Lecture 2. Applications 1: Scalar first order PDE, Lie Groups

2.1. The contact system

For any vector space V of dimension N over $\mathbb{R}$, let $G_n(V)$ denote the set of n-dimensional subspaces of V. When $0 < n < N$ (which I will assume from now on), the set $G_n(V)$ can naturally be regarded as a smooth manifold of dimension $n(N-n)$. To see this, set $s = N - n$ and, for any $E \in G_n(V)$ choose linear coordinates $(\mathbf{x}, \mathbf{u}) = (x^1, \ldots, x^n; u^1, \ldots, u^s)$ so that the x^i restrict to E to be linearly independent. Let $G_n(V, \mathbf{x}) \subset G_n(V)$ denote the set of $\tilde{E} \in G_n(V)$ to which the x^i restrict to be linearly independent. Then there are unique numbers $p^a_i(\tilde{E})$ so that the defining equations of $\tilde{E}$ are

$$u^a - p^a_i(\tilde{E})\, x^i = 0, \qquad 1 \le a \le s.$$

Give $G_n(V)$ the manifold structure so that the maps $(p^a_i) : G_n(V, \mathbf{x}) \to \mathbb{R}^{ns}$ are smooth coordinate charts.

Exercise 2.1: Check that this does work, i.e., that these charts are smooth on overlaps.

Now let X be a manifold of dimension N. The set of n-dimensional subspaces of the tangent spaces T_xX as x varies over X will be denoted by $G_n(TX)$. Any $E \in G_n(TX)$ is an n-dimensional subspace $E \subset T_{\pi(E)}X$ for a unique $\pi(E) \in X$. Obviously, the fiber of the map $\pi : G_n(TX) \to X$ over the point $x \in X$ is $G_n(T_xX)$. It should not be surprising, then, that there is a natural manifold structure on $G_n(TX)$ for which π is a submersion and for which $G_n(TX)$ has dimension $n+s+ns$.

In fact, consider a coordinate chart $(\mathbf{x}, \mathbf{u}) : U \to \mathbb{R}^n \times \mathbb{R}^s$ defined on some open set $U \subset X$, where $\mathbf{x} = (x^1, \ldots, x^n)$ and $\mathbf{u} = (u^1, \ldots, u^s)$. Let $G_n(TU, \mathbf{x}) \subset G_n(TU)$ denote the set of n-planes to which the differentials dx^i restrict to be independent. Then each $E \in G_n(TU, \mathbf{x})$ satisfies a set of linear relations of the form

$$du^a - p^a_i(E)\, dx^i = 0, \qquad 1 \le a \le s.$$

for some unique real numbers $p^a_i(E)$. Set $\mathbf{p} = (p^a_i) : G_n(TU, \mathbf{x}) \to \mathbb{R}^{ns}$. Then the map

$$(\mathbf{x}, \mathbf{u}, \mathbf{p}) : G_n(TU, \mathbf{x}) \to \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns}$$

embedds $G_n(TU, \mathbf{x})$ as an open subset of $\mathbb{R}^{n+s+ns}$. Give $G_n(TX)$ the manifold structure for which these maps are smooth coordinate charts.

Exercise 2.2: Check that this does work, i.e., that these charts are smooth on overlaps.

The coordinate chart $\bigl((\mathbf{x}, \mathbf{u}, \mathbf{p}), G_n(TU, \mathbf{x})\bigr)$ will be called the *canonical extension* of the coordinate chart $\bigl((\mathbf{x}, \mathbf{u}), U\bigr)$.

Any diffeomorphism $\phi : X \to Y$ lifts to a diffeomorphism $\phi^{(1)} : G_n(TX) \to G_n(TY)$ defined by the rule

$$\phi^{(1)}(E) = d\phi(E) \subset T_{\phi\left(\pi(E)\right)}Y.$$

Now $G_n(TX)$ comes endowed with a canonical exterior system $\mathcal{C}$ called the *contact system*. Abstractly, it can be defined as follows: There is a canonical $(n+ns)$-plane field $C \subset TG_n(TX)$ defined by

$$C_E = d\pi^{-1}(E) \subset T_E G_n(TX).$$

Then $\mathcal{C}$ is the ideal generated by the set of 1-forms on $G_n(TX)$ that vanish on C. From the canonical nature of the deifinition, it's clear that for any diffeomorphism $\phi : X \to Y$, the corresponding lift $\phi^{(1)} : G_n(TX) \to G_n(TY)$ will identify the two contact systems.

