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Application of Symmetry and Symmetry Analyses to Systems of First-Order Equations Arising from Mathematical Modelling in Epidemiology

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We examine the integrability, in terms of the Painlevé singularity analysis and of the Lie symmetry analysis, of systems of nonlinear first-order ordinary differential equations that arise in the particular area of Mathematical Modelling known as Rational Epidemiology. These analyses are presented as being complementary to the standard analysis using the methods of Dynamical Systems. The importance to obtain complete understanding of evolution of epidemics – one need think only of the potential for devastation by HIV-AIDS in African countries or the more recent threat posed by SARS – demands that all possible approaches of analysis be used. The concept of decomposed systems is introduced as illustrating some rather attractive mathematical properties of certain classes of systems of equations arising in Mathematical Modelling. Such systems allow the mathematician some opportunity to enjoy Mathematics even in the context of quite strict applications.

1 Introduction

There are four standard approaches to the analysis of a system of nonlinear first-order ordinary differential equations of the type which arise frequently in Mathematical Modelling. The approaches comprise numerical computation, dynamical systems, singularity analysis and symmetry analysis, all of which possess extensive literatures.

This work is concerned with Mathematical Modelling with particular emphasis on Rational Epidemiology. In particular our specialisation is to systems of first-order ordinary differential equations, i.e. compartmental models, in contrast to systems of partial differential equations. Thus we stress the evolutionary aspect of an epidemic in contrast to the dispersion in space, which happens when one uses partial differential equations. Obviously we can consider only a subset, and a simple one at that, of the class of models which can be investigated, but one can argue a case for the commencement with the simple. Here we concentrate upon the singularity and symmetry aspects of models. There are already many who have devoted much effort and ingenuity to the application of the methods of Dynamical Systems to and/or the determination of numerical solutions of these models.

A particular subset, the recently christened 'decomposed' systems, attracts our attention for the mathematical relief provided in the midst of the human – equally animal or vegetable should the system to be so modelled – misery underlying our studies.

There are some general and particular reasons for our interest. To general reasons are the need for sophisticated Mathematical Modelling in an environment of finite resources and rampant consumerism and the need to determine whether we live in the midst of potentially chaotic systems. In the particular case of Rational Epidemiology there is a need to develop the mathematical models and their analyses to the fullest extent of our ability. In this modern era we have the burden of crowding and widespread intermingling of diverse population groups with the consequent dangers of the catastrophic spreading of deadly diseases. Even in the slower times the catastrophe of a pandemic could make its mark for many centuries. In the context of the African Continent the spectre of depopulation consequent upon HIV-AIDS is a reality. This provides a case study for mathematical modelling which in the magnitude of its implications is probably a worthy successor of the Black Death that devastated Europe 700 years ago.

The structure of this paper is as follows. In Sections 2, 3 and 4 we examine certain specific epidemiological models in terms of the symmetry and/or singularity analyses. In Section 5 we introduce the notion of 'decomposition' and indicate some of the properties of a particular example of a decomposed system. We conclude with some remarks and observations in Section 6.

2 A model for the transmission of HIV-AIDS

The model [24],

$$\dot{u}_1 = -\frac{\beta c u_1 u_2}{u_1+u_2+u_3} - \mu u_1, \qquad \dot{u}_2 = \frac{\beta c u_1 u_2}{u_1+u_2+u_3} - (\mu+\nu)\, u_2, \qquad \dot{u}_3 = \nu u_2 - \alpha u_3, \quad (1)$$

has been quite successful in replicating results reported in studies of HIV transmission in the San Francisco area [12,27]. The variables $u_i(t)$, $i = 1, 3$ represent that part of population which is HIV negative, is HIV positive and has AIDS respectively. The parameter μ is the death rate from normal causes, α the death rate from AIDS and ν the rate at which HIV positives develop AIDS. The parameters β and c represent the rates of infection and change of partner respectively.

By inspection one notes that the system (1) possesses the two Lie point symmetries

$$\Gamma_1 = \partial_t \qquad \text{and} \qquad \Gamma_2 = u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}$$

with the Abelian algebra $2A_1$.

