Lecture 3. Integral Elements and the Cartan-Kähler Theorem

The lecture notes for this section will mostly be definitions, some basic examples, and exercises. In particular, I will not attempt to give the proofs of the various theorems that I state. The full details can be found in Chapter III of *Exterior Differential Systems*.

Before beginning the lecture proper, let me just say that our method for constructing integral manifolds of a given exterior differential system will be to do it by a process of successively 'thickening' p-dimensional integral manifolds to $(p{+}1)$-dimensional integral manifolds by solving successive initial value problems. This will require some tools from partial differential equations, phrased in a geometric language, but it will also require us to understand the geometry of certain 'infinitesimal' integral manifolds known as 'integral elements'. It is to this study that I will first turn.

3.1. Integral elements and their extensions

Let $(M,\mathcal{I})$ be an EDS. An n-dimensional subspace $E \subset T_xM$ is said to be an *integral element* of $\mathcal{I}$ if

$$\phi(v_1,\dots,v_n) = 0$$

for all $\phi \in \mathcal{I}^n$ and all $v_1,\dots,v_n \in E$. The set of all n-dimensional integral elements of $\mathcal{I}$ will be denoted $V_n(\mathcal{I}) \subset G_n(TM)$.

Our main interest in integral elements is that the tangent spaces to any n-dimensional integral manifold $N^n \subset M$ are integral elements. Our ultimate goal is to answer the 'converse' questions: When is an integral element tangent to an integral manifold? If so, in 'how many' ways?

It is certainly not always true that every integral element is tangent to an integral manifold.

Example 3.1: *Non-existence.* Consider

$$(M,\mathcal{I}) = \big(\mathbb{R}, \langle\, x\,dx\,\rangle\big).$$

The whole tangent space $T_o\mathbb{R}$ is clearly a 1-dimensional integral element of $\mathcal{I}$, but there can't be any 1-dimensional integral manifolds of $\mathcal{I}$.

For a less trivial example, do the following exercise.

Exercise 3.1: Show that the ideal $\mathcal{I}_1 = \langle\, dx{\wedge}dz, dy{\wedge}(dz - y\,dx)\,\rangle$ has exactly one 2-dimensional integral element at each point, but that it has no 2-dimensional integral manifolds. Compare this with the ideal $\mathcal{I}_2 = \langle\, dx{\wedge}dz, dy{\wedge}dz\,\rangle$.

Now, $V_n(\mathcal{I})$ is a closed subset of $G_n(TM)$. To see why this is so, let's see how the elements of $\mathcal{I}$ can be used to get defining equations for $V_n(\mathcal{I})$ in local coordinates. Let $(\mathbf{x},\mathbf{u}) : U \to \mathbb{R}^{n+s}$ be any local coordinate chart and let $(\mathbf{x},\mathbf{u},\mathbf{p}) : G_n(TX,\mathbf{x}) \to \mathbb{R}^{n+s+ns}$ be the canonical extension described in Lecture 2. Every $E \in G_n(TX,\mathbf{x})$ has a well-defined basis $\big(X_1(E),\dots,X_n(E)\big)$, where

$$X_i(E) = \frac{\partial}{\partial x^i} + p^a_i(E)\,\frac{\partial}{\partial u^a}.$$

(This is the basis of E that is dual to the basis $dx^1,\dots,dx^n$) of E^*.) Using this basis, we can define a function $\phi_{\mathbf{x}}$ on $G_n(TX,\mathbf{x})$ associated to any n-form ϕ by the rule

$$\phi_{\mathbf{x}}(E) = \phi\big(X_1(E),\dots,X_n(E)\big).$$

It's not hard to see that $\phi_{\mathbf{x}}$ will be smooth as long as ϕ is smooth.

Exercise 3.2: Prove this last statement.

With this notation, $V_n(\mathcal{I}) \cap G_n(TX,\mathbf{x})$ is seen to be the simultaneous zero locus of the set of functions $\{\,\phi_{\mathbf{x}} | \phi \in \mathcal{I}^n\}$. Thus $V_n(\mathcal{I}) \cap G_n(TX,\mathbf{x})$ is closed. It follows that $V_n(\mathcal{I})$ is a closed subset of $G_n(TX)$, as desired.

