Lecture 9. Applications 3: Geometric Systems Needing Prolongation

9.1. Orthogonal coordinates in dimension n.

In this example, I take up the question of orthogonal coordinates in general dimensions, as opposed to dimension 3, as was discussed in Lecture 5.

Let N be a manifold of dimension n endowed with a Riemannian metric g. If $U \subset N$ is an open set, a coordinate chart $(x^1, \dots, x^n) : U \to \mathbb{R}^n$ is said to be *orthogonal* if, on U,

$$g = g_{11}\,(dx^1)^2 + g_{22}\,(dx^2)^2 + \cdots + g_{nn}\,(dx^n)^2,$$

i.e., if the coordinate expression $g = g_{ij}\,dx^i\,dx^j$ satisfies $g_{ij} = 0$ for i different from j. This is $\binom{n}{2}$ equations for the n coordinate functions x^i. When $n > 3$, this is an overdetermined system and one should not expect there to be solutions. Indeed, very simple examples in dimension 4 show that there are metrics for which there are no orthogonal coordinates, even locally. (I'll say more about this below.)

I want to describe an EDS whose n-dimensional integral manifolds describe the solutions to this problem. Note that, if you have a solution, then the 1-forms $\eta_i = \sqrt{g_{ii}}\,dx^i$ form an orthonormal coframing, i.e.,

$$g = {\eta_1}^2 + {\eta_2}^2 + \cdots + {\eta_n}^2.$$

This coframing is not the most general orthonormal coframing, though, because it satisfies $\eta_i{\wedge}d\eta_i = 0$ since each η_i is a multiple of an exact 1-form. Conversely, any g-orthonormal coframing $(\eta_1, \dots, \eta_n)$ that satisfies $\eta_i{\wedge}d\eta_i = 0$ for $i = 1, \dots, n$ is locally of the form $\eta_i = A_i\,dx_i$ for some functions $A_i > 0$ and x^i, by the Frobenius Theorem. (Why?)

Thus, up to an application of the Frobenius Theorem, the problem of finding g-orthogonal coordinates is equivalent to finding g-orthonormal coframings $(\eta_1, \dots, \eta_n)$ satisfying $\eta_i{\wedge}d\eta_i = 0$. I now want to set up an exterior differential system whose integral manifolds reprsent these coframings.

To do this, let $\pi : F \to N$ be the g-orthonormal coframe bundle of N, i.e, a point of F is of the form $f = (x, u_1, \dots, u_n)$ where $x = \pi(f)$ belongs to N and $u_i \in T_xN$ are g-orthonormal. This is an $\mathrm{O}(n)$-bundle over N and hence is a manifold of dimension $n + \binom{n}{2}$. There are the canonical 1-forms $\omega_1, \dots, \omega_n$ on F that satisfy

$$\omega_i(v) = u_i\bigl(\pi'(v)\bigr), \qquad \text{for all } v \in T_fM \text{ with } f = (x, u_1, \dots, u_n).$$

These 1-forms have the 'reproducing property' that, if $\eta = (\eta_1, \dots, \eta_n)$ is a g-orthonormal coframing on $U \subset N$, then regarding η as a section of F over U via the map

$$\sigma_\eta(x) = \bigl(\,x,\,(\eta_1)_x,\,\dots\,(\eta_n)_x\,\bigr),$$

we have $\sigma_\eta^*(\omega_1, \dots, \omega_n) = (\eta_1, \dots, \eta_n)$.

Exercise 9.1: Prove this statement. Prove also that a n-dimensional submanifold $P \subset F$ can be locally represented as the graph of a local section $\sigma : U \to F$ if and only if $\omega_1{\wedge}\cdots{\wedge}\omega_n$ is nonvanishing on P.

Consider the ideal $\mathcal{I} = \langle\,\omega_1{\wedge}d\omega_1,\,\dots,\,\omega_n{\wedge}d\omega_n\,\rangle$ defined on F. The n-dimensional integral manifolds of $\mathcal{I}$ on which $\omega_1{\wedge}\cdots{\wedge}\omega_n$ is nonvanishing are then the desired local sections. We now want to describe these integral manifolds, so we start by looking at the integral elements.

Now, by the classical Levi-Civita existence and uniqueness theorem, there are unique 1-forms $\omega_{ij} = -\omega_{ji}$ that satisfy the structure equations

$$d\omega_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j\,.$$

The 1-forms ω_i, ω_{ij} $(i < j)$ are then a basis for the 1-forms on F.

