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Emmanuel Letellier

Fourier Transforms of Invariant Functions on Finite Reductive Lie Algebras

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To my parents

Preface

The present work is about the study of the *trigonometric sums* on finite reductive Lie algebras of Chevalley's type in the sense of [Spr76]. This subject has been introduced to me by my supervisors Gus Lehrer and Jean Michel in connection with [Leh96][Leh97] while I was starting my PhD under a co-tutelle agreement between the university Paris 6 and the university of Sydney.

The required background is the standard knowledge of the theory of connected reductive groups and finite groups of Lie type [Spr].

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Tokyo, July 2004 *Emmanuel Letellier*

Contents

1

Introduction

Here k is an algebraic closure of a finite field $\mathbb{F}_q$ with q a power of a prime p. Let G be a connected reductive algebraic group over k which is defined over $\mathbb{F}_q$. Let $\mathcal{G}$ be the Lie algebra of G. Then both $\mathcal{G}$ and the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathcal{G})$ are defined over $\mathbb{F}_q$. We denote by $F : G \to G$, $F : \mathcal{G} \to \mathcal{G}$ the Frobenius endomorphisms corresponding to these $\mathbb{F}_q$-structures. Assume that $\mu : \mathcal{G} \times \mathcal{G} \to k$ is a non-degenerate G-invariant symmetric bilinear form defined over $\mathbb{F}_q$ and let $\Psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ be a non-trivial additive character where $\overline{\mathbb{Q}}_\ell$ is an algebraic closure of the field of ℓ-adic numbers with ℓ a prime $\neq p$. We call *trigonometric sums* [Spr76] the G^F-invariant characters of the abelian group $\mathcal{G}^F$ which are of the form $y \mapsto \sum_{x \in \mathcal{O}} \Psi\big(\mu(y,x)\big)$ for some G^F-orbit $\mathcal{O}$ of $\mathcal{G}^F$. They form an orthogonal basis of the $\overline{\mathbb{Q}}_\ell$-vector space $\mathcal{C}(\mathcal{G}^F)$ of functions $\mathcal{G}^F \to \overline{\mathbb{Q}}_\ell$ which are constant on the G^F-orbits of $\mathcal{G}^F$. The *Fourier transform* $\mathcal{F}^{\mathcal{G}} : \mathcal{C}(\mathcal{G}^F) \to \mathcal{C}(\mathcal{G}^F)$ with respect to (μ, Ψ) is defined as follows:

$$\mathcal{F}^{\mathcal{G}}(f)(x) = |\mathcal{G}^F|^{-1} \sum_{y \in \mathcal{G}^F} \Psi\big(\mu(x,y)\big) f(y)$$

with $f \in \mathcal{C}(\mathcal{G}^F)$ and $x \in \mathcal{G}^F$. The trigonometric sums of $\mathcal{G}^F$ are thus (up to a scalar) the Fourier transforms of the characteristic functions of the G^F-orbits of $\mathcal{G}^F$.

The trigonometric sums were first studied by Springer [Spr71] [Spr76] in connection with the $\overline{\mathbb{Q}}_\ell$-character theory of finite groups of Lie type (i.e. finite groups of the form G^F): it was shown by Kazhdan [Kaz77], using the results of [Spr76], that the values of the Green functions of finite groups of Lie type [DL76] can be expressed (via the exponential map) in terms of the values of trigonometric sums of the form $y \mapsto \sum_{x \in \mathcal{O}} \Psi\big(\mu(y,x)\big)$ with $\mathcal{O}$ a semi-simple regular G^F-orbit of $\mathcal{G}^F$. In [Lus87] and [Lus92], Lusztig has outlined what should be the Lie algebra version of his character sheaves theory to study

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Fourier transforms and to give a general framework for computing the values of trigonometric sums. In particular, he has defined the "admissible complexes" on $\mathcal{G}$ as well as the generalized Green functions on $\mathcal{G}^F$ which coincide with the generalized Green functions on G^F [Lus85b] via any G-equivariant isomorphism from the nilpotent variety $\mathcal{G}_{nil}$ onto the unipotent variety G_{uni}. Within this framework, he was able to explain most of phenomena observed at first by Kawanaka like the existence of pairs (f, G) such that f is a nilpotently supported function on $\mathcal{G}^F$ invariant under the "modified" Fourier transforms [Kaw82].

In this book, we study trigonometric sums using the techniques developed principally by Lusztig to study the irreducible $\overline{\mathbb{Q}}_\ell$-characters of finite groups of Lie type. The first step is to define a "twisted" induction in the Lie algebra setting which fits to the study of trigonometric sums, that is, which commutes with Fourier transforms. Lehrer has proved [Leh96] that Harish-Chandra induction commutes with Fourier transforms, suggesting thus to define the required twisted induction as a generalization of Harish-Chandra induction. The definition of the twisted induction we give here (which is somehow a Lie algebra version of Deligne-Lusztig induction [DL76]) uses the "character formula" where the "two-variable Green functions" are defined in group theoretical terms and then transferred to the Lie algebra by means of a G-equivariant homeomorphism $\omega : \mathcal{G}_{nil} \rightarrow G_{uni}$ (see definition 3.2.13). Our definition of twisted induction (we call Deligne-Lusztig induction) is thus available if such a G-equivariant homeomorphism is well-defined which is the case if p is good for G [Spr69]. The author was informed that Lusztig already knew this definition at least when ω is the usual exponential map (unpublished). Let $\mathcal{L}$ be the Lie algebra of an F-stable Levi subgroup L of G and let $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}} : \mathcal{C}(\mathcal{L}^F) \rightarrow \mathcal{C}(\mathcal{G}^F)$ denote the Deligne-Lusztig induction. We conjecture the following commutation formula:

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}} \circ \mathcal{F}^{\mathcal{L}}(f) = \epsilon_G \epsilon_L \mathcal{F}^{\mathcal{G}} \circ \mathcal{R}^{\mathcal{G}}_{\mathcal{L}}(f) \qquad (*)$$

where $\epsilon_G = (-1)^{\mathbb{F}_q - rank(G)}$, $\mathcal{F}^{\mathcal{L}}$ is the Fourier transform with respect to $(\mu|_{\mathcal{L}\times\mathcal{L}}, \Psi)$, and $f \in \mathcal{C}(\mathcal{L}^F)$. If L is a Levi subgroup of an F-stable parabolic subgroup of G, then the formula (*) follows from a result of Lehrer [Leh96] since in that case $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}$ is the Harish-Chandra induction. If the function f is the characteristic function of a semi-simple regular orbit, then the formula (*) follows from Kazhdan-Springer's work [Spr76][Kaz77] assuming that p is large enough. When the prime p is acceptable (5.0.14), we define another twisted induction so-called geometrical induction (see 5.4.10) using the Lie algebra version of Lusztig's character sheaves theory. From the result of [Lus90], we prove that the geometrical induction coincides with the Deligne-Lusztig induction when q is large enough. Since the definition of geometrical induction does not involve any map $\mathcal{G}_{nil} \to G_{uni}$, it proves the independence of our definition of Deligne-Lusztig induction from the choice of such a map ω. Using the coincidence of these two twisted inductions, we prove the above commutation formula (*) in many cases (6.2.15, 6.2.17, 6.2.19). More precisely, we show (assuming that p is acceptable and q is large enough) that this commutation formula will be available in full generality (i.e. for any reductive group G) if we can verify that for any G such that G is either semi-simple of type A_n, or simple of type D_n,

(**) the constant coming from Fourier transforms [Lus87] (called Lusztig constant) attached to an F-stable "cuspidal pair" (C, ζ) [Lus84] with C a unipotent conjugacy class of G and ζ an irreducible G-equivariant local system on C, does not depend (up to a sign) on the $\mathbb{F}_q$-structure on G (see 6.2.18).

If $p > 3(h_o^G - 1)$ where h_o^G is the Coxeter number of G, then we express the Lusztig constant attached to (C, ζ) as a "generalized character sum" associated to the regular prehomogeneous vector space of Dynkin-Kostant type corresponding to C (see 6.2.25). Such a formula has been obtained by Digne-Lehrer-Michel [DLM97] in type A_n, by Kawanaka [Kaw86] in type E_8, F_4 and G_2, and by Waldspurger [Wal01] if G is of classical type, i.e. G is $SO_N(k)$ or $Sp_{2n}(k)$. Although this formula is not explicit enough to verify (**), it has been used by the previously named authors to compute explicitly the Lusztig constants in types A_n, E_8, F_4, G_2, and in the case where G is of classical type (see also [Gec99] for the simple adjoint case). In these cases, it is thus possible to verify the property (**).

Hence, to prove the conjecture (*) when $p > 3(h_o^G - 1)$, we are reduced to prove (**) in the case where G is the spin group $Spin_{2n}(k)$, i.e. the simply connected simple group of type D_n, which problem reduces to a problem on regular prehomogeneous vector spaces of Dynkin-Kostant type. As far as I know, the assertion (**) with $G = Spin_{2n}(k)$ is still an open problem.

Using the commutation formula (*), we reduce the computation of the trigonometric sums of $\mathcal{G}^F$ to the computation of the Lusztig constants (see above) attached to the F-stable cuspidal pairs on the F-stable Levi subgroups of G and the computation of the generalized Green functions. Lusztig has given an algorithm which reduces the computation of the values of the generalized Green functions to the computation of some roots of unity whose values are known in many cases. We thus have a method (up to the above conjecture) to compute the values of the trigonometric sums of $\mathcal{G}^F$. The commutation formula (*) has also other applications in the representation theory of finite groups of Lie type (see for instance [Let04]).

We shall now give a brief review of the content of the different chapters. In chapter 2, we give a review on algebraic groups and their Lie algebras of less accessible material. In particular we give for simple groups of type A_n or simply connected simple groups of type B_n, C_n or D_n, a necessary and sufficient condition on p to have a non-degenerate G-invariant bilinear form on $\mathcal{G}$. In chapter 3, we give the definition of Deligne-Lusztig induction in terms of two-variable Green functions, and we state its basic properties like transitivity, the Makey formula and commutation with duality. We also state our conjecture on the commutation formula (*) with Fourier transforms. In chapter 4, we give a review on perverse sheaves and local systems of what is needed. In chapter 5, we describe the theory of admissible complexes (character sheaves) on Lie algebras starting from [Lus87] [Lus92] and by adapting Lusztig's ideas [Lus84][Lus85b][Lus86a] to the Lie algebra case. While in [Lus87] [Lus92], the prime p is assumed to be large, here we give a strict bound on p using the results of chapter 2. Finally, using the theory of admissible complexes on the Lie algebra, we construct the geometrical induction and we prove (by transferring [Lus90, 1.14] to the Lie algebra case by means of a G-equivariant isomorphism $\mathcal{G}_{nil} \to G_{uni}$) its coincidence with the Deligne-Lusztig induction assuming that q is large enough. In chapter 6, we discuss the conjecture of chapter 3 and prove it in many cases. We first reduce the conjecture (*) to the case where f is a "cuspidal" function by using the coincidence of the two inductions. Then using a construction by Waldspurger [Wal01, chapter 2] to investigate the Frobenius action on the "parabolic induction" of "cuspidal orbital" perverse sheaves, the conjecture (*) is further reduced to the case where f is a cuspidal nilpotently supported function. From this, we reduce the conjecture (*) to the problem (**). We then state our main results on (*). Finally in chapter 7, we show how to compute the values of trigonometric sums.

2

Connected Reductive Groups and Their Lie Algebras

The geometrical objects considered are defined over an algebraically closed field k of characteristic p. In this chapter, we first introduce some notation which will be used throughout this book. We then discuss some properties about algebraic groups and their Lie algebras related to the characteristic p. These results will be used to give an explicit bound on p for which the main result of [Lus87] applies. For any prime r, we choose once for all an algebraic closure $\overline{\mathbb{F}}_r$ of the finite field $\mathbb{F}_r = \mathbb{Z}/r\mathbb{Z}$. Then we denote by $\mathbb{F}_{r^n}$ the unique extension of degree $n > 0$ of $\mathbb{F}_r$ in $\overline{\mathbb{F}}_r$.

2.1 Notation and Background

We denote by $\mathbb{G}_m$ the one-dimensional algebraic group $(k - \{0\}, \times)$, and by $\mathbb{G}_a$ the one-dimensional algebraic group $(k, +)$. Let H be a linear algebraic group over k, i.e. H is isomorphic to a closed subgroup of some $GL_n(k)$. We denote by 1_H the neutral element of H and by H^o the connected component of H containing 1_H. We denote by $\text{Lie}(H) = \mathcal{H}$ the Lie algebra of H (i.e. the tangent space of H^o at 1_H) and we denote by $[,]$ the Lie product on $\mathcal{H}$. The Lie algebras of $GL_n(k)$, $SL_n(k)$ and $PGL_n(k)$ are respectively denoted by $gl_n(k)$, $sl_n(k)$ and $pgl_n(k)$. Let $Z_H = \{x \in H | \forall y \in H, xy = yx\}$ be the center of H, and let $z(\mathcal{H}) = \{X \in \mathcal{H} | \forall Y \in \mathcal{H}, [X, Y] = 0\}$ be the center of $\mathcal{H}$. If $x \in H$, we denote by x_s the semi-simple part of x and by x_u its unipotent part. If $X \in \mathcal{H}$, then X_s denotes the semi-simple part of X and X_n denotes its nilpotent part.

For an arbitrary morphism $f : X \to Y$ of algebraic varieties, we denote by $d_x f$ the differential of f at x. If X is an algebraic group, we put $df = d_{1_X} f$.

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2.1.1 H-Varieties and Adjoint Action of H on $\mathcal{H}$

An algebraic variety on which H acts morphically is called an *H-variety.* If V is an H-variety and S a subset of V, we put $C_H(S) := \{h \in H | \forall x \in S, h.x = x\}$ and we denote by $C_H^o(S)$ instead of $C_H(S)^o$ its connected component. We also put $A_H(S) := C_H(S)/C_H^o(S)$. The normalizer $\{h \in H | h.S \subset S\}$ of S in H is denoted by $N_H(S)$. Let X be an homogeneous H-variety (i.e. H acts transitively on X). Then the choice of an element $x \in X$ defines an H-equivariant morphism $\pi_x : H \to X$, $h \mapsto h.x$ which factors through a bijective morphism $\overline{\pi}_x : H/C_H(x) \to X$. We have the following well-known proposition.

Proposition 2.1.2. *The following assertions are equivalent:*
(i) The morphism π_x is separable.
(ii) The natural inclusion Lie$(C_H(x)) \subset$ Ker$(d\pi_x)$ is an equality.
(iii) The morphism $\overline{\pi}_x$ is an isomorphism.

2.1.3. For any $h \in H$, let $\text{Int}_h : H \to H$ be the automorphism of H given by $g \mapsto hgh^{-1}$. Then the map $\text{Ad} : H \to \text{GL}(\mathcal{H})$, $h \mapsto d(\text{Int}_h)$ is a morphism of algebraic groups and is called the *adjoint action* of H on $\mathcal{H}$. We also have $[\text{Ad}(h)X, \text{Ad}(h)Y] = \text{Ad}(h)([X, Y])$ for any $h \in H$, $X, Y \in \mathcal{H}$. For a closed subgroup K of H, we use the terminology "K-orbit of $\mathcal{H}$" for the adjoint action of K on $\mathcal{H}$. If $X \in \mathcal{H}$, we denote by $\mathcal{O}_X^K$ the K-orbit of X and if $x \in H$, we denote by C_x^K the K-conjugacy class of x in H. If X, Y are two elements of $\mathcal{H}$, we say that they are *K-conjugate* if $X \in \mathcal{O}_Y^K$. The differential of $\text{Ad} : H \to \text{GL}(\mathcal{H})$ at 1 is denoted by ad. It satisfies $\text{ad}(X)(Y) = [X, Y]$ for any $X, Y \in \mathcal{H}$. Since the restriction of Ad to Z_H is trivial, we thus get that $\text{Lie}(Z_H) \subset z(\mathcal{H})$. We will see later that this inclusion is not always an equality.

Let K be a closed subgroup of H with Lie algebra $\mathcal{K}$. For $X \in \mathcal{H}$ and $x \in H$, we define

$$C_{\mathcal{K}}(X) := \{Y \in \mathcal{K} | [Y, X] = 0\},$$

$$C_{\mathcal{K}}(x) := \{Y \in \mathcal{K} | \text{Ad}(x)Y = Y\}.$$

Consider the orbit maps $\pi : K \to \mathcal{O}_X^K$, $h \mapsto \text{Ad}(h)X$ and $\rho : K \to C_x^K$, $h \mapsto hxh^{-1}$. Then by [Bor, III 9.1], we have $\text{Ker}(d\pi) = C_{\mathcal{K}}(X)$ and $\text{Ker}(d\rho) = C_{\mathcal{K}}(x)$. Hence, by 2.1.2 the orbit map π (resp. ρ) is separable if and only if $\text{Lie}(C_K(X)) = C_{\mathcal{K}}(X)$ (resp. $\text{Lie}(C_K(x)) = C_{\mathcal{K}}(x)$).

2.1.4 Reductive Groups

The letter G will always denote a **connected reductive algebraic group** over k and we will denote by $\mathcal{G}$ its Lie algebra. By a *semi-simple* algebraic group, we shall mean a **connected** reductive algebraic group whose radical is trivial, i.e. a connected reductive group whose center is finite.

Notation 2.1.5. We denote by G' the derived subgroup of G, i.e. the closed subgroup of G which is generated by the elements of the form $xyx^{-1}y^{-1}$ with $x, y \in G$, and by $\mathcal{G}'$ the Lie algebra of G'. We also denote by $\overline{G}$ the quotient G/Z_G^o and by $\overline{\mathcal{G}}$ the Lie algebra of $\overline{G}$.

Recall that G' and $\overline{G}$ are both semi-simple algebraic groups. Recall also that $\overline{\mathcal{G}} = \mathcal{G}/\mathrm{Lie}(Z_G^o)$. We will see that $\mathcal{G}'$ is not always the Lie subalgebra of $\mathcal{G}$ generated the elements of the form $[X, Y]$ with $X, Y \in \mathcal{G}$ (see 2.4.4).

Definition 2.1.6. *Let H be an algebraic group and let $H_1, ..., H_n$ be closed subgroups of H such that any two of them commute and each of them has a finite intersection with the product of the others. If $H = H_1...H_n$, then we say that H is the* almost-direct product *of the H_i.*

Theorem 2.1.7. *[DM91, 0.38] If G is a semi-simple algebraic group, then G has finitely many minimal non-trivial normal connected closed subgroups and G is the almost-direct product of them.*

Definition 2.1.8. *The minimal non-trivial normal connected closed subgroups of a semi-simple algebraic group G will be called the* simple components *of G. We shall say that G is simple if it has a unique simple component.*

The letter B will usually denote a Borel subgroup of G, the letter T a maximal torus of B and U the unipotent radical of B. Their respective Lie algebras will be denoted by $\mathcal{B}, \mathcal{T}$ and $\mathcal{U}$. The dimension of T is called the *rank* of G and is denoted by $rk(G)$. The rank of $\overline{G}$ is called the *semi-simple rank* of G and is denoted by $rk_{ss}(G)$. If P is an arbitrary parabolic subgroup of G, then we denote by U_P the unipotent radical of P and by $\mathcal{U}_P$ the Lie algebra of U_P. If $P = LU_P$ is a Levi decomposition of P with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$, then we denote by $\pi_P : P \to L$ and by $\pi_{\mathcal{P}} : \mathcal{P} \to \mathcal{L}$ the canonical projections. Throughout the book we will make the following abuse of language: by a "**Levi subgroup of** G", we shall mean a Levi subgroup of a parabolic subgroup of G.

We denote by $X(T)$ the group of algebraic group homomorphisms $T \to \mathbb{G}_m$. For any $\gamma \in X(T)$, put $\mathcal{G}_\gamma = \{v \in \mathcal{G} | \forall t \in T, \mathrm{Ad}(t)v = \gamma(t)v\}$ and $\Phi = \{\gamma \in X(T) - \{0\} | \mathcal{G}_\gamma \neq \{0\}\}$. We have

$$\mathcal{G} = \bigoplus_{\gamma \in \Phi \cup \{0\}} \mathcal{G}_\gamma = \mathcal{T} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{G}_\alpha.$$

For any $\alpha \in \Phi$, we denote by U_α the unique closed connected one-dimensional unipotent subgroup of G normalized by T such that $\mathrm{Lie}(U_\alpha) = \mathcal{G}_\alpha$.

It is known that Φ forms a (reduced) root system in the subspace V of $X(T) \otimes \mathbb{R}$ it generates. The set Φ is then called the *root system* of G with respect to T and the elements of Φ are called the roots of G with respect to T. If there is any ambiguity, we will write $\Phi(T)$ instead of Φ. We denote by $\Phi^\vee$ the set of coroots and by $X^\vee(T)$ the group of homomorphisms of algebraic groups $\mathbb{G}_m \to T$; the set $\Phi^\vee$ forms a root system in the subspace $V^\vee$ of $X^\vee(T) \otimes \mathbb{R}$ it generates. We denote by $Q(\Phi)$ the $\mathbb{Z}$-sublattice of $X(T)$ generated by Φ and by $Q(\Phi^\vee)$ the $\mathbb{Z}$-sublattice of $X^\vee(T)$ generated by $\Phi^\vee$. Recall that we have an exact pairing $\langle , \rangle : X(T) \times X^\vee(T) \to \mathbb{Z}$ such that for any $\alpha \in X(T)$, $\beta \in X^\vee(T)$ and $t \in \mathbb{G}_m$, we have $(\alpha \circ \beta^\vee)(t) = t^{\langle \alpha, \beta^\vee \rangle}$. By abuse of notation, we still denote by $\langle , \rangle$ the induced pairing between V and $V^\vee$. The $\mathbb{Z}$-lattice of weights $P(\Phi)$ is defined to be $\{x \in V | \langle x, \Phi^\vee \rangle \subset \mathbb{Z}\}$. The lattice $Q(\Phi)$ is then a $\mathbb{Z}$-sublattice of $P(\Phi)$ of finite index.

If G is semi-simple, we have the following inclusions of $\mathbb{Z}$-lattices $Q(\Phi) \subset X(T) \subset P(\Phi)$ and $Q(\Phi^\vee) \subset X^\vee(T) \subset P(\Phi^\vee)$; conversely if one these inclusions hold, then G is semi-simple. Moreover we have $|P(\Phi)/X(T)| = |X^\vee(T)/Q(\Phi^\vee)|$ and so

$$|X(T)/Q(\Phi)||X^\vee(T)/Q(\Phi^\vee)| = |P(\Phi)/Q(\Phi)|.$$

Definition 2.1.9. *We say that G is*

(i) adjoint *if $X(T) = Q(\Phi)$;*

(ii) simply connected *if $X^\vee(T) = Q(\Phi^\vee)$.*

It follows from Chevalley's classification theorem that each $\mathbb{Z}$-lattice between $Q(\Phi)$ and $P(\Phi)$ determines a unique (up to isomorphism) semi-simple algebraic group over k with root system Φ. We denote by G_{ad} the adjoint group corresponding to G and by G_{sc} the simply connected algebraic group corresponding to G. Their respective Lie algebras are denoted by $\mathcal{G}_{ad}$ and $\mathcal{G}_{sc}$. When G is semi-simple, the inclusions $Q(\Phi) \subset X(T) \subset P(\Phi)$ give rise to canonical isogenies (i.e surjective morphisms whose kernel is finite and so lies in the center) $\pi_{sc} : G_{sc} \to G$ and $\pi_{ad} : G \to G_{ad}$; the kernel of the later map is equal to Z_G (see [Ste68, page 45]). Moreover, the canonical isogenies π_{sc} and π_{ad} are central, that is $\mathrm{Ker}(d\pi_{sc}) \subset z(\mathcal{G}_{sc})$ and $\mathrm{Ker}(d\pi_{ad}) \subset z(\mathcal{G})$. In fact, for the later map we have $\mathrm{Ker}(d\pi_{ad}) = z(\mathcal{G})$.

The choice of the Borel subgroup B containing T defines an order on $\Phi \cup \{0\}$ such that any root is positive or negative by setting $\Phi^+ := \{\gamma \in \Phi | \, \mathcal{G}_\gamma \subset \mathcal{B}\}$. The set Π of positive roots that are indecomposable into a sum of other positive roots is called the *basis* of Φ with respect to B. The elements of Π are linearly independent and any root of Φ is a $\mathbb{Z}$-linear combination of elements of Π with coefficients all positive or all negative. If $\beta = \sum_{\alpha\in\Pi} n_\alpha \alpha \in \Phi$, then we define the *height* of β (with respect to Π) to be the integer $\sum_{\alpha\in\Pi} n_\alpha$. The *highest root* of Φ with respect to Π is defined to be the root of highest height. For any Levi subgroup L of G, we denote by $W_G(L)$ the group $N_G(L)/L$. The Weyl group of G relative to T is $W_G(T)$. We denote by h_o the Coxeter number of $W_G(T)$. It depends only on G, and so if there is any ambiguity, we will denote it h_o^G instead of h_o.

2.1.10 About Intersections of Lie Algebras of Closed Subgroups of *G*

Let M and N be two closed subgroups of G, then we have

2.1.11. $$\mathrm{Lie}(M \cap N) \subset \mathrm{Lie}(M) \cap \mathrm{Lie}(N).$$

In general this inclusion is not an equality; it becomes an equality exactly when the quotient morphism $\pi : G \to G/N$ induces a separable morphism $M \to \pi(M)$ (see [Bor, Proposition 6.12]).

2.1.12. *When $M \cap N$ contains a maximal torus of G, the inclusion 2.1.11 is an equality.*

The above assertion follows from [Bor, Proposition 13.20]; note that [Bor, Corollary 13.21], which asserts that 2.1.11 is an equality whenever M and N are normalized by a maximal torus of G, is not correct since in positive characteristic, the intersection of two subtori of a maximal torus of G may have finite intersection while their Lie algebras have an intersection of strictly positive dimension. For instance, let $G = SL_3(k)$ and let T be the maximal torus of G consisting of diagonal matrices, then the set Z_G is finite and is the intersection of the two subtori $T_\alpha = \mathrm{Ker}(\alpha)$ and $T_\beta = \mathrm{Ker}(\beta)$ of T where $\alpha : T \to k^\times$, $(t_1, t_2, t_1^{-1}t_2^{-1}) \mapsto t_1 t_2^{-1}$ and $\beta : T \to k^\times$, $(t_1, t_2, t_1^{-1}t_2^{-1}) \mapsto t_2^2 t_1$. The intersection of the Lie algebras of T_α and T_β is of dimension 0 unless $p = 3$ in which case the intersection is of dimension 1.

2.1.13. We will need to deal with the question of whether the inclusion 2.1.11 is an equality or not only in the cases where the closed subgroups M and N involved in 2.1.11 are parabolic subgroups, Levi subgroups or unipotent radicals of parabolic subgroups.

Let P and Q be two parabolic subgroups G. Let L and M be two Levi subgroups of P and Q respectively such that $L \cap M$ contains a maximal torus T of G (given P and Q, such Levi subgroups L and M always exists). We denote by $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{L}$ and $\mathcal{M}$ the corresponding Lie algebras of P, Q, L and M.

Proposition 2.1.14. *With the above notation, we have:*

(1) Lie$(P \cap Q) = \mathcal{P} \cap \mathcal{Q}$,
(2) Lie$(L \cap M) = \mathcal{L} \cap \mathcal{M}$,
(3) Lie$(L \cap U_Q) = \mathcal{L} \cap \mathcal{U}_Q$,
(4) Lie$(U_P \cap U_Q) = \mathcal{U}_P \cap \mathcal{U}_Q$.

Proof: The assertions (1) and (2) are clear from 2.1.12. Let us see (3). From 2.1.11, it is enough to prove that $\dim(L \cap U_Q) = \dim(\mathcal{L} \cap \mathcal{U}_Q)$. Since $L \cap U_Q$ is a closed unipotent subgroup of G normalized by T, by [DM91, 0.34], it is of dimension equal to the number of the U_α, with $\alpha \in \Phi$, it contains. On the other hand the torus T normalizes $\mathcal{L} \cap \mathcal{U}_Q$, therefore by full reducibility of the adjoint representation of T in $\mathcal{G}$, the space $\mathcal{L} \cap \mathcal{U}_Q$ is the direct sum of the $\mathcal{G}_\alpha$, $\alpha \in \Phi$, it contains. Hence the equality $\dim(L \cap U_Q) = \dim(\mathcal{L} \cap \mathcal{U}_Q)$ is a consequence of the fact that $\mathcal{G}_\alpha \subset \mathcal{L} \cap \mathcal{U}_Q$ if and only if $U_\alpha \subset L \cap U_Q$. The proof of (4) is completely similar. □

The above proposition together with [DM91, Proposition 2.1] has the following straightforward consequence.

Proposition 2.1.15. *With the above notation, we have*

$$\mathcal{P} \cap \mathcal{Q} = (\mathcal{L} \cap \mathcal{M}) \oplus (\mathcal{L} \cap \mathcal{U}_Q) \oplus (\mathcal{M} \cap \mathcal{U}_P) \oplus (\mathcal{U}_P \cap \mathcal{U}_Q).$$

2.1.16 $\mathbb{F}_q$-Structures

Notation 2.1.17. Let r be a prime and let X be an algebraic variety on $\overline{\mathbb{F}}_r$ defined over $\mathbb{F}_{r^n}$. If $F : X \to X$ denotes the corresponding Frobenius endomorphism, we say that $x \in X$ is *rational* if $F(x) = x$ and we denote by X^F the set of rational elements of X.

2.1.18. Let $k = \overline{\mathbb{F}}_p$, and let q be a power of p such that the group G is defined over $\mathbb{F}_q$. We then denote by $F : G \to G$ the corresponding Frobenius endomorphism. The Lie algebra $\mathcal{G}$ and the adjoint action of G on $\mathcal{G}$ are also defined over $\mathbb{F}_q$ and we still denote by $F : \mathcal{G} \to \mathcal{G}$ the Frobenius endomorphism on $\mathcal{G}$. Assume that the maximal torus T of G is F-stable, and denote by τ the unique automorphism on Φ such that for any root $\alpha \in \Phi$, we have $F(U_\alpha) = U_{\tau(\alpha)}$; it satisfies $(\tau\alpha)(F(t)) = (\alpha(t))^q$ for any $\alpha \in X(T)$ and $t \in T$. If B is also F-stable, then τ permutes the elements of the basis Π of Φ. Recall that an F-stable torus $H \subset G$ of rank n is said to be *split* if there exists an isomorphism $H \xrightarrow{\sim} (\mathbb{G}_m)^n$ defined over $\mathbb{F}_q$. The $\mathbb{F}_q$-rank of an F-stable maximal torus T of G is defined to be the rank of its maximum split subtori. An F-stable maximal torus T of G is said to be G-*split* if it is maximally split in G; recall that the G-split maximal torus of G are exactly those contained in some F-stable Borel subgroup of G. The $\mathbb{F}_q$-*rank* of G is defined to be the $\mathbb{F}_q$-rank of its G-split maximal tori. The *semi-simple* $\mathbb{F}_q$-*rank* of G is defined to be the $\mathbb{F}_q$-rank of $\overline{G}$. We say that an F-stable Levi subgroup L of G is G-split if it contains a G-split maximal torus; this is equivalent to say that there exists an F-stable parabolic subgroup P of G having L as a Levi subgroup.

