SERIES OF LIE GROUPS

J.M. LANDSBERG AND L. MANIVEL

Abstract. For various series of complex semi-simple Lie algebras $\mathfrak{g}(t)$ equipped with irreducible representations $V(t)$, we decompose the tensor powers of $V(t)$ into irreducible factors in a uniform manner, using a tool we call *diagram induction*. In particular, we interpret the decompostion formulas of Deligne [4] and Vogel [23] for decomposing $\mathfrak{g}^{\otimes k}$ respectively for the exceptional series and $k \leq 4$ and all simple Lie algebras and $k \leq 3$, as well as new formulas for the other rows of Freudenthal's magic chart. By working with Lie algebras augmented by the symmetry group of a marked Dynkin diagram, we are able to extend the list [2] of modules for which the algebra of invariant regular functions under a maximal nilpotent subalgebra is a polynomial algebra. Diagram induction applied to the exterior algebra furnishes new examples of distinct representations having the same Casimir eigenvalue.

1. Introduction

One way to define a collection of Lie algebras $\mathfrak{g}(t)$, parametrized by t, each equipped with a representation $V(t)$, as forming a "series" is, following Deligne, to require that the tensor powers of $V(t)$ should decompose into irreducible $\mathfrak{g}(t)$-modules in a manner independent of t, with formulas for the dimensions of the irreducible components of the form $P(t)/Q(t)$ with P, Q polynomials decomposing into products of integral linear factors. In this paper we study such decomposition formulas, which provides a companion to [17] where we study the corresponding dimension formulas. We connect the formulas to the geometry of the closed orbits $X(t) \subset \mathbb{P}V(t)$, and their unirulings by homogeneous subvarieties. We relate the linear unirulings to work of Kostant [12]. By studying such series, we determine new modules that, appropriately viewed, are *exceptional* in the sense of Brion [2], e.g., theorem 6.2.

The starting point of this paper was the work of Deligne et. al. [4, 5], containing uniform decomposition and dimension formulas for the tensor powers of the adjoint representations of the exceptional simple Lie algebras up to $\mathfrak{g}^{\otimes 5}$. Deligne's method for the decomposition formulas was based on comparing Casimir eigenvalues and he offered a conjectural explanation for the formulas via a categorical model based on bordisms between finite sets. Vogel [23] obtained similar formulas for all simple Lie super algebras based on his *universal Lie algebra*. We show that all *primitive* factors in the decomposition formulas of Deligne and Vogel can be accounted for using diagram inductions. (The non-primitive factors are those either inherited from lower degrees or arising from a bilinear form, thus knowledge of the primitive factors gives the full decomposition.) We also derive new decomposition formulas for other series of Lie algebras.

In §2, we describe a pictorial procedure using Dynkin diagrams for determining the decomposition of $V^{\otimes k}$. It was discovered by unifying several geometric observations about the closed orbit $X = G/P \subset \mathbb{P}V$. We also give an interpretation of this diagram induction in terms of vector bundles.

For example, in [14] we determined the Fano varieties $\mathbb{F}_k(X)$ of $\mathbb{P}^{k-1}$'s contained in X. Since $\mathbb{F}_k(X)$ is a subset of the Grassmannian of $(k+1)$-planes through the origin in V, $G(k+1, V) \subset \mathbb{P}\Lambda^{k+1}V$, $\mathbb{F}_k(X)$ determines a distinguished component of $\Lambda^{k+1}V$. This component has the property that it is a Casimir eigenspace and the corresponding Casimir eigenvalue is maximal among Casimir eigenvalues of $\Lambda^{k+1}V$. The components of $\Lambda^k\mathfrak{g}$ with maximal Casimir eigenvalue were considered by Kostant in [12]. In §9 we explain how Kostant's results can be extended to general fundamental representations, with special attention paid to minuscule representations.

These Casimir eigenspaces provide new examples of distinct modules with the same Casimir eigenvalue which are different from the examples in [11].

In §3 and §4, we distinguish and interpret the *primitive* components in the decomposition formulas of Deligne and Vogel.

The exceptional series of Lie algebras occurs as a line in Freudenthal's magic square (see e.g., [10], or the variant we use in [17]). The three other lines each come with preferred representations. Dimension formulas for all representations supported on the cone in the weight lattice generated by the weights of the preferred representations similar to those of the exceptional series were obtained in [17]. In §5, §6 and §7 we obtain the companion decomposition formulas. A very nice property shared by many of these prefered representations is that they are *exceptional* in the sense of [2], that is, their covariant algebras are polynomial algebras. We prove that in some cases where this is not naively true, it becomes so if we take into account the symmetry group of the associated marked Dynkin diagram.

In the course of revising the exposition of this paper for the referee, we ran across the closely related preprint [6].

1.1. **Notations and conventions.** For a given complex simple Lie algebra $\mathfrak{g}$ we fix a Cartan subalgebra and set of positive roots. The highest root of $\mathfrak{g}$ (resp. the sum of the positive roots of $\mathfrak{g}$) will be denoted $\tilde{\alpha}$ (resp. 2ρ).

Let $V = V_\lambda$ be be an irreducible representation of highest weight λ of $\mathfrak{g}$. To V we associate a marked Dynkin diagram $D(\mathfrak{g}, \lambda)$ where we identify the nodes of the diagram with the fundamental weights $\omega_1, ..., \omega_n$ and if $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$ we mark the node corresponding to ω_j with a_j. We freely interchange the terminology "marked Dynkin diagram" and "irreducible representation".

Let $D(\mathfrak{f}) \subset D(\mathfrak{g})$ be a subdiagram. We define the *border set* of $D(\mathfrak{f})$ in $D(\mathfrak{g})$ to be the nodes of $D(\mathfrak{g})\backslash D(\mathfrak{f})$ adjacent to the nodes of $D(\mathfrak{f})$. If $D(\mathfrak{g}, \lambda)$ is a marked diagram where all the nonzero markings lie on $D(\mathfrak{f})$, we say λ has *support* on $D(\mathfrak{f})$ and let W_λ denote the corresponding $\mathfrak{f}$-module.

If P denotes a partition of size k, we let $S_P(V)$ denote the corresponding Schur power, which can be considered as a submodule of $V^{\otimes k}$.

The Cartan product of two irreducible modules V_μ and V_ν is denoted $V_\mu V_\nu := V_{\mu+\nu} \subset V_\mu \otimes V_\nu$.

For a given irreducible $\mathfrak{g}$-module V_λ, we let $\theta_{V_\lambda} = (\lambda, \lambda + 2\rho)$ denote the Casimir eigenvalue for V with the normalization $(\tilde{\alpha}, \tilde{\alpha}) = 2$.

We use the ordering of the roots as in [1].

2. Diagram induction

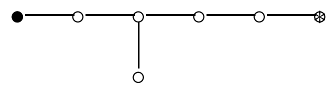
Theorem 2.1. *Let $\mathfrak{g}$ be a complex simple Lie algebra, with Dynkin diagram $D(\mathfrak{g})$ and let $D(\mathfrak{f}) \subset D(\mathfrak{g})$ be a subdiagram, and let λ be a weight of $\mathfrak{f}$ which we also consider as a weight of $\mathfrak{g}$ as explained above. We let $V = V_\lambda$ (resp. $W = W_\lambda$) denote the corresponding $\mathfrak{g}$ (resp. $\mathfrak{f}$-module). Let P be a partition of size k and let $\omega_1, ..., \omega_n$ denote the fundamental weights of $\mathfrak{g}$.*

1. If $\mathbb{C} \subset S_P(W)$ (i.e., the Schur power $S_P(W)$ contains a trivial representation) then the corresponding Schur power $S_P(V)$ contains an irreducible representation whose weight has support exactly the border set B, i.e., the support is contained in B and every weight of B appears with a nonzero multiplicity.

2. More generally if $W_\eta \subset S_P(W)$ is an irreducible submodule, write $\eta = k\lambda - \psi$, where ψ is a sum of simple roots of the root system of $\mathfrak{f}$. Consider ψ as a sum of simple roots of the root system of $\mathfrak{g}$, re-express ψ as a sum of fundamental weights of $\mathfrak{g}$, $\psi = a_1\omega_1 + \cdots + a_n\omega_n$ and let $\tilde{\eta} = k\lambda - (a_1\omega_1 + \cdots + a_n\omega_n)$ denote the corresponding weight of $\mathfrak{g}$. Then $\tilde{\eta}$ is a sum of weights from the border set of $\mathfrak{f}$ and the weight η considered as weights of $\mathfrak{g}$ and $V_{\tilde{\eta}}$ occurs as a submodule of $S_P(V)$.

3. If $\lambda_1, ..., \lambda_s$ are weights of $\mathfrak{f}$ and $W_{\lambda_1} \otimes \cdots \otimes W_{\lambda_s}$ contains a trival representation, then $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s}$ contains an irreducible representation whose weight has support exactly B and the analogue of 2. holds for irreducible submodules $W_\eta \subset W_{\lambda_1} \otimes \cdots \otimes W_{\lambda_s}$.

Example. Let $(\mathfrak{g}, V_\lambda) = (\mathfrak{e}_7, V_{\omega_7})$, let $(\mathfrak{f}, W_\lambda) = (\mathfrak{d}_6, W_{\omega_1})$. Then since S^2W contains the trivial representation, we have $V_{\omega_7} \subset S^2V_{\omega_1}$.



Example. Consider $V = V_{\omega_k} = \Lambda^k\mathbb{C}^n$ as a $\mathfrak{g} = \mathfrak{sl}_n$-module. There is a natural quadratic form on the $\mathfrak{h} = \mathfrak{sl}_{4p}$-module $\Lambda^{2p}\mathbb{C}^{4p}$. Thus for all $p < n$ the trivial representation in $S^2\Lambda^{2p}\mathbb{C}^{4p}$ induces a subspace of S^2V, namely $V_{\omega_{k-2p}+\omega_{k+2p}}$ and these give us the full decomposition

$$S^2(\Lambda^k\mathbb{C}^n) = S^2V_{\omega_k} = V_{2\omega_k} + V_{\omega_{k-2}+\omega_{k+2}} + V_{\omega_{k-4}+\omega_{k+4}} + \cdots$$

Similarly, there is a natural symplectic form on $\Lambda^{2p+1}\mathbb{C}^{4p+2}$ and the corresponding subdiagrams recover the complete decomposition:

$$\Lambda^2(\Lambda^k\mathbb{C}^n) = \Lambda^2V_{\omega_k} = V_{\omega_{k-1}+\omega_{k+1}} + V_{\omega_{k-3}+\omega_{k+3}} + \cdots$$

Proof. We have $\mathfrak{f} = \mathfrak{s} + [\mathfrak{s}, \mathfrak{s}]$, where $\mathfrak{s}$ is the sum of the root spaces of $\mathfrak{g}$ with support in $D(\mathfrak{f})$. Every dominant weight σ of $\mathfrak{g}$ with support in $D(\mathfrak{f})$ defines a $\mathfrak{g}$-module V_σ and an $\mathfrak{f}$-module W_σ.

The main observation is that if τ is a weight such that $\sigma - \tau$ has support on $D(\mathfrak{f})$, its multiplicity must be the same inside V_σ and W_σ. This is an easy consequence of Kostant's multiplicity formula:

$$\begin{aligned} \dim V_\sigma(\tau) &= \textstyle\sum_{w\in W(\mathfrak{g})} \varepsilon(w)P(w(\sigma+\rho) - (\tau+\rho)), \\ \dim W_\sigma(\tau) &= \textstyle\sum_{w\in W(\mathfrak{f})} \varepsilon(w)P'(w(\sigma+\rho') - (\tau+\rho')), \end{aligned}$$

where P and P' are the Kostant's partition functions in $\mathfrak{g}$ and $\mathfrak{f}$ respectively, and ρ, ρ' the half sums of the corresponding positive roots. First observe that since ρ is also the sum of the fundamental weights, if $w \notin W(\mathfrak{f})$, then $w(\rho) - \rho$ will have some negative coefficient on a simple root not in $D(\mathfrak{f})$. Then $\sigma - \tau + w(\rho) - \rho$, and a fortiori $w(\sigma+\rho) - (\tau+\rho)$, is not a sum of positive roots, and the partition function vanishes. Moreover, when $w \in W(\mathfrak{f})$, we have $w(\rho') - \rho' = w(\rho) - \rho$, which proves the two multiplicities are the same.

We give details for case 3 with two factors $W_\sigma \otimes W_{\sigma'}$, the other cases are similar:

Take two fundamental weights σ, σ' with support in $D(\mathfrak{f})$, and consider the decomposition

$$W_\sigma \otimes W_{\sigma'} = \bigoplus_{\tau\in\Xi} W_\tau.$$

We can in principle obtain this decomposition by the following algorithm: we consider all the sums of a weight of W_σ with a weight of $W_{\sigma'}$ (with multiplicities); then we choose a maximal element τ in this set. It must belong to Ξ. Then we subtract the weights of W_τ (with their multiplicities), and we continue until we are left with an empty set of weights.