By the structure equations, an alternative description of $\mathcal{I}$ is

$$\mathcal{I} = \Bigl\langle\,\omega_1 \wedge \bigl(\sum_{j=1}^{n} \omega_{1j} \wedge \omega_j\bigr), \dots, \omega_n \wedge \bigl(\sum_{j=1}^{n} \omega_{nj} \wedge \omega_j\bigr)\,\Bigr\rangle\,.$$

Let $G_n(TF,\omega)$ denote the set of tangent n-planes on which $\omega_1\wedge\cdots\wedge\omega_n$ is nonvanishing. Any $E\in G_n(TF,\omega)$ is defined by equations of the form

$$\omega_{ij}=\sum_{k=1}^{n} p_{ijk}\,\omega_k$$

Such an n-plane will be an integral element if and only if the $p_{ijk}=-p_{jik}$ (which are $n\binom{n}{2}$ in number) satisfy the equations

$$0=\omega_i\wedge\big(\sum_{j=1}^{n}\omega_{ij}\wedge\omega_j\big)=\omega_i\wedge\big(\sum_{j,k=1}^{n}p_{ijk}\,\omega_k\wedge\omega_j\big)\qquad\text{for } i=1,\dots,n.$$

Exercise 9.2: Show that these conditions imply that $p_{ijk}=0$ unless k is equal to i or j and then that every integral element is defined by equations of the form

$$\omega_{ij}=p_{ij}\,\omega_i-p_{ji}\,\omega_j$$

where the $n(n-1)$ numbers $\{\,p_{ij}\big|i\neq j\,\}$ are arbitrary. Explain why the p_{ii} don't matter, and conclude that the codimension of the space $V_n(\mathcal{I},\omega)$ in $G_n(TF,\omega)$ is

$$n\binom{n}{2}-n(n-1)=\tfrac{1}{2}n(n-1)(n-2).$$

Now, to check Cartan's Test, we need to compute the polar spaces of some flag in $E=E_n$. We already know from Lecture 5 that there are regular flags when $n=3$, so we might as well assume that $n>3$ from now on. I am going to argue that, in this case, there cannot be a regular flag, so Cartan-Kähler cannot be applied and we must prolong.

Let $F=(E_0,E_1,\dots,E_n)$ be any flag. Because $\mathcal{I}$ is generated by n 3-forms, it follows that $c(E_0)=c(E_1)=0$ and that $c(E_2)\leq n$. Moreover, because E_n has codimension $\frac{1}{2}n(n-1)$, it follows that $c(E_k)\leq\frac{1}{2}n(n-1)$ for all k. Combining these, we see that

$$c(F)\leq c(E_0)+\cdots+c(E_{n-1})\leq 0+0+n+(n-3)\cdot\tfrac{1}{2}n(n-1).$$

When $n>3$, this last number is strictly *less* than $\frac{1}{2}n(n-1)(n-2)$, the codimension of $V_n(\mathcal{I},\omega)$ in $G_n(TF,\omega)$ that we computed above. Thus Cartan's Test shows that the flag F is not regular.

Thus, if we want to find solutions, we will have to prolong. We make a new manifold $F^{(1)}=F\times\mathbb{R}^{n(n-1)}$, with $\{\,p_{ij}\big|i\neq j\,\}$ as coordinates on the second factor, and define $\mathcal{I}^{(1)}$ to be the ideal generated by the $\binom{n}{2}$ 1-forms

$$\theta_{ij}=\omega_{ij}-p_{ij}\,\omega_i+p_{ji}\,\omega_j\,.$$

Of course, if we are going to study the algebraic properties of this ideal, we are going to have to know $d\theta_{ij}$ and this will require that we know $d\omega_{ij}$. Now, the second structure equations of Cartan are

$$d\omega_{ij}=-\sum_{k=1}^{n}\omega_{ik}\wedge\omega_{kj}+\tfrac{1}{2}\sum_{k,l=1}^{n}R_{ijkl}\,\omega_k\wedge\omega_l\,,$$

where the functions R_{ijkl} are the Riemann curvature functions.

Now, using this, if you compute, you will get

$$d\theta_{ij}\equiv\tfrac{1}{2}\sum_{k,l=1}^{n}R_{ijkl}\,\omega_k\wedge\omega_l-\pi_{ij}\wedge\omega_i+\pi_{ji}\wedge\omega_j \mod \{\theta_{kl}\}_{k<l}$$

for some 1-forms π_{ij} $(i\neq j)$, with $\pi_{ij}\equiv dp_{ij}\mod\{\omega_1,\dots,\omega_n\}$.

