Cartan's spiral staircase in physics and, in particular, in the gauge theory of dislocations*

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Abstract

In 1922, Cartan introduced in differential geometry, besides the Riemannian curvature, the new concept of *torsion*. He visualized a homogeneous and isotropic distribution of torsion in *three dimensions* (3d) by the "helical staircase", which he constructed by starting from a 3d Euclidean space and by defining a new connection via helical motions. We describe this geometric procedure in detail and define the corresponding connection and the torsion. The interdisciplinary nature of this subject is already evident from Cartan's discussion, since he argued—but never proved—that the helical staircase should correspond to a continuum with constant

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pressure and constant internal *torque*. We discuss where in physics the helical staircase is realized: (i) In the continuum mechanics of Cosserat media, (ii) in (fairly speculative) 3d theories of gravity, namely a) in 3d Einstein-Cartan gravity—this is Cartan's case of constant pressure and constant intrinsic torque— and b) in 3d Poincaré gauge theory with the Mielke-Baekler Lagrangian, and, eventually, (iii) in the gauge field theory of *dislocations* of Lazar *et al.*, as we prove for the first time by arranging a suitable distribution of screw dislocations. Our main emphasis is on the discussion of dislocation field theory.

Keywords: Cartan's torsion, differential geometry, dislocations, Cosserat continuum, Einstein-Cartan theory, 3-dimensional theories of gravitation

1 Introduction: Homogeneous and isotropic torsion in three dimensions

1.1 Cartan's original idea

In 1922, when Élie Cartan [8, 9] analyzed Einstein's general relativity theory (GR), he introduced in this context the concept of torsion into differential geometry. Thereby he generalized Riemannian geometry, dominated by the metric tensor $g_{ij} = g_{ji}$ and the Riemannian curvature tensor $\widetilde{R}_{ijk}{}^{\ell}$, to Riemann-Cartan (RC) geometry carrying a generalized curvature $R_{ijk}{}^{\ell}$ and an additional fundamental third rank tensor $T_{ij}{}^{k} = -T_{ji}{}^{k}$, which was named torsion by Cartan; here $i, j, \ldots$ are coordinate indices running either over $1, 2, 3$ (space) or over $0, 1, 2, 3$ (spacetime).

Whereas it is simple to visualize say a 2-dimensional (2d) Riemannian space as a curved 2d surface imbedded in a (flat) 3d Euclidean space, no simple picture lends itself to a visualization of a space with torsion. Still, already in his first publication on the subject, Cartan [8], starting from 3d Euclidean space, gave a prescription of how to arrive at a specific 3d space with homogeneous and isotropic torsion. We refer to this space as *Cartan's spiral staircase* for reasons that will become clear in the next two paragraphs. This construction is largely forgotten[1] in spite of being quite helpful in explaining the characteristic features of a simple space with torsion.

The idea of Cartan was the following: Take a point A of a 3d Euclidean space E in Cartesian coordinates, as it is depicted in Figure 1. Consider a neighboring point A'. The vector linking A with A' will be denoted by $\overrightarrow{AA'}$. *Rotate* now the triad in A' in accordance with the vector $\vec{\omega} := \lambda \, \overrightarrow{AA'}$ in the right hand-sense, where λ is a prescribed constant. The new rotated triad serves as a basis for the space F with torsion: a vector in F at A' is said to be parallel to a vector at A, if its components in A with respect to the local triad are the same as in A' with respect to the *rotated* triad. Whereas the Euclidean connection $\widetilde{\Gamma}$ is zero with respect to the triads in E, the new Riemann-Cartan connection ${}^{\xi}\Gamma$ vanishes with respect to the rotated triads. This new "helical" connection ${}^{\xi}\Gamma$ carries a non-Riemannian piece that is proportional to the torsion. The space F is like our ordinary space as viewed by an observer whose perception has been twisted [8].

Now, the vector $\vec{\omega}$ can be decomposed into its components $\omega^1, \omega^2, \omega^3$, that is, into rotations around the x-, y-, and z-axis, see Figure 1. Accordingly, if A' first coincides with A and is then shifted further and further away from A, then the triad along each of the three axes undergoes a helical motion, that is, it is like going up a spiral staircase along each of the axes.

In Section 2, following the prescriptions of Cartan, we will construct, with the help of the calculus of exterior differential forms, the Riemann-Cartan connection ${}^{\xi}\Gamma$ of the spiral staircase F. In this way, we can put Cartan's idea in a very succinct form. We will determine the two-forms of torsion T^{α} and of curvature $R^{\alpha\beta}$ of the spiral staircase. This will allow us to understand the 3d Riemann-Cartan space under consideration.

[1] See, however, García et al. [14] and Hehl and Obukhov [22].

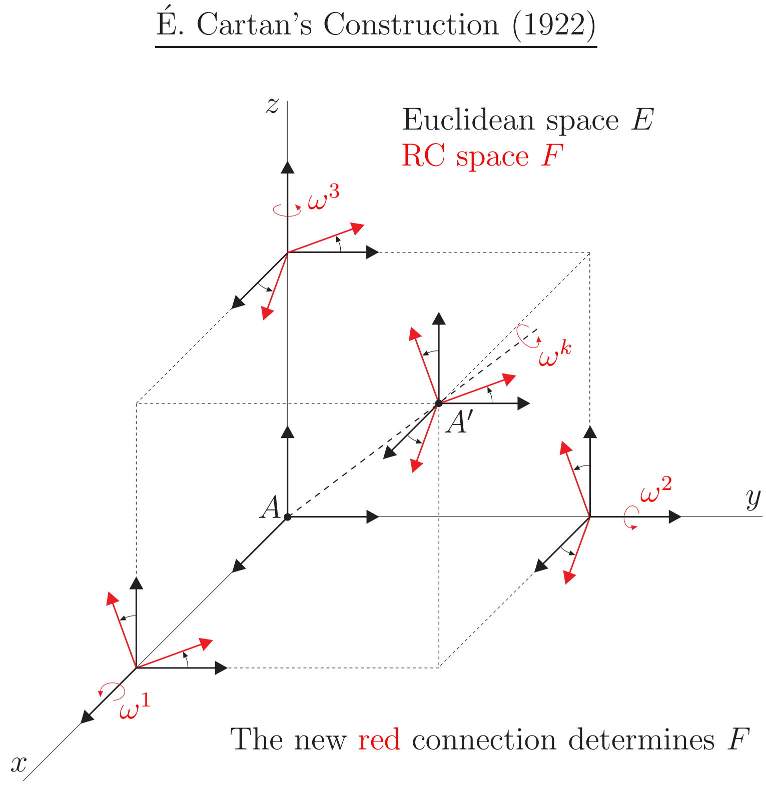

Figure 1: *Cartan's construction (1922) of a 3-dimensional space with homogeneous and isotropic torsion: the spiral staircase*

1.2 Cosserat medium

In his original article, Cartan [8] argued—and now we are back to physics—that the space F of the spiral staircase corresponds to a medium with constant *pressure* p and constant *internal torque* τ. As is clear from his acknowledgment, Cartan was influenced in his investigations by the work of the brothers Cosserat [10] on continuum mechanics. In classical continuum mechanics à la Euler and Cauchy—to name two main figures of the orthodoxy—a medium with internal torque does not exist, since the classical medium can only support (force-)stresses ${\sigma_i}^k$ (specifically with pressure $p := \frac{1}{3}{\sigma_k}^k$), but no internal torques.

However, already Voigt [63] had introduced the new concept of a (spin) moment stress tensor ${\tau_{ij}}^k = -{\tau_{ji}}^k$, which can exist in a suitable medium in addition to the (force) stress tensor ${\sigma_i}^k$. Then, specifically an internal torque $\tau := \frac{1}{2}\epsilon^{ijk}\tau_{ijk}$ can be defined, with $\epsilon^{ijk} = \pm 1, 0$ as totally antisymmetric Levi-Civita symbol. The notion of a (spin) moment stress tensor has been brought to fruition in the work of the brothers Cosserat.[2]

To get an intuitive feeling for their type of approach, we can start, as Cartan did, with

[2]For insight into more recently developed continuum theories with microstructure and with generalized stresses, see, for example, [24, 46, 15, 12, 47, 48, 43]; also the review of Neff [51] is very readable.

Euclidean geometry and its fundamental notions of *translation* and *rotation*. On the one hand, these two geometrical notions are—via a suitable variational principle— intrinsically related to the static notions of force stress and spin moment stress, on the other hand, if one restricts the validity of Euclidean geometry only to local neighborhoods ("gauging of the Euclidean group"), one ends up, guided by Cartan, with a Riemann-Cartan geometry with torsion and curvature.

This brings us back to Cartan's medium with constant pressure p and constant internal torque τ as image of his spiral staircase: Apparently a Cosserat medium, generalizing the medium of classical elasticity, fluid mechanics, etc., is appropriate for this purpose. Accordingly, Cartan's introduction of the torsion has as a consequence a more general conception of a continuum than the one taken for granted in classical continuum mechanics.

As far as we aware, nobody considered so far the implications of the spiral staircase to a Cosserat continuum. We will describe our corresponding results that the spiral staircase produces constant hydrostatic pressure and constant internal torque in a linear, incompatible isotropic Cosserat medium in Section 3.

1.3 Three-dimensional gravity

Let us recall that Cartan was in the process of analyzing GR, that is, a gravitational theory. In the course of these investigations, he developed the skeleton of a new, slightly generalized theory of gravity. This four-dimensional theory of gravitation, to which Cartan laid the foundations, was worked out by Sciama and Kibble around 1961, see [20, 62], and is called Einstein-Cartan (EC) theory of gravity. It is one of the very few viable theories of gravity.

In the meantime, however, also a somewhat speculative three-dimensional (3d) EC-type model of gravity, with a Lagrangian proportional to the curvature scalar of the RC-space, has been proposed. This 3d EC-model has an *exact solution* with a geometry described by the spiral staircase and *matter* of the Cosserat type carrying constant pressure and constant spin moment stress (torque). We believe that it is this solution that Cartan described in words. The corresponding derivations and the details will be presented in Section 4.