Now consider the decomposition of $V_\sigma \otimes V_{\sigma'}$. We apply the same procedure, only we begin with maximal weights which have support on $D(\mathfrak{f})$. For these weights, the multiplicities are the same as for the corresponding $\mathfrak{f}$-modules, thus we obtain the same set of dominant weights, except that at the end we are left with weights whose support is on $D(\mathfrak{g})\backslash D(\mathfrak{f})$ instead of the empty set. We conclude that

$$V_\sigma \otimes V_{\sigma'} \supset \bigoplus_{\tau\in\Xi} V_{\tilde{\tau}}.$$

Here $\tilde{\tau} = \sigma + \sigma' - \psi_{\mathfrak{g}}$, where $\psi_{\mathfrak{g}}$ is a non-negative linear combination of simple roots of $\mathfrak{g}$, is a weight of $\mathfrak{g}$ where the corresponding weight τ of $\mathfrak{f}$ is $\tau = \sigma + \sigma' - \psi$ where ψ is the same non-negative linear combination of simple roots of $\mathfrak{g}$ only now considered as simple roots of $\mathfrak{f}$.

Comparing the expressions rewritten in terms of fundamental weights, we see that $\tau - \tau'$ is a linear combination of fundamental weights coming from B. $\square$

2.1. **Diagram induction via vector bundles.** We explain the case of inducing a representation from a trival module $\mathbb{C} \subset W_1 \otimes W_2$, the other cases being similar, following the notation of above. (Here $W_1 = W_\sigma$ etc... in the notation above.) Say $\mathbb{C}$ induces a representation $U \subset V_1 \otimes V_2$. Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{g}$ whose semi-simple Levi factor is $\mathfrak{f}$. Pictorially, $D(\mathfrak{f})$ is the subdiagram of $D(\mathfrak{g})$ obtained by deleting the nodes corresponding to $\mathfrak{p}$, with the convention that $\mathfrak{p}$ is generated by the root vectors corresponding to the Borel and opposites of the undeleted simple roots.

Consider the rational homogeneous variety $G/P \subset \mathbb{P}U$, obtained by taking the projectivized orbit of a highest weight vector. We interpret diagram induction in terms of homogeneous bundles on G/P. First of all, $U = \Gamma(L)$, i.e., U is the space of sections of a homogeneous line bundle L over G/P and each V_j is $\Gamma(E_j)$ for some homogeneous vector bundle $E_j \to G/P$. (L is the tautological (hyperplane) line bundle on G/P.)

The homogenous vector bundles on G/P are in one to one correspondence with P or $\mathfrak{p}$-modules. Let W_j denote the irreducible $\mathfrak{p}$-module inducing E_j, i.e., $E_j = G \times_P W_j$. Write $\mathfrak{p} = \mathfrak{f} \oplus \mathfrak{z} \oplus \mathfrak{n}$, with $\mathfrak{f}$ semi-simple, $\mathfrak{n}$ nilpotent and $\mathfrak{z}$ the center of the reductive part $\mathfrak{f} \oplus \mathfrak{z}$.

$\mathfrak{n}$ acts trivially on W_j because W_j is an irreducible $\mathfrak{p}$-module and $\mathfrak{z}$ acts by some character. We have a nonzero multiplication map

$$m : \Gamma(E_1) \otimes \Gamma(E_2) \to \Gamma(E_1 \otimes E_2).$$

The occurence of $\mathbb{C}$ in $W_1 \otimes W_2$ as an $\mathfrak{f}$-module extends to a (nontrivial) $\mathfrak{p}$-submodule where $\mathfrak{f}$ and $\mathfrak{n}$ act trivially and the new character for $\mathfrak{z}$ is the sum of the characters for W_1 and W_2, thus we obtain a line subbundle $L \subset E_1 \otimes E_2$ and the desired inclusion $U = \Gamma(L) \subseteq m(\Gamma(E_1) \otimes \Gamma(E_2))$.

Example. Consider $(\mathfrak{g}, V) = (\mathfrak{sl}(W), V_{\omega_k} = \Lambda^k W)$. We have $U = V_{\omega_{k-1}+\omega_{k+1}} \subset \Lambda^2 V_{\omega_k}$ as mentioned above. Here $G/P = \mathbb{F}_{k-1,k+1}$, the variety of partial flags $\mathbb{C}^{k-1} \subset \mathbb{C}^{k+1} \subset W$, E is the bundle whose fiber over (W_{k-1}, W_{k+1}) is $det(W/W_{k+1}) \otimes W_{k+1}/W_{k-1}$. (Due to our convention, we actually have $\Gamma(E) = V^*_{\omega_k}$, not V_{ω_k}.) Then $L = \Lambda^2 E$ has fiber $det(W/W_{k+1}) \otimes det(W/W_{k-1})$ and $\Gamma(\Lambda^2 E) = V^*_{\omega_{k-1}+\omega_{k+1}}$.

We will apply diagram induction to study $\Lambda^k V$, $S^2 V$ and $S_{21} V = \mathrm{Ker}(S^2 V \otimes V \to S^3 V)$. We first review some notions of Tits:

2.2. **Tits' transforms and shadows.** For any simple Lie group G, with a fixed Borel subgroup, let S, S' be two sets of positive roots of G, and let P_S be the parabolic subgroup generated by the Borel and the root subgroups generated by $-S$. Consider the diagram

$$\begin{array}{ccc} & G & \\ {}^{\pi}\swarrow & & \searrow{}^{\pi'} \\ X = G/P_S & & X' = G/P_{S'} \end{array}$$

Let $x' \in X'$ and consider $Y := \pi(\pi'^{-1}(x')) \subset X$. Tits calls Y the *shadow* of x' in X. Then X is covered by such shadows Y. Tits shows [21] that $Y = H/R$ where $\mathcal{D}(H) = \mathcal{D}(G) \backslash (S \backslash S')$, and $R \subset H$ is the parabolic subgroup corresponding to $S' \backslash S$.

2.3. **Submodules of $\Lambda^k V$.** We deduce the existence of submodules of $\Lambda^k V_\lambda$ from marked subdiagrams of the marked Dynkin diagram $D(\mathfrak{g}, \lambda)$ isomorphic to $D(\mathfrak{a}_{k-1}, \omega_1)$ via the trivial representation $\Lambda^k \mathbb{C}^k$.

In [14] we showed that these subdiagrams describe linear unirulings of rational homogeneous varieties $X = G/P \subset \mathbb{P}V_\lambda$. If the subdiagram is of type $D(\mathfrak{a}_{k-1}, \omega_1)$, we get a family of $\mathbb{P}^{k-1}$'s on X that are linearly embedded, hence a linear uniruling of X. In the simply-laced case, all complete families of linear unirulings are obtained that way.

To recover a component of $\Lambda^k V$, let $\mathbb{F}_k(X) \subset G(k, V) \subset \mathbb{P}(\Lambda^k V)$ be the Fano variety of $\mathbb{P}^{k-1}$'s in X sitting inside the Plücker embedded Grassmannian. Our uniruling defines a homogeneous component of $\mathbb{F}_k(X)$, hence an irreducible submodule of $\Lambda^k V$ by taking the linear span.

In section §8 we determine the Casimir eigenvalues of these spaces. It turns out that the linear span $\langle \mathbb{F}_k(X) \rangle$ of the Fano variety is a Casimir eigenspace, with eigenvalue the largest possible.

For $k > 1$, $\langle \mathbb{F}_k(X) \rangle$ is usually not irreducible. When it is reducible, one obtains (pictorially!) distinct irreducible representations with the same Casimir eigenvalue. This construction of representations in the same Casimir eigenspace appears to be different from that in [11]. Some of the Casimir eigenspaces $\langle \mathbb{F}_k(X) \rangle$ were found in [24] via case by case checking. They were searching for such spaces because a homogeneous space G/H with its standard homogeneous metric is Einstein iff $T_{[e]}G/H$ is a Casimir eigenspace of H.

2.4. **Submodules of S^2V.** We deduce the existence of submodules of S^2V_λ from marked subdiagrams isomorphic to $D = D(\mathfrak{d}_k, \omega_1)$ or $D(\mathfrak{b}_k, \omega_1)$, thanks to the trivial representation given by the quadratic form.

In the case of such representations, the highest weight τ of the induced representation $V_\tau \subset S^2V$ can be computed as follows (recall we already have its support). Let W be the Weyl group of $\mathfrak{g}$, and $W(D) \subset W$ the subgroup corresponding to the subdiagram D so $W(D)$ is generated by the simple reflections corresponding to the nodes of D. Let $W_1(D) \subset W(D)$ be the stabilizer of λ, let $W^1(D)$ be the set of minimal length representatives of the cosets of $W_1(D)$ in $W(D)$. Then $W^1(D)$ has a unique element w_D^1 of maximal length, and $\tau = \lambda + w_D^1(\lambda)$.

We will use the notation $V_Q = V_\tau \subset S^2V$ to denote a representation induced from a subdiagram of quadric type. By Tits' transforms, the closed G-orbit $X_Q \subset \mathbb{P}V_Q$ is a parameter space for a uniruling of the closed orbit $X \subset \mathbb{P}V$ by quadrics, i.e., linear sections $X \cap L$ that are quadric hypersurfaces in the projective space L. We use the notation $Q = X \cap L$ to denote these quadrics. In the language of the section above, these quadrics are the shadows of points in X_Q on X.

Proposition 2.2. *Let $V = V_\lambda$ be a fundamental representation of $\mathfrak{g}$ such that there is a subdiagram of quadric type $\mathfrak{b}_l$ or $\mathfrak{d}_l$ and let $V_Q = V_\tau$ denote the induced submodule of S^2V. Then the Casimir eigenvalues are related by $\theta_{V_Q} = 2(\theta_V + (\lambda, \lambda) - (\dim Q + 2)(\alpha, \alpha))$, where α denotes the simple root dual to λ.*

In particular, if $\mathfrak{g}$ is simply-laced and $V = \mathfrak{g}$ is the adjoint representation, then $\theta_{\mathfrak{g}_Q} = 2(\theta_{\mathfrak{g}} - (\dim Q - 1)(\alpha, \alpha))$.

Proof. We treat the case of $\mathfrak{d}_l$, the case of $\mathfrak{b}_l$ is similar. Label the nodes of D as $\alpha_1, ..., \alpha_l$ and consider them as nodes of $D(\mathfrak{g})$ in what follows. Let $\sigma = \alpha_1 + \cdots + \alpha_{l-2} + \frac{1}{2}(\alpha_{l-1} + \alpha_l)$. Note that with our normalizations, $(\sigma, 2\rho) = \mathrm{dimQ}$, $(\lambda, \sigma) = 1$ and $(\sigma, \sigma) = 1$. We have $\tau = 2\lambda - 2\sigma$ so $\theta_{V_\tau} = (2\lambda - 2\sigma, 2\lambda - 2\sigma) + (2\lambda - 2\sigma, 2\rho) = 2(\theta_V + (\lambda, \lambda) - 4(\lambda, \sigma) + 2(\sigma, \sigma) - (\sigma, 2\rho))$ and the result follows. □

Several such subdiagrams may exist, each of them will provide us with a component of S^2V.

Example. For every simple Lie algebra $\mathfrak{g}$ whose adjoint representation is fundamental, $S^2\mathfrak{g}$ contains only $\mathfrak{g}^{(2)}$, a trivial factor corresponding to the Killing form, and factors of the form $\mathfrak{g}_Q$ (of which there are at most three, or two up to a symmetry of the Dynkin diagram).

Example. In the case of the subexceptional (see §5) and Scorza series (see §6), there is a unique V_Q and $S^2V = V^{(2)} \oplus V_Q$.

Example. In the case of (E_n, V_{ω_4}) there are three distinct subdiagrams of quadric type, but they furnish only a small part of $S^2V_{\omega_4}$.

Note that in this case a point of $X_Q \subset \mathbb{P}V_Q \subset \mathbb{P}S^2V$ produces both a quadric hypersurface in $\mathbb{P}V^*$ and a quadric section of $X \subset \mathbb{P}V$.

There is another characterization of maximal quadrics on $X = G/P \subset \mathbb{P}V$, at least in the case of minuscule and adjoint representations. Let $\sigma_+(X)$ denote a component of the set of points of $\mathbb{P}V \backslash X$ through which pass a family of secants of X of maximal dimension. If $p \in \sigma_+(X)$, its *entry locus* $\Sigma_p = \{x \in X, \exists y \in X - x, p \in \overline{xy}\}$, is a maximal quadric on X.

Example. Let $\mathfrak{g} = \mathfrak{so}_{2l}, V = V_{\omega_k}$, with $1 < k < l - 1$. Here $X = G_Q(k, \mathbb{C}^{2l})$ is the Grassmannian of Q-isotropic k-planes in $W = \mathbb{C}^{2l}$, where Q denotes the quadratic form preserved by $\mathfrak{g}$. The two families of quadrics given by Tits fibrations or diagram induction may be seen geometrically as follows: For the subdiagram corresponding to $\mathfrak{so}_{2l-2k}$, choose $E \in G_Q(k-1, W)$, then

$$q_E = \{P \in G_Q(k, W),\ E \subset P \subset E^\perp\} \simeq Q^{2l-2k}.$$

The second family comes from the $\mathfrak{a}_3$ subdiagram. Pick $E \in G_Q(k-2, W)$ and $F \in G_Q(k+2, W)$. Then

$$q_{E,F} = \{P \in G_Q(k, W),\ E \subset P \subset F\} \simeq Q^3.$$

We leave to the reader the pleasure of making the explicit correspondence with the quadric hypersurfaces as above. The correspondance with $\sigma_+(X)$ is straightforward: the line joining two distinct isotropic k-spaces U, U' is contained in X if and only if U and U' meet in codimension one, and $U + U'$ is isotropic. If this is not the case, points on the secant line between U and U' are contained in $\sigma_+(X)$ if either U, U' meet in codimension one but $U + U'$ is not isotropic - in this case the entry locus is $q_{U\cap U'}$, unless U, U' meet in codimension two and $U + U'$ is isotropic, in which case the entry locus is $q_{U\cap U', U+U'}$.

