## Lecture 4. The Cartan-Kähler Theorem: Ideas in the Proof

### 4.1. The Cauchy-Kowalewski Theorem

The basic PDE result that we will need is an existence and uniqueness theorem for initial value problems of a very special kind. You are probably familiar with the ODE existence and uniqueness theorem: If $D \subset \mathbb{R}\times\mathbb{R}^n$ is an open set and $F : D \to \mathbb{R}^n$ is a smooth map, then for any $(t_0, \mathbf{u}_0) \in D$, the initial value problem

$$\mathbf{u}'(t) = F\bigl(t, \mathbf{u}(t)\bigr), \qquad \mathbf{u}(t_0) = \mathbf{u}_0$$

has a solution $\mathbf{u} : I \to \mathbb{R}^n$ on some open interval $I \subset \mathbb{R}$ containing $t_0$, this solution is smooth, and this solution is unique in the sense that, if $\tilde{\mathbf{u}} : \tilde{I} \to \mathbb{R}^n$ is another solution for some interval $\tilde{I}$ containing $t_0$, then $\tilde{\mathbf{u}} = \mathbf{u}$ on the intersection $\tilde{I} \cap I$. Of course, smoothness of $F$ is a much more restrictive assumption than one actually needs; one can get away with locally Lipschitz, but the idea of the theorem is clear.

When one comes to initial value problems for PDE, the theorem we will need is the oldest known such result.

**Theorem 5:** (Cauchy-Kowalewski) Suppose that $D \subset \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^s\times\mathbb{R}^{ns}$ is an open set and suppose that $F : D \to \mathbb{R}^s$ is *real analytic*. Suppose that $U \subset \mathbb{R}^n$ is an open set and that $\phi : U \to \mathbb{R}^s$ is a *real analytic* function with the property that its '1-graph'

$$\left\{ \left( t_0, \mathbf{x}, \phi(\mathbf{x}), \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}) \right) \,\middle|\, \mathbf{x} \in U \right\}$$

lies in $D$ for some $t_0$. Then there exists a domain $V \subset \mathbb{R}\times\mathbb{R}^n$ for which $\{t_0\}{\times}U \subset V$ and a *real analytic* function $\mathbf{u} : V \to \mathbb{R}^s$ satisfying

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) &= F\bigl(t, \mathbf{x}, \mathbf{u}(t,\mathbf{x}), \frac{\partial \mathbf{u}}{\partial \mathbf{x}}(t,\mathbf{x})\bigr), \qquad \text{for } (t,\mathbf{x}) \in V \\ \mathbf{u}(t_0,\mathbf{x}) &= \phi(\mathbf{x}), \qquad \text{for } \mathbf{x} \in U. \end{aligned}$$

Moreover, $\mathbf{u}$ is unique as a *real analytic* solution in the sense that any other such $(\tilde{V}, \tilde{\mathbf{u}})$ with $\tilde{\mathbf{u}}$ real analytic satisfies $\tilde{\mathbf{u}} = \mathbf{u}$ on any component of $\tilde{V} \cap V$ that meets $\{t_0\}{\times}U$.

This may seem to be a complicated theorem, but it basically says that if the equation and initial data are real analytic and they have domains so that the initial data make sense, then you can find a solution $\mathbf{u}$ by expanding it out in a power series

$$\mathbf{u}(t,\mathbf{x}) = \phi(\mathbf{x}) + \phi_1(\mathbf{x})(t{-}t_0) + \tfrac{1}{2}\phi_2(\mathbf{x})(t{-}t_0)^2 + \cdots.$$

The equation will allow you to recursively solve for the sequence of analytic functions $\phi_k$ and the domains of convergence of the functions $F$ and $\phi$ give you estimates that allow you to show that the above series converges on some domain $V$ containing $\{t_0\}{\times}U$. (In fact, proving convergence of the series is the only really subtle point.)

Without the hypothesis of real analyticity, this theorem would not be true. The problem can fail to have a solution or can have more than one solution. There are even examples with $F$ smooth for which there are no solutions to the equation at all, whatever the initial conditions.

In any case, it is traditional to refer to a system of PDE written in the form

$$\frac{\partial \mathbf{u}}{\partial t} = F\bigl(t, \mathbf{x}, \mathbf{u}, \mathbf{v}, \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\bigr)$$

as a *system in Cauchy form*, 'underdetermined' if there are 'unconstrained' functions $\mathbf{v}$ present. In this case, we can always reduce to the determined case by simply specifying the functions $\mathbf{v}$ 'arbitrarily' (subject to the condition that the equations still make sense after the specification).