Right away, this says that there is trouble: If there is a point $f\in F$ for which there exist (i,j,k,l) distinct and $R_{ijkl}(f)\neq0$, then the prolonged ideal will not have any integral elements passing through f on which $\omega_1\wedge\ldots\wedge\omega_n$ is nonzero. (Why not?)

Now, it turns out that the functions R_{ijkl} with (i,j,k,l) distinct are all identically zero if and only if the Weyl curvature of the metric g vanishes, i.e., (since $n \geq 4$) if and only if g is conformally flat. Since orthogonal coordinates don't care about conformal factors (why not?), if we are going to restrict to the conformally flat case, then we might as well go whole hog and restrict to the flat case, i.e., the case where $R_{ijkl} = 0$ for all quadruples of indices. In this case, the structure equations of $\mathcal{I}^{(1)}$ become

$$d\theta_{ij} \equiv -\pi_{ij} \wedge \omega_i + \pi_{ji} \wedge \omega_j \mod \{\theta_{kl}\}_{k<l}$$

for some 1-forms π_{ij} $(i \neq j)$, with $\pi_{ij} \equiv dp_{ij} \mod \{\omega_1, \ldots, \omega_n\}$.

Exercise 9.3: Use these structure equations to show that $\bigl(F^{(1)}, \mathcal{I}^{(1)}\bigr)$ is involutive, with Cartan characters

$$(s_0, s_1, \cdots, s_n) = (\tfrac{1}{2}n(n-1), \tfrac{1}{2}n(n-1), \tfrac{1}{2}n(n-1), 0, 0, \ldots, 0).$$

In particular, the last nonzero Cartan character is $s_2 = \frac{1}{2}n(n-1)$. Explain the geometric meaning of this result: How much freedom do you get in constructing local orthogonal coordinates on $\mathbb{R}^n$?

Exercise 9.4: (*somewhat nontrivial*) Using the above analysis as starting point, show that the Fubini-Study metric g on $\mathbb{CP}^2$ does not allow any orthogonal coordinate systems, even locally.

9.2. Isometric Embedding of Surfaces with Prescribed Mean Curvature

Consider a given abstract oriented surface N^2 endowed with a Riemannian metric g and a choice of a smooth function H. The question we ask is this: When does there exist an isometric embedding $x : N^2 \to \mathbb{R}^3$ such that the mean curvature function of the immersion is H? If you think about it, this is four equations for the map x (which has three components), three of first order (the isometric embedding condition) and one of second order (the mean curvature restriction).

Since $H^2 - K = (\kappa_1 - \kappa_2)^2 \geq 0$ for any surface in 3-space, one obvious restriction coming from the Gauss equation is that $H^2 - K$ must be nonnegative, where K is the Gauss curvature of the metric g. I'm just going to treat the case where $H^2 - K$ is strictly positive, though there are methods for dealing with the 'umbilic locus' (I just don't want to bother with them here). In fact, set $r = \sqrt{H^2 - K} > 0$.

The simplest way to set up the problem is to begin by fixing an oriented, g-orthonormal coframing (η_1, η_2), with dual frame field (u_1, u_2). We know that there exists a unique 1-form η_{12} so that

$$d\eta_1 = -\eta_{12} \wedge \eta_2, \qquad d\eta_2 = \eta_{12} \wedge \eta_1, \qquad d\eta_{12} = K\, \eta_1 \wedge \eta_2\,.$$

Now, any solution $x : N \to \mathbb{R}^3$ of our problem will define a lifting $f : N \to F$ (the oriented orthonormal frame bundle of $\mathbb{R}^3$) via

$$f = \bigl[\; x \quad x'(u_1) \quad x'(u_2) \quad x'(u_1) \times x'(u_2) \;\bigr]$$

Of course, this will mean that

$$\begin{aligned} f^*\omega_3 &= 0 \\ f^*\omega_1 &= \eta_1 \\ f^*\omega_1 &= \eta_2 \\ f^*\omega_{31} &= h_{11}\, \eta_1 + h_{12}\, \eta_2 \\ f^*\omega_{32} &= h_{12}\, \eta_1 + h_{22}\, \eta_2 \end{aligned}$$

where $h_{11} + h_{22} = 2H$. We also know, by the uniqueness of the Levi-Civita connection, that

$$f^*\omega_{12} = \eta_{12}$$

and the Gauss equation tells us that $h_{11}h_{22} - {h_{12}}^2 = K$. This is two algebraic equations for the three h_{ij}. Because $H^2 - K = r^2 > 0$, these can be solved in terms of an extra parameter in the form

$$\begin{aligned} h_{11} &= H + r\, \cos\phi \\ h_{12} &= \phantom{H + {}} r\, \sin\phi \\ h_{22} &= H - r\, \cos\phi\,. \end{aligned}$$