2.5. **Linear syzygies and subdiagrams.** Consider diagram induction when $\mathfrak{f} = \mathfrak{a}_l$ with the trivial representation in $W_{\tau_1} \otimes W_{\tau_l}$. We obtain subrepresentations of $V_{\tau_1} \otimes V_{\tau_l}$. We will call such representations $(V_{\tau_1} V_{\tau_l})_{Aad}$. Changing notation, write $W_{\tau_1} = U_\lambda$, $W_{\tau_l} = W_\mu$, then $(UW)_{Aad}$ has highest weight $\tau = \lambda + \mu - \sigma$ where $\sigma = \alpha_1 + \cdots + \alpha_l$ where we have labelled the roots corresponding to the subdiagram $D(\mathfrak{a}_l)$. We can thus compute its Casimir as above.

Let $S \subset S_{21}(V)$ denote the space of linear syzygies among the generators of $I(X)$, the ideal of X (which are of degree two). We have $S = S_{21}(V) \cap (I_2(X) \otimes V)$. (We should really consider $X \subset \mathbb{P}V^*$ here, but our abuse of notation is harmless.)

Proposition 2.3. *Let V be a fundamental representation, let $X \subset \mathbb{P}V$ be the closed orbit and let $U \subset I_2(X)$ be an irreducible component of the space of quadrics containing X. Then $(VU)_{Aad} \subseteq S$.*

Unfortunately we have no general proof of this fact, but it can be checked case by case.

In the cases of the Severi and subexceptional series below we have equality.

3. The Vogel decompositions

Vogel [23] has proposed a *universal Lie algebra* $\mathfrak{g}([\alpha, \beta, \gamma])$, which allows one to parametrize all complex simple Lie superalgebras by a projective plane (over some extension of the rationals) quotiented out by $\mathfrak{S}_3$. Evaluating at particular points, one recovers all complex simple Lie algebras (and Lie superalgebras). He has given dimension and decomposition formulae for the irreducible modules in $\mathfrak{g}^{\otimes 2}, \mathfrak{g}^{\otimes 3}$ that, independent of the existence of the universal Lie algebra, give decomposition and dimension formulae for actual Lie algebras.

In order to connect his formulae to geometry, we break the $\mathfrak{S}_3$ symmetry. One reason for this is because inside $S^k\mathfrak{g}$ there is a preferred factor, the Cartan power $\mathfrak{g}^{(k)}$, which has the geometric interpretation of $I_k(X_{ad})^\perp$, the annhilator of the degree k component of the ideal of the closed orbit $X_{ad} \subset \mathbb{P}(\mathfrak{g})$. For example, $\mathfrak{g}^{(2)} \subset S^2\mathfrak{g}$ could be Y_2, Y_2' or Y_2'' for Vogel (following his notations). We fix it to be Y_2. This has the consequence of normalizing Vogel's parameter α to be $-(\tilde\alpha, \tilde\alpha)$, as according to Vogel, we have $2t = \theta_{\mathfrak{g}}$ and $2(\theta_{\mathfrak{g}} - \alpha) = 2\theta_{\mathfrak{g}^{(2)}} = 2\theta_{\mathfrak{g}} + 2(\tilde\alpha, \tilde\alpha)$.

In §2.3 we discussed the factor $\mathfrak{g}_Q \subset S^2\mathfrak{g}$, where we take the largest subdiagram of quadric type here. We break the remaining $\mathbb{Z}_2$ symmetry by requiring that this space be Y_2'. We obtain the following geometric interpretation of Vogel's parameter β, which follows from [23] and proposition 2.2:

Proposition 3.1. *Notations as above: $\beta = \mathrm{dimQ}$ where Q is the largest quadric contained in the adjoint variety $X \subset \mathbb{P}\mathfrak{g}$ obtained as a shadow as in §2.4.*

Example. In the adjoint representations of F_4, E_6, E_7 and E_8, there is a unique quadric type subdiagram of the marked Dynkin diagram, respectively of types B_3, D_4, D_5 and D_7, so that $\beta = 5, 6, 8$ and 12, respectively.

Example. If there is a second unextendable family of G-homogeneous quadrics on the adjoint variety (as is the case for the orthogonal groups), then this supplies a geometric interpretation of γ, namely γ is the dimension of a quadric in this second family. However, for adjoint representations, this occurs only for the orthogonal groups, and in this case we always have $\gamma = 4$.

Vogel describes three colinear collections of Lie algebras (in the sense that some choice of inverse images of the points associated with the Lie algebras are colinear in the projective plane). The three Vogel lines are the exceptional, Osp, and Sl. To these we add another line, the *subexceptional series*, see §5, which lies on the line $2\alpha - \beta + \gamma = 0$.

With the above normalizations:

	α	β	γ
Exceptional	-2	$m+4$	$2m+4$
Osp : $SO(m)$, $Sp(-m)$	-2	$m-4$	4
Sl : $Sl(m)$	-2	2	m
subexceptional	-2	m	$m+4$

In the exceptional series the values of m are $-\frac{2}{3}, 0, 1, 2, 4, 8$ for $G_2, D_4, F_4, E_6, E_7, E_8$. The subexceptional line is $A_1, A_1 \times A_1 \times A_1, C_3, A_5, D_6, E_7$, with parameter $m = -\frac{2}{3}, 0, 1, 2, 4, 8$. Although $A_1 \times A_1 \times A_1$ is not simple, one can check that the Vogel dimension and decomposition formulae still hold. The subexceptional line, unlike the three other lines osp, sl and exceptional, is generic to order three in the sense that none of the spaces that appear in Vogel's decomposition formulas are zero in $\mathfrak{g}^{\otimes k}$ for $k \leq 3$ except for the space Vogel labels X_3'' which is zero for all simple Lie algebras. So, by comparing Casimir eigenvalues we can obtain geometric interpretations for all the Vogel spaces. These interpretations (when such spaces exist) persist for other algebras not on the line.

Here are Vogel's decompositions with our interpretations of the spaces below. Recall our convention that $V_\mu V_\nu = V_{\mu+\nu}$.

$$\begin{aligned}
\Lambda^2\mathfrak{g} &= X_1 \oplus X_2 \\
&= \mathfrak{g} \oplus \mathfrak{g}_2 \\
S^2\mathfrak{g} &= Y_2 \oplus Y_2' \oplus Y_2'' \oplus X_0 \\
&= \mathfrak{g}^{(2)} \oplus \mathfrak{g}_Q \oplus \mathfrak{g}_{Q'} \oplus \mathbb{C}_B \\
\Lambda^3\mathfrak{g} &= X_3 \oplus X_3' \oplus X_3'' \oplus X_2 \oplus S^2\mathfrak{g} \\
&= \mathfrak{g}_3 \oplus \mathfrak{g}_2 \oplus S^2\mathfrak{g} \\
S^3\mathfrak{g} &= 2X_1 \oplus X_2 \oplus B \oplus B' \oplus B'' \oplus Y_3 \oplus Y_3' \oplus Y_3'' \\
&= 2\mathfrak{g} \oplus \mathfrak{g}_2 \oplus B \oplus \mathfrak{g}\mathfrak{g}_Q \oplus \mathfrak{g}\mathfrak{g}_{Q'} \oplus \mathfrak{g}^{(3)} \oplus \mathfrak{g}_{\mathbb{AP}^2} \oplus Y'' \\
S_{21}\mathfrak{g} &= 2X_2 \oplus 2X_2 \oplus Y_2 \oplus Y_2' \oplus Y_2'' \oplus B \oplus B' \oplus B'' \oplus C \oplus C' \oplus C'' \\
&= 2\mathfrak{g} \oplus 2\mathfrak{g}_2 \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}_Q \oplus \mathfrak{g}_{Q'} \oplus B \oplus \mathfrak{g}\mathfrak{g}_Q \oplus \mathfrak{g}\mathfrak{g}_{Q'} \oplus \mathfrak{g}\mathfrak{g}_2 \oplus (\mathfrak{g}\mathfrak{g}_Q)_{Aad} \oplus (\mathfrak{g}\mathfrak{g}_{Q'})_{Aad}
\end{aligned}$$

We have written $\mathfrak{g}_3 = X_3 \oplus X_3'$ as it is a Casimir eigenspace. We have no interpretation for X_3'' as it does not exist for actual Lie algebras, nor B because it does not exist for the exceptional series and it is $-\mathfrak{g}_2$ for the subexceptional series.

The other decompostions can be deduced from these, e.g., $X_1 \otimes X_2 = \mathfrak{g} \otimes \Lambda^2\mathfrak{g} - \mathfrak{g} \otimes \mathfrak{g} = (S_{21}\mathfrak{g} \oplus \Lambda^3\mathfrak{g}) - (S^2\mathfrak{g} \oplus \Lambda^2\mathfrak{g})$.

The only space not yet explained is $\mathfrak{g}_{\mathbb{AP}^2}$. It comes from diagram induction applied to a subdiagram in the *Severi series*, the distinguished representations in the second row of Freudenthal's magic chart (see §7) as there is an invariant cubic on the representation W. The subdiagram for $\mathfrak{g}_{\mathbb{A}\mathbb{P}^2}$ in the exceptional line is obtained by deleting the nodes for $\mathfrak{g}$ and $\mathfrak{g}_2$.

3.1. **Comparison with Freudenthal's magic square.** Normalizing $\alpha = -2$, Vogel's formula for $\dim\mathfrak{g}$ is

$$\dim\mathfrak{g} = \frac{(\beta+\gamma-1)(2\beta+\gamma-4)(2\gamma+\beta-4)}{\beta\gamma}$$

The triality model enables one to deduce the following two parameter formula for the dimensions of the Lie algebras occuring in Freudenthal's magic square (see [17]).

Proposition 3.2. *Let $\mathfrak{g}(a,b)$ denote the Lie algebra that Freudenthal associates to the pair $(\mathbb{A},\mathbb{B})$ of real division algebras of dimensions a and b, and let $p=a+4, q=b+4$ then*

$$\dim\mathfrak{g}(a,b) = 3\frac{(ab+4a+4b-4)(ab+2a+2b)}{(a+4)(b+4)}.$$

$$\dim\mathfrak{g}(p,q) = \frac{3(pq-20)(pq-2p-2q)}{pq}$$

The a,b parametrization is natural from the point of view of the composition algebras, the p,q parametrization is more natural from the point of view of Tit's fibrations. That the p,q parametrization might be simpler to work with was brought to our attention by B. Westbury.

4. The exceptional series

Using either Freudenthal's perspective of incidence geometry [10] or the triality model [17], one has four distinguished representations in the exceptional series, denoted X_1, X_2, X_3, Y_2^* in [4]. We refer the reader to [9] for the notations and decomposition formulae.

The spaces $Y_3^*, G^*, H^*, I^*, Y_4^*$ all contain virtual representations, i.e., negatives of actual representations, for the larger algebras in the series so it is not possible to assign direct geometric interpretations.

The primitive representations are as follows: $X_k = \mathfrak{g}_k, Y_2^* = \mathfrak{g}_Q, C^* = (\mathfrak{g}I_2)_{Aad} = S_1, F^* = (C^*\mathfrak{g})_{Aad} \subseteq S_2$. Here S_1, S_2 denote the first and second linear syzygies among the quadrics in the ideal for the closed orbit $X_{ad} \subset \mathbb{P}\mathfrak{g}$, where in general, for an algebraic variety $X \subset \mathbb{P}V$ defined by quadratic polynomials $I_2(X) \subset S^2V^*$, we let $S_1 := (V^*\otimes I_2(X)) \cap S_{21}V^*$ and $S_2 := (V^*\otimes S_1)\cap S_{211}V^*$ the second linear syzygies.

The others can be deduced from the primitive ones through Cartan products: $Y_k = \mathfrak{g}^{(k)}, A = \mathfrak{g}Y_2^*, C = \mathfrak{g}\mathfrak{g}_2, D = Y_2^*\mathfrak{g}^{(2)}, D^* = Y_3^*\mathfrak{g}, E = \mathfrak{g}C^*, F = \mathfrak{g}_2Y_2^*, G = \mathfrak{g}_2\mathfrak{g}^{(2)}, H = \mathfrak{g}_2{}^{(2)}, I = \mathfrak{g}\mathfrak{g}_3, J = Y_2^{*(2)}$.

5. Subminuscule representations

Recall that a $\mathfrak{g}$-module V is called of *type-θ* if there is a $\mathbb{Z}/m\mathbb{Z}$-grading (allowing the possibility of $\mathbb{Z}$-gradings as well) of a simple Lie algebra $\mathfrak{l}$ such that $\mathfrak{g}$ is the semi-simple part of $\mathfrak{l}_0$ and $V = \mathfrak{l}_k$ for some k. It is of *type I-θ* if $k=1$ and the grading is a $\mathbb{Z}$-grading. We define a further subclass, the *sub-minuscule* representations where the grading of $\mathfrak{l}$ is minuscule (i.e., three step). Geometrically, the subminuscule representations are the representations of semi-simple Lie algebras occuring as the isotropy representation on the tangent space of an irreducible compact Hermitian symmetric space, the type I-θ representations occur as the submodules $T_1 \subset T_{[e]}G/P$ where P is a maximal parabolic and T_1 is the (unique) irreducible P-submodule of $T_{[e]}G/P$ see [14].