4.2. Equations not in Cauchy form.

Many interesting equations cannot be put in Cauchy form by any choice of coordinates. For example, consider the equation familiar from vector calculus $\operatorname{curl}\mathbf{u} = \mathbf{f}$ where $\mathbf{f}$ is a known vector field in $\mathbb{R}^3$ and $\mathbf{u}$ is an unknown vector field. Certainly, by inspection of the equations

$$\frac{\partial u^2}{\partial x^3} - \frac{\partial u^3}{\partial x^2} = f^1, \quad \frac{\partial u^3}{\partial x^1} - \frac{\partial u^1}{\partial x^3} = f^2, \quad \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1} = f^3$$

it is hard to imagine how one might solve for all of the $u$-partials in some direction. This appears even more doubtful when you realize that there is no hope of uniqueness in this problem: If $\mathbf{u}$ is a solution, then so is $\mathbf{u} + \operatorname{grad} g$ for any function $g$. Even worse, assuming that $\mathbf{f}$ is real analytic doesn't help either since it is also clear that there can't be any solution at all unless $\operatorname{div}\mathbf{f} = 0$.

Of course, this is a very special equation, and we know how to treat it by ordinary differential equations means (e.g., the proof of Poincaré's Lemma).

**Example 4.1:** *Self-Dual Equations.* A more interesting problem is to consider the so-called 'self-dual equations' in dimension 4. Remember that there is the Hodge star operator $* : \Omega^p(\mathbb{R}^n) \to \Omega^{n-p}(\mathbb{R}^n)$, which is invariant under rigid motions in $\mathbb{R}^n$ and satisfies $**\alpha = (-1)^{p(n-p)}\,\alpha$. In particular, when $n = 4$ and $p = 2$, the 2-forms can be split into the forms that satisfy $*\alpha = \alpha$, the self-dual 2-forms $\Omega^2_+(\mathbb{R}^4)$, and the forms that satisfy $*\alpha = -\alpha$, the anti-self-dual 2-forms $\Omega^2_-(\mathbb{R}^4)$. For example, every $\phi \in \Omega^2_+(\mathbb{R}^4)$ is of the form

$$\begin{aligned}\phi = u^1\,(dx^2 \wedge dx^3 &+ dx^1 \wedge dx^4)\\ &+ u^2\,(dx^3 \wedge dx^1 + dx^2 \wedge dx^4) + u^3\,(dx^1 \wedge dx^2 + dx^3 \wedge dx^4).\end{aligned}$$

The equation $d\phi = 0$ then represents four equations for the three unknown coefficients $u^1, u^2, u^3$. Obviously, this overdetermined system cannot be put in Cauchy form. This raises the interesting question: How can one describe the space of local solutions of these equations? Well, let's look at the equations. They can be written in the form

$$\begin{aligned}0 &= \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3}\,,\\ \frac{\partial u^1}{\partial x^4} &= \frac{\partial u^2}{\partial x^3} - \frac{\partial u^3}{\partial x^2}\,,\\ \frac{\partial u^2}{\partial x^4} &= \frac{\partial u^3}{\partial x^1} - \frac{\partial u^1}{\partial x^3}\,,\\ \frac{\partial u^3}{\partial x^4} &= \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1}\,.\end{aligned}$$

Setting aside the first one, the remaining equations are certainly in Cauchy form and we could solve them (at least near $x^4 = 0$) for any real analytic initial conditions

$$u^i(x^1, x^2, x^3, 0) = f^i(x^1, x^2, x^3), \qquad \text{for } i = 1, 2, 3.$$

Unfortunately, there's no reason to believe that the resulting functions will satisfy the first equation. Indeed, unless the functions $f^i$ satisfy

$$0 = \frac{\partial f^1}{\partial x^1} + \frac{\partial f^2}{\partial x^2} + \frac{\partial f^3}{\partial x^3},$$

the resulting $u^i$ *can't* satisfy the first equation.