This suggests setting up the following exterior differential system for the 'graph' of f in $N \times F$. Let $M = N \times F \times S^1$, with ϕ being the 'coordinate' on the S^1 factor and consider the ideal $\mathcal{I}$ generated by the five 1-forms

$$\begin{aligned}
\theta_0 &= \omega_3 \\
\theta_1 &= \omega_1 - \eta_1 \\
\theta_2 &= \omega_2 - \eta_2 \\
\theta_3 &= \omega_{12} - \eta_{12} \\
\theta_4 &= \omega_{31} - (H{+}r\cos\phi)\,\eta_1 - r\,\sin\phi\,\eta_2 \\
\theta_5 &= \omega_{32} - r\,\sin\phi\,\eta_1 - (H{-}r\cos\phi)\,\eta_2
\end{aligned}$$

It's easy to see (and you should check) that

$$d\theta_0 \equiv d\theta_1 \equiv d\theta_2 \equiv d\theta_3 \equiv 0 \mod \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}.$$

The interesting case will come when we look at the other two 1-forms. In fact, the formula for these is simply

$$\left.\begin{aligned}
d\theta_4 &\equiv r\tau \wedge (\sin\phi\,\eta_1 - \cos\phi\,\eta_2) \\
d\theta_5 &\equiv -r\tau \wedge (\cos\phi\,\eta_1 + \sin\phi\,\eta_2)
\end{aligned}\right\} \mod \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$$

where, setting $dr = r_1\,\eta_1 + r_2\,\eta_2$ and $dH = H_1\,\eta_1 + H_2\,\eta_2$,

$$\begin{aligned}
\tau = d\phi - 2\,\eta_{12} - r^{-1}&(r_2 + H_2\cos\phi - H_1\sin\phi)\,\eta_1 \\
&+ r^{-1}(r_1 - H_1\cos\phi - H_2\sin\phi)\,\eta_2\,.
\end{aligned}$$

It is clear that there is a unique integral element at each point of M and that it is described by $\theta_0 = \cdots = \theta_5 = \tau = 0$. Thus, $M^{(1)} = M$ and

$$\mathcal{I}^{(1)} = \langle\, \theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \tau\,\rangle.$$

To get the structure of $\mathcal{I}^{(1)}$ is is only necessary to compute $d\tau$ now and the result of that is

$$d\tau \equiv r^{-2}\big(C\,\cos\phi + S\,\sin\phi + T\big)\,\eta_1 \wedge \eta_2 \mod \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \tau\}$$

where the functions C, S, and T are defined on the surface by

$$\begin{aligned}
C &= 2r_1H_1 - 2r_2H_2 - rH_{11} + rH_{22}\,, \\
S &= 2r_2H_1 + 2r_1H_2 - 2rH_{12}\,, \\
T &= 2r^4 - 2H^2r^2 + r(r_{11}{+}r_{22}) - {r_1}^2 - {r_2}^2 - {H_1}^2 - {H_2}^2\,.
\end{aligned}$$

and I have defined H_{ij} and r_{ij} by the equations

$$\begin{aligned}
dH_1 &= -H_2\,\eta_{12} + H_{11}\,\eta_1 + H_{12}\,\eta_2\,, \\
dH_2 &= H_1\,\eta_{12} + H_{12}\,\eta_1 + H_{22}\,\eta_2\,, \\
dr_1 &= -r_2\,\eta_{12} + r_{11}\,\eta_1 + r_{12}\,\eta_2\,, \\
dr_2 &= r_1\,\eta_{12} + r_{12}\,\eta_1 + r_{22}\,\eta_2\,.
\end{aligned}$$

Exercise 9.5: Why do such functions H_{ij} and r_{ij} exist? (What you need to explain is why H_{12} and r_{12} can appear in two places in these formulae.)

Clearly, there are no integral elements of $\mathcal{I}^{(1)}$ except along the locus where $C\,\cos\phi + S\,\sin\phi + T = 0$, so it's a question of what this locus looks like.