In [14] we showed that for subminuscule representations, the only G orbits in $\mathbb{P}V$ are the smooth points of the successive secant varieties of the closed orbit $X = G/P \subset \mathbb{P}V$, and moreover that the union of the secant $\mathbb{P}^{k-1}$'s, denoted $\sigma_k(X)$, is such that its ideal is generated in degree $k+1$ with $I_{k+1}\sigma_k(X) = I_2(X)^{(k-1)} := (I_2(X)\otimes S^{k-1}V^*)\cap S^{k+1}V^*$, where $I_2(X)^{(k-1)}$ is called the $(k-1)$-st *prolongation* of $I_2(X)$. Another way to phrase this prolongation property is that the spaces of generators are the sucessive Jacobian ideals of the highest degree space of generators. In practice these spaces are quite easy to compute and comparing with [2], we observe that the symmetric algebra is free and the prolongations of $I_2(X)$ furnish all the primitive factors for the symmetric algebra, so we obtain:

Theorem 5.1. *Let V be a sub-minuscule representation of a semi-simple Lie algebra $\mathfrak{g}$. With the notations above, and our convention $V_\mu V_\sigma = V_{\mu+\sigma}$, we have a uniform formula for the decomposition of the symmetric algebra into irreducible $\mathfrak{g}$-modules:*

$$\oplus_{k=1}^{\infty} t^k S^k V = \Pi_{j=2}^{\infty}(1 - t^j I_2(X)^{(j-1)})^{-1}$$

The product on the right hand side is finite.

Here the orbit closures exactly provide the primitive factors for the symmetric algebra. In general, the orbit closures will provide some, but not all primitive factors, see the examples of the subexceptional and sub-Severi series below.

Remark. A version of this result appears to have been known to Kostant as the "cascade of orthogonal vectors".

Example: The Scorza series. Zak established an upper bound on the codimension of a smooth variety $X^n \subset \mathbb{P}^{n+a}$ of a given secant defect. (The secant defect is the difference between the expected dimension of the secant variety of X ($\min\{n+a, 2n+1\}$) and its actual dimension.) He then went on to classify the varieties achieving this bound, which he calls the *Scorza varieties.* They are all closed orbits $G/P \subset \mathbb{P}V$ and give rise to the following two parameter (a, n) series: $(SL_n, V_{2\omega_1})$, $(SL_n \times SL_n, V_{\omega_1} \otimes W_{\eta_1})$, (SL_{2n}, V_{ω_2}), (E_6, V_{ω_1}), where a is respectively $1, 2, 4, 8$ and $a = 8$ only for the $n = 3$ case. This series is the second row of the generalized Freudenthal magic square, see [15]. We could add to this the finite group $\mathfrak{S}_n$, case $a = 0$, corresponding to the variety of n points in $\mathbb{P}^{n-1}$. In this case the symmetric algebra is generated by the "determinant" (see [16]), which has degree n, and the spaces of $k \times k$ minors. Here $I_2(X)^{(j-1)}$ respectively has highest weights $2\omega_{n-k}$, $\omega_{n-k} + \eta_{n-k}$, ω_{2n-2k}. We remark that dim $V(a, n) = n + a\frac{n(n-1)}{2}$.

6. The subexceptional series

This is the series coming from the third line of Freudenthal's square:

$$A_1, \quad A_1 \times A_1 \times A_1, \quad C_3, \quad A_5, \quad D_6, \quad E_7.$$

Let $m = -\frac{2}{3}, 0, 1, 2, 4, 8$ respectively. Freudenthal's perspective [10] or the triality model [17] uncovers three preferred irreducible representations, respectively denoted $V, V_Q = \mathfrak{g}, V_2$ in the table below and of dimensions $6m + 8, \frac{3(2m+3)(3m+4)}{(m+4)}, 9(m + 1)(2m + 3)$.

Let Γ_0 be the automorphism group of the Dynkin diagram, and $\Gamma \subset \Gamma_0$ be the subgroup preserving the marked Dynkin diagram ($\Gamma = \mathfrak{S}_3$ for $\mathfrak{g} = A_1 \times A_1 \times A_1$, $\Gamma = \mathfrak{S}_2$ for $\mathfrak{g} = A_5$ and is trivial otherwise). With the help of the programm LiE [8], we obtained the following decomposition formulae into $\mathfrak{g} \times \Gamma_0$-Casimir eigenspaces. Letting $V_0 = \mathbb{C}$, we have, up to at least degree six:

$$\begin{aligned} \Lambda^{2p} V &= V_{2p} \oplus V_{2p-2} \oplus \cdots \oplus V_0, \\ \Lambda^{2p+1} V &= V_{2p+1} \oplus V_{2p-1} \oplus \cdots \oplus V_1. \end{aligned}$$

Note that V_2, V_3 are irreducible.

This decomposition coincides with the decomposition into primitives for the symplectic form. In general the decomposition of a symplectic $\mathfrak{g}$ module W into primitives is not Casimir-irreducible. Consider the primitives in the A_9-module $\Lambda^2(\Lambda^5\mathbb{C}^{10}) = \mathbb{C} \oplus V_{\omega_2+\omega_8} \oplus V_{\omega_4+\omega_6}$. The last two factors consitute the primitive subspace but they have different Casimir eigenvalues.

In the last four cases of the series, V is *exceptional* in the sense of [2], that is, the algebra $\mathbb{C}[V]^{\mathfrak{u}}$ of invariant regular functions on V^* under a maximal nilpotent subalgebra $\mathfrak{u}$ of $\mathfrak{g}$, i.e., the covariant algebra, is a polynomial algebra. Such an invariant is a highest weight vector of some symmetric power of V, which allows one to decompose S^kV into irrreducible factors for all k. The results of [2] imply, again with our convention $V_\lambda V_\mu = V_{\lambda+\mu}$, that:

$$\bigoplus_{k\geq 0} t^k S^k V = (1-tV)^{-1}(1-t^2\mathfrak{g})^{-1}(1-t^3V)^{-1}(1-t^4)^{-1}(1-t^4V_2)^{-1}.$$

As with the subminuscule case, the spans of generators of ideals of each orbit closure in $\mathbb{P}V$ give primitive factors in $S^\bullet V$. In contrast, there is one additional primitive factor, $V_2 \subset S^4V$, which is also the primitive part of $\Lambda^2 V$. The presence of the primitive V_2 factor may be understood as follows: the symplectic form ω on V enables an equivariant identification $V \simeq V^*$. Polarizing the invariant quartic form gives a map $q : S^3V \to V^* \simeq V$. Finally, we obtain a natural map $s : S^4V \to V_2$ by letting $s(v^4) = v\wedge q(v^3)$ mod ω. This map is nonzero and exhibits V_2 as an irreducible component of S^4V.

The remaining decompositions for $V^{\otimes k}$ in degrees three and four are:

$$\begin{array}{rcl}
S_{21}V & = & V\oplus C\oplus V\mathfrak{g}\oplus VV_2\\
S_{31}V & = & V_2\oplus 2V^{(2)}\oplus 2\mathfrak{g}\oplus VC\oplus \mathfrak{g}V^{(2)}\oplus \mathfrak{g}V_2\oplus \mathfrak{g}_2\oplus V_2V^{(2)},\\
S_{22}V & = & \mathbb{C}\oplus 2V_2\oplus \mathfrak{g}V^{(2)}\oplus VC\oplus Q\oplus \mathfrak{g}^{(2)}\oplus {V_2}^{(2)},\\
S_{211}V & = & V_2\oplus V^{(2)}\oplus \mathfrak{g}\oplus VC\oplus \mathfrak{g}V_2\oplus \mathfrak{g}_2\oplus L\oplus VV_3,
\end{array}$$

Note that the only primitives up to degree three are $C, \mathfrak{g}$ and the V_k's and the only new primitives in degree four are Q and L.

The Casimir eigenvalues for the modules involved in these formulas are all of the form $\frac{am+b}{8m+8}$ with $a, b \in \mathbb{Z}$, and are linear functions of $(\lambda,\lambda) = \frac{3}{8m+8}$. Here are the Casimir eigenvalues:

$$\begin{array}{ccc}
\theta_{V^{(k)}} = \frac{k(6m+9)+3(k^2-k)}{8m+8}, & \theta_{V_k} = \frac{6km+(10k-k^2)}{8m+8}, & \theta_{\mathfrak{g}^{(k)}} = \frac{2km+(k^2+k)}{2(m+1)},\\
\theta_C = \frac{12m+9}{8m+8}, & \theta_Q = \frac{3m}{2m+2}, & \theta_L = \frac{2m+1}{m+1}
\end{array}$$

The dimensions of these modules are rational functions of m with simple denominators, see [17] for the dimension formulas with the exception of

$$\dim C = \frac{32(m+1)(2m+3)(3m+4)}{(m+4)(m+6)},$$

$$\begin{array}{rcl}
\dim Q & = & \dfrac{(8-m)(m+1)(2m+3)(3m+2)(3m+4)}{(m+4)^2(m+6)},\\
\dim L & = & \dfrac{9(8-m)(m+1)(2m+3)(3m+2)(3m+4)}{(m+4)(m+6)(m+8)}.
\end{array}$$

There is a geometric interpretation for the primitives C and L in terms of syzygies. We lack a geometric interpretation for Q as it is empty for $\mathfrak{e}_7$ and it does not appear in the minimal free resolutions.

Proposition 6.1. *Let S_k denote the space of linear syzygies of order k in the minimal resolution of a sub-exceptional variety, beginning with $S_0 = I_2(X)$. Then*

$$S_0 = \mathfrak{g}, \qquad S_1 = C, \qquad S_2 = L.$$

The decompositions of $\mathfrak{g}^{\otimes 2}$ and $\mathfrak{g}^{\otimes 3}$ are as with Vogel's formulas. Except in the case of $\mathfrak{e}_7$, where it is irreducible, $\mathfrak{g}_3$ decomposes into two irreducible representations that are called X_3 and X_3', by Vogel. Their dimensions have algebraic expressions which are not rational in m. (In Vogel's formulae, the expressions are not rational in α, β, γ either.) From Deligne's perspective, $\mathfrak{g}_3$ should not be considered a preferred representation as its dimension formula contains a quadratic factor in its numerator:

$$\dim \mathfrak{g}_3 = \frac{(2m+3)(3m+4)(9m+16)(m+1)(18m^2+43m+4)}{(m+4)^3}$$

We also have:

$$\begin{aligned}
\mathfrak{g}\otimes V &= V\oplus C\oplus V\mathfrak{g},\\
\mathfrak{g}^{(2)}\otimes V &= V\mathfrak{g}^{(2)}\oplus V\mathfrak{g}\oplus \mathfrak{g}C,\\
V^{(3)}\otimes \mathfrak{g} &= \mathfrak{g}V^{(3)}\oplus V^{(3)}\oplus VV_2\oplus CV^2,\\
\Lambda^2V_2 &= V^{(2)}\oplus \mathfrak{g}\oplus VC\oplus \mathfrak{g}V_2\oplus \mathfrak{g}_2\oplus L\oplus VV_3,\\
\Lambda^2V^{(2)} &= V^{(2)}\oplus \mathfrak{g}\oplus \mathfrak{g}V_2\oplus V_2V^{(2)},\\
S^2V^{(2)} &= \mathbb{C}\oplus V_2\oplus \mathfrak{g}V^{(2)}\oplus V^{(4)}\oplus \mathfrak{g}^{(2)}\oplus {V_2}^{(2)},\\
V_2\otimes V^{(2)} &= V_2\oplus V^{(2)}\oplus \mathfrak{g}\oplus VC\oplus \mathfrak{g}V^{(2)}\oplus \mathfrak{g}V_2\oplus \mathfrak{g}_2\oplus VV_3\oplus V_2V^{(2)},\\
V_2\otimes \mathfrak{g} &= V_2\oplus V^{(2)}\oplus \mathfrak{g}\oplus VC\oplus \mathfrak{g}V_2\oplus \mathfrak{g}_2\oplus L,\\
V^{(2)}\otimes \mathfrak{g} &= V_2\oplus V^{(2)}\oplus VC\oplus \mathfrak{g}V^{(2)},\\
V_2\otimes V &= C\oplus V\mathfrak{g}\oplus V_3\oplus VV_2\oplus V,\\
V^{(2)}\otimes V &= V\mathfrak{g}\oplus VV_2\oplus V^{(3)}\oplus VV_3,\\
C\otimes V &= V_2\oplus \mathfrak{g}\oplus VC\oplus \mathfrak{g}_2\oplus L\oplus Q,\\
V\mathfrak{g}\otimes V &= V_2\oplus V^{(2)}\oplus \mathfrak{g}\oplus VC\oplus \mathfrak{g}V^{(2)}\oplus \mathfrak{g}V_2\oplus \mathfrak{g}_2\oplus \mathfrak{g}^{(2)},\\
V_3\otimes V &= V_2\oplus VC\oplus \mathfrak{g}V_2\oplus \mathfrak{g}_2\oplus L\oplus VV_3\oplus V_4,\\
VV_2\otimes V &= V_2\oplus V^{(2)}\oplus VC\oplus \mathfrak{g}V^{(2)}\oplus \mathfrak{g}V_2\oplus VV_3\oplus V_2V^{(2)}\oplus {V_2}^{(2)},\\
V^{(3)}\otimes V &= V^{(2)}\oplus \mathfrak{g}V^{(2)}\oplus V_2V^{(2)}\oplus V^{(4)},\\
S^2V_2 &= {V_2}^{(2)}\oplus V_4\oplus \mathfrak{g}V^{(2)}\oplus \mathfrak{g}^{(2)}\oplus 2V_2\oplus \mathbb{C}\oplus CV
\end{aligned}$$