Moreover, in the 3d framework there also exist gravitational models with Lagrangians *quadratic* in torsion and curvature (Poincaré gauge models); the most general one of these is a model of Mielke & Baekler [49, 2]. It has a so-called BTZ (Bañados, Teitelboim, Zanelli) black hole *with* torsion as an exact vacuum solution [14]. Also in Section 4 we show that a subcase of the vacuum BTZ-solution with torsion (namely for $\Lambda_{\text{eff}} = 0$) carries the torsion of the spiral staircase—and for vanishing mass and angular momentum $M = 0$, $J = 0$, also its connection ${}^{\xi}\Gamma^{\alpha\beta}$. In contrast to the EC-solution mentioned above, it is an exact *vacuum* solution and, accordingly, was outside the scope of Cartan in 1922, as we shall see in Section 4.

1.4 Gauge theory of dislocations

Let us now turn to an important point of our investigations: During the 1950s it became clear that *crystal dislocations* can be described by Cartan's torsion [30, 32], basically since dislocations, similar as torsion, can break open infinitesimally small parallelograms, in this way inducing a *closure failure* of the parallelogram; for reviews, see Kröner [33] and, furthermore, Ruggiero and Tartaglia [54]. Since dislocation lines are discrete objects, it is helpful to consider in this context a "continuized crystal" [34].

During the last years a successful gauge field theory of dislocations has been developed by Lazar [35, 36, 38] and Lazar and Anastassiasis [39, 40, 41]; for different versions, see [26, 25, 45, 27] and the reviews [53, 29, 28]. Within this theory, the equivalence between torsion and dislocation density plays a leading role. Then immediately the question comes up whether Cartan's spiral staircase is an *exact solution* of Lazar's gauge theory of dislocations and whether one can find, indeed, constant pressure and constant internal torque in the corresponding medium.

It appears intuitively clear that the spiral staircases along the three Cartesian axes—with the same pitch!—must be implemented by three forests of parallel *screw dislocations*; the forests should be perpendicular to each other and each of them be of the same dislocation density, that is, their respective Burgers vectors should be the same and constant, see [19]. This distribution of screw dislocations should provide a constant torsion of the twisted type that, by means of the constitutive laws, should induce constant pressure and constant internal torque—provided Cartan's intuition was right and Lazar's theory appropriate.

We prove in Section 5 that with Lazar's highly non-trivial gauge theory of dislocations we can derive, in linear approximation, the constant pressure and the constant internal torque for the first time in a realistic setting.

In Section 6 we discuss our results and compare them with the literature.

2 The differential geometry of Cartan's spiral staircase

2.1 Mathematical framework, conventions[3]

The geometrical arena for our considerations consists of a three-dimensional manifold M together with a local Euclidean *metric* g. At each point we introduce a *coframe* field ϑ^α, with the anholonomic (or frame) indices $\alpha, \beta, ... = \hat{1}, \hat{2}, \hat{3}$, and, dual to it, the *frame* field e_β, with $e_\beta \rfloor \vartheta^\alpha = \delta^\alpha_\beta$, where $\rfloor$ denotes the interior product. In a trivial orthonormal coframe $\vartheta^\alpha = \delta^\alpha_i \, dx^i$, we have

$$g = g_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta , \qquad g_{\alpha\beta} \stackrel{*}{=} \mathrm{diag}(1,1,1)\,; \tag{1}$$

[3] A systematic presentation of our formalism can be found, for example, in Ref. [21].

the corresponding trivial frame field is $e_\beta = \delta^j_\beta \partial_j$. Thus, more explicitly, we have

$$\vartheta^{\hat 1} = \mathrm{d}x^1 \, , \; \vartheta^{\hat 2} = \mathrm{d}x^2 \, , \; \vartheta^{\hat 3} = \mathrm{d}x^1 \, , \quad e_{\hat 1} = \partial_1 \, , \; e_{\hat 2} = \partial_2 \, , \; e_{\hat 3} = \partial_3 \, , \tag{2}$$

the holonomic (or coordinate) indices $i, k, \ldots$ run over $1, 2, 3$. A ϑ-basis for 0-, 1-, 2-, and 3-forms is $\left\{1, \vartheta^\alpha, \, \vartheta^{\alpha\beta} := \vartheta^\alpha \wedge \vartheta^\beta, \, \vartheta^{\alpha\beta\gamma} := \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma\right\}$, the Hodge dual η-basis for 3-,2-,1-, and 0-forms is specified by

$$\eta := {}^\star 1 = \frac{1}{3!} \, \eta_{\alpha\beta\gamma} \, \vartheta^{\alpha\beta\gamma} \, , \tag{3}$$

$$\eta_\alpha := {}^\star \vartheta_\alpha = e_\alpha \rfloor \eta = \frac{1}{2} \, \eta_{\alpha\beta\gamma} \, \vartheta^{\beta\gamma} \, , \tag{4}$$

$$\eta_{\alpha\beta} := {}^\star(\vartheta_{\alpha\beta}) = e_\beta \rfloor \eta_\alpha = \eta_{\alpha\beta\gamma} \, \vartheta^\gamma \, , \tag{5}$$

$$\eta_{\alpha\beta\gamma} := {}^\star(\vartheta_{\alpha\beta\gamma}) = e_\gamma \rfloor \eta_{\alpha\beta} \, , \tag{6}$$

where * denotes the Hodge star and $\eta_{\alpha\beta\gamma} := \sqrt{\det(g_{\mu\nu})} \, \epsilon_{\alpha\beta\gamma}$, with $\epsilon_{\alpha\beta\gamma}$ as the totally antisymmetric Levi-Civita symbol with $\pm 1, 0$.

This formalism can be straightforwardly generalized to n dimensions and to a Lorentzian metric with, in $n = 4$, $g_{\alpha\beta} \stackrel{*}{=} \mathrm{diag}(-1, 1, 1, 1)$.

In the case of the spiral staircase, see Figure 1, we have an underlying three-dimensional Euclidean space E with *flat* metric (that is, vanishing Riemann curvature) and, accordingly, with a Euclidean connection 1-form $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha} = \Gamma_i{}^{\alpha\beta} \mathrm{d}x^i$ that has vanishing torsion and vanishing curvature.

2.2 Cartan's prescription

Cartan introduced, besides this Euclidean connection $\Gamma^{\alpha\beta}$, for the space F a non-Euclidean helical *Riemann-Cartan connection* 1-form,

$$^\xi\Gamma^{\alpha\beta} = -{}^\xi\Gamma^{\beta\alpha} = {}^\xi\Gamma_i{}^{\alpha\beta} \mathrm{d}x^i = {}^\xi\Gamma_1{}^{\alpha\beta} \mathrm{d}x^1 + {}^\xi\Gamma_2{}^{\alpha\beta} \mathrm{d}x^2 + {}^\xi\Gamma_3{}^{\alpha\beta} \mathrm{d}x^3 \, , \tag{7}$$

in the following way, see Figure 1: If we consider the non-Euclidean parallel displacement along the x^1-axis, then, according to Cartan's recipe, the corresponding orthonormal coframe rotates in the positive sense around the x^1-axis with the angle $\omega = \omega^{23} = -\omega^{32}$ per unit length. The analogous prescription applies to the parallel displacements along the x^2-axis and the x^3-axis. Then only the following connection components, up to their antisymmetry, are non-vanishing and equal:

$$^\xi\Gamma_1{}^{\hat 2 \hat 3} = {}^\xi\Gamma_2{}^{\hat 3 \hat 1} = {}^\xi\Gamma_3{}^{\hat 1 \hat 2} = \omega \, . \tag{8}$$

We transform the coordinate indices into frame indices and find,

$$^\xi\Gamma^{\gamma\alpha\beta} := g^{\gamma\delta} e^i{}_\delta \, {}^\xi\Gamma_i{}^{\alpha\beta} = \omega \eta^{\gamma\alpha\beta} \, , \tag{9}$$

with $e_\alpha = e^i{}_\alpha \partial_i \stackrel{*}{=} \delta^i_\alpha \partial_i$ and $g_{\alpha\beta} = \mathrm{diag}(1,1,1)$. In an anholonomic basis, the connection 1-form reads:

$$ {}^\xi\Gamma^{\alpha\beta} = {}^\xi\Gamma^{\gamma\alpha\beta}\vartheta_\gamma = \omega\eta^{\alpha\beta\gamma}\vartheta_\gamma = \omega\,\eta^{\alpha\beta}\,. \tag{10} $$

For the coframe we have, see Figure 1, $\vartheta^{\hat 1} = \mathrm{d}x^1\,, \vartheta^{\hat 2} = \mathrm{d}x^2\,, \vartheta^{\hat 3} = \mathrm{d}x^3$, which is a trivial coframe

$$ \vartheta^\alpha = \delta^\alpha_i\,\mathrm{d}x^i\,. \tag{11} $$

Accordingly, the torsion 2-form is constant and only its axial piece survives:

$$ T^\alpha := \mathrm{D}\vartheta^\alpha = \mathrm{d}\vartheta^a + {}^\xi\Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \omega\eta^{\alpha\beta\gamma}\vartheta_\beta \wedge \vartheta_\gamma = 2\omega\,\eta^\alpha\,. \tag{12} $$

For the Riemann-Cartan curvature 2-form we find[4]

$$ \begin{aligned} R_\alpha{}^\beta &:= \mathrm{d}\,{}^\xi\Gamma_\alpha{}^\beta - {}^\xi\Gamma_\alpha{}^\gamma \wedge {}^\xi\Gamma_\gamma{}^\beta = -\omega^2\eta_\alpha{}^{\delta\gamma}\eta^{\varepsilon\beta}{}_\gamma\,\vartheta_\delta \wedge \vartheta_\varepsilon \\ &= -\omega^2\left(\delta^\varepsilon_\alpha g^{\delta\beta} - \delta^\beta_\alpha g^{\delta\varepsilon}\right)\vartheta_\delta \wedge \vartheta_\varepsilon = \omega^2\,\vartheta_\alpha \wedge \vartheta^\beta \qquad \text{or} \\ R^{\alpha\beta} &= \omega^2\,\vartheta^{\alpha\beta}\,. \end{aligned} \tag{13} $$