Notation 2.1.19. Let H be a group with a morphism $\theta : H \to H$. We say that $x, y \in H$ are θ-*conjugate* if and only if there exists $h \in H$ such that $x = hy(\theta(h))^{-1}$. We denote by $H^1(\theta, H)$ the set of θ-conjugacy classes of H.

2.1.20. Let $k = \overline{\mathbb{F}}_q$ with q a power of p. Let H be a connected linear algebraic group acting morphically on a variety X. Assume that H, X and the action of H on X are all defined over $\mathbb{F}_q$. Let $F : X \to X$ and $F : H \to H$ be the corresponding Frobenius endomorphisms. Let $x \in X^F$ and let $\mathcal{O}$ be the H-orbit of x. The orbit $\mathcal{O}$ is thus F-stable and $\mathcal{O}^F$ is a disjoint union of H^F-orbits. By [SS70, I, 2.7] (see also [DM91, 3.21]) we have a well-defined parametrization of the H^F-orbits of $\mathcal{O}$ by $H^1(F, A_H(x))$. This parametrization is given as follows. Let $y \in \mathcal{O}^F$ and let $h \in H$ be such that $y = h.x$. Then to the H^F-orbit of y, we associate the F-conjugacy class of the image of $h^{-1}F(h)$ in $A_H(x)$.

2.2 Chevalley Formulas

For any $\alpha \in \Phi$, the symbol e_α denotes a non-zero element of $\mathcal{G}_\alpha$ and h_α denotes $[e_\alpha, e_{-\alpha}]$. When $p = 0$, we assume that the e_α are chosen such that the set $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is a Chevalley basis of $\mathcal{G}'$ (see [Car72, 4.2] or [Ste68]). When $p > 0$ and $\mathcal{G}' = \mathcal{G}_{sc}$, then $\mathcal{G}'$ is obtained by reduction modulo p from the

$\mathbb{Z}$-span of a Chevalley basis in the corresponding Lie algebra over $\mathbb{C}$. Hence in that case, we assume that the e_α are chosen such that $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is obtained from a Chevalley basis in the corresponding Lie algebra over $\mathbb{C}$; the set $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is then called a *Chevalley basis* of $\mathcal{G}'$. In the general case, let π denote the canonical central isogeny $G_{sc} \to G'$; the choice of the e_α is made such that $B_{\mathcal{G}} := \{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is the image by $d\pi$ of a Chevalley basis of $\mathcal{G}_{sc}$. When it is a basis of $\mathcal{G}'$, the set $B_{\mathcal{G}}$ is called a Chevalley basis of $\mathcal{G}'$. We will see in 2.4, that the existence of Chevalley basis on $\mathcal{G}' \neq \mathcal{G}_{sc}$ is subject to some restriction on p. With such a choice of the e_α, for any $r \in \Phi$, we have $dr(h_r) = 2$ and the vector h_r is a linear combination of the h_α with $\alpha \in \Pi$. The last fact can be deduced from the simply connected case by making the use of the canonical Lie algebra homomorphism $\mathcal{G}_{sc} \to \mathcal{G}'$.

2.2.1. *We then have the following well-known relations:*

(i) $[t, h] = 0,\ t, h \in \mathcal{T}$,
(ii) $[t, e_r] = dr(t)e_r,\ t \in \mathcal{T}, r \in \Phi$,
(iii) $[e_r, e_s] = 0,\ r \in \Phi, s \in \Phi, r + s \notin \Phi \cup \{0\}$,
(iv) $[e_r, e_s] \in \mathcal{G}_{r+s},\ r \in \Phi, s \in \Phi, r + s \in \Phi$.

Using the decomposition $\mathcal{G} = \mathcal{T} \oplus \bigoplus_\alpha \mathcal{G}_\alpha$ and the above formulas, we see that the subspace of $\mathcal{G}'$ generated by $\{h_\alpha, e_\gamma | \alpha, \gamma \in \Phi\}$ is $[\mathcal{G}, \mathcal{G}]$. But since the vectors h_r with $r \in \Phi$ are linear combinations of the h_α with $\alpha \in \Pi$, the Lie algebra $[\mathcal{G}, \mathcal{G}]$ is actually generated by $B_{\mathcal{G}}$. As a consequence, since $\mathcal{G}'$ is of dimension $|\Pi| + |\Phi| = |\mathcal{B}_{\mathcal{G}}|$, we see that $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ if and only if $B_{\mathcal{G}}$ is a basis of $\mathcal{G}'$, i.e. the elements of $\{h_\alpha | \alpha \in \Pi\}$ are linearly independent.

2.2.2. *For $r \in \Phi$, we fix an isomorphism of algebraic groups $x_r : \mathbb{G}_a \to U_r$ such that $dx_r(1) = e_r$. The following formulas give the action of U_r, with $r \in \Phi$, on $\mathcal{G}$:*

(i) $Ad(x_r(t))e_r = e_r$,

(ii) $Ad(x_r(t))e_{-r} = e_{-r} + th_r - t^2 e_r$,

(iii) $Ad(x_r(t))h = h - dr(h)te_r,\ h \in \mathcal{T}$,

(iv) $Ad(x_r(t))e_s = e_s + \sum_{\{i>0 | ir+s \in \Phi\}} c_{r,s,i} t^i e_{ir+s}$ *for some* $c_{r,s,i} \in k$, *if* $r \neq -s$.

2.3 The Lie Algebra of Z_G

Recall that by 2.1.3, we have an inclusion (*) $\mathrm{Lie}(Z_G) \subset z(\mathcal{G})$. In this subsection, we give among other things a necessary and sufficient condition on p for (*) to be an equality. We denote by $\overline{T}$ the maximal torus T/Z_G^o of $\overline{G}$ and by T' the maximal torus of G' which contains T.

We consider on $\mathrm{Lie}(Z_G)\oplus\overline{\mathcal{G}}$ the Lie product given by $[t\oplus v, h\oplus u] := [v, u]$.

2.3.1. *There is an isomorphism of Lie algebras $\mathcal{G} \simeq \mathrm{Lie}(Z_G) \oplus \overline{\mathcal{G}}$.*

Proof: It is enough to prove the existence of a k-subspace V of $\mathcal{G}$ such that $\mathcal{G} = \mathrm{Lie}(Z_G) \oplus V$ and $[V, V] \subset V$, so that $V \simeq \overline{\mathcal{G}}$. For $\alpha \in \Pi$, denote by $\overline{h}_\alpha \in \mathrm{Lie}(\overline{T})$ the image of h_α under the canonical projection $\mathcal{T} \to \mathrm{Lie}(\overline{T})$. We choose a subset I of Π such that $E = \{\overline{h}_\alpha | \alpha \in I\}$ is a basis of the subspace of $\mathrm{Lie}(\overline{T})$ generated by $\{\overline{h}_\alpha | \alpha \in \Pi\}$, and we complete E into a basis $E \cup \{\overline{x}_1, ..., \overline{x}_n\}$ of $\mathrm{Lie}(\overline{T})$. We choose $x_i \in \mathcal{T}$ such that its image in $\mathrm{Lie}(\overline{T})$ is $\overline{x}_i$. Now let V be the subspace of $\mathcal{G}$ generated by $X := \{x_1, ..., x_n, h_\alpha, e_\gamma | \alpha \in I, \gamma \in \Phi\}$. Since the image of X in $\overline{\mathcal{G}}$ is a basis of $\overline{\mathcal{G}}$, we have $\dim V = \dim \overline{\mathcal{G}}$ and $V \cap \mathrm{Lie}(Z_G) = \{0\}$. It follows that $\mathcal{G} = \mathrm{Lie}(Z_G) \oplus V$. From 2.2.1, we get that $[V, V] \subset V$. □

2.3.2. It follows from 2.2.1 that

$$z(\mathcal{G}) = \bigcap_{\alpha \in \Pi} \mathrm{Ker}(d\alpha), \tag{1}$$

and from [DM91, Proposition 0.35] that

$$Z_G = \bigcap_{\alpha \in \Pi} \mathrm{Ker}(\alpha). \tag{2}$$

2.3.3. The canonical morphism $\rho : T \to \overline{T}$ induces an injective group homomorphism $\rho^* : X(\overline{T}) \to X(T)$, $\gamma \mapsto \gamma \circ \rho$ mapping bijectively the roots of $\overline{G}$ with respect to $\overline{T}$ onto Φ. Hence we may identify the roots of $\overline{G}$ with respect to $\overline{T}$ with Φ. Under this identification, the lattice $Q(\Phi)$ is a $\mathbb{Z}$-sublattice of $X(\overline{T})$. We have the following proposition.

Proposition 2.3.4. *We have $|(X(T)/Q(\Phi))_{tor}| = |X(\overline{T})/Q(\Phi)|$. The following assertions are equivalent:*

(i) p does not divide $|(X(T)/Q(\Phi))_{tor}|$,

(ii) $\mathrm{Lie}(Z_G) = z(\mathcal{G})$.

Proof: For the sake of clarity, in this proof we prefer to differentiate the root system $\overline{\Phi}$ of $\overline{\mathcal{G}}$ with respect to $\overline{T}$ from Φ. Let r be the rank of G and s be the semi-simple rank of G. Let $\{\gamma_1, ..., \gamma_r\}$ be a basis of $X(T)$ such that for some integer s with $s \leq r$ and some non-zero integers $m_1, ..., m_s$, the set $\{m_1\gamma_1, ..., m_s\gamma_s\}$ is a basis of $Q(\Phi)$. We have $X(T)/Q(\Phi) = \mathbb{Z}^{r-s} \times \mathbb{Z}/m_1\mathbb{Z} \times ... \times \mathbb{Z}/m_s\mathbb{Z}$ and so $|(X(T)/Q(\Phi))_{tor}| = \prod_{i=1}^{s} m_i$. Now for $i \in \{1, .., s\}$, we have $m_i\gamma_i \in Q(\Phi)$ and so, by 2.3.2(2), we have $\gamma_i(z)^{m_i} = 1$ for any $z \in Z_G^o$. Hence, if μ_{m_i} denotes the group of m_i^{th} roots of unity, we get that $\gamma_i(Z_G^o) \subset \mu_{m_i}$. Since Z_G^o is connected, we deduce that $\gamma_i(Z_G^o) = \{1\}$. Thus, for $i \in \{1, .., s\}$, the morphism γ_i factors through a morphism $\overline{\gamma}_i : \overline{T} \to \mathbb{G}_m$. We see that $\{\overline{\gamma}_i\}_{i\in\{1,...,s\}}$ and $\{m_i\overline{\gamma}_i\}_i$ are respectively bases of the groups $X(\overline{T})$ and $Q(\overline{\Phi})$ (from which we see that $|(X(T)/Q(\Phi))_{tor}| = |X(\overline{T})/Q(\overline{\Phi})|$); this can be verified by using the fact that dim $X(\overline{T}) = s$ and the fact that ρ^* maps $\overline{\gamma_i}$ onto γ_i for $i \in \{1, ..., s\}$. From the fact that $\{\overline{\gamma}_i\}_i$ is a basis of $X(\overline{T})$, it results that the morphism $\overline{T} \to \mathbb{G}_m^s$ given by $t \mapsto (\overline{\gamma}_1(t), ..., \overline{\gamma}_s(t))$ is an isomorphism of algebraic groups. As a consequence, its differential Lie$(\overline{T}) \to k^s$ given by $t \mapsto (d\overline{\gamma}_1(t), ..., d\overline{\gamma}_s(t))$ is an isomorphism, i.e. the intersection of the s hyperplanes Ker$(d\overline{\gamma}_i)$ of Lie$(\overline{T})$ is $\{0\}$.

We deduce that the intersection of the s hyperplanes Ker$(m_i d\overline{\gamma}_i)$ of Lie$(\overline{T})$ is zero if and only if the m_i are invertible in k (i.e if p does not divide $|(X(T)/Q(\Phi))_{tor}|$). On the other hand, since $\{m_i\overline{\gamma}_i\}_i$ is a basis of $Q(\overline{\Phi})$, by 2.3.2 (1) we have $\bigcap_{i=1}^{i=s} \text{Ker}(m_i d\overline{\gamma}_i) = z(\overline{\mathcal{G}})$. We thus proved that the m_i are invertible in k if and only if $z(\overline{\mathcal{G}})$ is trivial.

We are now in position to see that the proposition is a consequence of the fact that any isomorphism of Lie algebras $\mathcal{G} \simeq \text{Lie}(Z_G) \oplus \overline{\mathcal{G}}$ as in 2.3.1 induces an isomorphism from $z(\mathcal{G})$ onto $\text{Lie}(Z_G) \oplus z(\overline{\mathcal{G}})$. □

Remark 2.3.5. If the assertion (i) (and so the assertion (ii)) of 2.3.4 holds for G, it does for any Levi subgroup of G.

Remark 2.3.6. Let $\pi : G \to G_{ad}$ be the composition morphism of the canonical projection $G \to \overline{G}$ with the canonical central isogeny $\overline{G} \to G_{ad}$, then we have $\text{Ker}(\pi) = Z_G$ and $\text{Ker}(d\pi) = z(\mathcal{G})$, so by 2.1.2, the morphism π is separable if and only if $\text{Lie}(Z_G) = z(\mathcal{G})$.

Using 2.3.6, we see that 2.3.4 has the following consequence.

Corollary 2.3.7. *The canonical morphism $G \to G_{ad}$ is separable if and only if p does not divide $|(X(T)/Q(\Phi))_{tor}|$.*

Corollary 2.3.8. *Assume that G is semi-simple and write $G = G_1 ... G_r$ where $G_1, ..., G_r$ are the simple components of G. Assume moreover that p does not divide $|(X(T)/Q(\Phi))_{tor}| = |X(T)/Q(\Phi)|$, then $\mathcal{G} = \bigoplus_i Lie(G_i)$.*

Proof: For any i, we denote by $\mathcal{G}_i$ the Lie algebra of G_i. We fix i and let I be a subset of $\{1, .., r\}$ which does not contain i. Let $x \in (\sum_{j \in I} \mathcal{G}_j) \cap \mathcal{G}_i$. Since for $i \neq j$ the group G_i commutes with G_j, we have $[\mathcal{G}_i, \mathcal{G}_j] = \{0\}$. Hence $x \in \sum_{j \in I} \mathcal{G}_j$ centralizes $\mathcal{G}_i$ and so $x \in z(\mathcal{G}_i)$. Since each element of $\mathcal{G}_i$ centralizes $\mathcal{G}_j$ for any $i \neq j$, we deduce that $x \in \mathcal{G}$. By 2.3.4 we have $x = 0$. We deduce that the sum $E = \sum_i \mathcal{G}_i$ is direct. Hence E is a subspace of $\mathcal{G}$ of dimension $\sum_i \dim \mathcal{G}_i$ and so since algebraic groups are smooth, we have $\dim E = \sum_i \dim G_i = \dim G$. We deduce that $\mathcal{G} = \bigoplus_{i=1}^{i=n} \mathcal{G}_i$. □

Using the canonical map $T' \to \overline{T}$, we identify $X(\overline{T})$ with a subgroup of $X(T')$ and the root system of $\mathcal{G}'$ with respect to T' with Φ. Then $|(X(T)/Q(\Phi))_{tor}| = |X(\overline{T})/Q(\Phi)|$ divides $|X(T')/Q(\Phi)|$.

Corollary 2.3.9. *Assume that p does not divide $|X(T')/Q(\Phi)|$, we have $\mathcal{G} = z(\mathcal{G}) \oplus \mathcal{G}'$.*

Proof: Since $\mathrm{Lie}(Z_G) \subset z(\mathcal{G})$, we have $\mathrm{Lie}(Z_G) \cap \mathcal{G}' \subset z(\mathcal{G}')$ and so by 2.3.4 applied to G', we have $\mathrm{Lie}(Z_G) \cap \mathcal{G}' = \{0\}$. Hence the sum $\mathrm{Lie}(Z_G) + \mathcal{G}'$ is direct and so it is a subspace of $\mathcal{G}$ of dimension $\dim Z_G + \dim G' = \dim G$; thus we get that $\mathrm{Lie}(Z_G) \oplus \mathcal{G}' = \mathcal{G}$. Now, since p does not divide $|X(T')/Q(\Phi)|$, it does not divide $|(X(T)/Q(\Phi))_{tor}|$, hence by 2.3.4, we get that $\mathcal{G} = z(\mathcal{G}) \oplus \mathcal{G}'$. □

Remark 2.3.10. The assumption " p does not divide $|(X(T)/Q(\Phi))_{tor}|$" is not sufficient for $\mathcal{G} = z(\mathcal{G}) \oplus \mathcal{G}'$ to hold. Indeed, consider $G = GL_n(k)$; the group $(X(T)/Q(\Phi))_{tor}$ is trivial while the group $X(T')/Q(\Phi)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Assume that p divides $|X(T')/Q(\Phi)| = n$. Then diagonal matrices $(a, ..., a)$ with $a \in k$ belong to the Lie algebra of Z_G but also to $sl_n = \mathcal{G}'$ since $na = 0$. Hence $\mathrm{Lie}(Z_G) \cap \mathcal{G}' \neq \{0\}$.

2.4 Existence of Chevalley Bases on $\mathcal{G}'$

We will need the following lemma.

Lemma 2.4.1. *[Bor, 8.5]*

(i) Let $T_i : (\mathbb{G}_m)^r \to \mathbb{G}_m$ be the i-th projection; the maps T_i form a basis of the abelian group $X(\mathbb{G}_m^r)$ of algebraic group homomorphisms $(\mathbb{G}_m)^r \to \mathbb{G}_m$, that is for any $f \in X(\mathbb{G}_m^r)$ there exists a unique tuple $(n_1, ..., n_r) \in \mathbb{Z}^r$ such

that $f = T_1^{n_1}...T_r^{n_r}$. *Let* $f = T_1^{n_1}...T_r^{n_r} \in X(\mathbb{G}_m^r)$, *then* $df : k^r \to k$ *is given by* $df(x_1, ..., x_r) = \sum_i n_i x_i$.

(ii) Let $T_i^\vee : \mathbb{G}_m \to (\mathbb{G}_m)^r$ *be given by* $T_i^\vee(t) = (1, .., 1, t, 1, .., 1)$ *(t being located at the* $i-th$ *rank); the maps* $T_i^\vee$ *form a basis of the abelian group* $X^\vee(\mathbb{G}_m^r)$ *of algebraic group homomorphisms* $\mathbb{G}_m \to (\mathbb{G}_m)^r$, *that is for any* $f \in X^\vee(\mathbb{G}_m^r)$ *there exists a unique uple* $(n_1, ..., n_r) \in \mathbb{Z}^r$ *such that* $f = (T_1^\vee)^{n_1}...(T_r^\vee)^{n_r}$. *If* $f = (T_1^\vee)^{n_1}...(T_r^\vee)^{n_r} \in X^\vee(\mathbb{G}_m^r)$, *then* $df : k \to k^r$ *is given by* $df(t) = (n_1 t, ..., n_r t)$.

Recall that T' denotes the maximal torus of G' contained in T and that $X(T')$ is a $\mathbb{Z}$-sublattice of $P(\Phi)$.

Definition 2.4.2. *The quotient* $P(\Phi)/X(T')$ *is called the* fundamental group *of* G *and is denoted by* $\pi_1(G)$.

Note that $\pi_1(G_{sc}) = 1$ and $\pi_1(G_{ad}) = P(\Phi)/Q(\Phi)$.

We assume that G is **semi-simple**.

By Chevalley's classification theorem, there exists a unique (up to isomorphism) connected reductive algebraic group G^* over k with a maximal torus T^* of G^* such that its root datum $(\Phi^*, X(T^*), (\Phi^*)^\vee, X^\vee(T^*))$ is $(\Phi^\vee, X^\vee(T), \Phi, X(T))$; we refer to [DM91] or [Car85] for the definition of root datum. We denote by $\mathcal{G}^*$ the Lie algebra of G^*. Since G is assumed to be semi-simple, the group G^* is also semi-simple. We denote by α^* the element of $X(T^*) = \mathrm{Hom}(T^*, \mathbb{G}_m)$ corresponding to $\alpha^\vee \in \Phi^\vee$ and by $\delta(\chi)$ the element of $X^\vee(T^*) = \mathrm{Hom}(\mathbb{G}_m, T^*)$ corresponding to $\chi \in X(T)$. Then for any $\chi \in X(T)$ and $\alpha \in \Phi$, we have

2.4.3.
$$\langle \chi, \alpha^\vee \rangle = \langle \alpha^*, \delta(\chi) \rangle.$$

Proposition 2.4.4. *The following assertions are equivalent:*

(i) $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$.

(ii) $B_{\mathcal{G}} = \{h_\alpha, e_\gamma |\ \alpha \in \Pi, \gamma \in \Phi\}$ *is a basis of* $\mathcal{G}$.

(iii) $z(\mathcal{G}^*) = \{0\}$.

(iv) p *does not divide* $|\pi_1(G)|$.

(v) The canonical central isogeny $G_{sc} \to G$ *is separable.*

Proof: The equivalence between (i) and (ii) follows from the fact that $[\mathcal{G},\mathcal{G}]$ is generated by $B_{\mathcal{G}}$. Let $\pi : G_{sc} \to G$ be the canonical central isogeny. The equivalence between the assertions (v) and (ii) follows from the fact that $B_{\mathcal{G}}$ is the image by $d\pi$ of a Chevalley basis of $\mathcal{G}_{sc}$ and that π is separable if and only if $d\pi$ is an isomorphism. Since we have $|P(\Phi)/X(T)| = |X^\vee(T)/Q(\Phi^\vee)|$, the equivalence between (iii) and (iv) is a consequence of 2.3.4 applied to G^*. We propose to prove the equivalence between the assertions (ii) and (iii).

We first prove that for any root α we have $d\alpha^\vee(k) = [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$ (this makes sense since $[\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] \subset \mathcal{T}$). It is known that for any root $\alpha \in \Phi$, the group $\alpha^\vee(\mathbb{G}_m)$ is contained in the subgroup H_α of G generated by U_α and $U_{-\alpha}$. But the group H_α is a semi-simple algebraic group of rank one with maximal torus $T \cap H_\alpha$; hence it is isomorphic to $SL_2(k)$ or $PGL_2(k)$. Now a simple computation in $SL_2(k)$ or in $PGL_2(k)$ shows that we have $d\alpha^\vee(k) = [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$. Hence $d\alpha^\vee(1) = \lambda h_\alpha$ for some $\lambda \in k$. Let us see that $\lambda = 1$. Since $\langle \alpha, \alpha^\vee \rangle = 2$, we have $d\alpha \circ d\alpha^\vee(1) = 2$, and by 2.2, we also have $d\alpha(h_\alpha) = 2$. Hence

$$d\alpha^\vee(1) = h_\alpha. \tag{*}$$

Let r be the rank of G. Let $(x_1, ..., x_r)$ be a basis of $X(T)$ and consider the isomorphisms of algebraic groups $\psi : T \to \mathbb{G}_m^r$ given by $t \mapsto (x_1(t), ..., x_r(t))$ and $\phi : \mathbb{G}_m^r \to T^*$ given by $(t_1, ..., t_r) \mapsto \prod_i \delta(x_i)(t_i)$.

Using ϕ and ψ to identify respectively T^* and T with $\mathbb{G}_m^r$, we identify (as suggested by 2.4.1) the abelian groups $X(T^*)$ and $X^\vee(T)$ with $\mathbb{Z}^r$. Under these identifications, for $\alpha \in \Phi$, both $\alpha^\vee$ and α^* correspond to the same element $(n_1^\alpha, ..., n_r^\alpha)$ of $\mathbb{Z}^r$. Indeed, for $i \in \{1, ..., r\}$, let $T_i^\vee : \mathbb{G}_m \to \mathbb{G}_m^r$ and $T_i : \mathbb{G}_m^r \to \mathbb{G}_m$ be the morphisms of 2.4.1; then we have $\delta(x_i) = \phi \circ T_i^\vee$ and $x_i = T_i \circ \psi$. Thus we get that $\langle \alpha^* \circ \phi, T_i^\vee \rangle = \langle \alpha^*, \delta(x_i) \rangle$ and $\langle T_i, \psi \circ \alpha^\vee \rangle = \langle x_i, \alpha^\vee \rangle$ for any $\alpha \in \Phi$. We deduce from 2.4.3 that $\langle \alpha^* \circ \phi, T_i^\vee \rangle = \langle T_i, \psi \circ \alpha^\vee \rangle = n_i^\alpha$.

Let $\{\alpha_1, ..., \alpha_r\} = \Pi$, then it is clear from (*) that $\{h_\alpha, \alpha \in \Pi\}$ is a basis of $\mathcal{T}$ if and only if the matrix $M = \begin{pmatrix} n_1^{\alpha_1} & ... & n_1^{\alpha_r} \\ : & & : \\ n_r^{\alpha_1} & ... & n_r^{\alpha_r} \end{pmatrix} \in M_r(k)$ is invertible. On the other hand, since $z(\mathcal{G}^*) = \bigcap_{\alpha \in \Pi} \mathrm{Ker}(d\alpha^*)$, we have $z(\mathcal{G}^*) = \{0\}$ if and only if the linear map $f : \mathrm{Lie}(T^*) \to k^r$ given by $f(t) = (d\alpha_1^*(t), ..., d\alpha_r^*(t))$ is injective, that is if and only if ${}^tM$ (and so M) is invertible. □

2.5 Existence of Non-degenerate G-Invariant Bilinear Forms on $\mathcal{G}$

By a G-invariant bilinear form $B(,)$ on $\mathcal{G}$ we shall mean a symmetric bilinear form $B(,)$ on $\mathcal{G}$ such that for any $g \in G$, $x, y \in \mathcal{G}$, we have $B(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y) = B(x, y)$. A well known example of such a form is the Killing form defined on $\mathcal{G} \times \mathcal{G}$ by $(x, y) \mapsto \mathrm{Trace}(ad(x) \circ ad(y))$. In this section, we want to discuss for which primes p there exists an G-invariant non-degenerate bilinear form on $\mathcal{G}$. The case of simple groups has been discussed among other things in [SS70] where it has been proved that the condition "p is good for G" (see 2.5.2) is enough to have non-degenerate invariant bilinear forms on $\mathcal{G}$ if G is not of type A_n. On the other hand, it is known that the condition "p is very good for G" (see 2.5.5) is sufficient if G is simple of type A_n. By making the use of 2.3.9 and 2.3.8, we will extend the above results to the case of connected reductive groups, that is, we will see that the condition "p is very good for G" is sufficient to have non-degenerate G-invariant bilinear forms on $\mathcal{G}$. However this is not completely satisfactory since if $G = GL_n(k)$, the "very good characteristics" for G are the characteristics which do not divide n, while the trace form $(X, Y) \mapsto \mathrm{Tr}(XY)$ is always non-degenerate on gl_n.

As far as I know, no necessary and sufficient condition on p for the existence of non-degenerate G-invariant bilinear forms on $\mathcal{G}$ has been given in the literature. While the above problem is not so important for reductive groups without component of type A_n (indeed the "very good characteristics for G" are then the "good ones for G", and there are only few "bad characteristics", see further), it becomes more important for the others. For this reason, we will give a necessary and sufficient condition on p in the case of simple groups of type A_n. We will also treat the cases of simply connected groups of type B_n, C_n or D_n since no extra work is required for these cases (see 2.5.11).

2.5.1. We start with some general properties of G-invariant bilinear forms on $\mathcal{G}$. Assume that $B(,)$ is a G-invariant bilinear form on $\mathcal{G}$. Then:
(1) For any $x, y, z \in \mathcal{G}$ we have

$$B(x, [y, z]) = B([x, y], z).$$

(2) Let $\alpha \in \Phi$. For any x in $\mathcal{T} \oplus (\bigoplus_{\gamma \in \Phi - \{-\alpha\}} \mathcal{G}_\gamma)$, we have $B(x, e_\alpha) = 0$.

Let us prove (2). Let $x \in \mathcal{T}$; since $B(,)$ is G-invariant, for any $t \in T$ we have

$$B(\mathrm{Ad}(t)x, \mathrm{Ad}(t)e_\alpha) = B(x, e_\alpha),$$

that is $\alpha(t)B(x, e_\alpha) = B(x, e_\alpha)$. But $\alpha \neq 0$, thus we get that $B(x, e_\alpha) = 0$. Now let $\beta \in \Phi - \{-\alpha\}$; we have $\alpha(t)\beta(t)B(e_\beta, e_\alpha) = B(e_\beta, e_\alpha)$ for any $t \in T$. Since $\beta \neq -\alpha$, we have $B(e_\beta, e_\alpha) = 0$. □

Definition 2.5.2 (good characteristics). *We say that p is* good *for G if p does not divide the coefficient of the highest root of Φ, otherwise p is said to be* bad *for G.*

Bad characteristics are $p = 2$ if the root system is of type B_n, C_n or D_n, $p = 2, 3$ in type G_2, F_4, E_6, E_7 and $p = 2, 3, 5$ in type E_8 (see [Bou, Ch. VI, 4]).

Definition 2.5.3. *[Ste75, Definition 1.3] We say that p is a* torsion prime *of Φ when there exists a closed root subsystem Φ' of Φ (i.e a root subsystem Φ' of Φ such that any element of Φ which is a $\mathbb{Z}$-linear combination of elements of Φ' is already in Φ') such that $Q(\Phi^\vee)/Q(\Phi'^\vee)$ has torsion of order p.*

Definition 2.5.4 (torsion primes of G). *We say that p is a* torsion prime *of G, when it is a torsion prime of Φ or when p divides $|\pi_1(G)|$.*

This definition is in fact [Ste75, Lemma 2.5]. For the original definition of torsion primes of G, see [Ste75, Definition 2.1].

Torsion primes of Φ are $p = 2$ when Φ is of type B_n, D_n or G_2, $p = 2, 3$ in type E_6, E_7, F_4, $p = 2, 3, 5$ in type E_8. The fundamental group $\pi_1(G)$ is a quotient of the biggest possible fundamental group $P(\Phi)/Q(\Phi)$ whose cardinal is $r + 1$ in types A_r, 2 in types B_n, C_n, E_7, 4 in types D_n, 3 in types E_6 and 1 in types E_8, F_4 or G_2 (see [Slo80, page 24]).

