_1. For $c_1 > 0$ we have

$$h_2 = \frac{1}{3}e^{-3x} \pm \frac{ye^{-2x}}{c_1}\sqrt{c_1 - 2y^{-1}e^{2x}} \pm c_1^{-3/2} \log\left|\frac{\sqrt{c_1 - 2y^{-1}e^{2x}} + \sqrt{c_1}}{\sqrt{c_1 - 2y^{-1}e^{2x}} - \sqrt{c_1}}\right|$$

and for $c_1 < 0$

$$h_2 = \frac{1}{3}e^{-3x} \pm \frac{ye^{-2x}}{c_1}\sqrt{c_1 - 2y^{-1}e^{2x}} \mp (-c_1)^{-3/2} \arctan\sqrt{\frac{2}{c_1}y^{-1}e^{2x} - 1}.$$

The general solution to equation (16) is thus given by implicit formula $h_2 = c_2 = \text{const}$. Graphs of these solutions for all values of c_1 and c_2 fill the region on the surface $\mathcal{E}$ where the algebra $\mathcal{G}$ is transversal to the Cartan's distribution. The complement of this region is defined by $\det \Xi = 0$, i.e. $c_1 = 0$ or, explicitely, $p = 2y \pm \sqrt{-2y^{-1}}$. The latter relation can be interpreted as an ordinary differential equation of first order whose solutions

$$y = -\frac{1}{2}(Ce^{3x} \mp 1)^{2/3}, \; C = \text{const},$$

are singular solutions of equation (16).

Remark 1. The famous MACSYMA system (Symbolics MacIvory, 1989 edition) cannot solve equation (16).

Remark 2. One could from the very beginning reduce equation (16) to a first order equation via the standard substitution $y' = z(y)$ (using translation invariance). However, this would lead to the Abel's equation

$$z\frac{dz}{dy} = z + y^{-2} + 2y, \tag{17}$$

which is rather difficult to integrate. The reason is that (17) does not inherit the other symmetry of (16), since the translation does not belong to the commutant subalgebra. It seems that the simplest way to solve equation (17) consists in passing over to second order equation (16) (which *ipso facto* has the translational symmetry), finding another invisible symmetry of the latter and then integrating it with the help of the two symmetries—taken in inverse order, according to the structure of the Lie algebra. After doing this, one has to filter the obtained two-parametric family of solutions of (16) thus chosing solutions of (17).

Of course, similar computations can be carried out for arbitrary values of parameters a, b, and c (see Theorem 6). Integration of equations of type (A) reduces finally to integration of a differential binomial, and Chebyshev's theorem implies that solutions are elementary functions whenever $c = -\frac{2}{n} - 1$, where n is integer—otherwise they are expressed through elliptic integrals. For two specific values of parameter c, $c = 2$ and $c = 3$, this equation was solved by Painlevé [Pa], who, however, gave no hint as to how he did that. Equations of type (B) have elementary solutions for all a and b.

6. Integration via overdetermination

It is well known that (systems of) partial differential equations (p.d.e.) are very reluctant towards explicit integration. Even finding one separate analytic solution often should be considered as a big success. Following the line of our previous argument, here we propose a method of searching for particular solutions which is in a sense 'perpendicular' to the standard procedure of computing invariant, or automodel, solutions. However, the method is also based on symmetries, and the equation under study should possess enough symmetries in order that the method be applicable.

The idea is the following. Given a p.d.e. or a system which is not of finite type but has an ample symmetry algebra, one should find complementary equations in such a way that the resulting system would have finite type and inherit as many symmetries of the initial system as are necessary for the integration procedure described in Section 2, i.e. in quantity equal to the codimension of Cartan's distribution of the overdetermined system.

Since our aim is only to sketch a certain method, in the subsequent considerations we will always mean generic situations, assertions which hold on some open sets, or for almost all values of parameters, etc., without explicitely mentioning that. We hope that the reader will excuse this liberty.