However, suppose that we choose the $f^i$ on $\mathbb{R}^3$ to satisfy the above equation on $\mathbb{R}^3$ (and to be real analytic, of course). Then do we have a hope that the resulting $u^i$ will satisfy the remaining equation? In fact, we do, for they will *always* satisfy it! Here is how you can see this: Define the 'error' to be

$$E = \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3}.$$

By the choice of $f$, we know that $E(x^1,x^2,x^3,0)=0$. Moreover, by the above equations and commuting partials, we have

$$\begin{aligned}\frac{\partial E}{\partial x^4} &= \frac{\partial}{\partial x^1}\left(\frac{\partial u^1}{\partial x^4}\right)+\frac{\partial}{\partial x^2}\left(\frac{\partial u^2}{\partial x^4}\right)+\frac{\partial}{\partial x^3}\left(\frac{\partial u^3}{\partial x^4}\right)\\ &= \frac{\partial}{\partial x^1}\left(\frac{\partial u^2}{\partial x^3}-\frac{\partial u^3}{\partial x^2}\right)+\frac{\partial}{\partial x^2}\left(\frac{\partial u^3}{\partial x^1}-\frac{\partial u^1}{\partial x^3}\right)+\frac{\partial}{\partial x^3}\left(\frac{\partial u^1}{\partial x^2}-\frac{\partial u^2}{\partial x^1}\right)\\ &= 0.\end{aligned}$$

Of course, this implies that $E(x^1,x^2,x^3,x^4)=0$, which is what we wanted to be true.

Thus, the solutions to the full system are found by choosing initial conditions $f^i$ to satisfy the single equation on $\mathbb{R}^3$

$$0=\frac{\partial f^1}{\partial x^1}+\frac{\partial f^2}{\partial x^2}+\frac{\partial f^3}{\partial x^3}.$$

Of course, this can be regarded as an equation in Cauchy form, now underdetermined, by writing it in the form

$$\frac{\partial f^3}{\partial x^3}=-\frac{\partial f^1}{\partial x^1}-\frac{\partial f^2}{\partial x^2}.$$

By Cauchy-Kowalewski, we can solve this equation uniquely by choosing $f^1$ and $f^2$ as arbitrary real analytic functions and then choosing the initial value $f^3(x^1,x^2,0)$ as a real analytic function on $\mathbb{R}^2$.

**Exercise 4.1:** Show that you don't need to invoke the Cauchy-Kowalewski Theorem for this problem on $\mathbb{R}^3$ and you also don't need real analyticity to solve the initial value problem. However, show that any solutions $u^i$ on $\mathbb{R}^4$ to the self-dual equations are harmonic and so must be real analytic. What does this tell you about the need for Cauchy-Kowalewski in the system for the $u^i$?

The upshot of all this discussion is that, although the system can't be put in Cauchy form, it can be regarded as a sequence of Cauchy problems. Moreover, this sequence has the unexpectedly nice property that, when you solve one of the Cauchy problems then use the solution as initial data for the next Cauchy problem, the satisfaction of the first set of equations is 'propagated' by the equations at the next level.

**Exercise 4.2:** Consider the overdetermined system

$$\begin{aligned} z_x &= F(x,y,z) \\ z_y &= G(x,y,z) \end{aligned} \qquad z(0,0)=z_0$$

for $z$ as a function of $x$ and $y$. Show that if you set it up as a sequence of Cauchy problems, first

$$w_x(x)=F\bigl(x,0,w(x)\bigr), \qquad w(0)=z_0$$

and then use the resulting function $w$ to consider the equation

$$z_y(x,y)=G\bigl(x,y,z(x,y)\bigr), \qquad z(x,0)=w(x),$$

then the resulting solutions will satisfy the equation $z_x=F(x,y,z)$ for all choices of $z_0$ only if $F$ and $G$ satisfy the condition needed for the system $\langle dz-F(x,y,z)\,dx-G(x,y,z)\,dy\rangle$ to be Frobenius.

**Exercise 4.3:** Go back to the equation $\operatorname{curl}\mathbf{u}=\mathbf{f}$ and show that you can write that as a sequence of Cauchy problems. Show also that they won't have this 'propagation' property unless $\operatorname{div}\mathbf{f}=0$.

**Exercise 4.4:** Now consider the equation $\operatorname{curl}\mathbf{u}=\mathbf{u}+\mathbf{f}$. Of course, this equation can't be put in Cauchy form either, since it differs from the previous one only by terms that don't involve any derivatives. However, show now that when you apply the divergence operator to both sides, you get, not a condition on $\mathbf{f}$, but *another* first order equation on $\mathbf{u}$. Show that you can write this system of four equations for the three unknowns as a sequence of Cauchy problems and that this system *does* have the good 'propagation' property. How much freedom do you get in specifying the initial data to determine a solution?