Now, why is $\mathcal{C}$ called a 'contact' system? Consider an immersion $f : N \to X$ where N has dimension n. This has a canonical 'tangential' lift $f^{(1)} : N \to G_n(TX)$ defined by

$$f^{(1)}(p) = df(T_pN) \subset T_{f(p)}X.$$

Almost by construction, $df^{(1)}(T_pN) \subset C_{f^{(1)}(p)}$, so that $f^{(1)} : N \to G_n(TX)$ is an integral manifold of $\mathcal{C}$. Conversely, if $F : N^n \to G_n(TX)$ is an integral manifold of $\mathcal{C}$ that is transverse to the fibration $\pi : G_n(TX) \to X$, i.e., $f = \pi \circ F : N^n \to M$ is an immersion, then $F = f^{(1)}$.

Exercise 2.3: Prove this last statement.

Thus, the contact system $\mathcal{C}$ essentially distinguishes the tangential lifts of immersions of n-manifolds into X from arbitrary immersions of n-manifolds into X. As for the adjective 'contact', it comes from the interpretation that two different immersions $f, g : N \to X$ will satisfy $f^{(1)}(p) = g^{(1)}(p)$ if and only if $f(p) = g(p)$ and the two image submanifolds share the same tangent n-plane at p. Intuitively, the two image submanifolds $f(N)$ and $g(N)$ have 'first order contact' at p.

Exercise 2.4: (Important!) Show that, in canonically extended coordinates $(\mathbf{x}, \mathbf{u}, \mathbf{p})$ on $G_n(TX, \mathbf{x})$,

$$\mathcal{C}_{G_n(TX,\mathbf{x})} = \langle\, du^1 - p^1_i\, dx^i,\ \ldots,\ du^s - p^s_i\, dx^i \,\rangle,$$

I.e., $\mathcal{C}$ is locally generated by the 1-forms $\theta^a = du^a - p^a_i\, dx^i$ for $1 \le a \le s$ in any canonically extended coordinate system.

As a consequence of the previous exercise, we see that the integral manifolds of $\mathcal{C}$ in $G_n(TX, \mathbf{x})$ to which $\mathbf{x}$ restricts to be a coordinate system are described by equations of the form

$$u^a = f^a(x^1, \ldots, x^n), \qquad p^a_i = \frac{\partial f^a}{\partial x^i}(x^1, \ldots, x^n)$$

for some differentiable functions f^a on an appropriate domain in $\mathbb{R}^n$.

Once the construction of the contact system $\big(G_n(TX), \mathcal{C}\big)$ is in place, it can be used to construct other canonical systems and manifolds. For example, Let X have dimension n and U have dimension s. Let $J^1(X, U) \subset G_n\big(T(X{\times}U)\big)$ denote the open (dense) set consisting of the n-planes $E \subset T_{(x,u)}X{\times}U$ that are transverse to the subspace $0 \oplus T_uU \subset T_{(x,u)}X{\times}U$. The graph $(\mathrm{id}, f) : X \to X{\times}U$ of any smooth map $f : X \to U$ then has the property that $j^1 f = (\mathrm{id}, f)^{(1)}$ lifts X into $J^1(X, U)$. In fact, two maps $f, g : X \to U$ satisfy $j^1 f(p) = j^1 g(p)$ if and only if f and g have the same 1-jet at p. Thus, $J^1(X, U)$ is canonically identified with the space of 1-jets of mappings of X into U. The contact system then restricts to $J^1(X, U)$ to be the usual contact system defined in the theory of jets.

If one chooses a submanifold $M \subset G_n(TX)$ and lets $\mathcal{I}$ be the differential ideal on M generated by the pullbacks to M of elements of $\mathcal{C}$, then the integral manifolds of $(M, \mathcal{I})$ can be thought of as representing the n-dimensional submanifolds of X whose tangent planes lie in M. In other words, M can be thought of as a system of first order partial differential equations for submanifolds of X. As we will see, this is a very useful point of view.

