Lecture 6. Applications 2: Weingarten Surfaces, etc.

This lecture will consist entirely of examples drawn from geometry, so that you can get some feel for the variety of applications of the Cartan-Kähler Theorem.

6.1. Weingarten surfaces

Let $x : N \to \mathbb{R}^3$ be an immersion of an oriented surface and let $u : N \to S^2$ be the associated oriented normal, sometimes known as the Gauss map. Recall that we have the two fundamental forms

$$\mathrm{I} = dx \cdot dx, \qquad \mathrm{I\!I} = -du \cdot dx.$$

The eigenvalues of $\mathrm{I\!I}$ with respect to I are known as the *principal curvatures* of the immersion. On the open set $N^* \subset N$ where the two eigenvalues are distinct, they are smooth functions on N. The complement $N \setminus N^*$ is known as the *umbilic locus*. For simplicity, I am going to suppose that $N^* = N$, though many of the constructions that I will do can, with some work, be made to go through even in the presence of umbilics.

Possibly after passing to a double cover, we can define vector-valued functions $e_1, e_2 : N \to \mathbb{S}^2$ so that $e_1 \times e_2 = u$ and so that, setting $\eta^i = e_i \cdot dx$, we can write

$$\begin{aligned} dx &= e_1 \quad \eta_1 + e_2 \quad \eta_2\,, \\ -du &= e_1\ \kappa_1\,\eta_1 + e_2\ \kappa_2\,\eta_2\,, \end{aligned}$$

where $\kappa_1 > \kappa_2$ are the principal curvatures. The immersion x defines a *Weingarten surface* if the principal curvatures satisfy a (non-trivial) relation of the form $F(\kappa_1, \kappa_2) = 0$. (For a generic immersion, the functions κ_i satisfy $d\kappa_1 \wedge d\kappa_2 \neq 0$, at least on a dense open set.) For example, the equations $\kappa_1 + \kappa_2 = 0$ and $\kappa_1\kappa_2 = 1$ define Weingarten relations, perhaps better known as the relations $H = 0$ (minimal surfaces) and $K = 1$, respectively.

I want to describe a differential system whose integral surfaces are the Weingarten surfaces. For underlying manifold M, I will take $G \times \mathbb{R}^2$ where G is the group of rigid motions of 3-space as described in Example 3.7 (I will mantain the notation established there) and the coordinates on the $\mathbb{R}^2$ factor will be κ_1 and κ_2. Consider the ideal $\mathcal{I} = \langle\, \theta_0, \theta_1, \theta_2, \Upsilon \,\rangle$, where

$$\theta_0 = \omega_3\,, \quad \theta_1 = \omega_{31} - \kappa_1\,\omega_1\,, \quad \theta_2 = \omega_{32} - \kappa_2\,\omega_2\,, \quad \Upsilon = d\kappa_1 \wedge d\kappa_2\,.$$

Exercise 6.1: Explain how every Weingarten surface without umbilic points gives rise to an integral 2-manifold of $(M, \mathcal{I})$ and, conversely why every integral 2-manifold of $(M, \mathcal{I})$ on which $\omega_1 \wedge \omega_2$ is nonvanishing comes from a Weingarten surface in $\mathbb{R}^3$ by the process you have described.

Now let's look a little closer at the algebraic structure of $\mathcal{I}$. First of all, by the structure equations

$$\begin{aligned} d\theta_0 = d\omega_3 &= -\omega_{31} \wedge \omega_1 - \omega_{32} \wedge \omega_2 \\ &= -(\theta_1 + \kappa_1\,\omega_1) \wedge \omega_1 - (\theta_2 + \kappa_2\,\omega_2) \wedge \omega_2 \\ &= -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2\,. \end{aligned}$$

Then, again, by the structure equations

$$\begin{aligned} d\theta_1 &= d\omega_{31} - d\kappa_1 \wedge \omega_1 - \kappa_1\,d\omega_1 \\ &= -\omega_{32} \wedge \omega_{21} - d\kappa_1 \wedge \omega_1 + \kappa_1\,(\omega_{12} \wedge \omega_2 + \omega_{13} \wedge \omega_3) \\ &= -(\theta_2 + \kappa_2\,\omega_2) \wedge \omega_{21} - d\kappa_1 \wedge \omega_1 + \kappa_1\,(-\omega_{21} \wedge \omega_2 + \omega_{13} \wedge \theta_0) \\ &\equiv -d\kappa_1 \wedge \omega_1 - (\kappa_1 - \kappa_2)\omega_{12} \wedge \omega_2 \mod \{\theta_0, \theta_1, \theta_2\}. \end{aligned}$$