The highest weights of the modules involved in the above formulas are as follows:

	A_1	$A_1^{\oplus 3}$	C_3	A_5	D_6	E_7
V	$[3]$	$[1,1,1]$	$[0,0,1]$	$[0,0,1,0,0]$	$[0,0,0,0,0,1]$	$[0,0,0,0,0,0,1]$
V_2	$[4]$	$[2,2,0]$	$[0,2,0]$	$[0,1,0,1,0]$	$[0,0,0,1,0,0]$	$[0,0,0,0,0,1,0]$
V_3		$[3,1,1]$	$[1,2,0]$	$[1,0,0,2,0]$	$[0,0,1,0,1,0]$	$[0,0,0,0,1,0,0]$
V_4	$-[4]$	$[4,0,0]$	$[0,3,0]$	$[0,3,0,0,0]$	$[0,1,0,0,2,0]$	$[0,0,0,1,0,0,0]$
		$[2,2,2]$	$[3,0,1]$	$[1,1,0,1,1]$	$[0,0,2,0,0,0]$	
V_5	$-[3]$		$[2,1,1]$	$[2,0,1,0,2]$	$[1,0,0,0,3,0]$	$[0,1,1,0,0,0,0]$
			$[5,0,0]$	$[1,2,0,1,0]$	$[0,1,1,0,1,0]$	
V_6	$-[0]$	$-[4,0,0]$	$[4,1,0]$	$[2,1,1,0,1]$	$[0,0,0,0,4,0]$	$[1,2,0,0,0,0,0]$
		$-[2,2,2]$	$[2,0,2]$	$[0,2,0,2,0]$	$[1,0,1,0,2,0]$	$[0,0,2,0,0,0,0]$
				$[3,0,0,0,3]$	$[0,2,0,1,0,0]$	
$\mathfrak{g}$	$[2]$	$[2,0,0]$	$[2,0,0]$	$[1,0,0,0,1]$	$[0,1,0,0,0,0]$	$[1,0,0,0,0,0,0]$
$\mathfrak{g}_2$		$[2,2,0]$	$[2,1,0]$	$[0,1,0,0,2]$	$[1,0,1,0,0,0]$	$[0,0,1,0,0,0,0]$
$\mathfrak{g}_3$		$[2,2,2]$	$[3,0,1]$	$[3,0,1,0,0]$	$[2,0,0,1,0,0]$	$[0,0,0,1,0,0,0]$
		$-[4,0,0]$	$[0,3,0]$	$[1,1,0,1,1]$	$[0,0,2,0,0,0]$	
C	$[1]$	$[1,1,1]\otimes\rho$	$[1,1,0]$	$[1,1,0,0,0]$	$[1,0,0,0,1,0]$	$[0,1,0,0,0,0,0]$
Q		$[0,0,0]\otimes\rho$	$[0,1,0]$	$[1,0,0,0,1]$	$[2,0,0,0,0,0]$	
L		$[2,0,0]$	$[1,0,1]$	$[0,1,0,1,0]$	$[0,0,0,0,2,0]$	

In the column corresponding to $A_1\times A_1\times A_1$, ρ denotes the two-dimensional irreducible representation of $\Gamma=\mathfrak{S}_3$.

The first two cases of the series deserve special care since they are slightly degenerate and we discuss them in the following two subsections.

6.1. **Binary cubics.** In the A_1 case $V_2=\mathfrak{g}^{(2)}$, and there is no factor $1-t^4V_2$ in the denominator. Moreover, V is not exceptional since there exists a relation in degree 6 between the fundamental covariants (see e.g. [7]). We have

$$\bigoplus_{k\geq 0} t^kS^kV=\frac{1-t^6V^{(2)}}{(1-tV)(1-t^2\mathfrak{g})(1-t^3V)(1-t^4)}.$$

6.2. $2\times 2\times 2$ **hypermatrices.** Write $A_1\times A_1\times A_1=\mathfrak{sl}(A)\times\mathfrak{sl}(B)\times\mathfrak{sl}(C)$, with $A,B,C\simeq\mathbb{C}^2$.

Introduce the symmetrization operator ϕ on formal power series with coefficients in $A_1\times A_1\times A_1$-modules, which associates to $S^aA\otimes S^bB\otimes S^cC$ its complete symmetrization, e.g., $\phi(S^2A\otimes B)=S^2A\otimes B\oplus S^2A\otimes C\oplus S^2B\otimes A\oplus S^2B\otimes C\oplus S^2C\otimes A\oplus S^2C\otimes B$, and $\phi(S^3A)=S^3A\oplus S^3B\oplus S^3C$.

Theorem 6.2. *The covariant algebra $\mathbb{C}[A\otimes B\otimes C]^{\mathfrak{n}\times\mathfrak{S}_3}$ is a polynomial algebra. More precisely,*

$$\bigoplus_{k\geq 0}t^kS^kV=\phi\frac{1}{(1-tV)(1-t^2\mathfrak{g})(1-t^3V)(1-t^4)(1-t^4V_2)},$$

where $V=A\otimes B\otimes C$, $\mathfrak{g}=S^2A\oplus S^2B\oplus S^2C$ and $V_2=\Lambda^2(A\otimes B\otimes C)/\mathbb{C}=S^2A\otimes S^2B\oplus S^2B\otimes S^2C\oplus S^2C\otimes S^2A$.

Here use the convention $\mathfrak{g}^{(k)}=S^{2k}A\oplus S^{2k}B\oplus S^{2k}C$, and similarly for $V_2^{(k)}$.

Thus although $A\otimes B\otimes C$ is not exceptional in the sense of [2], it does become exceptional when we take into account the $\mathfrak{S}_3$-symmetry.

Note that the generators of the symmetric algebra have the same degrees as in the other cases of the subexceptional series.

The theorem is a consequence of the following lemma:

Lemma 6.3. *Let $\mu(n;a,b,c)$ denote the multiplicity of $S_{n-a,a}A\otimes S_{n-b,b}B\otimes S_{n-c,c}C$ inside $S^n(A\otimes B\otimes C)$. Suppose that $c\geq a,b$ and $2c\leq n$. Then*

$$\mu(n;a,b,c)=\begin{cases}0 & \text{if } c>a+b,\\ E(\frac{a+b-c}{2})+1 & \text{if } c\leq a+b \text{ and } n\geq a+b+c,\\ E(\frac{a+b-c}{2})-E^+(\frac{a+b+c-n}{2})+1 & \text{if } c\leq a+b \text{ and } n\leq a+b+c.\end{cases}$$

Here $E(x)$ denotes the largest integer smaller than or equal to x, and $E^+(x)$ the smallest integer greater than or equal to x.

Recall that irreducible representations of $\mathfrak{S}_n$ are naturally indexed by partitions of n. We let $[\lambda]$ denote the representation associated to a partition λ, following the notation of [19].

By Schur duality, $\mu(n;a,b,c)$ can be interpreted in terms of representations of symmetric groups, as the dimension of the space of $\mathfrak{S}_n$-invariants in the triple tensor product $[n-a,a]\otimes[n-b,b]\otimes[n-c,c]$, or the multiplicity of $[n-a,a]$ inside $[n-b,b]\otimes[n-c,c]$. The behavior of the multiplicity of $[n+\lambda]$ inside $[n+\mu]\otimes[n+\nu]$ as a function of n was investigated in [3, 19], where it was proved to be non-decreasing, and constant for n sufficiently large.

Proof. We use Cauchy formula [13] for the symmetric powers of a tensor product:

$$\bigoplus_{k\geq 0}t^kS^k(A\otimes B\otimes C)=\bigoplus_{a\geq b\geq 0}t^{a+b}S_{a,b}A\otimes S_{a,b}(B\otimes C).$$

Since A is two dimensional, $S_{a,b}A=S^{a-b}A$ as $\mathfrak{sl}_2$-modules. Moreover, we can write $S_{a,b}(B\otimes C)=S^a(B\otimes C)\otimes S^b(B\otimes C)-S^{a+1}(B\otimes C)\otimes S^{b-1}(B\otimes C)$, so we first compute

$$\begin{aligned}&\textstyle\bigoplus_{a\geq b\geq 0}t^{a+b}S_{a,b}A\otimes S^a(B\otimes C)\otimes S^b(B\otimes C)=\\ &\quad=\textstyle\bigoplus_{\alpha\geq\beta,\gamma\geq\delta,\alpha+\beta\geq\gamma+\delta}t^{\alpha+\beta+\gamma+\delta}S_{\alpha+\beta,\gamma+\delta}A\otimes S_{\alpha,\beta}B\otimes S_{\gamma,\delta}B\otimes S_{\alpha,\beta}C\otimes S_{\gamma,\delta}C.\end{aligned}$$

The last equality follows from Cauchy formula. Now the Clebsh-Gordon formula implies that $S_{\alpha,\beta}B\otimes S_{\gamma,\delta}B=S^{\alpha-\beta}B\otimes S^{\gamma-\delta}B=\oplus_{0\leq k\leq\alpha-\beta,\gamma-\delta}S^{\alpha-\beta+\gamma-\delta-2k}B$. Define the formal series $P_{u,v,w}(t)$ by the identity

$$\bigoplus_{a\geq b\geq 0}t^{a+b}S_{a,b}A\otimes S^a(B\otimes C)\otimes S^b(B\otimes C)=\bigoplus_{u,v,w\geq 0}P_{u,v,w}(t)S^uA\otimes S^vB\otimes S^wC,$$

and observe that the coefficient of t^n inside $P_{u,v,w}(t)$ is equal to the number of solutions of the system of equations in nonnegative integers

$$\left\{\begin{array}{lcl} n & = & \alpha+\beta+\gamma+\delta, \\ u & = & \alpha+\beta-\gamma-\delta, \\ v & = & \alpha+\gamma-\beta-\delta-2k, \\ w & = & \alpha+\gamma-\beta-\delta-2l, \end{array}\right.$$

with $\alpha+\beta\geq\gamma+\delta$ and $0\leq k,l\leq\alpha-\beta,\gamma-\delta$. From these equations we first deduce that $u+v=2\alpha-2\delta-2k$ and $u+w=2\alpha-2\delta-2l$, which imply that u,v,w have the same parity. Let $2r=u+v$ and $2s=u+w$, so that $k=\alpha-\delta-r$ and $l=\alpha-\delta-s$. Suppose that $u\geq v\geq w$, so that in particular $r\geq s$. Then

$$P_{u,v,w}(t)=\sum_{\substack{\gamma+s\geq\alpha\geq\delta+r\\ \delta+s\geq\beta\geq 0\\ \alpha+\beta=\gamma+\delta+u}} t^{\alpha+\beta+\gamma+\delta}=\sum_{\substack{\alpha\geq\delta+r\\ \delta+s\geq\beta\geq 0\\ \beta+s=\delta+u}} t^{2\alpha+2\beta-u}=\frac{t^v}{1-t^2}\sum_{\substack{\delta+s\geq\beta\geq 0\\ \beta+s=\delta+u}} t^{2\delta+2\beta}=\frac{t^{u+v-w}(1-t^{2w+2})}{(1-t^2)^2(1-t^4)}.$$

A similar computation shows that

$$\bigoplus_{a\geq b>0} t^{a+b}S_{a,b}A\otimes S^{a+1}(B\otimes C)\otimes S^{b-1}(B\otimes C)=\bigoplus_{u,v,w\geq 0} Q_{u,v,w}(t)S^uA\otimes S^vB\otimes S^wC,$$

where $Q_{u,v,w}(t)=\frac{t^{u+v-w+2}(1-t^{2w+2})}{(1-t^2)^2(1-t^4)}$ for $u\geq v\geq w$. Thus

$$\bigoplus_{k\geq 0} t^kS^k(A\otimes B\otimes C)=\bigoplus_{u,v,w\geq 0}\frac{t^{u+v+w-2m}(1-t^{2m+2})}{(1-t^2)(1-t^4)}S^uA\otimes S^vB\otimes S^wC,$$

with the notation $m=\min(u,v,w)$. The lemma is now just a transcription of this formula. □

The lemma can be rewritten in the following form:

$$\begin{aligned}\textstyle\bigoplus_{k\geq 0} t^kS^k(A\otimes B\otimes C) &= \tfrac{1}{(1-tA\otimes B\otimes C)(1-t^3A\otimes B\otimes C)(1-t^4)}\times\\ &\times\Big(\tfrac{1}{1-t^4S^2A\otimes S^2B}\Big(\tfrac{1}{1-t^2S^2A}+\tfrac{1}{1-t^2S^2B}-1\Big)+\tfrac{1}{1-t^4S^2B\otimes S^2C}\Big(\tfrac{1}{1-t^2S^2B}+\tfrac{1}{1-t^2S^2C}-1\Big)+\\ &\quad+\tfrac{1}{1-t^4S^2C\otimes S^2A}\Big(\tfrac{1}{1-t^2S^2C}+\tfrac{1}{1-t^2S^2A}-1\Big)-\tfrac{1}{1-t^2S^2A}-\tfrac{1}{1-t^2S^2B}-\tfrac{1}{1-t^2S^2C}+1\Big).\end{aligned}$$

and the theorem follows.