Exercise 2.5: Let X^4 be an almost complex 4-manifold and let $M \subset G_2(TX)$ be the set of 2-planes that are invariant under complex multiplication. Show that M has (real) dimension 6 and describe the fibers of the projection $M \to X$. What can you say about the surfaces in X whose tangential lifts lie in M?

2.2. The method of characteristics

I now want to apply some of these ideas to the classical problem of solving a single, scalar first order PDE

$$F\big(x^1, \ldots, x^n, u, \tfrac{\partial u}{\partial x^1}, \ldots, \tfrac{\partial u}{\partial x^1}\big) = 0.$$

As explained before, I am going to regard this as an exterior differential system as follows: Using the standard coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ on $\mathbb{R}^n$ and $\mathbf{u} = (u)$ on $\mathbb{R}$, the canonical extended coordinates on $J^1(\mathbb{R}^n, \mathbb{R}) = G_n\big(T(\mathbb{R}^n{\times}\mathbb{R}), \mathbf{x}\big)$ become $(\mathbf{x}, \mathbf{u}, \mathbf{p})$ where $\mathbf{p} = (p_1, \ldots, p_n)$. The equation

$$F(x^1, \ldots, x^n, u, p_1, \ldots, p_n) = 0$$

then defines a subset $M \subset J^1(\mathbb{R}^n, \mathbb{R})$. I am going to suppose that F is smooth and that not all of the partials $\partial F/\partial p_i$ vanish at any single point of M. By the implicit function theorem, it follows then that M is a smooth manifold of dimension $2n$ and that the projection $(\mathbf{x}, \mathbf{u}) : M \to \mathbb{R}^n{\times}\mathbb{R}$ is a smooth submersion. Let $\mathcal{I}$ be the exterior differential system on M generated by the contact 1-form

$$\theta = du - p_i\, dx^i.$$

Note that, on M, the 1-forms dx^i, du, dp_i are not linearly independent (there are too many of them), but satisfy a single linear relation

$$0 = dF = \tfrac{\partial F}{\partial x^i}\, dx^i + \tfrac{\partial F}{\partial u}\, du + \tfrac{\partial F}{\partial p_i}\, dp_i\,.$$

Of course, $\theta{\wedge}(d\theta)^n = 0$, but $\theta{\wedge}(d\theta)^{n-1}$ is nowhere vanishing.

Exercise 2.6: Prove this last statement.

By the Pfaff theorem, it follows that every point in M has a neighborhood U on which there exist coordinates $(z, y^1, \ldots, y^{n-1}, v, q_1, \ldots, q_{n-1})$ so that

$$\langle \theta \rangle = \langle\, dv - q_1\, dy^1 - q_2\, dy^2 - \cdots - q_{n-1}\, dy^{n-1}\, \rangle.$$

I.e., there is a nonvanishing function μ on U so that

$$\theta = \mu\big(dv - q_1\, dy^1 - q_2\, dy^2 - \cdots - q_{n-1}\, dy^{n-1}\big).$$

Notice what this says about the vector field $Z = \frac{\partial}{\partial z}$. Not only does it satisfy $\theta(Z) = 0$, but it also satisfies

$$Z \lrcorner\, d\theta = d\mu(Z)\, \theta.$$

Moreover, up to a multiple, Z is the only vector field that satisfies $\theta(Z) = 0$ and $Z \lrcorner\, d\theta \equiv 0 \mod \theta$.

Exercise 2.7: Prove this last statement. Moreover, show that the vector field

$$Z = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial x^i} + p_i \frac{\partial F}{\partial p_i}\frac{\partial}{\partial u} - \left(\frac{\partial F}{\partial x^i} + p_i \frac{\partial F}{\partial u}\right)\frac{\partial}{\partial p_i}$$

defined on $J^1(\mathbb{R}^n, \mathbb{R})$ is tangent to the level sets of F (and $M = F^{-1}(0)$ in particular), satisfies $\theta(Z) = 0$, and satisfies $Z \lrcorner\, d\theta \equiv 0 \mod \{\theta, dF\}$. Conclude that this Z is, up to a multiple, equal to the Z described above in Pfaff coordinates on M. This vector field is known as the *Cauchy characteristic vector field* of the function F.