**Exercise 4.5:** Back to the self-dual equations: Now consider the $u^i$ as free coordinates and set $M = \mathbb{R}^4 \times \mathbb{R}^3$ with coordinates $x^1, x^2, x^3, x^4, u^1, u^2, u^3$. Define the 3-form

$$\Phi = du^1 \wedge (dx^2 \wedge dx^3 + dx^1 \wedge dx^4) \\ + du^2 \wedge (dx^3 \wedge dx^1 + dx^2 \wedge dx^4) + du^3 \wedge (dx^1 \wedge dx^2 + dx^3 \wedge dx^4).$$

Explain why the 4-dimensional integral manifolds in $M$ of $\mathcal{I} = \langle \Phi \rangle$ on which $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \neq 0$ can be thought of locally as representing closed self-dual 2-forms. Describe $V_4(\mathcal{I}) \cap G_4(TM, \mathbf{x})$. Are these ordinary or regular integral elements? What about $V_3(\mathcal{I}) \cap G_3\big(TM, (x^1, x^2, x^3)\big)$?

With all these examples in mind, I can now describe how the proof of the Cartan-Kähler Theorem goes: Remember that we start with a real analytic EDS $(M, \mathcal{I})$ and $P \subset M$ a connected, $k$-dimensional, real analytic, regular integral manifold of $\mathcal{I}$ with $r = r(P) \geq 0$. For each $p \in P$, the dimension of $H(T_pP) \subset T_pM$ is $r{+}k{+}1$ and the generic subspace $S \subset T_pM$ of codimension $r$ will intersect $H(T_pP)$ is a subspace $S \cap H(T_pP)$ of dimension $k+1$. Thus, choosing the 'generic' codimension $r$ submanifold $R \subset M$ that contains $P$ will have the property that $T_pR \cap H(T_pP)$ has dimension $k{+}1$ and so will be an integral element. So now suppose that we have a real analytic $R$ containing $P$ and satisfying this genericity condition. We now want to find a $(k{+}1)$-dimensional integral manifold $X$ satisfying $P \subset X \subset R$.

Because of the real analyticity assumption, it's enough to prove the existence and uniqueness of $X$ in a neighborhood of any point $p \in P$, so fix such a $p$ and let $e_1, \ldots, e_k$ be a basis of $T_pP$. Choose $\kappa^1, \ldots, \kappa^m \in \mathcal{I}^{k+1}$ so that

$$H(T_pP) = \{\ v \in T_pM \mid \kappa^\mu(v, e_1, \ldots, e_k) = 0,\ 1 \leq \mu \leq m\ \}$$

where $m = \dim T_pM - (r{+}k{+}1)$. Because of the regularity assumption, the forms $\kappa^1, \ldots, \kappa^m$ can be used to compute the polar space of any integral element $E \in V_k(\mathcal{I})$ that is sufficiently near $T_pP$.

Now $R$ has dimension $m{+}k{+}1$ and, when you pull back the forms $\kappa^\mu$ to $R$, they are 'independent' near $p$ because we assumed $T_pR \cap H(T_pP)$ to have dimension $k{+}1$. When you write them out in local coordinates, they become a system of $m$ PDE in Cauchy form for extending $P$ to a $(k{+}1)$-dimensional integral manifold of the system $\mathcal{J} = \langle \kappa^1, \ldots, \kappa^m \rangle$, and $P$ itself provides the initial condition. Thus, the Cauchy-Kowalewski Theorem applies: there is a unique, connected, real analytic $X$ of dimension $k{+}1$ satisfying $P \subset X \subset R$ that is an integral manifold of $\mathcal{J}$.

Now, all of the $k$-forms in $\mathcal{I}$ vanish when pulled back to $P$, but we need them to vanish when pulled back to $X$. Here, finally, is where the assumption that $\mathcal{I}$ be differentially closed comes in, as well as the need for the integral elements to be ordinary in the first place. What we do is show that the differential closure condition plus the ordinary assumption allows us to write down a system in Cauchy form for the coefficients of the $k$-forms in $\mathcal{I}$ pulled back to $X$. This system has 'zero' initial conditions since $P$ is an integral manifold of $\mathcal{I}$ and to have all of the coefficients be zero is a solution of the system. By the uniqueness part of the Cauchy-Kowalewski Theorem, it follows that 'zero' is the only solution, i.e., that all of the $k$-forms of $\mathcal{I}$ must vanish on $X$. However, this, coupled with the vanishing of the $\kappa^\mu$ and the fact that they determine the integral extensions (at least near $p$) forces all of the tangent spaces to $X$ to be integral elements of $\mathcal{I}$, i.e., forces $X$ to be an integral manifold of the whole ideal $\mathcal{I}$.

