Lecture 7. Prolongation

Almost all of the previous examples have been carefully chosen so that there will exist regular flags, so that the Cartan-Kähler theorem can be applied. Unfortunately, this is not always the case, in which case other methods must be applied. In this lecture, I'm going to describe those other methods.

7.1. When Cartan's Test fails.

We have already seen one case where the Cartan-Kähler approach fails, in the sense that it fails to find the integral manifolds that are actually there. That was in Exercise 3.1, where $M = \mathbb{R}^3$ and the ideal was

$$\mathcal{I} = \langle\, dx\wedge dz,\; dy\wedge dz\,\rangle.$$

Although there *are* 2-dimensional integral manifolds, namely the planes $z = c$, no 2-dimensional integral element is the terminus of a regular flag. It's not too surprising that this should happen, though, because the ideal

$$\mathcal{I}' = \langle\, dx\wedge dz,\; dy\wedge (dz{-}y,dx)\,\rangle$$

which is virtually indistinguisable from $\mathcal{I}$ in terms of the algebraic properties of the spaces of integral elements does not have any 2-dimensional integral manifolds. Some finer invariant of the ideals must be brought to light in order to distinguish the two cases.

Now, the above examples are admittedly a little artificial, so you might be surprised to see that they and their 'cousins' come up quite a bit.

Example 7.1: *Surfaces in $\mathbb{R}^3$ with constant principal curvatures.* You may already know how to solve this problem, but let's see what the naïve approach via differential systems will give. Looking back at the discussion of surface theory in Lecture 5, you can see that if we want to find the surfaces in $\mathbb{R}^3$ with principal curvatures equal to some fixed constants κ_1 and κ_2 (distinct), then we should look for integral manifolds of the ideal $\mathcal{I}$ on G that is generated by the three 1-forms

$$\theta_0 = \omega_3\,, \quad \theta_1 = \omega_{31} - \kappa_1\,\omega_1\,, \quad \theta_2 = \omega_{32} - \kappa_2\,\omega_2\,.$$

Now, if you compute the exterior derivatives of these forms, you'll get

$$\left.\begin{aligned} d\theta_0 &\equiv 0\\ d\theta_1 &\equiv -(\kappa_1{-}\kappa_2)\omega_{12}\wedge\omega_2\\ d\theta_2 &\equiv -(\kappa_1{-}\kappa_2)\omega_{21}\wedge\omega_1 \end{aligned}\right\}\ \operatorname{mod}\ \{\theta_0,\theta_1,\theta_2\}.$$

So

$$\mathcal{I} = \langle\, \theta_0,\theta_1,\theta_2,(\kappa_1{-}\kappa_2)\omega_{12}\wedge\omega_2,(\kappa_1{-}\kappa_2)\omega_{21}\wedge\omega_1\,\rangle)_{\text{alg}}.$$

Now there are two cases: One is that $\kappa_1 = \kappa_2$, in which case $\mathcal{I}$ is Frobenius.

Exercise 7.1: Explain why the integral manifolds in the case $\kappa_1 = \kappa_2$ correspond to the planes in $\mathbb{R}^3$ when $\kappa_1 = \kappa_2 = 0$ and to spheres in $\mathbb{R}^3$ when $\kappa_1 = \kappa_2 \neq 0$.

In the second case, where $\kappa_1 \neq \kappa_2$, you'll see that the 2-dimensional integral elements on which $\omega_1{\wedge}\omega_2$ is non-zero (and there are some) never contain any regular 1-dimensional integral elements, just as in our 'toy' example.

Exercise 7.2: Describe $V_2(\mathcal{I})$ when $\kappa_1 \neq \kappa_2$. Show that there is a unique 2-dimensional integral element of $\mathcal{I}$ at each point of G on which $\omega_1{\wedge}\omega_2$ is nonvanishing. Explain why this shows, via Cartan's Test, that such integral elements cannot be the terminus of a regular flag.

Exercise 7.3: Redo this problem assuming that the ambient 3-manifold is of constant sectional curvature c, not necessarily 0. You may want to recall that the structure equations in this case are of the form

$$d\omega_i = -\omega_{ij}\wedge\omega_j\,, \qquad d\omega_{ij} = -\omega_{ik}\wedge\omega_{kj} + c\,\omega_i\wedge\omega_j\,.$$

Does anything significant change?