A similar computation gives

$$d\theta_2 \equiv -(\kappa_1 - \kappa_2)\omega_{21} \wedge \omega_1 - d\kappa_2 \wedge \omega_2 \mod \{\theta_0, \theta_1, \theta_2\}.$$

Thus, setting $\pi_1 = d\kappa_1$, $\pi_2 = (\kappa_1{-}\kappa_2)\omega_{21}$, and $\pi_3 = d\kappa_2$, we have

$$\mathcal{I} = \langle\, \theta_0, \theta_1, \theta_2,\ \pi_1\wedge\omega_1{+}\pi_2\wedge\omega_2,\ \pi_2\wedge\omega_1{+}\pi_3\wedge\omega_2,\ \pi_1\wedge\pi_3\,\rangle_{\text{alg}}$$

Now, on the open set $M^+ \subset M$ where $\kappa_1 > \kappa_2$, the 1-forms

$$\omega_1,\ \omega_2,\ \theta_0,\ \theta_1,\ \theta_2,\ \pi_1,\ \pi_2,\ \pi_3$$

are linearly independent and are a basis for the 1-forms. For any $e \in TM^+$ we can write its components in this basis as

$$\begin{aligned}\omega_i(e) &= a_i\,,\ \ (i = 1,2),\\ \theta_j(e) &= t_j\,,\ \ (j = 0,1,2),\\ \pi_k(e) &= p_k\,,\ \ (k = 1,2,3).\end{aligned}$$

The vector e spans a 1-dimensional integral element E if and only if it is nonzero and satisfies $t_0 = t_1 = t_2 = 0$.

Exercise 6.2: Explain why this shows that all of the elements in $V_1(\mathcal{I})$ are ordinary.

Now, assuming e spans $E \in V_1(\mathcal{I})$, the polar space $H(E)$ is then defined as the set of vectors v that annihilate the 1-forms θ_i and the three 1-forms

$$\begin{aligned} e \lrcorner (\pi_1\wedge\omega_1{+}\pi_2\wedge\omega_2) &= p_1\,\omega_1 + p_2\,\omega_2 - a_1\,\pi_1 - a_2\,\pi_2\\ e \lrcorner (\pi_2\wedge\omega_1{+}\pi_3\wedge\omega_2) &= p_2\,\omega_1 + p_3\,\omega_2 \qquad\quad - a_1\,\pi_2 - a_2\,\pi_3\\ e \lrcorner (\pi_1\wedge\pi_3) &= \qquad\qquad\quad - p_3\,\pi_1 \qquad\quad + p_1\,\pi_3\,. \end{aligned}$$

Clearly, for any 'generic' choice of the quantities $(a_1, a_2, p_1, p_2, p_3)$, these three 1-forms will be linearly independent, so that $H(E)$ will have dimension 2. (Remember that M^+ has dimension 8.) In this case, the flag $\bigl(0, E, H(E)\bigr)$ will be regular with characters $(s_0, s_1, s_2) = (3,3,0)$. From the description of the generality of solutions given in the last Lecture, it follows that the 'general' Weingarten surface depends on 3 constants and 3 functions of one variable.

Exercise 6.3: Describe the set of $E_2 \subset V_2(\mathcal{I})$ on which $\omega_1{\wedge}\omega_2$ is nonzero. Show that this is not a smooth submanifold of $G_2(TM)$ and describe the singular locus. Show, however, that every $E_2 \in V_2^r(\mathcal{I})$ on which $\omega_1{\wedge}\omega_2$ is nonzero does contain a regular flag.

Exercise 6.4: Describe which curves $\bigl(x(t), e_1(t), e_2(t), e_3(t), \kappa_1(t), \kappa_2(t)\bigr)$ in M^+ represent regular 1-dimensional integral manifolds of $\mathcal{I}$.

Exercise 6.5: Suppose that you want to prescribe the relation $F(\kappa_1, \kappa_2) = 0$ beforehand and then describe all of the (umbilic-free) surfaces in $\mathbb{R}^3$ that satisfy $F(\kappa_1, \kappa_2) = 0$. How would you set this up as an exterior differential system? What are its characters?

