Lecture 1. Basic Systems

1.1. What is an exterior differential system?

An *exterior differential system* (EDS) is a pair $(M,\mathcal{I})$ where M is a smooth manifold and $\mathcal{I} \subset \Omega^*(M)$ is a graded ideal in the ring $\Omega^*(M)$ of differential forms on M that is closed under exterior differentiation, i.e., for any ϕ in $\mathcal{I}$, its exterior derivative $d\phi$ also lies in $\mathcal{I}$.

The main interest in an EDS $(M,\mathcal{I})$ centers around the problem of describing the submanifolds $f : N \to M$ for which all the elements of $\mathcal{I}$ vanish when pulled back to N, i.e., for which $f^*\phi = 0$ for all $\phi \in \mathcal{I}$. Such submanifolds are said to be *integral manifolds* of $\mathcal{I}$. (The choice of the adjective 'integral' will be explained shortly.)

In practice, most EDS are constructed so that their integral manifolds will be the solutions of some geometric problem one wants to study. Then the techniques to be described in these lectures can be brought to bear.

The most common way of specifying an EDS $(M,\mathcal{I})$ is to give a list of generators of $\mathcal{I}$. For $\phi_1, \ldots, \phi_s \in \Omega^*(M)$, the 'algebraic' ideal consisting of elements of the form

$$\phi = \gamma^1 \wedge \phi_1 + \cdots \gamma^s \wedge \phi_s$$

will be denoted $\langle \phi_1, \ldots, \phi_s \rangle_{\text{alg}}$ while the differential ideal $\mathcal{I}$ consisting of elements of the form

$$\phi = \gamma^1 \wedge \phi_1 + \cdots \gamma^s \wedge \phi_s + \beta^1 \wedge d\phi_1 + \cdots \beta^s \wedge d\phi_s$$

will be denoted $\langle \phi_1, \ldots, \phi_s \rangle$.

Exercise 1.1: Show that $\mathcal{I} = \langle \phi_1, \ldots, \phi_s \rangle$ really is a differentially closed ideal in $\Omega^*(M)$. Show also that a submanifold $f : N \to M$ is an integral manifold of $\mathcal{I}$ if and only if $f^*\phi_\sigma = 0$ for $\sigma = 1, \ldots, s$.

The p-th graded piece of $\mathcal{I}$, i.e., $\mathcal{I} \cap \Omega^p(M)$, will be denoted $\mathcal{I}^p$. For any $x \in M$, the evaluation of $\phi \in \Omega^p(M)$ at x will be denoted ϕ_x and is an element of $\Omega^p_x(M) = \Lambda^p(T^*_x M)$. The symbols $\mathcal{I}_x$ and $\mathcal{I}^p_x$ will be used for the corresponding concepts.

Exercise 1.2: Make a list of the possible ideals in $\Lambda^*(V)$ up to isomorphism, where V is a vector space over $\mathbb{R}$ of dimension at most 4. (Keep this list handy. We'll come back to it.)

1.2. Differential equations reformulated as EDSs

Élie Cartan developed the theory of exterior differential systems as a coordinate-free way to describe and study partial differential equations. Before I describe the general relationship, let's consider some examples:

Example 1.1: *An Ordinary Differential Equation.* Consider the system of ordinary differential equations

$$y' = F(x, y, z)$$
$$z' = G(x, y, z)$$

where F and G are smooth functions on some domain $M \subset \mathbb{R}^3$. This can be modeled by the EDS $(M,\mathcal{I})$ where

$$\mathcal{I} = \langle\, dy - F(x, y, z)\, dx,\ dz - G(x, y, z)\, dx \,\rangle.$$

It's clear that the 1-dimensional integral manifolds of $\mathcal{I}$ are just the integral curves of the vector field

$$X = \frac{\partial}{\partial x} + F(x, y, z)\,\frac{\partial}{\partial y} + G(x, y, z)\,\frac{\partial}{\partial z}.$$