Exercise 7.4: Set up an exterior differential system to model the solutions of the system $u_{xx} = u_{yy} = 0$ (where u is a function of x and y). Compare this to the analogous model of the system $u_{xx} = u_{xy} = 0$. In particular, compare the regular integral curves of the two systems and their 'thickenings' via the Cartan-Kähler Theorem.

Exercise 7.5: What can you say about the surfaces in $\mathbb{R}^3$ with the property that each principal curvature κ_i is constant on each of its corresponding principal curves?

7.2. Prolongation, systems in good form

In each of the examples in the previous subsection, we found an exterior differential system for which the interesting integral manifolds (if there are any) cannot be constructed by thickening along a regular flag. Cartan proposed a process of 'regularizing' these ideals which he called 'prolongation'. Intuitively, prolongation is just differentiating the equations you have and then adjoining those equations as new equations in the system. You can see why such a thing might work by looking at the following situation:

We know how to check whether the system

$$z_x = f(x, y), \qquad z_y = g(x, y)$$

is compatible. You just need to see whether or not $f_y = g_x$, a first order condition on the equations that is by looking at the exterior ideal generated by the 1-form $\zeta = dz - f(x, y)\,dx - g(x, y)\,dy$. On the other hand, if you consider the system

$$z_{xx} = f(x, y), \qquad z_{yy} = g(x, y),$$

the compatibility condition is not revealed until you differentiate twice, i.e, $f_{yy} = g_{xx}$. Now, it's not clear how to get to this condition by looking at the ideal on $xyzpqs$-space generated by

$$\begin{aligned}\theta_0 &= dz - p\,dx - q\,dy\\ \theta_1 &= dp - f(x, y)\,dx - s\,dy\\ \theta_2 &= dq - s\,dx - g(x, y)\,dy\end{aligned}$$

because the exterior derivatives of these forms will only contain first derivatives of the functions f and g. And, sure enough, Cartan's Test fails for this system.

However, if you differentiate the given equations once, you can see that they imply

$$z_{xxy} = f_y(x, y), \qquad z_{xyy} = g_x(x, y)$$

which suggests looking at the ideal on $xyzpqs$-space generated by

$$\begin{aligned}\theta_0 &= dz - p\,dx - q\,dy\\ \theta_1 &= dp - f(x, y)\,dx - s\,dy\\ \theta_2 &= dq - s\,dx - g(x, y)\,dy\\ \theta_3 &= ds - f_y(x, y)\,dx - g_x(x, y)\,dy.\end{aligned}$$

Now, this ideal is Frobenius if and only if $f_{yy} = g_{xx}$, so the obvious compatibility condition is the necessary and sufficient condition for there to exist solutions to the original problem.

Exercise 7.6: What can you say about the solutions of the system

$$z_{xx} = z \qquad\qquad z_{yy} = z?$$

A systematic way to 'adjoin derivatives as new variables' for the general exterior differential system $(M, \mathcal{I})$ is this: Suppose that you are interested in studying the n-dimensional integral manifolds of $(M, \mathcal{I})$ whose tangent planes lie in some smooth submanifold (usually a component)

$$Z \subset V_n(\mathcal{I}) \subset G_n(TM).$$

As explained in Lecture 2, every such integral manifold $f : N \hookrightarrow M$ has a canonical lift to a submanifold $f^{(1)} : N \hookrightarrow Z$ defined simply by

$$f^{(1)}(p) = f'(T_pN) \subset T_{f(p)}M.$$

Now, $f^{(1)} : N \hookrightarrow Z \subset G_n(TM)$ is an integral manifold of the contact system $\mathcal{C}$ and is transverse to the projection $\pi : Z \to M$. Conversely, if $F : N \to Z \subset G_n(TM)$ is an integral manifold of the contact system $\mathcal{C}$ that is transverse to the projection π, then $F = f^{(1)}$ where $f = \pi \circ F$, and so, *a fortiori*, the tangent spaces of the immersion $f : N \to M$ all lie in $Z \subset V_n(\mathcal{I})$. (In particular, $f : N \to M$ is an integral manifold of $\mathcal{I}$.)