Alternative to the torsion 2-form T^α, we can define the contortion 1-form $K_{\alpha\beta} = -K_{\beta\alpha}$ either implicitly by

$$ T^\alpha =: K^\alpha{}_\beta \wedge \vartheta^\beta \tag{14} $$

or explicitly by

$$ K_{\alpha\beta} = e_{[\alpha}\rfloor T_{\beta]} - \frac{1}{2}\,(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\vartheta^\gamma = 2e_{[\alpha}\rfloor T_{\beta]} - \frac{1}{2}\,e_\alpha\rfloor e_\beta\rfloor(T_\gamma \wedge \vartheta^\gamma)\,. \tag{15} $$

Simple algebra yields for Cartan's spiral staircase

$$ K_{\alpha\beta} = -\omega\,\eta_{\alpha\beta}\,. \tag{16} $$

In order to isolate the Riemannian part, we decompose the Riemann-Cartan connection into the Levi-Civita (or Christoffel) connection $\widetilde{\Gamma}^{\alpha\beta}$ and the contortion $K^{\alpha\beta}$ as follows

$$ {}^\xi\Gamma^{\alpha\beta} = \widetilde{\Gamma}^{\alpha\beta} - K^{\alpha\beta}\,. \tag{17} $$

Substituting (10) and (16) into (17), we find that the Levi-Civita connection vanishes

$$ {}^\xi\Gamma^{\alpha\beta} = -K^{\alpha\beta}, \qquad \widetilde{\Gamma}^{\alpha\beta} = 0\,. \tag{18} $$

Moreover, the Riemannian curvature 2-form $\widetilde{R}^{\alpha\beta}$ vanishes due to the trivial Riemannian geometry

$$ \widetilde{R}^{\alpha\beta} = \mathrm{d}\,\widetilde{\Gamma}^{\alpha\beta} - \widetilde{\Gamma}^{\alpha\gamma} \wedge \widetilde{\Gamma}_\gamma{}^\beta = 0\,. \tag{19} $$

[4] In [22] the curvature carries a different sign, since there the Lorentzian signature $-++$ was used.

Thus, the torsion, the Riemann-Cartan curvature, and the Riemannian curvature of Cartan's spiral staircase read, respectively,

$$\boxed{T^\alpha = 2\omega\,\eta^\alpha\,, \qquad R^{\alpha\beta} = \omega^2\,\vartheta^{\alpha\beta}\,, \qquad \widetilde{R}^{\alpha\beta} = 0\,.} \tag{20}$$

This is a very simple configuration. In 3d, we can decompose the torsion into three $SO(3)$-irreducible pieces according to $T^\alpha = {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha$ with the number of independent components $9 = 5 \oplus 3 \oplus 1$. These three pieces ${}^{(I)}T_\alpha$ are (see also [20])

$$ {}^{(1)}T^\alpha := T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha \qquad \text{(tentor)}, \tag{21}$$

$$ {}^{(2)}T^\alpha := \frac{1}{2}\,\vartheta^\alpha \wedge \left(e_\beta \rfloor T^\beta\right) \qquad \text{(trator)}, \tag{22}$$

$$ {}^{(3)}T_\alpha := \frac{1}{3}\,e_\alpha \rfloor \left(\vartheta^\beta \wedge T_\beta\right) \qquad \text{(axitor)}\,. \tag{23}$$

Simple algebra yields for our case that only the axial torsion part is nonvanishing,

$$ T^\alpha = {}^{(3)}T^\alpha = 2\omega\,\eta^\alpha \quad \text{or} \quad \mathcal{A} := \frac{1}{3}\,{}^\star\left(\vartheta^\alpha \wedge T_\alpha\right) = 2\omega\,. \tag{24}$$

In this case the autoparallels of the Riemann-Cartan space coincide with the Riemannian geodesics (extremals). This is obvious in the Cartan construction: here the geodesics are just the straight lines of the underlying Euclidean space—and they are at the same time the autoparallels in the newly constructed Riemann-Cartan space of constant axial torsion and constant Riemann-Cartan curvature.[5]

Of similar simplicity is the Riemann-Cartan curvature. In 3d, the curvature 2-form $R_\alpha{}^\beta$ is equivalent to the Ricci 1-form $\mathrm{Ric}_\alpha := e_\beta \rfloor R_\alpha{}^\beta = \mathrm{Ric}_{i\alpha} dx^i$. We have 9 components of the Ricci 1-form. By inspecting (20), we immediately recognize that only the curvature scalar $R := e^\alpha \rfloor \mathrm{Ric}_\alpha$ is non-vanishing:

$$ R^{\alpha\beta} = -\frac{1}{6} R\,\vartheta^{\alpha\beta} = \omega^2\,\vartheta^{\alpha\beta} \quad \text{or} \quad R = -6\omega^2\,. \tag{25}$$

Thus, Cartan's spiral staircase is characterized geometrically by the Riemann-Cartan quantities ($\mathcal{A} = 2\omega$, $R = -6\omega^2$) alone, $\mathcal{A}$ is a pseudoscalar, R a scalar.

If one decomposes the Riemann-Cartan curvature $R^{\alpha\beta}$ into its Riemannian part $\widetilde{R}^{\alpha\beta}$ and its rest and then multiplies with $\eta_{\alpha\beta}$, one finds the geometric identity, see [20, 23],

$$ R^{\alpha\beta} \wedge \eta_{\alpha\beta} = \widetilde{R}^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\,\mathrm{d}(\vartheta_\alpha \wedge {}^\star T^\alpha) + T^\alpha \wedge {}^\star\left(-{}^{(1)}T_\alpha + (n-2)\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha\right)\,, \tag{26}$$

[5]Let us stress that Cartan started from a *flat* 3d Euclidean space, that is, its curvature is *zero.* His mapping prescription yields the Riemann-Cartan connection ${}^\xi\Gamma^{\alpha\beta}$, which is depicted in Figure 1 by means of the constantly rotating triads whenever they move in a given direction. We find ${}^\xi\Gamma^{\alpha\beta} = \omega\,\eta^{\alpha\beta}$. Therefore, our image represents exactly the Cartan description. The criticism of Mielke and Maggiolo [50] that we "ignore[s] the constant curvature background" in [14] is incorrect; after all, a Euclidean space carries no nonvanishing curvature. The original space is Euclidean and *not* curved. However, the constant curvature (13) can be computed from the Riemann-Cartan connection ${}^\xi\Gamma^{\alpha\beta}$, which is depicted in our image.

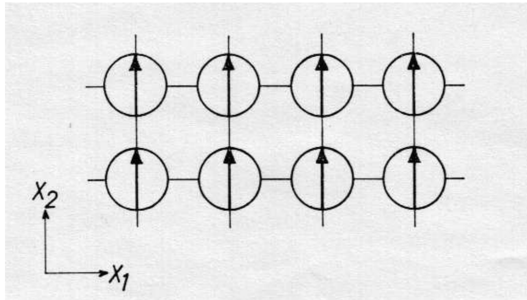

Figure 2: Schematic view on a two-dimensional Cosserat continuum: Undeformed initial state, see [22].

which is valid for any dimension n. For Cartan's spiral staircase, which carries a constant axial torsion, we are left with the 3-form

$$R^{\alpha\beta} \wedge \eta_{\alpha\beta} = \frac{1}{2}\, {}^{(3)}T^{\alpha} \wedge {}^{\star\,(3)}T_{\alpha}\,, \tag{27}$$

that is, the curvature is quadratic in the torsion.

3 The spiral staircase in a Cosserat continuum

3.1 Cosserat elasticity[6]

The classical continuum of elasticity and fluid dynamics consists of unstructured points, and the displacement vector u^α is the only quantity necessary for specifying the deformation. The Cosserats conceived a specific *medium with microstructure,* see [17, 6, 16, 12] and for a historical review [3], consisting of structured points such that, in addition to the displacement field u^α, it is possible to measure the rotation of such a structured point by the bivector field $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$, see Figure 2 for a schematic view.

The deformation 1-forms *distortion* $\beta^\alpha = \beta_i{}^\alpha\, \mathrm{d}x^i$ and *contortion* $\kappa^{\alpha\beta} = \kappa_i{}^{\alpha\beta}\, \mathrm{d}x^i$ of a linear Cosserat continuum are (D is the exterior covariant derivative of the Euclidean 3d space)

$$\begin{aligned} \beta^\alpha &= \mathrm{D}u^\alpha + \omega^{\alpha\beta}\vartheta_\beta\,, \qquad \omega^{\alpha\beta} = -\omega^{\beta\alpha}\,, && (28)\\ \kappa^{\alpha\beta} &= \mathrm{D}\omega^{\alpha\beta}\,, && (29) \end{aligned}$$

see Günther [17] and Schaefer [56]. For the components of the distortion, we have

$$\beta_i{}^\alpha = \mathrm{D}_i u^\alpha + \omega^{\alpha\beta} e_{i\beta} \qquad \text{or} \qquad \beta_{\alpha\beta} = \mathrm{D}_\alpha u_\beta - \omega_{\alpha\beta}\,. \tag{30}$$

In classical elasticity, the only deformation measure is the strain

$$\varepsilon_{\alpha\beta} := \frac{1}{2}(\beta_{\alpha\beta} + \beta_{\beta\alpha}) \equiv \beta_{(\alpha\beta)} = \mathrm{D}_{(\alpha} u_{\beta)}\,. \tag{31}$$

[6]In this subsection we follow in parts the presentation of [22].

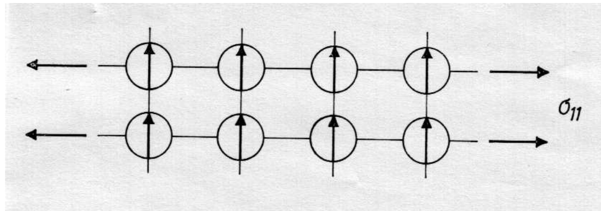

Figure 3: Conventional homogeneous strain ε_{11} of a Cosserat continuum: Distance changes of the "particles" caused by force stress σ_{11}, see [22].