A solution to the above equation is then represented by a function $f : \mathbb{R}^n \to \mathbb{R}$ so that $N = j^1 f(\mathbb{R}^n)$ lies in M. In other words, $j^1 f : \mathbb{R}^n \to M$ is an integral manifold of $\mathcal{I}$. Now, an n-dimensional integral manifolds of $\mathcal{I}$ is locally described in some Pfaff normal coordinates as above in the form

$$v = g(y^1, \ldots, g^{n-1}), \quad q_i = \frac{\partial g}{\partial y_i}(y^1, \ldots, g^{n-1})$$

for a suitable function g on a domain in $\mathbb{R}^{n-1}$. In particular, such an integral manifold is always tangent to the Cauchy characteristic vector field.

This gives a prescription for solving a given initial value problem for the above partial differential equation: Use initial data for the equation to find an $(n-1)$-dimensional integral manifold $P^{n-1} \subset M$ of $\mathcal{I}$ that is transverse to the Cauchy characteristic vector field Z. Then construct an n-dimensional integral manifold of $\mathcal{I}$ by taking the union of the integral curves of Z that pass through P.

This method of solving a single scalar PDE via ordinary differential equations (i.e., integrating the flow of a vector field) is known as the *method of characteristics*. For some explicit examples, consult pp. 25–27 of the EDS notes.

2.3. Maps into Lie groups – existence and uniqueness

Let G be a Lie group with Lie algebra $\mathfrak{g} = T_e G$, and let η be its canonical left-invariant 1-form. Thus, η is a 1-form on G with values in $\mathfrak{g}$ that satisfies the conditions that, first $\eta_e : T_e G = \mathfrak{g} \to \mathfrak{g}$ is the identity, and, second, that η is left invariant, i.e., $L_a^*(\eta) = \eta$ for all $a \in G$, where $L_a : G \to G$ is left multiplication by a.

Exercise 2.8: Show that if G is a matrix group, with $g : G \to M_n(\mathbb{R})$ the inclusion into the n-by-n matrices, then

$$\eta = g^{-1}\, dg.$$

It is well-known (and easy to prove) that η satisfies the Maurer-Cartan equation

$$d\eta = -\tfrac{1}{2}\,[\eta, \eta].$$

(In the matrix case, this is equivalent to the perhaps-more-familiar equation $d\eta = -\eta{\wedge}\eta$.)

There are many cases in differential geometry where a geometric problem can be reduced to the following problem: Suppose given a manifold N and a $\mathfrak{g}$-valued 1-form γ on N that satisfies the Maurer-Cartan equation $d\gamma = -\frac{1}{2}\,[\gamma, \gamma]$. Prove that there exists a smooth map $g : N \to G$ so that $\gamma = g^*(\eta)$.

The fundamental result concerning this problem is due to Elie Cartan and is the foundation of the method of the moving frame:

Theorem 3: (Maurer-Cartan) If N is connected and simply connected and γ is a smooth $\mathfrak{g}$-valued 1-form on N that satisfies $d\gamma = -\frac{1}{2}\,[\gamma, \gamma]$, then there exists a smooth map $g : N \to G$, unique up to composition with a constant left translation, so that $g^*\eta = \gamma$.

I want to sketch the proof as an application of the Frobenius theorem. Here are the ideas: Let $M = N{\times}G$ and consider the $\mathfrak{g}$-valued 1-form

$$\theta = \eta - \gamma.$$

It's easy to compute that

$$d\theta = -\tfrac{1}{2}\,[\theta, \theta] - [\theta, \gamma].$$

In particular, writing $\theta = \theta^1\, x_1 + \cdots + \theta^s\, x_s$ where $x_1, \ldots, x_s$ is a basis of $\mathfrak{g}$, the differential ideal

$$\mathcal{I} = \langle\, \theta^1, \ldots, \theta^s \,\rangle$$

satisfies $\mathcal{I} = \langle\, \theta^1, \ldots, \theta^s \,\rangle_{\text{alg}}$. Moreover, the θ^a are manifestly linearly independent since they restrict to each fiber $\{n\}{\times}G$ to be linearly independent. Thus, the hypotheses of the Frobenius theorem are satisfied, and M is foliated by maximal connected integral manifolds of $\mathcal{I}$, each of which can be shown to project onto the first factor N to be a covering map.