6.3. **Isotropy representations of orthogonal adjoint varieties.** The set of semi-simple parts of the isotropy groups for all fundamental adjoint varieties consists of the subexceptional series plus $\mathfrak{sl}_2\times\mathfrak{so}_n$ acting on $V=A\otimes B=\mathbb{C}^2\otimes\mathbb{C}^n$. This new case is quite similar as:

Proposition 6.4.

$$\bigoplus_{k\geq 0} t^kS^kV=\frac{1}{(1-tV)(1-t^2S_{[1,1]}B)(1-t^3V)(1-t^4)}\Big(\frac{1}{1-t^2S_{[2]}A}+\frac{1}{1-t^4S_{[2]}B}-1\Big).$$

Thus the covariant algebra $\mathbb{C}[V]^{\mathfrak{u}}$ is not a polynomial algebra as in the subexceptional cases, although it has generators of exactly the same degrees. The fact that we no longer obtain a polynomial algebra seems to be related, first to the nonsimplicity of $\mathfrak{g}=S^2A\oplus S_{[1,1]}B$, and also to the fact that $V_2=S_{[2]}B\oplus S^2A\otimes S_{[1,1]}B$ partly comes from $\mathfrak{g}$, since its second factor is just the tensor product of the two components of $\mathfrak{g}$. Unlike the subexceptional case, the orbit closures here are not nested.

Proof. The Cauchy formula gives

$$S^k(A\otimes B)=\bigoplus_{\substack{l\geq m\geq 0\\ l+m=k}} S_{l,m}A\otimes S_{l,m}B.$$

Here the Schur power $S_{l,m}B$ is not irreducible as a $\mathfrak{so}_n$-module, its decomposition into irreducibles can be found in [13] and is given by

$$S_{l,m}B=\bigoplus_{\substack{a\geq b\geq 0,\\ p\geq q\geq 0}} c^{l,m}_{(2a,2b),(p,q)}S_{[p,q]}B,$$

where $S_{[p,q]}B$ denotes the irreducible $\mathfrak{so}_n$-module indexed by the two-parts partition (p,q), and the Littlewood-Richardson coefficient $c^{l,m}_{(2a,2b),(p,q)}$ is the multiplicity of the $GL(C)$-module $S_{l,m}C$ inside the tensor product $S_{2a,2b}C\otimes S_{p,q}C$, where C is some vector space of dimension at least two. By the Littlewood-Richardson rule, this multiplicity equals the number of triples of non-negative integers α,β,γ such that $0\leq\beta\leq 2a-2b$ and $0\leq\gamma\leq\alpha$, $l=2a+\alpha$, $m=2b+\beta+\gamma$, $p=\alpha+\beta$ and $q=\gamma$. Letting $a=b+c$, we get

$$\bigoplus_{k\geq 0}t^kS^kV=\bigoplus_{\substack{b,c,\alpha,\beta,\gamma\geq 0\\ 0\leq\beta\leq 2c,0\leq\gamma\leq\alpha}} t^{4b+2c+\alpha+\beta+\gamma}S_{2c+\alpha-\beta-\gamma}A\otimes S_{[\alpha+\beta,\gamma]}B.$$

We let $\alpha=\gamma+\delta$, and for a we distinguish two cases: either $\beta=2\rho$ is even and we let $a=\rho+\sigma$, or $\beta=2\rho+1$ is odd and we let $a=\rho+\sigma+1$. Then

$$\begin{aligned}\bigoplus_{k\geq 0}t^kS^kV &= \tfrac{1}{1-t^4}\Big(\bigoplus_{\rho,\sigma,\gamma,\delta\geq 0}t^{4\rho+2\sigma+2\gamma+\delta}S_{2\sigma+\delta}A\otimes S_{[\gamma+\delta+2\rho,\gamma]}B+\\ &\qquad +\bigoplus_{\rho,\sigma,\gamma,\delta\geq 0}t^{4\rho+2\sigma+2\gamma+\delta+3}S_{2\sigma+\delta+1}A\otimes S_{[\gamma+\delta+2\rho+1,\gamma]}B\Big),\end{aligned}$$

giving the rational expressions

$$\begin{aligned}\bigoplus_{k\geq 0}t^kS^kV &= \frac{1+t^3A\otimes B}{(1-t^4)(1-tA\otimes B)(1-t^2S_2A)(1-t^2S_{[1,1]}B)(1-t^4S_{[2]}B)}\\ &= \frac{1-t^6S_2A\otimes S_{[2]}B}{(1-tA\otimes B)(1-t^4)(1-t^3A\otimes B)(1-t^2S_2A)(1-t^2S_{[1,1]}B)(1-t^4S_{[2]}B)}\\ &= \frac{1}{(1-tA\otimes B)(1-t^4)(1-t^3A\otimes B)(1-t^2S_{[1,1]}B)}\Big(\frac{1}{1-t^2S_2A}+\frac{1}{1-t^4S_{[2]}B}-1\Big).\end{aligned}$$

□

7. The Severi series

Zak proved Hartshorne's conjecture that a smooth subvariety $X^n\subset\mathbb{P}^{n+a}$ not contained in a hyperplane cannot have a degenerate secant variety if $a<\frac{n}{2}+2$, and then classified the boundary case. The answer gives rise to the series corresponding to the second line in Freudenthal's square:

$$A_2,\quad A_2\times A_2,\quad A_5,\quad E_6$$

which we parametrize by $m=1,2,4,8$. We could add the finite group $\mathfrak{S}_3$ with $m=0$. (In the case $m=0$ that V, defined below, has the correct dimension, but $\mathfrak{g}$ does not.)

Freudenthal's incidence geometries [10] or the triality model [17] distinguishes two isomorphic representations of dimension $3m+3$. We choose one, call it V and call its dual V^*. In fact $V^*=V_Q=I_2(X)$ with respect to our previous notations, where $X\subset\mathbb{P}V$ denotes the unique closed orbit. While not singled out by the triality model, $\mathfrak{g}$ does occur as $\mathfrak{g}=(VV^*)_{Aad}$, i.e., as a space of linear syzygies. Its dimension is $\dim\mathfrak{g}=\frac{4(m+1)(3m+2)}{m+4}$.

Let Γ_0 be the automorphism group of the Dynkin diagram, and $\Gamma\subset\Gamma_0$ be the subgroup preserving the marked Dynkin diagram (Γ is trivial except for $\mathfrak{g}=A_2\times A_2$, for which $\Gamma=\mathfrak{S}_2$). We obtain the following decomposition formulae into $\mathfrak{g}\times\Gamma_0$-Casimir eigenspaces:

$$
\begin{array}{rcl}
\Lambda^k V &=& V_k, \quad 2 \le k \le 6, \\
\mathfrak{g} \otimes V &=& V \oplus V_2{}^* \oplus V\mathfrak{g} \oplus J, \\
S_{21}V &=& \mathfrak{g} \oplus VV^* \oplus VV_2, \\
S_{31}V &=& V \oplus V_2{}^* \oplus V^{(2)}V^* \oplus V\mathfrak{g} \oplus V^*V_2 \oplus V^{(2)}V_2, \\
S_{22}V &=& V \oplus V^{(2)*} \oplus V^{(2)}V^* \oplus V\mathfrak{g} \oplus V_2{}^{(2)}, \\
S_{211}V &\supset& V_2{}^* \oplus V\mathfrak{g} \oplus J \oplus V^*V_2.
\end{array}
$$

The Severi series is sub-minuscule, so Theorem 5.1 applies. There are only three orbits:

$$\bigoplus_{k \ge 0} t^k S^k V = (1-tV)^{-1}(1-t^2V^*)^{-1}(1-t^3)^{-1}.$$

The Casimir eigenvalues for these modules are all of the form $\frac{am+b}{9m}$ with $a, b \in \mathbb{Z}$ and are linear functions of (λ, λ) as before.

The dimensions of these modules are rational functions of m with simple denominators. The formulas can be found in [17], with the exceptions of V_k which is obvious, $\mathfrak{g}$ given above and

$$\dim J = \frac{3(m+1)(8-m)(3m+2)}{2(m+4)(m+6)}.$$

As explained in §2, $\mathfrak{g} \subset S_1$ is a subspace of the space of linear syzygies of $X \subset \mathbb{P}V$. In fact more is true:

Proposition 7.1. *Let S_k denote the chain of linear syzygies in the minimal resolution of a Severi variety, beginning with $S_0 = I_2(X)$. Then*

$$S_0 = V^*, \qquad S_1 = \mathfrak{g}, \qquad S_2 = J.$$

The highest weights of the modules involved in the decomposition formulas are given in the following table. Note that for $A_1^{\oplus 2}$ each time a representation occurs, its mirror occurs as well which we supress in the list. In particular, the adjoint representation is not irreducible:

	A_2	$A_2^{\oplus 2}$	A_5	E_6
V	$[2,0]$	$[0,1\vert 1,0]$	$[0,1,0,0,0]$	$[1,0,0,0,0,0]$
$\mathfrak{g}$	$[1,1]$	$[1,1\vert 0,0]$	$[1,0,0,0,1]$	$[0,1,0,0,0,0]$
V_2	$[2,1]$	$[1,0\vert 2,0]$	$[1,0,1,0,0]$	$[0,0,1,0,0,0]$
V_3	$[3,0]$	$[0,0\vert 3,0]$	$[0,0,2,0,0]$	$[0,0,0,1,0,0]$
	$[0,3]$		$[2,0,0,1,0]$	
		$[1,1\vert 1,1]$		
V_4	$[1,2]$	$[2,0\vert 0,2]$	$[3,0,0,0,1]$	$[0,1,0,0,1,0]$
		$[1,0\vert 1,2]$	$[1,0,1,1,0]$	
V_5	$[0,2]$	$[2,0\vert 0,2]$	$[4,0,0,0,0]$	$[0,2,0,0,0,1]$
		$[1,0\vert 1,2]$	$[2,0,1,0,1]$	$[0,0,0,0,2,0]$
		$[2,1\vert 0,1]$	$[0,1,0,2,0]$	
V_6	$[0,0]$	$[0,0\vert 3,0]$	$[1,1,0,1,1]$	$[0,3,0,0,0,0]$
		$[0,3\vert 0,0]$	$[3,0,1,0,0]$	$[0,1,0,0,1,1]$
		$[1,1\vert 1,1]$	$[0,0,0,3,0]$	
J	$[0,1]$	$[1,0\vert 0,1]$	$[2,0,0,0,0]$	

8. The Severi-section series

This is the series of the first line in Freudenthal's square:

$$A_1, \quad A_2, \quad C_3, \quad F_4.$$

Again let $m = 1, 2, 4, 8$ respectively. This series does not correspond to a line in Vogel's plane, but $B_1 = A_1, A_2, C_3$ are on a line, their parameters being $(7m-8, -2m, 4)$ for $m = 1, 2, 4$. In particular the sum of these coefficients is $5m-4$, which is precisely the denominator in the Casimir eigenvalues below. There is a distinguished $\mathfrak{g}$-module V of dimension $3m+2$.

We have a uniform decomposition

$$\Lambda^2 V = \mathfrak{g} \oplus V_2$$

where the presence of both factors is easily understood, the first because $\mathfrak{g}$ preserves a quadratic form on V and thus lies in $\mathfrak{so}(V)$, the second by diagram induction.

Let Γ_0 be the automorphism group of the Dynkin diagram, and $\Gamma \subset \Gamma_0$ be the subgroup preserving the marked Dynkin diagram (Γ is trivial except for $\mathfrak{g} = A_2$, for which $\Gamma = \mathfrak{S}_2$). We obtain the following decomposition formulae into irreducible $\mathfrak{g} \times \Gamma$-modules:

Proposition 8.1. *Let $\varepsilon_m = 1$ for $m = 1$, $\varepsilon_m = 0$ for $m = 2, 4, 8$. Then*

$$\sum_{k \geq 0} t^k S^k V = \frac{1 - \varepsilon_m t^6 V^{(3)}}{(1-tV)(1-t^2)(1-t^2V)(1-t^3)(1-t^3V_2)}.$$

All the generators except V_2 and the quadratic form are generators of ideals of orbits. The presence of V_2 can be understood as follows: the polarization of the cubic invariant gives a map $r : S^2V \to V^* \simeq V$, hence a map $s : S^3V \to V_2$ by letting $s(v^3) = p(v, r(v))$, where we identify $V \simeq V^*$ using the quadratic form.

For $m = 4$ or 8 the representation V is again exceptional in the sense of [2], whose results imply the Proposition in those cases.