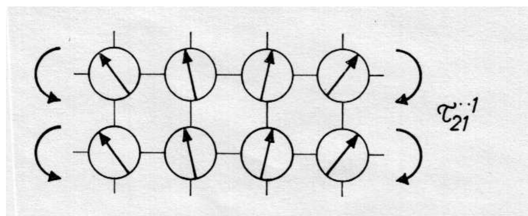

Figure 4: Homogeneous contortion κ_{112} of a Cosserat continuum: Orientation changes of the "particles" caused by spin moment stress ${\tau_{21}}^1$, see [22].

Let us visualize these deformations. If the displacement field $u_1 \sim x$ and the rotation field $\omega_{\alpha\beta} = 0$, we find $\beta_{11} = \varepsilon_{11} = const$ and $\kappa_{\alpha\beta\gamma} = 0$, see Figure 3. This homogeneous strain is created by ordinary force stresses. In contrast, if we put $u_\alpha = 0$ and $\omega_{12} \sim x$, then $\beta_{12} = \omega_{12} \sim x$ and $\kappa_{112} \sim const$, see Figure 4. This homogeneous contortion is induced by applied spin moment stresses. Figure 5 depicts the pure constant antisymmetric stress with $\omega_{12} = const$ and Figure 6 the conventional rotation of the particles according to ordinary elasticity. This has to be distinguished carefully from the situation in Figure 4.

Apparently, in addition to the force stress $\overline{\Sigma}_\alpha \equiv \Sigma_{\alpha\beta}\eta^\beta \sim \partial\mathcal{H}/\partial\beta^\alpha$ (here $\mathcal{H}$ is an elastic potential), which is asymmetric in a Cosserat continuum, i.e., $\Sigma_{\alpha\beta} \neq \Sigma_{\beta\alpha}$, we have

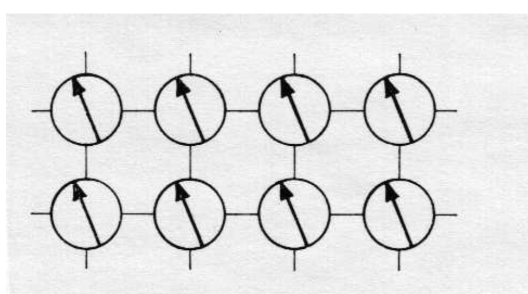

Figure 5: Homogeneous Cosserat rotation ω_{12} of the "particles" of a Cosserat continuum caused by the antisymmetric piece of the stress $\Sigma_{[12]}$, see [22].

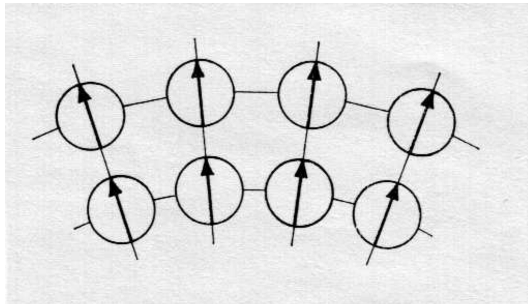

Figure 6: Conventional rotation $\partial_{[1}u_{2]}$ of the "particles" of a Cosserat continuum caused by an inhomogeneous strain, see [22].

as new response the *spin moment* stress $\overline{\tau}_{\alpha\beta} \equiv \tau_{\alpha\beta}{}^{\gamma}\eta_{\gamma} \sim \partial\mathcal{H}/\partial\kappa^{\alpha\beta}$. Hence the 2-forms of (force) stress $\overline{\Sigma}_{\alpha}$ and of *spin moment* stress $\overline{\tau}_{\alpha\beta}$ characterize a Cosserat continuum from the static side. We used bars for denoting the Cosserat stress and spin moment stress 2-forms specifically in 3d.

The equilibrium conditions for forces and moments read

$$\mathrm{D}\overline{\Sigma}_{\alpha} + f_{\alpha} = 0\,, \quad \mathrm{D}\overline{\tau}_{\alpha\beta} + \vartheta_{[\alpha} \wedge \overline{\Sigma}_{\beta]} + m_{\alpha\beta} = 0\,. \tag{32}$$

where f_{α} are the volume forces and $m_{\alpha\beta} = -m_{\beta\alpha}$ volume moments. These relations are valid in all dimensions $n \geq 1$, see [15]. In 3 dimensions we have $3+3$ and in 4 dimensions $4+6$ independent components of the "equilibrium" conditions. They correspond to translational and rotational Noether identities. In classical elasticity and in fluid dynamics, $\overline{\tau}_{\alpha\beta} = 0$ and $m_{\alpha\beta} = 0$; thus, the stress is symmetric, $\vartheta_{[\alpha} \wedge \overline{\Sigma}_{\beta]} = 0$, and then denoted by $\overline{\sigma}_{\alpha}$; for early investigations of asymmetric stress and energy-momentum tensors, see Costa de Beauregard [11].

For a local, linear, isotropic Cosserat continuum we have the following constitutive relations

$$\overline{\Sigma}_{\alpha} = g_{\alpha\beta}\,{}^{\star}\sum_{I=1}^{3} \overline{c}_{I}\,{}^{(I)}\beta^{\beta}\,, \tag{33}$$

$$\overline{\tau}_{\alpha\beta} = g_{\alpha\gamma}g_{\beta\delta}\,{}^{\star}\sum_{I=1}^{3} \overline{a}_{I}\,{}^{(I)}\kappa^{\gamma\delta}\,, \tag{34}$$

where the 3 irreducible pieces of the distortion 1-form β are given by

$$\beta^{\alpha} = {}^{(1)}\beta^{\alpha} + {}^{(2)}\beta^{\alpha} + {}^{(3)}\beta^{\alpha}\,, \tag{35}$$

with the number of independent components $9 = 5 \oplus 3 \oplus 1$. In component notation, they are given by

$$
\begin{aligned}
{}^{(1)}\beta_{\alpha\beta} &:= \beta_{\alpha\beta} - {}^{(2)}\beta_{\alpha\beta} - {}^{(3)}\beta_{\alpha\beta}\,, && (36)\\
{}^{(2)}\beta_{\alpha\beta} &:= \frac{1}{2}\left(\beta_{\alpha\beta} - \beta_{\beta\alpha}\right), && (37)\\
{}^{(3)}\beta_{\alpha\beta} &:= \frac{1}{3}\,\delta_{\alpha\beta}\,\beta_{\gamma}{}^{\gamma}\,. && (38)
\end{aligned}
$$

In addition, the 3 irreducible pieces of the contortion 1-form κ are given by

$$
\kappa^{\alpha\beta} = {}^{(1)}\kappa^{\alpha\beta} + {}^{(2)}\kappa^{\alpha\beta} + {}^{(3)}\kappa^{\alpha\beta}\,, \tag{39}
$$

with the number of independent pieces $9 = 5 \oplus 3 \oplus 1$. In component notation, they are given by

$$
\begin{aligned}
{}^{(1)}\kappa_{\alpha\beta\gamma} &:= \kappa_{\alpha\beta\gamma} - {}^{(2)}\kappa_{\alpha\beta\gamma} - {}^{(3)}\kappa_{\alpha\beta\gamma}\,, && (40)\\
{}^{(2)}\kappa_{\alpha\beta\gamma} &:= \frac{1}{2}\left(\delta_{\alpha\beta}\,\kappa^{\delta}{}_{\delta\gamma} - \delta_{\alpha\gamma}\kappa^{\delta}{}_{\delta\beta}\right), && (41)\\
{}^{(3)}\kappa_{\alpha\beta\gamma} &:= \frac{1}{3}\left(\kappa_{\alpha\beta\gamma} + \kappa_{\beta\gamma\alpha} + \kappa_{\gamma\alpha\beta}\right). && (42)
\end{aligned}
$$

The nonnegative moduli $\overline{c}_1$, $\overline{c}_2$, $\overline{c}_3$ have the dimension: $[\overline{c}_I] = \text{force}/(\text{length})^2$ and $\overline{a}_1$, $\overline{a}_2$, $\overline{a}_3$ have the dimension: $[\overline{a}_I] = \text{force}$. In this way, the elastic potential for local, linear and isotropic Cosserat theory reads

$$
\mathcal{H} = \frac{1}{2}\,\overline{\Sigma}_{\alpha} \wedge \beta^{\alpha} + \frac{1}{2}\,\overline{\tau}_{\alpha\beta} \wedge \kappa^{\alpha\beta}\,. \tag{43}
$$

Nowadays the Cosserat continuum finds many applications. As one example we may mention the work of Zeghadi et al. [64] who take the grains of a metallic polycrystal as (structured) Cosserat particles and develop a linear Cosserat theory with the constitutive laws $\overline{\Sigma}_{\alpha} \sim \beta_{\alpha}$ and $\overline{\tau}_{\alpha\beta} \sim \kappa_{\alpha\beta}$.

The Riemannian space is the analogue of the body of classical continuum theory: points and their relative distances is all what is needed to describe it geometrically; the analogue of the strain ε_{ij} of classical elasticity is the difference between the metric tensor g_{ij} of the Riemannian space and a flat background metric. In GR, a symmetric "stress" $\sigma_{ij} = \sigma_{ji}$ is the response of the matter Lagrangian to a variation of the metric g_{ij}.

A RC-space can be realized by a generalized Cosserat continuum. The "deformation measures" $\vartheta^{\alpha} = e_i{}^{\alpha}\, dx^i$ and $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta} dx^i = -\Gamma^{\beta\alpha}$ of a RC-space correspond to those of a Cosserat continuum according to the transcription[7]

$$\delta e_i{}^\alpha \to \beta^\alpha\,, \qquad \delta\Gamma_i{}^{\alpha\beta} \to \kappa^{\alpha\beta}\,. \tag{46}$$

However, in general, the coframe ϑ^α and the connection $\Gamma^{\alpha\beta}$ cannot be derived from a displacement field u^α and a rotation field $\omega^{\alpha\beta}$, as in (28) and (29). Such a generalized Cosserat continuum is called incompatible, since the deformation measures β^α and $\kappa^{\alpha\beta}$ don't fulfill the so-called compatibility conditions

$$\mathrm{D}\beta^\alpha + \vartheta^\beta \wedge \kappa_\beta{}^\alpha = 0\,, \qquad \mathrm{D}\kappa^{\alpha\beta} = 0\,, \tag{47}$$

see Günther [17] and Schaefer [55, 56]. They guarantee that the "potentials" u^α and $\omega^{\alpha\beta}$ can be introduced in the way as it is done in (28) and (29). Still, also in the RC-space, as *incompatible* Cosserat continuum, we have, besides the force stress $\overline{\Sigma}_\alpha \sim \partial\mathcal{H}/\partial\beta^\alpha$, the spin moment stress $\overline{\tau}_{\alpha\beta} \sim \partial\mathcal{H}/\partial\Gamma^{\alpha\beta}$. And in the geometro-physical interpretation of the structures of the RC-space, Cartan apparently made use of these results of the brothers Cosserat.