Example 1.2: *A Pair of Partial Differential Equations.* Consider the system of partial differential equations

$$z_x = F(x,y,z)$$
$$z_y = G(x,y,z)$$

where F and G are smooth functions on some domain $M \subset \mathbb{R}^3$. This can be modeled by the EDS $(M, \mathcal{I})$ where

$$\mathcal{I} = \langle\, dz - F(x,y,z)\, dx - G(x,y,z)\, dy\,\rangle.$$

On any 2-dimensional integral manifold $N^2 \subset M$ of $\mathcal{I}$, the differentials dx and dy must be linearly independent (Why?). Thus, N can be locally represented as a graph $\bigl(x, y, u(x,y)\bigr)$ The 1-form

$$dz{-}F(x,y,z)\, dx{-}G(x,y,z)\, dy$$

vanishes when pulled back to such a graph if and only if the function u satisfies the differential equations

$$u_x(x,y) = F\bigl(x,y,u(x,y)\bigr)$$
$$u_y(x,y) = G\bigl(x,y,u(x,y)\bigr)$$

for all (x,y) in the domain of u.

Exercise 1.3: Check that a surface $N \subset M$ is an integral manifold of $\mathcal{I}$ if and only if each of the vector fields

$$X = \frac{\partial}{\partial x} + F(x,y,z)\,\frac{\partial}{\partial z} \qquad \text{and} \qquad Y = \frac{\partial}{\partial y} + G(x,y,z)\,\frac{\partial}{\partial z}$$

is tangent to N at every point of N. In other words, N must be a union of integral curves of X and also a union of integral curves of Y. By considering the special case $F = y$ and $G = -x$, show that there need not be any 2-dimensional integral manifolds of $\mathcal{I}$ at all.

Example 1.3: *Complex Curves in* $\mathbb{C}^2$. Consider $M = \mathbb{C}^2$, with coordinates $z = x + i\,y$ and $w = u + i\,v$. Let $\mathcal{I} = \langle \phi_1, \phi_2 \rangle$ where ϕ_1 and ϕ_2 are the real and imaginary parts, respectively, of

$$dz \wedge dw = dx \wedge du - dy \wedge dv + i\,(dx \wedge dv + dy \wedge du).$$

Since $\mathcal{I}^1 = (0)$, any (real) curve in $\mathbb{C}^2$ is an integral curve of $\mathcal{I}$. A (real) surface $N \subset \mathbb{C}^2$ is an integral manifold of $\mathcal{I}$ if and only if it is a complex curve. If dx and dy are linearly independent on N, then locally N can be written as a graph $\bigl(x, y, u(x,y), v(x,y)\bigr)$ where u and v satisfy the Cauchy-Riemann equations: $u_x - v_y = u_y + v_x = 0$. Thus, $(M, \mathcal{I})$ provides a model for the Cauchy-Riemann equations.

In fact, any 'reasonable' system of partial differential equations can be described by an exterior differential system. For concreteness, let's just stick with the first order case. Suppose, for example, that you have a system of equations of the form

$$F^\rho\bigl(\mathbf{x}, \mathbf{z}, \frac{\partial \mathbf{z}}{\partial \mathbf{x}}\bigr) = 0, \qquad \rho = 1, \ldots, r,$$

where $\mathbf{x} = (x^1, \ldots, x^n)$ are the independent variables, $\mathbf{z} = (z^1, \ldots, z^s)$ are the dependent variables, and $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$ is the Jacobian matrix of $\mathbf{z}$ with respect to $\mathbf{x}$. The hypotheses that I want to place on the functions F^ρ is that they are smooth on some domain $D \subset \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns}$ and that, at every point $(\mathbf{x}, \mathbf{z}, \mathbf{p}) \in D$ at which all of the F^ρ vanish, one can smoothly solve the above equations for r of the $\mathbf{p}$-coordinates in terms of $\mathbf{x}$, $\mathbf{z}$, and the $ns{-}r$ remaining $\mathbf{p}$-coordinates. If we then let $M^{n+s+ns-r} \subset D$ be the common zero locus of the F^ρ, set