Well, that, in outline, is the proof of the Cartan-Kähler Theorem. The full details are in Chapter III of the EDS book. I encourage you to look at them at some point, probably after you have been convinced, by seeing its applications, that the Cartan-Kähler Theorem is worth knowing.

**Exercise 4.6:** How would you describe the 2-forms on $\mathbb{R}^5$ that are both closed and coclosed? What I'm asking for is an analysis of the local solutions to the equations $d\alpha = d({*}\alpha) = 0$ for $\alpha \in \Omega^2(\mathbb{R}^5)$. If you think you have a handle on this, you might want to go ahead and try the general case: $d\alpha = d({*}\alpha) = 0$ for $\alpha \in \Omega^p(\mathbb{R}^n)$.

## 4.3. Integral Flags and Cartan's Test

In light of the Cartan-Kähler Theorem, there is a simple sufficient condition for the existence of an integral manifold tangent to $E \in V_n(\mathcal{I})$.

**Theorem 6:** Let $(M, \mathcal{I})$ be a real analytic EDS. If $E \in V_n(\mathcal{I})$ contains a flag of subspaces

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

where $E_i \in V_i^r(\mathcal{I})$ for $0 \leq i < n$, then there is a real analytic $n$-dimensional integral manifold $P \subset M$ passing through $p$ and satisfying $T_pP = E$.

The proof is the obvious one: Just apply the Cartan-Kähler Theorem one step at a time, noting that, because $V_k^r(\mathcal{I})$ is an open subset of $V_k(\mathcal{I})$, any $k$-dimensional integral manifold of $\mathcal{I}$ that is tangent to $E_k \in V_k^r(\mathcal{I})$ will perforce be a regular integral manifold in some neighborhood of $p$.

Now this is a nice result but it leaves a few things to be desired. First of all, this sufficient condition is not necessary. As we will see, there are quite a few cases in which the integral manifolds we are interested in cannot be constructed by the above process, simply because the integral elements to which they would be tangent are not the terminus of a flag of regular integral elements. Second, as things stand, it is a lot of work to check whether or not a given integral element *is* the terminus of a flag of regular integral elements.

**Exercise 4.7:** Look back at the two ideals of Exercise 3.1. Show that in neither case does any $E \in V_2(\mathcal{I}_i)$ contain a $E_1 \in V_1^r(\mathcal{I}_i)$. Now, $\mathcal{I}_1$ has no 2-dimensional integral manifolds anyway. For $\mathcal{I}_2$, however, ...

**Exercise 4.8:** For Exercise 4.5, determine which integral elements of $\mathcal{I}$ are the terminus of a flag of regular integral elements.

As you can see, computing with flags of subspaces can be a bit of work. I am now going to describe a simplification of this process that will make these computations almost routine. First, though, some simplifications and terminology.

A flag of integral elements

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

where $E_i \in V_i^r(\mathcal{I})$ for $0 \leq i < n$ and $E_n \in V_n(\mathcal{I})$ will be known as a *regular flag* for short. (Note that the terminus $E_n$ of a regular flag is not required to be regular and, in fact, it can fail to be. However, it does turn out that $E_n$ is ordinary.)

Note that the assumption that $E_0 = 0_p \subset T_pM$ be regular implies, in particular, that is it ordinary, i.e., $E_0$ is an ordinary zero of the set of functions $\mathcal{I}^0 \subset \Omega^0(M)$. Now, the set $V_0^o(\mathcal{I})$ is a smooth submanifold of $G_0(TM) = M$.

**Exercise 4.9:** Explain why any $n$-dimensional integral element $E \subset T_pM$ with $p \in V_0^o(\mathcal{I})$ must be tangent to $V_0^o(\mathcal{I})$. Is this necessarily true if $p$ does not lie in $V_0^o(\mathcal{I})$?

Obviously, every integral manifold of $\mathcal{I}$ that is constructed by the 'regular flag' approach will lie in $V_0^o(\mathcal{I})$ anyway. Thus, at least on theoretical grounds, nothing will be lost if we simply replace $M$ by $V_0^o(\mathcal{I})$, i.e., restrict to the ordinary part of the zero locus of the functions in $\mathcal{I}$. I am going to do this for the rest of this section. This amounts to the blanket assumption that $\mathcal{I}^0 = (0)$, i.e., that $\mathcal{I}$ is generated in positive degree.