6.2. Orthogonal Coordinates on 3-manifolds

Suppose now that N^3 is a 3-manifold and that $g : TN \to \mathbb{R}$ is a Riemmanian metric, i.e., a smooth function on TN with the property that, on each T_xN, g is a positive definite quadratic form. A coordinate chart $(x^1, x^2, x^3) : U \to \mathbb{R}^3$ is said to be *g-orthogonal* if, on U,

$$g = g_{11}\,(dx^1)^2 + g_{22}\,(dx^2)^2 + g_{33}\,(dx^3)^2,$$

i.e., if the coordinate expression $g = g_{ij}\,dx^i\,dx^j$ satisfies $g_{ij} = 0$ for i different from j. This is three equations for the three coordinate functions x^i. I now want to describe an exterior differential system whose 3-dimensional integral manifolds describe the solutions to this problem.

First, note that, if you have a solution, then the 1-forms $\eta_i = \sqrt{g_{ii}}\,dx^i$ form a g-orthonormal coframing, i.e.,

$$g = {\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2.$$

This coframing is not the most general coframing, though, because it satisfies

$$\eta_1 \wedge d\eta_1 = \eta_2 \wedge d\eta_2 = \eta_3 \wedge d\eta_3 = 0,$$

since each η_i is a multiple of an exact 1-form. Conversely, any g-orthonormal coframing (η_1, η_2, η_3) that satisfies $\eta_i \wedge d\eta_i = 0$ for $i = 1, 2, 3$ is locally of the form $\eta_i = A_i\, dx_i$ for some functions $A_i > 0$ and x^i, by the Frobenius Theorem. (Why?)

Thus, up to an application of the Frobenius Theorem, the problem of finding g-orthogonal coordinates is equivalent to finding g-orthonormal coframings (η_1, η_2, η_3) satisfying $\eta_i \wedge d\eta_i = 0$. I now want to set up an exterior differential system whose integral manifolds reprsent these coframings.

To do this, let $\pi : F \to N$ be the g-orthonormal coframe bundle of N, i.e, a point of F is a quadruple $f = (x, u_1, u_2, u_3)$ where $x = \pi(f)$ belongs to N and $u_i \in T_x N$ are g-orthonormal. This is an O(3)-bundle over N and hence is a manifold of dimension 6. There are canonical 1-forms $\omega_1, \omega_2, \omega_3$ on F that satisfy

$$\omega_i(v) = u_i\big(\pi'(v)\big), \qquad \text{for all } v \in T_f M \text{ with } f = (x, u_1, u_2, u_3).$$

These 1-forms have the 'reproducing property' that, if $\eta = (\eta_1, \eta_2, \eta_3)$ is a g-orthonormal coframing on $U \subset M$, then regarding η as a section of F over U via the map

$$\sigma_\eta(x) = \big(\, x,\, (\eta_1)_x,\, (\eta_2)_x,\, (\eta_3)_x \,\big),$$

we have $\sigma_\eta^*(\omega_1, \omega_2, \omega_3) = (\eta_1, \eta_2, \eta_3)$.

Exercise 6.6: Prove this statement. Prove also that $\pi^*(*1) = \omega_1 \wedge \omega_2 \wedge \omega_3$, and that a 3-dimensional submanifold $P \subset F$ can be locally represented as the graph of a local section $\sigma : U \to F$ if and only if $\omega_1 \wedge \omega_2 \wedge \omega_3$ is nonvanishing on P.

Consider the ideal $\mathcal{I} = \langle\, \omega_1 \wedge d\omega_1,\ \omega_2 \wedge d\omega_2,\ \omega_3 \wedge d\omega_3 \,\rangle$. The 3-dimensional integral manifolds of $\mathcal{I}$ on which $\omega_1 \wedge \omega_2 \wedge \omega_3$ is nonvanishing are then the desired local sections. We now want to describe these integral manifolds.

First, it is useful to note that, just as for the orthonormal (co-)frame bundle of Euclidean space, there are unique 1-forms $\omega_{ij} = -\omega_{ji}$ that satisfy the structure equations

$$d\omega_i = -\sum_{j=1}^{3} \omega_{ij} \wedge \omega_j\ .$$

The 1-forms $\omega_1, \omega_2, \omega_3, \omega_{23}, \omega_{31}, \omega_{12}$ are then a basis for the 1-forms on F.

