RICCI SOLITON SOLVMANIFOLDS

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Abstract. All known examples of nontrivial homogeneous Ricci solitons are left-invariant metrics on simply connected solvable Lie groups whose Ricci operator is a multiple of the identity modulo derivations (called *solsolitons*, and *nilsolitons* in the nilpotent case). The tools from geometric invariant theory used to study Einstein solvmanifolds, turned out to be useful in the study of solsolitons as well. We prove that, up to isometry, any solsoliton can be obtained via a very simple construction from a nilsoliton N together with any abelian Lie algebra of symmetric derivations of its metric Lie algebra $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$. The following uniqueness result is also obtained: a given solvable Lie group can admit at most one solsoliton up to isometry and scaling. As an application, solsolitons of dimension ≤ 4 are classified.

1. Introduction

It has recently been proved by Lott in [Lt] that if $g(t)$ is a Ricci flow solution on a 3-dimensional compact manifold M, with sectional curvatures that are $\mathrm{O}(t^{-1})$ and diameter that is $\mathrm{O}(\sqrt{t})$, then the pullback Ricci flow solution on the simply connected cover $\tilde{M}$ approaches a homogeneous expanding Ricci soliton. Among many others, this is certainly a good motivation to study Ricci solitons in the homogeneous case. A natural question we are particularly interested in is how much stronger is, for homogeneous metrics, the Einstein condition compared with the condition of being a Ricci soliton.

From results due to Ivey, Naber, Perelman and Petersen-Wylie, it follows that any nontrivial homogeneous Ricci soliton must be noncompact, expanding and non-gradient (see Section 2). Up to now, all known examples are isometric to a left-invariant metric g on a simply connected Lie group G, which when identified with an inner product on the Lie algebra $\mathfrak{g}$ of G satisfies

$$\mathrm{Ric}(g) = cI + D, \qquad \text{for some} \quad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mathfrak{g}), \tag{1}$$

where $\mathrm{Ric}(g)$ is the Ricci operator of g. On the other hand the converse is true: any left invariant metric which satisfies (1) is automatically a Ricci soliton. For G nilpotent, these metrics are called *nilsolitons* and have been extensively studied in the last decade, mainly because of the strong connection with Einstein solvmanifolds (see the survey [L4]). Examples with G solvable but non-nilpotent have explicitly appeared in [BD] ($\dim G = 3$) and [IJL] ($\dim G = 4$).

The aim of this paper is to study the structure of solvable Lie groups admitting a left invariant metric for which (1) holds; these metrics will be called *solsolitons* from now on. The tools from geometric invariant theory used in [L3] to prove that

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any Einstein solvmanifold is standard (see Section 3), turned out to be useful in the study of solsolitons as well.

It is really easy to get examples of solsolitons from only a nilsoliton and an abelian Lie algebra of symmetric derivations of its metric Lie algebra. Our main result asserts that any solsoliton can actually be obtained (up to isometry) by such a simple construction (see Section 4). In particular, any solsoliton is standard, and if not Einstein, it admits a one-dimensional extension which is an Einstein solvmanifold, just as for nilsolitons. We are therefore showing that most of the structural results proved for Einstein solvmanifolds in [H, L3] are still valid for solsolitons. We also obtain a uniqueness result that generalizes the known results for Einstein solvmanifolds and nilsolitons: among all left invariant metrics on a given solvable Lie group, there is at most one solsoliton up to isometry and scaling (see Section 5). All this is used in Section 6 to classify solvable Lie groups admitting solsolitons in dimension ≤ 4.

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2. Homogeneous Ricci solitons

A complete Riemannian metric g on a differentiable manifold M is said to be a *Ricci soliton* if its Ricci tensor satisfies

$$\operatorname{Ric}(g) = cg + \mathrm{L}_X\, g, \qquad \text{for some} \quad c \in \mathbb{R}, \quad X \in \chi(M) \text{ complete}, \tag{2}$$

where $\chi(M)$ denotes the space of all differentiable vector fields on M and L_X the Lie derivative (see [C] for further information on Ricci solitons). Recall that if θ_t is the one-parameter group associated to X then $\mathrm{L}_X\, g = \frac{\mathrm{d}}{\mathrm{d}t}|_0 \theta_t^* g$, and hence the Ricci soliton condition may be rephrased as follows: $\operatorname{Ric}(g)$ is tangent at g to the space of all metrics which are *homothetic* (i.e. isometric up to scaling) to g. If in addition X is the gradient field of a smooth function $f : M \longrightarrow \mathbb{R}$, then (2) becomes $\operatorname{Ric}(g) = cg + 2\operatorname{Hess}(f)$ and g is called a *gradient* Ricci soliton. In any case, we see that Ricci solitons are very natural generalizations of *Einstein* metrics (i.e. $\operatorname{Ric}(g) = cg$).

The main significance, though, of the concept is that g is a Ricci soliton if and only if the curve of metrics

$$g(t) = (-2ct + 1)\varphi_t^* g, \tag{3}$$

is a solution to the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\operatorname{Ric}(g(t)), \tag{4}$$

for some one-parameter group φ_t of diffeomorphisms of M. In other words, the Ricci flow starting at g stays forever in the space of metrics which are homothetic to g; it is unable to 'improve' g. According to (3), Ricci solitons are called *expanding*, *steady*, or *shrinking* depending on whether $c < 0$, $c = 0$, or $c > 0$.

We are interested in the following question:

> Which homogeneous manifolds G/K admit a G-invariant Ricci soliton metric?

Unfortunately, even Einstein homogeneous manifolds are still not well understood (see [BWZ] and [L4] for the compact and noncompact cases, respectively). But let us first review to what extent the Ricci soliton condition is weaker than the Einstein condition for homogeneous manifolds.

Let (M, g) be a Ricci soliton and let us assume it is homogeneous, i.e. its isometry group acts transitively on M. In particular, g has bounded curvature. If g is steady, it is easy to see from the ODE that the scalar curvature $\mathrm{sc}(g(t))$ satisfies that $\mathrm{Ric}(g) = 0$, and consequently g must be flat (see [AK]). In the shrinking case, it follows from [N, Theorem 1.2] that g is of gradient type, and it is proved in [PW] that any homogeneous gradient Ricci soliton is isometric to a quotient of $N \times \mathbb{R}^k$, where N is some homogeneous Einstein manifold with positive scalar curvature and so compact and with $\Pi_1(N)$ finite (see also [W]). Finally, if g is expanding then M must be noncompact (see [I]). Recall also that it follows from [P] that on a compact manifold all Ricci solitons are of gradient type.

We conclude that,

> the noncompact expanding case is the only one allowing nontrivial homogeneous Ricci solitons, and furthermore, they can not be of gradient type.

All known examples so far of nontrivial homogeneous Ricci solitons are isometric to a left-invariant metric g on a simply connected Lie group G (see Remark 4.12 concerning non-simply connected Lie groups), and can be obtained in the following way. Assume that g, which is identified with an inner product on the Lie algebra $\mathfrak{g}$ of G, satisfies

$$\mathrm{Ric}(g) = cI + D, \qquad \text{for some} \quad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mathfrak{g}), \tag{5}$$

where $\mathrm{Ric}(g)$ also denotes the *Ricci operator* of g (i.e. $\mathrm{Ric}(g) = g(\mathrm{Ric}(g)\cdot, \cdot)$). If $X_D \in \chi(G)$ is defined by

$$X_D(p) = \tfrac{\mathrm{d}}{\mathrm{d}t}|_0 \varphi_t(p), \qquad p \in G, \tag{6}$$

where $\varphi_t \in \mathrm{Aut}(G)$ is the unique automorphism such that $\mathrm{d}\,\varphi_t|_e = e^{tD} \in \mathrm{Aut}(\mathfrak{g})$ (the existence of φ_t follows from G being simply connected), then

$$\mathrm{L}_{X_D}\, g = \tfrac{\mathrm{d}}{\mathrm{d}t}|_0 \varphi_t^* g = \tfrac{\mathrm{d}}{\mathrm{d}t}|_0 g(e^{-tD}\cdot, e^{-tD}\cdot) = -2g(D\cdot, \cdot).$$

This implies that the Ricci tensor equals $\mathrm{Ric}(g) = cg - \frac{1}{2}\mathrm{L}_{X_D}\, g$, and henceforth g is a Ricci soliton. These vector fields X_D's can be viewed as a generalization to any Lie group of the so called *linear vector fields* on $\mathbb{R}^n$ (i.e. $X(p) = Ap$, $A \in \mathfrak{gl}_n(\mathbb{R})$), and they play a nice and important role in control theory (see [AT]). We notice that for the Gaussian soliton on $\mathbb{R}^n$ one uses the linear vector field $X(p) = cp$.

Condition (5) nicely combines the geometric and algebraic features of a left-invariant metric on a Lie group, providing a neat way to find examples of homogeneous Ricci solitons. These examples first appeared in [L1] (G nilpotent), [BD] (G solvable, $\dim G = 3$) and [IJL] (G solvable, $\dim G = 4$).