$$\theta^\alpha = dz^\alpha - p^\alpha_i\, dx^i$$

and let $\mathcal{I} = \langle\, \theta^1, \ldots, \theta^s \rangle$. Then any n-dimensional integral manifold $N \subset M$ of $\mathcal{I}$ on which the $\{dx^i\}_{1 \le i \le n}$ are linearly independent is locally the graph of a solution to the original system of first order PDE.

Obviously, one can 'encode' higher order PDE as well, by simply regarding the intermediate partial derivatives as dependent variables in their own right, constrained by the obvious PDE needed to make them be the partials of the lower order partials. For example, in the classical literature, one frequently sees a second order scalar PDE

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

written in the standard classical notation

$$\begin{aligned} 0 &= F(x, y, u, p, q, r, s, t) \\ 0 &= du - p\,dx - q\,dy \\ 0 &= dp - r\,dx - s\,dy \\ 0 &= dq - s\,dx - t\,dy \end{aligned}$$

We would interpret this to mean that the equation $F = 0$ defines a smooth hypersurface M^7 in $xyupqrst$-space and the differential equation is then modeled by the differential ideal $\mathcal{I} \subset \Omega^*(M)$ given by

$$\mathcal{I} = \langle\, du{-}p\,dx{-}q\,dy,\;\; dp{-}r\,dx{-}s\,dy,\;\; dq{-}s\,dx{-}t\,dy\,\rangle.$$

The assumption that the PDE be 'reasonable' is then that not all of the partials (F_r, F_s, F_t) vanish along the locus $F = 0$, so that x, y, u, p, q, and two of r, s, and t can be taken as local coordinates on M.

Exercise 1.4: Show that a second order scalar equation of the form

$$\begin{aligned} A(x, y, u, p, q)\, r + 2B(x, y, u, p, q)\, s + C(x, y, u, p, q)\, t \\ +D(x, y, u, p, q)\,(rt - s^2) + E(x, y, u, p, q) = 0 \end{aligned}$$

(in the classical notation described above) can be modeled on $xyupq$-space (i.e., $M = \mathbb{R}^5$) via the ideal $\mathcal{I}$ generated by $\theta = du{-}p\,dx{-}q\,dy$ together with the 2-form

$$\Upsilon = A\,dp\wedge dy + B\,(dq\wedge dy{-}dp\wedge dx) - C\,dq\wedge dx + D\,dp\wedge dq + E\,dx\wedge dy.$$

(Equations of this kind are known as *Monge-Ampere equations*. They come up very frequently in differential geometry.)

Example 1.4: *Linear Weingarten Surfaces.* This example assumes that you know some differential geometry. Let $M^5 = \mathbb{R}^3 \times S^2$ and let $\mathbf{x} : M \to \mathbb{R}^3$ and $\mathbf{u} : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ be the projections on the two factors. Notice that the isometry group G of Euclidean 3-space acts on M in a natural way, with translations acting only on the first factor and rotations acting 'diagonally' on the two factors together.

Consider the 1-form $\theta = \mathbf{u}\cdot d\mathbf{x}$, which is G-invariant. If $\iota : N \hookrightarrow \mathbb{R}^3$ is an oriented surface, then the lifting $f : N \to M$ given by $f(p) = \big(\iota(p), \nu(p)\big)$ where $\nu(p) \in S^2$ is the oriented unit normal to the immersion ι at p, is an integral manifold of θ. (Why?) Conversely, any integral 2-manifold $f : N \to M$ of θ for which the projection $\mathbf{x}{\circ}f : N \to \mathbb{R}^3$ is an immersion is such a lift of a canonically oriented surface $\iota : N \hookrightarrow \mathbb{R}^3$.