By the structure equations, an alternative description of $\mathcal{I}$ is

$$\mathcal{I} = \langle\, \omega_2 \wedge \omega_3 \wedge \omega_{23},\ \omega_3 \wedge \omega_1 \wedge \omega_{31},\ \omega_1 \wedge \omega_2 \wedge \omega_{12} \,\rangle.$$

Let $G_3(TF, \omega)$ denote the set of tangent 3-planes on which $\omega_1 \wedge \omega_2 \wedge \omega_3$ is nonvanishing. Any $E \in G_3(TF, \omega)$ is defined by equations of the form

$$\begin{aligned}
\omega_{23} - p_{11}\, \omega_1 - p_{12}\, \omega_2 - p_{13}\, \omega_3 &= 0 \\
\omega_{31} - p_{21}\, \omega_1 - p_{22}\, \omega_2 - p_{23}\, \omega_3 &= 0 \\
\omega_{12} - p_{31}\, \omega_1 - p_{32}\, \omega_2 - p_{33}\, \omega_3 &= 0
\end{aligned}$$

Such a plane E is an integral element of $\mathcal{I}$ if and only the coefficients p_{ij} satisfy $p_{11} = p_{22} = p_{33} = 0$, which shows that $V_3(\mathcal{I}) \cap G_3(TF, \omega)$ consists entirely of ordinary integral elements. (Why?) Since $\mathcal{I}$ is generated in degree 3, each 1-plane or 2-plane is an ordinary integral element of $\mathcal{I}$. Moreover, since $\mathcal{I}$ is generated by three 3-forms, it follows that for any $E_2 \in V_2(\mathcal{I})$, the codimension of $H(E_2)$ in $T_p F$ is at most 3. In particular, every such E_2 has at least one extension to a 3-dimensional integral element, so that $r(E_2) \geq 0$ for every $E_2 \in V_2(\mathcal{I})$.

If the metric g is real analytic, then the Cartan-Kähler Theorem applies and it follows that there will be 3-dimensional integral manifolds of $\mathcal{I}$ and that, in fact, the generic real analytic surface in F lies in such an integral manifold.

This would be enough to solve our problem, but it is useful to determine the explicit condition that makes a surface in F be a regular integral manifold. To do this, we need to determine $V_2^r(\mathcal{I})$. Now, suppose that E_2 is spanned by two vectors a and b and set $a_i = \omega_i(a)$ and $b_i = \omega_i(b)$. A vector v will lie in the polar space of E_2 if and only if it is annihilated by the three 1-forms

$$\left.\begin{aligned}(a\wedge b) \lrcorner (\omega_2\wedge\omega_3\wedge\omega_{23}) &\equiv (a_2b_3-a_3b_2)\,\omega_{23}\\ (a\wedge b) \lrcorner (\omega_3\wedge\omega_1\wedge\omega_{31}) &\equiv (a_3b_1-a_1b_3)\,\omega_{31}\\ (a\wedge b) \lrcorner (\omega_1\wedge\omega_2\wedge\omega_{12}) &\equiv (a_1b_2-a_2b_1)\,\omega_{12}\end{aligned}\right\}\ \mathrm{mod}\ \{\omega_1,\omega_2,\omega_3\}\,.$$

In particular, $r(E_2) = 0$ and $H(E_2) = E_3$ lies in $G_3(TF, \omega)$ when all of the numbers

$$\{(a_2b_3-a_3b_2), (a_3b_1-a_1b_3), (a_1b_2-a_2b_1)\}$$

are nonzero.

Exercise 6.7: Show that this computation leads to the following geometric description of the regular integral surfaces of $\mathcal{I}$. A regular integral surface can be seen as a surface $S \subset M$ and a choice of a g-orthonormal coframing $\eta = (\eta_1, \eta_2, \eta_3)$ along S such that none of the $\eta_i\wedge\eta_j$ $(i \neq j)$ vanish on the tangent planes to the surface S. By the Cartan-Kähler Theorem, a real analytic coframing satisfying this nondegeneracy condition defined along a real analytic surface S can be 'thickened' uniquely to a real analytic coframing in a neighborhood of S in such a way that each of the η_i become integrable (i.e., locally exact up to multiples).