The presence of the symmetry Γ_2, representing homogeneity in the dependent variables, precludes the possibility that the system (1) possess the Painlevé Property.

These two symmetries are insufficient for integrability in the sense of Lie. A third symmetry, which would constitute a three-dimensional solvable Lie algebra with these two, is required.

As a system of first-order equations (1) has an infinite number of Lie point symmetries. Infinity is not a satisfactory number. We use the second part of the method of reduction of order [18–20] to replace the system of three first-order equations by a system of one second-order equation and one first-order equation.

In the analysis of this system, obtained by the elimination of u_3 from (1a), one finds that the system is separable and linearisable in the case that $\alpha = \mu + \beta c$. The symmetry

$$\Gamma_3 = \exp\left[-(\mu+\nu)t\right]\partial_{u_2} + \frac{u_1+u_3}{u_2}\exp\left[-(\mu+\nu)t\right]\partial_{u_3}$$

provides the third element of the required solvable algebra.

The explicit solution to the system (1) is found to be

$$u_1 = \frac{e^{\nu t} c_2}{e^{\mu t}\left[e^{\nu t}\left(\beta\, c - \nu\right) c_1 + e^{\beta\, c\, t}\beta\, c\,\right]},$$

$$u_2 = \frac{\left(\beta\, c - \nu\right)\int \frac{e^{\beta\, c\, t + 2\nu t}}{\left(e^{\beta\, c\, t}\beta\, c + e^{\nu t}\beta\, c\, c_1 - e^{\nu t} c_1 \nu\right)^2}\, \mathrm{d}\, t}{\beta\, c\, c_2 + c_3} e^{\mu t + \nu t},$$

$$
\begin{aligned}
u_3 &= \frac{\left[e^{\nu t}\left(\beta\, c-\nu\right) c_1+e^{\beta\, c\, t}\nu\right] c_3}{e^{\beta\, c\, t+\mu t+\nu t}\left(\beta\, c-\nu\right)}+\frac{e^{\nu t}\, c_2}{e^{\mu t}\left[e^{\nu t}\left(\beta\, c-\nu\right) c_1+e^{\beta\, c\, t}\beta\, c\right]}\\
&\quad+\frac{\beta\, c\, c_2\left[e^{\nu t}\left(\beta\, c-\nu\right) c_1+e^{\beta\, c\, t}\nu\right]\int e^{\beta\, c\, t+2\nu t}\left(e^{\beta\, c\, t}\beta\, c\,+e^{\nu t}\beta\, c\, c_1-e^{\nu t}c_1\nu\right)^2 \mathrm{d}\, t}{\exp[\beta\, c\, t+\mu t+\nu t]}
\end{aligned}
$$

for general values of the parameters subject to the constraint $\alpha=\mu+\beta c$.

There is a considerable simplification if in addition $\beta c=2\nu$. The failure of the system to possess the Painlevé Property is quite manifest in this case for the solution is given in closed-form as

$$
\begin{aligned}
u_1 &= \frac{c_1}{e^{\mu t}\left(2e^{\nu t}+c_1c_2\right)},\\
u_2 &= \left[2e^{\nu t}\log\left(2e^{\nu t}+c_1c_2\right)c_1-2e^{\nu t}c_1+4e^{\nu t}c_3\right.\\
&\quad\left.+\log\left(2e^{\nu t}+c_1c_2\right)c_1^2c_2+2c_1c_2c_3\right]\big/\left[2e^{\mu t+\nu t}\left(2e^{\nu t}+c_1c_2\right)\right],\\
u_3 &= \left[e^{\nu t}\log\left(2e^{\nu t}+c_1c_2\right)c_1-2e^{\nu t}c_1+2e^{\nu t}c_3\right.\\
&\quad\left.+\log\left(2e^{\nu t}+c_1c_2\right)c_1^2c_2+2c_1c_2c_3\right]\big/\left[2e^{\mu t+2\nu t}\right].
\end{aligned}
$$

One notes with interest that the constraint, $\alpha=\mu+\beta c$, which leads to the integrability of the system (1) in the sense of Lie, fits quite well with observational data accumulated over the last twenty years in the United States.