Exercise 1.5: Prove this last statement.

In the classical literature, the elements of M are called the (first order) contact elements of (oriented) surfaces in $\mathbb{R}^3$. (The adjective 'contact' refers to the image from mechanics of two oriented surfaces making contact to first order at a point if and only if they pass through the point in question and have the same unit normal there.)

It is not hard to show that any G-invariant 1-form on M is a constant multiple of θ. However, there are several G-invariant 2-forms (in addition to $d\theta$). For example, the 2-forms

$$\Upsilon_0 = \tfrac{1}{2}\mathbf{u}\cdot(d\mathbf{x}\times d\mathbf{x}), \quad \Upsilon_1 = \tfrac{1}{2}\mathbf{u}\cdot(d\mathbf{u}\times d\mathbf{x}), \quad \Upsilon_2 = \tfrac{1}{2}\mathbf{u}\cdot(d\mathbf{u}\times d\mathbf{u}).$$

are all manifestly G-invariant.

Exercise 1.6: For any oriented surface $\iota : N \hookrightarrow \mathbb{R}^3$ with corresponding contact lifting $f : N \to M$, show that

$$f^*(\Upsilon_0) = dA, \qquad f^*(\Upsilon_1) = -H\,dA, \qquad f^*(\Upsilon_2) = K\,dA.$$

where dA is the induced area form of the immersion ι and H and K are its mean and Gauss curvatures, respectively. Moreover, an integral 2-manifold of θ is a contact lifting if and only if Υ_0 is nonvanishing on it.

From this exercise, it follows, for example, that the contact liftings of minimal surfaces in $\mathbb{R}^3$ are integral manifolds of $\mathcal{I} = \langle \theta, \Upsilon_1 \rangle$. As another example, it follows that the surfaces with Gauss curvature $K = -1$ are integral manifolds of the ideal $\mathcal{I} = \langle \theta, \Upsilon_2 + \Upsilon_0 \rangle$. In fact, any constant coefficient linear equation of the form $a\,K + b\,H + c = 0$ is modeled by $\mathcal{I} = \langle \theta, a\,\Upsilon_2 - b\,\Upsilon_1 + c\,\Upsilon_0 \rangle$. Such equations are called *linear Weingarten equations* in the literature.

Exercise 1.7: Fix a constant r and consider the mapping $\Phi_r : M \to M$ satisfying $\Phi_r(x,u) = (x + ru, u)$. Show that $\Phi^*\theta = \theta$ and interpret what this means with regard to the integral surfaces of θ. Compute $\Phi^*\Upsilon_i$ for $i = 0, 1, 2$ and interpret this in terms of surface theory. In particular, what does this say about the relation between surfaces with $K = +1$ and surfaces with $H = \pm\frac{1}{2}$?

Exercise 1.8: Show that the cone $z^2 = x^2 + y^2$ is the projection to $\mathbb{R}^3$ of an embedded smooth cylinder in M that is an integral manifold of $\langle \theta, \Upsilon_2 \rangle$. Show that the double tractrix (or pseudosphere), a rotationally invariant singular 'surface' with Gauss curvature $K = -1$ at its smooth points, is the projection to $\mathbb{R}^3$ of an embedded cylinder in M that is an integral manifold of $\langle \theta, \Upsilon_2 + \Upsilon_0 \rangle$.

1.3. The Frobenius Theorem

Of course, reformulating a system of PDE as an EDS might not necessarily be a useful thing to do. It will be useful if there are techniques available to study the integral manifolds of an EDS that can shed light on the set of integral manifolds and that are not easily applicable to the original PDE system. The main techniques of this type will be discussed in lectures later in the week, but there are a few techniques that are available now.

The first of these is when the ideal $\mathcal{I}$ is algebraically as simple as possible.