6.3. The existence of local Lie groups

As you know, every Lie group G has an associated Lie algebra structure on the tangent space $\mathfrak{g} = T_eG$. This Lie algebra structure is a skewsymmetric bilinear pairing $[,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ that satisfies the *Jacobi identity*

$$\big[u, [v, w]\big] + \big[v, [w, u]\big] + \big[w, [u, v]\big] = 0$$

for all $u, v, w \in \mathfrak{g}$. One way this shows up in the geometry of G (there are many ways) is that, as discussed in Lecture 2, the canonical left invariant 1-form η on G satisfies the Mauer-Cartan equation $d\eta = -\frac{1}{2}[\eta, \eta]$.

We have already seen Cartan's Theorem, which says that any $\mathfrak{g}$-valued 1-form ω on a connected and simply connected manifold M that satisfies $d\omega = -\frac{1}{2}[\omega, \omega]$ is of the form $\omega = g^*\eta$ for some $g : M \to G$, unique up to composition with left translation. This implies, in particular, that there is at most one connected and simply connected Lie group associated to each Lie algebra.

I now want to consider the existence question: Suppose that we are given a Lie algebra, i.e., a vector space $\mathfrak{g}$ over $\mathbb{R}$ with a skewsymmetric bilinear pairing $[,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity. Does there exist a Lie group G with Lie algebra $\mathfrak{g}$? Now, the answer is known to be 'yes', but it's rather delicate because of certain global topological issues that I don't want to get into here. What I want to do instead is use the Cartan-Kähler Theorem to give a quick, simple proof that there exists a *local* Lie group with Lie algebra $\mathfrak{g}$.

What this amounts to is showing that there exists a $\mathfrak{g}$-valued 1-form η on a neighborhood U of $0 \in \mathfrak{g}$ with the property that $\eta_0 : T_0\mathfrak{g} \to \mathfrak{g}$ is the identity and that it satisfies the Maurer-Cartan equation $d\eta = -\frac{1}{2}[\eta, \eta]$.

Exercise 6.8: Assuming such an η exists, prove that there exists some 0-neighborhood $V \subset U$ and a smooth (in fact, real analytic) map $\mu : V \times V \to U$ satisfying

(1) (*Identity*) $\mu(0, v) = \mu(v, 0) = v$ for all $v \in V$,

(2) (*Inverses*) For each $v \in V$, there is a $v^* \in V$ so that $\mu(v, v^*) = \mu(v^*, v) = 0$.

(3) (*Associativity*) For $u, v, w \in V$, $\mu\big(\mu(u, v), w\big) = \mu\big(u, \mu(v, w)\big)$ when both sides make sense,

and so that, if L_v is defined by $L_v(u) = \mu(v, u)$, then $(L_v)'\big(\eta_v(w)\big) = w$ for all $w \in T_v\mathfrak{g} = \mathfrak{g}$.

To prove the existence of η, we proceed as follows: First identify $\mathfrak{g}$ with $\mathbb{R}^n$ by choosing linear coordinates $\mathbf{x} = (x^i)$. Now, let $M = \mathrm{GL}(n,\mathbb{R}) \times \mathbb{R}^n$, with $\mathbf{u} : M \to \mathrm{GL}(n,\mathbb{R})$ and $\mathbf{x} : M \to \mathbb{R}^n$ being the projections onto the first and second factors. Now set

$$\Theta = d(\mathbf{u}\,d\mathbf{x}) + \tfrac{1}{2}\,\big[\mathbf{u}\,d\mathbf{x}, \mathbf{u}\,d\mathbf{x}\big] = \big(\Theta^i\big).$$

Exercise 6.9: Show that the Jacobi identity implies that (in fact, is equivalent to the fact that) $\bigl[\psi,[\psi,\psi]\bigr] = 0$ for any $\mathfrak{g}$-valued 1-form ψ. Conclude that Θ satisfies $d\Theta = \frac{1}{2}\,\bigl[\Theta, \mathbf{u}\,d\mathbf{x}\bigr] - \frac{1}{2}\,\bigl[\mathbf{u}\,d\mathbf{x}, \Theta\bigr]$.

From this exercise, it follows that the ideal $\mathcal{I}$ generated by the n component 2-forms Θ^i is generated algebraically by these 2-forms.

Exercise 6.10: If $\mathbf{u} = (u^i_j)$, then show that there exist (linearly independent) 1-forms π^i_j satisfying

$$\pi^i_j \equiv du^i_j \mod \{dx^1, \ldots, dx^n\}$$

for which $\Theta^i = \pi^i_j{\wedge}dx^j$.