Exercise 3.3: Describe $V_1(\mathcal{I})$ and $V_2(\mathcal{I})$ for

i. $(M,\mathcal{I}) = \big(\mathbb{R}^4, \langle\, dx^1{\wedge}dx^2 + dx^3{\wedge}dx^4 \,\rangle\big)$.

ii. $(M,\mathcal{I}) = \big(\mathbb{R}^4, \langle\, dx^1{\wedge}dx^2,\ dx^3{\wedge}dx^4 \,\rangle\big)$.

iii. $(M,\mathcal{I}) = \big(\mathbb{R}^4, \langle\, dx^1{\wedge}dx^2 + dx^3{\wedge}dx^4,\ dx^1{\wedge}dx^4 - dx^3{\wedge}dx^2 \,\rangle\big)$.

Now, there are some relations among the various $V_k(\mathcal{I})$. An easy one is that if E belongs to $V_n(\mathcal{I})$, then every p-dimensional subspace of E is also an integral element, i.e, $G_p(E) \subset V_p(\mathcal{I})$. This follows because $\mathcal{I}$ is an ideal. The point is that if $E' \subset E$ were a p-dimensional subspace and $\phi \in \mathcal{I}^p$ did not vanish when pulled back to E', then there would exist an $(n{-}p)$-form α so that $\alpha{\wedge}\phi$ (which belongs to $\mathcal{I}$) did not vanish when pulled back to E.

Exercise 3.4: Prove this last statement.

On the other hand, obviously not every extension of an integral element is an integral element. In fact, from the previous exercise, you can see that the topology of the space of integral elements of a given degree can be surprisingly complicated. However, describing the integral extensions one dimension at a time turns out to be reasonably simple:

Let $E \in V_k(\mathcal{I})$ be an integral element and let $(e_1, \ldots, e_k)$ be a basis for $E \subset T_xM$. The set

$$H(E) = \{\, v \in T_xM \,\big|\, \kappa(v, e_1, \ldots, e_k) = 0, \forall \kappa \in \mathcal{I}^{k+1} \} \subseteq T_xM$$

is known as the *polar space* of E, though it probably ought to be called the *extension space* of E, since a vector $v \in T_xM$ lies in $H(E)$ if and only if either it lies in E (the trivial case) or else $E^+ = E + \mathbb{R}v$ lies in $V_{k+1}(\mathcal{I})$. In other words, a $(k{+}1)$-plane E^+ containing E is an integral element of $\mathcal{I}$ if and only if it lies in $H(E)$.

Now, from the very definition of $H(E)$, it is a vector space and contains E. It is traditional to define the function $r : V_k(\mathcal{I}) \to \{-1, 0, 1, 2, \ldots\}$ by the formula

$$r(E) = \dim H(E) - k - 1.$$

The reason for subtracting 1 is that then $r(E)$ is the dimension of the set of $(k{+}1)$-dimensional integral elements of $\mathcal{I}$ that contain E, with $r(E) = -1$ meaning that there are no such extensions. When $r(E) \geq 0$, we have

$$\{\, E^+ \in V_{k+1}(\mathcal{I}) \,\big|\, E \subset E^+ \,\} \simeq \mathbb{P}\big(H(E)/E\big) \simeq \mathbb{RP}^{r(E)}.$$

Exercise 3.5: Compute the function $r : V_1(\mathcal{I}) \to \{-1, 0, 1, 2, \ldots\}$ for each of the examples in Exercise 3.3. Show that $V_3(\mathcal{I})$ is empty in each of these cases. What does this say about r on $V_2(\mathcal{I})$?

3.2. Ordinary and Regular Elements

Right now, we only have that $V_k(\mathcal{I})$ is a closed subset of $G_n(TX)$ and closed subsets can be fairly nasty objects in the eyes of a geometer. We want to see if we can put a nicer structure on $V_k(\mathcal{I})$.

First, some terminology. If $S \subset C^\infty(M)$ is some set of smooth functions on M, we can look at the common zero set of S, i.e.,

$$Z_S = \{\ x \in M \,\big|\, f(x) = 0,\ \forall f \in S\ \}.$$

Of course, this is a closed set, but we'd like to find conditions that will make it be a smooth manifold. One such case is provided by the implicit function theorem: Say that $z \in Z_S$ is an *ordinary zero* of S if there is an open neighborhood U of z in M and a set of functions $f_1, \ldots, f_c \in S$ so that

(1). $df_1{\wedge}df_2{\wedge}\cdots{\wedge}df_c \neq 0$ on U, and

(2). $Z_S \cap U = \{\, y \in U \mid f_1(y) = \cdots = f_c(y) = 0 \,\}$.