First, off, note that if $T^2 > S^2 + C^2$, then this locus is empty. Now, this inequality is easily seen not to depend on the choice of coframing (η_1, η_2) that we made to begin with. It depends only on the metric g and the function H. One way to think of this is that the condition $T^2 \leq S^2 + C^2$ is a differential inequality any g and H satisfy if they are the metric and mean curvature of a surface in $\mathbb{R}^3$.

Now, when $T^2 < C^2 + S^2$, there will be exactly two values of ϕ (mod 2π) that satisfy $C\,\cos\phi + S\,\sin\phi + T = 0$, say ϕ_+ and ϕ_-, thought of as functions on the surface N. If you restrict to this double cover $\phi = \phi_\pm$, we now have an ideal $\mathcal{I}^{(1)}$ on an 8-manifold that is generated by seven 1-forms. In fact, $\theta_0, \ldots, \theta_5$ are clearly independent, but now

$$\tau = E_1\,\eta_1 + E_2\,\eta_2$$

where E_1 and E_2 are functions on the surface $\tilde{N} \subset N \times S^1$ defined by the equation $C\,\cos\phi + S\,\sin\phi + T = 0$. Wherever either of these functions is nonzero, there is clearly no solution. On the other hand, if $E_1 = E_2 = 0$ on $\tilde{N}$, then there are exactly two geometrically distinct ways for the surface to be isometrically embeded with mean curvature H. If you unravel this, you will see that it is a pair of fifth order equations on the pair (g, H). (The expressions T and $S^2 + C^2$ are fourth order in g and second order in H. Why?)

Exercise 9.6: (*somewhat nontrivial*) See if you can reproduce Cartan's result that the set of surfaces that admit two geometrically distinct isometric embeddings with the same mean curvature depend on four functions of one variable. (In the literature, such pairs of surfaces are known as *Bonnet pairs* after O. Bonnet, who first studied them.)

Another possibility is that $T = C = S = 0$, in which case $\mathcal{I}^{(1)}$ becomes Frobenius.

Exercise 9.7: Explain why $T = C = S = 0$ implies that the surface admits a one-parameter family (in fact, a circle) of geometrically distinct isometric embeddings with mean curvature H.

Of course, this raises the question of whether there exist any pairs (g, H) satisfying these equations. One way to try to satisfy the equations is to look for special solutions. For example, if H were constant, then H_1, H_2, H_{11}, H_{12}, and H_{22} would all be zero, of course, so this would automatically make $C = S = 0$ and then there is only one more equation to satisfy, which can now be reëxpressed, using $K = H^2 - r^2$, as

$$T = r^2\big(\Delta_g \ln(H^2 - K) - 4K\big) = 0$$

where Δ_g is the Laplacian associated to g.

It follows that any metric g on a simply connected surface N that satisfies the fourth order differential equation $\Delta_g \ln(H^2 - K) - 4K = 0$ can be isommetrically embedded in $\mathbb{R}^3$ as a surface of constant mean curvature H in a 1-parameter family (in fact, an S^1) of ways. In particular, we have Bonnet's Theorem: *Any simply connected surface in $\mathbb{R}^3$ with constant mean curvature can be isometrically deformed in an circle of ways preserving the constant mean curvature.*

However, the cases where H is constant give only one special class of solutions of the three equations $C = S = T = 0$. Could there be others?

Well, let's restrict to the open set $U \subset N$ where $dH \neq 0$, i.e., where ${H_1}^2 + {H_2}^2 > 0$. Remember, the original coframing (η_1, η_2) we chose was arbitrary, so we might as well use the nonconstancy of H to tack this down. In fact, let's take our coframing so that the dual frame field (u_1, u_2) has the property that u_1 points in the direction of steepest increase for H, i.e., in the direction of the gradient of H. This means that, for this coframing $H_2 = 0$ and $H_1 > 0$.