Remark 2.1. It is an open question whether any Ricci soliton left invariant metric will satisfy (5), and concerning existence, we do not know of any non-solvable Lie group admitting a nontrivial Ricci soliton.

In the case when G is nilpotent, metrics for which (5) holds are called *nilsolitons* and are known to satisfy the following properties (see the recent survey [L4] for further information on nilsolitons):

(a) Any left invariant Ricci soliton on G is a nilsoliton.
(b) A given G can admit at most one nilsoliton up to isometry and scaling among all its left-invariant metrics.
(c) Nilsolitons are also characterized by the following extremal property:

$$||\operatorname{Ric}(g)|| = \min \{||\operatorname{Ric}(g')|| : g' \text{ left-invariant on } G,\ \operatorname{sc}(g') = \operatorname{sc}(g)\}.$$

Furthermore, they are the critical points of the functional square norm of the Ricci tensor on the space of all nilmanifolds of a given dimension and scalar curvature.
(d) Nilsolitons are precisely the nilpotent parts of Einstein solvmanifolds.

Nevertheless, the existence, structural and classification problems on nilsolitons seem to be far from being satisfactory solved, if at all possible.

Definition 2.2. A left-invariant metric g on a simply connected solvable Lie group is called a *solsoliton* if the corresponding Ricci operator satisfies (5).

The name is inspired by the 3-dimensional homogeneous geometry Sol from the Geometrization Conjecture. It is natural to ask, in the case of solsolitons, for properties analogous to (a)-(d) above. We will consider properties (b) and (d) here and leave (a) and (c) for a forthcoming paper. Concerning property (a), it is worth mentioning that any left invariant Ricci soliton on a *completely solvable* Lie group (i.e. the eigenvalues of any $\operatorname{ad} X$ are all real) is necessarily a solsoliton. This follows analogously to the proof of [L1, Proposition 1.1] by using that two left-invariant metrics on one of these groups are isometric if and only if there is an isomorphism which is an isometry between them (see [A]).

3. Variety of nilpotent Lie algebras

Let G be a real reductive group acting linearly on a finite dimensional real vector space V via $(g, v) \mapsto g.v$, $g \in G, v \in V$. The precise setting is the one in [RS]. We also refer to [EJ, HSS], where many results from geometric invariant theory are adapted and proved over $\mathbb{R}$. The Lie algebra $\mathfrak{g}$ of G also acts linearly on V by the derivative of the above action, which will be denoted by $(\alpha, v) \mapsto \pi(\alpha)v$, $\alpha \in \mathfrak{g}$, $v \in V$. We consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K of G. Endow V with a from now on fixed K-invariant inner product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{p}$ acts by symmetric operators, and endow $\mathfrak{p}$ with an $\operatorname{Ad}(K)$-invariant inner product, which will be also denoted by $\langle \cdot, \cdot \rangle$.

The function $m : V \smallsetminus \{0\} \longrightarrow \mathfrak{p}$ implicitly defined by

$$\langle m(v), \alpha \rangle = \tfrac{1}{||v||^2} \langle \pi(\alpha)v, v \rangle, \qquad \forall \alpha \in \mathfrak{p},\ v \in V, \tag{7}$$

is called the *moment map* for the representation V of G. It is easy to see that $m(cv) = m(v)$ for any nonzero $c \in \mathbb{R}$ and m is K-equivariant: $m(k.v) = \operatorname{Ad}(k)m(v)$ for all $k \in K$. In the complex case (i.e. for a complex representation of a complex reductive algebraic group), under the natural identifications $\mathfrak{p} = \mathfrak{p}^* = (\mathtt{i}\mathfrak{k})^* = \mathfrak{k}^*$, the function m is precisely the moment map from symplectic geometry, corresponding to the Hamiltonian action of K on the symplectic manifold $\mathbb{P}V$ (see [MFK, Chapter 8] for further information).

The functional square norm of the moment map,

$$F : V \smallsetminus \{0\} \mapsto \mathbb{R}, \qquad F(v) = ||m(v)||^2, \tag{8}$$

is scaling invariant, so it can actually be viewed as a function on any sphere of V or on the projective space $\mathbb{P}V$. If $\mathcal{C}$ denotes the set of critical points of $F : V \setminus \{0\} \longrightarrow \mathbb{R}$, then it is proved in [M] (see [K, Ns] for the complex case) that $v \in \mathcal{C}$ (or equivalently, v is a minimum for $F|_{G.v}$) if and only if v is an eigenvector of $\pi(m(v))$, and in that case, the following uniqueness result holds: $G.v \cap \mathcal{C} = K.v$ (up to scaling). This was previously proved in [RS] (see [KN] for the complex case) for the zeroes of m (or equivalently, minimal vectors), which can only appear inside closed orbits (see [L4, Section 11] for a more complete overview).

Let us consider the space of all skew-symmetric algebras of dimension n, which is parameterized by the vector space

$$V := \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n = \{\mu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n : \mu \text{ bilinear and skew-symmetric}\}.$$

Then

$$\mathcal{N} = \{\mu \in V : \mu \text{ satisfies Jacobi and is nilpotent}\}$$

is an algebraic subset of V as the Jacobi identity and the nilpotency condition can both be written as zeroes of polynomial functions. $\mathcal{N}$ is often called the *variety of nilpotent Lie algebras* (of dimension n). There is a natural linear action of $G = \mathrm{GL}_n(\mathbb{R})$ on V given by

$$g.\mu(X,Y) = g\mu(g^{-1}X, g^{-1}Y), \qquad X,Y \in \mathbb{R}^n, \quad g \in \mathrm{GL}_n(\mathbb{R}), \quad \mu \in V. \tag{9}$$

Recall that $\mathcal{N}$ is $\mathrm{GL}_n(\mathbb{R})$-invariant and the Lie algebra isomorphism classes are precisely the $\mathrm{GL}_n(\mathbb{R})$-orbits. The action of $\mathfrak{gl}_n(\mathbb{R})$ on V obtained by differentiation of (9) is given by

$$\pi(\alpha)\mu = \alpha\mu(\cdot,\cdot) - \mu(\alpha\cdot,\cdot) - \mu(\cdot,\alpha\cdot), \qquad \alpha \in \mathfrak{gl}_n(\mathbb{R}), \quad \mu \in V. \tag{10}$$

We note that $\pi(\alpha)\mu = 0$ if and only if $\alpha \in \mathrm{Der}(\mu)$, the Lie algebra of derivations of the algebra μ. The canonical inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ determines an $\mathrm{O}(n)$-invariant inner product on V, also denoted by $\langle\cdot,\cdot\rangle$, as follows:

$$\langle\mu,\lambda\rangle = \sum \langle\mu(e_i,e_j),\lambda(e_i,e_j)\rangle = \sum \langle\mu(e_i,e_j),e_k\rangle\langle\lambda(e_i,e_j),e_k\rangle, \tag{11}$$

and also the standard $\mathrm{Ad}(\mathrm{O}(n))$-invariant inner product on $\mathfrak{gl}_n(\mathbb{R})$ given by

$$\langle\alpha,\beta\rangle = \operatorname{tr}\alpha\beta^{\mathrm{t}} = \sum\langle\alpha e_i,\beta e_i\rangle = \sum\langle\alpha e_i,e_j\rangle\langle\beta e_i,e_j\rangle, \qquad \alpha,\beta \in \mathfrak{gl}_n(\mathbb{R}), \tag{12}$$

where $\{e_1,...,e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. We have made several abuses of notation concerning inner products. Recall that $\langle\cdot,\cdot\rangle$ has been used to denote inner products on $\mathbb{R}^n$, V and $\mathfrak{gl}_n(\mathbb{R})$ indistinctly. We note that $\pi(\alpha)^t = \pi(\alpha^t)$ and $(\operatorname{ad}\alpha)^t = \operatorname{ad}\alpha^t$ for any $\alpha \in \mathfrak{gl}_n(\mathbb{R})$, due to the choice of these canonical inner products everywhere.