In 4d, the stress $\overline{\Sigma}_\alpha$ corresponds to energy-momentum[8] Σ_α and the spin moment stress $\overline{\tau}_{\alpha\beta}$ to spin angular momentum $\tau_{\alpha\beta}$. Accordingly, Cartan enriched the 4d Riemannian space of GR geometrically by the *torsion* 2-form T^α and statically (or dynamically) by the *spin angular momentum* 3-form $\tau_{\alpha\beta}$ of matter.

3.2 Incompatible Cosserat elasticity

In order to include the torsion tensor and the curvature tensor into the framework of Cosserat elasticity, we have to generalize compatible Cosserat elasticity to incompatible Cosserat elasticity. Such an extension is necessary for Cartan's spiral staircase in the framework of Cosserat theory since we have already seen that the Cartan spiral staircase is related to the notion of torsion (12).

In the case of incompatible Cosserat elasticity, the distortion and the contortion do not satisfy any longer the compatibility conditions (47). Then the elastic distortion and the elastic contortion are given as

$$\beta^\alpha = \mathrm{D}u^\alpha + \omega^{\alpha\beta}\vartheta_\beta - {}^{\mathrm{P}}\beta^\alpha\,, \tag{48}$$

$$\kappa^{\alpha\beta} = \mathrm{D}\omega^{\alpha\beta} - {}^{\mathrm{P}}\kappa^{\alpha\beta}\,. \tag{49}$$

[7] This can be seen from the response of the coframe $e_i{}^\alpha$ and the Lorentz connection $\Gamma_i{}^{\alpha\beta}$ in a RC-space to a local Poincaré gauge transformation consisting of small translations ϵ^α and small Lorentz transformations $\omega^{\alpha\beta}$,

$$\mathrm{d}e_i{}^\alpha = -\mathrm{D}_i\epsilon^a + e_i{}^\gamma\omega_\gamma{}^\alpha - \epsilon^\gamma T_{\gamma i}{}^\alpha\,, \tag{44}$$

$$\mathrm{d}\Gamma_i{}^{\alpha\beta} = -\mathrm{D}_i\omega^{\alpha\beta} - \epsilon^\gamma R_{\gamma i}{}^{\alpha\beta}\,, \tag{45}$$

see [18], Eqs.(4.33),(4.32); here $\mathrm{D}_i := \partial_i \rfloor \mathrm{D}$ are the components of the exterior covariant derivative. The second term on the right-hand-side of (44) is due to the semi-direct product structure of the Poincaré group. If we put torsion and curvature to zero, these formulas are analogous to (28),(29).

[8] This is well-known from classical electrodynamics: The 3d Maxwell stress generalizes, in 4d, to the energy-momentum tensor of the electromagnetic field, see [21].

It can be seen that the plastic distortion ${}^{\rm P}\beta^{\alpha}$ and the plastic contortion ${}^{\rm P}\kappa^{\alpha\beta}$ are the causes of the incompatibility. The failure of the elastic and plastic fields to be compatible gives rise to incompatibility conditions [17, 56]:

$$T^{\alpha} = {\rm D}\beta^{\alpha} + \kappa^{\beta\alpha} \wedge \vartheta_{\beta}\,, \tag{50}$$

$$R^{\alpha\beta} = {\rm D}\kappa^{\alpha\beta}\,, \tag{51}$$

and for the plastic fields

$$T^{\alpha} = -{\rm D}\,{}^{\rm P}\beta^{\alpha} - {}^{\rm P}\kappa^{\beta\alpha} \wedge \vartheta_{\beta}\,, \tag{52}$$

$$R^{\alpha\beta} = -{\rm D}\,{}^{\rm P}\kappa^{\alpha\beta}\,. \tag{53}$$

The measures of incompatibilities (50)–(53) may be identified with the torsion 2-form and the curvature 2-form of the incompatible Cosserat medium in linear approximation.

3.3 Cartan's spiral staircase as a solution in incompatible Cosserat elasticity

In this subsection we want to show, that the solutions (9) and (11) of Cartan's spiral staircase are also solutions in incompatible Cosserat elasticity. If we use the identification (46), we find for the elastic distortion and the elastic contortion

$$\beta_{\alpha\beta} = \delta_{\alpha\beta}\,, \qquad \kappa_{\alpha\beta\gamma} = \omega\,\eta_{\alpha\beta\gamma}\,. \tag{54}$$

Thus, the elastic distortion and the elastic contortion are constant. The irreducible pieces are

$${}^{(1)}\beta_{\alpha\beta} = 0\,, \qquad {}^{(2)}\beta_{\alpha\beta} = 0\,, \qquad {}^{(3)}\beta_{\alpha\beta} = \delta_{\alpha\beta}\,, \tag{55}$$

$${}^{(1)}\kappa_{\alpha\beta\gamma} = 0\,, \qquad {}^{(2)}\kappa_{\alpha\beta\gamma} = 0\,, \qquad {}^{(3)}\kappa_{\alpha\beta\gamma} = \omega\,\eta_{\alpha\beta\gamma}\,. \tag{56}$$

If we substitute (54) into (48) and (49) and integrate, we find for the displacement and the microrotation bivector

$$u_{\alpha} = x_{\alpha}\,, \qquad \omega_{\alpha\beta} = \omega\,\epsilon_{\alpha\beta\gamma}x^{\gamma} \tag{57}$$

and for the plastic distortion and plastic contortion

$${}^{\rm P}\beta_{\alpha\beta} = -\omega_{\alpha\beta} \qquad {}^{\rm P}\kappa_{\alpha\beta\gamma} = 0\,. \tag{58}$$

Thus, the plastic rotation is equal to the negative microrotation bivector and the plastic contortion is zero. From (50)–(53) we calculate the torsion and the curvature produced by Cartan's spiral staircase in linear, incompatible Cosserat elasticity as

$$T^{\alpha} = 2\omega\,\eta^{\alpha}\,, \qquad R^{\alpha\beta} = 0\,. \tag{59}$$

The vanishing of $R^{\alpha\beta}$ identifies the corresponding RC-space as a teleparallel one.

Now we may substitute (55) and (56) into the constitutive relations (33) and (34) and we find for the force and internal moment stresses caused by Cartan's spiral staircase

$$\Sigma_{\alpha\beta} = -p\,\delta_{\alpha\beta} = \overline{c}_3\,\delta_{\alpha\beta}\,, \qquad \tau_{\alpha\beta\gamma} = \overline{a}_3\omega\,\eta_{\alpha\beta\gamma}\,. \tag{60}$$

Mechanically, we have found a constant hydrostatic *pressure* $-\overline{c}_3$ and a constant *torque* $\overline{a}_3\omega$ as predicted by Cartan. Thus, we conclude that Cartan's spiral staircase is a solution in linear, incompatible Cosserat theory producing constant pressure and constant internal torque in a Cosserat medium.

4 The spiral staircase in three-dimensional theories of gravity

In the realm of quantum gravity, people are interested in (1+2)-dimensional theories of gravity, basically since (1+3)-dimensional theories, like GR or the Einstein-Cartan theory, in some high 'temperature' limit, may effectively reduce to (1+2)-dimensional theories. A good reference describing this approach is Carlip [7]. We concentrate here purely on the classical aspect of these theories.

The conventional gravitational Lagrangian in 4d is the Hilbert-Einstein Lagrangian $\sim \eta_{\alpha\beta}\wedge R^{\alpha\beta}$. This term also works in 3d. However, in 3d there exist topological, connection-dependent terms, namely the Chern-Simons 3-form for the curvature

$$C_{\rm RR} := -\frac{1}{2}\left(\Gamma_\alpha{}^\beta\wedge d\Gamma_\beta{}^\alpha - \frac{2}{3}\Gamma_\alpha{}^\beta\wedge\Gamma_\beta{}^\gamma\wedge\Gamma_\gamma{}^\alpha\right)\,. \tag{61}$$

This equation is correct in a Riemann or a Riemann-Cartan space, for details, see, e.g., [20], Sec.3.9. In a Riemann-Cartan space we can define an analogous 3-form for the torsion, namely

$$C_{\rm TT} := \frac{1}{2\ell^2}g_{\alpha\beta}\vartheta^\alpha\wedge T^\beta\,, \tag{62}$$

where ℓ is some constant with the dimension of a length. Introducing additionally a cosmological term with Λ as cosmological constant, we end up with the Mielke-Baekler Lagrangian [49, 2], see also [50],

$$\begin{aligned} V_{\rm MB} \;=\; & -\frac{\chi}{2\ell}R^{\alpha\beta}\wedge\eta_{\alpha\beta} - \frac{\Lambda}{\ell}\eta + \frac{\theta_{\rm T}}{2\ell^2}\vartheta^\alpha\wedge T_\alpha \\ & -\frac{\theta_{\rm L}}{2}\left(\Gamma_\alpha{}^\beta\wedge d\Gamma_\beta{}^\alpha - \frac{2}{3}\Gamma_\alpha{}^\beta\wedge\Gamma_\beta{}^\gamma\wedge\Gamma_\gamma{}^\alpha\right) + L_{\rm mat}\,, \end{aligned} \tag{63}$$

with some coupling constants $\chi, \theta_{\rm T}, \theta_{\rm L}$ (here 'T' stands for translation and 'L' for Lorentz). Theories with this general Lagrangian will be considered.