Let $\mathcal{I}^{(1)} \subset \Omega^*(Z)$ denote the exterior ideal on Z induced by pulling back $\mathcal{C}$ on $G_n(TM)$ via the inclusion $Z \subset G_n(TM)$. The pair $\big(Z, \mathcal{I}^{(1)}\big)$ is known as the *Z-prolongation* of $\mathcal{I}$. Our argument in the above paragraph has established that the integral manifolds of $\mathcal{I}$ whose tangent planes lie in Z are in one-to-one correspondance with the integral manifolds of $\big(Z, \mathcal{I}^{(1)}\big)$ that are transverse to the projection $\pi : Z \to M$.

Usually, there is only one component of $V_n^o(\mathcal{I})$ of interest anyway. In this case, it is common to refer to it as $M^{(1)} \subset V_n^o(\mathcal{I})$ and then simply say that $\big(M^{(1)}, \mathcal{I}^{(1)}\big)$ is the prolongation of $\mathcal{I}$, imprecise though this is.

Example 7.2: *The toy model again.* Look at the EDS

$$\big(\mathbb{R}^3, \langle\, dx \wedge dz,\ dy \wedge dz \rangle\,\big).$$

There is exactly one 2-dimensional integral element at each point, namely, the 2-plane defined by $dz = 0$. Since these 2-dimensional integral elements define a Frobenius system on $\mathbb{R}^3$, there is a unique integral surface passing through each point of $\mathbb{R}^3$.

It's instructive to go through the above prolongation process explicitly here: Using coordinates (x, y, z), consider the open set $G_2\big(T\mathbb{R}^3, (x, y)\big)$ consisting of the 2-planes on which $dx \wedge dy$ is nonzero. This 5-manifold has coordinates (x, y, z, p, q) so that $E \in G_2\big(T\mathbb{R}^3, (x, y)\big)$ is spanned by

$$\left\{ \frac{\partial}{\partial x} + p(E)\,\frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial y} + q(E)\,\frac{\partial}{\partial z} \right\}.$$

In these coordinates, the contact system $\mathcal{C}$ is generated by the 1-form

$$\theta = dz - p\,dx - q\,dy.$$

Now, $Z = V_2(\mathcal{I}) \subset G_2\big(T\mathbb{R}^3, (x, y)\big)$ is defined by the equations $p = q = 0$, so pulling back the form θ to this locus yields that (x, y, z) are coordinates on Z and that $\mathcal{I}^{(1)} = \langle\, dz\,\rangle$, an ideal to which the Frobenius Theorem applies.

Exercise 7.7: Repeat this analysis for the EDS

$$\big(\mathbb{R}^3, \langle dx \wedge dz,\ dy \wedge (dz - y\,dx)\,\rangle\big).$$

Frequently, M has a coframing $(\omega^1, \ldots, \omega^n, \pi^1, \ldots, \pi^s)$ (i.e., a basis for the 1-forms on M) and one is interested in the n-dimensional integral manifolds of some $\mathcal{I}$ on which the 1-forms $(\omega^1, \ldots, \omega^n)$ are linearly independent. Let $V_n(\mathcal{I}, \omega)$ denote the integral elements on which $\omega = \omega^1 \wedge \ldots \wedge \omega^n$ is nonvanishing. The usual procedure is then to describe $V_n(\mathcal{I}, \omega)$ as the set of n-planes defined by equations of the form

$$\pi^a - p^a_i\,\omega^i = 0$$

where the p^a_i are subject to the constraints that make such an n-plane be an integral element. In this way, the p^a_i become functions on $V_n(\mathcal{I}, \omega)$. Moreover, the contact ideal $\mathcal{C}$ pulls back to $V_n(\mathcal{I}, \omega)$ to be generated by the 1-forms

$$\theta^a = \pi^a - p^a_i\,\omega^i\,,$$

thus giving us an explicit expression for the ideal $\mathcal{I}^{(1)}$ as

$$\mathcal{I}^{(1)} = \langle\ \theta^1, \ldots, \theta^s\ \rangle.$$

Example 7.3: *Constant principal curvatures.* Look back at Example 7.1, with $\kappa_1 \neq \kappa_2$, where we found an ideal

$$\mathcal{I} = \langle\, \omega_3, \omega_{31}{-}\kappa_1\,\omega_1, \omega_{32}{-}\kappa_2\,\omega_2, (\kappa_1{-}\kappa_2)\omega_{12}\wedge\omega_2, (\kappa_1{-}\kappa_2)\omega_{21}\wedge\omega_1\,\rangle_{\text{alg}}.$$