Definition 2.5.5 (very good characteristics). *We say that p is* very good *for G when p is good for G and p does not divide $|P(\Phi)/Q(\Phi)| = |\pi_1(G_{ad})|$.*

Remark 2.5.6. (a) If Φ does not have any component of type A_n, then p is very good for G if and only if p is good for G.

(b) If p is very good for G then it is not a torsion prime of G.

(c) If p is very good for G and G has a component of type A_n, then it is not necessarily very good for Levi subgroups of G,

(d) If G is of type A_n, B_n, C_n or D_n, then p is very good if and only if p does not divide $|P(\Phi)/Q(\Phi)|$.

Proposition 2.5.7. *[SS70, I, 5.3] Let G be either an adjoint simple group not of type A_n or $G = GL_n(k)$. We assume that p is good for G. Then there exists a faithful rational representation (ρ, V) of G or a group isogenous to G (i.e a simple group with same Dynkin diagram as G) such that the symmetric bilinear form $B(,)$ on $\mathcal{G}$ defined by $B(x,y) = Trace(d\rho(x) \circ d\rho(y))$ is non-degenerate. Moreover $B(,)$ is G-invariant.*

Corollary 2.5.8. *Let G be simple not of type A_n and assume that p is good for G. Then there exists a non-degenerate G-invariant bilinear form on $\mathcal{G}$.*

Proof: Let H be a group isogenous to G (with Lie algebra $\mathcal{H}$) and let $(H_{sc} = G_{sc}, \pi)$ be the simply connected cover of H. Since p is very good for G (and so for H), it is not a torsion prime of H (see 2.5.6 (b)), and so by 2.4.4, the differential $d\pi : \mathcal{H}_{sc} \to \mathcal{H}$ of $\pi : H_{sc} \to H$ is an isomorphism. Moreover it satisfies $d\pi \circ \mathrm{Ad}(h) = \mathrm{Ad}(\pi(h)) \circ d\pi$ for any $h \in H_{sc}$. Hence we deduce that any H_{sc}-invariant non-degenerate bilinear form on $\mathcal{H}_{sc}$ induces an H-invariant non-degenerate bilinear form on $\mathcal{H}$ and conversely. Hence, the corollary follows from 2.5.7. □

In order to do a more accurate study of the type A_n we need the following well known result.

Proposition 2.5.9. *Let G be simple of type A_{n-1} $(n > 1)$. Then recall that $\mathcal{G}_{sc} = sl_n$ and $\mathcal{G}_{ad} = pgl_n$. Then we have the following assertions:*

(1) We always have $sl_n = [sl_n, sl_n]$. Moreover $dim\, z(sl_n) \neq 0$ if and only if p is not very good, in which case $dim\, z(sl_n) = 1$.

(2) We always have $z(pgl_n) = \{0\}$, moreover $pgl_n = [pgl_n, pgl_n]$ if and only if p is very good. When p is not very good, the Lie algebra pgl_n is of the form $k.\sigma \oplus [pgl_n, pgl_n]$ where σ is a semi-simple element.

(3) The three following situations occur:

(3.1) p does not divide $|P(\Phi)/X(T)|$, then $\mathcal{G} \simeq sl_n$,

(3.2) p does not divide $|X(T)/Q(\Phi)|$, then $\mathcal{G} \simeq pgl_n$,

(3.3) p divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$, then $\mathcal{G}$ is neither isomorphic to pgl_n nor to sl_n, and has a one-dimensional center. In fact $\mathcal{G}$ is of the form $z(\mathcal{G}) \oplus [\mathcal{G}, \mathcal{G}] \simeq z(\mathcal{G}) \oplus (sl_n/z(sl_n))$.

Proof: The assertions (1) and the first sentence of (2) follow from 2.4.4 and 2.3.4; the fact that dim $z(sl_n) \leq 1$ is easy.

Now we prove the second assertion of (2). Assume that p is not very good. Recall that for any semi-simple algebraic group G, the Lie algebra $[\mathcal{G},\mathcal{G}]$ is generated by $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$, moreover by (1), we have $sl_n = [sl_n, sl_n]$. Hence if $\rho : SL_n \to PGL_n$ denotes the canonical central isogeny then $d\rho(sl_n) = [pgl_n, pgl_n]$. On the other hand since we always have $\mathrm{Ker}(d\rho) = z(sl_n)$, we have $d\rho(sl_n) \simeq sl_n/z(sl_n)$. We deduce that $[pgl_n, pgl_n] \simeq sl_n/z(sl_n)$. Now since p is not very good, by (1), we have $\dim z(sl_n) = 1$ and so $[pgl_n, pgl_n]$ is of codimension one in pgl_n; the fact that σ in (2) can be chosen semi-simple follows from the fact that for any connected reductive group G, the Lie algebra $[\mathcal{G},\mathcal{G}]$ contains all the nilpotent elements of $\mathcal{G}$.

Now we describe the situation (3.3). First note that the situations (3.1) and (3.2) have been already studied, see equivalence between (iv) and (v) in 2.4.4 for (3.1) and in 2.3.7 for (3.2). Let $\pi : SL_n \to G$ be the canonical central isogeny.

Assume that p divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$.

(i) Since p divides $|P(\Phi)/X(T)|$, the map $d\pi$ is not injective. Moreover by (1), the Lie algebra $z(sl_n)$ is one-dimensional, thus we deduce from $\mathrm{Ker}(d\pi) \subset z(sl_n)$ that $\mathrm{Ker}(d\pi) = z(sl_n)$. As a consequence we have $d\pi(sl_n) \simeq sl_n/z(sl_n)$ and so $[\mathcal{G},\mathcal{G}]$, which is equal to $d\pi([sl_n, sl_n]) = d\pi(sl_n)$, is of codimension one in $\mathcal{G}$ and has a trivial center.

(ii) Now since p divides $|X(T)/Q(\Phi)|$, the Lie algebra $\mathcal{G}$ has a non-trivial center (see 2.3.4). Hence by (i), the Lie algebra $z(\mathcal{G})$ must be one-dimensional.

□

We are now in position to discuss the existence of non-degenerate invariant bilinear forms on the Lie algebras of simple algebraic groups of type A_n. We have the following proposition.

Proposition 2.5.10. *Assume that G is simple of type A_n. Then $\mathcal{G}$ is endowed with a non-degenerate G-invariant bilinear form if and only if p is very good for G or p divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$.*

Proof: Assume that G is of type A_{n-1} with $n > 1$ and that p is very good for G. Then p does not divide n and so the SL_n-invariant bilinear form $(X,Y) \mapsto \mathrm{Tr}(XY)$ on sl_n is non-degenerate. Moreover the canonical morphism $sl_n \to \mathcal{G}$ is an isomorphism, hence we can proceed as in the proof of 2.5.8 to show the existence of a non-degenerate G-invariant bilinear form on $\mathcal{G}$.

Assume now that p divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$. Then by 2.5.9 (3.3), we have $\mathcal{G} = z(\mathcal{G}) \oplus [\mathcal{G}, \mathcal{G}]$. Since G acts trivially on $z(\mathcal{G})$, any G-invariant non-degenerate bilinear form on $[\mathcal{G}, \mathcal{G}]$ can be extended to a non-degenerate G-invariant bilinear form on $\mathcal{G}$. Hence, it is enough to show the existence of a non-degenerate G-invariant bilinear form on $[\mathcal{G}, \mathcal{G}] \simeq sl_n/z(sl_n)$. Define $\langle,\rangle$ on $sl_n/z(sl_n)$ by $\langle x + z(sl_n), y + z(sl_n)\rangle = \mathrm{Tr}(xy)$. This is well defined since $z(sl_n) \simeq k$ and for any $X \in sl_n$, $a \in k$, $\mathrm{Tr}(aX) = a\mathrm{Tr}(X) = 0$. Let $\pi : SL_n \to G$ be the canonical central isogeny, then for any $g \in SL_n$, we have $d\pi \circ \mathrm{Ad}(g) = \mathrm{Ad}(\pi(g)) \circ d\pi$ so it is not difficult to check that $\langle,\rangle$ is G-invariant. It remains to check that it is non-degenerate. Let $x \in sl_n$ and assume that for any $y \in sl_n$, we have $\mathrm{Tr}(xy) = 0$. Then an easy calculation shows that $x \in z(sl_n)$, that is, its image in $sl_n/z(sl_n)$ is zero. We thus proved the non-degeneracy of $\langle,\rangle$ on $[\mathcal{G}, \mathcal{G}]$.

Now assume that there exists a non-degenerate G-invariant bilinear form $\langle,\rangle$ on $\mathcal{G}$ and that p does not divide $|X(T)/Q(\Phi)|$ or $|P(\Phi)/X(T)|$. We want to prove that p is very good for G. Two situations occur,

(1) p does not divide $|P(\Phi)/X(T)|$, then we may assume that $G = SL_n$. Let $z \in z(\mathcal{G})$, then by 2.5.1(2), for any $\alpha \in \Phi$, we have $\langle z, e_\alpha\rangle = 0$, and since z is central, by 2.5.1(1) we have $\langle z, h_\alpha\rangle = 0$. But by 2.4.4, the set $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is a basis of $\mathcal{G}$, hence the non-degeneracy of $\langle,\rangle$ implies that $z = 0$. We thus proved that $z(\mathcal{G}) = \{0\}$. By 2.3.4, we deduce that p does not divide $|(X(T)/Q(\Phi))_{tor}| = |P(\Phi)/Q(\Phi)|$ and so that p is very good for G.

(2) p does not divide $|X(T)/Q(\Phi)|$, then we may assume that $G = PGL_n$. Assume that p is not very good (i.e p divides $|P(\Phi)/X(T)|$).

Let $T \subset G$ be the set of diagonal matrices modulo Z_{GL_n} and let B be the set of upper triangular matrices modulo Z_{GL_n}. For $i \in \{1, ..., n-1\}$, let $\alpha_i : T \to k^\times$ be defined by $\alpha_i(t_1, ..., t_n) = t_i t_{i+1}^{-1}$; note that $d\alpha_i(t_1, ..., t_{n-1}) = t_i - t_{i+1}$. The basis Π of Φ is equal to $\{\alpha_1, ..., \alpha_{n-1}\}$. Since p is not very good, by 2.5.9 (2), the Lie algebra $[\mathcal{G}, \mathcal{G}]$ is of codimension one in $\mathcal{G}$. As a consequence, the vectors h_{α_i} with $i \in \{1, ..., n-1\}$ are linearly dependent i.e. there exists $\lambda_1, .., \lambda_{n-1}$ not all equal to zero such that $h := \lambda_1 h_{\alpha_1} + ... + \lambda_{n-1} h_{\alpha_{n-1}} = 0$. Let r be the smallest integer such that $\lambda_r \neq 0$ and let σ be the $n \times n$ matrix $(a_{ij})_{i,j}$ (modulo $z(gl_n)$) with $a_{rr} = 1$ and $a_{ij} = 0$ for $i, j \neq r$. Since $h = 0$, we have

$$\langle \sigma, h\rangle = 0. \qquad (*)$$

On the other hand, since $\langle , \rangle$ is G-invariant, we have $\langle \sigma, h_\alpha \rangle = \langle \sigma, [e_\alpha, e_{-\alpha}] \rangle = \langle [\sigma, e_\alpha], e_{-\alpha} \rangle = \langle d\alpha(\sigma) e_\alpha, e_{-\alpha} \rangle = d\alpha(\sigma) \langle e_\alpha, e_{-\alpha} \rangle$ for any $\alpha \in \Phi$. Since $d\alpha_r(\sigma) = 1$ and $d\alpha_i(\sigma) = 0$ for any $i > r$, we deduce that $\langle \sigma, h \rangle = \lambda_r \langle e_{\alpha_r}, e_{-\alpha_r} \rangle$. But the bilinear form $\langle , \rangle$ is non-degenerate, hence by 2.5.1(2), we have $\langle e_{\alpha_r}, e_{-\alpha_r} \rangle \neq 0$ which contradicts (*). □

Remark 2.5.11. Assume that G is simply connected. Then we can proceed as in (1) of the proof of 2.5.10 to show that the existence of a non-degenerate G-invariant bilinear form on $\mathcal{G}$ implies that p does not divide $|P(\Phi)/Q(\Phi)|$. Hence, when G is of type B_n, C_n or D_n, by 2.5.6 (a), (d) and by 2.5.8, the Lie algebra $\mathcal{G}$ admits a non-degenerate G-invariant bilinear form if and only if p is good for G.

Proposition 2.5.12. *Let G be a connected reductive group. Assume that p is very good for G, then there exists a non-degenerate G-invariant bilinear form on $\mathcal{G}$.*

Proof: By assumption, the prime p does not divide $|(X(T)/Q(\Phi))_{tor}|$. Thus, by 2.3.4 and 2.3.1, we may identify $\mathcal{G}$ with $z(\mathcal{G}) \oplus \overline{\mathcal{G}}$. Since G acts trivially on $z(\mathcal{G})$, any non-degenerate $\overline{G}$-invariant bilinear form on $\overline{\mathcal{G}}$ can be extended to a non-degenerate G-invariant bilinear form on $z(\mathcal{G}) \oplus \overline{\mathcal{G}} \simeq \mathcal{G}$. So it is enough to show the existence of a non-degenerate $\overline{G}$-invariant bilinear form on $\overline{\mathcal{G}}$. Let $\overline{G} = G_1 ... G_n$ be a decomposition of $\overline{G}$ as the almost-direct product of its simple components. By 2.5.8 and 2.5.10, for any simple component G_i of $\overline{G}$, there exists an G_i-invariant non-degenerate bilinear form B_i on $\mathcal{G}_i := \mathrm{Lie}(G_i)$. Since p is very good for $\overline{G}$, by 2.3.8, we have a decomposition $\overline{\mathcal{G}} = \bigoplus_i \mathcal{G}_i$ and so the form $B = \bigoplus_i B_i$ provides a non-degenerate $\overline{G}$-invariant bilinear form on $\overline{\mathcal{G}}$. □

Remark 2.5.13. We saw in the proof of 2.5.12 that a non-degenerate $\overline{G}$-invariant bilinear form on $\overline{\mathcal{G}}$ can be extended to a non-degenerate G-invariant bilinear form on $\mathcal{G}$. However it is not true that all non-degenerate G-invariant bilinear forms on $\mathcal{G}$ are obtained in this way. Indeed, the trace form $(X, Y) \mapsto \mathrm{Tr}(XY)$ is always non-degenerate on gl_n while (see 2.5.10) there is no non-degenerate PGL_n-invariant bilinear form on pgl_n unless p is very good.

We have the following lemma.

Lemma 2.5.14. *[Leh96, proof of 4.3] If $\mathcal{G}$ admits an G-invariant non-degenerate bilinear form $B(,)$, then the restriction of $B(,)$ to any Levi subalgebra of $\mathcal{G}$ is still non-degenerate.*

2.5.15. Now we assume that p is very good for G. By 2.5.12, the Lie algebra $\mathcal{G}$ is endowed with a G-invariant non-degenerate bilinear form $B(,)$ and in view of 2.3.1 and 2.3.4, we may write $\mathcal{G} = z(\mathcal{G}) \oplus \overline{\mathcal{G}}$. We have the following lemma.

Lemma 2.5.16. *The vector space $\overline{\mathcal{G}}$ is the orthogonal complement of $z(\mathcal{G})$ in $\mathcal{G}$ with respect to $B(,)$. In particular, the restrictions of $B(,)$ to $z(\mathcal{G})$ and to $\overline{\mathcal{G}}$ remain non-degenerate.*

Proof: Since p is very good for G (and so for $\overline{G}$), by 2.4.4, we have $[\overline{\mathcal{G}}, \overline{\mathcal{G}}] = \overline{\mathcal{G}}$. Thus, by 2.5.1(1), the vector space $\overline{\mathcal{G}}$ is orthogonal to $z(\mathcal{G})$. Hence the lemma follows from the non-degeneracy of $B(,)$. □

Remark 2.5.17. Note that if $G = GL_n(k)$, the restriction of $B(,)$ to $z(\mathcal{G})$ is non-degenerate if and only if the condition "p is very good for G" is satisfied. However, it is not a necessary condition in the general case. For instance, if G is simple of type A_n and p divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$. Then $z(\mathcal{G})$ is a one-dimensional vector space and we have $\mathcal{G} = z(\mathcal{G}) \oplus [\mathcal{G}, \mathcal{G}]$, see 2.5.9 (3.3). By 2.5.10, there exists a non-degenerate G-invariant bilinear form $B(,)$ on $\mathcal{G}$. The G-invariance of $B(,)$ implies that $z(\mathcal{G})$ is orthogonal to $[\mathcal{G}, \mathcal{G}]$ with respect to $B(,)$. Thus the non-degeneracy of $B(,)$ implies that its restriction to $z(\mathcal{G})$ is still non-degenerate.

2.6 Centralizers

Let H be a closed subgroup of G with Lie algebra $\mathcal{H}$. For any $X \in \mathcal{G}$, recall (see 2.1.2(iii) and 2.1.3) that we have

2.6.1. $$\mathrm{Lie}(C_H(x)) \subset C_{\mathcal{H}}(x).$$

When $H = G$, this inclusion is known to be an equality when x is semi-simple [Bor, 9.1]. Due to Richardson-Springer-Steinberg, it is also known to be an equality for any $x \in \mathcal{G}$ when $H = G = GL_n$ or when $H = G$ is simple and p is very good for G [SS70, I, 5.6] [Slo80, 3.13]. In the following lemma, we extend the above result of R-S-S to the case where G is an arbitrary reductive group and p is very good for G.

Lemma 2.6.2. *Let $x \in \mathcal{G}$, then the inclusion 2.6.1 with $H = G$ is an equality (i.e the morphism $G \to \mathcal{O}_x^G$, $g \mapsto Ad(g)x$ is separable) in the following cases:*

(i) x is semi-simple,

(ii) p is very good for G or $G = GL_n$.

Proof: As noticed above, the lemma is already established in the case where X is semi-simple and the cases where $G = GL_n$, or G is simple and p is very good for G.

To show that $\mathrm{Lie}(C_G(x)) \subset C_{\mathcal{G}}(x)$ is an equality, it is enough to prove that $\dim(C_G(x)) = \dim(C_{\mathcal{G}}(x))$.

(a) Assume first that G is semi-simple and write $G = G_1...G_n$ where $G_1, ..., G_n$ are the simple components of G. Since p is very good for G, by 2.3.8 we have the following corresponding Lie algebras decomposition $\mathcal{G} = \bigoplus_i \mathcal{G}_i$ where $\mathcal{G}_i$ is the Lie algebra of G_i. Let $x = \sum_i x_i \in \bigoplus_i \mathcal{G}_i$, then $C_G(x)$ is the almost-direct product of the $C_{G_i}(x_i)$. We thus have

$$\dim(C_G(x)) = \sum_i \dim(C_{G_i}(x_i)). \tag{1}$$

On the other hand, let $y = y_1 + ... + y_n$ be the decomposition of $y \in \mathcal{G}$ in $\bigoplus_i \mathcal{G}_i$. Since $[\mathcal{G}_i, \mathcal{G}_j] = 0$ for $i \neq j$, we have $[y, x] = \sum_i [y_i, x_i]$. Hence $y \in C_{\mathcal{G}}(x)$ if and only if $y_i \in C_{\mathcal{G}_i}(x_i)$ for any i. Thus we have $C_{\mathcal{G}}(x) = \bigoplus_i C_{\mathcal{G}_i}(x_i)$ and so

$$\dim(C_{\mathcal{G}}(x)) = \sum_i \dim(C_{\mathcal{G}_i}(x_i)). \tag{2}$$

Since for any i, the group G_i is simple and p is very good for G_i, we have $\mathrm{Lie}(C_{G_i}(x_i)) = C_{\mathcal{G}_i}(x_i)$ and so $\dim(C_{G_i}(x_i)) = \dim(C_{\mathcal{G}_i}(x_i))$. Then we deduce from (1) and (2) that $\dim(C_{\mathcal{G}}(x)) = \dim(C_G(x))$.

(b) Assume now that G is reductive. Since p is very good for G, by 2.3.9 we have a decomposition

$$\mathcal{G} = z(\mathcal{G}) \oplus \mathcal{G}'. \tag{1}$$

Write $x = z + y$ with $z \in z(\mathcal{G})$ and $y \in \mathcal{G}'$. Since Z_G acts trivially on $\mathcal{G}$ we have $C_G(x) = Z_G^o.C_{G'}(x)$. But G acts trivially on $z(\mathcal{G})$, hence $C_{G'}(x) = C_{G'}(y)$. We deduce that $C_G(x) = Z_G^o.C_{G'}(y)$. Since $Z_G^o \cap C_{G'}(y)$ is finite we have

$$\dim(C_G(x)) = \dim Z_G^o + \dim(C_{G'}(y)). \tag{2}$$

On the other hand, from (1) we see that $C_{\mathcal{G}}(x) = z(\mathcal{G}) \oplus C_{\mathcal{G}'}(y)$ and so that

$$\dim(C_{\mathcal{G}}(x)) = \dim z(\mathcal{G}) + \dim(C_{\mathcal{G}'}(y)). \tag{3}$$

The group G' is semi-simple, so using (a) we have $\dim(C_{\mathcal{G}'}(y)) = \dim(C_{G'}(y))$. Hence the equality $\dim(C_G(x)) = \dim(C_{\mathcal{G}}(x))$ follows from (2), (3) and 2.3.4.

□

Remark 2.6.3. Note that 2.6.1 with $H = G$ may not be an equality if p is not very good for G. Indeed, consider $G = PGL_2$ with $p = 2$ and $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. A simple calculation shows that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ commutes with e. Hence $\dim(C_{\mathcal{G}}(e)) = 2$ while $C_G(e)$ is of dimension one.

Now we give some various results on centralizers of elements of $\mathcal{G}$ which will be used later. We first start with the following well-known characterization of the centralizers of semi-simple elements of $\mathcal{G}$ (see [SS70, II, 4.1]).

Proposition 2.6.4. *For each element $w \in W_G(T)$, we choose a representative $\dot{w}$ of w in $N_G(T)$. Let $x \in \mathcal{T}$.*

(i) The group $C_G(x)$ is generated by T, the U_α such that $d\alpha(x) = 0$ and the $\dot{w}$ such that $Ad(\dot{w})x = x$.

(ii) The group $C_G^o(x)$ is generated by T, and the U_α such that $d\alpha(x) = 0$.

(iii) The algebraic group $C_G^o(x)$ is reductive.

Lemma 2.6.5. *[HS85, Proposition 3] Let $P = LU_P$ be a Levi decomposition in G and let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition. Then the centralizer $C_{U_P}(x)$ is connected for any element x of $\mathcal{L}$.*

We have the following standard result.

Lemma 2.6.6. *Let L be a Levi subgroup of a parabolic subgroup P of G and let $\mathcal{L}$ be the Lie algebra of L. For any element z of $\mathcal{L}$, we have $\mathcal{O}_z^{U_P} \subset z + \mathcal{U}_P$. If $C_G^o(z_s) \subset L$, then the map $U_P \to z + \mathcal{U}_P$ given by $u \mapsto Ad(u)z$ is an isomorphism.*

Proof: Let $z \in \mathcal{L}$, we assume that $L \supseteq T$ and that $z_s \in \mathcal{T}$ so that we can use the notation of 2.2.2. Let $\alpha, \beta \in \Phi$ be such that $\alpha \neq -\beta$ and let $u_\alpha \in U_\alpha$, then by 2.2.2 we have

(2) $\text{Ad}(u_\alpha)z_s \in z_s + k.e_\alpha$,

(3) $\text{Ad}(u_\alpha)e_\beta = e_\beta + \sum_{\{i>o|\beta+i\alpha\in\Phi\}} c_i e_{\beta+i\alpha}$, for some $c_i \in k$.

Note also that

2.6.7. *if β is a root of L with respect to T (this makes sense since we assumed that $T \subset L$) and if $\alpha \in \Phi$ is a root of U_P (i.e $\mathcal{G}_\alpha \subset \mathcal{U}_P$), then for any $i > 0$ such that $i\alpha + \beta \in \Phi$, the root $i\alpha + \beta$ is a root of U_P.*

From (2), (3) and 2.6.7, we observe that $\mathcal{O}_z^{U_P} - Z$ is a subvariety of $\mathcal{U}_P$.

Assume now that $C^o_G(z_s) \subset L$, then since $C_{U_P}(z)$ is connected by 2.6.5 and $C_{U_P}(z) \subset C_G(z_s)$, we have $C_{U_P}(z) = \{1\}$ and so $\mathcal{O}^{U_P}_z - z$ is of dimension $\dim U_P$. On the other hand, by [Bor, Proposition 4.10], the variety $\mathcal{O}^{U_P}_z$ is closed in $\mathcal{G}$. We deduce that $\mathcal{O}^{U_P}_z = z + \mathcal{U}_P$. We thus have proved that the map $U_P \to z + \mathcal{U}_P$ given by $u \mapsto \mathrm{Ad}(u)z$ is a bijective morphism. To verify that it is an isomorphism, it is sufficient to verify that it is separable, that is, we have to show that $\mathrm{Lie}\big(C_{U_P}(z)\big) = C_{\mathcal{U}_P}(z)$. Since $C_{U_P}(z) = \{1\}$, we have to show that $C_{\mathcal{U}_P}(z) = \{0\}$. Let $v \in \mathcal{U}_P$ be such that $[v, z] = 0$. We have $[v, z_s] = -[v, z_n]$. Let B_L be a Borel subgroup of L such that $z \in \mathrm{Lie}(B_L)$; we may assume without loss of generality that B_L contains T. Then $B = B_L U_P$ is a Borel subgroup of G containing T and we denote by Φ^+ the positive roots of Φ with respect to B. We also denote by Φ^+_L the positive roots (with respect to B_L) of the root system Φ_L of L (with respect to T). Then we may write $v = \sum_{\alpha \in \Phi^+ - \Phi_L} \lambda_\alpha e_\alpha$ and $z_n = \sum_{\alpha \in \Phi^+_L} \beta_\alpha e_\alpha$. Assume that $v \neq 0$ and let $\alpha_o \in \Phi^+ - \Phi_L$ be such that $\lambda_{\alpha_o} \neq 0$ and the height of α_o (with respect to B) is minimal among the heights of the roots $\alpha \in \Phi^+ - \Phi_L$ such that $\lambda_\alpha \neq 0$. Since $C^o_G(z_s) \subset L$, from 2.6.4(ii), we have $d\alpha_o(z_s) \neq 0$, and so, from 2.2.1(ii), the vector $[v, z_s]$ has a non-zero coefficient in e_{α_o} while from the Chevalley relations 2.2.1(iii)(iv), we see that the vector $[v, z_n]$ does not have non-zero coefficients in e_α if α is of same height as α_o. Hence we have $v = 0$. □

Notation 2.6.8. For any set J contained in a basis of Φ, we denote by Φ_J the subroot system of Φ generated by J, by L_J the Levi subgroup of G corresponding to Φ_J (i.e the subgroup of G generated by T and the U_α such that $\alpha \in \Phi_J$) and by $\mathcal{L}_J$ the Lie algebra of L_J. If I is a subset of a basis of Φ, we denote by $B(I)$ the subset of $\Phi - \Phi_I$ consisting of the elements γ such that the set $I \cup \{\gamma\}$ is contained in a basis of Φ.

Proposition 2.6.9. *Let I be a subset of a basis of Φ. The minimal Levi subgroups of G strictly containing L_I are the $L_{\Phi_{I\cup\{\alpha\}}}$ with $\alpha \in B(I)$.*

Proof: Let M be a Levi subgroup of G containing L_I and let Φ_M be the root system of M with respect to T. Let P be a parabolic subgroup of M such that $P = L_I U_P$ is a Levi decomposition of P. Let B be a Borel subgroup of P containing T, then it defines a basis θ of Φ_M and since L_I is the unique Levi subgroup of P containing T, the group L_I must be of the form L_J for some subset J of θ (cf. [DM91, Propositions 1.6, 1.15]). Now, if $\gamma \in \Phi_M$ is a $\mathbb{Q}$-linear combination of elements of Φ_I, it is a $\mathbb{Z}$-linear combination of elements of θ. We deduce that γ is a $\mathbb{Z}$-linear combination of elements of J. We thus have $\gamma \in \Phi_I$. We proved that Φ_I is $\mathbb{Q}$-closed root subsystem of Φ_M (i.e any element of Φ_M which is a $\mathbb{Q}$-linear combination of elements of Φ_I is already in Φ_I). By [Bou, VI, 1, 7, Proposition 24], we deduce that we can extend I to a basis I' of Φ_M. Using the same argument, we can also prove that I' can be extended to a basis of Φ. Hence, we proved that any Levi subgroup of G containing strictly L_I contains a Levi subgroup of the form $L_{I\cup\{\alpha\}}$ with $\alpha \in B(I)$. It is then clear that minimal Levi subgroups containing strictly L_I are of the form $L_{I\cup\{\alpha\}}$ for some $\alpha \in B(I)$. The fact that the Levi subgroups $L_{I\cup\{\alpha\}}$ with $\alpha \in B(I)$ are minimal is clear. □

Definition 2.6.10. *Let L be a Levi subgroup of G, then we say that $x \in \mathcal{G}$ is L-*regular *in $\mathcal{G}$ if $L = C_G^o(x)$.*

Lemma 2.6.11. *Let L be a Levi subgroup of G and let $\mathcal{L}$ be its Lie algebra, then the L-regular elements in $\mathcal{G}$ belong to $z(\mathcal{L})$.*

Proof: Let x be L-regular in $\mathcal{G}$, then $C_G(x)$ contains a maximal torus T of L. Write $x = t + \sum_\alpha \lambda_\alpha e_\alpha \in \mathcal{G} = \mathcal{T} \oplus \bigoplus_{\alpha\in\Phi(T)} \mathcal{G}_\alpha$. Since T centralizes x, we must have $\lambda_\alpha = 0$ for all $\alpha \in \Phi(T)$, i.e. $x \in \mathcal{T}$. Since $C_G^o(x) = L$ we deduce from 2.6.4(ii) and 2.3.2(1) that $x \in z(\mathcal{L})$. □

Definition 2.6.12. *Let L be a Levi subgroup of G and let $\mathcal{L}$ be its Lie algebra. If $x \in z(\mathcal{L})$ is not L-regular in $\mathcal{G}$, then x is said to be L-*irregular*.*

Lemma 2.6.13. *(i) Assume that p is good for G and that p does not divide $|(X(T)/Q(\Phi))_{tor}|$, then if L is a Levi subgroup of G, the Lie algebra $\mathcal{G}$ contains L-regular elements in $\mathcal{G}$.*

(ii) If p is good for G, then for any semi-simple element $x \in \mathcal{G}$, the group $C_G^o(x)$ is a Levi subgroup of G.