4.1 3d Einstein-Cartan theory

In the simplest case we just have, for $\chi = 1$, the 3d Einstein-Cartan Lagrangian without cosmological constant,

$$V_{\rm EC} = -\frac{\chi}{2\ell} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + L_{\rm mat}\,. \tag{64}$$

Variations with respect to coframe ϑ^α and Lorentz connection $\Gamma^{\alpha\beta}$, yield the field equations of the 3d Einstein-Cartan theory (with Euclidean signature) [20, 14, 62]:

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\, R^{\beta\gamma} = \ell\, \Sigma_\alpha\,, \tag{65}$$

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\, T^{\gamma} = \ell\, \tau_{\alpha\beta}\,, \tag{66}$$

where Σ_α and $\tau_{\alpha\beta}$ are the 2-forms of (force) stress and of (spin) moment stress, respectively. Moreover, ℓ is a characteristic length.[9]

Substituting (13) and (12) into (65) and (66), respectively, and using simple algebra, we find the force stress 2-form and the moment stress 2-form,[10]

$$\Sigma_\alpha = \frac{\omega^2}{\ell}\,\eta_\alpha\,, \qquad \tau_{\alpha\beta} = \frac{\omega}{\ell}\,\vartheta_{\alpha\beta}\,. \tag{67}$$

In order to find the tensor components, we develop the 2-forms Σ_α and $\tau_{\alpha\beta}$ with respect to the 2-form η_α:

$$\Sigma_\alpha =: \mathfrak{t}_\alpha{}^\beta\,\eta_\beta\,, \qquad \tau_{\alpha\beta} =: \mathfrak{s}_{\alpha\beta}{}^\gamma\,\eta_\gamma\,. \tag{68}$$

Inversion of (68) and use of (67) yields for the force stress tensor and the moment stress tensor

$$\mathfrak{t}_\alpha{}^\beta = -p\delta^\beta_\alpha = \frac{\omega^2}{\ell}\,\delta^\beta_\alpha\,, \qquad \mathfrak{s}_{\alpha\beta\gamma} = \frac{\omega}{\ell}\,\eta_{\alpha\beta\gamma}\,. \tag{69}$$

We have found a constant hydrostatic *pressure* $-\omega^2/\ell$ and a constant *torque* ω/ℓ, exactly as foreseen by Cartan. Thus, the spiral staircase is an exact solution of the 3d Einstein-Cartan theory (with Euclidean signature) carrying constant pressure and constant torque as sources.

4.2 3d Poincaré gauge theory of gravity with Mielke-Baekler Lagrangian

The EC-theory has a Lagrangian linear in the curvature. As a consequence, the Lorentz connection is non-propagating. If one allows for higher order terms, as in the Mielke-Baekler Lagrangian, the Lorentz connection becomes 'liberated'. Such theories are Poincaré

[9] Roughly speaking, we could imagine ℓ as the distance between neighboring dislocation lines of the dislocation forests mentioned above, see [19]; depending on the state of the crystal, this length ℓ could typically be of the order of some 50 nm.

[10] Also here the signs in [22] are opposite.

gauge theories. By variation of the Mielke-Baekler Lagrangian one arrives at the field equations

$$\frac{\chi}{2}\eta_{\alpha\beta\gamma}\,R^{\beta\gamma}+\Lambda\,\eta_\alpha-\frac{\theta_{\rm T}}{\ell}T_\alpha \;=\; \ell\Sigma_\alpha\,, \tag{70}$$

$$\frac{\chi}{2}\eta_{\alpha\beta\gamma}\,T^{\gamma}-\frac{\theta_{\rm T}}{2\ell}\vartheta_{\alpha\beta}-\theta_{\rm L}\ell\,R_{\alpha\beta} \;=\; \ell\tau_{\alpha\beta}\,. \tag{71}$$

García et al. [14] looked for static circularly symmetric *vacuum* solutions of these field equation. In fact, for the 3d Einsteinian case in a Riemannian space such a solution had been found by Bañados et al. [4]. García et al. generalized this so-called BTZ-solution (Bañados, Teitelboim, Zanelli) to a 'BTZ-solution with torsion' [14]. The details for this solution can be found in [14], Table I. If one puts the effective cosmological constant to zero, $\Lambda_{\rm eff}=0$, this vacuum solution has the torsion and curvature

$$T^\alpha=2\frac{\mathcal{T}}{\ell}\,\eta^\alpha\,,\qquad R^{\alpha\beta}=-\left(\frac{\mathcal{T}}{\ell}\right)^2\vartheta^{\alpha\beta}\,,\qquad \widetilde{R}^{\alpha\beta}=0\,. \tag{72}$$

Here the constant $\mathcal{T}$ can be expressed in terms of the coupling constants according to

$$\mathcal{T}=-\frac{\theta_{\rm T}\chi}{2(\chi^2+2\theta_{\rm T})\theta_{\rm L}}\,. \tag{73}$$

If we compare (72) with (20), we see that (apart from a probably signature dependent sign) the torsion and the curvatures coincide. Consequently, a subcase of the vacuum BTZ-solution with torsion carries the torsion and the curvature of Cartan's spiral staircase. We stress that, in contrast to the solution (65) with (66), where we only have to assume constant sources, in (72) we have an exact *vacuum* solution. This was outside the scope of Cartan in 1922.

5 The translation gauge theory of dislocations in three dimensions

Let be given a solid body with crystalline structure. Often such solids contain lattice defects that may be created during the growing of the crystal or during plastic deformation. One-dimensional lattice defects are dislocation lines that (in a cubic primitive crystal) are of two types: Edge dislocations (see Figure 7) and screw dislocations (see Figure 8). From comparing Figure 8 with the spiral staircase Figure 1, it is clear that the geometry of Cartan's spiral staircase can be represented by a set of three perpendicular constant 'forests' of *screw dislocations* of equal strength. It is our goal to show that the spiral staircase emerges as a solution in the framework of the gauge theory of dislocations. In this theory the real crystal, containing dislocations, is modeled as a 3-dimensional space with teleparallelism (Weitzenböck space) the torsion of which represents the dislocation

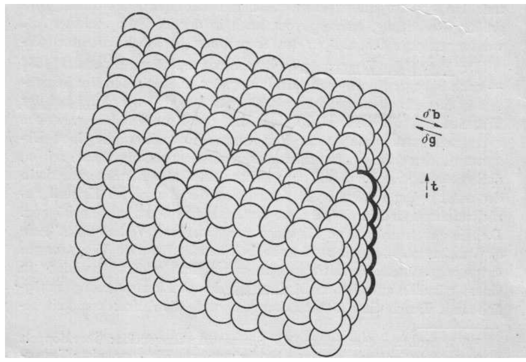

Figure 7: *Edge dislocation* after Kröner [31]: The dislocation line is parallel to the vector **t**. The Burgers vector $\delta\mathbf{b}$, characterizing the missing half-plane, is perpendicular to **t**. The vector $\delta\mathbf{g}$ characterizes the gliding of the dislocation as it enters the ideal crystal.

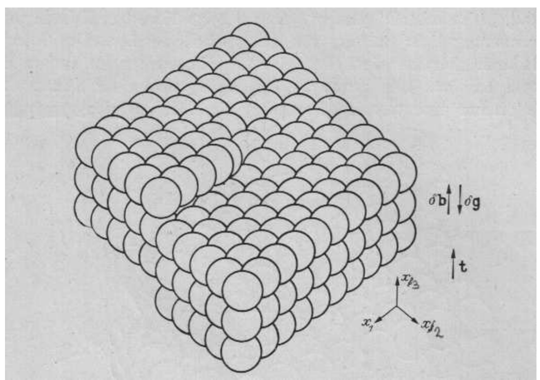

Figure 8: *Screw dislocation* after Kröner [31]: Here the Burgers vector is parallel to **t**.

density. By a suitable choice of the frames, it is always possible to 'gauge' the connection 1-form of the Weitzenböck space globally to zero [52],

$$\Gamma^{\alpha\beta} = 0 \qquad \text{(in suitable frames)}\,. \tag{74}$$

We dropped here the ξ that designated in Section 2 the non-Riemannian connection.

5.1 Foundations

Let us display the structure of the three-dimensional translational gauge theory of dislocations as proposed by Lazar [35, 36]. This theory follows the basic features of the metric-affine gauge theory of gravity, see the review [20].

At the outset we identify the torsion 2-form T^α with the *dislocation density*. Since subsequently we always assume orthonormal frames that nullify the connection according to (74), the torsion 2-form reads

$$T^\alpha = \mathrm{d}\vartheta^\alpha = \frac{1}{2}\, T_{\beta\gamma}{}^\alpha\, \vartheta^\beta \wedge \vartheta^\gamma . \tag{75}$$

For the physical interpretation of quantities, the knowledge of the physical dimensions is decisive. The dimension of the coframe ϑ^α is $[\vartheta^\alpha]$ = length and $[e_\alpha]$ = 1/length. Thus, the dimension of T^α is $[T^\alpha]$ = length—it is called the *absolute* dimension of T— and the *physical* dimension (of its components) turns out to be $[T_{\beta\gamma}{}^\alpha]$ = 1/length. The torsion 2-form satisfies the first (or translational) Bianchi identity

$$\mathrm{d}T^\alpha = 0 . \tag{76}$$

In a second step, the coframe 1-form ϑ^α is identified with the (incompatible) *elastic distortion* 1-form known from continuum mechanics. Accordingly, we take ϑ^α and T^α as field variables of the dislocation gauge theory.

Now we can set up the total Lagrangian 3-form L_{tot} describing dislocations in an incompatible elastic continuum, with absolute dimension $[L_{\mathrm{tot}}]$ = energy. It is given by the sum of the *elastic* Lagrangian L of the material continuum and the *gauge* Lagrangian V of the dislocation fields:

$$L_{\mathrm{tot}} = L(\vartheta^\alpha) + V(\vartheta^\alpha, T^\alpha) . \tag{77}$$

The covector-valued 2-form of the elastic (force) stress is defined by

$$\Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha} . \tag{78}$$

It has the absolute dimension $[\Sigma_\alpha]$ = force. In general, this stress Σ_α is asymmetric. Analogously to (78), we can define the dislocation stress, a covector-valued 2-form, as

$$E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \rfloor V + \left(e_\alpha \rfloor T^\beta\right) \wedge H_\beta . \tag{79}$$

the absolute dimension of which is $[E_\alpha]$ = force. Eshelby [13] called such a type of expression an "elastic energy-momentum"; if E_α is integrated over a 2-dimensional closed surface, it yields the force on defects (here dislocations) within the surface. Similar to Σ_a, the dislocation stress E_α is asymmetric in general.