In the case $m = 1$ the invariant algebra $\mathbb{C}[V]^{\mathfrak{g}}$ is free, but there exists a (unique) relation in degree six between the fundamental covariants in $\mathbb{C}[V]^{\mathfrak{u}}$. This is the classical case of quartic binary forms (see [7] and references therein for covariants of binary forms).

The Casimir eigenvalues for these modules are all of the form $\frac{am+b}{5m-4}$ with $a, b \in \mathbb{Z}$ and are linear functions of (λ, λ).

We have

$$\dim \mathfrak{g} = \frac{3m(3m+2)}{m+4}, \quad \dim V_2 = \frac{(3m+2)(3m+4)(m+1)}{2(m+4)}$$

and highest weights are given in the following table:

	A_1	A_2	C_3	F_4
V	$[4]$	$[1,1]$	$[0,1,0]$	$[0,0,0,1]$
$\mathfrak{g}$	$[2]$	$[1,1]$	$[2,0,0]$	$[1,0,0,0]$
V_2	$[6]$	$[3,0]$	$[1,0,1]$	$[0,0,1,0]$

8.1. **The adjoint representation of $\mathfrak{sl}_3$.** . The case $m = 2$ deserves some explanation since V_2 is irreducible as a $\mathfrak{g} \times \Gamma$-module, but has two components as a $\mathfrak{g}$-module: $V_2 = V_{3\omega_1} \oplus V_{3\omega_2}$, the nontrivial element of $\Gamma = \mathbb{Z}/2\mathbb{Z}$ permutes the two components. The identity above should be understood as

$$\sum_{k \geq 0} t^k S^k V = \frac{1}{(1-tV)(1-t^2)(1-t^2V)(1-t^3)} \Big(\frac{1}{1-t^3 V_{3\omega_1}} + \frac{1}{1-t^3 V_{3\omega_2}} - 1 \Big).$$

Note that $(1-t^3V_{3\omega_1})^{-1}+(1-t^3V_{3\omega_2})^{-1}-1 = 1+\sum_{k>0} t^k(V_{3k\omega_1}\oplus V_{3k\omega_2})$, so that the preceeding identity means that $\mathfrak{sl}_3$ is exceptional in the sense that $\mathbb{C}[\mathfrak{sl}_3]^{\mathfrak{u}\times\Gamma}$ is a polynomial algebra, although $\mathbb{C}[\mathfrak{sl}_3]^{\mathfrak{u}}$ is not.

We briefly explain how one obtains the preceeding generating function $g_{\mathfrak{sl}_3}(t)$ for the symmetric powers of $\mathfrak{sl}_3$. If U denotes the natural three-dimensional module, first note that $U^*\otimes U = \mathfrak{sl}_3\oplus\mathbb{C}$, so that $g_{\mathfrak{sl}_3}(t) = (1-t)g_{U^*\otimes U}(t)$. Again the symmetric powers of a tensor product are given by the Cauchy formula:

$$S^k(U^*\otimes U) = \sum_{a+2b+3c=k} S_{a+b+c,b+c,c}U\otimes S_{a+b+c,b+c,c}U^*.$$

But as $\mathfrak{sl}_3$-modules, $S_{a+b+c,b+c,c}U^* = S_{a+b,b}U^* = S_{a+b,a}U$, and we get

$$g_{\mathfrak{sl}_3}(t) = \frac{1-t}{1-t^3}\sum_{a,b\geq 0} t^{a+2b}S_{a+b,a}U\otimes S_{a+b,b}U.$$

Now we use the Littlewood-Richardson rule to compute these scalar products: we refer the reader to [18] for the statement and the terminology we use in the sequel. Following this rule, the irreducible components of $S_{a+b,a}U\otimes S_{a+b,b}U$ are encoded by skew-tableaux of the following type:

						1	1	1	1
		1	1	1	2	2	2		
1	2	2	2						

We have $a+b$ empty boxes on the first line, b on the second line. We add α_i boxes numbered 1 on the i-th line, $i=1,2,3$, with total number $a+b$, and β_j boxes numbered 2 on the j-th line, $j=2,3$, with total number a. Moreover, there are two types of constraints. First we must get a semistandard skew-tableau, which means that below a box numbered 1 there can be no box also numbered 1, and below a box numbered 2 there can be no box at all. This means that

$$\alpha_2\leq a,\quad \alpha_2+\beta_2\leq a+\alpha_1,\quad \alpha_3\leq b,\quad \alpha_3+\beta_3\leq b+\alpha_2.$$

Second, the word one obtains by reading the numbered boxes right to left and top to bottom must be Yamanouchi (or a lattice word), which means that

$$\beta_2\leq\alpha_1\quad\text{and}\quad\beta_2+\beta_3\leq\alpha_1+\alpha_2.$$

When these conditions are fulfilled, we have $S_{a+b+\alpha_1,b+\alpha_2+\beta_2,\alpha_3+\beta_3}U\subset S^k(U^*\otimes U)$.

Recalling that $a=\alpha_1+\alpha_2+\alpha_3$ and $b=\beta_2+\beta_3$, it is easy to see that this set of inequalities actually reduces to:

$$\beta_2\leq\alpha_1,\quad \alpha_2\leq\beta_2+\beta_3,\quad \beta_2+2\beta_3\leq\alpha_1+2\alpha_2.$$

The first of theses implies that we can write $\alpha_1=\beta_2+u$ for some non-negative integer u. Then we have two cases.

If $\alpha_2\geq\beta_3$, we let $\alpha_2=\beta_3+v$ for some non-negative integer v, then the third inequality is automatically true and the second one reduces to $\beta_2\geq v$, so that $\beta_2=v+w$ for some non-negative integer w. The $\mathfrak{sl}_3$-module we obtain this way is $S_{2u+3v+2w+\alpha_3+\beta_3,u+3v+w+\alpha_3+\beta_3,\alpha_3+\beta_3}U = S_{2u+3v+2w,u+3v+w}U$, and the overall contribution of this case is

$$\sum_{u,v,w,\alpha_3,\beta_3\geq 0} t^{2u+3v+w+2\alpha_3+\beta_3}S_{2u+3v+2w,u+3v+w}U = \frac{1}{(1-t^2S_{21}U)(1-t^3S_{33}U)(1-tS_{21}U)(1-t^2)(1-t^3)}.$$

If $\alpha_2\leq\beta_3$, we let $\alpha_2=\beta_3-v$ for some non-negative integer v; then the second inequality is automatically true and the third one reduces to $u\geq 2v$, so that $u=2v+w$ for some non-negative integer w. The $\mathfrak{sl}_3$-module we obtain this way is $S_{4v+2w+\alpha_2+\alpha_3+2\beta_2,v+w+\alpha_2+\alpha_3+\beta_2,v+\alpha_2+\alpha_3}U = S_{3v+2w+2\beta_2,w+\beta_2}U$, and the overall contribution of this case is

$$\sum_{v,w,\alpha_2,\alpha_3,\beta_2\geq 0} t^{3v+2w+2\alpha_2+\alpha_3+\beta_2}S_{3v+2w+2\beta_2,w+\beta_2}U = \frac{1}{(1-t^3S_3U)(1-t^2S_{21}U)(1-t^2)(1-t)(1-tS_{21}U)}.$$

Finally, we counted the case $\alpha_2 = \beta_3$ twice, whose contribution is easily calculated to be

$$\sum_{u,\alpha_2,\alpha_3,\beta_2\geq 0} t^{2u+\alpha_2+2\alpha_3+\beta_2} S_{2u+2\beta_2,u+\beta_2}U = \frac{1}{(1-t^2S_{21}U)(1-t)(1-t^2)(1-tS_{21}U)}.$$

Putting together theses three contributions we easily obtain the expression we claimed for the generating series $g_{\mathfrak{sl}_3}(t)$.

9. The highest possible Casimir eigenspace of $\Lambda^k V$

Let V be a fundamental representation of a simple Lie algebra $\mathfrak{g}$ with highest weight λ and Casimir eigenvalue θ_V. Let α denote the simple root whose coroot is Killing-dual to λ. Define $V_k \subseteq \Lambda^k V$ to be the (possibly empty) subspace with Casimir eigenvalue

$$\theta_{V_k} := k\theta_V + k(k-1)\left[(\lambda,\lambda) - (\alpha,\alpha)\right].$$

We expect that V_k, when nonempty, is the highest Casimir eigenspace in $\Lambda^k V$. We show below that this is the case when V is minuscule, it is true when V is adjoint by [12], and we extend it to other fundamental representations in low degrees in proposition 9.3. Let k_0 denote the largest k for which V_k is nonempty.

Remark. In [12], a beautiful characterization of V_k is given in the case $V = \mathfrak{g}$ is the adoint representation: the components of $\mathfrak{g}_k$ correspond to abelian ideals of a fixed Borel $\mathfrak{b}$. Our answer in the general case is not as elegant and it would be nice to have a simpler characterization.

For the adjoint representations k_0 is explicitly known. Also note that $\theta_{\mathfrak{g}_k} = k$ as our formula predicts.

In the standard representations of classical series and the Severi series, we have $V_k = \Lambda^k V$ in low degrees. In the subexceptional series, in low degrees V_k is the primitive subspace for the symplectic form ω, i.e., $\Lambda^k V = V_k \oplus (\omega \wedge \Lambda^{k-2}V)$. For the exceptional series, at least in low degrees, $\mathfrak{g}_k$ is the primitive part of $\Lambda^k\mathfrak{g}$ For example, $\Lambda^2\mathfrak{g} = \mathfrak{g}\oplus\mathfrak{g}_2$, but the inclusion $\mathfrak{g}\to\Lambda^2\mathfrak{g}$ is just the Lie bracket, so the only primitive piece is $\mathfrak{g}_2$.

Let W_k denote the Casimir eigenspace of $\Lambda^k V$ of maximal eigenvalue. The discussion of [12] implies that W_k is decomposably generated, i.e., its highest weight vectors are all of the form $v_1\wedge\cdots\wedge v_k$ for some weight vectors $v_1,\dots,v_k$ of V. (Kostant only considered the case where $V = \mathfrak{g}$ is the adjoint representation, but his arguments apply to any irreducible module.) Note that the set of vectors $v_1, ..., v_k$ is B-stable and conversely a B-stable set of vectors wedged together furnishes a highest weight vector. Here B denotes the Borel compatible with our choices.

We will call a B-stable set of weight vectors *complete*. We will also call the corresponding set of weights complete. A subset S of the weights of V is complete if and only if for all $\mu\in S$ and each β that is a sum of positive roots, if $\mu+\beta$ is a weight of V, then $\mu+\beta\in S$.

Thus the problem of characterizing W_k is to characterize which complete subsets of weights (possibly with multiplicities, bounded by their multiplicities in V) determine a maximal Casimir eigenvalue.

Let H_j be an orthonormal basis of the Cartan subalgebra of $\mathfrak{g}$, and X_β a generator of the root space $\mathfrak{g}_\beta$. Let Θ denote the Casimir operator. We have (see [16])

$$\begin{aligned}\Theta(v_1\wedge\cdots\wedge v_k) &= \sum_i H_iH_i(v_1\wedge\cdots\wedge v_k) + \sum_{\beta\in\Delta} X_\beta X_{-\beta}(v_1\wedge\cdots\wedge v_k)\\ &= k\theta_V v_1\wedge\cdots\wedge v_k + 2\sum_l\sum_{i<j} v_1\wedge\cdots\wedge H_lv_i\wedge\cdots\wedge H_lv_j\wedge\cdots\wedge v_k\\ &\quad + \sum_{\beta\in\Delta}\sum_{i\neq j}\frac{4}{(X_\beta,X_{-\beta})} v_1\wedge\cdots\wedge X_\beta v_i\wedge\cdots\wedge X_{-\beta}v_j\wedge\cdots\wedge v_k.\end{aligned}$$

In order to state the main result of this section, we define the *diameter* of a subset S of the weights of V to be the minimal number δ such that $\|\mu-\mu'\|^2 \leq \delta(\alpha,\alpha)$ for all $\mu,\mu' \in S$. The diameter of V is obtained for $\mu=\lambda$ and $\mu'=w_0(\lambda)$, where w_0 denotes the longest element of the Weyl group. Thus, we easily compute that $\delta=i$ for the i-th fundamental representation of A_l, $\delta=2$ for the natural representations of C_l, $\delta=l$ for the spin representation of B_l, $\delta=[l/2]$ for a spin representation of D_l, $\delta=2$ for the minuscule representation of E_6, and $\delta=3$ for that of E_7. Note that when $\delta=2$, any decomposably generated component of $\Lambda^k V$ has maximal Casimir eigenvalue.