The existence of η will be established if we can show that there exists an n-dimensional integral manifold $N \subset M$ of $\mathcal{I}$ passing through $p = (I_n, 0)$ on which the n-form $dx^1{\wedge}\cdots{\wedge}dx^n$ is nonvanishing.

To do this, consider the integral element $E_n \subset T_pM$ defined by the equations $\pi^i_j = 0$, and let $E_k \subset E_n$ be defined by the additional equations $dx^j = 0$ for $j > k$ for $0 \le k \le n$. Since the π^i_j are linearly independent, it follows that

$$H(E_k) = \{\, v \in T_pM \bigm| \pi^i_j(v) = 0 \text{ for } 1 \le j \le k \,\},$$

so $c(E_k) = nk$ for $0 \le k \le n$. Thus, for the flag $F = (E_0, \ldots, E_n)$, we have

$$c(F) = 0 + n + 2n + \cdots + (n{-}1)n = \tfrac{1}{2}n^2(n{-}1).$$

On the other hand an n-plane $E \in G_n(TM, \mathbf{x})$ is defined by equations of the form

$$\pi^i_j - p^i_{jk}\,dx^k = 0$$

and it will be an integral element if and only if the $\frac{1}{2}n^2(n{-}1)$ linear equations $p^i_{jk} = p^i_{kj}$ hold. Consequently, Cartan's Test is satisfied and the flag is regular. The Cartan-Kähler Theorem now implies that there is an integral manifold of $\mathcal{I}$ tangent to E_n. QED

Exercise 6.11: If you are familiar with the proof of this theorem that uses only ODE techniques (see, for example, [Helgason]), compare that proof with this one. Can you see how the two are related?

6.4. Hyper-Kähler metrics

This example is somewhat more advanced that the previous ones. I'm including it for the sake of those who might be interested in seeing how the Cartan-Kähler theorem can be used to study more advanced problems in differential geometry.

A hyper-Kähler structure on a manifold M^{4n} is a quadruple (g, I, J, K) where g is a Riemannian metric and $I, J, K : TM \to TM$ are g-parallel and orthogonal skewcommuting linear transformations of TM that satisfy

$$I^2 = J^2 = K^2 = -1, \qquad IJ = -K, \quad JK = -I, \quad KI = -J.$$

In other words (I, J, K) define a *right* quaternionic structure on the tangent bundle of M that is orthogonal and parallel with respect to g.

Suppose we have such a structure on M. Set

$$\omega_1(v, w) = g(Iv, w), \qquad \omega_2(v, w) = g(Jv, w), \qquad \omega_3(v, w) = g(Kv, w).$$

Then these three 2-forms are g-parallel and hence closed. Moreover, these three 2-forms are enough data to recover I, J, K and even g. For example, I is the unique map that satisfies $\omega_3(Iv, w) = -\omega_2(v, w)$ and then $g(v, w) = \omega_1(v, Iw)$.

There remains the question of 'how many' such hyper-Kähler metrics there are locally. One obvious example is to take $M = \mathbb{H}^n$ with its standard metric and let I, J, and K be the usual multiplication (on the *right*) by the obvious unit quaternions. However, this is not a very interesting example.

Two of these 2-forms at a time can indeed be made flat in certain coordinates: If we set $\Omega = \omega_2 - i\,\omega_3$, it is easy to compute that

$$\Omega(Ix, y) = \Omega(x, Iy) = i\,\Omega(x, y)$$

for all tangent vectors $x, y \in T_pM$. Thus, Ω is a closed 2-form of type $(2, 0)$ with respect to the complex structure I. Moreover, it is easy to compute that Ω^n is nowhere vanishing but that $\Omega^{n+1} = 0$. It follows from the complex version of the Darboux Theorem that every $p \in M$ has a neighborhood U on which there exist complex coordinates $z^1, \ldots, z^{2n}$ that are holomorphic for the complex structure I and for which

$$\Omega = dz^1 \wedge dz^{n+1} + dz^2 \wedge dz^{n+2} + \cdots + dz^n \wedge dz^{2n}.$$

These coordinates are unique up to a holomorphic symplectic transformation. Meanwhile, the 2-form ω_1 in these coordinates takes the form

$$\omega_1 = \tfrac{\sqrt{-1}}{2}\, u_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

where $U = (u_{i\bar{j}})$ is a positive definite Hermitian matrix of functions that satisfies the equation ${}^tU\,Q\,U = Q$ where

$$Q = \begin{pmatrix} \mathrm{O}_n & \mathrm{I}_n \\ -\mathrm{I}_n & \mathrm{O}_n \end{pmatrix}.$$

One cannot generally choose the coordinates to make U be the identity matrix. Indeed, this is the necessary and sufficient condition that the hyper-Kähler structure be locally equivalent to the flat structure mentioned above.