Torsion T^α has the status of a gauge field strength in the formalism. Accordingly, we can define the attached *excitation* 1-form as

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha} , \tag{80}$$

with $[H_\alpha]$ = force. It is the response of the gauge Lagrangian V to the torsion 2-form T^α.

The moment stress $\tau_{\alpha\beta} = -\tau_{\beta\alpha}$ is coupled to the contortion 1-form $K_{\alpha\beta} = -K_{\beta\alpha}$ according to

$$\tau_{\alpha\beta} = \frac{\partial V}{\partial K^{\alpha\beta}} , \qquad \text{and} \qquad \tau_{\alpha\beta} = \vartheta_{[\alpha} \wedge H_{\beta]} , \tag{81}$$

with the dimension $[\tau_{ab}] = \text{force} \times \text{length} = \text{moment}$, since $[K_{\alpha\beta}] = 1$. The last formula in (81), analogously to (15), can be inverted as follows:

$$H_\alpha = -2e_\beta \rfloor \tau_\alpha{}^\beta + \frac{1}{2}\,\vartheta_\alpha \wedge (e_\beta \rfloor e_\gamma \rfloor \tau^{\beta\gamma})\ . \tag{82}$$

Since, according to (75), T^α depends on ϑ^α, the independent variable of the variational principle is ϑ^α. Independent variation yields the *Euler-Lagrange equation* of dislocation gauge theory:

$$\frac{\delta L_{\text{tot}}}{\delta\vartheta^\alpha} \equiv \mathrm{d}\,\frac{\partial L_{\text{tot}}}{\partial T^\alpha} + \frac{\partial L_{\text{tot}}}{\partial\vartheta^\alpha} = 0\,. \tag{83}$$

By means of (80), (78), and (79), we can rewrite it as:

$$\mathrm{d}H_\alpha - E_\alpha = \Sigma_\alpha\,. \tag{84}$$

This is a Yang-Mills type field equation. The sum of the two stresses Σ_α and E_α constitutes the source of the excitation H_α.

The field equation implies the *force equilibrium*: we differentiate (84) and find the law

$$\mathrm{d}(E_\alpha + \Sigma_\alpha) = 0\,. \tag{85}$$

The total stress is apparently in equilibrium. From this equation we can read off the covector-valued *Peach-Koehler* 3-form as

$$f_\alpha := \mathrm{d}\Sigma_\alpha = -\mathrm{d}E_\alpha = (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta\,. \tag{86}$$

It represents the force density acting on a dislocation.

The *moment equilibrium* requires a bit of algebra. We start with the field equation (84) and compute the antisymmetric piece of the total stress, use (81), and find

$$\mathrm{d}\tau_{\alpha\beta} - T_{[\alpha} \wedge H_{\beta]} + \vartheta_{[\alpha} \wedge \big(E_{\beta]} + \Sigma_{\beta]}\big) = 0\,. \tag{87}$$

Apart from the nonlinear term $-T_{[\alpha} \wedge H_{\beta]}$, this is exactly the expected law.

This represents the general framework of the dislocation gauge theory. Now we have to specify a *constitutive laws.* For the *excitation* in a local, linear, isotropic continuum we have

$$H_\alpha = {}^\star\sum_{I=1}^{3} a_I\,{}^{(I)}T_\alpha\,, \tag{88}$$

wherein ${}^{(I)}T_\alpha$ are the irreducible pieces (21), (22), and (23) of the torsion and a_1, a_2, and a_3 nonnegative constitutive moduli with dimension: $[a_I] = \text{force}$. For the elastic (force) *stress*, we assume a Hooke type law

$$\Sigma_\alpha = {}^\star\sum_{I=1}^{3} c_I\,{}^{(I)}\vartheta_\alpha\,, \tag{89}$$

where c_1, c_2, and c_3 are nonnegative constitutive moduli with the dimension: $[c_I]$ = force/(length)^2. With (88) and (89) and Euler's theorem for homogeneous functions, we can rewrite the gauge (or dislocation) Lagrangian and the elastic Lagrangian, respectively, as

$$V = -\frac{1}{2}\,H_\alpha \wedge T^\alpha = \frac{1}{2}\,\tau_{\alpha\beta} \wedge K^{\alpha\beta} \quad \text{and} \quad L = \frac{1}{2}\,\Sigma_\alpha \wedge \vartheta^\alpha\,. \tag{90}$$

Summing up, the dislocation theory we displayed encompasses two kinds of asymmetric force stresses (namely E_α and Σ_α) and one type of moment stress $\tau_{\alpha\beta}$. Equivalent to $\tau_{\alpha\beta}$ is the excitation H_α that plays a fundamental role in the Yang-Mills type field equation (84). Compactly written, we have

$$T^\alpha = \mathrm{d}\vartheta^\alpha\,, \tag{91}$$

$$\mathrm{d}H_\alpha - E_\alpha = \Sigma_\alpha\,, \tag{92}$$

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha} \qquad \text{or} \qquad H_\alpha \approx g_{\alpha\beta}\,{}^\star \sum_{I=1}^{3} a_I\,{}^{(I)}T^\beta\,, \tag{93}$$

$$E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \rfloor V + \left(e_\alpha \rfloor T^\beta\right) \wedge H_\beta\,, \tag{94}$$

$$\Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha} \qquad \text{or} \qquad \Sigma_\alpha \approx g_{\alpha\beta}\,{}^\star \sum_{I=1}^{3} c_I\,{}^{(I)}\vartheta^\beta\,, \tag{95}$$

$$f_\alpha = (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta \tag{96}$$

(ϑ^α = distortion, T^α = torsion = dislocation density, V = gauge Lagrangian $\sim$ torsion^2, H_α = excitation, Σ_α = force stress, E_α = dislocation stress, L = matter Lagrangian $\sim$ distortion^2, f_α = Peach-Koehler force density).

We could add simple consequences of the scheme, namely the homogeneous field equation and an alternative version of the inhomogeneous field equation,

$$\mathrm{d}T^\alpha = 0\,, \tag{97}$$

$$\mathrm{d}\tau_{\alpha\beta} - T_{[\alpha} \wedge H_{\beta]} + \vartheta_{[\alpha} \wedge E_{\beta]} = \vartheta_{[\alpha} \wedge \Sigma_{\beta]}\,, \tag{98}$$

with the spin moment stress $\tau_{\alpha\beta} = \vartheta_{[\alpha} \wedge H_{\beta]}$.

In a more recent development, one of us ref. [37] investigated the Higgs mechanism in the framework of the translation gauge theory of dislocations. At the same time, he discussed an anisotropic version of the dislocation gauge theory as well as a Chern-Simons type theory of dislocation.

5.2 Cartan's spiral staircase as a solution of dislocation gauge theory

We want to model Cartan's spiral staircase in the gauge theory of dislocations as a homogeneous distributions of three perpendicular forests of screw dislocations of equal strength.

Hence, for the dislocation density, we make the ansatz

$$T^\alpha \equiv {}^{(3)}T^\alpha = \mathcal{A}\eta^\alpha \,. \tag{99}$$

The pitch of the helices is proportional to the constant $\mathcal{A}$ with the dimension $[\mathcal{A}] =$ 1/length. Thus, in the gauge (74), Eq.(17) yields the Levi-Civita connection

$$\widetilde{\Gamma}_{\alpha\beta} = K_{\alpha\beta} = -\frac{\mathcal{A}}{2}\,\eta_{\alpha\beta} \,. \tag{100}$$

and, as a consequence therefrom, the Riemannian curvature

$$\widetilde{R}^{\alpha\beta} = -\frac{\mathcal{A}^2}{4}\,\vartheta^{\alpha\beta} \,. \tag{101}$$

Due to (74), we find

$$T^\alpha = -\widetilde{\Gamma}_\beta{}^\alpha \wedge \vartheta^\beta \,. \tag{102}$$

Note that (101) *deviates* from the original result (20) of the spiral staircase.

In order to determine the excitation, we insert (99) into the constitutive law (88):

$$H_\alpha = a_3\mathcal{A}\,{}^\star\eta_\alpha = a_3\mathcal{A}\,{}^{\star\star}\vartheta_\alpha = a_3\mathcal{A}\,\vartheta_\alpha \,. \tag{103}$$

In turn, the *moment stress* reads help of (81):

$$\tau_{\alpha\beta} = \vartheta_{[\alpha} \wedge H_{\beta]} = a_3\mathcal{A}\vartheta_{\alpha\beta} \,. \tag{104}$$

Inversion yields the components of the moment stress tensor, see (68),

$$\mathfrak{s}_{\alpha\beta\gamma} = \mathfrak{s}_{[\alpha\beta\gamma]} = a_3\mathcal{A}\eta_{\alpha\beta\gamma} \,. \tag{105}$$

This is apparently a pure constant torque.

Turning now to the *force stress,* we first compute the dislocation stress E_α which, for a quadratic Lagrangian, can be rewritten as

$$E_\alpha = \frac{1}{2}\left[\left(e_\alpha \rfloor T^\beta\right) \wedge H_\beta - \left(e_\alpha \rfloor H_\beta\right) T^\beta\right] \,. \tag{106}$$

We substitute (99) and (103) and find

$$E_\alpha = \frac{1}{2}\,a_3\mathcal{A}^2\left[\left(e_\alpha \rfloor \eta^\beta\right) \wedge \vartheta_\beta - \left(e_\alpha \rfloor \vartheta_\beta\right)\eta^\beta\right] = \frac{1}{2}\,a_3\mathcal{A}^2\eta_\alpha \,. \tag{107}$$

This is a hydrostatic pressure quadratic in $\mathcal{A}$, as an inversion à la (68) shows explicitly:

$$\mathfrak{t}_\alpha{}^\beta = \frac{1}{2}\,a_3\mathcal{A}^2\delta_\alpha^\beta \,. \tag{108}$$

Note that $\mathcal{A}$ enters this equation quadratically.