By the implicit function theorem, $Z_S \cap U$ is an embedded submanifold of U of codimension c. Let $Z_S^o \subset Z_S$ denote the set of ordinary zeros of S.

Exercise 3.6: Show that Z_S and Z_S^o depend only on the ideal generated by S in $C^\infty(M)$. Also, show that for $z \in Z_S^o$, the integer c described above is well-defined, so that one can speak without ambiguity of the codimension of Z_S^o at z.

This idea can now be applied to the $V_n(\mathcal{I})$. Say that $E \in V_n(\mathcal{I})$ is an *ordinary* integral element if it is an ordinary zero of the set

$$S_{\mathbf{x}} = \{\, \phi_{\mathbf{x}} \mid \phi \in \mathcal{I}^n \}$$

for some local coordinate chart $(\mathbf{x}, \mathbf{u}) : U \to \mathbb{R}^{n+s}$ with E in $G_n(TM, \mathbf{x})$.

Exercise 3.7: Show that on the intersection $G_n(TM, \mathbf{x}) \cap G_n(TM, \mathbf{y})$, the two sets of functions $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ generate the same ideal. Conclude that this notion of ordinary does not depend on the choice of a coordinate chart, only on the ideal $\mathcal{I}$.

Let $V_n^o(\mathcal{I}) \subset V_n(\mathcal{I})$ be the set of ordinary integral elements of dimension n. By the implicit function theorem, the connected components of $V_n^o(\mathcal{I})$ are smooth embedded submanifolds of $G_n(TM)$. They may not be closed or even all have the same dimension, but at least they are smooth manifolds and are cut out 'cleanly' by the condition that the n-forms vanish on them.

Exercise 3.8: Find an example of an integral element that is not ordinary. Now find a non-trivial example.

Exercise 3.9: Check to see whether or not all the integral elements you found in Exercise 3.3 are ordinary.

Even the ordinary integral elements aren't quite as nice as you could want. For example, the function $r : V_n(\mathcal{I}) \to \{-1, 0, 1, \ldots\}$ might not be locally constant on $V_n^o(\mathcal{I})$.

Example 3.2: *Polar Jumping.* Look back to the first ideal given in Exercise 3.1. There, $V_1(\mathcal{I}_1) = G_1(T\mathbb{R}^3)$ because ${\mathcal{I}_1}^1 = (0)$. Now a 1-dimensional integral element E based at (x, y, z) will be spanned by a vector

$$e_1 = a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y} + c\,\frac{\partial}{\partial z}$$

where not all of a, b, and c vanish. Using the definition of the polar space, we see that

$$v = f\,\frac{\partial}{\partial x} + g\,\frac{\partial}{\partial y} + h\,\frac{\partial}{\partial z}$$

lies in $H(E)$ if and only if $dx{\wedge}dz(v, e_1) = \big(dy{\wedge}(dz - y\,dx)\big)(v, e_1) = 0$, i.e.,

$$c\,f - a\,h = -yb\,f - (c - y\,a)\,g + b\,h = 0.$$

These two linear equations for (f, g, h) will be linearly independent, forcing $H(E) = E$ and $r(E) = -1$, unless $c - y\,a = 0$, in which case the two equations are linearly dependent and $\dim H(E) = 2$, so that $r(E) = 0$.

We say that an ordinary integral element $E \in V_n^o(\mathcal{I})$ is *regular* if r is locally constant in a neighborhood of E in $V_n^o(\mathcal{I})$. Denote the set of regular integral elements by $V_n^r(\mathcal{I}) \subset V_n^o(\mathcal{I})$.

The regular integral elements are extremely nice. Not only do they 'vary smoothly', but their possible extensions 'vary smoothly' as well.