Conversely, if one can find a smooth function U on a domain $D \subset \mathbb{C}^{2n}$ with values in positive definite Hermitian $2n$-by $2n$ matrices satisfying the algebraic condition ${}^tU\,Q\,U = Q$ as well as the differential condition that the 2-form

$$\omega_1 = \tfrac{\sqrt{-1}}{2}\, u_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

be closed, then setting

$$\omega_2 - i\,\omega_3 = dz^1 \wedge dz^{n+1} + dz^2 \wedge dz^{n+2} + \cdots + dz^n \wedge dz^{2n}$$

defines a triple $(\omega_1, \omega_2, \omega_3)$ on D that determines a hyper-Kähler structure on D.

This suggests the construction of a differential ideal whose integral manifolds will represent the desired functions U. First, define

$$Z = \{ H \in \mathrm{GL}(2n, \mathbb{C}) \mid H = {}^t\bar{H} > 0,\ {}^tH\,Q\,H = Q \}.$$

Exercise 6.12: Show that Z can also be described as the space of matrices $H = {}^t\bar{A}\,A$ with $A \in \mathrm{Sp}(n, \mathbb{C}) = \{ A \in \mathrm{GL}(2n, \mathbb{C}) \mid {}^tA\,Q\,A = Q \}$ and hence that Z is just the Riemannian symmetric space $\mathrm{Sp}(n, \mathbb{C})/\mathrm{Sp}(n)$, whose dimension is $2n^2{+}n$. In particular, Z is a smooth submanifold of $\mathrm{GL}(2n, \mathbb{C})$.

Now define $M = Z \times \mathbb{C}^{2n}$ and let $H = (h_{i\bar{j}}) : M \to Z$ be the projection onto the first factor and $z : M \to \mathbb{C}^{2n}$ be the projection onto the second factor. Let $\mathcal{I}$ be the ideal generated by the (real) 3-form

$$\Theta = \tfrac{\sqrt{-1}}{2}\, dh_{i\bar{j}} \wedge dz^i \wedge d\bar{z}^j = d\left(\tfrac{\sqrt{-1}}{2}\, h_{i\bar{j}}\, dz^i \wedge d\bar{z}^j \right).$$

Obviously $\mathcal{I}$ is generated algebraically by Θ, since Θ is closed. One integral manifold of Θ is given by the equations $H = \mathrm{I}_{2n}$, which corresponds to the flat solution. We want to determine how general the space of solutions is near this solution.

First, let me note that the group $\mathrm{Sp}(n, \mathbb{C})$ acts on M preserving Θ via the action

$$A \cdot (H,\, z) = \left({}^t\bar{A}\,H\,A,\ \bar{A}^{-1}z\right).$$

The additive group $\mathbb{C}^{2n}$ also acts on M via translations in the $\mathbb{C}^{2n}$-factor, and this action also preserves Θ. These two actions combined generate a transitive action on M preserving Θ, so the ideal $\mathcal{I}$ is homogeneous. Thus, we can do our computations at any point, say $p = (\mathrm{I}_{2n}, 0)$, which I fix from now on.

Let $E_{4n} \subset T_pM$ be the tangent space at p to the flat solution $H = \mathrm{I}_{2n}$. Let $F = (E_0, \ldots, E_{4n})$ be any flag. Because $\mathcal{I}$ is generated by a single 3-form, it follows that

$$c(E_k) \le \binom{k}{2}$$

for all k. (Why?) On the other hand, since the codimension of E_{4n} in T_pM is $2n^2{+}n = \dim Z$, equality cannot hold for $k > 2n{+}1$.