Now, corresponding to any integral flag

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

(regular or not), there is the *descending* flag of corresponding polar spaces

$$T_pM \supseteq H(E_0) \supseteq H(E_1) \supseteq \cdots \supseteq H(E_{n-1}) \supseteq H(E_n) \supseteq E_n\,.$$

It will be convenient to keep track track of the dimensions of these spaces in terms of their codimension in $T_pM$. For $k < n$, set

$$c(E_k) = \dim(T_pM) - \dim H(E_k) = n + s - k - 1 - r(E_k)$$

where $\dim M = n{+}s$. It works out best to make the special convention that $c(E_n) = s$. (In practice, it is usually the case that $H(E_n) = E_n$, in which case, the above formula for $c(E_k)$ works even when you set $k = n$.) Since $\dim H(E_k) \geq \dim E_n = n$, we have $c(E_k) \leq s$. Because of the nesting of these spaces, we have

$$0 \leq c(E_0) \leq c(E_1) \leq \cdots \leq c(E_n) \leq s.$$

For notational convenience, set $c(E_{-1}) = 0$. The *Cartan characters* of the flag $F = (E_0, E_1, \ldots, E_n)$ are the numbers

$$s_k(F) = c(E_k) - c(E_{k-1}) \geq 0.$$

They will play an important role in what follows.

I'm now ready to describe *Cartan's Test*, a necessary and sufficient condition for a given flag to be regular. First, let me introduce some terminology: A subset $X \subset M$ will be said to have *codimension at least $q$ at $x \in X$* if there is an open $x$-neighborhood $U \subset M$ and a codimension $q$ submanifold $Q \subset U$ so that $X \cap U$ is a subset of $Q$. In the other direction, $X$ will be said to have *codimension at most $q$ at $x \in X$* if there is an open $x$-neighborhood $U \subset M$ and a codimension $q$ submanifold $Q \subset U$ containing $x$ so that $Q \subset X \cap U$.

**Theorem 7:** (Cartan's Test) Let $(M, \mathcal{I})$ be an EDS and let $F = (E_0, E_1, \ldots, E_n)$ be an integral flag of $\mathcal{I}$. Then $V_n(\mathcal{I})$ has codmension at least

$$c(F) = c(E_0) + c(E_1) + \cdots + c(E_{n-1})$$

in $G_n(TM)$ at $E_n$. Moreover, $V_n(\mathcal{I})$ is a smooth submanifold of $G_n(TM)$ of codimension $c(F)$ in a neighborhood of $E_n$ if and only if the flag $F$ is regular.

This is a very powerful result, because it allows one to test for regularity of a flag by simple linear algebra, computing the polar spaces $H(E_k)$ and then checking that $V_n(\mathcal{I})$ is smooth near $E_n$ and of the smallest possible codimension, $c(F)$. In many cases, these two things can done by inspection.

**Example 4.2:** *Self-Dual 2-Forms.* Look back at Exercise 4.5. Any integral element $E \in V_4(\mathcal{I}) \cap G_4(T\mathbb{R}^7, d\mathbf{x})$ is defined by linear equations of the form

$$\pi^a = du^a - p^a_i(E)\,dx^i = 0.$$

In order that $\Phi$ vanish on such a 4-plane, it suffices that the $p^a_i(E)$ satisfy four equations:

$$p^1_1{+}p^2_2{+}p^3_3 = p^1_4{-}p^2_3{+}p^3_2 = p^2_4{-}p^3_1{+}p^1_3 = p^3_4{-}p^1_2{+}p^2_1 = 0$$

It's clear from this that $V_4(\mathcal{I}) \cap G_4(T\mathbb{R}^7, d\mathbf{x})$ is a smooth manifold of codimension 4 in $G_4(T\mathbb{R}^7)$. On the other hand, if we let $E_k \subset E$ be defined by the equation $dx^{k+1} = dx^{k+2} = \cdots = dx^4 = 0$ for $0 \leq k < 4$, then it is easy to see that

$$\begin{aligned} H(E_0) = H(E_1) &= T_p(M) \\ H(E_2) &= \{ v \in T_p(M) \mid \pi_3(v) = 0 \} \\ H(E_3) &= \{ v \in T_p(M) \mid \pi_1(v) = \pi_2(v) = \pi_3(v) = 0 \} \\ H(E_4) &= \{ v \in T_p(M) \mid \pi_1(v) = \pi_2(v) = \pi_3(v) = 0 \} \end{aligned}$$

so $c(E_0) = c(E_1) = 0$, $c(E_2) = 1$, $c(E_3) = 3$, and $c(E_4) = 3$. Since $c(F) = 0{+}0{+}1{+}3 = 4$, which is the codimension of $V_4(\mathcal{I})$ in $G_4(T\mathbb{R}^7)$, Cartan's Test is verified and the flag is regular.