Proof: We first prove (ii). We may assume that $x \in \mathcal{T}$. Since p is good for G, it follows that the set $\Phi_x := \{\alpha \in \Phi |\, d\alpha(x) = 0\}$ is a $\mathbb{Q}$-closed root subsystem of Φ. Hence by [Bou, VI, 1, 7, Proposition 24], the set Φ_x is of the form Φ_J for some subset J of some basis of Φ. Thus by 2.6.4(ii), we have $C_G^o(x) = L_J$.

We now prove (i).

We may assume without loss of generality that L is a Levi subgroup of the form L_I for some subset I of some basis of Φ. We want to prove that $\mathcal{L}_I$ contains L_I-regular elements in $\mathcal{G}$. Recall first that if J is a set contained in a basis of Φ, then we have

$$z(\mathcal{L}_J) = \bigcap_{\alpha \in J} \mathrm{Ker}(d\alpha). \tag{1}$$

From (ii), 2.6.11, 2.6.9 and 2.6.4 (ii), we see that x is L_I-regular in $\mathcal{G}$ if and only if

2.6.14.

$$x \in z(\mathcal{L}_I) - \bigcup_{\gamma \in B(I)} (z(\mathcal{L}_I) \cap \mathrm{Ker}(d\gamma)).$$

Since $B(I)$ is finite, the assertion (i) will follow from the fact that the subspaces $z(\mathcal{L}_I) \cap \mathrm{Ker}(d\gamma)$, where γ runs over $B(I)$, are of dimension strictly less than $\dim z(\mathcal{L}_I)$. Hence from (1), it is enough to prove that for any basis Π' of Φ and any inclusions $\Pi' \supseteq J \supsetneq I$ we have $z(\mathcal{L}_J) \subsetneq z(\mathcal{L}_I)$.

Consider the following inclusions $\Pi' \supseteq J \supsetneq I$ with Π' a basis of Φ, then it follows from [DM91, Proposition 1.21] that $Z^o_{L_J} \subsetneq Z^o_{L_I}$ and so (because tori are smooth) we get that $\mathrm{Lie}(Z^o_{L_J}) \subsetneq \mathrm{Lie}(Z^o_{L_I})$. From 2.3.5, we get that p satisfies 2.3.4(i) applied to L_I and L_J; thus $\mathrm{Lie}(Z^o_{L_I}) = z(\mathcal{L}_I)$ and $\mathrm{Lie}(Z^o_{L_J}) = z(\mathcal{L}_J)$. We deduce that $z(\mathcal{L}_J) \subsetneq z(\mathcal{L}_I)$. □

Remark 2.6.15. Let L be a Levi subgroup of G and let $\mathcal{L}$ be the Lie algebra of L. Assume that the set of L-regular elements in $\mathcal{G}$ is non-empty, then from 2.6.14 we see that it is an open dense subset of $z(\mathcal{L})$.

Lemma 2.6.16. *Let L be a Levi subgroup of G (with Lie algebra $\mathcal{L}$) and let $x \in \mathcal{G}$ be L-regular in $\mathcal{G}$. Let $g \in G$ be such that $Ad(g)x \in z(\mathcal{L})$, then $Ad(g)x$ is also L-regular in $\mathcal{G}$ and we have $g \in N_G(L)$.*

Proof: It is enough to show that $g \in N_G(L)$. We have $C^o_G(z(\mathcal{L})) \subset C^o_G(\mathrm{Ad}(g)x)$, that is $C^o_G(z(\mathcal{L})) \subset gC^o_G(x)g^{-1} = gLg^{-1}$. Since $C^o_G(z(\mathcal{L})) \supseteq L$, we deduce that $L \subset gLg^{-1}$, i.e. $L = gLg^{-1}$. □

Lemma 2.6.17. *We assume that $k = \overline{\mathbb{F}}_q$, that p is good for G and that p does not divide $|(X(T)/Q(\Phi))_{tor}|$. We also assume that T and B are both F-stable. Let I be a subset of Π such that the Levi subgroup L_I of G is F-stable, i.e. the set I is τ-stable where τ is as in 2.1.18. If $q > |B(I)|$, then $\mathcal{L}_I^F$ contains L_I-regular elements in $\mathcal{G}$.*

Proof: Recall that the subset of $z(\mathcal{L}_I)$ consisting of the L_I-irregular elements of $\mathcal{G}$ is

$$\bigcup_{\gamma\in B(I)}(z(\mathcal{L}_I)\cap \operatorname{Ker}(d\gamma)).$$

Let $V = z(\mathcal{L}_I)\cap \operatorname{Ker}(d\gamma)$ for some $\gamma\in B(I)$, then for i large enough, the set $V\cap F(V)\cap ...\cap F^i(V)$ is F-stable and contains all the rational elements of V. On the other hand, from the proof of 2.6.13, we have $\dim V < \dim z(\mathcal{L}_I)$. Thus by [DM91, 3.7], the number of rational L_I-irregular elements of $\mathcal{G}$ is $\leq |B(I)|q^{\dim(z(\mathcal{L}_I))-1}$. Hence, if $q > |B(I)|$, the number of L_I-irregular elements is less than $|z(\mathcal{L}_I)^F|$ and so rational L_I-regular elements must exist. □

Proposition 2.6.18. *[Ste75, Theorem 3.14] The centralizers in G of the semi-simple elements of $\mathcal{G}$ are connected if and only if p is not a torsion prime for G.*

2.7 The Varieties G_{uni} and $\mathcal{G}_{nil}$

Let G_{uni} be the subvariety of G consisting of unipotent elements and let $\mathcal{G}_{nil}$ be the subvariety of $\mathcal{G}$ formed by nilpotent elements. For any $X\subset G$ and $Y\subset\mathcal{G}$, put $X_{uni} = X\cap G_{uni}$ and $Y_{nil} = Y\cap\mathcal{G}_{nil}$. Recall that the subvarieties $G_{uni}\subset G$ and $\mathcal{G}_{nil}\subset\mathcal{G}$ are closed, irreducible of codimension $rk(G)$. It has been proved [Lus76] that the number of unipotent classes of G is finite for any p. By 2.7.5, this implies that the number of nilpotent orbits of $\mathcal{G}$ is also finite if p is good for G. In the case of bad characteristics, the finiteness of nilpotent orbits results from a case by case argument (see [Car72, 5.11] for the classification of nilpotent orbits in bad characteristics).

The following propositions are well-known.

Proposition 2.7.1. *[Leh79] Let $P = LU_P$ be a Levi decomposition of a parabolic subgroup P of G and let $\mathcal{P} = \mathcal{L}\oplus\mathcal{U}_P$ be the corresponding Lie algebra decomposition.*

(i) Let $l\in L$, then the semi-simple part of any element of lU_P is U_P-conjugate to the semi-simple part of l.

(ii) Let $x\in\mathcal{L}$, then the semi-simple part of any element of $x+\mathcal{U}_P$ is U_P-conjugate to the semi-simple part of x. That is, for any $v\in\mathcal{U}_P$, we have $(x+v)_s = Ad(u)(x_s)$ for some $u\in U_P$.

The following result is a straightforward consequence of the above proposition.

Corollary 2.7.2. *If $P = LU_P$ is a Levi decomposition in G with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$, then for any unipotent element $l \in L$ and any nilpotent element $x \in \mathcal{L}$, we have $lU_P \subseteq G_{uni}$ and $x+\mathcal{U}_P \subseteq \mathcal{G}_{nil}$.*

Proposition 2.7.3. *[Spr69] If the canonical morphism $\pi : G \to G_{ad}$ is separable (which by 2.3.7 is equivalent to p does not divide the torsion of $X(T)/Q(\Phi)$), then the bijective morphism $\pi_{uni} : G_{uni} \to (G_{ad})_{uni}$ given by restricting π to G_{uni} is an isomorphism.*

Remark 2.7.4. Consider $G = SL_2(k)$ and assume that the morphism $\pi : G \to G_{ad}$ is not separable (i.e. $p = 2$). Then the morphism $\pi_{uni} : G_{uni} \to (G_{ad})_{uni}$ is not an isomorphism. To see that, it is enough to see that its differential $d(\pi_{uni})$ at $1 = 1_G$ is not an isomorphism. Note that $d(\pi_{uni}) : T_1(G_{uni}) \to T_1((G_{ad})_{uni})$ is the restriction morphism of $d\pi$ to the tangent space $T_1(G_{uni})$ of G_{uni} at 1. On the other hand, $\dim T_1(G_{uni}) > 2$; indeed $\dim \mathcal{G}_{nil} = 2$ and the inclusion $T_1(G_{uni}) \supset \mathcal{G}_{nil}$ is strict since $\mathcal{G}_{nil}$ is not a vector space. Hence $T_1(G_{uni}) = \mathcal{G}$ since $\dim \mathcal{G} = 3$, and so we deduce that $d(\pi_{uni}) = d\pi$. Since π is not separable, the morphism $d\pi = d(\pi_{uni})$ is not an isomorphism.

2.7.5. By a G-equivariant morphism $\pi : G_{uni} \to \mathcal{G}_{nil}$, we shall mean a morphism $\pi : G_{uni} \to \mathcal{G}_{nil}$ such that $\pi(gxg^{-1}) = \mathrm{Ad}(g)\pi(x)$ for all $g \in G$ and $x \in G_{uni}$. The existence of G-equivariant isomorphisms $G_{uni} \to \mathcal{G}_{nil}$ is discussed in [Spr69] and in [BR85]. It is proved that if p is good for G, resp. very good for G, then G-equivariant homeomorphisms, resp. isomorphisms, $G_{uni} \to \mathcal{G}_{nil}$ exist.

We have the following lemma.

Lemma 2.7.6. *[Bon04, Proposition 6.1] Let $f : G_{uni} \to \mathcal{G}_{nil}$ be a G-equivariant homeomorphism, then for any Levi decomposition $P = LU_P$ in G with $\mathcal{L} = Lie(L)$, we have*

(i) $f(L_{uni}) = \mathcal{L}_{nil}$,

(ii) for any $x \in L_{uni}$, $f(xU_P) = f(x) + \mathcal{U}_P$.

3

Deligne-Lusztig Induction

From now we assume that $k = \overline{\mathbb{F}}_q$ with q a power of p and that G is defined over $\mathbb{F}_q$. Let ℓ denote a prime not equal to p and $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers. In this chapter, we first recall some facts about the $\overline{\mathbb{Q}}_\ell$-space $\mathcal{C}(\mathcal{G}^F)$ of all functions $\mathcal{G}^F \to \overline{\mathbb{Q}}_\ell$ which are invariant under the adjoint action of G^F on $\mathcal{G}^F$. We then define, when p is good for G, a Lie algebra version of Deligne-Lusztig induction [DL76], that is for any F-stable Levi subgroup L of G with Lie algebra $\mathcal{L}$, we define a $\overline{\mathbb{Q}}_\ell$-linear map $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}} : \mathcal{C}(\mathcal{L}^F) \to \mathcal{C}(\mathcal{G}^F)$ which satisfies analogous properties to the group case, like transitivity, the Mackey formula and commutation with the duality map. We finally formulate as a conjecture a property which has no counterpart in the group setting, namely that the Deligne-Lusztig induction commutes with Fourier transforms.

3.1 The Space of G^F-Invariant Functions on $\mathcal{G}^F$

3.1.1. Let H be an F-stable closed subgroup of G with Lie algebra $\mathcal{H}$. We denote by $\mathcal{C}(\mathcal{H}^F)$ the $\overline{\mathbb{Q}}_\ell$-space of all functions $f : \mathcal{H}^F \to \overline{\mathbb{Q}}_\ell$ which are H^F-invariant i.e. for any $h \in H^F$ and any $x \in \mathcal{H}^F$, $f(\mathrm{Ad}(h)x) = f(x)$. We denote by $\mathcal{C}(\mathcal{H}^F)_{nil}$ the subspace of $\mathcal{C}(\mathcal{H}^F)$ consisting of functions which are nilpotently supported. If $x \in \mathcal{H}^F$, we denote by ξ_x^H the characteristic functions of $\mathcal{O}_x^{H^F}$, i.e. $\xi_x^H(y) = 1$ if $y \in \mathcal{O}_x^{H^F}$ and $\xi_x^H(y) = 0$ otherwise. The functions ξ_x^H, with $x \in \mathcal{H}^F$, form a $\overline{\mathbb{Q}}_\ell$-basis of $\mathcal{C}(\mathcal{H}^F)$, and the functions ξ_x^H, with $x \in \mathcal{H}_{nil}^F$, form a $\overline{\mathbb{Q}}_\ell$-basis of $\mathcal{C}(\mathcal{H}^F)_{nil}$. Sometimes, it will be more convenient to use the functions $\gamma_x^H := |C_H(x)^F|\xi_x^H$ instead of ξ_x^H. We denote by $\eta_o^{\mathcal{H}}$ the function which takes the value 1 on $\mathcal{H}_{nil}^F$ and the value 0 on $\mathcal{H}^F - \mathcal{H}_{nil}^F$.

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We choose once for all an automorphism $\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell, x \mapsto \overline{x}$ such that $\overline{\zeta} = \zeta^{-1}$ for any root of unity $\zeta \in \overline{\mathbb{Q}}_\ell$. We define a non-degenerate bilinear form $(,)_{\mathcal{H}^F}$ on $\mathcal{C}(\mathcal{H}^F)$ by

$$(f, g)_{\mathcal{H}^F} = |H^F|^{-1} \sum_{x \in \mathcal{H}^F} f(x)\overline{g(x)}.$$

Note that for $x \in \mathcal{H}^F$ and $f \in \mathcal{C}(\mathcal{H}^F)$, we have $(f, \gamma_x^H)_{\mathcal{H}^F} = f(x)$ and $(\gamma_x^H, f)_{\mathcal{H}^F} = \overline{f(x)}$.

Definition 3.1.2. *Let P be an F-stable parabolic subgroup of G and L be an F-stable Levi subgroup of P. Let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition. Recall that $\pi_{\mathcal{P}} : \mathcal{P} \to \mathcal{L}$ denotes the canonical projection.*

(i) The Harish-Chandra restriction ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}} : \mathcal{C}(\mathcal{G}^F) \to \mathcal{C}(\mathcal{L}^F)$ is defined by the formula

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}(f)(x) = |U_P^F|^{-1} \sum_{y \in \mathcal{U}_P^F} f(x + y), \quad \textit{where } f \in \mathcal{C}(\mathcal{G}^F),\, x \in \mathcal{L}^F.$$

(ii) The Harish-Chandra induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}} : \mathcal{C}(\mathcal{L}^F) \to \mathcal{C}(\mathcal{G}^F)$ is defined by

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}(f)(x) = |P^F|^{-1} \sum_{\{g \in G^F | Ad(g)x \in \mathcal{P}^F\}} f\big(\pi_{\mathcal{P}}(Ad(g)x)\big),$$

where $f \in \mathcal{C}(\mathcal{L}^F)$, $x \in \mathcal{G}^F$.

We have the following proposition (see [Leh96]).

Proposition 3.1.3. *The maps ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}$ and $\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}$ are adjoint with respect to the forms $(,)_{\mathcal{G}^F}$ and $(,)_{\mathcal{L}^F}$, that is, for any $f \in \mathcal{C}(\mathcal{G}^F)$, $g \in \mathcal{C}(\mathcal{L}^F)$, we have $\big(\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}(g), f\big)_{\mathcal{G}^F} = \big(g, {}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}(f)\big)_{\mathcal{L}^F}$. Moreover they are independent of P.*

Notation 3.1.4. Since the map $\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}$ is independent of P, we write $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}$ instead of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}$.

3.1.5. We define (following Kawanaka [Kaw82] in the Lie algebra case and Lusztig, Curtis and Alvis in the group case) the "duality map" $\mathcal{D}_{\mathcal{G}} : \mathcal{C}(\mathcal{G}^F) \to \mathcal{C}(\mathcal{G}^F)$. For any F-stable parabolic subgroup P of G, we denote by $r(P)$ the semi-simple $\mathbb{F}_q$-rank of P/U_P.

Definition 3.1.6. *Let B be an F-stable Borel subgroup of G. For $f \in \mathcal{C}(\mathcal{G}^F)$, we define $\mathcal{D}_\mathcal{G}(f)$ by*

$$\mathcal{D}_\mathcal{G}(f) = \sum_{P \supset B} (-1)^{r(P)} \mathcal{R}^\mathcal{G}_{\mathcal{L}_P} \circ {}^*\mathcal{R}^\mathcal{G}_{\mathcal{L}_P}(f)$$

where the summation is over the set of the F-stable parabolic subgroups P of G containing B and where $\mathcal{L}_P$ denotes the Lie algebra of an arbitrarily chosen F-stable Levi subgroup of P.

Recall that the map $\mathcal{D}_\mathcal{G}$ does not depend on the F-stable Borel subgroup B and on the choice of the $\mathcal{L}_P$.

Proposition 3.1.7. *[Kaw82] We have the following assertions.*

(i) The duality map $\mathcal{D}_\mathcal{G}$ is an isometry with respect to the form $(,)_{\mathcal{G}^F}$.
(ii) $\mathcal{D}_\mathcal{G}$ is an involution, i.e. $\mathcal{D}_\mathcal{G} \circ \mathcal{D}_\mathcal{G} = Id_{\mathcal{C}(\mathcal{G}^F)}$.

Proposition 3.1.8. *[Leh96, Proposition 3.15] Let L be an G-split F-stable Levi subgroup of G and let $\mathcal{L} = Lie(L)$. Then*

$$\mathcal{D}_\mathcal{G} \circ \mathcal{R}^\mathcal{G}_\mathcal{L} = \mathcal{R}^\mathcal{G}_\mathcal{L} \circ \mathcal{D}_\mathcal{L}.$$

3.1.9. Now we assume the existence of a G-invariant non-degenerate bilinear form on $\mathcal{G}$ defined over $\mathbb{F}_q$ which we denote by μ. We fix (throughout this book) a non-trivial additive character $\Psi : \mathbb{F}_q^+ \to \overline{\mathbb{Q}}_\ell^\times$. Let $\mathcal{H}$ be an F-stable Lie subalgebra of $\mathcal{G}$ such that the restriction of μ to $\mathcal{H} \times \mathcal{H}$ remains non-degenerate. Let $Fun(\mathcal{H}^F)$ be the $\overline{\mathbb{Q}}_\ell$-space of all functions $\mathcal{H}^F \to \overline{\mathbb{Q}}_\ell$. The Fourier transform $\mathcal{F}^\mathcal{H} : Fun(\mathcal{H}^F) \to Fun(\mathcal{H}^F)$ with respect to (μ, Ψ) is defined as follows:

For any $f \in Fun(\mathcal{H}^F)$ and any $x \in \mathcal{H}^F$, define

$$\mathcal{F}^\mathcal{H}(f)(x) = |\mathcal{H}^F|^{-\frac{1}{2}} \sum_{y \in \mathcal{H}^F} \Psi\left(\mu(x,y)\right) f(y).$$

Clearly if H is an F-stable closed subgroup of G having $\mathcal{H}$ as a Lie algebra, then $\mathcal{F}^\mathcal{H}$ induces a linear map $\mathcal{C}(\mathcal{H}^F) \to \mathcal{C}(\mathcal{H}^F)$ denoted again by $\mathcal{F}^\mathcal{H}$.

For $f, g \in Fun(\mathcal{H}^F)$, we denote by $f.g$ the pointwise multiplication of f and g, i.e. $(f.g)(x) = f(x)g(x)$ for $x \in \mathcal{H}^F$, and we denote by $f * g$ the *convolution product* of f and g, i.e.

$$(f * g)(x) = |\mathcal{G}^F|^{-\frac{1}{2}} \sum_{y \in \mathcal{G}^F} f(x-y)g(y).$$

Lemma 3.1.10. *[Leh96, Lemma 4.2] Let $\mathcal{H}$ be the Lie algebra of an F-stable closed subgroup H of G such that the restriction of μ to $\mathcal{H} \times \mathcal{H}$ remains non-degenerate. Let $f, g \in Fun(\mathcal{H}^F)$ and put $\mathcal{F} = \mathcal{F}^{\mathcal{H}}$.*

(i) $\mathcal{F}$ is an isometry of $Fun(\mathcal{H}^F)$ with respect to the form $(,)_{\mathcal{H}^F}$,
(ii) $\mathcal{F}^2 f = f^-$, where $f^-(x) = f(-x)$ for $x \in \mathcal{H}^F$,
(iii) $\mathcal{F}^4 = Id$,
(iv) $\mathcal{F}(f * g) = (\mathcal{F}f).(\mathcal{F}g)$,
(v) $\mathcal{F}(f.g) = (\mathcal{F}f) * (\mathcal{F}g)$.

We have the following theorems.

Theorem 3.1.11. *[Leh96, Theorem 4.5] Let $\mathcal{L}$ be the Lie algebra of a G-split F-stable Levi subgroup of G, then*

(i) $\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}^{\mathcal{G}}_{\mathcal{L}} = \mathcal{R}^{\mathcal{G}}_{\mathcal{L}} \circ \mathcal{F}^{\mathcal{L}}$,
(ii) $\mathcal{F}^{\mathcal{L}} \circ {}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}} = {}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}} \circ \mathcal{F}^{\mathcal{G}}$.

Theorem 3.1.12. *[Leh96, Theorem 4.6] The isometries $\mathcal{D}_{\mathcal{G}}$ and $\mathcal{F}^{\mathcal{G}}$ of $\mathcal{C}(\mathcal{G}^F)$ commute.*

3.2 Deligne-Lusztig Induction: Definition and Basic Properties

3.2.1 Deligne-Lusztig Induction: The Group Case

If X is a variety over k, then we denote by $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ the i-th group of ℓ-adic cohomology with compact support as in [Del77]. All what we need to know (in this chapter) about these groups can be found in [DM91, Chapter 10].

3.2.2. Let L be an F-stable Levi subgroup of G, let $P = LU_P$ be a Levi decomposition of a (possibly non F-stable) parabolic subgroup P of G and let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition. We denote by $\mathcal{L}_G$ the Lang map $G \to G, x \mapsto x^{-1}F(x)$. The variety $\mathcal{L}_G^{-1}(U_P)$ is endowed with an action of G^F on the left and with an action of L^F on the right. By [DM91, Proposition 10.2], these actions induce actions on the cohomology and so make $H_c^i(\mathcal{L}_G^{-1}(U_P), \overline{\mathbb{Q}}_\ell)$ into a G^F-module-L^F. The virtual $\overline{\mathbb{Q}}_\ell$-vector space $H_c^*(\mathcal{L}_G^{-1}(U_P)) := \sum_i (-1)^i H_c^i(\mathcal{L}_G^{-1}(U_P), \overline{\mathbb{Q}}_\ell)$ is thus a virtual G^F-module-L^F.

Notation 3.2.3. If $(g, l) \in G^F \times L^F$, define:

$$S_{L\subset P}^G(g, l) := \text{Trace}\left((g, l^{-1}) | H_c^*(\mathcal{L}_G^{-1}(U_P))\right).$$

To each L^F-module M, corresponds thus a virtual G^F-module $R_{L\subset P}^G(M) := H_c^*(\mathcal{L}_G^{-1}(U_P)) \otimes_{L^F} M$ (see [Lus76]). Hence, using the basis of the $\overline{\mathbb{Q}}_\ell$-vector space of class functions on L^F formed by the irreducible characters of L^F, the map $R_{L\subset P}^G$ gives rise to a natural $\overline{\mathbb{Q}}_\ell$-linear map, so-called *Deligne-Lusztig induction* and still denoted by $R_{L\subset P}^G$, from the $\overline{\mathbb{Q}}_\ell$-vector space of class functions on L^F onto the $\overline{\mathbb{Q}}_\ell$-vector space of class functions on G^F. More precisely if f is a class function on L^F, the class function $R_{L\subset P}^G(f)$ on G^F is given by the following formula:

3.2.4. $$R_{L\subset P}^G(f)(g) = |L^F|^{-1} \sum_{h\in L^F} S_{L\subset P}^G(g, h) f(h) \qquad \text{for any } g \in G^F.$$

Remark 3.2.5. It is conjectured and proved for q large enough that $R_{L\subset P}^G$ is independent of the parabolic subgroup P having L as a Levi subgroup (see 3.2.25 and 3.2.27 for more details).

We now define the two-variable Green functions; they appear naturally in the computation of the values of the Deligne-Lusztig induction of class functions (see 3.2.7 below).

Definition 3.2.6. *The function* $Q_{L\subset P}^G : G^F \times L^F \to \overline{\mathbb{Q}}_\ell$ *defined by*

$$Q_{L\subset P}^G(u, v) = \begin{cases} |L^F|^{-1} \, \textit{Trace}\left((u, v^{-1}) | \; H_c^*(\mathcal{L}_G^{-1}(U_P))\right) & \textit{if } (u, v) \in G_{uni}^F \times L_{uni}^F, \\ 0 & \textit{otherwise.} \end{cases}$$

is called a two-variable Green function.

In the case where L is a maximal torus of G, the two-variable Green functions become one-variable functions and are the ordinary Green functions introduced for any reductive groups by Deligne-Lusztig [DL76]. In the case of $G = GL_n(\mathbb{F})$, they were first introduced by Green [Gre55].

The following formula [DM91, 12.2][DM87][Lus86b], so-called the character formula for $R_{L\subset P}^G$, expresses the values of the functions $R_{L\subset P}^G(f)$, where f is a class function on L^F, in terms of the values of f and in terms of the values of some two-variable Green functions.

3.2.7. For any $x \in G^F$,

$$R^G_{L\subset P}(f)(x) = |L^F|^{-1}|C^o_G(x_s)^F|^{-1}\times$$
$$\sum_{\{h\in G^F | x_s \in {}^hL\}} |C^o_{{}^hL}(x_s)^F| \sum_{v\in (C^o_{{}^hL}(x_s)_{uni})^F} Q^{C^o_G(x_s)}_{C^o_{{}^hL}(x_s)}(x_u, v)^h f(x_s v),$$

where ${}^hL := hLh^{-1}$ and ${}^hf(y) := f(h^{-1}yh)$.

To simplify the notation, we usually omit the parabolic subgroup ${}^hP\cap C^o_G(x_s)$ from the notation $Q^{C^o_G(x_s)}_{C^o_{{}^hL}(x_s)}$.

3.2.8 Deligne-Lusztig Induction: The Lie Algebra Case

To define the Lie algebra analogue of Deligne-Lusztig induction, we use the Lie algebra version of the character formula 3.2.7 where the two-variable Green functions are transfered to the Lie algebra by means of a G-equivariant homeomorphism $\mathcal{G}_{nil} \to G_{uni}$ (see 2.7.5).

Assumption 3.2.9. *From now we assume that p is good for G so that there exists a G-equivariant homeomorphism $\omega : \mathcal{G}_{nil} \to G_{uni}$ which commutes with the Frobenius F.*

Remark 3.2.10. Since p is good for G, by 2.6.13(ii), the connected components of the centralizer in G of the semi-simple elements of $\mathcal{G}$ are Levi subgroups of G. Hence by 2.6.2 and 2.7.6, for any semi-simple element $\sigma \in \mathcal{G}$, the morphism ω induces a $C^o_G(\sigma)$-equivariant isomorphism $C_{\mathcal{G}}(\sigma)_{nil} \to C^o_G(\sigma)_{uni}$.

Definition 3.2.11. *With the notation of 3.2.2, the* two-variable Green function $\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} : \mathcal{G}^F \times \mathcal{L}^F \to \overline{\mathbb{Q}}_\ell$ *is defined by* $\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(u, v) =$

$$\begin{cases} |L^F|^{-1}\, Trace\left((\omega(u), \omega(v)^{-1})|\ H^*_c(\mathcal{L}^{-1}_G(U_P))\right) & \text{if } (u, v) \in \mathcal{G}^F_{nil} \times \mathcal{L}^F_{nil}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2.12. Assume that ω is the exponential map (which is well-defined if $p > 3(h^G_o - 1)$). Let T be an F-stable maximal torus of G contained in a (possibly non F-stable) Borel subgroup B of G. Assume that $\sigma \in \mathcal{T}^F$ is T-regular in $\mathcal{G}$, i.e. $C^o_G(\sigma) = T$. By a result of Kazhdan-Springer [Kaz77][Spr76], we have

$$\mathcal{Q}^{\mathcal{G}}_{\mathcal{T}\subset\mathcal{B}} = \epsilon_G \epsilon_T q^{\frac{rk(G)}{2}} \mathcal{F}^{\mathcal{G}}(\xi^G_\sigma).\eta^{\mathcal{G}}_o$$

where $\epsilon_G = (-1)^{\mathbb{F}_q - rank(G)}$.

Definition 3.2.13. *Let L be an F-stable Levi subgroup of G and let $P = LU_P$ be a Levi decomposition in G with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$.*

(i) Let $f \in \mathcal{C}(\mathcal{L}^F)$, then the Deligne-Lusztig induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f) \in \mathcal{C}(\mathcal{G}^F)$ *of f is defined by*

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(x_s + x_n) = |L^F|^{-1}|C^o_G(x_s)^F|^{-1}\times$$

$$\sum_{\{h\in G^F | x_s \in {}^h\mathcal{L}\}} |C^o_{{}^hL}(x_s)^F| \sum_{v\in C_{{}^h\mathcal{L}}(x_s)^F_{nil}} \mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{{}^h\mathcal{L}}(x_s)}(x_n, v) Ad_h(f)(x_s + v)$$

where for any $g \in G^F$, ${}^gL := gLg^{-1}$, ${}^g\mathcal{L} = Ad(g)\mathcal{L}$ and $Ad_g : \mathcal{C}(\mathcal{L}^F) \to \mathcal{C}(Ad(g)\mathcal{L}^F)$ is given by, $Ad_g(f)(x) = f(Ad(g^{-1})x)$.

(ii) Let $f \in \mathcal{C}(\mathcal{G}^F)$, then the Deligne-Lusztig restriction ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f) \in \mathcal{C}(\mathcal{L}^F)$ *of f is defined by*

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(x_s+x_n) = |C^o_L(x_s)^F||C^o_G(x_s)^F|^{-1} \sum_{u\in C_{\mathcal{G}}(x_s)^F_{nil}} \mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{\mathcal{L}}(x_s)}(u, x_n) f(x_s+u).$$

The group version of 3.2.13(ii) is due to Digne-Michel [DM87].

Remark 3.2.14. The notation $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ is used both for Deligne-Lusztig induction and Harish-Chandra induction; this is justified by 3.2.23. The independence of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ from the choice of ω will be proved (under some assumptions on p and q) in chapter 5 (see 5.5.17).

Open problem 3.2.15. Define Deligne-Lusztig induction using ℓ-adic cohomology but without using a G-equivariant homeomorphism $\mathcal{G}_{nil} \to G_{uni}$.