Exercise 3.10: Show that $V_n^r(\mathcal{I})$ is a dense open subset of $V_n^o(\mathcal{I})$. Hint: Show that if $E \subset T_xM$ is regular, then one can choose a fixed set of $(n{+}1)$-forms, say $\kappa^1, \ldots, \kappa^m \in \mathcal{I}^{n+1}$, where m is the codimension of $H(E)$ in T_xM, so that

$$H(E^*) = \{\, v \in T_xM \mid \kappa^\mu(v, e_1, \ldots, e_k) = 0,\ 1 \le \mu \le m \,\}$$

for all E^* in a neighborhood of E in $V_n^o(\mathcal{I})$. This shows that it is open. To get denseness, explain why r is upper semicontinuous and use that.

One more bit of terminology: An integral manifold $N^k \subset M$ of $\mathcal{I}$ will be said to be *ordinary* if all of its tangent planes are ordinary integral elements and *regular* if all of its tangent planes are regular integral elements. Note that if $N \subset M$ is a connected regular integral manifold of $\mathcal{I}$ then the numbers $r(T_xN)$ are all the same, so it makes sense to define $r(N) = r(T_xN)$ for any $x \in N$.

3.3. The Cartan-Kähler Theorem

I can now state one of the fundamental theorems in the subject. A discussion of the proof will be deferred to the next lecture. Here, I am just going to state the theorem, discuss the need for the hypotheses, and do a few examples. In the next lecture, I'll try to give you a feeling for why it works.

Theorem 4: (Cartan-Kähler) Let $(M, \mathcal{I})$ be a real analytic EDS and suppose that

(1) $P \subset M$ is a connected, k-dimensional, real analytic, regular integral manifold of $\mathcal{I}$ with $r(P) \geq 0$ and

(2) $R \subset M$ is a real analytic submanifold of codimension $r(P)$ containing P and having the property that $T_pR \cap H(T_pP)$ has dimension $k{+}1$ for all $p \in P$.

There exists a unique, connected, $(k{+}1)$-dimensional, real analytic integral manifold X of $\mathcal{I}$ that satisfies $P \subset X \subset R$.

The sudden appearance of the hypothesis of real analyticity is somewhat unexpected. However the PDE results that the enter in the proof of the Cartan-Kähler theorem require this assumption and, as will be seen, the theorem is not even true without this hypothesis in the generality stated.

Example 3.3: *The importance of regularity for existence.* Consider the case of Exercise 3.1. For either of the ideals, the line L defined by $x = z = 0$ is an integral curve of the ideal with the property that $r(T_pL) = 0$ for all $p \in L$. However, $\mathcal{I}_1$ has no integral surfaces while $\mathcal{I}_2$ has the integral surface $z = 0$ that contains L. In both cases, however, L is an ordinary integral manifold but not a regular one, so the Cartan-Kähler Theorem does not apply.

Example 3.4: *The importance of regularity for existence.* Consider the case of Exercise 3.3,*ii*. The line L defined by $x^2 = x^3 = x^4 = 0$ is a non-regular integral curve of this ideal, and has $r(T_pL) = 1$ for all $p \in L$, with the polar space $H(T_pL)$ being spanned by the vectors

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}.$$

for all $p \in L$. If you take R to be the 3-plane defined by $x^3 = 0$, then $T_pR \cap H(T_pL)$ has dimension 2 for all $p \in L$, but there is no integral surface X of $\mathcal{I}$ satisfying $L \subset X \subset R$, even though there are integral surfaces of $\mathcal{I}$ that contain L.

Example 3.5: *The meaning of R.* The manifold R that appears in the Cartan-Kähler Theorem is sometimes known as the 'restraining manifold'. You need it when $r(P) > 0$ because then the extension problem is actually underdetermined in a certain sense. (I'll try to make that precise in the next lecture.) However, you can see a little bit of why you need it by looking at the case of Exercise 3.3, (*i*). There, you should have computed that all of the integral elements $E \in V_1(\mathcal{I}) = G_1(T\mathbb{R}^4)$ are regular, with $r(E) = 1$. This means that every integral element has a 1-dimensional family of possible extensions to a 2-dimensional integral element. Suppose, for example, that you start with the curve $P \subset \mathbb{R}^4$ defined by the equations $x^2 = x^3 = x^4 = 0$. Then it is easy to compute that $H(T_pP)$ is spanned by the vectors