Let us discuss in more detail a possible way of hunting for particular solutions of one second order equation imposed on the function $u(x, y)$. In Monge's notation such an equation is written as

$$f(x, y, u, p, q, r, s, t) = 0. \tag{18}$$

Suppose that we know a 4-dimensional solvable Lie algebra $\mathcal{G}$ of symmetries of this equation. We may try to find another equation of the same kind

$$g(x, y, u, p, q, r, s, t) = 0 \tag{19}$$

which is compatible with the first one and admits the same algebra $\mathcal{G}$. Geometrically, the system (18)-(19) represents a 6-dimensional submanifold of the Monge's jet space $J^2(2, 1)$. The relevant Cartan's distribution is given by the set of 1-forms

$$\begin{aligned} &du - pdx - qdy \\ &dp - rdx - sdy \\ &dq - sdx - tdy \end{aligned} \tag{20}$$

so that its restriction to the system is 3-dimensional, which does not meet our needs, because the solutions we are looking for should be represented by 2-dimensional surfaces.

However, after one prolongation the system will consist of 6 equations so that the dimension of the corresponding submanifold of $J^3(2,1)$ will be 6. If we choose the system of coordinates on this submanifold to be x, y, u, p, q, s, then to obtain a basis of the Cartan's distribution, we can add the 1-form $ds - u_{xxy}dx - u_{xyy}dy$ to the previously written set, which means that the Cartan's distribution of the prolongated equation is exactly 2-dimensional.

Let us try to run the entire procedure, finding explicitely a 4-parametric family of solutions for the Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = u^2,$$

or in Monge's notation,

$$rt - s^2 = u^2. \tag{21}$$

This equation possesses a huge amount of symmetries. Already its algebra of point symmetries has dimension 6. Let us take four point symmetries which are most 'visible', two translations with generating functions $\varphi_1 = p$ and $\varphi_2 = q$ and two scale transformations with generating functions $\varphi_3 = xp - yq$ and $\varphi_4 = u$. The lifts of the corresponding contact vector fields to the space of 2-jets are

$$\begin{aligned}
X_1 &= -\frac{\partial}{\partial x}, \\
X_2 &= -\frac{\partial}{\partial y}, \\
X_3 &= -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} + 2r\frac{\partial}{\partial r} - 2t\frac{\partial}{\partial t}, \\
X_4 &= u\frac{\partial}{\partial u} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} + r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t},
\end{aligned}$$

and they form a solvable Lie algebra $\mathcal{G}$.

The general second order equation invariant under this algebra is

$$F(u^{-2}pq, up^{-2}r, u^{-1}s, uq^{-2}t) = 0, \tag{22}$$

F being an arbitrary function of specified arguments. One may try to determine all equations (22) compatible with equation (21) in the sense that they have a 4-parametric family of common solutions. This notion of *complete compatibility* is equivalent to complete integrability of the Cartan's distribution after the first prolongation and can be handled in the following way.

In the generic situation the system (21)—(22) can be rewritten as

$$\begin{aligned}
r &= \frac{u(u^2 + s^2)}{q^2\, f(u^{-2}pq, u^{-1}s, u^2p^{-2}q^{-2}(u^2 + s^2))}, \\
t &= u^{-1}q^2\, f(u^{-2}pq, u^{-1}s, u^2p^{-2}q^{-2}(u^2 + s^2)),
\end{aligned}$$

where f is an arbitrary function of its arguments. These equations should be once differentiated giving some expressions for the third derivatives $K = u_{xxx}$, $L = u_{xxy}$, $M = u_{xyy}$, $N = u_{yyy}$ in terms of x, y, u, p, q, s. All geometrical objects here and below are assumed to be defined on the resulting 6-dimensional submanifold in $J^3(2,1)$. We will use variables x, y, u, p, q and s as coordinates on this manifold, understanding all the other involved letters as their functions.