Theorem 1: (The Frobenius Theorem) Let $(M, \mathcal{I})$ be an EDS with the property that $\mathcal{I} = \langle \mathcal{I}^1 \rangle_{\text{alg}}$ and so that $\dim \mathcal{I}^1_p$ is a constant r indepdendent of $p \in M$. Then for each point $p \in M$ there is a coordinate system $\mathbf{x} = (x^1, \ldots, x^{n+r})$ on a p-neighborhood $U \subset M$ so that

$$\mathcal{I}_U = \langle dx^{n+1}, \ldots, dx^{n+r} \rangle.$$

In other words, if $\mathcal{I}$ is algebraically generated by 1-forms and has constant 'rank', then $\mathcal{I}$ is locally equivalent to the obvious 'flat' model. In such a case, the n-dimensional integral manifolds of $\mathcal{I}$ are described locally in the coordinate system $\mathbf{x}$ as 'slices' of the form

$$x^{n+1} = c^1, \qquad x^{n+2} = c^2, \qquad \ldots, \qquad x^{n+r} = c^r.$$

In particular, each connected integral manifold of $\mathcal{I}$ lies in a unique maximal integral manifold, which has dimension n. Moreover, these maximal integral manifolds foliate the ambient manifold M.

If you look back at Example 1.2, you'll notice that $\mathcal{I}$ is generated algebraically by $\mathcal{I}^1$ if and only if it is generated algebraically by

$$\zeta = dz - F(x,y,z)\,dx - G(x,y,z)\,dy,$$

and this, in turn, is true if and only if $\zeta \wedge d\zeta = 0$. (Why?) Now

$$\zeta \wedge d\zeta = \big(F_y - G_x + G\,F_z - F\,G_z\big)\,dx \wedge dy \wedge dz.$$

Thus, by the Frobenius Theorem, if the two functions F and G satisfy the PDE $F_y - G_x + G\,F_z - F\,G_z = 0$, then for every point $(x_0, y_0, z_0) \in M$, there is a function u defined on an open neighborhood of $(x_0, y_0) \in \mathbb{R}^2$ so that $u(x_0, y_0) = z_0$ and so that u satisfies the equations $u_x = F(x, y, u)$ and $u_y = G(x, y, u)$.

Exercise 1.9: State and prove a converse to this last statement.

Note, by the way, that it may not be easy to actually *find* the 'flat' coordinates $\mathbf{x}$ for a given $\mathcal{I}$ that satisfies the Frobenius condition.

Exercise 1.10: Suppose that u and v are functions of x and y that satisfy the equations

$$u_x - v_y = e^u \sin v, \qquad\qquad u_y + v_x = e^u \cos v.$$

Show that $u_{xx}+u_{yy} = e^{2u}$ and that $v_{xx}+v_{yy} = 0$. Conversely, show that if $u(x,y)$ satisfies $u_{xx}+u_{yy} = e^{2u}$, then there exists a one parameter family of functions v so that the pair (u,v) satisfies the displayed equations. Prove a similar existence theorem for a given arbitrary solution of $v_{xx}+v_{yy} = 0$. (This peculiar system is an elementary example of what is known as a *Bäcklund transformation*. More on this later.)

1.4. The Pfaff Theorem

There is another case (or rather, sequence of cases) in which there is a simple local normal form.

Theorem 2: (The Pfaff Theorem) Let $(M,\mathcal{I})$ be an EDS with the property that $\mathcal{I} = \langle\omega\rangle$ for some nonvanishing 1-form ω. Let $r \geq 0$ be the smallest integer for which $\omega{\wedge}(d\omega)^{r+1} \equiv 0$. Then for each point $p \in M$ at which $\omega{\wedge}(d\omega)^r$ is nonzero, there is a coordinate system $\mathbf{x} = (x^1,\ldots,x^{n+2r+1})$ on a p-neighborhood $U \subset M$ so that $\mathcal{I}_U = \langle dx^{n+1}\rangle$ if $r=0$ and, if $r>0$, then

$$\mathcal{I}_U = \langle\, dx^{n+1} - x^{n+2}\, dx^{n+3} - x^{n+4}\, dx^{n+5} - \cdots x^{n+2r}\, dx^{n+2r+1}\,\rangle.$$

Note that the case where $r=0$ is really a special case of the Frobenius Theorem. Points $p \in M$ for which $\omega{\wedge}(d\omega)^r$ is nonzero are known as the *regular* points of the ideal $\mathcal{I}$. The regular points are an open set in M.