We use $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{sym}(n)$ as a Cartan decomposition for the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of $\mathrm{GL}_n(\mathbb{R})$, where $\mathfrak{so}(n)$ and $\mathrm{sym}(n)$ denote the subspaces of skew-symmetric and symmetric matrices, respectively. It is proved in [L2, Proposition 3.5] that if ad_μ denotes left multiplication for the algebra μ then the moment map $m : V \setminus \{0\} \longrightarrow \mathrm{sym}(n)$ for the action (9) is given by

$$m(\mu) = \tfrac{1}{||\mu||^2}\left(-2\sum(\operatorname{ad}_\mu e_i)^t \operatorname{ad}_\mu e_i + \sum \operatorname{ad}_\mu e_i(\operatorname{ad}_\mu e_i)^t\right),$$

or equivalently, for all $X \in \mathbb{R}^n$,

$$\langle m(\mu)X,X\rangle = \tfrac{1}{||\mu||^2}\left(-2\sum\langle\mu(X,e_i),e_j\rangle^2 + \sum\langle\mu(e_i,e_j),X\rangle^2\right). \tag{13}$$

Let $\mathfrak{t}$ denote the set of all diagonal $n \times n$ matrices. If $\{e_1', ..., e_n'\}$ is the basis of $(\mathbb{R}^n)^*$ dual to the canonical basis $\{e_1, ..., e_n\}$ then

$$\{v_{ijk} = (e_i' \wedge e_j') \otimes e_k : 1 \leq i < j \leq n,\ 1 \leq k \leq n\}$$

is a basis of weight vectors of V for the action (9), where v_{ijk} is actually the bilinear form on $\mathbb{R}^n$ defined by $v_{ijk}(e_i, e_j) = -v_{ijk}(e_j, e_i) = e_k$ and zero otherwise. The corresponding weights $\alpha_{ij}^k \in \mathfrak{t}$, $i < j$, are given by

$$\pi(\alpha)v_{ijk} = (a_k - a_i - a_j)v_{ijk} = \langle \alpha, \alpha_{ij}^k \rangle v_{ijk}, \quad \forall \alpha = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathfrak{t}, \tag{14}$$

where $\alpha_{ij}^k = E_{kk} - E_{ii} - E_{jj}$ and $\langle \cdot, \cdot \rangle$ is the inner product defined in (12). As usual E_{rs} denotes the matrix with 1 at entry rs and 0 elsewhere.

From now on, we will always denote by μ_{ij}^k the structure constants of a vector $\mu \in V$ with respect to the basis $\{v_{ijk}\}$:

$$\mu = \sum \mu_{ij}^k v_{ijk}, \qquad \mu_{ij}^k \in \mathbb{R}, \qquad \text{i.e.} \quad \mu(e_i, e_j) = \sum_{k=1}^n \mu_{ij}^k e_k, \quad i < j.$$

Each nonzero $\mu \in V$ uniquely determines an element $\beta_\mu \in \mathfrak{t}$ given by

$$\beta_\mu := \operatorname{mcc}\left\{\alpha_{ij}^k : \mu_{ij}^k \neq 0\right\},$$

where $\operatorname{mcc}(X)$ denotes the unique element of minimal norm in the convex hull $\operatorname{CH}(X)$ of a subset $X \subset \mathfrak{t}$. We note that β_μ is always nonzero since $\operatorname{tr} \alpha_{ij}^k = -1$ for all $i < j$ and consequently $\operatorname{tr} \beta_\mu = -1$.

Let $\mathfrak{t}^+$ denote the Weyl chamber of $\mathfrak{gl}_n(\mathbb{R})$ given by

$$\mathfrak{t}^+ = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathfrak{t} : a_1 \leq ... \leq a_n \right\}. \tag{15}$$

In [L3], a $\mathrm{GL}_n(\mathbb{R})$-invariant stratification for $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ has been defined by adapting to this context the construction given in [K, Section 12] for reductive group representations over an algebraically closed field. We summarize in the following theorem the main properties of the stratification, which will become our main tool in the study of the structure of solsolitons in next section (see [L4, Section 7] for a detailed overview).

Theorem 3.1. [L3, LW] *There exists a finite subset $\mathcal{B} \subset \mathfrak{t}^+$, and for each $\beta \in \mathcal{B}$ a $\mathrm{GL}_n(\mathbb{R})$-invariant subset $\mathcal{S}_\beta \subset V$ (a stratum) such that*

$$V \setminus \{0\} = \bigcup_{\beta \in \mathcal{B}} \mathcal{S}_\beta \qquad \textit{(disjoint union)},$$

and $\operatorname{tr} \beta = -1$ for any $\beta \in \mathcal{B}$. For $\mu \in \mathcal{S}_\beta$ we have that

$$\beta + ||\beta||^2 I \quad \textit{is positive definite for all } \beta \in \mathcal{B} \textit{ such that } \mathcal{S}_\beta \cap \mathcal{N} \neq \emptyset, \textit{ and} \tag{16}$$

$$||\beta|| \leq ||m(\mu)|| \qquad \textit{(here, equality holds} \Leftrightarrow m(\mu) \textit{ is conjugate to } \beta). \tag{17}$$

If in addition, $\mu \in \mathcal{S}_\beta$ satisfies $\beta_\mu = \beta$ (or equivalently, $\min\left\{\langle \beta, \alpha_{ij}^k \rangle : \mu_{ij}^k \neq 0\right\} = ||\beta||^2$), which always holds for some $g.\mu$, $g \in \mathrm{O}(n)$, then

$$\langle [\beta, D], D \rangle \geq 0 \qquad \forall\, D \in \operatorname{Der}(\mu) \qquad \textit{(here, equality holds} \Leftrightarrow [\beta, D] = 0), \tag{18}$$

$$\operatorname{tr} \beta D = 0 \quad \forall\, D \in \operatorname{Der}(\mu), \ \textit{and} \tag{19}$$

(20) $\langle \pi\left(\beta+||\beta||^2 I\right)\mu,\mu\rangle \geq 0$ (*here, equality holds* $\Leftrightarrow \beta+||\beta||^2 I \in \mathrm{Der}(\mu)$).

This stratification is based on instability results and is strongly related to the moment map in many ways other than (17) (see [L4]).

4. Structure of solsolitons

Let S be a *solvmanifold*, that is, a simply connected solvable Lie group endowed with a left invariant Riemannian metric. S will be often identified with its metric Lie algebra $(\mathfrak{s},\langle\cdot,\cdot\rangle)$, where $\mathfrak{s}$ is the Lie algebra of S and $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathfrak{s}$ which determines the metric. We consider the orthogonal decomposition

$$\mathfrak{s}=\mathfrak{a}\oplus\mathfrak{n},$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$ (i.e. maximal nilpotent ideal). The *mean curvature vector* of S is the only element $H\in\mathfrak{a}$ such that $\langle H,A\rangle=\mathrm{tr}\,\mathrm{ad}\,A$ for any $A\in\mathfrak{a}$. If B denotes the symmetric map defined by the Killing form of $\mathfrak{s}$ relative to $\langle\cdot,\cdot\rangle$ (i.e. $\langle BX,X\rangle=\mathrm{tr}\,(\mathrm{ad}\,X)^2$ for all $X\in\mathfrak{s}$) then $B(\mathfrak{a})\subset\mathfrak{a}$ and $B|_{\mathfrak{n}}=0$. The Ricci operator Ric of S is given by (see for instance [B, 7.38]):

$$\mathrm{Ric}=R-\tfrac{1}{2}B-S(\mathrm{ad}\,H), \tag{21}$$

where $S(\mathrm{ad}\,H)=\frac{1}{2}(\mathrm{ad}\,H+(\mathrm{ad}\,H)^t)$ is the symmetric part of $\mathrm{ad}\,H$ and R is the symmetric operator defined by

$$\langle RX,X\rangle=-\tfrac{1}{2}\sum\langle[X,X_i],X_j\rangle^2+\tfrac{1}{4}\sum\langle[X_i,X_j],X\rangle^2,\qquad \forall X\in\mathfrak{s}, \tag{22}$$

where $\{X_i\}$ is any orthonormal basis of $(\mathfrak{s},\langle\cdot,\cdot\rangle)$.

It follows from (13) that this anonymous tensor R in the formula of the Ricci operator satisfies

$$m([\cdot,\cdot])=\tfrac{4}{||[\cdot,\cdot]||^2}R, \tag{23}$$

where $m:\Lambda^2\mathfrak{s}^*\otimes\mathfrak{s}\longrightarrow\mathrm{sym}(\mathfrak{s})$ is the moment map for the natural action of $\mathrm{GL}(\mathfrak{s})$ on $\Lambda^2\mathfrak{s}^*\otimes\mathfrak{s}$. In other words, R may be alternatively defined as follows:

$$\mathrm{tr}\,RE=\tfrac{1}{4}\langle\pi(E)[\cdot,\cdot],[\cdot,\cdot]\rangle,\qquad \forall E\in\mathrm{End}(\mathfrak{s}), \tag{24}$$

where we are considering the Lie bracket $[\cdot,\cdot]$ of $\mathfrak{s}$ as a vector in $\Lambda^2\mathfrak{s}^*\otimes\mathfrak{s}$, $\langle\cdot,\cdot\rangle$ is the inner product defined in (11) and π is the representation given in (10) (see the notation in Section 3 and replace $\mathbb{R}^n$ by $\mathfrak{s}$).

Remark 4.1. In particular, R is orthogonal to any derivation of $\mathfrak{s}$. It is easy to see that the same holds for B.

Remark 4.2. If $\mathfrak{s}$ is nilpotent then $\mathrm{Ric}=R$ and hence the scalar curvature is simply given by $\mathrm{sc}=\mathrm{tr}\,R=-\frac{1}{4}||[\cdot,\cdot]||^2$.