Now, I claim that there exists a flag F for which $c(E_k) = \binom{k}{2}$ for $k \le 2n{+}1$ while $c(E_k) = 2n^2{+}n$ when $2n{+}1 < k \le 4n$. Moreover, I claim that Cartan's Test is satisfied for such a flag, i.e., $V_{4n}(\mathcal{I}) \cap G_{4n}(TM, \mathbf{z})$ is a smooth submanifold of $G_{4n}(TM, \mathbf{z})$ of codimension

$$c(F) = c(E_0) + \cdots + c(E_{4n-1}) = \frac{4}{3}n(2n-1)(2n+1).$$

Consequently, such a flag is regular.

Since $s_k(F) = k{-}1$ for $0 < k \le 2n{+}1$ and $s_k(F) = 0$ for $k > 2n{+}1$, the description of the generality of solutions near the flat solution now shows that the solutions depend on $2n$ 'arbitrary' functions of $2n{+}1$ variables and that a solution is determined by its restriction to a generic (real analytic) submanifold of dimension $2n{+}1$. Since the symplectic biholomorphisms depend only on arbitrary functions of $2n$ variables (why?), it follows that the generic hyper-Kähler structure is not flat. In fact, as we shall see in the next lecture, this calculation will yield much more detailed information about the local solutions.

I'm only going to sketch out the proof of these claims and leave much of the linear algebra to you.

The first thing to do is to get a description of the relations among the components of dH_p. Computing the exterior derivatives of the defining relations $H = {}^t\bar{H}$ and ${}^tH\,Q\,H = Q$ gives

$$dH = {}^t\overline{dH}, \qquad {}^t(dH)\,Q\,H + {}^tH\,Q\,dH = 0.$$

Evaluating this at $p = (\mathrm{I}_{4n}, 0)$, gives

$$dH_p = {}^t\overline{dH_p}, \qquad {}^t(dH_p)\,Q + Q\,dH_p = 0,$$

so it follows that

$$dH_p = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\bar{\alpha} \end{pmatrix}$$

where $\alpha = {}^t\bar{\alpha}$ and $\beta = {}^t\beta$ are n-by-n matrices of complex-valued 1-forms. Writing $z^i = u^i$ and $z^{i+n} = v^i$ for $1 \le i \le n$, it follows that

$$\Theta_p = \tfrac{\sqrt{-1}}{2}\,\alpha_{i\bar{j}} \wedge \big(du^i \wedge d\bar{u}^j - dv^j \wedge d\bar{v}^i\big) + \tfrac{\sqrt{-1}}{2}\,\big(\beta_{ij} \wedge du^i \wedge d\bar{v}^j + \bar{\beta}_{ij} \wedge dv^i \wedge d\bar{u}^j\big)$$

Now the only relations among the α-components and the β-components are $\alpha_{i\bar{j}} - \overline{\alpha_{j\bar{\imath}}} = \beta_{ij} - \beta_{ji} = 0$. Using this information, you can verify the computation of the $c(E_k)$ simply by finding an $E_{2n+1} \subset E_{4n}$ for which $c(E_{2n+1}) = 2n^2{+}n$, since this forces all the rest of the formulae for $c(E_k)$. (Why?) (Such an E_{2n+1} shouldn't be hard to find, since the generic element of $G_{2n+1}(E_{4n})$ works.)

Now, to verify the codimension of $V_{4n}(\mathcal{I})$, note that any $E^*_{4n} \subset T_pM$ that is transverse to the Z factor can be defined by equations of the form

$$\begin{aligned}
\alpha_{i\bar{j}} &= A_{i\bar{j}k}\,du^k - \overline{A_{j\bar{\imath}k}}\,d\bar{u}^k + B_{i\bar{j}k}\,dv^k - \overline{B_{j\bar{\imath}k}}\,d\bar{v}^k \\
\beta_{ij} &= P_{ijk}\,du^k + Q_{ij\bar{k}}\,d\bar{u}^k + R_{ijk}\,dv^k + S_{ij\bar{k}}\,d\bar{v}^k
\end{aligned}$$

where the coefficients are arbitrary subject to the relations $P_{ijk} = P_{jik}$, $Q_{ij\bar{k}} = Q_{ji\bar{k}}$, $R_{ijk} = R_{jik}$, $S_{ij\bar{k}} = S_{ji\bar{k}}$. Now you just need to check that the condition that Θ_p vanish on E^*_{4n} is exactly $\frac{4}{3}n(2n{+}1)(2n{-}1)$ linear relations on the coefficients A, B, P, Q, R, and S.

Exercise 6.13: Fill in the details in this proof.