Laplace Invariants via Vessiot Equivalence Method

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Outline

Introduction – Laplace Example

- Linear partial differential operators (LPDOs) of order 2:

$$L = \partial_x \partial_y + a\,\partial_x + b\,\partial_y + c$$

- Gauge transformations:

$$L \mapsto g^{-1} L g, \qquad g = g(x, y).$$

- Laplace invariants:

$$h = c - a_x - ab, \qquad k = c - b_y - ab.$$

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- Laplace invariants:

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- The operator L can be factorised if $h = 0$ or $k = 0$:

$$\begin{aligned} L &= (\partial_x + b)(\partial_y + a) + h, \\ &= (\partial_y + a)(\partial_x + b) + k. \end{aligned}$$

Introduction – General Situation

- Consider arbitrary LPDOs of order d:

$$L = \sum_{|\mu| \leq d} a_\mu(x) \partial^\mu, \qquad \partial^\mu = \partial_{x_1}^{\mu_1} \cdots \partial_{x_n}^{\mu_n}$$

  with symbol $\sum_{|\mu|=d} a_\mu(x) X^\mu$.
- The factorisation of LPDOs is gauge invariant:

$$g^{-1} L g = g^{-1} L_1 L_2 \, g = (g^{-1} L_1 \, g)(g^{-1} L_2 \, g).$$

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Methods to compute invariants:

- Partial factorisation (obstacles) [SW07b], [SW07a]
- Moving frames [MS08]
- ...
- Vessiot equivalence method [Ves03]

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Third Order LPDOs

The number of invariants in a generating set:

Symbol, order	[MS08] 0	1	2	3	total	Vessiot 0	1	2	total
X^3	2	2	1		5	2	3		5
X^3, (a)	2	0	2		4	2	1	1	4
X^3, (b)	1	1	0	1	3	1	1	1	3
X^3, (c)	0	1	1		2	0	2		2
$X^2 Y$	1	3	1		5	1	5		6
$X\,Y\,(pX+qY)$	3	3	1		7	3	4	1	8
full						5	4	1	10

$$L_{X^3} = \partial_x^3 + a_{20}\partial_x^2 + a_{02}\partial_y^2 + a_{11}\partial_x\partial_y + a_{10}\partial_x + a_{01}\partial_y + a_{00}.$$

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$X Y (pX + qY)$	3	3	1		7	3	4	1	8
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- Moving frames: small invariants of higher order,
- Vessiot: large invariants of minimal order.

In future: Combine both methods!

Invariants for $L_{X^3,(c)} = \partial_x^3 + a_{20}\partial_x^2 + a_{10}\partial_x + a_{00}$

- Moving frames [MS08]:

$$
\begin{aligned}
I^{a_{10}} &= a_{10} - \frac{1}{3}a_{20}^2 - a_{20,x}, \\
I_x^{a_{00}} &= a_{00} - \frac{1}{3}a_{10}a_{20} + \frac{2}{27}a_{20}^3 - \frac{1}{3}a_{20,xx}.
\end{aligned}
$$

- Vessiot:

$$
\begin{aligned}
I_1^1 &= -a_{10} + \frac{1}{3}a_{20}^2 + a_{20,x}, \\
I_2^1 &= a_{10,x} - 3a_{00} + a_{20}a_{10} - \frac{2}{3}a_{20}a_{20,x} - \frac{2}{9}a_{20}^3.
\end{aligned}
$$

- Comparison:

$$
I^{a_{10}} = -I_1^1 \qquad I_x^{a_{00}} = -\frac{1}{3}(I_1^2 + I_{1,x}^1)
$$

Invariants for Fourth Order LPDOs

Results of the Vessiot equivalence method:

Symbol, order	0	1	2	3	4
X^4	5	5	1		
$X^4\,(a)$	3	6			
$X^4\,(d)$	2	2	1	0	2
$\vdots$					
$X^3 Y$	4	7	1		
$X^2 Y^2$	3	10			
$X^3 (pX + qY)$	5	7	1		
$X^2 Y (pX + qY)$	4	9	1		
$X^2 (pX + qY)(rX + sY)$	5	9	1		
$X Y (pX + qY)(rX + sY)$	5	9			
$X Y (pX^2 + qY^2)$	5	6	1		

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Natural Bundles

Let X be a manifold, coordinates $(x) = (x^1, \ldots, x^n)$.

- $\mathsf{Diff}_{\mathsf{loc}}(X, X)$: local diffeomorphisms $\varphi : X \to X$.
- Pseudogroup $\Theta \subseteq \mathsf{Diff}_{\mathsf{loc}}(X, X)$.

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- A natural Θ-bundle is a fibre bundle

$$\pi : \mathcal{F} \to X : (x, v) \to (x)$$

such that each $\tilde{x}(x) \in \Theta$ lifts to $\Phi : \mathcal{F} \to \mathcal{F}$ as:

$$\tilde{x} = \tilde{x}(x), \qquad v = \Phi_{\tilde{v}}(\tilde{x}, \tilde{x}_q).$$

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In other words: Θ acts on $\mathcal{F}$.

- A section of $\mathcal{F}$ is called geometric object:

$$\omega : X \to \mathcal{F} : (x) \mapsto (x, v = \omega(x)).$$

- $\psi : \mathcal{F} \to \mathbb{R}$ is an invariant if $\psi \circ \Phi = \psi \quad \forall\, \tilde{x}(x) \in \Theta$.