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}.$$

for all $p \in P$. In particular, any (real analytic) hypersurface R given by an equation $x^4 = F(x^1, x^2, x^3)$ where F satisfies $F(x^1, 0, 0) = 0$ will satisfy the conditions of the Theorem. If we pull the ideal $\mathcal{I}$ back to this hypersurface and use x^1, x^2, x^3 as coordinates on R, then the ideal on R is generated by the 2-form

$$dx^1 \wedge dx^2 + dx^3 \wedge (F_1\, dx^1 + F_2\, dx^2) = \big(dx^1 + F_2\, dx^3\big) \wedge \big(dx^2 - F_1\, dx^3\big)$$

Of course, this is a closed 2-form on R and its integral surfaces are swept out by integral curves of the vector field

$$X = -F_2 \frac{\partial}{\partial x^1} + F_1 \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}.$$

(Why?). Thus, to get the integral surface X, we take the union of these integral curves that pass through the initial curve P. Clearly, x^1 and x^3 are independent coordinates on a neighborhood of P in X, so X can also be written locally as a graph

$$x^2 = f(x^1, x^3), \qquad x^4 = g(x^1, x^3).$$

where f and g are functions that satisfy $f(x^1, 0) = g(x^1, 0) = 0$. The condition that these define an integral surface then turns out to be that there is another function h so that

$$x^2 = \frac{\partial h}{\partial x^1}(x^1, x^3), \qquad x^4 = \frac{\partial h}{\partial x^3}(x^1, x^3).$$

On the other hand, any such function works as long as its first partials vanish along the line $x^3 = 0$. This shows why you don't usually get uniqueness without a restraining manifold.

Example 3.6: *The importance of real analyticity.* Consider the case of Exercise 3.3, (*iii*). You'll probably recognize this as the ideal generated by the real and imaginary parts of the complex 2-form

$$\left(dx^1 - i\, dx^3\right) \wedge \left(dx^2 + i\, dx^4\right),$$

so the 2-dimensional integral manifolds are complex curves in $\mathbb{R}^4 \simeq \mathbb{C}^2$. Now, if you have done the exercises up to this point, you know that all of the 1-dimensional elements $E \in V_1(\mathcal{I}) = G_1(TM)$ are regular and satisfy $r(E) = 0$, so that each one can be extended uniquely to a 2-dimensional integral element. The Cartan-Kähler theorem then says that any real analytic curve in M lies in a unique connected, real analytic integral surface of $\mathcal{I}$ (i.e., a complex curve). As you know, a complex curve is necessarily real analytic when considered as a surface in $\mathbb{R}^4$. Now suppose that you had a curve described by

$$x^2 = f(x^1), \qquad x^3 = 0, \qquad x^4 = g(x^1),$$

where f and g are smooth, but not real analytic. Then I claim that there is no complex curve that can contain this curve, because if there were, it could be described locally in the form $x^2 + i\,x^4 = F(x^1 - i\,x^3)$ where F is a holomorphic function of one variable. However, setting $x^3 = 0$ in this equation shows that the original curve would be described by $x^2 + i\,x^4 = F(x^1)$, which is absurd because the real and imaginary parts of a holomorphic function are themselves real analytic.

Example 3.7: *Linear Weingarten Surfaces, again.* I now want to return to Example 1.4 and compute the integral elements, determine the notions of ordinary and regular, etc., and see what the Cartan-Kähler Theorem tells us about the integral manifolds.

For example, I claim that, for the EDS $\bigl(M, \langle\theta, \Upsilon_1\rangle\bigr)$, the space $V_1(\mathcal{I})$ is a smooth bundle over M, whose fiber at every point is diffeomorphic to $\mathbb{RP}^3$, that $V_1(\mathcal{I})$ consists entirely of regular integral elements, and that $r(E) = 0$ for all $E \in V_1(\mathcal{I})$. By the Cartan-Kähler Theorem, it will then follow that every real analytic integral curve of $\mathcal{I}$ lies in a unique real analytic integral surface.

Now, the integral curves of $\mathcal{I}$ are easy to describe: They are just of the form $\bigl(x(t), u(t)\bigr)$, where $x : (a, b) \to \mathbb{R}^3$ is a space curve and $u : (a, b) \to S^2$ is a unit length curve with $u(t) \cdot x'(t) = 0$. The condition that this describe an immersed curve in M is, of course, that x' and u' do not simultaneously vanish.