This coincidence of a relationship between the parameters of a system leading to better properties in terms of integrability and the practical realisation of that relationship in terms of the modelling of reality is intriguing and is sufficient to prompt one to pursue the possibility of a similar coincidence in other models.

3 The classic $S-I-R$ model

The model introduced by Kermack and McKendrick [13] to describe the effects of the Black Death in the seventeenth century is

$$
\dot S=-rSI,\qquad \dot I=rSI-aI,\qquad \dot R=aI, \tag{2}
$$

where S is the proportion of the population susceptible to the infection, I the proportion of population infected by the infection and R that proportion of population removed from consideration either through recovery with immunity or death. The parameters r and a represent the rate of infection and the rate of removal respectively.

When we add the three equations comprising (2), we have

$$
(S+I+R)^{\cdot}=0,
$$

i.e. $\dot N=0$, where $N=S+I+R$.

Obviously $\dot N=0 \implies N=\text{const}=1$.

The parameter-free form of the system (2) is

$$
\dot x=-xy,\qquad \dot y=xy-y,\qquad \dot z=y, \tag{3}
$$

where

$$
S=\frac{a}{r}x,\qquad I=\frac{a}{r}y,\qquad R=\frac{a}{r}z,\qquad t=\frac{t}{r},
$$

with the obvious summation $(x+y+z)^{\cdot}=0$.

The Painlevé analysis of the system (3) fails immediately since the exponents of the leading order terms are not all negative. An attempt to rescue the Painlevé analysis by the dropping of the third member of (3) is encouraging since the exponent of the leading order term of the remaining two variables is -1 and the resonances are at ± 1. Unfortunately the system fails the test for consistency. Consequently a logarithmic term must be introduced and this destroys the integrability in the sense of Painlevé.

In the system (3) the variable z is ignorable and the system is a candidate for the application of the technique of reduction of order. The quotients (3a)/(3c) and (3b)/(3c) are

$$\frac{\mathrm{d}x}{\mathrm{d}z} = -x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = x - 1. \tag{4}$$

We ignore the trivial route to the solution of the system (4) and persist with the spirit of the method of reduction of order to reduce the pair of first-order equations to a single second-order equation through the elimination of the dependent variable x. We obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \frac{\mathrm{d}y}{\mathrm{d}z} = 1, \tag{5}$$

which is a linear second-order ordinary differential equation in $y(z)$, indeed, it is a quasi-first-order equation in the first derivative of the function.

The integration of (5) and hence (4) is trivial. However, in the case of (3c) we have

$$\dot{z} = A + z - B\mathrm{e}^{-z},$$

where A and B are the constants of integration introduced at the solution of (5), the quadrature of which in closed-form is not possible.

We note that in the reduction of the system (3) to (5) we have reduced the nonlinear system to a linear system and also that (5), as a linear second-order ordinary differential equation, possesses eight Lie point symmetries.

4 Some curious facets of gonorrhoea

A somewhat simplified model [16] for the sexually transmitted disease gonorrhoea is described by the system

$$\dot{S} = -sSI^* + aI, \qquad \dot{I} = rSI^* - aI, \qquad \dot{S}^* = -r^*S^*I + a^*I^*, \qquad \dot{I}^* = r^*S^*I - a^*I^*, \tag{6}$$

in which the lower case letters are the rate constants and the symbols for susceptibles and infectives are obvious. The population is divided into two groups, male and female, and each of these two groups is divided into susceptibles and infectives. The females are distinguished by the possession of an asterisk. The model assumes free mixing which is generally not a valid assumption for sexually transmitted diseases. However, as one is informed, the properties of gonorrhoea are such to permit this assumption. The simplification in the model is that, since gonorrhoea does not confer immunity and so a person who has been treated returns to the classes of susceptibles, the time of treatment, during which one hopes the interaction between infectives and susceptibles is suspended, is regarded as negligible. The populations of males and females are assumed to be constant.