The following more explicit formula for the Ricci operator of S follows from a straightforward computation by using (21) and (22):

$$
\begin{aligned}
\langle \operatorname{Ric} A, A\rangle &= -\tfrac{1}{2}\sum ||[A,A_i]||^2 - \operatorname{tr} S(\operatorname{ad} A|_{\mathfrak{n}})^2,\\
\langle \operatorname{Ric} A, X\rangle &= -\tfrac{1}{2}\sum\langle [A,A_i],[X,A_i]\rangle - \tfrac{1}{2}\operatorname{tr}(\operatorname{ad} A|_{\mathfrak{n}})^t \operatorname{ad} X|_{\mathfrak{n}}\\
&\quad -\tfrac{1}{2}\langle [H,A],X\rangle, \qquad (25)\\
\langle \operatorname{Ric} X, X\rangle &= \tfrac{1}{4}\sum\langle [A_i,A_j],X\rangle^2 + \tfrac{1}{2}\sum\langle [\operatorname{ad} A_i|_{\mathfrak{n}}, (\operatorname{ad} A_i|_{\mathfrak{n}})^t](X),X\rangle\\
&\quad -\tfrac{1}{2}\sum\langle [X,X_i],X_j\rangle^2 + \tfrac{1}{4}\sum\langle [X_i,X_j],X\rangle^2 - \langle [H,X],X\rangle,
\end{aligned}
$$

for all $A \in \mathfrak{a}$ and $X \in \mathfrak{n}$, where $\{A_i\}$, $\{X_i\}$, are any orthonormal basis of $\mathfrak{a}$ and $\mathfrak{n}$, respectively, and $S(\operatorname{ad} A|_{\mathfrak{n}}) = \frac{1}{2}(\operatorname{ad} A|_{\mathfrak{n}} + (\operatorname{ad} A|_{\mathfrak{n}})^t)$. It is now clear from (25) that there is a substantial simplification of the expression of Ric under the assumptions $[\mathfrak{a},\mathfrak{a}] = 0$ and $\operatorname{ad} A$ symmetric for all $A \in \mathfrak{a}$. This gives rise to the following natural construction of solsolitons starting from a nilsoliton.

Proposition 4.3. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle_1)$ be a nilsoliton, say with Ricci operator $\operatorname{Ric}_1 = cI + D_1$, $c < 0$, $D_1 \in \operatorname{Der}(\mathfrak{n})$, and consider $\mathfrak{a}$ any abelian Lie algebra of symmetric derivations of $(\mathfrak{n}, \langle\cdot,\cdot\rangle_1)$. Then the solvmanifold S with Lie algebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ (semidirect product) and inner product given by*

$$
\langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}} = \langle\cdot,\cdot\rangle_1, \qquad \langle\mathfrak{a},\mathfrak{n}\rangle = 0, \qquad \langle A,A\rangle = -\tfrac{1}{c}\operatorname{tr} A^2 \qquad \forall A \in \mathfrak{a},
$$

is a solsoliton with $\operatorname{Ric} = cI + D$, where $D \in \operatorname{Der}(\mathfrak{s})$ is defined by $D|_{\mathfrak{a}} = 0$, $D|_{\mathfrak{n}} = D_1 - \operatorname{ad} H|_{\mathfrak{n}}$ and H is the mean curvature vector of S. Furthermore, S is Einstein if and only if $D_1 \in \mathfrak{a}$.

Remark 4.4. The aim of this section is to show that this very simple procedure actually yields all solsolitons up to isometry.

Remark 4.5. If $\mathfrak{n}$ is abelian then $\operatorname{Ric}_1 = 0$ and so we can take $D_1 = -cI$ for any $c < 0$.

Proof. It follows directly from the hypotheses and (25) that $\operatorname{Ric}|_{\mathfrak{a}} = cI$ and

$$
\operatorname{Ric}|_{\mathfrak{n}} = \operatorname{Ric}_1 - H = cI + D_1 - \operatorname{ad} H|_{\mathfrak{n}}.
$$

If $\mathfrak{n} = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_r$ is the orthogonal decomposition with $[\mathfrak{n},\mathfrak{n}] = \mathfrak{n}_2 \oplus ... \oplus \mathfrak{n}_r$, $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = \mathfrak{n}_3 \oplus ... \oplus \mathfrak{n}_r$, and so on, then for any $A \in \mathfrak{a}$, $\operatorname{ad} A|_{\mathfrak{n}} = A$ leaves the subspaces $\mathfrak{n}_i$ invariant as it is a symmetric derivation (it is actually enough for this that $(\operatorname{ad} A|_{\mathfrak{n}})^t$ be a derivation as well) and $\operatorname{ad} X(\mathfrak{n}_i) \subset \mathfrak{n}_{i+1} \oplus ... \oplus \mathfrak{n}_r$ for all i. This implies that $\operatorname{tr}(\operatorname{ad} A|_{\mathfrak{n}})^t \operatorname{ad} X|_{\mathfrak{n}} = 0$, and thus $\langle \operatorname{Ric}\mathfrak{a},\mathfrak{n}\rangle = 0$. It only remains to prove that $D \in \operatorname{Der}(\mathfrak{s})$, for which it is enough to show that $[\mathfrak{a}, D_1] = 0$, but this follows by using that any symmetric derivation of $(\mathfrak{n}, \langle\cdot,\cdot\rangle_1)$ (and more generally any derivation whose transpose is also a derivation) commutes with Ric_1 (see [H, Lemma 2.2]).

Let us now prove the Einstein assertion. If S is Einstein then $D_1 = \operatorname{ad} H|_{\mathfrak{n}} = H \in \mathfrak{a}$. Conversely, if $D_1 \in \mathfrak{a}$, then since $\operatorname{tr}\operatorname{Ric}_1 A = 0$ for any $A \in \mathfrak{a}$ (see Remarks 4.2 and 4.1) we have that

$$
\operatorname{tr} D_1 A = -c\operatorname{tr} A = -c\operatorname{tr}\operatorname{ad} A = -c\langle H,A\rangle = \operatorname{tr} HA = \operatorname{tr}\operatorname{ad} H|_{\mathfrak{n}} A, \qquad \forall A \in \mathfrak{a}.
$$

This implies that $D_1 = \operatorname{ad} H|_{\mathfrak{n}}$ (i.e. $\operatorname{Ric} = cI$), completing the proof of the proposition. □

We now show that $c \geq 0$ is actually not allowed for nontrivial solsolitons. This gives an alternative proof of the fact that any nonflat solsoliton must be expanding (see Section 2).

Proposition 4.6. *Let S be a solsoliton, say with $\operatorname{Ric} = cI + D$, $c \in \mathbb{R}$, $D \in \operatorname{Der}(\mathfrak{s})$. If $c \geq 0$ then $\operatorname{Ric} = 0$.*

Proof. Since D is a symmetric derivation we have that $D|_{\mathfrak{a}} = 0$, $D(\mathfrak{n}) \subset \mathfrak{n}$. It then follows from just the first line in (25) that $c = 0$, $[\mathfrak{a}, \mathfrak{a}] = 0$ and $\operatorname{ad} A^t = -\operatorname{ad} A$ for any $A \in \mathfrak{a}$. We also obtain from (25) that $D|_{\mathfrak{n}} = \operatorname{Ric}_1$, the Ricci operator of $(\mathfrak{n}, \langle \cdot, \cdot \rangle_{\mathfrak{n} \times \mathfrak{n}})$, and so $\operatorname{Ric}_1 = 0$ by Remarks 4.2 and 4.1, concluding the proof. □

We will need in what follows the following technical lemma valid for general solvmanifolds.

Lemma 4.7. *Let S be a solvmanifold. Then, for any $A \in \mathfrak{a}$, the following conditions are equivalent:*

(i) $(\operatorname{ad} A)^t$ *is a derivation of* $\mathfrak{s}$.
(ii) $\operatorname{ad} A$ *is a normal operator (i.e.* $[\operatorname{ad} A, (\operatorname{ad} A)^t] = 0$*).*

Proof. It follows from (24) that

$$\begin{aligned}
\operatorname{tr} R[\operatorname{ad} A, (\operatorname{ad} A)^t] &= \tfrac{1}{4} \langle \pi([\operatorname{ad} A, (\operatorname{ad} A)^t])[\cdot, \cdot], [\cdot, \cdot] \rangle \\
&= \tfrac{1}{4} \langle \pi(\operatorname{ad} A) \pi((\operatorname{ad} A)^t)[\cdot, \cdot], [\cdot, \cdot] \rangle \\
&= \tfrac{1}{4} \langle \pi((\operatorname{ad} A)^t)[\cdot, \cdot], \pi(\operatorname{ad} A)^t [\cdot, \cdot] \rangle \\
&= \tfrac{1}{4} ||\pi((\operatorname{ad} A)^t)[\cdot, \cdot]||^2,
\end{aligned}$$

and so if $\operatorname{ad} A$ is normal then $\pi((\operatorname{ad} A)^t)[\cdot, \cdot] = 0$, that is, $(\operatorname{ad} A)^t \in \operatorname{Der}(\mathfrak{s})$.