Remark 3.2.16. It follows easily from the formulas of 3.2.13 that

(i) for any $f \in \mathcal{C}(\mathcal{L}^F)$, we have

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f.\eta^{\mathcal{L}}_o) = \mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f).\eta^{\mathcal{G}}_o,$$

(ii) for any $g \in \mathcal{C}(\mathcal{G}^F)$, we have

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g.\eta^{\mathcal{G}}_o) = {}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g).\eta^{\mathcal{L}}_o.$$

3.2.17 Basic Properties of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$

We now state some properties of Deligne-Lusztig induction in the Lie algebra setting. These properties, which are all know in the group case, are proved in [Let].

As it can be seen from 3.2.4, the function $S^G_{L\subset P}: G^F\times L^F\to\overline{\mathbb{Q}}_\ell$ plays a fundamental role in Deligne-Lusztig's theory. We would like to have such a function in the Lie algebra setting; this is possible thanks to [DM91, Lemma 12.3] which gives an expression of $S^G_{L\subset P}(g,l)$ (where $g\in G^F, l\in L^F$) in terms of the values of some two-variable Green functions. More precisely the function $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}:\mathcal{G}^F\times\mathcal{L}^F\to\overline{\mathbb{Q}}_\ell$ we are looking for is defined as follows.

Definition 3.2.18. *For $x\in\mathcal{G}^F, y\in\mathcal{L}^F$, we define $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)$ by*

$$S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)=\sum_{\{h\in G^F\,|\,Ad(h)y_s=x_s\}}|C^o_L(y_s)^F||C^o_G(y_s)^F|^{-1}\mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{L}}(y_s)}(Ad(h^{-1})x_n,y_n).$$

Remark 3.2.19. Note that $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)=|L^F|\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)$ for any $(x,y)\in\mathcal{G}^F_{nil}\times\mathcal{L}^F_{nil}$.

The following lemma is the Lie algebra version of 3.2.4.

Lemma 3.2.20. *Let $f\in\mathcal{C}(\mathcal{G}^F)$, $g\in\mathcal{C}(\mathcal{L}^F)$, we have*

(1) $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g)(x)=|L^F|^{-1}\sum_{y\in\mathcal{L}^F}S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)g(y)$,

(2) ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y)=|G^F|^{-1}\sum_{x\in\mathcal{G}^F}S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)f(x)$.

Proposition 3.2.21. *The maps $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ and ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ are adjoint with respect to the forms $(,)_{\mathcal{G}^F}$ and $(,)_{\mathcal{L}^F}$.*

Proof: Let $g\in\mathcal{C}(\mathcal{L}^F)$ and $f\in\mathcal{C}(\mathcal{G}^F)$. We have

$$\begin{aligned}\left(f,\mathcal{R}^G_{L\subset P}(g)\right)_{\mathcal{G}^F}&=|G^F|^{-1}\sum_{x\in\mathcal{G}^F}f(x)\overline{\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g)(x)}\\&=|L^F|^{-1}|G^F|^{-1}\sum_{x\in\mathcal{G}^F}\sum_{y\in\mathcal{L}^F}f(x)\overline{S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)g(y)}\text{ by 3.2.20(1)}\\&=|L^F|^{-1}|G^F|^{-1}\sum_{y\in\mathcal{L}^F}\sum_{x\in\mathcal{G}^F}S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)f(x)\overline{g(y)}.\end{aligned}$$

The last equality follows from the fact that $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)\in\mathbb{Q}$. From 3.2.20 (2) we get that

$$\left(f,\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g)\right)_{\mathcal{G}^F}=\left({}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f),g\right)_{\mathcal{L}^F}.$$

□

We now state the transitivity property of Deligne-Lusztig induction. Let $M\subset L$ be an inclusion of F-stable Levi subgroups of G with respective Lie algebras $\mathcal{M}$ and $\mathcal{L}$. Let P and Q be two parabolic subgroups of G, having respectively L and M as Levi subgroups, such that $Q\subset P$.

Proposition 3.2.22. *We have* $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}\circ\mathcal{R}^{\mathcal{L}}_{\mathcal{M}\subset\mathcal{L}\cap\mathcal{Q}}=\mathcal{R}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}$.

We have the following proposition.

Proposition 3.2.23. *If the parabolic subgroup P is F-stable, then the Deligne-Lusztig induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ coincides with Harish-Chandra induction.*

Proposition 3.2.24. *Let L be an F-stable Levi subgroup of G and P be a parabolic subgroup of G having L as a Levi subgroup. Let $\mathcal{L}:=Lie(L)$ and $\mathcal{P}:=Lie(P)$. Let $x\in\mathcal{L}^F$ be such that $C^o_G(x_s)\subseteq L$, then $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x)=\gamma^G_x$.*

Proof: We compute the values of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x)$. Let $y\in\mathcal{G}^F$, then

$$\left(\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x),\gamma^G_y\right)_{\mathcal{G}^F}=\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x)(y).$$

From 3.2.21 we have

$$\left(\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x),\gamma^G_y\right)_{\mathcal{G}^F}=\left(\gamma^L_x,{}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)\right)_{\mathcal{L}^F}.$$

Combining the above two equations we get that

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x)(y)=\overline{{}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x)}. \tag{1}$$

Now, by definition we have

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x)=|C^o_L(x_s)^F||C^o_G(x_s)^F|^{-1}\sum_{n\in C_{\mathcal{G}}(x_s)^F_{nil}}\mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{\mathcal{L}}(x_s)}(n,x_n)\gamma^G_y(x_s+n).$$

Since by assumption $C^o_G(x_s)\subseteq L$, we have $C^o_G(x_s)=C^o_L(x_s)$, and so we get that

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x)=\sum_{n\in C_{\mathcal{G}}(x_s)^F_{nil}}\mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{\mathcal{G}}(x_s)}(n,x_n)\gamma^G_y(x_s+n).$$

This formula shows that if x_s is not G^F-conjugate to y_s, then ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x)=0$. Hence we may assume that $y_s=x_s$, and we have

$$^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = |C_G(y)^F|^{-1} \sum_{n\in\mathcal{O}^{C_G(y_s)^F}_{y_n}} \mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{G}}(y_s)}(n, x_n). \qquad (2)$$

We now compute the quantity $\mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{G}}(y_s)}(n, x_n)$. By definition of Green functions, we have

$$\mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{G}}(y_s)}(n, x_n) = |C^o_G(y_s)^F|^{-1}\text{Trace}\left((\omega(n), \omega(x_n)^{-1})|H^*_c(C^o_G(y_s)^F)\right).$$

From [DM91, Proposition 10.8], we deduce that

$$\begin{aligned}\mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{G}}(y_s)}(n, x_n) &= |C^o_G(y_s)^F|^{-1}\text{Trace}\left((\omega(n), \omega(x_n)^{-1})|\overline{\mathbb{Q}}_\ell[C^o_G(y_s)^F]\right) \\ &= |C^o_G(y_s)^F|^{-1}\sharp\{g \in C^o_G(y_s)^F | \omega(n)g\omega(x_n)^{-1} = g\} \\ &= |C^o_G(y_s)^F|^{-1}\sharp\{g \in C^o_G(y_s)^F | \text{Ad}(g)x_n = n\}.\end{aligned}$$

From the last formula and (2), we deduce that $^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = |C_G(y)^F|$ if x is G^F-conjugate to y and $^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = 0$ otherwise. From (1), it follows that $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x) = \gamma^G_x$. □

3.2.25. We now discuss the validity of the Mackey formula for $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$. In the group case, this has been discussed by many authors including Deligne-Lusztig [DL83], and Bonnafé [Bon98][Bon00]. According to Bonnafé (personal communication), the Mackey formula holds if $q > 3$. Now in [Bon98], it is proved that the Mackey formula (in the group case) is equivalent to a formula on two-variable Green functions. In [Let], we prove a similar result in the Lie algebra setting using the same arguments as in [Bon98]. As a consequence, we get that the Mackey formula holds in the Lie algebra case if and only if it does in the group case. We thus have the following theorem.

Theorem 3.2.26. *Assume that $q > 3$, and let $P = LU_P$ and $Q = MU_Q$ be two Levi decompositions in G such that L and M are F-stable Levi subgroups of G. Then the Mackey formula with respect to (G, L, P, M, Q) holds, that is*

$$^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} \circ \mathcal{R}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}} = \sum_{x\in L^F\backslash\mathcal{S}_G(L,M)^F/M^F} \mathcal{R}^{\mathcal{L}}_{\mathcal{L}\cap^x\mathcal{M}\subset\mathcal{L}\cap^x\mathcal{Q}} \circ {}^*\mathcal{R}^{^x\mathcal{M}}_{\mathcal{L}\cap^x\mathcal{M}\subset\mathcal{P}\cap^x\mathcal{M}} \circ Ad_x$$

where $\mathcal{S}_G(L, M)$ denotes the set of $x \in G$ such that $L\cap^x M$ contains a maximal torus of G.

3.2.27. As in the group case, the Mackey formula as the following consequences (see [Let] for more details):

(1) If $q > 3$, the Deligne-Lusztig induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ does not depend on the choice of the parabolic subgroup P of G having L as a Levi subgroup.

(2) If $q > 3$, we have $\mathcal{D}_{\mathcal{G}} \circ \mathcal{R}^{\mathcal{G}}_{\mathcal{L}} = \epsilon_G\epsilon_L\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}$ where $\epsilon_G = (-1)^{\mathbb{F}_q-rank(G)}$.

Notation 3.2.28. Because of 3.2.27(1), we write $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}$ instead of $\mathcal{R}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}$.

3.2.29. We now state our conjecture about a commutation formula between Fourier transforms and Deligne-Lusztig induction. Let $\mathcal{F}^{\mathcal{G}} : \mathcal{C}(\mathcal{G}^F) \to \mathcal{C}(\mathcal{G}^F)$ be as in 3.1.9.

Conjecture 3.2.30. *For any F-stable Levi subgroup L of G and any function $f \in \mathcal{C}(\mathcal{L}^F)$, we have*

$$\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f) = \epsilon_G \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}(f)$$

where $\epsilon_G = (-1)^{\mathbb{F}_q - rank(G)}$.

Note that if L is G-split, in which case $\epsilon_G \epsilon_L = 1$ and $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}$ is the Harish-Chandra induction, the above commutation formula holds (see 3.1.11(i)). The conjecture 3.2.30 will be discussed in chapter 6.

4

Local Systems and Perverse Sheaves

In this chapter we introduce the results on local systems and perverse sheaves which will be used in the following next chapters.

Throughout this chapter, the letter X denotes an algebraic variety over $\overline{\mathbb{F}}_q$. As in the previous chapter, we denote by ℓ a prime not equal to p and by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$. In the following, the finite extensions of $\mathbb{Q}_\ell$ considered are in $\overline{\mathbb{Q}}_\ell$.

4.0.31. For any finite field extension E of $\mathbb{Q}_\ell$, we have the notion of E-sheaves as in [Del77, p.85] (see also [FK88][KW01]). By a $\overline{\mathbb{Q}}_\ell$-sheaf (or *sheaf*), we shall mean an E-sheaf for some finite extension E of $\mathbb{Q}_\ell$. We denote by $\mathcal{S}h(X)$ the abelian category of $\overline{\mathbb{Q}}_\ell$-sheaves on X. The constant sheaf on X is denoted by $\overline{\mathbb{Q}}_\ell$. If $\mathcal{F}$ is a sheaf on X, the support of $\mathcal{F}$ will be denoted by $Supp\ (\mathcal{F})$.

4.0.32. By a *local system* $\mathcal{E}$ on X we shall mean a locally constant $\overline{\mathbb{Q}}_\ell$-sheaf on X for which each stalk $\mathcal{E}_x$ at $x \in X$ is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space. We denote by $ls(X)$ the full subcategory of $\mathcal{S}h(X)$ consisting of local systems on X. Recall that a pro-finite group is the projective limit of finite groups, each given the discrete topology. Pro-finite groups are compact and one says that a pro-finite group π acts continuously on a set if the stabilizer of any point is an open subgroup of π. An *ℓ-adic* representation of a pro-finite group π on a $\overline{\mathbb{Q}}_\ell$-vector space V is a group homomorphism $f : \pi \to GL(V)$ such that there exists a finite extension E of $\mathbb{Q}_\ell$ and an E-structure V_E on V such that f factors through a continuous homomorphism $\pi \to GL(V_E)$. We denote by $Rep_{\ell-adic}(\pi)$ the category of ℓ-adic representations of π (the morphisms being the obvious ones). For a base point $x \in X$, we denote by $\pi_1(X, x)$ the *fundamental étale group* of X at x. This is a pro-finite group and when X is connected we have an equivalence of categories $ls(X) \to Rep_{\ell-adic}(\pi_1(X, x))$

E. Letellier: LNM 1859, pp. 45–60, 2005.

mapping $\mathcal{E}$ onto the $\pi_1(X,x)$-module $\mathcal{E}_x$. Under this equivalence, irreducible local systems on X correspond to irreducible representations of $\pi_1(X,x)$.

4.0.33. As in [BBD82, 2.2.18], we denote by $\mathcal{D}^b_c(X)$ the bounded "derived category" of $\overline{\mathbb{Q}}_\ell$-(constructible) sheaves. By a *complex* on X we shall mean an object of $\mathcal{D}^b_c(X)$. For $K \in \mathcal{D}^b_c(X)$, the i-th cohomology sheaf of K is denoted by $\mathcal{H}^i K$. If $f : X \to Y$ is a morphism of varieties, we have the usual functors $f_* : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ (direct image), $f_! : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ (direct image with compact support), $f^* : \mathcal{S}h(Y) \to \mathcal{S}h(X)$ (inverse image) and the functors $Rf_* : \mathcal{D}^b_c(X) \to \mathcal{D}^b_c(Y)$, $Rf_! : \mathcal{D}^b_c(X) \to \mathcal{D}^b_c(Y)$ and $Rf^* : \mathcal{D}^b_c(Y) \to \mathcal{D}^b_c(X)$ as in [Gro73, Exposé XVII]; if f is a proper morphism, then we have $f_* = f_!$ and $Rf_* = Rf_!$. The right adjoint of $Rf_!$ is denoted by $f^!$ and is called the exceptional inverse image; if f is an open immersion, then $f^! = Rf^*$. The functors Rf_*, $Rf_!$, Rf^* and $f^!$ commute with the shift operations $[m]$ (if $K \in \mathcal{D}^b_c(X)$, the m-th shift of K is denoted by $K[m]$; for any integer i, we have $\mathcal{H}^i(K[m]) = \mathcal{H}^{i+m}K$). We will use freely the well-known properties of the functors f_*, $f_!$, f^*, Rf_*, $Rf_!$, Rf^* and $f^!$ (such as base change theorems and the adjunction properties). If there is no ambiguity we will denote by f_*, $f_!$ and f^* the functors Rf_*, $Rf_!$ and Rf^*.

If $j : F \hookrightarrow X$ is a closed immersion and if K denotes an object in $\mathcal{D}^b_c(F)$, then the object $j_!(K) \in \mathcal{D}^b_c(X)$ will be called the extension of K by zero on $X - F$.

4.0.34. We denote by $D_X : \mathcal{D}^b_c(X) \to \mathcal{D}^b_c(X)$ the Verdier dual operator; recall that $D_X \circ D_X$ is isomorphic to the identity functor. From [Ara01, 1.6.6], if $f : X \to Y$ is a morphism, we have the following functor isomorphisms $D_Y \circ Rf_! \simeq Rf_* \circ D_X$ and $f^! \circ D_Y \simeq D_X \circ Rf^*$. In particular if f is proper we have $D_Y \circ Rf_! \simeq Rf_! \circ D_X$, if f is an open immersion we have $Rf^* \circ D_Y \simeq D_X \circ Rf^*$ and if f is smooth with connected fibers of same dimension d, we have $Rf^*[2d] \circ D_Y \simeq D_X \circ Rf^*$ since in that case we have $f^! \simeq f^*[2d]$ by [BBD82, 4.2.4].

4.0.35. Recall that a perverse sheaf K over X is an object of $\mathcal{D}^b_c(X)$ which satisfies the two following conditions.

(i) $\dim\big(Supp\,(\mathcal{H}^i K)\big) \leq -i$,
(ii) $\dim\big(Supp\,(\mathcal{H}^i D_X K)\big) \leq -i$ for all $i \in \mathbb{Z}$.

We denote by $\mathcal{M}(X)$ the full subcategory of $\mathcal{D}^b_c(X)$ consisting of perverse sheaves on X. The category $\mathcal{M}(X)$ is abelian ([BBD82, Théorème 1.3.6]) and its objects are all of finite length (see [BBD82, Théorème 4.3.1 (i)]). If ξ is a local system on X, then we will denote by $\xi[d] \in \mathcal{D}^b_c(X)$ the complex $K^\bullet$

concentrated in degree $-d$ such that $K^{-d} = \xi$ and $K^i = 0$ if $i \neq -d$. The functor $ls(X) \to \mathcal{D}_c^b(X)$, $\xi \mapsto \xi[d]$ is fully faithful for any integer d. Note that if X is smooth of pure dimension, then for any $\xi \in ls(X)$, the complex $\xi[\dim X]$ is a perverse sheaf on X.

4.1 Simple Perverse Sheaves, Intersection Cohomology Complexes

Let $Y \subset X$ be a locally closed smooth irreducible subvariety of X. Let $\overline{Y}$ be the Zariski closure of Y in X. Then, for a local system ξ on Y let $\mathrm{IC}(\overline{Y}, \xi) \in \mathcal{D}_c^b(\overline{Y})$ be the corresponding intersection cohomology complex defined by Goresky-MacPherson and Deligne.

4.1.1. *The complex $K = IC(\overline{Y}, \xi)[dim Y]$ is characterized by the following properties.*

(i) $\mathcal{H}^i K = 0$ if $i < -dim Y$,
(ii) $\mathcal{H}^{-dim Y} K|_Y = \xi$,
(iii) dim $\big(Supp\,(\mathcal{H}^i K)\big) < -i$ if $i > -dim Y$,
(iv) dim $\big(Supp\,(\mathcal{H}^i D_X K)\big) < -i$ if $i > -dim Y$.

The complex $\mathrm{IC}(\overline{Y}, \xi)[\dim Y]$ is thus clearly a perverse sheaf on $\overline{Y}$. It follows from 4.1.1 that the restriction of $\mathrm{IC}(\overline{Y}, \xi)[\dim Y]$ to Y is $\xi[\dim Y]$. Moreover, if U is any smooth open subset of $\overline{Y}$ and ζ is any local system on U such that $\xi|_{Y\cap U} \simeq \zeta|_{Y \cap U}$, then $\mathrm{IC}(\overline{U}, \zeta) = \mathrm{IC}(\overline{Y}, \xi)$.

4.1.2. Let $j : Y \hookrightarrow X$ and $\overline{j} : \overline{Y} \hookrightarrow X$ denote the inclusions. There exists a fully faithful functor $j_{!*} : \mathcal{M}(Y) \to \mathcal{M}(X)$ (see [BBD82]) that takes $\xi[\dim Y]$ to $\overline{j}_!(\mathrm{IC}(\overline{Y}, \xi)[\dim Y])$ for each local system ξ on Y; we say that $\overline{j}_!(\mathrm{IC}(\overline{Y}, \xi)[\dim Y])$ is the *perverse extension* of ξ on X. If ζ is an irreducible local system on Y, then $K = \mathrm{IC}(\overline{Y}, \zeta)[\dim Y]$ is a simple object in $\mathcal{M}(\overline{Y})$ and $\overline{j}_! K$ is a simple object in $\mathcal{M}(X)$. Moreover by [BBD82, 4.3.1], all the simple objects in $\mathcal{M}(X)$ are obtained in this way for some pair (Y, ζ) as above.

Proposition 4.1.3. *[BBD82, 4.2.5, 4.2.6] Let $f : X \to Y$ be a smooth morphism with connected fibers of dimension d.*

(a) Assume that X is irreducible (and so Y) and let V be an open smooth subset of Y, we have the following commutative diagram.

$$\begin{array}{ccc} f^{-1}(V) & \hookrightarrow & X \\ \downarrow{\scriptstyle f_V} & & \downarrow{\scriptstyle f} \\ V & \hookrightarrow & Y \end{array}$$

Let $\mathcal{E}$ be an irreducible local system on V, then the local system $f_V^*(\mathcal{E})$ is irreducible and we have

$$IC(\overline{f^{-1}(V)}, f_V^*(\mathcal{E}))[dim X] \simeq f^*[d](IC(\overline{V}, \mathcal{E})[dim Y]).$$

(b) The functor $f^*[d]$ induces a fully faithful functor $\mathcal{M}(Y) \to \mathcal{M}(X)$.

4.1.4. Let $Y \times Z$ be the product of two varieties. We denote by $\boxtimes$ the bifunctor (called the external tensor product) $\mathcal{S}h(Y) \times \mathcal{S}h(Z) \to \mathcal{S}h(Y \times Z)$ that takes (ζ, ξ) to $pr_1^*(\zeta) \otimes pr_2^*(\xi)$ where pr_1, pr_2 are respectively the projections on the first coordinate and on the second coordinate. We also have a bifunctor (also called the external tensor product) $\boxtimes : \mathcal{D}_c^b(Y) \times \mathcal{D}_c^b(Z) \to \mathcal{D}_c^b(Y \times Z)$ that takes (K_1, K_2) to $pr_1^*(K_1) \otimes pr_2^*(K_2)$ where $\otimes$ denotes [1] the left derived functor of the tensor product $\otimes : \mathcal{S}h(Y) \times \mathcal{S}h(Z) \to \mathcal{S}h(Y \times Z)$. The external tensor product (either for sheaves or complexes) commutes with the usual operations: if $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are morphisms, then

$$(f_1)_*K_1 \boxtimes (f_2)_*K_2 \xrightarrow{\sim} (f_1 \times f_2)_*(K_1 \boxtimes K_2),$$

similarly for $(f_1 \times f_2)^*$, $(f_1 \times f_2)_!$ and $(f_1 \times f_2)^!$.

We have the following proposition.

Proposition 4.1.5. *[BBD82, Proposition 4.2.8] Let $Y \times Z$ be the product of two varieties and let K_1 and K_2 be two perverse sheaves respectively on Y and Z. Then the complex $K_1 \boxtimes K_2$ is a perverse sheaf on $Y \times Z$.*

Lemma 4.1.6. *Consider the product $Z \times X$ of a smooth irreducible variety Z with an arbitrary algebraic variety X. Assume that $U \subset X$ is a smooth irreducible locally closed subvariety of X. Let ξ and $\mathcal{E}$ be two local systems*

[1] In the literature, it is usually denoted by $\overset{L}{\otimes}$.

respectively on Z and U and let $\overline{U}$ be the Zariski closure of U in X. Then we have an isomorphism in $\mathcal{D}_c^b(Z \times \overline{U})$ of perverse sheaves

$$IC(Z \times \overline{U}, \xi \boxtimes \mathcal{E})[dim\,(Z \times U)] \simeq \xi[dim\,Z] \boxtimes IC(\overline{U}, \mathcal{E})[dim U].$$

Proof: Let $K_1 = \mathrm{IC}(\overline{U}, \mathcal{E})[\dim U]$ and $K = \xi[\dim Z] \boxtimes K_1$. We need to check that K satisfies the four axioms of 4.1.1 which characterize the complex $\mathrm{IC}(Z \times \overline{U}, \xi \boxtimes \mathcal{E})[\dim\,(Z \times U)]$. We have a canonical morphism

$$\sum_{p+q=n} \mathcal{H}^p(\xi[\dim Z]) \boxtimes \mathcal{H}^q(K_1) \rightarrow \mathcal{H}^n K$$

which is in fact an isomorphism since the complex $\xi[\dim Z]$ is a local system concentrated in degree $-\dim Z$. We thus have, $\mathcal{H}^n K = \xi \boxtimes \mathcal{H}^{n+\dim Z} K_1$. It is then easy to check (i), (ii) and (iii) of 4.1.1. By [BBD82, 4.2.7 (b)] we have $D_{Z \times \overline{U}} K = D_Z(\xi[\dim Z]) \boxtimes D_{\overline{U}} K_1$. Since Z is smooth we have

$$D_Z(\xi[\dim Z]) = \xi^\vee[\dim Z]$$

where for a local system $\mathcal{L}$ on a variety V, $\mathcal{L}^\vee$ denotes the dual local system of $\mathcal{L}$ on V. The axiom (iv) of 4.1.1 follows easily. □

4.2 *H*-Equivariance

Let H denote a **connected** linear algebraic group over $\overline{\mathbb{F}}_q$ acting algebraically on X. We have the notion of H-*equivariant sheaves* on X defined as follows:

Let $\pi : H \times X \rightarrow X$ be the morphism given by the second projection and let $\rho : H \times X \rightarrow X$ given by the action of H on X.

Definition 4.2.1. *We say that $\mathcal{E} \in \mathcal{S}h(X)$ is an H-equivariant sheaf on X if there exists an isomorphism $\pi^*(\mathcal{E}) \xrightarrow{\sim} \rho^*(\mathcal{E})$.*

We have the following lemma.

Lemma 4.2.2. *Let $f : X \rightarrow Y$ be an H-equivariant morphism between two H-varieties. Then the following assertions hold,*

(i) If $\mathcal{E}$ is an H-equivariant sheaf on Y, then $f^*(\mathcal{E})$ is an H-equivariant sheaf on X.

(ii) If ξ is an H-equivariant sheaf on X, then $f_*(\xi)$ and $f_!(\xi)$ are H-equivariant.

Proof: We first prove (i). Let $\mathcal{E}$ be an H-equivariant sheaf on Y. Since f is H-equivariant the following diagram commutes:

$$\begin{array}{ccc} H \times X & \xrightarrow{\rho_X} & X \\ {\scriptstyle Id_H \times f}\downarrow & & \downarrow{\scriptstyle f} \\ H \times Y & \xrightarrow{\rho_Y} & Y \end{array}$$

where ρ_X denotes the action of H on X. We deduce that

$$(\rho_X)^* \circ f^*(\mathcal{E}) \simeq (Id_H \times f)^* \circ (\rho_Y)^*(\mathcal{E}). \tag{1}$$

Since $\mathcal{E}$ is H-equivariant, we have $(\rho_Y)^*\mathcal{E} \simeq (pr_Y)^*(\mathcal{E})$ (where $pr_Y : H \times Y \to Y$ is the projection on the second coordinate). But $(pr_Y)^*(\mathcal{E}) \simeq \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{E}$, thus (1) leads to an isomorphism $(\rho_X)^* \circ f^*(\mathcal{E}) \simeq (Id_H \times f)^*(\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{E})$, and so we have an isomorphism $(\rho_X)^* \circ f^*(\mathcal{E}) \simeq (\overline{\mathbb{Q}}_\ell \boxtimes f^*\mathcal{E}) \simeq (pr_X)^* \circ (f^*\mathcal{E})$ (where $pr_X : H \times X \to X$ is the projection on the second coordinate). The proof of assertion (ii) is also easy and involves the base change theorems. □

We have the following lemma.

Lemma 4.2.3. *Let H act on $H \times X$ by left translation on the first coordinate and on X trivially so that the morphism π is H-equivariant. The functor $\pi^* : \mathcal{S}h(X) \to \mathcal{S}h(H \times X)$ induces an equivalence of categories between $\mathcal{S}h(X)$ and the full subcategory of $\mathcal{S}h(H \times X)$ whose objects are the H-equivariant sheaves. Its inverse functor is given by i^* where $i : X \to H \times X$, $x \mapsto (1, x)$.*

Proof: Let $p, m : H \times H \times X \to H \times X$ be defined by $m(h, h', x) = (hh', x)$ and $p(h, h', x) = (h', x)$. Consider $j : H \times X \to H \times H \times X$ defined by $j(h, x) = (h, 1, x)$. Let $\mathcal{E}$ be an H-equivariant sheaf on $H \times X$ with an isomorphism $\phi : m^*(\mathcal{E}) \simeq p^*(\mathcal{E})$. Since $i \circ \pi = p \circ j$ and $m \circ j = Id_{H \times X}$, the isomorphism $j^*(\phi)$ is an isomorphism $\mathcal{E} \xrightarrow{\sim} \pi^* \circ i^*(\mathcal{E})$. We thus have proved that the functor π^* is essentially surjective. It is also fully faithful because π is smooth with connected fibers; this is proved in [BBD82, 4.2.5] when reducing the proof of 0.1.7(b) to the case of sheaves. We thus deduce that π^* is an equivalence of categories between $\mathcal{S}h(X)$ and the full subcategory of $\mathcal{S}h(H \times X)$ consisting of H-equivariant sheaves. Moreover since $\pi \circ i = Id_X$, we have $i^*\pi^*(\mathcal{L}) = \mathcal{L}$ for any sheaf $\mathcal{L}$ on X and so the inverse functor of π^* is given by the restriction of i^* to the full subcategory of $\mathcal{S}h(H \times X)$ formed by H-equivariant sheaves. □

In the following lemma we denote by $\alpha : H \times H \to H$ the multiplication and by $p_2 : H \times H \to H$ the projection on the second coordinate. As in 4.2.3, we denote by $i : X \to H \times X$ the map $x \mapsto (1, x)$.

Lemma 4.2.4. *Let $\mathcal{E}$ be an H-equivariant sheaf on X. Then there exists a unique isomorphism $\phi_{\mathcal{E}} : \pi^*(\mathcal{E}) \xrightarrow{\sim} \rho^*(\mathcal{E})$ such that*

(i) $i^*(\phi_{\mathcal{E}}) : \mathcal{E} \to \mathcal{E}$ is the identity.

(ii) $(\alpha \times Id_X)^*(\phi_{\mathcal{E}}) = (Id_H \times \rho)^*(\phi_{\mathcal{E}}) \circ (p_2 \times Id_X)^*(\phi_{\mathcal{E}})$.