There are three conservation laws, *videlicet*

$$\dot{N} = 0, \qquad \dot{N}^* = 0 \qquad \text{and} \qquad \dot{P} = 0, \tag{7}$$

where $N = S + I$, $N^* = S^* + I^*$ and $P = N + N^*$.

The leading order behaviour is given by

$$S = -\frac{1}{r^*}\tau^{-1}, \qquad I = \frac{1}{r^*}\tau^{-1}, \qquad S^* = -\frac{1}{r}\tau^{-1}, \qquad I^* = \frac{1}{r}\tau^{-1} \tag{8}$$

and the resonances occur at -1 and $1(3)$.

The consistency of the system is determined by the substitution of

$$S = a_i\tau^{i-1}, \qquad I = b_i\tau^{i-1}, \qquad S^* = c_i\tau^{i-1}, \qquad I^* = d_i\tau^{i-1}$$

into the full system (6) with the leading order terms as given in (8). Fortunately the location of the resonance at $+1$ makes the calculation fairly easy. At the positive resonance we obtain the system

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{r}{r^*} \\ -1 & 0 & 0 & \frac{r}{r^*} \\ 0 & -\frac{r^*}{r} & 1 & 0 \\ 0 & \frac{r^*}{r} & -1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} -\frac{a}{r^*} \\ \frac{a}{r^*} \\ -\frac{a^*}{r} \\ \frac{a^*}{r} \end{bmatrix}.$$

Performance of the row operations $R_2' = R_2 + R_1$ and $R_4' = R_4 + R_3$ leads to the simpler system

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{r}{r^*} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{r^*}{r} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} -\frac{a}{r^*} \\ 0 \\ -\frac{a^*}{r} \\ 0 \end{bmatrix}$$

in which the consistency is self-evident.

We have the explicit relations

$$a_1 = \frac{1}{r^*}\left(a + rd_1\right) \qquad \text{and} \qquad c_1 = \frac{1}{r}\left(a^* + r^*b_1\right). \tag{9}$$

There are two arbitrary constants introduced at the resonance, i.e. the geometric multiplicity of the eigenvector is two. However, the algebraic multiplicity of the eigenvalue is three.

The Laurent expansion contains only three arbitrary constants – the two introduced in (9) and the location of the movable pole – so that it does not represent the general solution of the system (6).

To obtain the general solution one must introduce a logarithmic term at the resonance and this destroys the analytic nature of the solution.

We continue our analysis of the system (6) by transforming the system of first-order equations to a scalar higher-order equation. From (7) we obtain

$$I = N - S \qquad \text{and} \qquad I^* = N^* - S^*. \tag{10}$$

With (10) equations (6b) and (6d) may be written as

$$-\dot{S} = rS(N^* - S^*) - a(N - S), \tag{11}$$

$$-\dot{S}^* = r^* S^*(N - S) - a^*(N^* - S^*). \tag{12}$$

We substitute for S^* from (11) into (12) to obtain a single second-order equation for S, *videlicet*

$$S\ddot{S} - \dot{S}^2 - r^* S^2 \dot{S} + (r^* N + a^*) S\dot{S} + aN\dot{S} - r^* (N^* r + a) S^3 \\ + (NN^* r r^* + 2Nar^* + aa^*) S^2 - aN (r^* N + a^*) S = 0. \tag{13}$$

The first three terms in (13) are dominant. The exponent of the leading order term is -1 and the resonances are at ± 1. To establish that there is consistency when the coefficient of the resonant term firstly enters we examine the coefficients in (summation on repeated indices is implied)

$$(i-1)(i-2)a_i a_j \tau^{i+j-4} - (i-1)(j-1)a_i a_j \tau^{i+j-4} - (i-1)r^* a_i a_j a_k \tau^{i+j+k-4} \\ + (r^* N + a^*)(i-1)a_i a_j \tau^{i+j-3} + aN(i-1)a_i \tau^{i-2} - r^* (rN^* + a) a_i a_j a_k \tau^{i+j+k-3} \\ + (NN^* r r^* + 2aNr^* + aa^*) a_i a_j \tau^{i+j-2} - aA (r^* N + a^*) a_i \tau^{i-1} = 0.$$