By means of the *field equation* (84) we can determine the elastic stress Σ_α. We first differentiate (103) and find

$$\mathrm{d}H_\alpha = a_3\mathcal{A}\,\mathrm{d}\vartheta_\alpha = a_3\mathcal{A}^2\,\eta_\alpha\,. \tag{109}$$

Now we turn to (84) and substitute (109) and (107) into it:

$$\Sigma_\alpha = \mathrm{d}H_\alpha - E_\alpha = \frac{1}{2}\,a_3\mathcal{A}^2\,\eta_\alpha\,. \tag{110}$$

Therefore, we have for both stress 2-forms

$$\Sigma_\alpha = E_\alpha = \frac{1}{2}\,a_3\mathcal{A}^2\,\eta_\alpha\,, \tag{111}$$

and the total stress add up to

$$\Sigma_\alpha + E_\alpha = a_3\mathcal{A}^2\,\eta_\alpha\,. \tag{112}$$

If we denote the total stress tensor of the left-hand-side of (112) by ${}^{\rm tot}\mathfrak{t}_\alpha{}^\beta$, compare (68), then we find again a hydrostatic pressure, namely

$${}^{\rm tot}\mathfrak{t}_\alpha{}^\beta = -\,{}^{\rm tot}p\delta_\alpha^\beta = a_3\,\mathcal{A}^2\,\delta_\alpha^\beta\,. \tag{113}$$

Accordingly, collecting our results, the force and moment stresses turn out to be

$${}^{\rm tot}\Sigma_\alpha = a_3\mathcal{A}^2\eta_\alpha\,, \qquad \tau_{\alpha\beta} = a_3\mathcal{A}\vartheta_{\alpha\beta}\,. \tag{114}$$

As a check we differentiate the total force stress

$$\mathrm{d}\,{}^{\rm tot}\Sigma_\alpha = a_3\mathcal{A}^2\mathrm{d}\eta_\alpha = -a_3\mathcal{A}^2\eta_{\alpha\beta}\wedge T^\beta = -a_3\mathcal{A}^3\eta_{\alpha\beta}\wedge\eta^\beta = 0\,, \tag{115}$$

since $\eta_{\alpha\beta}\wedge\eta^\beta = {}^\star\vartheta_{\alpha\beta}\wedge{}^\star\vartheta^\beta = \vartheta^\beta\wedge\vartheta_{\alpha\beta} = 0$. Hence the force equilibrium law (85) is guaranteed. Moreover, even the Peach-Koehler 3-form (86) itself is zero:

$$f_\alpha = (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta = -\frac{1}{2}\,a_3\mathcal{A}^3\eta_{\alpha\beta}\wedge\eta^\beta = 0\,. \tag{116}$$

Thus, the elastic stress and the dislocation stress are conserved, separately, $\mathrm{d}\Sigma_\alpha = 0$ and $\mathrm{d}E_\alpha$. Similarly, we obtain

$$\mathrm{d}\tau_{\alpha\beta} = a_3\mathcal{A}\mathrm{d}\vartheta_{\alpha\beta} = a_3\mathcal{A}\,T_{[\alpha}\wedge\vartheta_{\beta]} = a_3\mathcal{A}^2\,\eta_{[\alpha}\wedge\vartheta_{\beta]} = 0\,. \tag{117}$$

Thus, the moment equilibrium law (87) is also fulfilled since ${}^{\rm tot}\Sigma_\alpha$ is symmetric and $T_{[\alpha}\wedge H_{\beta]} = 0$.

If we look back to our scheme (91) to (96), then we recognize that all equations are now fulfilled with the exception of (91) and the constitutive law in (95). Let us first turn to the former equation. Since the torsion is known, we can write down (91) explicitly:

$$T^\alpha = \mathcal{A}\eta^\alpha = \mathcal{A}\,{}^\star\vartheta^\alpha = \mathrm{d}\vartheta^\alpha = (\partial_i e_j{}^\alpha)\,\mathrm{d}x^i\wedge\mathrm{d}x^j\,. \tag{118}$$

Applying the star to the equation, we find

$$\mathcal{A}\vartheta^\alpha = (\partial_i e_j{}^\alpha)\,{}^\star(\mathrm{d}x^i \wedge \mathrm{d}x^j)\,. \tag{119}$$

With ${}^\star\,(\mathrm{d}x^i \wedge \mathrm{d}x^j) = \eta^{ijk}\mathrm{d}x_k$, we have

$$\mathcal{A}e_i{}^\alpha = \eta_i{}^{jk}(\partial_j e_k{}^\alpha)\,. \tag{120}$$

In symbolic notation we can write

$$\mathcal{A}\,\vartheta^\alpha = (\mathrm{curl}\,\vartheta)^\alpha\,. \tag{121}$$

This means that the object of anholonomity is constant, that is, we have a constant 'vorticity' field. This coincides with our intuition of the distribution of screw dislocations.

In order to solve (120) approximately, we linearize the coframe, $e_i{}^\alpha = \delta_i^\alpha + h_i{}^\alpha + \ldots$. We substitute it into (120) and find

$$\mathcal{A}\delta_i^\alpha = \eta_i{}^{jk}(\partial_j h_k{}^\alpha)\,. \tag{122}$$

Then we can read off the result $h_k{}^\alpha = \frac{\mathcal{A}}{2}\eta_k{}^{\alpha\ell}x_\ell$ or, in terms of the distortion 1-form

$$\vartheta^\alpha = \left(\delta_i^\alpha - \frac{\mathcal{A}}{2}\eta^\alpha{}_{ij}x^j\right)\mathrm{d}x^i\,. \tag{123}$$

The distortion describes a rotation perpendicular to the (αi)-plane.

For the 3 irreducible pieces of the distortion 1-form we write

$$\vartheta^\alpha = F_i{}^\alpha\,\mathrm{d}x^i = {}^{(1)}\vartheta^\alpha + {}^{(2)}\vartheta^\alpha + {}^{(3)}\vartheta^\alpha\,, \tag{124}$$

with the number of independent components $9 = 5 \oplus 3 \oplus 1$

$$ {}^{(1)}\vartheta^\alpha := \vartheta^\alpha - {}^{(2)}\vartheta^\alpha - {}^{(3)}\vartheta^\alpha\,, \tag{125}$$

$$ {}^{(2)}\vartheta^\alpha := \frac{1}{2}\left(F_i{}^\alpha - F^\alpha{}_i\right)\mathrm{d}x^i\,, \tag{126}$$

$$ {}^{(3)}\vartheta^\alpha := \frac{1}{3}\,\delta_i^\alpha\,F_k{}^k\,\mathrm{d}x^i\,. \tag{127}$$

With the help of (123) we find the three different pieces of the distortion as

$$ {}^{(1)}\vartheta^\alpha = 0\,, \qquad {}^{(2)}\vartheta^\alpha = -\frac{\mathcal{A}}{2}\eta^\alpha{}_{ij}x^j\,, \qquad {}^{(3)}\vartheta^\alpha = \delta_i^\alpha\,\mathrm{d}x^i\,. \tag{128}$$

In turn, the constitutive law (89) expresses the stress Σ_α in terms of the distortion (123). If we substitute (128) into (89) and compare it with (111), we find for the Hooke type moduli

$$c_2 = 0\,, \qquad c_3 = \frac{1}{2}\,a_3\mathcal{A}^2\,. \tag{129}$$

The elastic modulus c_3 corresponds to the compression modulus κ, with $c_3 = \kappa/3$. In this way, the dislocation modulus a_3 can be expressed in terms of the modulus of compression and the pitch of helices:

$$a_3 = \frac{2\kappa}{3\mathcal{A}^2} \,. \tag{130}$$

Thus, we conclude that Cartan's spiral staircase is a solution in the gauge theory of dislocations provided the moduli of the underlying material obey the conditions (129).

Summing up: In the framework of dislocation gauge theory, we found

$$T^\alpha = 2\omega\,\eta^\alpha\,, \qquad R^{\alpha\beta} = 0\,, \qquad \widetilde{R}^{\alpha\beta} = -\omega^2\,\vartheta^{\alpha\beta}\,, \quad \text{with} \quad \omega = \frac{\mathcal{A}}{2}\,. \tag{131}$$

If we compare with Cartan's spiral staircase (20), we observe that there is a difference to second order in ω: the Riemann and the Riemann-Cartan curvature exchange places. For Cartan's spiral staircase we have a teleparallelism with respect to the Riemannian (or Levi-Civita) connection, for the forests of the screw dislocations the underlying Riemann-Cartan space is teleparallel. Insofar the dislocation theory led to a slightly different result as compared to the spiral staircase. This is not unexpected: Let us consider the case of a constant dislocations density in a real crystal. Then dislocation theory for geometrical reasons predicts that the underlying connection, in terms of which the torsion is defined, has to be teleparallel, see [33].

A second remark is in order: The torque stress $\tau_{\alpha\beta} = a_3\mathcal{A}\vartheta_{\alpha\beta}$ is linear in the pitch $\mathcal{A}$ and the hydrostatic pressure $\Sigma_\alpha = \frac{1}{2}a_3\mathcal{A}^2\eta_\alpha$ quadratic in $\mathcal{A}$. Thus, the pressure corresponds to a nonlinear effect. This is consistent with the screw dislocation distribution specified. In linear elasticity, the stress fields of screw dislocations are represented by pure shear stresses. Therefore, a constant pressure, caused by screw dilations, can only occur in the nonlinear regime. Hence our picture is apparently consistent.

6 Discussion

As we have seen, Cartan's spiral staircase corresponds to a homogeneous and isotropic torsion distribution in *three* dimensions. Is it also possible to have such torsion distributions in *two* dimensions? Geometrically this has been demonstrated by Schuecking and Surowitz [59], Sec.14 (see also [57, 58, 44]). We expect that it is likewise possible in dislocation theory.

Thus, we have methods to visualize two- and three-dimensional distributions of homogeneous and isotropic torsion, and this may help to understand the corresponding situations in gravitational physics, in particular in the framework of the Poincaré gauge theory of gravitation. We wonder whether one can find in this framework a simple cosmological model[11] with constant and isotropic torsion.

[11]For such models one should compare, for instance, [1] and [60].

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