Conversely, if $(\operatorname{ad} A)^t$ is a derivation of $\mathfrak{s}$ then both $\operatorname{ad} A$ and $(\operatorname{ad} A)^t$ must vanish on $\mathfrak{a}$, since they leave $\mathfrak{n}$ invariant and their images are contained in $\mathfrak{n}$ (this last statement is well-known to be true for any derivation, see for instance [GW, Lemma 2.6]). This implies that

$$(\operatorname{ad} A)^t([A, X]) = [(\operatorname{ad} A)^t(A), X] + [A, (\operatorname{ad} A)^t(X)] = [A, (\operatorname{ad} A)^t(X)], \qquad \forall X \in \mathfrak{n},$$

which is equivalent to saying that $[\operatorname{ad} A, (\operatorname{ad} A)^t] = 0$. □

The following structural theorem for solsolitons is the main result of this paper.

Theorem 4.8. *Let S be a solvmanifold with metric Lie algebra $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ and consider the orthogonal decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$. Then $\operatorname{Ric} = cI + D$ for some $c < 0$, $D \in \operatorname{Der}(\mathfrak{s})$, i.e. S is a solsoliton, if and only if the following conditions hold:*

(i) $(\mathfrak{n}, \langle \cdot, \cdot \rangle|_{\mathfrak{n} \times \mathfrak{n}})$ *is a nilsoliton with Ricci operator* $\operatorname{Ric}_1 = cI + D_1$*, for some* $D_1 \in \operatorname{Der}(\mathfrak{n})$.
(ii) $[\mathfrak{a}, \mathfrak{a}] = 0$.
(iii) $(\operatorname{ad} A)^t \in \operatorname{Der}(\mathfrak{s})$ *(or equivalently,* $[\operatorname{ad} A, (\operatorname{ad} A)^t] = 0$*) for all* $A \in \mathfrak{a}$.
(iv) $\langle A, A \rangle = -\frac{1}{c} \operatorname{tr} S(\operatorname{ad} A)^2$ *for all* $A \in \mathfrak{a}$*, where* $S(\operatorname{ad} A) = \frac{1}{2}(\operatorname{ad} A + (\operatorname{ad} A)^t)$.

Proof. If conditions (i)-(iv) are satisfied by S then we can argue exactly as in the proof of Proposition 4.3 to obtain that $\mathrm{Ric} = cI + D$, where D is defined by $D|_{\mathfrak{a}} = 0$, $D|_{\mathfrak{n}} = D_1 - S(\mathrm{ad}\, H|_{\mathfrak{n}})$, H the mean curvature vector of S.

Let us then prove the converse assertion. Let S be a solvmanifold with $\mathrm{Ric} = cI + D$, $c < 0$, $D \in \mathrm{Der}(\mathfrak{s})$. If $F := S(\mathrm{ad}\, H) + D$ then we obtain from (21) and (24) that

$$\mathrm{tr}\left(cI + \tfrac{1}{2}B + F\right)E = \tfrac{1}{4}\langle \pi(E)[\cdot,\cdot],[\cdot,\cdot]\rangle, \qquad \forall E \in \mathrm{End}(\mathfrak{s}). \tag{26}$$

By letting $E = \mathrm{ad}\, H + D$ in (26) and using Remark 4.1 we get

$$c\,\mathrm{tr}\, F + \mathrm{tr}\, F^2 = 0. \tag{27}$$

In particular, $\mathrm{tr}\, F \geq 0$ and equality holds if and only if $F = 0$. Also, by applying (26) to E defined by $E|_{\mathfrak{a}} = 0$ and $E|_{\mathfrak{n}} = I$, it is easy to see that

$$cn + \mathrm{tr}\, F = \tfrac{1}{4}\sum ||[A_i, A_j]||^2 + \tfrac{1}{4}||[\cdot,\cdot]_{\mathfrak{n}\times\mathfrak{n}}||^2, \qquad n = \dim \mathfrak{n}. \tag{28}$$

We first consider the case when $\mathfrak{n}$ is abelian. It follows from (28) that $cn + \mathrm{tr}\, F \geq 0$, and so by (27) we get that $\mathrm{tr}\, F^2 \leq \frac{1}{n}(\mathrm{tr}\, F)^2$. This implies that $[\mathfrak{a},\mathfrak{a}] = 0$, since equality must hold in (28), and also that $F|_{\mathfrak{n}} = tI$ for some $t \geq 0$, but therefore $F|_{\mathfrak{a}} = 0$ and $F|_{\mathfrak{n}} = -cI$. We now obtain from (25) that the restrictions to $\mathfrak{n}$ satisfy

$$cI + D = \tfrac{1}{2}\sum [\mathrm{ad}\, A_i, (\mathrm{ad}\, A_i)^t] - S(\mathrm{ad}\, H),$$

from which it follows that $\sum[\mathrm{ad}\, A_i, (\mathrm{ad}\, A_i)^t] = 0$. By arguing as in the proof of Lemma 4.7 we get that

$$0 = \mathrm{tr}\left(R\sum[\mathrm{ad}\, A_i, (\mathrm{ad}\, A_i)^t]\right) = \tfrac{1}{4}\sum ||\pi((\mathrm{ad}\, A_i)^t)[\cdot,\cdot]||^2, \tag{29}$$

and therefore (iii) follows.

Let us assume from now on that $\mathfrak{n}$ is not abelian. In order to apply the results in Section 3, we identify $\mathfrak{n}$ with $\mathbb{R}^n$ via an orthonormal basis $\{e_1, ..., e_n\}$ of $\mathfrak{n}$. In this way, $\mu := [\cdot,\cdot]|_{\mathfrak{n}\times\mathfrak{n}}$ can be viewed as a nonzero element of $\mathcal{N} \subset V$. Thus $\mu \in \mathcal{S}_\beta$ for some $\beta \in \mathcal{B} \subset \mathfrak{t}^+$ and there exists $g \in \mathrm{O}(n)$ such that $\tilde{\mu} := g.\mu$ satisfies $\beta_{\tilde{\mu}} = \beta$, so that in addition (19) and (20) hold for $\tilde{\mu}$ (see Theorem 3.1). Let $\tilde{g}$ denote the orthogonal map of $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ defined by $\tilde{g}|_{\mathfrak{a}} = I$, $\tilde{g}|_{\mathfrak{n}} = g$, and let $\tilde{\mathfrak{s}}$ be the Lie algebra which is $\mathfrak{s}$ as a vector space and has Lie bracket

$$\tilde{g}.[\cdot,\cdot] = \tilde{g}[\tilde{g}^{-1}\cdot, \tilde{g}^{-1}\cdot].$$

We therefore have that $(\tilde{\mathfrak{s}}, \langle\cdot,\cdot\rangle)$ is isometric to $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$, as $\tilde{g} : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}$ is an isometric isomorphism between the two metric Lie algebras. Since conditions (i)-(iv) hold for $(\tilde{\mathfrak{s}}, \langle\cdot,\cdot\rangle)$ if and only if they hold for $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$, we can assume from now on that all properties (18)-(20) in Theorem 3.1 hold for μ.

Consider $E_\beta \in \mathrm{End}(\mathfrak{s})$ defined by

$$E_\beta := \begin{bmatrix} 0 & 0 \\ 0 & \beta + ||\beta||^2 I \end{bmatrix}, \qquad \text{that is,} \quad E_\beta|_{\mathfrak{a}} = 0, \quad E_\beta|_{\mathfrak{n}} = \beta + ||\beta||^2 I.$$

Lemma 4.9. $\langle \pi(E_\beta)[\cdot,\cdot],[\cdot,\cdot]\rangle \geq 0$ *and it equals the sum of the following three nonnegative terms:*

$$\begin{aligned} &\tfrac{1}{4}\sum \langle (\beta + \|\beta\|^2 I)[A_i,A_j],[A_i,A_j]\rangle \\ &+\tfrac{1}{2}\sum \langle [\beta, \operatorname{ad} A_i], \operatorname{ad} A_i\rangle \\ &+\tfrac{1}{4}\langle \pi(\beta + \|\beta\|^2 I)\mu, \mu\rangle. \end{aligned} \tag{30}$$

Proof. If $\{A_r\}$ is an orthonormal basis of $\mathfrak{a}$ then

$$\begin{aligned} \langle \pi(E_\beta)[\cdot,\cdot],[\cdot,\cdot]\rangle \;=\; &\tfrac{1}{4}\sum \langle E_\beta [A_r,A_s],[A_r,A_s]\rangle \\ &+\tfrac{1}{2}\sum \langle E_\beta [A_r,e_i],[A_r,e_i]\rangle - \tfrac{1}{2}\sum \langle [A_r, E_\beta e_i],[A_r,e_i]\rangle, \\ &+\tfrac{1}{4}\sum \langle E_\beta [e_i,e_j] - [E_\beta e_i, e_j] - [e_i, E_\beta e_j], [e_i,e_j]\rangle, \end{aligned}$$

which in turn equals

$$\begin{aligned} &\tfrac{1}{4}\sum \langle (\beta + \|\beta\|^2 I)[A_r,A_s],[A_r,A_s]\rangle \\ &+\tfrac{1}{2}\sum \langle (\beta \operatorname{ad} A_r - \operatorname{ad} A_r \beta)(e_i), \operatorname{ad} A_r(e_i)\rangle + \tfrac{1}{4}\langle \pi(\beta + \|\beta\|^2 I)\mu, \mu\rangle, \end{aligned} \tag{31}$$

and so (30) follows. The three terms in (30) are ≥ 0 by (16), (18) and (20), respectively. □