Exercise 2.9: Prove this. (You will need to use the fact that the foliation is invariant under the maps $\mathrm{id}{\times}L_a : N{\times}G \to N{\times}G$.)

Since N is connected and simply connected, each integral leaf projects diffeomorphically onto N and hence is the graph of a map $g : N \to G$. This g has the desired property. QED

Exercise 2.10: Use Cartan's Theorem to prove that for every Lie algebra $\mathfrak{g}$, there is, up to isomorphism, at most one connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$. (Such a Lie group does exist for every Lie algebra, but this is proved by other techniques.) Hint: If G_1 and G_2 satisfy these hypotheses, consider the map $g : G_1 \to G_2$ that satisfies $g^*\eta_2 = \eta_1$ and $g(e_1) = e_2$.

2.4. The Gauss and Codazzi equations

As another typical application of the Frobenius Theorem, I want to consider one of the fundamental theorems of surface theory in Euclidean space.

Let $x : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface Σ and let $u : \Sigma \to S^2$ be its Gauss map. In particular $u \cdot dx = 0$. The two quadratic forms

$$\mathrm{I} = dx \cdot dx\,, \qquad\qquad \mathrm{I\!I} = -\,du \cdot dx$$

are known as the first and second fundamental forms of the oriented immersion x.

It is evident that if $y = Ax + b$ where A lies in $\mathrm{O}(3)$ and b lies in $\mathbb{R}^3$, then y will be an immersion with the same first and second fundamental forms. (NB. The Gauss map of y will be $v = \det(A)\, Au = \pm Au$.) One of the fundamental results of surface theory is a sort of converse to this statement, namely that if $x, y : \Sigma \to \mathbb{R}^3$ have the same first and second fundamental forms, then they differ by an ambient isometry. (Note that the first or second fundamental form alone is not enough to determine the immersion up to rigid motion.) This is known as Bonnet's Theorem, although it appears to have been accepted as true long before Bonnet's proof appeared.

The standard argument for Bonnet's Theorem goes as follows: Let $\pi : F \to \Sigma$ be the oriented orthonormal frame bundle of Σ endowed with the metric I. Elements of F consist of triples (p, v_1, v_2) where (v_1, v_2) is an oriented, I-orthonormal basis of $T_p\Sigma$ and $\pi(p, v_1, v_2) = p$. There are unique 1-forms on F, say $\omega_1, \omega_2, \omega_{12}$ so that

$$d\pi(w) = v_1\,\omega_1(w) + v_2\,\omega_2(w)$$

for all $w \in T_{(p,v_1,v_2)}F$ and so that

$$d\omega_1 = -\omega_{12} \wedge \omega_2\,, \qquad d\omega_2 = \omega_{12} \wedge \omega_1\,.$$

Then

$$\pi^*\mathrm{I} = {\omega_1}^2 + {\omega_2}^2, \qquad \pi^*\mathrm{I\!I} = h_{11}\,{\omega_1}^2 + 2h_{12}\,\omega_1\omega_2 + h_{22}\,{\omega_2}^2,$$

for some functions h_{11}, h_{12}, and h_{22}. Defining $\omega_{31} = h_{11}\,\omega_1 + h_{12}\,\omega_2$ and $\omega_{32} = h_{12}\,\omega_1 + h_{22}\,\omega_2$, it is not difficult to see that the $\mathbb{R}^3$-valued functions x, $e_1 = x'(v_1)$, $e_2 = x'(v_2)$, and $e_3 = e_1 \times e_2$ must satisfy the matrix equation

$$d\begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \omega_{12} & -\omega_{31} \\ \omega_2 & -\omega_{12} & 0 & -\omega_{32} \\ 0 & \omega_{31} & \omega_{32} & 0 \end{pmatrix}.$$