The basic vector fields that span the Cartan's distribution of the prolongated system are

$$X = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + r\frac{\partial}{\partial p} + s\frac{\partial}{\partial q} + L\frac{\partial}{\partial s},$$
$$Y = \frac{\partial}{\partial y} + q\frac{\partial}{\partial u} + s\frac{\partial}{\partial p} + t\frac{\partial}{\partial q} + M\frac{\partial}{\partial s},$$

and the integrability condition $[X, Y] = \lambda X + \mu Y$ is equivalent to

$$X(M) = Y(L). \tag{23}$$

This is a first order nonlinear partial differential equation imposed on the function f. Of course, it is rather complicated (we do not even write it explicitly in terms of f!) and the problem of finding its general solution is difficult. But every particular solution of this equation gives rise to a specific auxiliary equation (22) and thus to a 4-parametric family of solutions to the initial equation (21). One may try various ansatzes for the function f in order to find a solution of (24). We have tried $f = 1$ and found out that it just complies with the integrability condition (24).

Let us execute the integration procedure for the system

$$\begin{aligned} r &= us^2q^{-2} + u^3q^{-2}, \\ t &= u^{-1}q^2, \end{aligned} \tag{24}$$

corresponding to $f = 1$. We will use the algebra $X_1, \ldots, X_4$ and the following set of basic 1-forms for the Cartan's distribution:

$$\begin{aligned} \omega_1 &= du - pdx - qdy \\ \omega_2 &= dp - rdx - sdy \\ \omega_3 &= dq - sdx - tdy \\ \omega_4 &= ds - Ldx - Mdy, \end{aligned}$$

where

$$\begin{aligned} L &= 3q^{-1}s^2 - 2u^{-1}ps + q^{-1}u^2, \\ M &= 2u^{-1}qs - u^{-2}pq^2. \end{aligned}$$

Since the commutator subalgebra of $\mathcal{G}$ is spanned by X_1 and X_2, by theorem 2 we know that in the new basis of 1-forms $\omega'_1, \ldots, \omega'_4$, such that $\omega'_i(X_j) = \delta_{ij}$, two forms, ω'_3 and ω'_4, will be closed. A simple calculation yields

$$\omega'_3 = \frac{1}{2uqs - 2pq^2}[qsdu - q^2dp + (pq - 2us)dp + uqds]$$

and

$$\int \omega_3' = -\frac{1}{2}\log(us - pq) + \log q.$$

The function $h = (us - pq)/q^2$ is thus a first integral of the system (23). According to the general scheme, we now have to integrate ω_4' and then, restricting everything to the mutual level surfaces of the two first integrals, compute the integrals of the two remaining basic 1-forms. But profiting by the apparent simplicity of function h, we might as well interrupt the smooth flow of the algorithm here and start it anew in a simpler environment. Here is what we mean by that.

Existence of the first integral h implies that the system (23) is equivalent to a 1-parametric family of equations

$$\begin{aligned} r &= us^2q^{-2} + u^3q^{-2}, \\ s &= Cu^{-1}q^2 + u^{-1}pq, \\ t &= u^{-1}q^2 \end{aligned} \tag{25}$$

for arbitrary $C \in \mathbf{R}$. For any specific value of C equation (25) represents a 3-dimensional submanifold in $J^2(2,1)$ and the corresponding Cartan's distribution is 2-dimensional, so that there is no need for prolongation and everything can be done within a 5-dimensional manifold with coordinates x, y, u, p, q, using the set of 1-forms (20) and only three symmetries, say X_1, X_2 and X_4. This algebra is commutative, so that all the three 1-forms of the new recalculated basic set are closed. Their integration gives the following first integrals:

$$\begin{aligned} &\frac{au}{q} + x = \text{const}, \\ &\frac{a^2u^5}{5q^5} + \frac{aup}{q^2} + \frac{u}{q} - y = \text{const}, \\ &\frac{a^2u^4}{4q^4} + \frac{ap}{q} - \log q = \text{const}, \end{aligned}$$

where $a = C^{-1}$ (which almost always makes sense!), and after eliminating p and q we arrive at the explicit formula for a 4-parametric family of solutions to the initial equation (21):

$$u = \frac{x + C_1}{a}\exp\Big[\frac{(x + C_1)^4}{20a^2} + \frac{a(y + C_2)}{x + C_1} + C_3\Big].$$

Remark. One could try to handle the system (23) by first solving its second equation. This way leads, however, to a rather intricate nonlinear second order o.d.e. which the authors were not smart enough to cope with. This is yet another confirmation of S.Lie's motto

Use symmetries to solve differential equations!

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