We have already said that the integral surfaces of $\mathcal{I}$ are 'generalized' minimal surfaces, so what the Cartan-Kähler Theorem says in this case is the geometric theorem that every real analytic 'framed curve', $\bigl(x(t), u(t)\bigr)$ in space lies on a unique, oriented minimal surface S for which $u(t)$ is the unit normal.

Exercise 3.11: Use this result to show that every nondegenerate real analytic space curve is a geodesic on a unique connected minimal surface. Also, use this result to prove the existence of a minimal Möbius band. (You'll have to think of a trick to get around the non-orientability of the Möbius band.)

Now, here is how this computation can be done. The principal difficulty in working with $M = \mathbb{R}^3 \times S^2$ is that, unlike $\mathbb{R}^4$ and other simple manifolds that we have been mostly dealing with, there is no obvious basis of 1-forms in which to compute. However, we can remedy this situation by regarding M as a homogeneous space of the group G of rigid motions of $\mathbb{R}^3$. Recall that

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix} \,\middle|\, b \in \mathbb{R}^3, A \in \mathrm{SO}(3) \right\}.$$

and that G acts on $\mathbb{R}^3$ by

$$\begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix} \cdot y = Ay + b.$$

Writing out the columns of the inclusion map $g : G \to \mathrm{GL}(4, \mathbb{R})$ as

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix},$$

we have the structure equations

$$d \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \omega_{12} & -\omega_{31} \\ \omega_2 & -\omega_{12} & 0 & -\omega_{32} \\ \omega_3 & \omega_{31} & \omega_{32} & 0 \end{pmatrix},$$

i.e., the classical structure equations

$$d\mathbf{x} = \mathbf{e}_j \, \omega_j, \qquad d\mathbf{e}_i = \mathbf{e}_j \, \omega_{ji}$$

where ω_i and $\omega_{ij} = -\omega_{ji}$ satisfy

$$d\omega_i = -\omega_{ij} \wedge \omega_j, \qquad d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj}.$$

Now, consider the map $\pi : G \to M = \mathbb{R}^3 \times S^2$ given by

$$\pi(g) = (\mathbf{x}, \mathbf{e}_3).$$

This map is a smooth submersion and its fibers are the circles that are the left cosets of the circle subgroup H consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, the 1-forms $\omega_1, \omega_2, \omega_3, \omega_{12}, \omega_{31}, \omega_{32}$ are a convenient basis for the left-invariant 1-forms on G, so we should be able to express the pullbacks of the various forms we have constructed on M in terms of these.

Exercise 3.12: Prove the formulae:

$$\begin{aligned} \pi^*\theta &= \omega_3\,, \\ \pi^*\Upsilon_0 &= \omega_1 \wedge \omega_2\,, \\ \pi^*\Upsilon_1 &= -\tfrac{1}{2}\big(\omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32}\big)\,, \\ \pi^*\Upsilon_2 &= \omega_{31} \wedge \omega_{32}\,. \end{aligned}$$

In particular, it follows from this exercise that

$$\begin{aligned} \pi^*\big(\langle \theta, \Upsilon_1 \rangle\big) &= \langle \omega_3, \omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32} \rangle \\ &= \langle\ \omega_3,\ \omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2,\ \omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32}\ \rangle_{\mathrm{alg}}. \end{aligned}$$

Now, let $E \subset T_{(x,u)}M$ be a 1-dimensional integral element of $\langle \theta, \Upsilon_1 \rangle = \langle \theta, d\theta, \Upsilon_1 \rangle_{\text{alg}}$. I want to compute the polar space $H(E)$. If $e_1 \in E$ is a basis element, then

$$H(E) = \{\ v \in T_{(x,u)} \bigm| \ \theta(v) = d\theta(v, e_1) = \Upsilon_1(v, e_1) = 0\ \},$$

so, *a priori*, the dimension of $H(E)$ could be anywhere from 2 (if the three equations on v are all linearly independent) to 4 (if the three equations on v are all multiples of $\theta(v) = 0$, which we know to be nontrivial). To see what actually happens, fix a $g \in G$ so that $\pi(g) = (x, u)$ and choose vectors $\tilde{e}_1$ and $\tilde{v}$ in T_gG so that $\pi_*(\tilde{e}_1) = e_1$ and $\pi_*(\tilde{v}) = v$. Define $a_i = \omega_i(\tilde{e}_1)$ and $a_{ij} = \omega_{ij}(\tilde{e}_1)$ and define $v_i = \omega_i(\tilde{v})$ and $v_{ij} = \omega_{ij}(\tilde{v})$. Then by the formulae from the exercise, we have