Laplace Example I

- Pseudogroup Θ of gauge transformations:

$$X \to X : \begin{pmatrix} x \\ y \\ u \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} = x \\ \tilde{y} = y \\ \tilde{u} = e^{g(x,y)} u \end{pmatrix}.$$

- The natural Θ-bundle $\mathcal{F}$ for the Laplace operators

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$$\begin{aligned} a &= \tilde{a} + g_y \\ b &= \tilde{b} + g_x \\ c &= \tilde{c} + g_{xy} + \tilde{a}\, g_x + \tilde{b}\, g_y. \end{aligned}$$

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- A section $a(x, y)$, $b(x, y)$, $c(x, y)$ specifies an LPDO.

Prolongation and Projection

- Choosing $v = v(x)$ and $\tilde{v} = \tilde{v}(\tilde{x})$, the Θ-action on $\mathcal{F}$

$$v = \Phi_{\tilde{v}}(\tilde{x}, \tilde{x}_q)$$

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by eliminating derivatives of order $q + 1$.

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- Vessiot structure equations: Integrability conditions.

Laplace Example II

- The Θ-action on $\mathcal{F}$ is (with $q = 2$):

$$\begin{aligned} a &= \tilde{a} + g_y \\ b &= \tilde{b} + g_x \\ c &= \tilde{c} + g_{xy} + \tilde{a}\, g_x + \tilde{b}\, g_y + g_x g_y. \end{aligned}$$

- First prolongation to $J_1(\mathcal{F})$:

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- Projection: $\mathcal{F}_{(1)}$ has the improved coordinates:

$$h = a_x - c + ab, \quad , a_y, \quad b_x, \quad k = b_y - c + ab.$$

- Invariants: Projection to order zero.

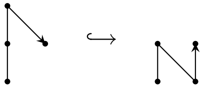
Embedding Theorem

Theorem

If the symbol of $\Phi_{\tilde{v}}(\tilde{x}, \tilde{x}_q) = v$ *is* 2*-acyclic for generic* $\tilde{v}(\tilde{x})$*, then*

$$\iota : J_2(\mathcal{F})/K_{q+2} \to J_1(\mathcal{F}_{(1)})$$

is an embedding.

- Visualisation: 

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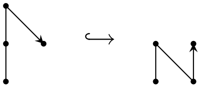
- Visualisation: 
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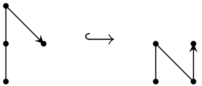
- Visualisation: 
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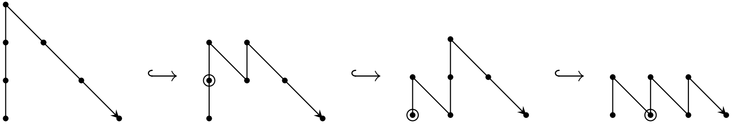
- More general situation:



Laplace Example III

- The bundle $\mathcal{F}_{(1)}$ has the coordinates $(x, y, u;\ a, b, c)$ and

$$h = a_x - c + ab, \quad , d = a_y, \quad e = b_x, \quad k = b_y - c + ab.$$

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- Prolongation to $J_1(\mathcal{F}_{(1)})$:

$$\begin{array}{llll} a_x & a_y & b_x & b_y \\ c_x & c_y & & \\ d_x & d_y & e_x & e_y \\ h_x & h_y & k_x & k_y \end{array}$$

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- Prolongation to $J_1(\mathcal{F}_{(1)})$ and the embedding $\mathrm{im}(\iota)$:

$$\begin{array}{llll} a_x = h + c - ab & a_y = d & b_x = e & b_y = k + c - ab \\ c_x & c_y & & \\ d_x = h_y + \dots & d_y & e_x & e_y = k_x + \dots \\ h_x & h_y & k_x & k_y \end{array}$$

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- Projection to $\mathcal{F}_{(2)}$:

$$h_x, \qquad h_y, \qquad k_x, \qquad k_y.$$

- All new coordinates on $\mathcal{F}_{(2)}$ are invariants
  $\Rightarrow\ \{h, k\}$ is a generating set of invariants.

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- Invariants on $J_i(\mathcal{F}) =$ invariants on $\mathcal{F}_{(i)}$.
- All new coordinates of $\mathcal{F}_{(i)}$ are invariants
  $\Rightarrow$ generating set of invariants on $\mathcal{F}_{(i)}$.
- Computation of invariants:
  - Moving frames on $\mathcal{F}_{(i)}$ or
  - Linear PDEs on $\mathcal{F}_{(i)}$.
- Sucessfully treated fourth order LPODs.
- Even a fifth order example (X^3Y^2) was computable...

The end.

Done!

Literature

M. Barakat.
Jets. A MAPLE-package for formal differential geometry.
In *Computer algebra in scientific computing (Konstanz, 2001)*, pages 1–12. Springer, Berlin, 2001.

A. Lorenz.
On local integrability conditions of jet groupoids.
Acta Appl. Math., 101:205–213, 2008.

E. L. Mansfield and E. Shemyakova.
Moving frames for Laplace invariants.
ISSAC'08, 2008.

J.-F. Pommaret.
Systems of partial differential equations and Lie pseudogroups, volume 14 of *Mathematics and its Applications*.
Gordon & Breach Science Publishers, New York, 1978.
With a preface by André Lichnerowicz.

E. S. Shemyakova and F. Winkler.
A Full System of Invariants for Third-Order Linear Partial Differential Operators in General Form.
Lecture Notes in Comput. Sci., 4770:360–369, 2007.

E. S. Shemyakova and F. Winkler.
Obstacles to the factorization of partial differential operators into several factors.
Programmirovanie, (2):17–25, 2007.

E. Vessiot.
Sur la théorie des groupes continus.
Ann. Sci. École Norm. Sup. (3), 20:411–451, 1903.