There is a unique 2-dimensional integral element at each point of G, defined by the equations

$$\omega_3 = \omega_{31}{-}\kappa_1\,\omega_1 = \omega_{32}{-}\kappa_2\,\omega_2, = \omega_{12} = 0.$$

Thus $V_2(\mathcal{I})$ is diffeomorphic to G. By the same reasoning employed above, we have that

$$\mathcal{I}^{(1)} = \langle\, \omega_3,\ \omega_{31}{-}\kappa_1\,\omega_1,\ \omega_{32}{-}\kappa_2\,\omega_2,\ \omega_{12}\,\rangle$$

Computing exterior derivatives and using the structure equations, we find that

$$\mathcal{I}^{(1)} = \langle\, \omega_3,\ \omega_{31}{-}\kappa_1\,\omega_1,\ \omega_{32}{-}\kappa_2\,\omega_2,\ \omega_{12},\ \kappa_1\kappa_2\,\omega_1\wedge\omega_2\,\rangle.$$

Now we can see a distinction: If $\kappa_1\kappa_2 \neq 0$, then this ideal has no 2-dimensional integral elements at all, and hence no integral surfaces. On the other hand, if $\kappa_1\kappa_2 = 0$ (i.e., one of the κ_i is zero), then $\mathcal{I}^{(1)}$ is a Frobenius system and is foliated by 2-dimensional integral manifolds.

Exercise 7.8: Repeat this analysis of the surfaces with constant principal curvatures for the other 3-dimensional spaces of constant sectional curvature. What changes? (You may want to look back at Exercise 7.3, for the structure equations.)

I want to do one more example of this kind of problem so that you can get some sense of what the process can be like. (I warn you that this is a rather involved example.)

Example 7.4: *Restricted Principal Curvatures.* Consider the surfaces described in Exercise 7.5, i.e., the surfaces with the property that each principal curvature κ_i is constant along each of its corresponding principal curves. A little thought, together with reference to the discussion of Weingarten surfaces in Lecture 6 should convince you that these surfaces are the integral manifolds in $M = G{\times}\mathbb{R}^2$ of the ideal

$$\mathcal{I} = \langle\, \theta_0,\ \theta_1,\ \theta_2,\ \pi_1\wedge\omega_1{+}\pi_2\wedge\omega_2,\ \pi_2\wedge\omega_1{+}\pi_3\wedge\omega_2,\ \pi_1\wedge\omega_2,\ \pi_3\wedge\omega_1\,\rangle_{\text{alg}}\,.$$

(I am maintaining the notation established in Lecture 6.1.) Now, each 2-dimensional integral element on which $\omega_1{\wedge}\omega_2$ is non-vanishing is defined by equations of the form

$$\theta_0 = \theta_1 = \theta_2 = \pi_1 - p_1\,\omega_2 = \pi_2 - p_1\,\omega_1 - p_2\,\omega_2 = \pi_3 - p_2\,\omega_1 = 0.$$

where p_1 and p_2 are arbitrary parameters. I'll leave it to you to check that these integral elements are *not* the terminus of any regular flag. Consequently, we cannot apply the Cartan-Kähler Theorem to construct examples of such surfaces.

It is computationally advantageous to parametrize the integral elements by $q_1 = p_1/(\kappa_1{-}\kappa_2)$ and $q_2 = p_2/(\kappa_1{-}\kappa_2)$ rather than by p_1 and p_2 as defined above, so that is what we will do. (This change of scale avoids having to divide by $(\kappa_1{-}\kappa_2)$ several times later.)

Now, following the prescription already given, construct $\big(M^{(1)}, \mathcal{I}^{(1)}\big)$ as follows: We let $M^{(1)} = M \times \mathbb{R}^2$, with p_1 and p_2 being the coordinates on the $\mathbb{R}^2$-factor and set

$$\mathcal{I}^{(1)} = \langle\, \theta_0,\ \theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5\,\rangle$$

where

$$\begin{aligned} \theta_3 &= \pi_1 \qquad\qquad - p_1\,\omega_2 &&= \qquad d\kappa_1 \qquad\qquad - p_1\,\omega_2\\ (\kappa_1{-}\kappa_2)\,\theta_4 &= \pi_2 - p_1\,\omega_1 - p_2\,\omega_2 &&= (\kappa_1{-}\kappa_2)\,\omega_{21} - p_1\,\omega_1 - p_2\,\omega_2\\ \theta_5 &= \pi_3 - p_2\,\omega_1 &&= \qquad d\kappa_2 - p_2\,\omega_1 \end{aligned}$$

or, in terms of the q_i, we have

$$\theta_3 = d\kappa_1 - (\kappa_1{-}\kappa_2)q_1\,\omega_2\,,$$
$$\theta_4 = \omega_{21} - q_1\,\omega_1 - q_2\,\omega_2\,,$$
$$\theta_5 = d\kappa_2 - (\kappa_1{-}\kappa_2)q_2\,\omega_1\,.$$