$$\begin{aligned}
\theta(v) &= \omega_3(\tilde{v}) \\
&= v_3 \\
d\theta(v, e_1) &= -(\omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2)(\tilde{v}, \tilde{e}_1) \\
&= a_{31}\, v_1 \ + a_{32}\, v_2 \ - a_1 \ v_{31} - a_2 \ v_{32} \\
-2\Upsilon_1(v, e_1) &= (\omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32})(\tilde{v}, \tilde{e}_1) \\
&= a_{32}\, v_1 \ - a_{31}\, v_2 \ + a_2 \ v_{31} - a_1 \ v_{32}
\end{aligned}$$

Now, unless $a_1 = a_2 = a_{31} = a_{32} = 0$, these are three linearly independent relations for $(v_1, v_2, v_3, v_{31}, v_{32})$. However, since e_1 is nonzero, we cannot have $a_1 = a_2 = a_{31} = a_{32} = 0$ (Why?). Thus, the three relations are linearly independent and it follows that $H(E)$ has dimension 2 for all $E \in V_1(\mathcal{I})$, as I wanted to show.

Exercise 3.13: Show that the same conclusion holds for all of the ideals of the form $\mathcal{I} = \langle\, \theta,\ \Upsilon_1 + c\,\Upsilon_0\, \rangle$. Thus, every real analytic framed curve $\big(x(t), u(t)\big)$ lies in a unique (generalized) surface S with mean curvature $H = c$. Do the same for the ideal $\mathcal{I} = \langle\, \theta,\ \Upsilon_2 - c^2\,\Upsilon_0\, \rangle$, and give a geometric interpretation of this result.

However, it is not always true that every integral element is regular, even for the linear Weingarten ideals.

Example 3.8: *Surfaces with* $K = -1$. Consider $\mathcal{I} = \langle\, \theta,\ \Upsilon_2 + \Upsilon_0\, \rangle$, whose integrals correspond to surfaces with $K \equiv -1$. If you go through the same calculation as above for this ideal, everything runs pretty much the same until you get to

$$\begin{aligned}
\theta(v) &= \omega_3(\tilde{v}) \\
&= v_3 \\
d\theta(v, e_1) &= -(\omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2)(\tilde{v}, \tilde{e}_1) \\
&= a_{31}\, v_1 \ + a_{32}\, v_2 \ - a_1 \ v_{31} - a_2 \ v_{32} \\
(\Upsilon_2 + \Upsilon_1)(v, e_1) &= (\omega_{31} \wedge \omega_{32} + \omega_1 \wedge \omega_2)(\tilde{v}, \tilde{e}_1) \\
&= a_2 \ v_1 \ - a_1 \ v_2 \ + a_{32}\, v_{31} - a_{31}\, v_{32}
\end{aligned}$$

These three relations on $(v_1, v_2, v_3, v_{31}, v_{32})$ will be independent except when $(a_{31}, a_{32}) = \pm(a_2, -a_1)$, when the last two relations become dependent. For such integral elements $E \in V_1(\mathcal{I})$, we have $r(E) = 1$ but for all the other integral elements, we have $r(E) = 0$.

Exercise 3.14: Show that if $\big(x(t), u(t)\big)$ is an integral curve of θ, then its tangent vectors are all irregular if and only if $x : (a, b) \to \mathbb{R}^3$ is an immersed space curve of torsion $\tau = \pm 1$ and $u(t)$ is its binormal (up to a sign). Thus, these are the framed curves for which we cannot say that there exists a surface with $K = -1$ containing the curve with u as the surface normal along the curve. Even if there exists one, we cannot claim that it is unique.

Exercise 3.15: Determine which of the ideals $\mathcal{I} = \langle\, \theta,\ a\,\Upsilon_2 + b\,\Upsilon_1 + c\,\Upsilon_0 \rangle$ (where a, b, and c are constants, not all zero) have irregular integral elements in $V_1(\mathcal{I})$.