Exercise 1.11: Explain why the integer r is well-defined, i.e, if $\mathcal{I} = \langle\omega\rangle = \langle\eta\rangle$, then you will get the same integer r if you use η as the generator and you will get the same notion of regular points.

In fact, the Pfaff Theorem has a slightly stronger form. It turns out that the maximum dimension of an integral manifold of $\mathcal{I}$ that lies in the regular set is $n{+}r$. Moreover, if $N^{n+r} \subset M$ is such a maximal dimensional integral manifold and N is embedded, then for every $p \in N$, one can choose the coordinates $\mathbf{x}$ so that $N \cap U$ is described by the equations

$$x^{n+1} = x^{n+2} = x^{n+4} = \cdots = x^{n+2r} = 0.$$

Any integral manifold in U near this one on which the $n{+}r$ functions $x^1,\ldots,x^n,x^{n+3},x^{n+5},\ldots,x^{n+2r+1}$ form a coordinate system can be described by equations of the form

$$\begin{aligned} x^{n+1} &= f(x^{n+3},x^{n+5},\ldots,x^{n+2r+1}),\\ x^{n+2k} &= \frac{\partial f}{\partial y^k}(x^{n+3},x^{n+5},\ldots,x^{n+2r+1}), \quad 1 \leq k \leq r \end{aligned}$$

for some suitable function $f(y^1,\ldots,y^r)$. Thus, one can informally say that the integral manifolds of maximal dimension depend on one arbitrary function of r variables.

Exercise 1.12: Consider the contact ideal $\big(\mathbb{R}^3{\times}S^2, \langle \mathbf{u}{\cdot}d\mathbf{x}\rangle\big)$ introduced in Example 1.4. Show that one can introduce local coordinates (x, y, z, p, q) in a neighborhood of any point of $M^5 = \mathbb{R}^3{\times}S^2$ so that

$$\langle \mathbf{u}{\cdot}d\mathbf{x}\rangle = \langle\, dz{-}p\,dx{-}q\,dy \,\rangle$$

and conclude that $\theta = \mathbf{u}{\cdot}d\mathbf{x}$ satisfies $\theta{\wedge}(d\theta)^2 \neq 0$. Explain how this shows that each of the ideals $\mathcal{I} = \langle \theta, a\,\Upsilon_2 - b\,\Upsilon_1 + c\,\Upsilon_0\rangle$ is locally equivalent to the ideal associated to a Monge-Ampere equation, as defined in Exercise 1.4.

1.5. Jørgen's Theorem

I want to conclude this lecture by giving one example of the advantage one gets by looking at even a very classical problem from the point of view of an exterior differential system.

Consider the Monge-Ampere equation

$$z_{xx}\,z_{yy} - {z_{xy}}^2 = 1.$$

It is easy to see that this has solutions of the form

$$z = u(x, y) = a\,x^2 + 2b\,xy + c\,y^2 + d\,x + e\,y + f$$

for any constants $a, \dots, f$ satisfying $4(ac - b^2) = 1$. According to a theorem of Jørgen, these are the only solutions whose domain is the entire xy-plane. I now want to give a proof of this theorem.