Proof: Let h be an isomorphism $\pi^*(\mathcal{E}) \xrightarrow{\sim} \rho^*(\mathcal{E})$. Let $\phi_{\mathcal{E}} = h \circ \pi^*(i^*h)^{-1}$, then $\phi_{\mathcal{E}} : \pi^*(\mathcal{E}) \to \rho^*(\mathcal{E})$ is an isomorphism which satisfies $i^*(\phi_{\mathcal{E}}) = Id_{\mathcal{E}}$. Let $f_1, f_2 : \pi^*(\mathcal{E}) \to \rho^*(\mathcal{E})$ be two isomorphisms such that $i^*(f_1) = i^*(f_2) = Id_{\mathcal{E}}$. Then $f_1^{-1} \circ f_2 : \pi^*(\mathcal{E}) \to \pi^*(\mathcal{E})$ is an isomorphism. Since the functor π^* is fully faithful (see proof of 4.2.3) , there exists a morphism $g : \mathcal{E} \to \mathcal{E}$ such that $\pi^*(g) = f_1^{-1} \circ f_2$. Hence $i^* \circ \pi^*(g) = i^*(f_1^{-1}) \circ i^*(f_2)$, so $g = Id_{\mathcal{E}}$, that is $f_1 = f_2$. We thus proved the existence of a unique isomorphism $\phi_{\mathcal{E}} : \pi^*(\mathcal{E}) \simeq \rho^*(\mathcal{E})$ which satisfies 4.2.4 (i). First note that $(\alpha \times \mathrm{Id}_X)^*(\phi_{\mathcal{E}})$ and $(\mathrm{Id}_H \times \rho)^*(\phi_{\mathcal{E}}) \circ (p_2 \times \mathrm{Id}_X)^*(\phi_{\mathcal{E}})$ are morphisms between the same sheaves. Let $j : X \to H \times H \times X$, $x \mapsto (1, 1, x)$, then $j^*\big((\alpha \times \mathrm{Id}_X)^*(\phi_{\mathcal{E}})\big) = j^*\big((\mathrm{Id}_H \times \rho)^*(\phi_{\mathcal{E}}) \circ (p_2 \times \mathrm{Id}_X)^*(\phi_{\mathcal{E}})\big) = Id_{\mathcal{E}}$. Let $\tilde{j} : H \times X \to H \times H \times X$, $(g, x) \mapsto (1, g, x)$ so that $j = \tilde{j} \circ i$ and $(\alpha \times \mathrm{Id}_X) \circ \tilde{j} = Id_{H \times X}$. Let H acts on $H \times X$ by left multiplication on the first coordinate. Then the sheaves $\rho^*(\mathcal{E})$ and $\pi^*(\mathcal{E})$ are H-equivariant on $H \times X$ since ρ is naturally H-equivariant and π is H-equivariant if we let H acts trivially on X, hence by 4.2.3 we get that $\tilde{j}^*\big((\alpha \times \mathrm{Id}_X)^*(\phi_{\mathcal{E}})\big) = \tilde{j}^*\big((\mathrm{Id}_H \times \rho)^*(\phi_{\mathcal{E}}) \circ (p_2 \times \mathrm{Id}_X)^*(\phi_{\mathcal{E}})\big)$. Applying again 4.2.3, we get that $\tilde{j}^*$ is an equivalence of categories from the full subcategory of $\mathcal{S}h(H \times H \times X)$ of H-equivariant sheaves on $H \times H \times X$ (H acting by left multiplication on the first coordinate) onto $\mathcal{S}h(H \times X)$. Hence it remains to see that $(\alpha \times Id_X)^*(\phi_{\mathcal{E}})$ is a morphism between H-equivariant sheaves. But this follows from the fact that the map $\alpha \times Id_X$ is H-equivariant if we let H act by left multiplication on the first coordinate of $H \times H \times X$ and $H \times X$. □

Proposition 4.2.5. *Let $\mathcal{E}$ be an H-equivariant sheaf on X and let $\phi_{\mathcal{E}} : \pi^*(\mathcal{E}) \xrightarrow{\sim} \rho^*(\mathcal{E})$ be the unique isomorphism such that $i^*(\phi_{\mathcal{E}})$ is the identity on $\mathcal{E}$. For $h \in H$, define $i_h : X \to H \times X$, $x \mapsto (h, x)$. Let H_1 be a closed subgroup of H acting trivially on X, then the map $H_1 \to Aut(\mathcal{E})$ given by $h \mapsto i_h^*(\phi_{\mathcal{E}})$ is a group homomorphism and factors through a morphism $H_1/H_1^o \to Aut(\mathcal{E})$.*

Proof: For $h \in H_1$, we have $\rho \circ i_h = \pi \circ i_h = Id_X$ from which we see that the map $H_1 \to \mathrm{Aut}(\mathcal{E})$, $h \mapsto i_h^*(\phi_{\mathcal{E}})$ is well defined. It is a group homomorphism because of 4.2.4(ii). Note that the restriction of ρ to $H_1^o \times X$ is equal to the restriction π_1 of π to $H_1^o \times X$. Hence if $f : H_1^o \times X \hookrightarrow H \times X$ is the inclusion and $i_1 : X \to H_1^o \times X$, $x \mapsto (e, x)$, then $f^*(\phi_{\mathcal{E}})$ is an isomorphism $\pi_1^*(\mathcal{E}) \to \pi_1^*(\mathcal{E})$ such that $i_1^* f^*(\phi_{\mathcal{E}})$ is the identity on $\mathcal{E}$. Since H_1^o is connected, by 4.2.3 the functor $i_1^* : \mathcal{S}h(X) \to \mathcal{S}h_{H_1^o}(H_1^o \times X)$ is an equivalence of categories. Thus it remains to see that the sheaf $\pi_1^*(\mathcal{E})$ is H_1^o-equivariant, but this follows from the fact that the morphism π_1 is H_1^o-equivariant if we let H_1^o act on X trivially. □

Remark 4.2.6. If $\mathcal{E}$ is irreducible, the group $\mathrm{Aut}(\mathcal{E})$ is canonically isomorphic to $\overline{\mathbb{Q}}_\ell^\times$, and so $\mathcal{E}$ defines a character of H_1/H_1^o.

The category $\mathcal{S}h_H(X)$ is defined to be the category whose objects are H-equivariant sheaves on X and whose morphisms are defined as follows.

Let $\mathcal{E}$ and $\mathcal{L}$ be two H-equivariant sheaves on X and let $\phi : \pi^*(\mathcal{E}) \to \rho^*(\mathcal{E})$ and $\psi : \pi^*(\mathcal{L}) \to \rho^*(\mathcal{L})$ be the two isomorphisms such that $i^*(\phi) = i^*(\psi) = Id$. Then a morphism $\mathcal{E} \to \mathcal{L}$ in $\mathcal{S}h_H(X)$ is a morphism of sheaves $\Psi : \mathcal{E} \to \mathcal{L}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \pi^*(\mathcal{E}) & \xrightarrow{\pi^*(\Psi)} & \pi^*(\mathcal{L}) \\ \phi \big\downarrow & & \psi \big\downarrow \\ \rho^*(\mathcal{E}) & \xrightarrow{\rho^*(\Psi)} & \rho^*(\mathcal{L}) \end{array}$$

Proposition 4.2.7. *$\mathcal{S}h_H(X)$ is a full subcategory of $\mathcal{S}h(X)$.*

Proof: Let $\mathcal{E}$ and $\mathcal{L}$ be two H-equivariant sheaves on X and let $\phi : \pi^*(\mathcal{E}) \to \rho^*(\mathcal{E})$ and $\psi : \pi^*(\mathcal{L}) \to \rho^*(\mathcal{L})$ be the two isomorphisms such that $i^*(\phi) = i^*(\psi) = Id$. Let $\Psi : \mathcal{E} \to \mathcal{L}$ be a morphism in $\mathcal{S}h(X)$. We want to show that $\psi \circ \pi^*(\Psi) = m^*(\Psi) \circ \phi$. We have $i^*(\psi \circ \pi^*(\Psi)) = \Psi = i^*(m^*(\Psi) \circ \psi)$. Moreover ρ is clearly H-equivariant and π becomes H-equivariant if H acts on X trivially, hence the sheaves $\pi^*(\mathcal{E})$, $\pi^*(\mathcal{L})$, $\rho^*(\mathcal{E})$ and $\rho^*(\mathcal{L})$ are H-equivariant sheaves on $H \times X$. We deduce from 4.2.3 that $\rho^*(\Psi) \circ \phi = \psi \circ \pi^*(\Psi)$. □

We denote by $ls_H(X)$ the full subcategory of $\mathcal{S}h_H(X)$ consisting of H-equivariant local systems on X.

Definition 4.2.8. *Let $K \in \mathcal{M}(X)$, K is said to be H-equivariant if there is an isomorphism $\phi : \pi^*(K) \xrightarrow{\sim} \rho^*(K)$ in $\mathcal{D}_c^b(X)$.*

Lemma 4.2.9. *Let $f : X \to Y$ be a H-equivariant morphism between two H-varieties. Then the following assertions hold.*

(i) If K is a H-equivariant perverse sheaf on Y, and if $f^*[d]K$ (resp. $f^![d]K$) is a perverse sheaf for some integer d, then $f^*[d]K$ (resp. $f^![d]K$) is also H-equivariant.

(ii) If K is a H-equivariant perverse sheaf on X, and if $f_*[d]K$ (resp. $f_![d]K$) is a perverse sheaf for some integer d, then $f_*[d]K$ (resp. $f_![d]K$) is also an H-equivariant perverse sheaf on Y.

Proof: The proof is entirely similar to that of 4.2.2. □

4.2.10. If K is an H-equivariant perverse sheaf on X, then as for sheaves, see 4.2.4, we can show the existence of a unique isomorphism $\phi_K : \pi^*(K) \xrightarrow{\sim} \rho^*(K)$ such that $i^*(\phi_K)$ is the identity and ϕ_K satisfies 4.2.3(ii). We denote by $\mathcal{M}_H(X)$ the subcategory of $\mathcal{M}(X)$ consisting of H-equivariant perverse sheaves and whose morphisms are defined in the same way as we defined morphisms in $\mathcal{S}h_H(X)$. As we did for sheaves, we can prove that $\mathcal{M}_H(X)$ is in fact a full subcategory of $\mathcal{M}(X)$. The proof of proposition 4.2.5 works also for perverse sheaves.

Remark 4.2.11. The definition 4.2.8 on H-equivariance is not the appropriate one for the case where $K \in \mathcal{D}_c^b(X)$ is not a perverse sheaf or when H is not connected. For the general definition of H-equivariance see [BL94].

Remark 4.2.12. Since the morphisms π and ρ are smooth with connected fibers of same dimension, by 4.0.34, the Verdier dual of an H-equivariant perverse sheaf on X is H-equivariant, hence the restriction of D_X to $\mathcal{M}_H(X)$ is an equivalence of categories $\mathcal{M}_H(X) \to \mathcal{M}_H(X)$.

Proposition 4.2.13. *Let $K \in \mathcal{M}(X)$ be H-equivariant, then any subquotient of K is also H-equivariant.*

Proof: Let K' be a subquotient of K. Then $\rho^*(K')$ is a subquotient of $\rho^*(K)$ and so is a subquotient of $\pi^*(K)$ since K is H-equivariant. By [BBD82, 4.2.6.2], there exists a complex K'' on X such that $\pi^*(K'') \simeq \rho^*(K')$. Applying the functor i^* (where $i : X \to H \times X$, $x \mapsto (1, x)$) to both side, we get that $K'' \simeq K'$. □

The following proposition is the H-equivariant analogue of 4.1.2.

Proposition 4.2.14. *The simple objects of $\mathcal{M}_H(X)$ are the perverse extensions of H-equivariant irreducible local systems on H-stable locally closed smooth irreducible subvarieties of X.*

4.2.15. Let $\mathcal{O}$ be an homogeneous H-variety. We are going to describe the well-known bijection between the isomorphic classes of H-equivariant irreducible local systems on $\mathcal{O}$ and the irreducible characters of $A(x) := A_H(x)$. Since the bijective morphism $f : H/C_H(x) \to \mathcal{O}$ induces an equivalence of categories $f^* : ls(\mathcal{O}) \to ls(H/C_H(x))$, we may assume that $\mathcal{O} = H/C_H(x)$. Let $\pi : H/C_H^o(x) \to H/C_H(x) = \mathcal{O}$ be the projection. Then π is a Galois covering with Galois group $A(x)$, hence the local system $\pi_*(\overline{\mathbb{Q}}_\ell)$ is a semi-simple local system on $H/C_H(x)$ whose endomorphism algebra is isomorphic to the group algebra of $A(x)$. The local system $\pi_*(\overline{\mathbb{Q}}_\ell)$ decomposes as follows.

$$\pi_*(\overline{\mathbb{Q}}_\ell) = \bigoplus_{\chi \in A(x)^\vee} (\mathcal{L}_\chi)^{\chi(1)}$$

where for a group F, we denote by $F^\vee$ the set of irreducible $\overline{\mathbb{Q}}_\ell$-characters of F and where $\mathcal{L}_\chi$ is the irreducible local system on $\mathcal{O}$ associated to χ as in 4.0.32. Since π is H-equivariant, by 4.2.13, the local systems $\mathcal{L}_\chi$ are also H-equivariant. Hence we have defined a map $\chi \mapsto \mathcal{L}_\chi$ from the set of irreducible characters of $A(x)$ onto the set of H-equivariant irreducible local systems on $\mathcal{O}$. Conversely, let $\mathcal{E}$ be an irreducible H-equivariant local system on $\mathcal{O}$. Then the homomorphism $\rho : A(x) \to \mathrm{Aut}(\mathcal{E}_x)$ (see 4.2.5) is an irreducible representation of $A(x)$ such that if χ is the character of ρ, then $\mathcal{E} = \mathcal{L}_\chi$.

4.2.16. Let C be an H-stable locally closed smooth irreducible subvariety of X and let ξ be an irreducible H-equivariant local system on C. Then (C, ξ) is called a *pair* of X. We say that a pair (C, ξ) of X is *orbital* if C is an H-orbit of X. If H acts by Ad on $\mathcal{H} := \mathrm{Lie}(H)$, then an orbital pair (C, ξ) of $\mathcal{H}$ is said to be *nilpotent* if C is a nilpotent orbit. If H acts by conjugation on itself, we say that an orbital pair (C, ξ) of H is *unipotent* if C is a unipotent conjugacy class of H.

If Y is locally closed smooth irreducible subvariety of X and if $\mathcal{E}$ is a local system on Y, then we denote by $K^X(Y, \mathcal{E}) \in \mathcal{M}(X)$ the complex $\mathrm{IC}(\overline{Y}, \mathcal{E})[\dim Y]$ extended by zero on $X - \overline{Y}$. We say that a simple perverse sheaf on X is *orbital* if its is of the form $K^X(C, \xi)$ for some orbital pair (C, ξ) of X. If $X = \mathcal{H}$ and H acts by Ad, or if $X = H$ and H acts by conjugation, then we will write $K(C, \xi)$ instead of $K^X(C, \xi)$ if there is no ambiguity.

4.3 Locally (Iso)trivial Principal *H*-Bundles

Definition 4.3.1. *[BR85, 5.2][Ser58] Let H be an algebraic group and let H act morphically on a variety X on the right. Let $\pi : X \to Y$ be a morphism which is constant on H-orbits. Then π is a* trivial principal H-bundle *if there exists an H-isomorphism $\phi : H \times Y \to X$ (H acts on $H \times Y$ on the right by $(h', x).h = (h'h, x)$) such that $\pi \circ \phi = pr_2$.*

(i) We say that π is a locally trivial principal H-bundle *if for any $x \in Y$, there exists an open neighborhood U of x such that $\pi^{-1}(U)$ is a trivial principal H-bundle over U.*

(ii) We say that π is a locally isotrivial principal H-bundle *if for any $x \in Y$, there exists an open neighborhood U of x and an étale covering $V \to U$ such that the pull back $V \times_Y X$ is a trivial principal H-bundle over V.*

Proposition 4.3.2. *[Ser58] Let H be a linear algebraic group and K be a closed subgroup of H. The projection $\pi : H \to H/K$ is a locally isotrivial principal K-bundle.*

Proposition 4.3.3. *[Ser58] Let H and K be as in 4.3.2 with K reductive and let V be an affine K-variety. We define a right action of K on $H \times V$ by $(h, v).k = (hk, k^{-1}.v)$. We denote by $H \times^K V$ the quotient $(H \times V)/K$ (it exists since K is reductive). Let $\pi : H \times V \to H \times^K V$ be the canonical projection, then π is a locally isotrivial principal K-bundle.*

Proposition 4.3.4. *Let H be a connected algebraic group and let X be an H-variety. Let $f : X \to Y$ be a locally isotrivial principal H-bundle. Then the functor $f^* : \mathcal{S}h(Y) \to \mathcal{S}h_H(X)$ is an equivalence of categories with inverse functor $f_* : \mathcal{S}h_H(X) \to \mathcal{S}h(Y)$. In particular f_* maps H-equivariant irreducible local systems over X onto irreducible local systems over Y.*

Proof: We may assume without loss of generality that $X = H \times Y$ and f is the projection on the second coordinate.

Let $i : Y \to H \times Y$ be the injection given by $y \mapsto (1, y)$. By 4.2.3, $f^* : \mathcal{S}h(Y) \to \mathcal{S}h_H(X)$ is an equivalence of categories whose inverse functor is given by $i^* : \mathcal{S}h_H(X) \to \mathcal{S}h(Y)$. It remains to prove that the functors f_* and i^* are isomorphic. Since f_* is a right adjoint to f^*, it is enough to show that i^* is also a right adjoint to f^*. Since $f \circ i = Id_Y$, the functor i^* defines a map of bifunctors $Hom(f^*(.), .) \to Hom(., i^*(.))$ which is clearly an isomorphism of bifunctors. □

Theorem 4.3.5. *Let H be a connected algebraic group, let X be an H-variety and let $f : X \to Y$ be a locally trivial principal H-bundle. Let $d = dim H$. Then the functor $f^*[d] : \mathcal{M}(Y) \to \mathcal{M}_H(X)$ which sends $K \in \mathcal{M}(Y)$ onto $f^*K[d]$ is an equivalence of categories.*

Proof: Since f is a locally trivial principal H-bundle and H is connected, the morphism f is smooth with connected fibers and so the theorem follows from 4.1.3 and [Lus85a, 1.9.3]. □

We will use the following proposition in chapter 7.

Proposition 4.3.6. *Let G be a connected reductive group over an algebraically closed field k and let $x \in \mathcal{G} := Lie(G)$. We assume that p is very good for G so that $L := C_G(x_s)$ is a connected Levi subgroup of G (see 2.6.13(ii)), 2.6.18), and $Lie\big(C_G(y)\big) = C_{\mathcal{G}}(y)$ for any $y \in \mathcal{G}$ (see 2.6.2). Let $j : \mathcal{O}_x^L \hookrightarrow \mathcal{O}_x^G$ and $\overline{j} : \overline{\mathcal{O}_x^L} \hookrightarrow \overline{\mathcal{O}_x^G}$ be the inclusions. Then for any G-equivariant local system $\mathcal{E}$ on $\mathcal{O}_x^G$, we have $\overline{j}^*\big(IC(\overline{\mathcal{O}_x^G}, \mathcal{E})\big) \simeq IC(\overline{\mathcal{O}_x^L}, j^*(\mathcal{E}))$.*

Proof: Let $y \in \mathcal{G}$ be such that $y_s = x_s$. Since $L = C_G(x_s)$, we have $C_G(y) = C_L(y)$ and we also have $C_{\mathcal{G}}(y) = C_{\mathcal{L}}(y)$. Hence we have $\mathrm{Lie}\big(C_L(y)\big) = \mathrm{Lie}\big(C_G(y)\big) = C_{\mathcal{G}}(y) = C_{\mathcal{L}}(y)$, and so we may identify $\mathcal{O}_y^G$ and $\mathcal{O}_y^L$ respectively with $G/C_G(y)$ and $L/C_L(y)$. By 4.3.2, the morphism $G \to G/L$ is a locally isotrivial principal L-bundle. So let $V \to G/L$ be an étale open set of G/L such that the projection on the second coordinate $G \times_{G/L} V \to V$ is a trivial principal L-bundle (we take V smooth and irreducible), and let $f_V : L \times V \to G \times_{G/L} V$ be an L-isomorphism such that the following diagram commutes.

$$\begin{array}{ccc} L \times V & \xrightarrow{f_V} & G \times_{G/L} V \\ & {\scriptstyle pr_2}\searrow \quad \swarrow{\scriptstyle pr_2} & \\ & V & \end{array}$$

where pr_2 denotes the projection on the second coordinate. The map f_V is thus of the form $(h, v) \mapsto (g_v h, v)$ for some morphism $V \to G$, $v \mapsto g_v$. Since $C_G(y) = C_L(y)$, the map f_V gives rise to an isomorphism $f_{V,y} : L/C_L(y) \times V \xrightarrow{\sim} \big(G/C_G(y)\big) \times_{G/L} V$. Let $v \in V$, we have the following commutative diagram.

$$\begin{array}{ccc} L/C_L(y) \times V & \xrightarrow{f_{V,y}} & G/C_G(y) \times_{G/L} V \\ \uparrow{\scriptstyle i_v} & & \downarrow{\scriptstyle pr_1} \\ L/C_L(y) & \xrightarrow{f_{g_v}} & G/C_G(y) \end{array}$$

where $i_v(X) = (X, v)$ and $f_{g_v}(X) = g_v X$. Now let $x_1, ..., x_r \in \mathcal{L}$ be such that $\overline{\mathcal{O}_x^L} = \coprod_i \mathcal{O}_{x_i}^L$. Then we have $\overline{\mathcal{O}_x^G} = \coprod_i \mathcal{O}_{x_i}^G$ (see 7.1.7). Since $(x_i)_s = x_s$ for any $i \in \{1, ..., r\}$, the above diagram is available if we replace y by any x_i. Hence by identifying $\mathcal{O}_{x_i}^G$ and $\mathcal{O}_{x_i}^L$ respectively with $G/C_G(x_i)$ and $L/C_L(x_i)$ we get the following commutative diagram.

$$\begin{array}{ccc} \overline{\mathcal{O}_x^L}\times V & \xrightarrow{\;F_V\;} & \overline{\mathcal{O}_x^G}\times_{G/L} V \\ \uparrow{\scriptstyle i_v} & & \downarrow{\scriptstyle pr_1} \\ \overline{\mathcal{O}_x^L} & \xrightarrow{\;f_{g_v}\;} & \overline{\mathcal{O}_x^G} \end{array}$$

where $i_v(x)=(x,v)$, pr_1 is the projection on the first coordinate, $f_{g_v}=\mathrm{Ad}(g_v)$ and where $F_V(x,v)=(\mathrm{Ad}(g_v)x,v)$. The morphism $\phi:=pr_1\circ F_V$ is smooth with connected fibers of same dimension, hence from 4.1.3, we have

$$\phi^*\big(\mathrm{IC}\,(\overline{\mathcal{O}_x^G},\mathcal{E})\big)\simeq \mathrm{IC}(\overline{\mathcal{O}_x^L\times V},\psi^*(\mathcal{E})) \tag{1}$$

where $\psi:\mathcal{O}_x^L\times V\to\mathcal{O}_x^G$ is the restriction of ϕ to $\mathcal{O}_x^L\times V$. Now we decompose ϕ as follows.

$$\begin{array}{ccc} \overline{\mathcal{O}_x^L}\times V & \xrightarrow{\;\alpha\;} & \overline{\mathcal{O}_x^L}\times G \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle i} \\ \overline{\mathcal{O}_x^G} & \xleftarrow{\;\rho\;} & \overline{\mathcal{O}_x^G}\times G \end{array}$$

where $\alpha(x,v)=(x,g_v)$, i is the inclusion and ρ is given by the adjoint action of G on $\mathcal{G}$. Hence if we put $K=\mathrm{IC}\,(\overline{\mathcal{O}_x^G},\mathcal{E})$, then using the G-equivariance of K we get that, $\phi^*(K)\simeq\overline{j}^*(K)\boxtimes\overline{\mathbb{Q}}_\ell$. Similarly we see that $\psi^*(\mathcal{E})\simeq j^*(\mathcal{E})\boxtimes\overline{\mathbb{Q}}_\ell$. Hence from 4.1.6 and (1), we deduce that $\overline{j}^*(K)\boxtimes\overline{\mathbb{Q}}_\ell\simeq\mathrm{IC}\,(\overline{\mathcal{O}_x^L},j^*(\mathcal{E}))\boxtimes\overline{\mathbb{Q}}_\ell$. Applying the functor i_v^*, we prove the proposition. □

4.4 F-Equivariant Sheaves and Complexes

Assume now that X is defined over $\mathbb{F}_q$ and denote by F the corresponding Frobenius on X.

Definition 4.4.1. *A complex $K\in\mathcal{D}_c^b(X)$ is said to be F-stable if $F^*(K)$ is isomorphic to K.*

Definition 4.4.2. *An F-equivariant complex on X is a pair (K,ϕ) where $K\in\mathcal{D}_c^b(X)$ and $\phi:F^*(K)\xrightarrow{\sim}K$ is an isomorphism.*

A morphism $f:(K,\phi_K)\to(K',\phi_{K'})$ of F-equivariant complexes is a morphism $f:K\to K'$ in $\mathcal{D}_c^b(X)$ which makes the following diagram commutative,

$$\begin{array}{ccc} F^*(K) & \xrightarrow{F^*(f)} & F^*(K') \\ \Big\downarrow{\scriptstyle \phi_K} & & \Big\downarrow{\scriptstyle \phi_{K'}} \\ K & \xrightarrow{f} & K' \end{array}$$

Similarly, we define the notion of F-stable sheaves, F-equivariant sheaves and morphisms of F-equivariant sheaves on X.

Definition 4.4.3. *If (K, ϕ) is an F-equivariant complex on X, we define the* characteristic function $\mathbf{X}_{K,\phi} : X^F \to \overline{\mathbb{Q}}_l$ *of (K, ϕ) by*

$$\mathbf{X}_{K,\phi}(x) = \sum_i (-1)^i \, Trace(\phi_x^i, \mathcal{H}_x^i K)$$

where ϕ_x^i denotes the automorphism of $\mathcal{H}_x^i K$ induced by ϕ.

The characteristic function $\mathbf{X}_{\mathcal{E},\phi} : X^F \to \overline{\mathbb{Q}}_\ell$ of an F-equivariant sheaf $(\mathcal{E}, \phi)$ on X is defined as follows,

$$\mathbf{X}_{\mathcal{E},\phi}(x) = Trace(\phi_x, \mathcal{E}_x)$$

where $\phi_x : \mathcal{E}_x \to \mathcal{E}_x$ is the isomorphism induced by ϕ.

Remark 4.4.4. If (K, ϕ) and (K', ϕ') are two isomorphic F-equivariant complexes (or sheaves), then their characteristic functions are equal.

Notation 4.4.5. If K is a complex (or a sheaf) on X, then for any integer r, we denote by $K(r)$ the r-th *Tate twist* of K.

Remark 4.4.6. If (K, ϕ) is an F-equivariant complex (or sheaf), then for any integer n, recall that $\mathbf{X}_{K(n),\phi(n)} = q^{-n}\mathbf{X}_{K,\phi}$.

We have the following fact.

Lemma 4.4.7. *Let (K, ϕ), (K', ϕ') be two F-equivariant simple perverse sheaves on X such that $K \simeq K'$ in $\mathcal{M}(X)$. Then there is a unique element $c \in \overline{\mathbb{Q}}_\ell^\times$ such that $\mathbf{X}_{K,\phi} = c\mathbf{X}_{K',\phi}$. Moreover if $c = 1$, then (K, ϕ) and (K', ϕ') are isomorphic.*

Notation 4.4.8. Let (K, ϕ) be an F-equivariant complex on X, then for any integer $n \geq 1$, we denote by $\phi^{(n)} : (F^n)^*K \xrightarrow{\sim} K$ the isomorphism defined by $\phi^{(n)} = F^{n-1}(\phi) \circ ... \circ F(\phi) \circ \phi$.

We have (see [Sho88, 18.8]):

Lemma 4.4.9. *Let (K,ϕ) and (K',ϕ') be two F-equivariant semi-simple perverse sheaves on X such that for any $n\geq 1$, we have $\mathbf{X}_{K,\phi^{(n)}}=\mathbf{X}_{K',\phi'^{(n)}}$. Then K and K' are isomorphic in $\mathcal{M}(X)$.*

4.4.10. Let H be a connected linear algebraic group defined over $\mathbb{F}_q$ acting morphically on X. We assume that this action is defined over $\mathbb{F}_q$ and we still denote by $F: H\rightarrow H$ the Frobenius on H.

Remark 4.4.11. Let K be an H-equivariant F-stable complex (or sheaf) on X and let $\phi: F^*(K)\rightarrow K$ be an isomorphism. Then the function $X_{K,\phi}$ on X^F is H^F-invariant i.e. for any $h\in H^F$ and any $x\in X^F$, we have $X_{K,\phi}(h.x)=X_{K,\phi}(x)$.

We say that a pair $(Z,\mathcal{E})$ of X, see 4.2.16, is F-stable if Z and $\mathcal{E}$ are both F-stable. Two orbital pairs $(\mathcal{O},\mathcal{E})$ and (C,ζ) of X are said to be isomorphic if $\mathcal{O}=C$ and $\mathcal{E}$ is isomorphic to ζ. Let I be a set of representatives of the isomorphic classes of orbital pairs of X and we denote by I^F the subset of I corresponding to F-stable pairs. For each F-stable H-orbit $\mathcal{O}$ of X, we choose an element $x_{\mathcal{O}}\in\mathcal{O}^F$ and we put $A(x_{\mathcal{O}}):=A_H(x_{\mathcal{O}})$. Let $(X/H)^F$ be the set of F-stable H-orbits of X, then I^F is in bijection with the set $\coprod_{\mathcal{O}\in(X/H)^F}H^1(F,A(x_{\mathcal{O}}))$ which by 2.1.20 is in bijection with the H^F-orbits of X^F.

Indeed, let $\mathcal{O}\in(X/H)^F$, then under the bijection of 4.2.15 between isomorphic classes of H-equivariant irreducible local systems and irreducible characters of $A(x_{\mathcal{O}})$, the F-stable local systems corresponds to the F-stable characters. Moreover we have:

4.4.12. *Let H be a finite group and $\theta: H\rightarrow H$ an automorphism of finite order. Then the number of θ-stable irreducible $\overline{\mathbb{Q}}_\ell$-characters of H is equal to the number of elements of $H^1(\theta,H)$.*

Since $A(x_{\mathcal{O}})$ is finite, there is a power q^n for which all the elements of $A(x_{\mathcal{O}})$ are defined over $\mathbb{F}_{q^n}$, i.e. such that $F^n=\mathrm{Id}_{A(x_{\mathcal{O}})}$. Hence we can apply 4.4.12 to $(A(x_{\mathcal{O}}),F)$ and we get that the set of isomorphic classes of F-stable H-equivariant irreducible local systems on $\mathcal{O}$ is in bijection [2] with $H^1(F,A(x_{\mathcal{O}}))$ and so we get a bijection between I^F and $\coprod_{\mathcal{O}\in(X/H)^F}H^1(F,A(x_{\mathcal{O}}))$.