The equations $C = S = 0$ now simplify to

$$H_{12} = (r_2/r)\,H_1, \qquad H_{11} - H_{22} = (2r_1/r)H_1\,.$$

Moreover, looking back at the structure equations found so far, this implies that $dH = H_1\,\eta_1$ and that there is a function P so that

$$\begin{aligned} {H_1}^{-1} dH_1 &= (rP + r_1/r)\,\eta_1 + (r_2/r)\,\eta_2, \, , \\ -\eta_{12} &= (r_2/r)\,\eta_1 + (rP - r_1/r)\,\eta_2\,. \end{aligned}$$

The first equation can be written in the form

$$d\big(\ln(H_1/r)\big) = rP\,\eta_1.$$

Differentiating this and using the structure equations we have so far then yields that $dP{\wedge}\eta_1 = 0$, so that there is some λ so that $dP = \lambda\,\eta_1$. On the other hand, differentiating the second of the two equations above and using $T = 0$ to simplify the result, we see that the multiplier λ is determined. In fact, we must have

$$dP = (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)\,\eta_1\,.$$

Differentiating this relation and using the equations we have found so far yields

$$0 = 2r^{-4}\big({H_1}^2 + r^2H^2\big)\,r_2\,\eta_1\wedge\eta_2\,.$$

In particular, we must have $r_2 = 0$. Of course, this simplifies the equations even further. Taking the components of $0 = dr_2 = r_1\eta_{12} + r_{11}\,\eta_1 + r_{22}\,\eta_2$ together with the equation $T = 0$ allows us to solve for r_{11}, r_{12}, and r_{22} in terms of $\{r, H, r_1, H_1, P\}$.

In fact, collecting all of this information, we get the following structure equations for any solution of our problem:

$$\begin{aligned}
d\eta_1 &= 0\\
d\eta_2 &= (rP - r_1/r)\,\eta_1\wedge\eta_2\\
dr &= r_1\,\eta_1\\
dH &= H_1\,\eta_1\\
dr_1 &= (2r^3 - 2H^2r + r_1rP - 2{r_1}^2/r - {H_1}^2/r)\,\eta_1\\
dH_1 &= H_1(rP + r_1/r)\,\eta_1\\
dP &= (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)\,\eta_1
\end{aligned}$$

These may not look promising, but, in fact, they give a rather complete description of the pairs (g, H) that we are seeking. Suppose that N is simply connected. The first structure equation then says that $\eta_1 = dx$ for some function x, uniquely defined up to an additive constant. The last 5 structure equations then say that the functions (r, H, r_1, H_1, P) are solutions of the ordinary differential equation system

$$\begin{aligned}
r' &= r_1\\
H' &= H_1\\
r_1' &= (2r^3 - 2H^2r + r_1rP - 2{r_1}^2/r - {H_1}^2/r)\\
H_1' &= H_1(rP + r_1/r)\\
P' &= (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)
\end{aligned}$$

Obviously, this defines a vector field on the open set in $\mathbb{R}^5$ defined by $r > 0$, and there is a four parameter family of integral curves of this vector field. Given a solution of this ODE system on some maximal x-interval, there will be a function F uniquely defined up to an additive constant so that

$$F' = (rP - r_1/r).$$

Now by the second structure equation, we have $d(e^{-F}\eta_2) = 0$, so that there must exist a function y on the surface N so that $\eta_2 = e^F\,dy$. Thus, in the (x, y)-coordinates, the metric is of the form

$$g = dx^2 + e^{2F(x)}\,dy^2$$

where (r, H, r_1, H_1, P, F) satisfy the above equations.

Exercise 9.8: Explain why this shows that the space of inequivalent solutions (g, H) with H nonconstant can be thought of as being of dimension 4. Also, note that the metric g has a symmetry, namely translation in y. Can you use this to understand the circle of isometric embeddings of (N, g) into $\mathbb{R}^3$ with mean curvature H? (Hint: Look back at the EDS analysis we did earlier and apply Bonnet's Theorem.

Exercise 9.9: Redo this analysis for isometric immersion with prescribed curvature in a 3-dimensional space form of constant sectional curvature c. Does anything significant change?

Exercise 9.10: (*somewhat nontrivial*) Regarding the equations $S = C = T = 0$ as a set of three partial differential equations for the pair (g, H), show that they are not involutive as they stand, carry out the prolongation process and show how the space of integral manifolds breaks into two distinct pieces because the space of integral elements has two distinct components at a certain level of prolongation. Show that one of these (the one corresponding to the case where H is constant) goes into involution right away, but that the other (corresponding to the Bonnet surfaces that we found above) takes considerably longer.

Exercise 9.11: (*also somewhat nontrivial*) Suppose that we want to isometrically embedd (N^2, g) into $\mathbb{R}^3$ in such a way that a given g-orthogonal coframing (η_1, η_2) defines the principal coframing. Set up the exterior differential system and carry out the prolongations to determine how many solutions to this problem there are in general and whether there are any special metrics and coframings for which there is a larger than expected space of solutions.