Now, it should not be a surprise that

$$d\theta_0 \equiv d\theta_1 \equiv d\theta_2 \equiv 0 \mod \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5$$

(you should check this if you are surprised). Moreover, using the structure equations and the definitions of the forms given so far, we can compute that

$$\begin{aligned} d\theta_3 &\equiv -(\kappa_1{-}\kappa_2)\, dq_1 \wedge \omega_2 \\ d\theta_4 &\equiv\; - dq_1 \wedge \omega_1 - dq_2 \wedge \omega_2 - ({q_1}^2{+}{q_2}^2{+}\kappa_1\kappa_2)\, \omega_1 \wedge \omega_2 \\ d\theta_5 &\equiv -(\kappa_1{-}\kappa_2)\, dq_2 \wedge \omega_1 \end{aligned}$$

where the congruences are taken modulo θ_0, θ_1, θ_2, θ_3, θ_4, θ_5. It follows from this computation that the 2-dimensional integral elements on which the 2-form $\omega_1{\wedge}\omega_2$ is nonzero are all of the form

$$\begin{aligned} \theta_0 = \cdots = \theta_5 &= 0 \\ dq_1 - (q_3 + {q_1}^2 + \tfrac{1}{2}\kappa_1\kappa_2)\,\omega_2 &= 0 \\ dq_2 - (q_3 - {q_2}^2 - \tfrac{1}{2}\kappa_1\kappa_2)\,\omega_1 &= 0 \end{aligned}$$

for some q_3. Thus, there is a 1-parameter family of such integral elements at each point. Unfortunately, none of these integral elements are the terminus of a regular flag, so the Cartan-Kähler Theorem *still* cannot be applied.

There's nothing to do now, but do it again: We now parametrize the space of these integral elements of $\bigl(M^{(1)}, \mathcal{I}^{(1)}\bigr)$ as $M^{(2)} = M^{(1)} \times \mathbb{R}$ with q_3 being the coordinate on the $\mathbb{R}$-factor and we consider the ideal

$$\mathcal{I}^{(2)} = \langle\, \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5,\, \theta_6,\, \theta_7 \,\rangle$$

where

$$\theta_6 = dq_1 - (q_3 + {q_1}^2 + \tfrac{1}{2}\kappa_1\kappa_2)\,\omega_2,$$
$$\theta_7 = dq_2 - (q_3 - {q_2}^2 - \tfrac{1}{2}\kappa_1\kappa_2)\,\omega_1.$$

Now we get

$$d\theta_0 \equiv \cdots \equiv d\theta_5 \equiv 0 \mod \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5,\, \theta_6,\, \theta_7$$

(again, this should be no surprise), but we must still compute $d\theta_6$ and $d\theta_7$. Well, using the structure equations, we can do this and we get

$$\left.\begin{aligned} d\theta_6 &\equiv -\theta_8 \wedge \omega_2, \\ d\theta_7 &\equiv -\theta_8 \wedge \omega_1, \end{aligned}\right\} \mod \theta_0,\, \ldots\, \theta_7$$

where

$$\theta_8 = dq_3 + (q_3{+}{q_1}^2{+}\tfrac{1}{2}{\kappa_1}^2)q_2\,\omega_1 - (q_3{-}{q_2}^2{-}\tfrac{1}{2}{\kappa_2}^2)q_1\,\omega_2\,.$$

(Whew!) At this point, it is clear that there is only one 2-dimensional integral element of $\mathcal{I}^{(2)}$ on which $\omega_1{\wedge}\omega_2$ is nonzero at each point of $M^{(2)}$ and it is defined by

$$\theta_0 = \theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = \theta_7 = \theta_8 = 0.$$

Thus $M^{(3)} = M^{(2)}$ and we can take

$$\mathcal{I}^{(3)} = \langle\, \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5,\, \theta_6,\, \theta_7,\, \theta_8 \,\rangle$$