We therefore obtain, by applying (26) to $E = E_\beta$ and using Lemma 4.9, that

$$c \operatorname{tr} E_\beta + \operatorname{tr} F E_\beta \geq 0. \tag{32}$$

Recall that $\operatorname{tr}\beta = -1$ (see Theorem 3.1) and so

$$\begin{aligned} \operatorname{tr} E_\beta^2 &= \operatorname{tr}(\beta^2 + \|\beta\|^4 I + 2\|\beta\|^2\beta) = \|\beta\|^2(1 + n\|\beta\|^2 - 2) \\ &= \|\beta\|^2(-1 + n\|\beta\|^2) = \|\beta\|^2 \operatorname{tr} E_\beta. \end{aligned} \tag{33}$$

On the other hand, we have that

$$\operatorname{tr} F E_\beta = \operatorname{tr} F|_{\mathfrak{n}}(\beta + \|\beta\|^2) = \|\beta\|^2 \operatorname{tr} F \tag{34}$$

by (19). We now use (27), (32), (33) and (34) and obtain by a straightforward manipulation that

$$\operatorname{tr} F^2 \operatorname{tr} E_\beta^2 \leq (\operatorname{tr} F E_\beta)^2,$$

that is, we get a 'backwards' Cauchy-Schwartz inequality. This has many strong consequences. A first one is that $F = tE_\beta$ for some $t \geq 0$, and thus by (27),

$$F|_{\mathfrak{n}} = -\tfrac{c}{\|\beta\|^2} E_\beta. \tag{35}$$

Recall that $\operatorname{tr} F > 0$ since $F \neq 0$ by (32) and (16). Secondly, the three nonnegative terms in (30) must vanish, which respectively implies that $[\mathfrak{a},\mathfrak{a}] = 0$ by (16), $[\beta, \operatorname{ad}\mathfrak{a}|_{\mathfrak{n}}] = 0$ by (18) and $\beta + \|\beta\|^2 I \in \operatorname{Der}(\mathfrak{n})$ by (20). Thus (ii) holds and so (iv) follows from (25).

We also obtain from (25) that the restrictions to $\mathfrak{n}$ satisfy

$$cI + D = \tfrac{1}{2}\sum [\operatorname{ad} A_i, (\operatorname{ad} A_i)^t] + \operatorname{Ric}_1 - S(\operatorname{ad} H), \tag{36}$$

and hence it follows from (35) that

$$\tfrac{1}{2}\sum [\operatorname{ad} A_i, (\operatorname{ad} A_i)^t] + \operatorname{Ric}_1 = -\tfrac{c}{\|\beta\|^2}\beta.$$

By taking traces we get $-\frac{1}{4}||\mu||^2 = \frac{c}{||\beta||^2}$ (see Remark 4.2). This implies that

$$\tfrac{1}{8}\sum ||\pi((\operatorname{ad} A_i)^t)[\cdot,\cdot]||^2 + \operatorname{tr}\operatorname{Ric}_1^2 = \operatorname{tr}\operatorname{Ric}_1 \sum [\operatorname{ad} A_i, (\operatorname{ad} A_i)^t] + \operatorname{tr}\operatorname{Ric}_1^2$$

$$= -\tfrac{c}{||\beta||^2}\operatorname{tr}\operatorname{Ric}_1 \beta = \tfrac{1}{4}||\mu||^2 \operatorname{tr}\operatorname{Ric}_1 \beta = \tfrac{1}{16}||\mu||^4\langle m(\mu),\beta\rangle \leq \tfrac{1}{16}||\mu||^4||m(\mu)||||\beta||$$

$$\leq \tfrac{1}{16}||\mu||^4||m(\mu)||^2 = \operatorname{tr}\operatorname{Ric}_1^2.$$

The first equality above holds by (29) and the last inequality follows from (17) and the fact that $m(\mu) = \frac{4}{||\mu||^2}\operatorname{Ric}_1$ (see (23) and Remark 4.2). We therefore obtain that

$$\sum ||\pi((\operatorname{ad} A_i)^t)[\cdot,\cdot]||^2 = 0,$$

and so (iii) follows. We now use (36) and Lemma 4.7 to conclude that

$$\operatorname{Ric}_1 = cI + D + S(\operatorname{ad} H) \in \mathbb{R}I + \operatorname{Der}(\mathfrak{n}),$$

and therefore part (i) holds. □

It is well-known that a solvmanifold which satisfies conditions (ii) and (iii) in Theorem 4.8 is isometric to the one obtained by just changing the Lie bracket into

$$[A,X]' = S(\operatorname{ad} A)X, \qquad [X,Y]' = [X,Y], \qquad \forall A \in \mathfrak{a},\ X,Y \in \mathfrak{n}, \tag{37}$$

and keeping the same $\langle\cdot,\cdot\rangle$ (see for instance [H, Proposition 2.5]).

Thus the next result follows directly from Theorem 4.8.

Corollary 4.10. *Up to isometry, any solsoliton can be constructed as in* Proposition 4.3.

As a byproduct of the proof of Theorem 4.8, the following extra structural properties for solsolitons have been obtained.

Proposition 4.11. *Let S be a solsoliton, say with $\operatorname{Ric} = cI + D$, $c < 0$, $D \in \operatorname{Der}(\mathfrak{s})$, and let us assume that $\mathfrak{n}$ is not abelian, $\mu := [\cdot,\cdot]|_{\mathfrak{n}\times\mathfrak{n}} \in \mathcal{S}_\beta$ and $\beta_\mu = \beta$. If $E_\beta \in \operatorname{End}(\mathfrak{s})$ is defined by*

$$E_\beta := \begin{bmatrix} 0 & 0 \\ 0 & \beta + ||\beta||^2 I \end{bmatrix}, \qquad i.e. \quad E_\beta|_{\mathfrak{a}} = 0, \quad E_\beta|_{\mathfrak{n}} = \beta + ||\beta||^2 I,$$

then the following conditions hold:

(i) $E_\beta \in \operatorname{Der}(\mathfrak{s})$ *(or equivalently, $[\beta, \operatorname{ad}\mathfrak{a}] = 0$ and $\beta + ||\beta||^2 I \in \operatorname{Der}(\mathfrak{n})$).*

(ii) $S(\operatorname{ad} H) + D = -\frac{c}{||\mu||^2}E_\beta$. *In particular, S is Einstein if and only if $S(\operatorname{ad} H) = -\frac{c}{||\mu||^2}E_\beta$.*

(iii) $c = -\frac{1}{4}||\mu||^2||\beta||^2$ *and $m(\mu) = \beta$.*

Recall that condition $\beta_\mu = \beta$ can be assumed to hold up to isometry since always $\beta_{g.\mu} = \beta$ for some $g \in \operatorname{O}(\mathfrak{n})$ (see Theorem 3.1 and the paragraph before Lemma 4.9).

Remark 4.12. Can we use condition (5) to find examples of Ricci solitons on solvable Lie groups which are not simply connected? The answer is no, such as for Einstein solvmanifolds. Indeed, for the field X_D (see (6)) to be defined on a Lie group S/Γ covered by a simply connected solvable Lie group S, where Γ is a discrete subgroup of the center of S, it is necessary that the one-parameter group of $\varphi_t \in \operatorname{Aut}(S)$ with $\operatorname{d}\varphi_t|_e = e^{tD}$ satisfies $\varphi_t(\Gamma) = \Gamma$ for all $t \in \mathbb{R}$. But since Γ is discrete this implies that $\varphi_t(\gamma) = \gamma$ for all $\gamma \in \Gamma$ and t, and consequently $DX = 0$ for some nonzero X

in the center $\mathfrak{z}(\mathfrak{s})$ of $\mathfrak{s}$. It follows from Theorem 4.8 (ii) and (iii) that $X \in \mathfrak{n}$, and since $D|_{\mathfrak{n}} = D_1 - S(\operatorname{ad} H)|_{\mathfrak{n}}$ we obtain that

$$0 = \langle DX, X\rangle = \langle D_1X, X\rangle - \langle [H, X], X\rangle = \langle D_1X, X\rangle,$$

which contradicts the fact that D_1 is positive definite.

5. Uniqueness of solsolitons

The structural results obtained in Theorem 4.8 pave the way for the following uniqueness result for solsolitons, which is the analogous of the one already known for nilsolitons.