The coefficient of τ^{-4} gives

$$2a_0^2 - a_2^2 + r^* a_0^3 = 0 \implies a_0 = -\frac{1}{r^*} \tag{14}$$

and the coefficient of τ^{-3} gives

$$2a_0 a_1 + r^* a_0(2a_2 a_1) - (r^* N + a^*)a_0^2 - r^*(rN^* + a)a_0^3 = 0. \tag{15}$$

When the value of a_0 from (14) is substituted into (15), the coefficient of a_1 is identically zero as one expects. The terms remaining impose the condition

$$r^* N + a = rN^* + a^*. \tag{16}$$

Subject to the constraint (16) on the parameters in (13) the latter equation has an analytic solution for $S(t)$. It follows from (11) that $S^*(t)$ is also analytic and from (10) that $I(t)$ and $I^*(t)$ are also analytic.

This analysis simply reinforces the conclusion reached above that the three-parameter solution obtained for the original system, (6), is analytic away from the movable polelike singularity.

This type of integrability, which occurs for specific values of the first integrals of the base system determined by the relationship (16), is something of a generalisation of the integrability which occurs when an integral takes a particular value, i.e. in the context of a configurational invariant [8, 23].

The solution obtained is not the general solution of (6) since it lacks the requisite four arbitrary constants of integration. However, we have demonstrated the existence of an analytic solution of (6) containing three arbitrary constants of integration. Such solutions are not unknown [11, p. 355].

In the past this type of solution has been interpreted as indicating that the system is at least integrable on a surface in the space of initial conditions [5, 22]. This interpretation has not been universally accepted [4]. Since the inference of an analytic solution has been the basis for the interpretation in the past, there has been room for disagreement over the interpretation. However, in this instance the interpretation is based upon explicit demonstration of the analytic nature of the solutions for each of the component functions of the system.

One instance does not prove a universal fact. On the other hand one instance does disprove the denial of the possibility.

The system (6) and the scalar equation (13) have the obvious Lie symmetry, ∂_t, which reflects their autonomy. Even with the constraint (16) there appears to be no other point symmetries. This is a little curious as one would expect some additional symmetry to match the better quality of the system in terms of integrability. Perhaps this is to be found as either a generalised or a nonlocal symmetry.

5 Decomposed systems

There are several standard models for the rate of change of a population which is of sufficient magnitude to be treated as a continuum. They are

$$\textbf{1.}\quad \frac{\mathrm{d}N}{\mathrm{d}t} = 0 \quad \Leftrightarrow \quad N(t) = N_0. \tag{17}$$

This is the trivial model and applies if one is considering a population over a period of time sufficiently small for the normal means of increase and decrease of population not to have any effect. The famous SIR model of Kermack and McKendrick [13], *videlicet*

$$\dot{S} = -rSI, \qquad \dot{I} = rSI - aI, \qquad \dot{R} = aI, \tag{18}$$

is an example for, if we sum the three equations, we obtain

$$(S + I + R)^{\cdot} = 0$$

which is just (17). One recalls that the system (18) was used to model the Great Plague of 1666-7 in London and did it very well.

$$\textbf{2.}\quad \frac{\mathrm{d}N}{\mathrm{d}t} = K \quad \Leftrightarrow \quad N(t) = N_0 + Kt. \tag{19}$$

The constant rate of increase of the population described by (19) enables one to do a little more than with (17) without a serious increase in mathematical difficulty. An integrable model of this genre is the $S - I - R$ system

$$\dot{S} = -\beta SI - \mu S + \gamma I + \mu K, \qquad \dot{I} = \beta SI - (\mu + \gamma)I, \qquad \dot{R} = \mu(S + I). \tag{20}$$