As before, it is clear that

$$d\theta_0 \equiv \cdots \equiv d\theta_7 \equiv 0 \text{ mod } \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5,\, \theta_6,\, \theta_7,\, \theta_8\,.$$

What is surprising (perhaps) is that

$$d\theta_8 \equiv 0 \text{ mod } \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5,\, \theta_6,\, \theta_7,\, \theta_8\,!$$

In other words, $\mathcal{I}^{(3)}$ is a Frobenius system! Consequently, $M^{(3)}$, a manifold of dimension 11 (count it up) is foliated by 2-dimensional integral manifolds of $\mathcal{I}^{(3)}$.

Now this rather long example is meant to convince you that the process of prolongation can actually lead you to some answers. Unfortunately, although we now know that there is a 9-parameter family of such surfaces (i.e., the solutions depend on $s_0 = 9$ constants), we don't know what the surfaces are in any explicit way.

Exercise 7.9: Show that cylinders, circular cones and tori of revolution where the profile curve is a standard circle are examples of such surfaces. How do you know that this is not all of them? Does every such surface have at least a 1-parameter family of symmetries?

Exercise 7.10: Note that the forms $\theta_0, \ldots, \theta_8$ are well defined on the locus $\kappa_1 - \kappa_2 = 0$. Show that any leaf of $\mathcal{I}^{(3)}$ that intersects this locus stays entirely in this locus. What do these integral surfaces mean? (After all, an umbilic surface does not have well-defined principal curvatures.)

Exercise 7.11: What would have happened if, instead, we had looked for surfaces for which each principal curvature was constant on each principal curve belonging to the *other* principal curvature? Write down the appropriate exterior differential system and analyse it.

7.3. The Cartan-Kuranishi Theorem

Throughout this section, I am going to assume that all the ideals in question are generated in positive degrees, i.e., that they contain no nonzero functions. This is just to simplify the statements of the results. I'll let you worry about what to do when you have functions in the ideal.

Let $(M, \mathcal{I})$ be an EDS and let $Z \subset V_n^o(\mathcal{I})$ be a connected open subset of $V_n^o(\mathcal{I})$. We say that Z is *involutive* if every $E \in Z$ is the terminus of a regular flag. Usually, in applications, there is only one such Z to worry about anyway, or else the 'interesting' component Z is clear from context, in which case we simply say that $(M, \mathcal{I})$ is involutive.

The first piece of good news about the prolongation process is that it doesn't destroy involutivity:

Theorem 8: (Persistence of Involutivity) Let $(M, \mathcal{I})$ be an EDS with $\mathcal{I}^0 = (0)$ and let $M^{(1)} \subset V_n^o(\mathcal{I})$ be a connected open subset of $V_n^o(\mathcal{I})$ that is involutive. Then the character sequence $\big(s_0(F), \ldots, s_n(F)\big)$ is the same for all regular flags $F = (E_0, \ldots, E_n)$ with $E_n \in M^{(1)}$. Moreover, the EDS $\big(M^{(1)}, \mathcal{I}^{(1)}\big)$ is involutive on the set $M^{(2)} \subset V_n\big(\mathcal{I}^{(1)}\big)$ of elements that are transverse to the projection $\pi : M^{(1)} \to M$ and its character sequence $\big(s_0^{(1)}, \ldots, s_n^{(1)}\big)$ is given by

$$s_k^{(1)} = s_k + s_{k+1} + \cdots + s_n\,.$$

Exercise 7.12: Define $\big(M^{(k)}, \mathcal{I}^{(k)}\big)$ by the obvious induction, starting with $\big(M^{(0)}, \mathcal{I}^{(0)}\big) = (M, \mathcal{I})$ and show that

$$\dim M^{(k)} = n + s_0 + \binom{k{+}1}{1} s_1 + \binom{k{+}2}{2} s_2 + \cdots + \binom{k{+}n}{n} s_n\,.$$

Explain why $M^{(k)}$ can be interpreted as the space of k-jets of integral manifolds of $\mathcal{I}$ whose tangent planes lie in $M^{(1)}$.