For each $\iota\in I$, put $\iota=(\mathcal{O}_\iota,\mathcal{E}_\iota)$. If $\iota\in I^F$, we choose an F-equivariant local system $(\mathcal{E}_\iota,\phi_\iota)$ and we denote by $\mathcal{Y}_\iota: X^F\rightarrow\overline{\mathbb{Q}}_\ell$ the characteristic

[2] This bijection is not canonical.

function of $(\mathcal{E}_\iota, \phi_\iota)$ extended by zero on $X^F - \mathcal{O}_\iota^F$. The F-equivariant local system $(\mathcal{E}_\iota, \phi_\iota)$ leads to a canonical F-equivariant complex (K_ι, ϕ_ι) where $K_\iota = K^X(\mathcal{O}_\iota, \mathcal{E}_\iota)$ and where ϕ_ι still denotes the isomorphism $F^*(K_\iota) \to K_\iota$ induced by $\phi_\iota : F^*(\mathcal{E}_\iota) \to \mathcal{E}_\iota$. The characteristic function of (K_ι, ϕ_ι) is denoted by $\mathcal{X}_\iota$. We have:

Proposition 4.4.13. *The sets $\{\mathcal{Y}_\iota, \iota \in I^F\}$ and $\{\mathcal{X}_\iota, \iota \in I^F\}$ are both bases of the space $\mathcal{C}(X^F)$ of $\overline{\mathbb{Q}}_\ell$-valued H^F-invariant functions on X^F.*

Proof: We already saw that these two sets have the right number of elements to be bases of $\mathcal{C}(X^F)$. We thus have to verify that both sets consist of linearly independent elements. The case of the functions $\mathcal{X}_\iota$ reduces easily to the case of the functions $\mathcal{Y}_\iota$. Let $(\mathcal{O}, \mathcal{E})$ be a pair of X and let $x \in \mathcal{O}$. By 4.2.15, the local system $\mathcal{E}$ corresponds to an irreducible representation $\rho : A\big(F(x)\big) \to GL(\mathcal{E}_{F(x)})$ and the local system $F^*(\mathcal{E})$ corresponds to an irreducible representation $\rho' : A(x) \to GL(\mathcal{E}_{F(x)})$. Hence $\rho \circ F$ is also a representation of $A(x)$ corresponding to the local system $F^*(\mathcal{E})$, it is thus isomorphic to ρ'. We thus assume that $\rho \circ F = \rho'$. Assume now that $(\mathcal{O}, \mathcal{E}, x)$ is F-stable and let $\phi : F^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ be an isomorphism. Then the representations $\rho \circ F$ and ρ of $A(x)$ are isomorphic, that is there exists a $\overline{\mathbb{Q}}_\ell$-linear isomorphism $\alpha_x : \mathcal{E}_x \xrightarrow{\sim} \mathcal{E}_x$ such that $\alpha_x \circ \rho(t) = \rho(F(t)) \circ \alpha_x$ for all $t \in A(x)$. Now let $h \in H$ be such that $h.x \in \mathcal{O}^F$ and let $t \in A(x)$ be such that $h^{-1}F(h)$ is a representative of t in $C_H(x)$; then $\mathrm{Trace}\,(\alpha_{h.x}) = \mathrm{Trace}\,(\alpha_x \circ \rho(t))$. This defines a function $\alpha : \mathcal{O}^F \to \overline{\mathbb{Q}}_\ell$, $y \mapsto \mathrm{Trace}(\alpha_y)$ which is equal to the function $\mathcal{O}^F \to \overline{\mathbb{Q}}_\ell$, $y \mapsto \mathrm{Trace}\,(\phi_y, \mathcal{E}_y)$ for an appropriate choice of ϕ. Now put $(\mathcal{O}_\iota, \mathcal{E}_\iota, x_\iota, \rho_\iota) = (\mathcal{O}, \mathcal{E}, x, \rho)$ for some $\iota \in I^F$, and let $\gamma_\iota : A(x_\iota) \to \overline{\mathbb{Q}}_\ell$ be defined by $\gamma_\iota(t) = \mathrm{Trace}\,(\alpha_{x_\iota} \circ \rho_\iota(t))$. We may assume that the $x_\iota \in \mathcal{O}_\iota^F$ are chosen such that if $\mathcal{O}_\iota = \mathcal{O}_\mu$ for $\iota, \mu \in I^F$, then $x_\iota = x_\mu$. To prove the independence of the functions $\mathcal{Y}_\iota$, $\iota \in I^F$, we are thus reduced to show that for any $\iota \in I^F$, the functions γ_μ with $\mu \in A_\iota := \{\mu \in I^F | \mathcal{O}_\mu = \mathcal{O}_\iota\}$ are linearly independent. Define $\gamma_\iota^{-1} : A(x_\iota) \to \overline{\mathbb{Q}}_\ell$, $t \mapsto \mathrm{Trace}\big((\alpha_{x_\iota} \circ \rho_\iota(t))^{-1}\big)$. It is sufficient to show that for any $\mu \in A_\iota$, we have $\sum_{t \in A(x_\iota)} \gamma_\mu^{-1}(t)\gamma_\iota(t) = 0$ if and only if $\iota \neq \mu$. The proof of such a fact is similar to that of the orthogonality formula for irreducible characters (see for instance [Ser78]). □

5

Geometrical Induction

In the group case, Deligne-Lusztig induction (see 3.2.1) is defined using the basis formed by characters. By making the use of Lusztig's character sheaves, it is possible to define another "twisted" induction using the basis formed by the characteristic functions of some simple perverse sheaves so-called character sheaves. In [Lus90], it is proved, under some restrictions on p and q, that the two inductions coincide.

Starting from [Lus87] and by adapting Lusztig's ideas to the Lie algebra case, we write down a character sheaves theory for reductive Lie algebras adapted to the study of Fourier transforms. Using the character sheaves on Lie algebras, we define a "twisted" induction for invariant functions we call "geometrical induction". By transferring [Lus90, 1.14] to the Lie algebra case by means of a G-equivariant isomorphism $G_{uni} \rightarrow \mathcal{G}_{nil}$, we show, as in the group case, that Deligne-Lusztig induction coincides with geometrical induction. The coincidence of these two definitions will be used (see next chapter) to study the commutation of Deligne-Lusztig induction with Fourier transforms.

The reader will be able to notice that when establishing the results of 5.1.9, 5.1.14, 5.1.26, 5.1.41 and 5.1.51, analogous to [Lus84], we do not assume, unlike in [Lus84], that the pair $(\Sigma, \mathcal{E})$ is "cuspidal" and that $\mathcal{Z}$ is the whole center. Indeed, the proofs of these results do not require such assumptions and it will be useful here to state these results in that more general context.

Throughout this chapter we make the following assumption, where by a "cuspidal pair" of G, we mean a cuspidal pair $(S, \mathcal{E})$ of G in the sense of [Lus84, 2.4] such that S contains a unipotent conjugacy class of G.

E. Letellier: LNM 1859, pp. 61–113, 2005.

Assumption 5.0.14. *The prime p is acceptable, that is, it satisfies the following conditions:*

(i) p is good for G.

(ii) p does not divide $|(X(T)/Q(\Phi))_{tor}|$.

(iii) There exists a non-degenerate G-invariant bilinear form on $\mathcal{G}$.

(iv) p is very good for any Levi subgroup of G supporting a cuspidal pair.

(v) There exists a G-equivariant isomorphism $G_{uni} \to \mathcal{G}_{nil}$.

Remark 5.0.15. Although 5.0.14 (v) might not be necessary, it will be useful to transfer some results from the group case to the Lie algebra case. The assumption 5.0.14(iv) will be used to apply 2.5.16.

We have the following properties which can be deduced easily from the results of the second chapter and the classification of the cuspidal data of G [Lus84].

Lemma 5.0.16. *(i) If p is acceptable for G, then it is acceptable for any Levi subgroup of G.*

(ii) If p is very good for G, it is acceptable for G.

(iii) All primes are acceptable for $G = GL_n(k)$.

(iv) If G is simple, the very good primes are the acceptable ones for G.

We choose once for all a Lie algebra isomorphism $\mathcal{G} \simeq z(\mathcal{G}) \oplus \overline{\mathcal{G}}$ as in 2.3.1 (note that under our assumption on p, we have $\mathrm{Lie}(Z_G^o) = z(\mathcal{G})$).

5.1 Admissible Complexes and Orbital Perverse Sheaves on $\mathcal{G}$

Following [Lus87] we introduce a kind of Harish-Chandra theory for a subclass of the class of G-equivariant perverse sheaves on $\mathcal{G}$. This will be achieved through the definition of cuspidal G-equivariant perverse sheaves on $\mathcal{G}$ together with a functor $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} : \mathcal{M}_L(\mathcal{L}) \to \mathcal{D}_c^b(\mathcal{G})$ defined for any Levi decomposition $P = LU_P$ in G with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$.

The above subclass will consist of so-called "admissible complexes" (or character-sheaves) on $\mathcal{G}$ and the cuspidal perverse sheaves on $\mathcal{G}$ will be those admissible complexes which can not be obtained as a direct summand of some $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K)$ with L a proper Levi subgroup of G and K an admissible complex of $\mathcal{L}$.

5.1.1 Parabolic Induction of Equivariant Perverse Sheaves

5.1.2. Let P be a parabolic subgroup of G, and LU_P a Levi decomposition of P. Let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition. Recall that $\pi_{\mathcal{P}} : \mathcal{P} \to \mathcal{L}$ denotes the canonical projection. Define

$$V_1 = \{(x, h) \in \mathcal{G} \times G |\, \mathrm{Ad}(h^{-1})x \in \mathcal{P}\},$$

$$V_2 = \{(x, hP) \in \mathcal{G} \times (G/P) |\, \mathrm{Ad}(h^{-1})x \in \mathcal{P}\}.$$

We have the following diagram

$$\mathcal{L} \xleftarrow{\pi} V_1 \xrightarrow{\pi'} V_2 \xrightarrow{\pi''} \mathcal{G}$$

where $\pi''(x, hP) = x$, $\pi'(x, h) = (x, hP)$, $\pi(x, h) = \pi_{\mathcal{P}}(\mathrm{Ad}(h^{-1})x)$.

5.1.3. Let K be an object in $\mathcal{M}_L(\mathcal{L})$. The morphism π is smooth with connected fibers of dimension $m = \dim G + \dim U_P$ and is P-equivariant with respect to the action of P on V_1 and on $\mathcal{L}$ given respectively by $p.(x, h) = (x, hp^{-1})$ and $p.x = \mathrm{Ad}(\pi_P(p))x$. Hence $\pi^* K[m]$ is a P-equivariant perverse sheaf on V_1 (see 4.1.3 (b)). But π' is a locally trivial principal P-bundle (see [Jan87, page 183, (5)]), hence by 4.3.5, there exists a unique perverse sheaf $\tilde{K}$ on V_2 such that

$$\pi^* K[m] = (\pi')^* \tilde{K}[\dim P].$$

Now we define the induced complex $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}(K)$ of K by

$$\mathrm{ind}_{\mathcal{L} \subset \mathcal{P}}^{\mathcal{G}}(K) = (\pi'')_! \tilde{K} \in \mathcal{D}_c^b(\mathcal{G}).$$

This process defines a functor $\mathrm{ind}_{\mathcal{L} \subset \mathcal{P}}^{\mathcal{G}} : \mathcal{M}_L(\mathcal{L}) \to \mathcal{D}_c^b(\mathcal{G})$.

Remark 5.1.4. Assume that P, L and K are all F-stable and let $\phi : F^*(K) \to K$ be an isomorphism. Then ϕ induces a canonical isomorphism $\psi : F^*(\mathrm{ind}_{\mathcal{L} \subset \mathcal{P}}^{\mathcal{G}}(K)) \xrightarrow{\sim} \mathrm{ind}_{\mathcal{L} \subset \mathcal{P}}^{\mathcal{G}}(K)$ such that,

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(X_{K,\phi}) = \mathbf{X}_{\mathrm{ind}_{\mathcal{L} \subset \mathcal{P}}^{\mathcal{G}}(K), \psi} \qquad (*)$$

where $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}$ is the Harish-Chandra induction (see 3.1.2). Indeed, if we denote by F_2 the Frobenius on V_2 defined by $F_2(x, hP) = (F(x), F(h)P)$ and by $\tilde{\phi} : F_2^*(\tilde{K}) \xrightarrow{\sim} \tilde{K}$ the isomorphism induced by ϕ, then (*) follows easily from the formula

$$\mathbf{X}_{\pi''_!(\tilde{K}), \psi}(y) = \sum_{x \in (\pi''^{-1}(y))^{F_2}} \mathbf{X}_{\tilde{K}, \tilde{\phi}}(x)$$

which is a consequence of the Grothendieck trace formula.

Lemma 5.1.5. *We have an isomorphism of functors*

$$ind^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} \circ D_{\mathcal{L}} \simeq D_{\mathcal{G}} \circ ind^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}.$$

Proof: Since the morphisms π and π' are smooth with connected fibers of same dimension and since the morphism π'' is proper, we get the following relations:
(i) $D_{V_1} \circ (\pi^*[m]) \simeq \pi^*[m] \circ D_{\mathcal{L}}$,

(ii) $D_{V_1} \circ ((\pi')^*[\dim P]) \simeq (\pi')^*[\dim P] \circ D_{V_2}$,

(iii) $D_{\mathcal{G}} \circ (\pi'')_! \simeq (\pi'')_! \circ D_{V_2}$.
Hence the lemma follows from 4.2.12. □

Remark 5.1.6. If $K \in \mathcal{M}_L(\mathcal{L})$ is such that $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(K) \in \mathcal{M}(\mathcal{G})$ then $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(K)$ is automatically a G-equivariant perverse sheaf on $\mathcal{G}$; this follows from 4.2.9 since the morphisms π, π' and π'' are all G-equivariant if we let G act on V_1 and V_2 by Ad on the first coordinate and by left translation on the second coordinate, and on $\mathcal{L}$ trivially.

5.1.7. We now state a transitivity property of induction. Let $P = LU_P$ and $Q = MU_Q$ be two Levi decompositions in G, with corresponding Lie algebra decompositions $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ and $\mathcal{Q} = \mathcal{M} \oplus \mathcal{U}_Q$, such that $L \subset M$ and $P \subset Q$. Then we have the following proposition (see [Lus85a, Proposition 4.2]).

Proposition 5.1.8. *Let $K \in \mathcal{M}_L(\mathcal{L})$ and assume that $ind^{\mathcal{M}}_{\mathcal{L}\subset\mathcal{P}\cap\mathcal{M}}(K)$ is a perverse sheaf. Then $ind^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}\big(ind^{\mathcal{M}}_{\mathcal{L}\subset\mathcal{M}\cap\mathcal{P}}(K)\big) = ind^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(K)$.*

Proof: The proof is entirely similar to that of [Lus85a, Proposition 4.2]. □

5.1.9 The Complexes $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} K(\Sigma, \mathcal{E})$

Let $(P, L, \Sigma, \mathcal{E})$ be a tuple where P is a parabolic subgroup of G, L is a Levi subgroup of P and where $(\Sigma, \mathcal{E})$ is a pair of $\mathcal{L} = \mathrm{Lie}(L)$ (see 4.2.16) such that $\Sigma = \mathcal{Z} + C$ with C a nilpotent orbit of $\mathcal{L}$ and $\mathcal{Z}$ is a closed irreducible smooth subvariety of $z(\mathcal{L})$. Let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the Lie algebra decomposition corresponding to the decomposition $P = LU_P$.

The admissible complexes will be defined as the simple direct summand of the complexes of the form $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} K(\Sigma, \mathcal{E})$ where $(\Sigma, \mathcal{E})$ is a "cuspidal" pair of

$\mathcal{L}$ (in which case $\mathcal{Z} = z(\mathcal{L})$). However, we will also need to use the complexes of the form $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})$ where $(\Sigma, \mathcal{E})$ is a (non-necessarily "cuspidal") orbital pair of $\mathcal{L}$, i.e. $\mathcal{Z} = \{\sigma\}$ with $\sigma \in z(\mathcal{L})$.

We keep the notation of 5.1.1.

Define

$$X_1 := \{(x, g) \in \mathcal{G} \times G \mid \mathrm{Ad}(g^{-1})x \in \overline{\Sigma} + \mathcal{U}_P\},$$

$$X_2 := \{(x, gP) \in \mathcal{G} \times (G/P) \mid \mathrm{Ad}(g^{-1})x \in \overline{\Sigma} + \mathcal{U}_P\}.$$

Remark 5.1.10. Note that the definition of X_2 makes sense since by 2.6.6, the group P normalizes $\overline{\Sigma} + \mathcal{U}_P$. Note that X_1 is closed in V_1 since it is the inverse image of $\overline{\Sigma} + \mathcal{U}_P$ by the morphism $\mathcal{G} \times G \to \mathcal{G}, (x, g) \mapsto \mathrm{Ad}(g^{-1})x$. Therefore X_2 is closed in V_2; indeed the morphism π' of 5.1.2 is open and $\pi'^{-1}(X_2) = X_1$. Moreover, the morphism $X_1 \to (\overline{\Sigma} + \mathcal{U}_P) \times G$, $(x, g) \mapsto (\mathrm{Ad}(g^{-1})x, g)$ being an isomorphism, X_1 and X_2 are both irreducible.

We have the following commutative diagram.

5.1.11.

$$\begin{array}{ccccccc}
\overline{\Sigma} & \xleftarrow{\rho} & X_1 & \xrightarrow{\rho'} & X_2 & \xrightarrow{\rho''} & \mathcal{G} \\
{\scriptstyle i}\downarrow & & {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow & & \|\downarrow \\
\mathcal{L} & \xleftarrow{\pi} & V_1 & \xrightarrow{\pi'} & V_2 & \xrightarrow{\pi''} & \mathcal{G}
\end{array}$$

where i, i_1, i_2 are the natural inclusions and ρ, ρ' and ρ'' are given by the respective restrictions of π, π' and π''.

Remark 5.1.12. Note that ρ and ρ' being obtained respectively from π and π' by base change, ρ is smooth with connected fibers of dimension $m = \dim G + \dim U_P$ and ρ' is a locally trivial principal P-bundle.

The variety Σ is open in its Zariski closure $\overline{\Sigma}$, hence $X_{1,o} = \rho^{-1}(\Sigma)$ is an open subset of X_1. Using the fact that ρ' is a quotient map we deduce that $X_{2,o} = \rho'(X_{1,o})$ is open in X_2. We have,

$$X_{1,o} = \{(x, g) \in \mathcal{G} \times G \mid \mathrm{Ad}(g^{-1})x \in \Sigma + \mathcal{U}_P\},$$

$$X_{2,o} = \{(x, gP) \in \mathcal{G} \times (G/P) \mid \mathrm{Ad}(g^{-1})x \in \Sigma + \mathcal{U}_P\}.$$

Remark 5.1.13. Since ρ is smooth (see 5.1.12), as well as Σ, we get that $X_{1,o}$ is also smooth. Hence, from the fact that the restriction $\rho'_o : X_{1,o} \to X_{2,o}$ of ρ' is a locally trivial principal P-bundle (see 5.1.12), we deduce that $X_{2,o}$ is also smooth. Note also that $X_{2,o}$ is irreducible since X_2 is irreducible.

Now let P act on X_1 by right translation on the second coordinate and on Σ by $p.x = \mathrm{Ad}(\pi_P(p))x$ with $p \in P$, $x \in \Sigma$. These actions make ρ into a P-equivariant morphism from which we deduce that $X_{1,o}$ is P-stable and since the above action of P on Σ factors through L we also deduce that the local system $\mathcal{E}$ is P-equivariant. As a consequence we get that $\rho_o^*(\mathcal{E})$ is a P-equivariant irreducible local system on $X_{1,o}$. By 5.1.12, the morphism ρ'_o is a locally trivial principal P-bundle, therefore, by 4.3.4, there exists a unique irreducible local system $\mathcal{E}_2$ on $X_{2,o}$ such that $\rho_o^*(\mathcal{E}) = (\rho'_o)^*(\mathcal{E}_2)$. We consider the complex $K_2 = \mathrm{IC}(\overline{X_{2,o}}, \mathcal{E}_2)[\dim X_{2,o}]$ on X_2. We have

$$\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E}) = (\rho'')_! K_2. \tag{*}$$

Indeed, if $K = K(\Sigma, \mathcal{E})$, we have $K = i_!(\mathrm{IC}(\overline{\Sigma}, \mathcal{E})[\dim \Sigma])$ and so by using the fact [1] that

$$(\rho')^*[\dim P]K_2 = \rho^*[m](\mathrm{IC}(\overline{\Sigma}, \mathcal{E})[\dim \Sigma])$$

and by applying the proper base change theorem successively to the left square and the middle square (which are cartesian) of the diagram 5.1.11, we see that $(i_2)_! K_2 = \tilde{K}$ (see 5.1.1 for the definition of $\tilde{K}$) and so we have $(\pi'')_! \circ (i_2)_! K_2 = (\pi'')_! \tilde{K}$. Since $\rho'' = \pi'' \circ i_2$, we see that $(\rho'')_! K_2 = \mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})$.

5.1.14 The Complexes $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})$ Are G-Equivariant Perverse Sheaves

We first establish some intermediate results.

Let (P, L, Σ) be as in 5.1.9. We denote by $z(\mathcal{L})_{reg}$ the set of L-regular elements in $\mathcal{G}$; by 2.6.13 (i), this set is a non-empty open subset of $z(\mathcal{L})$. We define

$$Z = \{(x, gP, hP) \in \mathcal{G} \times (G/P) \times (G/P) |\ x \in \mathrm{Ad}(g)(\Sigma + \mathcal{U}_P) \cap \mathrm{Ad}(h)(\Sigma + \mathcal{U}_P)\}.$$

We consider the action of G on $(G/P) \times (G/P)$ by left multiplication on both coordinates, then we have a partition $Z = \bigcup_{\mathcal{O}} Z_{\mathcal{O}}$ according to the G-orbits $\mathcal{O}$ on $(G/P) \times (G/P)$. A G-orbit $\mathcal{O}$ is said to be *good* if for $(gP, hP) \in (G/P) \times (G/P)$, there is a common Levi subgroup of gPg^{-1} and hPh^{-1}; otherwise $\mathcal{O}$ is said to be *bad*. Let $d = \dim G - \dim L + \dim \Sigma$.

[1] Both $(\rho')^*[\dim P]K_2$ and $\rho^*[m](\mathrm{IC}(\overline{\Sigma}, \mathcal{E})[\dim \Sigma])$ are canonically isomorphic to $\mathrm{IC}(\overline{X_{1,o}}, \rho_o^*\mathcal{E})[\dim X_{1,o}] = \mathrm{IC}(\overline{X_{1,o}}, (\rho'_o)^*\mathcal{E}_2)[\dim X_{1,o}]$ in view of 4.1.3 (a) which by 5.1.12 and 5.1.13 can be applied to (ρ, Σ) and $(\rho', X_{2,o})$.

We have the following proposition which is slightly more general than the Lie algebra version of [Lus84, Proposition 1.2 (a), (c)] since in our case, $\mathcal{Z}$ is not necessarily the whole center.

Proposition 5.1.15. *With the above notation we have:*

(1) For any nilpotent G-orbit O of $\mathcal{G}$ and $u \in \mathcal{L}_{nil}$ we have

$$dim\,(O \cap \pi_{\mathcal{P}}^{-1}(u)) \leq \frac{1}{2}(dim\,O - dim\,\mathcal{O}_u^L).$$

(2) For any G-orbit $\mathcal{O}$ we have

$$dim\,Z_{\mathcal{O}} \leq d. \qquad (*)$$

If $\mathcal{Z} \cap z(\mathcal{L})_{reg} \neq \emptyset$ then the inequality (*) is strict for bad $\mathcal{O}$. In any case we have $dim\,Z \leq d$.

Proof: We get (1) as a consequence of its group version (see [Lus84, Proposition 1.2 (a)]) via a G-equivariant isomorphism $G_{uni} \to \mathcal{G}_{nil}$. We now prove (2) by adapting the proof of [Lus84, Proposition 1.2 (c)] to the Lie algebra case.

Let T be a maximal torus contained in P. By the Bruhat decomposition of G, any G-orbit $\mathcal{O}$ is the G-orbit of $(P, \dot{w}P)$ for some $\dot{w} \in N_G(T)$. Let $w \in W_G(T)$ and let $\mathcal{O}_w$ be the G-orbit of $(P, \dot{w}P)$ in $(G/P) \times (G/P)$ where $\dot{w} \in N_G(T)$ denotes a representative of w. The fibers of the morphism $Z_{\mathcal{O}_w} \to \mathcal{O}_w$ given by $(X, gP, g\dot{w}P) \mapsto (gP, g\dot{w}P)$ are all isomorphic to $(\Sigma + \mathcal{U}_P) \cap \mathrm{Ad}(\dot{w})(\Sigma + \mathcal{U}_P)$; the map $Z_{\mathcal{O}_w} \to \mathcal{O}_w$ is in fact a locally trivial fibration. It follows that

$$\dim\big((\Sigma + \mathcal{U}_P) \cap \mathrm{Ad}(\dot{w})(\Sigma + \mathcal{U}_P)\big) = \dim Z_{\mathcal{O}_w} - \dim \mathcal{O}_w.$$

Hence to prove the proposition, it is enough to prove that

$$\dim\big((\Sigma + \mathcal{U}_P) \cap \mathrm{Ad}(\dot{w})(\Sigma + \mathcal{U}_P)\big) \leq \dim G - \dim L + \dim \Sigma - \dim \mathcal{O}_w \quad \text{(a)}$$

with strict inequality if $\mathcal{Z} \cap z(\mathcal{L})_{reg} \neq \emptyset$ and $\mathcal{O}_w$ is bad.

An element of $(\Sigma + \mathcal{U}_P) \cap \mathrm{Ad}(\dot{w})(\Sigma + \mathcal{U}_P)$ can be written both in the form $x + u$ with $x \in \Sigma$, $u \in \mathcal{U}_P$ and in the form $y + v$ with $y \in \mathrm{Ad}(\dot{w})\Sigma$, $v \in \mathrm{Ad}(\dot{w})\mathcal{U}_P$. By decomposing $x + u = y + v \in \mathcal{P} \cap \mathrm{Ad}(\dot{w})\mathcal{P}$ with respect to the formula of 2.1.15 with $\mathcal{Q} := \mathrm{Ad}(\dot{w})\mathcal{P}$ and $\mathcal{M} := \mathrm{Ad}(\dot{w})\mathcal{L}$, we have $x = z + u'$ for some unique $z \in \mathcal{L} \cap \mathrm{Ad}(\dot{w})\mathcal{L}$, $u' \in \mathcal{L} \cap \mathrm{Ad}(\dot{w})\mathcal{U}_P$ and $y = z + v'$ for some unique $v' \in \mathrm{Ad}(\dot{w})\mathcal{L} \cap \mathcal{U}_P$.

Note that we have $u + u' = v + v'$. Let $L^w = \dot{w}L\dot{w}^{-1}$, $P^w = \dot{w}P\dot{w}^{-1}$, $\mathcal{L}^w = \mathrm{Ad}(\dot{w})\mathcal{L}$, $\mathcal{P}^w = \mathrm{Ad}(\dot{w})\mathcal{P}$, $\Sigma^w = \mathrm{Ad}(\dot{w})\Sigma$, $\mathcal{Z}^w = \mathrm{Ad}(\dot{w})\mathcal{Z}$, $C^w = \mathrm{Ad}(\dot{w})C$. Let

X_w be the subvariety of $\mathcal{U}_P \times \mathcal{U}_{P^w} \times (\mathcal{U}_{P^w} \cap \mathcal{L}) \times (\mathcal{U}_P \cap \mathcal{L}^w) \times (\mathcal{L} \cap \mathcal{L}^w)$ consisting of (u, v, u', v', z) such that $u + u' = v + v'$, $z + u' \in \Sigma$ and $z + v' \in \Sigma^w$. The inequality (a) is then equivalent to

$$\dim X_w \leq \dim G - \dim L + \dim \Sigma - \dim \mathcal{O}_w \tag{b}$$

with strict inequality if $\mathcal{Z} \cap z(\mathcal{L})_{reg} \neq \emptyset$ and $\mathcal{O}_w$ is bad. Let

$$Y_w := \{(u', v', z) \in (\mathcal{U}_{P^w} \cap \mathcal{L}) \times (\mathcal{U}_P \cap \mathcal{L}^w) \times (\mathcal{L} \cap \mathcal{L}^w) | z + u' \in \Sigma, z + v' \in \Sigma^w\}$$

We have an isomorphism $X_w \to Y_w \times \{(x, y) \in \mathcal{U}_P \times \mathcal{U}_{P^w} |\ x = y\}$ given by $(u, v, u', v', z) \mapsto ((u', v', z), (u-v', v-u'))$, thus $\dim X_w = \dim Y_w + \dim (\mathcal{U}_P \cap \mathcal{U}_{P^w})$. From the fact that $\mathcal{O}_w$ and $G/(P \cap P^w)$ have the same dimension, we see that $\dim (\mathcal{U}_P \cap \mathcal{U}_{P^w}) = \dim G - \dim L - \dim \mathcal{O}_w$. We deduce that (b) is equivalent to

$$\dim Y_w \leq \dim \Sigma$$

with strict inequality if $\mathcal{Z} \cap z(\mathcal{L})_{reg} \neq \emptyset$ and $\mathcal{O}_w$ is bad.

Let $(u', z) \in (\mathcal{U}_{P^w} \cap \mathcal{L}) \times (\mathcal{L} \cap \mathcal{L}^w)$ such that $z + u' \in \Sigma$. The product $(L^w \cap L).(U_{P^w} \cap L)$ being a Levi decomposition of the parabolic subgroup $P^w \cap L$ of L (with corresponding Lie algebra decomposition $\mathcal{P}^w \cap \mathcal{L} = (\mathcal{L} \cap \mathcal{L}^w) \oplus (\mathcal{U}_{P^w} \cap \mathcal{L})$), by 2.7.1(ii) we get that $(z + u')_s$ is $(U_{P^w} \cap L)$-conjugate to z_s. But since $z + u' \in \Sigma = \mathcal{Z} + C$, we have $(z + u')_s \in \mathcal{Z}$. We deduce that $z_s \in \mathcal{Z}$. Similarly, if $v' \in (\mathcal{U}_P \cap \mathcal{L}^w)$ is such that $z + v' \in \Sigma^w$, then $z_s \in \mathcal{Z}^w$. By the finiteness of the number of nilpotent orbits in $\mathcal{L} \cap \mathcal{L}^w = \mathrm{Lie}(L \cap L^w)$, we see that the image of the projection $pr_3 : Y_w \to \mathcal{L} \cap \mathcal{L}^w$ on the third coordinate is thus contained in $((\mathcal{Z} \cap \mathcal{Z}^w) + C_1) \cup ... \cup ((\mathcal{Z} \cap \mathcal{Z}^w) + C_n)$ for a finite set of nilpotent $(L \cap L^w)$-orbits C_i of $\mathcal{L} \cap \mathcal{L}^w$ such that for $i \in \{1, ..., n\}$, the image of pr_3 intersects $(\mathcal{Z} \cap \mathcal{Z}^w) + C_i$. Now note that $L \cap L^w$ acts on Y_w by the adjoint action on the three coordinates and so pr_3 is naturally $(L \cap L^w)$-equivariant, hence its image must be $(L \cap L^w)$-invariant. As a consequence, we see that the image of pr_3 is $((\mathcal{Z} \cap \mathcal{Z}^w) + C_1) \cup ... \cup ((\mathcal{Z} \cap \mathcal{Z}^w) + C_n)$. Now if $z \in (\mathcal{Z} \cap \mathcal{Z}^w) + C_i$ for some $i \in \{1, ..., n\}$, then $pr_3^{-1}(z)$ is isomorphic to $\{u' \in \mathcal{U}_{P^w} \cap \mathcal{L} |\ z_n + u' \in C\} \times \{v' \in \mathcal{U}_P \cap \mathcal{L}^w |\ z_n + v' \in C^w\}$ which is isomorphic to $(\pi^{-1}_{\mathcal{L} \cap \mathcal{P}^w}(z_n) \cap C) \times (\pi^{-1}_{\mathcal{L}^w \cap \mathcal{P}}(z_n) \cap C^w)$.