**Exercise 4.10:** Show that, for $E \in V_4(\mathcal{I}) \cap G_4(T\mathbb{R}^7, d\mathbf{x})$, every flag is regular. (Hint: Rotations in $\mathbb{R}^4$ preserve the self-dual equations.)

**Exercise 4.11:** Write the exterior derivative $d : \Omega^1(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$ as a sum $d_+ + d_-$ where $d_\pm : \Omega^1(\mathbb{R}^4) \to \Omega^2_\pm(\mathbb{R}^4)$. Show that if a 1-form $\lambda$ satisfies $d_+\lambda = 0$, then locally it can be written in the form $\lambda = df + \psi$, where $\psi$ is real analytic. Use this result to show that if $d_+\lambda = 0$, then there exist non-vanishing self-dual 2-forms $\Upsilon$ satisfying $d\Upsilon = \lambda{\wedge}\Upsilon$. (Hint: You will want to recall that any closed self-dual or anti-self-dual 2-form is real analytic and also that if $\Upsilon$ is self-dual while $\Lambda$ is anti-self dual, then $\Upsilon{\wedge}\Lambda$ vanishes identically. What can you say about the local solvability of the equation $d\Upsilon = \lambda{\wedge}\Upsilon$ for $\Upsilon \in \Omega^2_+(\mathbb{R}^4)$ if you don't have $d_+\lambda = 0$? (I don't expect a complete answer to this yet. I just want you to think about the issue. We'll come back to this later.

**Example 4.3:** *Special Lagrangian Manifolds in* $\mathbb{C}^n$. Let $M = \mathbb{C}^n$ with standard complex coordinates $z^1, \ldots, z^n$. Write $z^k = x^k + i\,y^k$, as usual. Let $\mathcal{I}$ be the ideal generated by the Kähler 2-form

$$\omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$$

and the $n$-form

$$\begin{aligned}\Phi &= \mathrm{Im}(dz^1 \wedge \cdots \wedge dz^n)\\ &= dy^1 \wedge dx^2 \wedge \cdots \wedge dx^n + dx^1 \wedge dy^2 \wedge \cdots \wedge dx^n + \cdots\\ &\quad + dx^1 \wedge dx^2 \wedge \cdots \wedge dy^n + (\text{higher order terms in } \{dy^k\})\end{aligned}$$

The $n$-dimensional integral manifolds of $\mathcal{I}$ are known as *special Lagrangian*. They and their generalizations to the special Lagrangian submanifolds of Kähler-Einstein manifolds are the subject of much interest now in mathematical physics.

Consider the integral element $E \in V_n(\mathcal{I})$ based at $0 \in \mathbb{C}^n$ defined by the relations

$$dy^1 = dy^2 = \cdots = dy^n = 0.$$

Let $E_k \subset E$ be defined by the additional relations $dx^j = 0$ for $j > k$. Then, for $k < n{-}1$, the polar space for $E_k$ is easily seen to be defined by the relations $dy^j = 0$ for $j \le k$. In particular, $c(E_k) = k$ for $k < n{-}1$. However, for $k = n{-}1$, the form $\Phi$ enters into the computation of the polar equations, showing that $H(E_{n-1}) = E_n$. Consequently, $c(E_{n-1}) = n$. It follows that $V_n(\mathcal{I})$ must have codimension at least

$$0 + 1 + \cdots + (n{-}2) + n = \tfrac{1}{2}(n^2{-}n{+}2).$$

On the other hand, on any nearby integral element $E^*$, the 1-forms $dx^i$ are linearly independent, so it can be described by relations of the form

$$dy^a - p^a_i\, dx^i = 0.$$

The condition that $\omega$ vanish on $E^*$ is just that $p^a_i = p^i_a$, while the condition that $\Phi$ vanish on $E^*$ is a polynomial equation in the $p^a_i$ of the form

$$0 = p^1_1{+}p^2_2{+}\cdots{+}p^n_n + (\text{higher order terms in } \{p^a_i\}).$$

This equation has independent differential from the equations $p^a_i = p^i_a$ at the integral element $E$ (defined by $p^a_i = 0$). Consequently, $V_n(\mathcal{I})$ is smooth near $E$ and of codimension $\frac{1}{2}(n^2{-}n{+}2)$ in $G_n(T\mathbb{C}^n)$. Thus, by Cartan's Test, the flag is regular.