The sum of the three equations in the system (20) gives

$$(S + I + R)^{\cdot} = \mu K$$

which is precisely the form of (19).

$$\textbf{3.}\quad \frac{\mathrm{d}N}{\mathrm{d}t} = \sigma N \quad \Leftrightarrow \quad N(t) = N_0 \exp[\sigma t]. \tag{21}$$

This is the famous model publicised by the Englishman, Thomas Malthus, in 1798 [17]. An integrable model of this genre is the $S - I - S$ system [21]

$$\dot{S} = -\beta SI - \mu S + \gamma I + \mu K, \tag{22}$$
$$\dot{I} = \beta SI - (\mu + \gamma)I, \tag{23}$$

i.e. (20) when the third equation of the system is ignored in the analysis. We sum these two equations to obtain

$$(S + I)^{\cdot} = \mu K - \mu(S + I)$$

which is precisely of the type (21) if we set $S + I - K = N$.

$$\textbf{4.}\quad \frac{\mathrm{d}N}{\mathrm{d}t} = \sigma N\left(1 - \frac{N}{C}\right) \quad \Leftrightarrow \quad N(t) = \frac{CN_0}{N_0 + (C - N_0)\exp[-\sigma t]} \tag{24}$$

in which the additional parameter C is of the nature of a 'carrying capacity'. This variation was introduced in 1838 [25] by Verhulst to obviate the excesses to which the model of Malthus led.

These four models may be regarded as successive approximations of the same phenomenon. For a short period of time one could treat the population as a constant. For a longer period of time a constant rate of increase is not unreasonable. The next model is to assume that the rate is proportional to the existing population and finally one must take into account the limitations of the environment in which the population is growing by adding the additional term due to Verhulst.

Because these different models have validity for different regimes it is possible to find a mixture of growth rates in a population containing varied components. An early example of this is found in an extension to the model proposed by Volterra [26,3] of a predator-prey system of two types of fish in the Adriatic. The model of Volterra is

$$\dot{N} = aN - bPN, \qquad \dot{P} = -cP + dPN. \tag{25}$$

A similar model, connected to chemical reactions, was proposed by Lotka [15] about the same time.

The bilinear term PN is typical of Lotka–Volterra models. In this model the prey N is taken to have Malthusian growth with the implicit assumption that the predator P consumes sufficient of the prey for the Malthusian growth not to test the limits of the carrying capacity of the local environment. If this be not the case, the Malthusian term should be replaced by a logistic term – for some reason generally Verhulst is forgotten in the naming – so that (25a) is replaced by

$$\dot{N} = aN\left(1 - \frac{N}{C}\right) - bPN.$$

Since the Malthusian term in (25b) gives a reduction in the population of the predator, there is no need to introduce a logistic term here.

The four model equations, *videlicet* (17), (19), (21) and (24), are trivially integrable and possess the Painlevé Property.

The systems (18), (20) and (22)–(23) have the property that their sums are simply (17), (19) and (21) respectively.

The same behaviour is not found with the Lotka–Volterra model (25) except for unexpectedly agreeable values of the parameters.

Systems such as the $S - I - R$ model represented by (18) which can be summed to give a single scalar equation in one variable are called 'decomposed systems' [2] and may be viewed as decomposition of the single equation, (17) in the case of (18), according to some rule.

Under which rules of decomposition are the decomposed systems integrable?

The general decomposition cannot be expected to be integrable. We consider an example [6, 7, 14] which has its origin in the operator Yang–Baxter equations of Mathematical Physics, but we do so in reverse. The two-dimensional system

$$\dot{r} = 2rw, \qquad \dot{w} = r^2 + w^2 \tag{26}$$

can be considered to be the decomposition of the Riccati equation

$$\dot{z} = z^2, \tag{27}$$

where $z = r + w$. A further decomposition is

$$\dot{p} = 2pw + \lambda qw, \qquad \dot{q} = 2qw - \lambda wp, \qquad \dot{w} = (p+q)^2 + w^2, \tag{28}$$

where $r = p + q$.