As in Exercise 1.4, every (local) solution $z = u(x, y)$ of this equation gives rise to an integral manifold of an ideal $\mathcal{I}$ on $xyupq$-space where

$$\begin{aligned}\mathcal{I} &= \langle\, du - p\,dx - q\,dy,\ dp\wedge dq - dx\wedge dy \,\rangle \\ &= \langle\, du - p\,dx - q\,dy,\ dp\wedge dx + dq\wedge dy,\ dp\wedge dq - dx\wedge dy \,\rangle_{\text{alg}}.\end{aligned}$$

Now, consider the mapping $\Phi : \mathbb{R}^5 \to \mathbb{R}^5$ defined by

$$\Phi(x, y, u, p, q) = (x, q, u{-}qy, p, -y).$$

Then Φ is a smooth diffeomorphism of $\mathbb{R}^5$ with itself and it is easy to check that

$$\Phi^*(\mathcal{I}) = \langle\, du - p\,dx - q\,dy,\ dp\wedge dy + dx\wedge dq \,\rangle$$

However, this latter ideal is the ideal associated to $u_{xx} + u_{yy} = 0$! In other words, 'solutions' to the Monge-Ampere equation are transformed into 'solutions' of Laplace's equation by this mapping.

The reason for the scare quotes around the word 'solution' is that, while we know that the integral surfaces of the two ideals correspond under Φ, not all of the integral surfaces actually represent solutions, since, for example, some of the integral surfaces of $\mathcal{I}$ won't even have dx and dy be linearly independent, and these must somehow be taken into account.

Still, the close contact with the harmonic equation and thence the Cauchy-Riemann equations suggests an argument: Namely, the integral surface $N \subset \mathbb{R}^5$ of a solution to the Monge-Ampere equation must satisfy

$$0 = dp\wedge dx + dq\wedge dy + i(dp\wedge dq - dx\wedge dy) = (dp + i\,dy)\wedge(dx + i\,dq).$$

Thus the projection of N into $xypq$-space is a complex curve when $p{+}i\,y$ and $x{+}i\,q$ are regarded as complex coordinates on this $\mathbb{R}^4$. In particular, N can be regarded as a complex curve for which each of $p{+}i\,y$ and $x{+}i\,q$ are holomorphic functions.

Since dx and dy are linearly independent on N, it follows that neither of the 1-forms $dp{+}i\,dy$ nor $dq{-}i\,dx$ can vanish on N. Thus, there exists a holomorphic function λ on N so that

$$dp + i\,dy = \lambda\,(dx + i\,dq).$$

Because $dx \wedge dy$ is nonvanishing on N, the real part of λ can never vanish.

Suppose that the real part of λ is always positive. (I'll leave the other case to you.) Then $|\lambda+1|^2 > |\lambda-1|^2$, which implies that

$$|\lambda+1|^2(dx^2+dq^2) > |\lambda-1|^2(dx^2+dq^2) > 0$$

and, by the above relation, this is

$$|(dp+i\,dy)+(dx+i\,dq)|^2 > |(dp+i\,dy)-(dx+i\,dq)|^2$$

or, more simply,

$$d(p+x)^2 + d(q+y)^2 > d(p-x)^2 + d(q-y)^2.$$

In particular, the left hand quadratic form is greater than the average of the left and right hand quadratic forms, i.e.,

$$d(p+x)^2 + d(q+y)^2 > dp^2 + dx^2 + dq^2 + dy^2 > dx^2 + dy^2.$$

If the solution is defined on the whole plane, then the right hand quadratic form is complete on N, so the left hand quadratic form must be complete on N also. It follows from this that the holomorphic map

$$(p+x) + i(y+q) : N \to \mathbb{C}$$

is a covering map and hence must be a biholomorphism, so that N is equivalent to $\mathbb{C}$ as a Riemann surface. By Liouville's Theorem, it now follows that λ (which takes values in the right half plane) must be constant. The constancy of λ implies that dp and dq are constant linear combinations of dx and dy, which forces u to be a quadratic function of x and y. QED.

Exercise 1.13: Is it necessarily true that any entire solution of

$$u_{xx}\,u_{yy} - {u_{xy}}^2 = -1$$

must be a quadratic function of x and y? Prove or give a counterexample.