Now, the matrix

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \omega_{12} & -\omega_{31} \\ \omega_2 & -\omega_{12} & 0 & -\omega_{32} \\ 0 & \omega_{31} & \omega_{32} & 0 \end{pmatrix}$$

takes values in the Lie algebra of the group $G \subset \mathrm{SL}(4, \mathbb{R})$ of matrices of the form

$$\begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix}, \qquad b \in \mathbb{R}^3,\ A \in \mathrm{SO}(3),$$

while the mapping $g : F \to G$ defined by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix}$$

clearly satisfies $g^{-1}\,dg = \gamma$. Thus, by the uniqueness in Cartan's Theorem, the map g is uniquely determined up to left multiplication by a constant in G.

Exercise 2.11: Explain why this implies Bonnet's Theorem as it was stated.

Perhaps more interesting is the application of the existence part of Cartan's Theorem. Given any pair of quadratic forms $(\mathrm{I}, \mathrm{I\!I})$ on a surface Σ with I being positive definite, the construction of F and the accompanying forms $\omega_1, \omega_2, \omega_{12}, \omega_{31}, \omega_{32}$ and thence γ can obviously be carried out. However, it won't necessarily be true that $d\gamma = -\gamma \wedge \gamma$. In fact,

$$d\gamma + \gamma \wedge \gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_{12} & -\Omega_{31} \\ 0 & -\Omega_{12} & 0 & -\Omega_{32} \\ 0 & \Omega_{31} & \Omega_{32} & 0 \end{pmatrix}$$

where, for example,

$$\Omega_{12} = \left(K - h_{11}h_{22} + {h_{12}}^2\right)\omega_1 \wedge \omega_2$$

where K is the Gauss curvature of the metric I. Thus, a necessary condition for the pair $(\mathrm{I}, \mathrm{I\!I})$ to come from an immersion is that the *Gauss equation* hold, i.e.,

$$\det_{\mathrm{I}} \mathrm{I\!I} = K.$$

The other two expressions $\Omega_{31} = h_1\,\omega_1{\wedge}\omega_2$ and $\Omega_{32} = h_2\,\omega_1{\wedge}\omega_2$ are such that there is a well-defined 1-form η on Σ so that $\pi^*\eta = h_1\,\omega_1 + h_2\,\omega_2$. The mapping δ_{I} from quadratic forms to 1-forms that $\mathrm{I\!I} \mapsto \eta$ defines is a first order linear differential operator. Thus, another necessary condition that the pair $(\mathrm{I}, \mathrm{I\!I})$ come from an immersion is that the *Codazzi equation* hold, i.e.,

$$\delta_{\mathrm{I}}(\mathrm{I\!I}) = 0.$$

By Cartan's Theorem, if a pair $(\mathrm{I}, \mathrm{I\!I})$ on a surface Σ satisfy the Gauss and Codazzi equations, then, at least locally, there will exist an immersion $x : \Sigma \to \mathbb{R}^3$ with $(\mathrm{I}, \mathrm{I\!I})$ as its first and second fundamental forms.

Exercise 2.12: Show that this immersion can be defined on all of Σ if Σ is simply connected. (Be careful: Just because Σ is simply connected, it does not follow that F is simply connected. How do you deal with this?) Is this necessarily true if Σ is not simply connected?

Exercise 2.13: Show that the quadratic forms on $\Sigma = \mathbb{R}^2$ defined by

$$\mathrm{I} = \cos^2 u\, dx^2 + \sin^2 u\, dy^2$$
$$\mathrm{I\!I} = \cos u \sin u\left(dx^2 - dy^2\right)$$

satisfy the Gauss and Codazzi equations if and only if the function $u(x, y)$ satisfies $u_{xx} - u_{yy} = \sin u\,\cos u \neq 0$. What sorts of surfaces in $\mathbb{R}^3$ correspond to these solutions? What happens if u satisfies the differential equation but either $\sin u$ or $\cos u$ vanishes? Does Cartan's Theorem give anything?