As a Riccati equation (27) possesses the Painlevé Property and is integrable in terms of analytic functions. The first decomposed system (26) is equally integrable. The second decomposed system (28) is integrable in the sense that the dependent variables can be written as explicit expressions of the independent variable, but the system does not possess the Painlevé Property since its solution is not in terms of analytic functions.

A whole class of problems has been excluded from even the possibility of consideration here. We are concerned with populations with a rate of change which is either independent of the population or proportional in some sense to the total population. In some populations, microbes come to mind, this is not unreasonable as microbes become capable of replication rather quickly. On the other hand populations comprising more complicated species can be divided into the three classes of nonreproductive due to immaturity, reproductive and nonreproductive due to – shall we politely put it – postmaturity.

This class of population lies outside the class of decomposible systems considered here.

Even models for which the reproductive aspect is not of major relevance can still exhibit various ways to be decomposed from a simpler system. A model for HIV infection in a homosexual population,

$$\begin{aligned} &\dot{X} = B - \mu X - \lambda c X, \qquad \dot{Y} = \lambda c X - (\nu + \mu) Y, \\ &\dot{A} = p\nu Y - (d + \mu) A, \qquad \dot{Z} = (1 - p)\nu Y - \mu Z, \end{aligned} \tag{29}$$

sums to give the rate of change of the total population as

$$\dot{N} = B - \mu N - dA,$$

where B represents the recruitment rate of susceptibles into the population, μ is the natural rate of demise and d the additional rate of demise of the AIDS-afflicted portion of the population A.

In the absence of AIDS the system (29) fits into the scheme of decomposition. In the absence of AIDS perhaps the population being modelled would not attract so much interest.

Our present interest in decomposible systems is motivated by the existence of decomposed systems as models of actual phenomena, in particular epidemiology.

Decomposed systems entered the literature from a more mathematical direction. In 2002 Imai and Hirata [9] published their investigations of the existence of autonomous symmetries of the form

$$\Gamma = \phi_i(x)\partial_{x_i}$$

for the autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(x),$$

where $\mathbf{g}(x)$ and the coefficient functions $\phi_i(x)$ are analytic in the neighbourhood of some fixed point. Their interest was in the determination of integrable systems through an algebraic approach. As applications they considered n-dimensional homogeneous, that is to say quadratic, Lotka–Volterra systems of the form

$$\dot{x} = x_i \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, n. \tag{30}$$

Imai and Hirata considered some very specific examples of (30) which they termed the ladder system and the generalised ladder system [10].

The n-dimensional ladder system is the system (30) with the specific coefficient matrix

$$A = (a_{ij}) = \begin{bmatrix} 1 & 0 & \cdots & -n+2 \\ 2 & 1 & \cdots & -n+3 \\ \vdots & \vdots & & \vdots \\ n & n-1 & \cdots & 1 \end{bmatrix},$$

i.e. the elements of the matrix A are given by

$$a_{ij} = a_i + 1 - a_j \qquad \text{and} \qquad a_{i+1,j} - a_{ij} = 1.$$

The ladder system has certain general properties [1].

Proposition 1.

$$\det A = \begin{cases} 1, & n = 1, 2, \\ 0, & n > 2. \end{cases}$$

Corollary. *The rank of the matrix A is two for $n > 2$.*

Proposition 2. *The n-dimensional ladder system can be considered as the simplest Riccati equation for $\sum\limits_{i=1}^{n} x_i$.*

Proposition 3. *The ratio of two successive solutions of the ladder system is*

$$\frac{x_{i+1}}{x_i} = \frac{K_i}{t - t_0}.$$

Proposition 4. *The solution of the n-dimensional ladder system is given by*

$$x_{i+1} = x_1 \frac{\prod\limits_{j=1}^{i} K_j}{(t - t_0)^i}, \tag{31}$$

where

$$x_1 = -\frac{(t - t_0)^{n-2}}{\sum\limits_{j=0}^{n-1} \left(\prod\limits_{k=0}^{j} K_k \right) (t - t_0)^{n-1-j}}. \tag{32}$$

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