Now Theorem 8 is quite useful, as we will see in the next lecture, but what we'd really like to know is whether prolongation will help with components $Z \subset V_n(\mathcal{I})$ that are not involutive. The answer is a sort of qualified 'yes':

Theorem 9: (Cartan-Kuranishi) Suppose that one has a sequence of manifolds M_k for $k \geq 0$ together with embeddings $\iota_k : M_k \hookrightarrow G_n(TM_{k-1})$ for $k > 0$ with the properties

(1) The composition $\pi_{k-1} \circ \iota_k : M_k \to M_{k-1}$ is a submersion,
(2) For all $k \geq 2$, $\iota_k(M_k)$ is a submanifold of $V_n(\mathcal{C}_{k-2}, \pi_{k-2})$, the integral elements of the contact system $\mathcal{C}_{k-2}$ on $G_n(TM_{k-2})$ transverse to the fibers of $\pi_{k-2} : G_n(TM_{k-2}) \to M_{k-2}$.

Then there exists a $k_0 \geq 0$ so that for $k \geq k_0$, the submanifold $\iota_{k+1}(M_{k+1})$ is an involutive open subset of $V_n\big(\iota_k^* \mathcal{C}_{k-1}\big)$, where $\iota_k^* \mathcal{C}_{k-1}$ is the EDS on M_k pulled back from $G_n(TM_{k-1})$.

A sequence of manifolds and immersions as described in the theorem is sometimes known as a *prolongation sequence*.

Now, you can imagine how this theorem might be useful. When you start with an EDS $(M, \mathcal{I})$ and some submanifold $\iota : Z \hookrightarrow V_n(\mathcal{I})$ that is not involutive, you can start building a prolongation sequence by setting $M_1 = Z$ and looking for a submanifold $M_2 \subset V_n(\iota^* \mathcal{C}_0)$ that is some component of $V_n(\iota^* \mathcal{C}_0)$. You keep repeating this process until either you get to a stage M_k where $V_n(\iota^* \mathcal{C}_{k-1})$ is empty, in which case there aren't any integral manifolds of this kind, or else, eventually, this will have to result in an involutive system, in which case you can apply the Cartan-Kähler Theorem (if the system that you started with is real analytic).

The main difficulty that you'll run into is that the spaces $V_n(\mathcal{I})$ can be quite wild and hard to describe. I don't want to dismiss this as a trivial problem, but it really is an algebra problem, in a sense. The other difficulty is that the the components $M_1 \subset V_n(\mathcal{I})$ might not submerse onto $M_0 = M$, but onto some proper submanifold, in which case, you'll have to restrict to that submanifold and start over.

In the case that the original EDS $(M, \mathcal{I})$ is real analytic, the set $V_n(\mathcal{I}) \subset G_n(TM)$ will also be real analytic and so has a canonical stratification into submanifolds

$$V_n(\mathcal{I}) = \bigcup_{\beta \in B} Z_\beta \,.$$

One can then consider the family of prolongations $(Z_\beta, \mathcal{I}_\beta^{(1)})$ and analyse each one separately. (Fortunately, in all the interesting cases I'm aware of, the number of strata is mercifully small.)

Now, there are precise, though somewhat technical, hypotheses that will ensure that this *prolongation Ansatz*, when iterated and followed down all of its various branches, terminates after a finite number of steps, with the result being a finite (possibly empty) set of EDSs $\big\{ (M_\gamma, \mathcal{I}_\gamma) \bigm| \gamma \in \Gamma \big\}$ that are involutive. This result (with the explicit technical hypotheses) is due to Kuranishi and is known as the Cartan-Kuranishi Prolongation Theorem. (Cartan had conjectured/stated this result in his earlier writings, but never provided adequate justification for his claims.) In practice, though, Kuranishi's result is used more as a justification for carrying out the process of prolongation as part of the analysis of an EDS, when it is necessary.

Exercise 7.13: Analyse the system

$$\frac{\partial^n z}{\partial x^n} = f(x,y), \qquad \frac{\partial^n z}{\partial y^n} = g(x,y),$$

and explain why you'll have to prolong it $(n-1)$ times before you reach either a system with no 2-dimensional integral elements or one that has 2-dimensional integral elements that can be reached by a regular flag. In the latter case, do you actually need the full Cartan-Kähler Theorem to analyse the solutions?

Exercise 7.14: Analyse the system for $u(x,y,z)$ given by

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = u, \qquad \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = u.$$

Show that the natural system generated by four 1-forms you would write down on $\mathbb{R}^{11}$ to model the solutions is not involutive but that its first prolongation is. How much data do you get to specify in a solution?