Theorem 5.1. *Let S and S' be two solsolitons which are isomorphic as Lie groups. Then S is isometric to S' up to scaling.*

Remark 5.2. In particular, a given solvable Lie group can admit at most one Ricci soliton left invariant metric up to isometry and scaling.

Proof. Without any loss of generality, we can assume that $\mathfrak{s} = \mathfrak{s}'$, $\mathfrak{a} = \mathfrak{a}'$ and $\mathfrak{n} = \mathfrak{n}'$ as vector spaces and that $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle'$. Thus the solvmanifolds S and S' are respectively determined only by their Lie brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]'$, which must satisfy all structural properties in Theorem 4.8. From now on, such properties will be referred to just by (i),...,(iv). Since S and S' are isomorphic, there exists a Lie algebra isomorphism between $[\cdot,\cdot]$ and $[\cdot,\cdot]'$ of the form

$$f = \begin{bmatrix} g & 0 \\ j & h \end{bmatrix}, \qquad g \in \mathrm{GL}(\mathfrak{a}), \quad h \in \mathrm{GL}(\mathfrak{n}), \quad j : \mathfrak{a} \longrightarrow \mathfrak{n},$$

such that

$$h.\mu = \mu', \qquad \mu := [\cdot,\cdot]|_{\mathfrak{n}\times\mathfrak{n}}, \quad \mu' := [\cdot,\cdot]'|_{\mathfrak{n}\times\mathfrak{n}}, \tag{38}$$

and

$$h \operatorname{ad}(g^{-1}A)|_{\mathfrak{n}} h^{-1} = \operatorname{ad}' A|_{\mathfrak{n}} + \operatorname{ad}_{\mu'}(jg^{-1}A), \qquad \forall A \in \mathfrak{a}. \tag{39}$$

These two conditions are also sufficient for f being an isomorphism by (ii). We can assume up to scaling that the scalars c and c' in the definition of a solsoliton satisfy $c = c'$, and therefore $||\mu|| = ||\mu'||$ by Proposition 4.11, (iii) and the fact that μ and μ' belong to the same $\mathrm{GL}(\mathfrak{n})$-orbit and consequently they must lie in the same stratum $\mathcal{S}_\beta$ (see Theorem 3.1). It then follows from (i) and the uniqueness for nilsolitons (see either [L1, Theorem 3.5] or [L4, Theorem 4.2]) that $h \in \mathrm{O}(\mathfrak{n})$.

If $\mathfrak{n} = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_r$ is the orthogonal decomposition with $[\mathfrak{n},\mathfrak{n}] = \mathfrak{n}_2 \oplus ... \oplus \mathfrak{n}_r$, $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = \mathfrak{n}_3 \oplus ... \oplus \mathfrak{n}_r$, and so on, then by (iii) we have that for all i, $\mathfrak{n}_i$ is an invariant subspace for any $\operatorname{ad} A|_{\mathfrak{n}}, \operatorname{ad}' A|_{\mathfrak{n}}$, $A \in \mathfrak{a}$, and also for h. But since $\operatorname{ad}_{\mu'}(jg^{-1}A)\mathfrak{n}_i \subset \mathfrak{n}_{i+1} \oplus ... \oplus \mathfrak{n}_r$ for all i, condition (39) implies that $\operatorname{ad}_{\mu'}(jg^{-1}A) = 0$ for all $A \in \mathfrak{a}$ and consequently

$$\tilde{f} = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$$

is also an isomorphism between $[\cdot,\cdot]$ and $[\cdot,\cdot]'$. We finally use (39) and (iv) to obtain that

$$-c\langle g^{-1}A, g^{-1}A\rangle = \operatorname{tr} S(\operatorname{ad}(g^{-1}A)|_{\mathfrak{n}})^2 = \operatorname{tr} S(\operatorname{ad}' A|_{\mathfrak{n}})^2 = -c\langle A, A\rangle,$$

that is, $g \in \mathrm{O}(\mathfrak{a})$. Thus $\tilde{f}$ is an orthogonal isomorphism and so it determines an isometry between S and S', which concludes the proof of the theorem. □

It follows from Corollary 4.10 that to classify solsolitons up to isometry one firstly has to classify nilsolitons and for each of these to consider all possible abelian Lie algebras of symmetric derivations. The following result gives us the precise equivalence relation that must be considered between such algebras.

Proposition 5.3. *Let* $(\mathfrak{n}, \langle\cdot,\cdot\rangle_1)$ *be a nilsoliton and consider two solsolitons* S *and* S' *constructed as in* Proposition 4.3 *for abelian subalgebras*

$$\mathfrak{a}, \mathfrak{a}' \subset \operatorname{Der}(\mathfrak{n}) \cap \operatorname{sym}(\mathfrak{n}, \langle\cdot,\cdot\rangle_1),$$

respectively. Then S *is isometric to* S' *if and only if there exists* $h \in \operatorname{Aut}(\mathfrak{n}) \cap \operatorname{O}(\mathfrak{n}, \langle\cdot,\cdot\rangle_1)$ *such that* $\mathfrak{a}' = h\mathfrak{a}h^{-1}$.

Proof. If S and S' are isometric then we can argue as in the proof of Theorem 5.1 to obtain that $h \in \operatorname{Aut}(\mathfrak{n}) \cap \operatorname{O}(\mathfrak{n}, \langle\cdot,\cdot\rangle_1)$ (see (38)) and that $\mathfrak{a}' = h\mathfrak{a}h^{-1}$ by (39).

Conversely, if we define $g : \mathfrak{a} \longrightarrow \mathfrak{a}'$ by $gA = hAh^{-1}$, then

$$\langle gA, gA\rangle' = -\tfrac{1}{c}\operatorname{tr}(gA)^2 = -\tfrac{1}{c}\operatorname{tr}(hAh^{-1})^2 = -\tfrac{1}{c}\operatorname{tr} A^2 = \langle A, A\rangle.$$

This implies that $f := \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$ is an isometric isomorphism between $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ and $(\mathfrak{s}', \langle\cdot,\cdot\rangle')$ and thus f defines an isometry between S and S', as was to be shown. □

6. Examples of solsolitons

Once we have chosen our favorite nilsoliton, it is quite easy to get examples of solsolitons by using Proposition 4.3. The results in Section 4 even tell us that any solsoliton can be constructed in this simple way. We consider in this section the problem of which simply connected solvable Lie groups of dimension ≤ 4 admit a solsoliton.

6.1. **Dimension** 3. It is well known that for any 3-dimensional real solvable Lie algebra which is not nilpotent there exists a basis $\{A, X_1, X_2\}$ such that

$$\text{(40)} \qquad \begin{aligned} \mathfrak{r}_\alpha : &\quad [A, X_1] = X_1, \quad [A, X_2] = \alpha X_2, \qquad -1 \leq \alpha \leq 1, \\ \mathfrak{s}_\beta : &\quad [A, X_1] = X_2, \quad [A, X_2] = -X_1 + \beta X_2, \qquad 0 \leq \beta \leq 2. \end{aligned}$$

The constraints on the parameters α and β guarantee that these Lie algebras are in addition pairwise non-isomorphic (see [Jc]). It follows from Proposition 4.3 that the inner product $\langle\cdot,\cdot\rangle$ for which $\{A, X_1, X_2\}$ is orthonormal is a solsoliton on $\mathfrak{r}_\alpha$ for all α. Recall that in all cases we have $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ for $\mathfrak{a} = \mathbb{R}A$ and $\mathfrak{n} = \mathbb{R}X_1 + \mathbb{R}X_2$, $\mathfrak{n}$ abelian. For $\mathfrak{s}_\beta$, the matrix of $\operatorname{ad} A|_{\mathfrak{n}}$ relative to $\{X_1, X_2\}$ equals $\left[\begin{smallmatrix} 0 & -1 \\ 1 & \beta \end{smallmatrix}\right]$, which has eigenvalues $\frac{\beta}{2} \pm i\sqrt{1 - \frac{\beta^2}{4}}$, and thus for any $0 \leq \beta < 2$, there is a basis $\{Y_1, Y_2\}$ of $\mathfrak{n}$ with respect to which $\operatorname{ad} A|_{\mathfrak{n}}$ has the normal matrix

$$\begin{bmatrix} \frac{\beta}{2} & -\sqrt{1-\frac{\beta^2}{4}} \\ \sqrt{1-\frac{\beta^2}{4}} & \frac{\beta}{2} \end{bmatrix}.$$

This implies that the inner product $\langle\cdot,\cdot\rangle_1$ given by $\langle A, A\rangle_1 = \operatorname{tr} S(\operatorname{ad} A)^2 = \frac{\beta^2}{2}$ and with $\{Y_1, Y_2\}$ as an orthonormal basis for $\mathfrak{n}$ is a solsoliton for $0 < \beta < 2$ (see Theorem 4.8). But hence no new example appears, since by (37) the solsolitons $(\mathfrak{s}_\beta, \langle\cdot,\cdot\rangle_1)$ are either isometric to the hyperbolic space $H^3 = (\mathfrak{r}_1, \langle\cdot,\cdot\rangle)$ for $0 < \beta < 2$ or to the euclidean space $\mathbb{R}^3$ for $\beta = 0$.