Proposition 9.1. *Let V be a minuscule representation. Then the irreducible components of $\Lambda^k V$ have Casimir eigenvalue less than or equal to θ_{V_k}. Those with Casimir eigenvalue equal to θ_{V_k} are in correspondance with complete cardinality k subsets S of the set of weights of V of diameter at most 2. In the case of the minuscule representation of B_l, we require additionally that the difference between two elements of S cannot be a root strictly longer than α_l.*

Proof. Let U denote a component of $\Lambda^k V$ of maximal Casimir eigenvalue, let $v_1\wedge\cdots\wedge v_k$ be a highest weight vector. Let μ_i denote the weight of v_i, and suppose that V is endowed with an invariant Hermitian product $\langle\ ,\ \rangle$, such that the v_i are part of a unitary basis. Then the eigenvalue of the Casimir operator on U is

$$\theta_U=\langle\Theta(v_1\wedge\cdots\wedge v_k),v_1\wedge\cdots\wedge v_k\rangle=k\theta_V+\sum_{i\neq j}(\mu_i,\mu_j)+\sum_{\beta\in\Delta}\sum_{i\neq j}\frac{\langle X_\beta v_i\wedge X_{-\beta}v_j,v_i\wedge v_j\rangle}{(X_\beta,X_{-\beta})}.$$

Since weight vectors of distinct weights are orthogonal, $\langle X_\beta v_i\wedge X_{-\beta}v_j,v_i\wedge v_j\rangle$ can be nonzero only if $\mu_i=\mu_j-\beta$ and there exist scalars s and t such that $X_\beta v_i=sv_j$ and $X_{-\beta}v_j=tv_i$. Assuming this, we compute

$$stv_i=sX_{-\beta}v_j=X_{-\beta}X_\beta v_i=[X_{-\beta},X_\beta]v_i+X_\beta X_{-\beta}v_i.$$

The latter term is zero since, V being minuscule, $X_{-\beta}^2v_j=0$ ([1], page 128). Moreover, we may suppose that $[X_{-\beta},X_\beta]=H_\beta$ is the coroot of β (see [1], page 82), and note that in this case $2(X_\beta,X_{-\beta})=-(H_\beta,H_\beta)$, and we get

$$\frac{\langle X_\beta v_i\wedge\cdots\wedge X_{-\beta}v_j,v_i\wedge v_j\rangle}{(X_\beta,X_{-\beta})}=\frac{2}{(H_\beta,H_\beta)}\mu_i(H_\beta)=(\mu_i,\beta)=(\mu_i,\mu_j-\mu_i).$$

Hence

$$\begin{aligned}\theta_U &= k\theta_V+\textstyle\sum_{i\neq j}(\mu_i,\mu_j)+\sum_{\mu_j-\mu_i\in\Delta}(\mu_i,\mu_j-\mu_i)\\ &= k\theta_V+\textstyle\sum_{\mu_j-\mu_i\notin\Delta}(\mu_i,\mu_j)+\sum_{\mu_j-\mu_i\in\Delta}(\mu_i,2\mu_j-\mu_i).\end{aligned}$$

Note that since V is minuscule, the weights μ_i are all conjugate under the Weyl group, in particular they have the same norm as λ. We need the following observation:

Lemma 9.2. *For $i\neq j$, either $\|\mu_i-\mu_j\|^2=(\alpha,\alpha)$ and $\mu_i-\mu_j\in\Delta$, or $\|\mu_i-\mu_j\|^2\geq 2(\alpha,\alpha)$ and $\mu_i-\mu_j\notin\Delta$.*

Proof. We may suppose that $\mu_i=\lambda$, since the Weyl group acts transitively on the weights of V. Since λ is the highest weight of V, we can write $\mu_j=\lambda-\sum_k n_k\alpha_k$ for some non-negative integers n_k, where the α_k are the simple roots. Since λ is fundamental it is orthogonal to every simple root except $\alpha=\alpha_l$, say, and we get $(\mu_i,\mu_j)=(\lambda,\lambda)-n_l(\alpha_l,\omega_l)=(\lambda,\lambda)-n_l(\alpha,\alpha)/2$, hence $\|\mu_i-\mu_j\|^2=n_l(\alpha,\alpha)$.

Suppose that $n_l=1$. The highest weight of V after λ is $\lambda-\alpha$. Since every nonzero weight of V is obtained by a sequence of simple reflections in λ, there is a sequence ν_i, $1\leq i\leq k+1$ of weights of V such that $\nu_0=\lambda$, $\nu_1=s_\alpha(\lambda)=\lambda-\alpha$, $\nu_t=\mu_j$ for some t and $\nu_{k+1}=s_{\beta_k}(\nu_k)$ for some simple root β_k, which is different from α if $k\neq 0$ because $n_l=1$. But then $s_{\beta_k}(\lambda-\nu_k)=\lambda-\nu_{k+1}$, thus $\lambda-\nu_k$ is a root if and only if $\lambda-\nu_{k+1}$ is also a root. Since $\lambda-\nu_1=\alpha$ is indeed a root, we conclude that $\mu_i-\mu_j=\lambda-\nu_t$ is a root. This argument is reversible, proving the lemma if we remember the formula $\|\mu_i-\mu_j\|^2=n_l(\alpha,\alpha)$. ∎

To conclude the proof of the proposition, we just need, for each pair μ_i, μ_j, to choose an element w of the Weyl group such that $w(\mu_i) = \lambda$, and define the integer $n_{i,j}$ to be the coefficient of $\lambda - w(\mu_j)$ on the simple root α. Then $(\mu_i, \mu_j) = (\lambda, \lambda) - n_{i,j}(\alpha, \alpha)/2$ and we get the formula

$$\theta_U = k\theta_V + \sum_{\mu_j - \mu_i \notin \Delta} ((\lambda, \lambda) - n_{i,j}(\alpha, \alpha)/2) + \sum_{\mu_j - \mu_i \in \Delta} ((\lambda, \lambda) - n_{i,j}(\alpha, \alpha)).$$

The $n_{i,j}$ are all positive, and they are at least equal to two in the first sum. We conclude that $\langle \Theta(v_1 \wedge \cdots \wedge v_k), v_1 \wedge \cdots \wedge v_k \rangle$ will be maximal when $n_{i,j}$ is always equal to two in the first sum, meaning that two weights whose difference is not a root have the square of their distance equal to $2(\alpha, \alpha)$, and always equal to one in the second sum (which means that their difference is a root which is not longer than α). Then we get $\theta_U = k\theta_V + k(k-1)((\lambda, \lambda) - (\alpha, \alpha))$, and the proposition is proved. □

In general, it is clear from the Proposition that V_k is nonzero when k is not too big, but the maximal integer k_0 for which this is true is not so easy to compute. At least can we say that k_0 can be quite large. Indeed, for the i-th fundamental representation of A_l, the set of weights $\mu_{j,k} = \omega_i - \varepsilon_j + \varepsilon_k$, where $1 \leq j \leq i$ and $i < k \leq l+1$, form, with ω_i, a set of weights with the required properties, so that $k_0 > i(l+1-i)$. We suspect that $k_0 = i(l+1-i)+1$ in that case but we have not proved it. Note also that the number of irreducible components in V_k can be arbitrary large, as easily follows from Proposition 9.1.

A nice consequence of the fact that Proposition 9.1 above holds for the fundamental representations of A_l is that we can extend its validity as follows:

Proposition 9.3. *Let V be a fundamental representation of the simple Lie algebra $\mathfrak{g}$. Suppose that the corresponding node of the Dynkin diagram of $\mathfrak{g}$ is on an A_l-chain in $D(\mathfrak{g})$, at distance at least k_1 from an extremity of the diagram. Then for $k \leq k_1 + 2$, θ_{V_k} is the largest Casimir eigenvalue of $\Lambda^k V$, and the irreducible components of V_k can be described in exactly the same way as in the preceeding proposition.*

Proof. An irreducible component of $\Lambda^k V$ with maximal Casimir eigenvalue is decomposably generated, hence generated by the wedge product of weight vectors whose set of weights form a complete subset of the set of weights of V. Moreover, there are at most k distinct weights in this set (possibly less if V has weights with multiplicity greater than one). But for $k \leq k_1 + 2$, every weight of a complete k-set of weights of V is of the form $\lambda - \theta$, where θ is a sum of simple roots corresponding to nodes on the A_l-chain only, and such that $\lambda - \theta$ is also a weight of the corresponding fundamental representation of A_l. Indeed, we know that the weights of V are the weights of the convex hull of the translates of λ by the Weyl group, which are congruent to λ modulo the root lattice. We can obtain the translates of λ by applying successively the simple reflections of the Weyl group so that the distance to λ increases (if we measure that distance by the sum of the coefficients of the difference, expressed in terms of simple roots). At the beginning of this process, the simple reflections involved are those associated to nodes of the A_l-chain only, and the weights one obtains are formally the same as for the corresponding fundamental representation of A_l. More precisely, this is the case until we do not apply more than $k_1 + 1$ simple reflections. Moreover, we obtain no new weight by considering the convex hull of those, and we conclude that the weights of V, at a distance at most $k_1 + 1$ from λ, are formally the same as those of the corresponding representation of A_l, with the same multiplicity, one. The scalar products of two such weights can be computed in terms of (λ, λ) and (α, α), and a part of the Cartan matrix which only involves the A_l-chain, thus the computation is formally the same as in the weight lattice of A_l, and therefore the computation of the Casimir eigenvalue of a k-set of weights will again be formally identical. Finally, since we only need to consider the same k-sets of weights and the same Casimir eigenvalues as in the A_l-case, the conclusions of the preceeding proposition for the fundamental representations of A_l directly apply to V, and this concludes the proof. □

Returning to geometry, we arrive at the following statement, which could also be deduced from [15].

Corollary 9.4. *Let V be a fundamental representation of $\mathfrak{g}$, and let $X \subset \mathbb{P}V$ be the closed orbit. If the Fano variety $\mathbb{F}_k(X)$ of $\mathbb{P}^{k-1}$'s in X is nonempty, then its linear span $\langle \mathbb{F}_k(X) \rangle$ is contained in V_k and in this case V_k is the highest Casimir eigenspace.*

Proof. We know from [14] that the closed orbits in $\mathbb{F}_k(X)$ are in correspondance with marked subdiagrams of type $(\mathfrak{a}_{k-1}, \omega_1)$. By the proposition above, such subdiagrams detect components of $V_k \subset \Lambda^k V$. □

We can be more precise for $k = 2$:

Corollary 9.5. *Let V be a fundamental representation of $\mathfrak{g}$. Then V_2 is irreducible and coincides with $\langle \mathbb{F}_2(X) \rangle$, the linear span in $\Lambda^2 V$ of the set of lines contained in the closed orbit of $\mathbb{P}V$.*

References

[1] Bourbaki N.: Groupes et algèbres de Lie, Hermann, Paris 1968.

[2] Brion M.: *Invariants d'un sous-groupe unipotent maximal d'un groupe semi-simple*, Ann. Inst. Fourier **33**, 1-27 (1983).

[3] Brion M.: *Stable properties of plethysm : On two conjectures of Foulkes*, Manuscripta Math. **80**, 347-371 (1993).

[4] Deligne P.: *La série exceptionnelle des groupes de Lie*, C.R.A.S **322**, 321-326 (1996).

[5] Deligne P., de Man R.: *The exceptional series of Lie groups*, C.R.A.S **323**, 577-582 (1996).

[6] Deligne P., Gross, B.: *The exceptional series and its descendents*, to appear in C.R.A.S.

[7] Dixmier J.: *Quelques aspects de la théorie des invariants*, Gaz.Math. **43**, 39-64 (1990).

[8] Cohen A.M., van Leeuwen M.A., Lisser B.: LiE, a package for Lie group computations, CAN, Amsterdam, 1992.

[9] Cohen A.M., de Man R.: *Computational evidence for Deligne's conjecture regarding exceptional Lie groups*, C.R.A.S **322**, 427-432 (1996).

[10] Freudenthal H.: *Lie groups in the foundations of geometry*, Adv. Math. **1**, 145-190 (1964).

[11] Gross B., Kostant B., Ramond P., Sternberg S.: *The Weyl character formula, the half-spin representations, and equal rank subgroups*, Proc. Natl. Acad. Sci. USA **95** (1998), no. 15, 8441–8442.

[12] Kostant B. : *Eigenvalues of a Laplacian and commutative Lie subalgebras*, Topology **3**, 147-159 (1965).

[13] Littlewood : The Theory of Group Characters and Matrix Representations of Groups, Oxford University Press, New York 1940.

[14] Landsberg J.M., Manivel L.: *On the projective geometry of homogeneous spaces*, preprint math.AG/9810140, to appear in Comm. Math. Helv.

[15] Landsberg J.M., Manivel L.: *The projective geometry of Freudenthal's magic square*, J. Algebra **239**, 477-512 (2001).

[16] Landsberg J.M., Manivel L.: *Classification of complex simple Lie algebras via projective geometry*, Selecta Mathematica **8** (2002) 137-159.

[17] Landsberg J.M., Manivel L.: *Triality, exceptional Lie algebras and Deligne dimension formulas*, Advances in Math. **171** (2002) 59-85

[18] Manivel L.: Symmetric functions, Schubert polynomials and Degeneracy loci, SMF/AMS **6** (2001).

[19] Manivel L.: *Application de Gauss et pléthysme*, Ann. de l'Institut Fourier **47**, 715-773 (1997).

[20] Onischik A.L., Vinberg E.B.: Lie groups and Lie algebras III, Encyclopaedia of Mathematical Sciences 41, Springer-Verlag, Berlin, 1994.

[21] Tits J., *Les groupes de Lie exceptionnels et leur interprétation géométrique*, Bull. Soc. Math. Belg. **8**, 48-81 (1956).

[22] Vogel P.: *Algebraic structures on modules of diagrams*, preprint 1995.

[23] Vogel P.: *The universal Lie algebra*, preprint 1999.

[24] Wang M., Ziller, W. *On normal homogeneous Einstein manifolds*, Ann. scient. Ec. Norm. Sup. **18**, 563-633 (1985).