### 4.4. The notion of generality of integral manifolds

It is very useful to know not only that integral manifolds exist, but 'how many' integral manifolds exist. I now want to make this into a precise notion and give the answer.

Suppose that $F = (E_0, E_1, \ldots, E_n)$ is a regular flag of a real analytic EDS $(M, \mathcal{I})$. By the Cartan-Kähler Theorem, there exists at least one real analytic integral manifold $N^n \subset M$ containing the basepoint $p$ of $E_n$ and satisfying $T_pN = E_n$. Set

$$c_k = \begin{cases} 0 & \text{for } k = -1;\\ c(E_k) & \text{for } 0 \le k < n; \text{ and}\\ s & \text{for } k = n.\end{cases}$$

and define $s_k = c_k - c_{k-1}$ for $0 \leq k \leq n$.

Choose a real analytic coordinate system

$$(\mathbf{x}, \mathbf{u}) = (x^1, \ldots, x^n, u^1, \ldots u^s) : U \to \mathbb{R}^{n+s}$$

centered on $p \in U$ with the following properties:

$(i)$ $N \cap U \subset U$ is defined by $u^a = 0$.

$(ii)$ $E_k \subset E_n$ is defined by $dx^j = 0$ for $j > k$.

$(iii)$ $H(E_k)$ is defined by $du^a = 0$ for $a \leq c_k$ when $0 \leq k < n$.

**Exercise 4.12:** Explain why such a coordinate system must exist.

Define the *level* $\lambda(a)$ of an integer $a$ between 1 and $s$ to be the smallest integer $k \geq 0$ for which $a \leq c_k$. Note that $0 \leq \lambda(a) \leq n$. Note that there are exactly $s_k$ indices of level $k$.

Now, let $\mathcal{C}$ denote the collection of real analytic integral manifolds of $(U, \mathcal{I})$ that are 'near' $N$ in the following sense: An integral manifold $N^*$ belongs to $\mathcal{C}$ if it can be represented by equations of the form

$$u^a = F^a(x^1, \ldots, x^n)$$

where the $F^a$ are real analytic functions defined on a neighborhood of $\mathbf{x} = 0$ and, moreover, these functions and their first partial derivatives are 'sufficiently small' near $\mathbf{x} = 0$. ('Sufficiently small' can be made precise in terms of a connected neighborhood of the flag $F = (E_0, \ldots, E_n)$ in the space of regular flags.)

If the index $a$ has level $k$, define the function $f^a$ on a neighborhood of 0 in $\mathbb{R}^k$ by

$$f^a(x^1, \ldots, x^k) = F^a(x^1, \ldots, x^k, 0, \ldots, 0).$$

Then $f^a$ is a function of $k$ variables. (By convention, we will sometimes refer to a constant as a function of 0 variables.) We then have a mapping

$$N^* \longmapsto \{f^a\}_{1 \leq a \leq s}.$$

A close analysis of the proof of the Cartan-Kähler Theorem then shows that this correspondance between the elements of $\mathcal{C}$ and collections of 'small' functions $\{f^a\}_{1 \leq a \leq s}$ consisting of

$s_0$ constants,
$s_1$ functions of one variable,
$s_2$ functions of two variables,
$\vdots$
$s_n$ functions of $n$ variables.

is one-to-one and onto.

**Example 4.4:** *Self-Dual* 2*-Forms again.* Looking at the self-dual 2-forms example, one sees the real analytic functions

$$f^1(x^1, x^2) = u^3(x^1, x^2, 0, 0)$$
$$f^2(x^1, x^2, x^3) = u^1(x^1, x^2, x^3, 0)$$
$$f^3(x^1, x^2, x^3) = u^2(x^1, x^2, x^3, 0)$$

can be specified arbitrarily and that there is only one solution with any such triple of functions $f^i$ as it's 'initial data'.

**Exercise 4.13:** Consider the EDS $(\mathbb{R}^{2n}, \langle dx^1 \wedge dy_1 + \cdots + dx^n \wedge dy_n \rangle)$ whose $n$-dimensional integral manifolds are the Lagrangian submanifolds of $\mathbb{R}^{2n}$. Compare and contrast the Cartan-Kähler description of these integral manifolds near the $n$-plane $N$ defined by $y_1 = y_2 = \cdots = y_n = 0$ with the more common description as the solutions of the equations

$$y_i = \frac{\partial f}{\partial x^i}$$

where $f$ is an arbitrary differentiable function of $n$ variables. Is there a contradiction here?