	$\mathfrak{n}$	$\operatorname{ad} A\|_{\mathfrak{n}}$	*constraints*	*unimodular*	*solsoliton*	*Einstein*
$\mathfrak{r}_3$	$\mathbb{R}^2$	$\left[\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right]$	–	–	–	–
$\mathfrak{r}_{3,\lambda}$	$\mathbb{R}^2$	$\left[\begin{smallmatrix}1&\\&\lambda\end{smallmatrix}\right]$	$-1 \leq \lambda \leq 1$	$\lambda = -1$	✓	$\lambda = 1$
$\mathfrak{r}'_{3,\lambda}$	$\mathbb{R}^2$	$\left[\begin{smallmatrix}\lambda&1\\-1&\lambda\end{smallmatrix}\right]$	$0 \leq \lambda$	$\lambda = 0$	✓	✓

Table 1. Classification of 3-dimensional solvable Lie algebras admitting solsolitons

The only remaining case is $\mathfrak{s}_2$, for which we have that $\operatorname{ad} A|_{\mathfrak{n}}$ is not diagonalizable over $\mathbb{C}$ and so it can not be normal with respect to any inner product on $\mathfrak{n}$. We conclude from Theorem 4.8 that $\mathfrak{s}_2$ is the only 3-dimensional solvable Lie group which does not admit a solsoliton.

There are many ways to parametrize 3-dimensional solvable Lie algebras over $\mathbb{R}$ other than (40). It is in fact more convenient for our purpose to use the description given in [ABDO, Theorem 1.1], which we show in Table 1 together with the information about the existence of solsolitons. In this case, if $\{X_1, X_2\}$ is the basis of $\mathfrak{n} = \mathbb{R}^2$ that we use to write $\operatorname{ad} A|_{\mathfrak{n}}$, then the inner product $\langle \cdot, \cdot \rangle$ making of $\{A, X_1, X_2\}$ an orthonormal basis is always a solsoliton.

We note that for any $0 \leq \lambda$, $\mathfrak{r}'_{3,\lambda}$ is isomorphic to $\mathfrak{s}_\beta$ for $\beta = \frac{2\lambda}{\sqrt{\lambda^2+1}}$ and $\mathfrak{r}_3$ is isomorphic to $\mathfrak{s}_2$. A third description can be found in [Ml].

6.2. **Dimension 4.** It is enough to consider 4-dimensional real solvable Lie algebras which are not either nilpotent or a direct sum of two Lie algebras. If $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{n}$ the nilradical of $\mathfrak{s}$, then the only one with $\dim \mathfrak{a} = 2$ is $\mathfrak{aff}(\mathbb{C})$, which is defined for a basis $\{A_1, A_2\}$ of $\mathfrak{a}$ by

$$\operatorname{ad} A_1|_{\mathfrak{n}} = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \quad \operatorname{ad} A_2|_{\mathfrak{n}} = \left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right].$$

It follows from Theorem 4.8 that $\mathfrak{aff}(\mathbb{C})$ admits a solsoliton, which is isometric to $H^3 \times \mathbb{R}$ by (37).

Any other has $\dim \mathfrak{a} = 1$, say $\mathfrak{a} = \mathbb{R}A$, and let $\{X_1, X_2, X_3\}$ be a basis of $\mathfrak{n}$, which will be assumed to satisfy $[X_1, X_2] = X_3$ when $\mathfrak{n}$ is not abelian (i.e. $\mathfrak{n}$ isomorphic to the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$). We have used the classification given in [ABDO, Theorem 1.5] to summarize all the relevant information in Table 2, which has been obtained as in the 3-dimensional case above by a direct application of Proposition 4.3 and Theorem 4.8. The map $\operatorname{ad} A|_{\mathfrak{n}}$ is always written in terms of the basis $\{X_1, X_2, X_3\}$ and the inner product $\langle \cdot, \cdot \rangle$ for which $\{A, X_1, X_2, X_3\}$ is an orthonormal basis is always a solsoliton, when there is one.

By using (37), we get that $\mathfrak{r}'_{4,\mu,\lambda}$, $\lambda \neq 0$, is isometric to $\mathfrak{r}_{4,\lambda/\mu,\lambda/\mu}$ and that $\mathfrak{s}'_{4,\lambda}$ is isometric to $\mathfrak{s}_{4,1/2}$ for any $\lambda > 0$.

One can see in Table 2 that $\mathfrak{r}_{4,-1/2}$ is the only unimodular solvable Lie algebra of dimension 4 which does not admit a solsoliton. It follows however from the results in [Hn] that $\mathfrak{r}_{4,-1/2}$ does not either admit a *lattice* (i.e. cocompact discrete

	$\mathfrak{n}$	$\operatorname{ad} A\|_{\mathfrak{n}}$	*constraints*	*unimodular*	*solsoliton*	*Einstein*
$\mathfrak{r}_4$	$\mathbb{R}^3$	$\begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$	–	–	–	–
$\mathfrak{r}_{4,\lambda}$	$\mathbb{R}^3$	$\begin{bmatrix} 1 & & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$	$-\infty < \lambda < \infty$	$\lambda = -\frac{1}{2}$	–	–
$\mathfrak{r}_{4,\mu,\lambda}$	$\mathbb{R}^3$	$\begin{bmatrix} 1 & & \\ & \mu & \\ & & \lambda \end{bmatrix}$	$-1 < \mu \leq \lambda \leq 1$; $-1 = \mu \leq \lambda < 0$	$\mu = -1 - \lambda$	✓	$\mu = \lambda = 1$
$\mathfrak{r}'_{4,\mu,\lambda}$	$\mathbb{R}^3$	$\begin{bmatrix} \mu & & \\ & \lambda & 1 \\ & -1 & \lambda \end{bmatrix}$	$0 < \mu$	$\mu = -2\lambda$	✓	$\mu = \lambda$
$\mathfrak{s}_4$	$\mathfrak{h}_3$	$\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$	–	✓	✓	–
$\mathfrak{s}_{4,\lambda}$	$\mathfrak{h}_3$	$\begin{bmatrix} \lambda & & \\ & 1-\lambda & \\ & & 1 \end{bmatrix}$	$\frac{1}{2} \leq \lambda$	–	✓	$\lambda = \frac{1}{2}$
$\mathfrak{s}'_{4,\lambda}$	$\mathfrak{h}_3$	$\begin{bmatrix} \lambda & 1 & \\ -1 & \lambda & \\ & & 2\lambda \end{bmatrix}$	$0 \leq \lambda$	$\lambda = 0$	$\lambda \neq 0$	$\lambda \neq 0$
$\mathfrak{h}_4$	$\mathfrak{h}_3$	$\begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 2 \end{bmatrix}$	–	–	–	–

TABLE 2. Classification of 4-dimensional solvable Lie algebras with a 3-dimensional nilradical admitting a solsoliton

subgroup). Thus the universal cover of any 4-dimensional compact solvmanifold S/Γ does admit a solsoliton, a result which has already been proved in [IJL]. This is essentially due to the fact from algebraic number theory that any $A \in \mathrm{SL}_3(\mathbb{Z})$ with positive real eigenvalues and at least one of them different from 1 is necessarily diagonalizable over $\mathbb{C}$. The next example shows that this is no longer true for compact solvmanifolds of dimension ≥ 5.

Example 6.1. Let $\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{n}$ be the solvable Lie algebra defined by: $\mathfrak{n}$ is abelian, $\dim \mathfrak{n} = 4$ and

$$\operatorname{ad} A|_{\mathfrak{n}} = \begin{bmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & \ln \lambda & \\ & & & -\ln \lambda \end{bmatrix}, \qquad \lambda = \tfrac{3+\sqrt{5}}{2}.$$

Thus there exists $\sigma \in \mathrm{GL}_4(\mathbb{R})$ such that

$$\sigma e^{\operatorname{ad} A|_{\mathfrak{n}}} \sigma^{-1} = \begin{bmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 2 & 1 \\ & & 1 & 1 \end{bmatrix} \in \mathrm{SL}_4(\mathbb{Z}),$$

and it is therefore easy to check that $\Gamma := \mathbb{Z} \ltimes \sigma^{-1}\mathbb{Z}^4$ is a lattice of the solvable Lie group S with Lie algebra $\mathfrak{s}$. Recall that $S = \mathbb{R} \ltimes \mathbb{R}^4$ with multiplication given by

$$(t, X).(s, Y) = (t + s, X + e^{t \operatorname{ad} A|_{\mathfrak{n}}} Y), \qquad t, s \in \mathbb{R}, \quad X, Y \in \mathbb{R}^4.$$

However, since $\operatorname{ad} A|_{\mathfrak{n}}$ is not diagonalizable over $\mathbb{C}$, we conclude from Theorem 4.8 that S can never admit a solsoliton.

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