We deduce from 5.1.15(1) that $\dim \left(pr_3^{-1}(z)\right) \leq \frac{1}{2}(\dim C - \dim C_i) + \frac{1}{2}(\dim C - \dim C_i)$ and so that

$$\dim \left(pr_3^{-1}((\mathcal{Z} \cap \mathcal{Z}^w) + C_i)\right) \leq \dim (\mathcal{Z} \cap \mathcal{Z}^w) + \dim C. \tag{*}$$

Now $Y_w = \coprod_{i \in \{1,...,n\}} pr_3^{-1}((\mathcal{Z} \cap \mathcal{Z}^w) + \overline{C_i})$. Since the above union of closed sets is finite, we deduce from (*) that $\dim Y_w \leq \dim (\mathcal{Z} \cap \mathcal{Z}^w) + \dim C$. We deduce that

$$\dim Y_w \leq \dim \mathcal{Z} + \dim C$$

with strict inequality if $\mathcal{Z} \cap z(\mathcal{L})_{reg} \neq \emptyset$ and $\mathcal{O}_w$ is bad since in that case we always have $\mathcal{Z} \cap \mathcal{Z}^w \subsetneq \mathcal{Z}$. □

We are now going to see a first consequence of 5.1.15 (see proposition 5.1.18 below). We use the notation of 5.1.9 relatively to $(P, L, \Sigma, \mathcal{E})$.

Remark 5.1.16. Note that the morphism ρ'' is proper. Indeed the projection $pr_2 : \mathcal{G} \times (G/P) \to G/P$ on the second coordinate is proper (since the variety G/P is complete) and as we did for $X_2 \subset V_2$ (see 5.1.10), we can show that V_2 is closed in $\mathcal{G} \times (G/P)$. We deduce that X_2 is closed in $\mathcal{G} \times (G/P)$ and so that the restriction of pr_2 to X_2 (which is ρ'') is proper.

Notation 5.1.17. Let $X_3 = \rho''(X_2)$. By the above remark, X_3 is closed in $\mathcal{G}$.

Proposition 5.1.18. *The varieties X_2 and X_3 are both irreducible of dimension $d = dim\, G - dim\, L + dim\, \Sigma$.*

Proof: We already saw that X_2 (and thus X_3) is irreducible, see 5.1.10. The fibers of the map $X_2 \to G/P$ are all isomorphic to $\overline{\Sigma} + \mathcal{U}_P$. Hence, $\dim X_2 = \dim (G/P) + \dim \Sigma + \dim \mathcal{U}_P$, that is $\dim X_2 = d$.

From $\dim X_2 = d$ we deduce that $\dim X_3 \leq d$. It remains to see that $\dim X_3 \geq d$; this is in fact a corollary of 5.1.15. Indeed since $\rho'' : X_2 \to X_3$ is a morphism between irreducible varieties, there exists an open subset U of X_3 such that for any $x \in U$, $\dim\ \rho''^{-1}(x) = \dim X_2 - \dim X_3 = d - \dim X_3$. To apply 5.1.15, let us introduce $f : \rho''^{-1}(U) \times_U \rho''^{-1}(U) \to \rho''^{-1}(U)$ the projection onto the second coordinate. For $x \in \rho''^{-1}(U)$, the fiber of f at x is isomorphic to $\rho''^{-1}(\rho''(x))$, hence is of dimension $d - \dim X_3$. We deduce that $\dim (\rho''^{-1}(U) \times_U \rho''^{-1}(U)) - \dim (\rho''^{-1}(U)) = d - \dim X_3$, i.e. $\dim (\rho''^{-1}(U) \times_U \rho''^{-1}(U)) = 2d - \dim X_3$. On the other hand, $\rho''^{-1}(U) \times_U \rho''^{-1}(U)$ is open dense in $X_2 \times_{X_3} X_2$, thus its dimension is equal to $\dim (X_2 \times_{X_3} X_2) = \dim (X_{2,o} \times_{X_3} X_{2,o})$. But $X_{2,o} \times_{X_3} X_{2,o} = Z$ where Z is as in 5.1.15, therefore by 5.1.15, we deduce that $2d - \dim X_3 \leq d$, i.e. $d \leq \dim X_3$. □

We have the following proposition.

Proposition 5.1.19. *The complex $ind_{\mathcal{L} \subset \mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})$ is a G-equivariant perverse sheaf.*

Proof: We denote by K^G the complex $\mathrm{ind}_{\mathcal{L} \subset \mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E}) = \rho''_! K_2$. To show that K^G is a perverse sheaf, we have to show that for any $i \in \mathbb{Z}$,

(i) dim $\left(Supp\,(\mathcal{H}^i K^G)\right) \leq -i$ and
(ii) dim $\left(Supp\,(\mathcal{H}^i D_{\mathcal{G}} K^G)\right) \leq -i$.
Recall that $D_{\mathcal{G}}$ denotes the Verdier dual operator on $\mathcal{D}_c^b(\mathcal{G})$.

Since ρ'' is proper, we have $D_{\mathcal{G}} \circ \rho''_! = \rho''_! \circ D_{X_2}$, hence $Supp\,(\mathcal{H}^i D_{\mathcal{G}} K^G) = Supp\,(\mathcal{H}^i \rho''_! D_{X_2} K_2)$. By [Ara01, Corollaire 3.1.4 (d)], we have $D_{X_2} K_2 = \mathrm{IC}(\overline{X_{2,o}}, \mathcal{E}_2^\vee)[\dim X_2]$ where $\mathcal{E}_2^\vee$ denotes the dual local system of $\mathcal{E}_2$ on X_2. Hence, since the proof of (i) applies with $\mathcal{E}_2$ replaced by any local system on $X_{2,o}$, we only prove (i). The proof is inspired from that of [Lus84, Proposition 4.5].

Since ρ'' is proper, we get that for any $x \in X_3$, the stalk $\mathcal{H}_x^i(\rho''_! K_2)$ at x is the hypercohomology with compact support $\mathbb{H}_c^i(\rho''^{-1}(x), K_2|_{\rho''^{-1}(x)})$ of $\rho''^{-1}(x)$ with coefficient in $K_2|_{\rho''^{-1}(x)}$ (since $\rho''^{-1}(x)$ is closed in X_2 and so is a complete variety, this is in fact the hypercohomology).

Following the proof of [Lus84, Proposition 4.5], we first exhibit for $x \in X_3$ a stratification of $\rho''^{-1}(x)$, i.e. a partition of $\rho''^{-1}(x)$ into locally closed nonempty subsets. Let $\{C_\alpha | \alpha \in A\}$ be the nilpotent orbits of $\mathcal{L}$ contained in $\overline{C}$; this provides a stratification $\overline{\Sigma} = \coprod_{\alpha \in A} \Sigma_\alpha$ with $\Sigma_\alpha = \mathcal{Z} + C_\alpha$. By taking the inverse images of these strata under the map ρ, we get a stratification $X_1 = \coprod_{\alpha \in A} X_{1,\alpha}$. Since the $X_{1,\alpha}$ are P-invariant for the P-action on X_1 given by right translation on the second coordinate, their images $X_{2,\alpha} = \rho'(X_{1,\alpha})$ provides a stratification for X_2. Note that $X_{2,\alpha} = \{(X, gP) \in \mathcal{G} \times (G/P) |\ \mathrm{Ad}(g^{-1})X \in \Sigma_\alpha + \mathcal{U}_P\}$. For $x \in X_3$, we stratify $\rho''^{-1}(x)$ by $\rho''^{-1}(x)_\alpha = \rho''^{-1}(x) \cap X_{2,\alpha}$.

Now if for $x \in X_3$, $\mathbb{H}^i(\rho''^{-1}(x), K_2|_{\rho''^{-1}(x)}) \neq 0$, then there exists a stratum $\rho''^{-1}(x)_\alpha$ such that $\mathbb{H}_c^i(\rho''^{-1}(x)_\alpha, K_2|_{\rho''^{-1}(x)_\alpha}) \neq 0$. Therefore to show (i), it is enough to show that for any i, and any $\alpha \in A$,

5.1.20. $$\dim \{x \in X_3 |\ \mathbb{H}_c^i(\rho''^{-1}(x)_\alpha, K_2|_{\rho''^{-1}(x)_\alpha}) \neq 0\} \leq -i.$$

Following [Lus84, p.221] (see the case $\alpha \neq \alpha_o$) we are reduced to prove the following assertion.

For any $\alpha \in A$ and any $i \in \mathbb{Z}$,

5.1.21. $$\dim \{x \in X_3 |\ \dim \rho''^{-1}(x)_\alpha \geq \tfrac{i}{2} - \tfrac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha)\} \leq \dim X_3 - i.$$

We denote by $X_3^{i,\alpha}$ the set $\{x \in X_3 |\ \dim \rho''^{-1}(x)_\alpha \geq \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha)\}$. To prove 5.1.21, it is enough to prove the following inequality [2] for any $i \in \mathbb{Z}$ and $\alpha \in A$,

5.1.22. $$\dim (X_{2,\alpha} \times_{X_3} X_{2,\alpha}) \geq \dim X_3^{i,\alpha} + i - \dim \Sigma + \dim \Sigma_\alpha.$$

[2] This inequality is used implicitly in [Lus84] without proof.

Indeed, if 5.1.21 is false, then for some i and some $\alpha \in A$, we have $\dim X_3^{i,\alpha} > \dim X_3 - i$ and so from 5.1.22, we deduce that

$$\dim (X_{2,\alpha} \times_{X_3} X_{2,\alpha}) > \dim X_3 - i + i - \dim \Sigma + \dim \Sigma_\alpha.$$

By 5.1.18, this gives

$$\dim (X_{2,\alpha} \times_{X_3} X_{2,\alpha}) > \dim G - \dim L + \dim \Sigma_\alpha$$

but the last inequality contradicts 5.1.15 (2) applied to (P, L, Σ_α).

It remains to prove the inequality 5.1.22 for any $i \in \mathbb{Z}$ and $\alpha \in A$.

Let $i \in \mathbb{Z}$, $\alpha \in A$. Note that $X_3^{i,\alpha}$ is a constructible subset of X_3, i.e. a finite union of locally closed subsets of X_3.

Indeed, for any morphism $f : X \to X'$ of algebraic varieties, the set $\{x \in X |\ \dim_x f^{-1}(f(x)) \geq k\}$ is closed in X for any integer k, therefore its image under f, which is $\{x' \in X' |\ \dim f^{-1}(x') \geq k\}$ (note that $\dim f^{-1}(x') = \mathrm{Max}_{f(x)=x'} \dim_x f^{-1}(x')$) is a constructible subset of X'.

We choose a locally closed subset $V^{i,\alpha}$ of X_3 contained in $X_3^{i,\alpha}$ and of maximal dimension, i.e. of same dimension as $X_3^{i,\alpha}$.

Since for any $z \in X_3^{i,\alpha}$, dim $\rho''^{-1}(z)_\alpha \neq -\infty$, i.e. $\rho''^{-1}(z) \cap X_{2,\alpha} \neq \emptyset$, we have $X_3^{i,\alpha} \subseteq \rho''(X_{2,\alpha})$ and the fiber at $x \in X_3^{i,\alpha}$ of the restriction ρ''_α of ρ'' to $X_{2,\alpha}$ are all of dimension $\geq \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha)$. Hence the fibers of the morphism (of varieties) ${\rho''_\alpha}^{-1}(V^{i,\alpha}) \to V^{i,\alpha}$ induced by ρ''_α are all of dimension $\geq \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha)$.

We deduce that

$$\dim {\rho''_\alpha}^{-1}(V^{i,\alpha}) - \dim V^{i,\alpha} \geq \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha),$$

that is,

$$\dim {\rho''_\alpha}^{-1}(V^{i,\alpha}) - \dim X_3^{i,\alpha} \geq \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha). \tag{1}$$

Moreover the fiber at $x \in {\rho''_\alpha}^{-1}(V^{i,\alpha})$ of the projection

$${\rho''_\alpha}^{-1}(V^{i,\alpha}) \times_{X_3} {\rho''_\alpha}^{-1}(V^{i,\alpha}) \to {\rho''_\alpha}^{-1}(V^{i,\alpha})$$

on the second coordinate is isomorphic to ${\rho''_\alpha}^{-1}(\rho''_\alpha(x))$, hence

$$\dim ({\rho''_\alpha}^{-1}(V^{i,\alpha}) \times_{X_3} {\rho''_\alpha}^{-1}(V^{i,\alpha})) - \dim {\rho''_\alpha}^{-1}(V^{i,\alpha}) \geq \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha).$$

We sum this inequality with (1) and we get the inequality 5.1.22.

We proved that $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma,\mathcal{E})$ is a perverse sheaf. The G-equivariance of $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma,\mathcal{E})$ follows from 5.1.6. □

Definition 5.1.23. *We say that two triples $(L,\Sigma,\mathcal{E})$ and $(L',\Sigma',\mathcal{E}')$ as in 5.1.9 are* G-conjugate *if there exists $g\in G$ such that, $L'=gLg^{-1}$, $\Sigma'=Ad(g)\Sigma$ and $\mathcal{E}'$ is isomorphic to $Ad(g^{-1})^*\mathcal{E}$. If $(L,\Sigma,\mathcal{E})$ is a triple as in 5.1.9, then the* G-conjugacy class *of $(L,\Sigma,\mathcal{E})$ is the set of triples $(L',\Sigma',\mathcal{E}')$ which are G-conjugate to $(L,\Sigma,\mathcal{E})$.*

Remark 5.1.24. Let $(L,\Sigma,\mathcal{E})$ and $(L',\Sigma',\mathcal{E}')$ be two triple as in 5.1.9 such that for some $g\in G$, we have $L'=gLg^{-1}$, $\Sigma'=\mathrm{Ad}(g)\Sigma$ and $\mathcal{E}'$ is isomorphic to $\mathrm{Ad}(g^{-1})^*\mathcal{E}$. Let $P=LU_P$ be a Levi decomposition in G with corresponding Lie algebra decompositions $\mathcal{P}=\mathcal{L}\oplus\mathcal{U}_P$, let $P'=gPg^{-1}$ and let $\mathcal{P}'=\mathcal{L}'\oplus\mathcal{U}_{P'}$ be the Lie algebra decomposition corresponding to the decomposition $P'=L'U_{P'}$. Then the complex $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma,\mathcal{E})$ is isomorphic to $\mathrm{ind}_{\mathcal{L}'\subset\mathcal{P}'}^{\mathcal{G}} K(\Sigma',\mathcal{E}')$.

In the following proposition we use the notion of "perverse sheaves of geometrical origin" as in [BBD82, 6.2.4].

Proposition 5.1.25. *Assume that $\mathcal{E}$ is of the form $\zeta\boxtimes\xi$ with $\xi\in ls_L(C)$ and $\zeta\in ls(\mathcal{Z})$ is such that $\zeta[dim\,\mathcal{Z}]$ is of geometrical origin. Then the perverse sheaf $ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma,\mathcal{E})$ is semi-simple.*

Proof: Since ρ'' is proper, from the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber (see [BBD82]) it is enough to show that K_2 is of geometrical origin. Since "being of geometrical origin" is stable by the functors $j_{!*}$ (see [BBD82, 6.2.4]) we need to show that $\mathcal{E}_2$ is of geometrical origin, i.e that $\zeta[\dim\mathcal{Z}]\boxtimes\xi[\dim C]$ is of geometrical origin. From [BBD82, 6.2.4 (c)], it is thus enough to see that $\xi[\dim C]$ is of geometrical origin. Let u be an element of C, then the morphism $L\to C$, $g\mapsto\mathrm{Ad}(g)u$ factors through a bijective morphism $f:L/C_L(u)\to C$. Since $f^*:ls_L(C)\to ls_L(L/C_L(u))$ is an equivalence of categories , we are reduced to show that $f^*\xi[\dim C]$ is of geometrical origin. But this is a consequence of the fact that $f^*(\xi)$ is a simple direct summand of $\pi_*(\overline{\mathbb{Q}}_\ell)$ if π is the Galois covering $L/C_L(u)^o\to L/C_L(u)$, and that the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $L/C_L(u)^o$ is the inverse image of $\overline{\mathbb{Q}}_\ell$ on a point. □

5.1.26 When the Complexes $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma,\mathcal{E})$ Are Intersection Cohomology Complexes

Let $(P, L, \Sigma, \mathcal{E})$ be as in 5.1.9. Recall that $z(\mathcal{L})_{reg}$ denotes the set of L-regular elements in $\mathcal{G}$. In this subsection, we show that if $\mathcal{Z}\cap z(\mathcal{L})_{reg} \neq \emptyset$, then the complex $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma,\mathcal{E})$ is an intersection cohomology complex.

We assume that $\mathcal{Z}\cap z(\mathcal{L})_{reg} \neq \emptyset$, and we denote by $\mathcal{Z}_{reg}$ the set $\mathcal{Z}\cap z(\mathcal{L})_{reg}$ and by Σ_{reg} the set $\mathcal{Z}_{reg} + C$. Then Σ_{reg} is open dense in $\overline{\Sigma}$. Let

$$Y = Y_{(L,\Sigma)} = \bigcup_{g\in G} \mathrm{Ad}(g)(\Sigma_{reg})$$

and

$$Y_2 := \{(x, gL) \in \mathcal{G} \times (G/L) | \,\mathrm{Ad}(g^{-1})x \in \Sigma_{reg}\}.$$

We have the following lemma (see [Lus84, Lemma 4.3 (c)]).

Lemma 5.1.27. *The map $\gamma : Y_2 \to \rho''^{-1}(Y)$ defined by $\gamma(x, gL) = (x, gP)$ is an isomorphism.*

Proof: Only the surjectivity of γ is proved in [Lus84], which proof simplifies in the Lie algebra case (essentially because Σ can not have elements with non-central semi-simple part). Before proving the surjectivity, we prove that γ is an isomorphism onto its image. For that, we first have to check that the image of γ is a variety.

By 2.6.6, the image of γ is $\{(x, gP) \in \mathcal{G} \times (G/P)|\mathrm{Ad}(g^{-1})x \in \Sigma_{reg} + \mathcal{U}_P\}$, thus it is an open subset of X_2: indeed, it is the image of

$$\{(x, g) \in \mathcal{G} \times G|\mathrm{Ad}(g^{-1})x \in \Sigma_{reg} + \mathcal{U}_P\}$$

by the quotient map ρ' and $\{(x, g) \in \mathcal{G} \times G|\mathrm{Ad}(g^{-1})x \in \Sigma_{reg} + \mathcal{U}_P\}$ is the inverse image of $\Sigma_{reg} + \mathcal{U}_P$ under the morphism $X_1 \to \overline{\Sigma} + \mathcal{U}_P$, $(x, g) \mapsto \mathrm{Ad}(g^{-1})x$.

Let P act on $G \times (\Sigma + \mathcal{U}_P)$ by $p.(g, x) = (gp^{-1}, \mathrm{Ad}(p)x)$ and L acts on $G\times\Sigma_{reg}$ by $l.(x, g) = (gl^{-1}, \mathrm{Ad}(l)x)$. Then we may identify Y_2 with $G\times^L \Sigma_{reg}$ and the image of γ with $G\times^P (\Sigma_{reg}+\mathcal{U}_P)$. By 2.6.6, we have an isomorphism $U_P \times \Sigma_{reg} \to \Sigma_{reg} + \mathcal{U}_P$ given by the adjoint action. Via this isomorphism, the P-variety $G \times (\Sigma_{reg} + \mathcal{U}_P)$ can be identified with the P-variety $G \times (\Sigma_{reg} \times U_P)$ where $P = L \times U_P$ acts on $G \times (\Sigma_{reg} \times U_P)$ by $(l, v).(g, (x, u)) = (g(lv)^{-1}, (\mathrm{Ad}(l)x, lvul^{-1})$. Then we have a natural map $G \times (\Sigma_{reg} \times U_P) \to$

$G \times \Sigma_{reg}$ given by $(g, (x, u)) \mapsto (gu, x)$ which induces a morphism $G \times (\Sigma_{reg} \times U_P) \to G \times^L \Sigma_{reg}$. This morphism is constant on the P-orbits, hence gives rise to a morphism $G \times^P (\Sigma_{reg} \times U_P) \to G \times^L \Sigma_{reg}$ which is the inverse of γ.

We show now that γ is surjective. Let

$$Z = \{(x, gP) \in \mathcal{G} \times (G/P) | \operatorname{Ad}(g^{-1})x \in \Sigma_{reg} + \mathcal{U}_P\}.$$

We have to show that $\rho''^{-1}(Y) = Z$. If we let G act on X_2 by Ad on the first coordinate and by left translation on the second coordinate, and by Ad on $\mathcal{G}$, then the morphism ρ'' is G-equivariant. This implies that to prove the inclusion $\rho''^{-1}(Y) \subset Z$, it is enough to prove that $\rho''^{-1}(\Sigma_{reg}) \subset Z$. Let $x \in \Sigma_{reg}$ and $g \in G$ be such that $\operatorname{Ad}(g^{-1})x \in \overline{\Sigma} + \mathcal{U}_P$; let us show that $\operatorname{Ad}(g^{-1})x \in \Sigma_{reg} + \mathcal{U}_P$. Write $\operatorname{Ad}(g^{-1})x = l + u$ with $l \in \overline{\Sigma}$, $u \in \mathcal{U}_P$. We have to show that $l_s \in \mathcal{Z}_{reg}$ and $l_n \in C$ (note that $l_n \in \overline{C}$ and l_s is already in $\mathcal{Z}$). By 2.7.1, there exists $v \in U_P$ such that $\operatorname{Ad}(v^{-1}g^{-1})x_s = l_s \in \mathcal{Z}$. Since $x_s \in \mathcal{Z}_{reg}$, we deduce from 2.6.16 that $\operatorname{Ad}(v^{-1}g^{-1})x_s \in z(\mathcal{L})_{reg}$, i.e. that $l_s \in \mathcal{Z}_{reg} := z(\mathcal{L})_{reg} \cap \mathcal{Z}$. Therefore, from 2.6.6, there exist an element $v' \in U_P$ such that $\operatorname{Ad}(v')l = l + u$. Such an element v' satisfies

(i) $\operatorname{Ad}(v'^{-1}g^{-1})x_s = l_s$ and

(ii) $\operatorname{Ad}(v'^{-1}g^{-1})x_n = l_n$.

From (i) we deduce that $v'^{-1}g^{-1} \in N_G(L)$, and so since $x_n \in C$ by assumption, we deduce from (ii) and the relation $l_n \in \overline{C}$ that $\operatorname{Ad}(v'^{-1}g^{-1})C$ is a an L-orbit of $\mathcal{L}$ which intersects $\overline{C}$, hence we have $\operatorname{Ad}(v'^{-1}g^{-1})C = C$ and so by (ii), $l_n \in C$. □

Lemma 5.1.28. *The subset Y is locally closed in $\mathcal{G}$, irreducible and smooth of dimension $\dim G - \dim L + \dim \Sigma$.*

Proof: We saw in the proof of 5.1.27 that $\rho''^{-1}(Y)$ is an open subset U of X_2. Moreover we have $\rho''^{-1}(\rho''(U)) = U$. Therefore, from the fact that ρ'' is a closed morphism, we get that $\rho''(U) = Y$ is open in its Zariski closure in $\mathcal{G}$.

Now Y is the image by $G \times \mathcal{G} \to \mathcal{G}$, $(g, x) \mapsto \operatorname{Ad}(g)x$ of the irreducible subvariety $G \times \Sigma_{reg}$ of $G \times \mathcal{G}$, therefore Y is irreducible.

We consider the morphism $f : G \times^L \Sigma_{reg} \to Y$ given by $f(g, x) = \operatorname{Ad}(g)x$. Then we see that f is a Galois covering with Galois group, the stabilizer of Σ in $W_G(L)$. Since $G \times^L \Sigma_{reg}$ is smooth, by [Gro71, exposé I, Corollaire 9.2], we deduce that Y is smooth of dimension $\dim G - \dim L + \dim \Sigma$. □

Remark 5.1.29. If we denote by $W_G(\Sigma)$ the Galois group of the Galois covering $G\times^L\Sigma_{reg}\to Y$, $(g,x)\mapsto \mathrm{Ad}(g)x$, then by [Gro71, exposé V, Proposition 2.6] the canonical bijective morphism $(G\times^L\Sigma_{reg})/W_G(\Sigma)\to Y$ is an isomorphism.

We have the following lemma (see [Lus84, Lemma 4.3]).

Lemma 5.1.30. *(i) X_2 is an irreducible variety of dimension $\dim Y$,*
(ii) ρ'' is proper and $\rho''(X_2)=\overline{Y}$.

Proof: We already saw that X_2 is irreducible and that ρ'' is proper. From 5.1.18, we know that X_2 and $\rho''(X_2)$ are irreducible of same dimension $\dim G-\dim L+\dim\Sigma$. Moreover from 5.1.28, Y is an irreducible subvariety of $\rho''(X_2)$ of dimension $\dim G-\dim L+\dim\Sigma$, we deduce that $\overline{Y}=\rho''(X_2)$ and that $\dim X_2=\dim Y$. □

Now we are going to construct a G-equivariant semi-simple local system on Y whose perverse extension on $\mathcal{G}$ will be canonically isomorphic to $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\Sigma,\mathcal{E})$.

We consider the following diagram

5.1.31. $$\Sigma\xleftarrow{\alpha}Y_1\xrightarrow{\alpha'}Y_2\xrightarrow{\alpha''}Y$$

where

$$Y_1:=\{(x,g)\in\mathcal{G}\times G|\mathrm{Ad}(g^{-1})x\in\Sigma_{reg}\}$$

$$Y_2:=\{(x,gL)\in\mathcal{G}\times(G/L)|\mathrm{Ad}(g^{-1})x\in\Sigma_{reg}\}$$

and $\alpha(x,g)=\mathrm{Ad}(g^{-1})x$, $\alpha'(x,g)=(x,gL)$, $\alpha''(x,g)=x$.

Denote by ξ_1 the local system $\alpha^*(\mathcal{E})$ on Y_1; it is an irreducible local system since Σ_{reg} is open dense in Σ and $Y_1\simeq\Sigma_{reg}\times G$. The map α being L-equivariant (with respect to the adjoint action of L on Σ and the action of L on Y_1 given by $l.(x,g)=(x,gl^{-1})$), the local system ξ_1 is L-equivariant. Now, the map $(x,g)\mapsto(g,\mathrm{Ad}(g^{-1})x)$ defines an isomorphism $Y_1\to G\times\Sigma_{reg}$ which is L-equivariant for the action of L on $G\times\Sigma_{reg}$ given by $l.(g,x)=(gl^{-1},\mathrm{Ad}(l)x)$. Since $Y_2\simeq G\times^L\Sigma_{reg}$, by 4.3.3 the triple (Y_1,Y_2,α') is a locally isotrivial principal L-bundle. Thus by 4.3.4 the L-equivariance of ξ_1 implies the existence of a unique irreducible local system ξ_2 on Y_2 such that $(\alpha')^*\xi_2=\xi_1$. We consider the direct image $(\alpha'')_*\xi_2$ on Y. Since α'' is a Galois covering with Galois group $W_G(\Sigma)$, the stabilizer of Σ in $W_G(L)$, the sheaf $(\alpha'')_*\xi_2$ is a semi-simple local system on Y. Now G acts on Y by Ad, on Y_1 and Y_2 by Ad on the first coordinate and by left translation on the second coordinate, and on Σ trivially; the morphisms α, α' and α'' are then G-equivariant. We deduce that the local system $(\alpha'')_*\xi_2$ is G-equivariant. Then the complex $K(Y,(\alpha'')_*\xi_2)$ is G-equivariant semi-simple and by 4.2.13, each of its direct summand is G-equivariant.

Notation 5.1.32. Denote $K(Y,(\alpha'')_*\xi_2)$ by $\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$.

The following proposition which is the Lie algebra version of [Lus84, Proposition 4.5].

Proposition 5.1.33. *We consider a triple $(L,\Sigma,\mathcal{E})$ as in 5.1.9 such that $\mathcal{Z}\cap z(\mathcal{L})_{reg}\neq\emptyset$. Then the complex $ind_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ defined above is canonically isomorphic to the complex $ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\Sigma,\mathcal{E})$ for any parabolic subgroup P of G containing L as a Levi subgroup.*

Proof: Let $(P,L,\Sigma,\mathcal{E})$ be as in 5.1.9. We use the notation related to $(P,L,\Sigma,\mathcal{E})$ introduced before (5.1.9 and 5.1.26). Let $K=K(\Sigma,\mathcal{E})\in\mathcal{M}_L(\mathcal{L})$. To show that the complex $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K)$ is isomorphic to $\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$, we need to check that $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K)$ satisfies the axioms 4.1.1 which characterize the complex $\mathrm{IC}\big(\overline{Y},(\alpha'')_*\xi_2\big)[\dim Y]$. We first show that

(1) $\mathcal{H}^{-\dim Y}(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K)|_Y$ is a local system canonically isomorphic to $(\alpha'')_*\xi_2$.

The following diagram is clearly commutative.

5.1.34.

$$\begin{array}{ccccccc}
\Sigma & \xleftarrow{\alpha} & Y_1 & \xrightarrow{\alpha'} & Y_2 & \xrightarrow{\alpha''} & Y \\
\| \downarrow & & i\downarrow & & \gamma\downarrow & & i\downarrow \\
\Sigma & \xleftarrow{\rho_o} & X_{1,o} & \xrightarrow{\rho'_o} & X_{2,o} & \xrightarrow{\rho''_o} & \overline{Y}
\end{array}$$

where i denotes the inclusions and γ is given as in 5.1.27. Since the middle square is commutative we have $i^*\circ(\rho'_o)^*(\mathcal{E}_2)=(\alpha')^*\circ\gamma^*(\mathcal{E}_2)$. But $\rho_o^*(\mathcal{E})=(\rho'_o)^*(\mathcal{E}_2)$, thus we have $i^*\circ\rho_o^*(\mathcal{E})=(\alpha')^*\circ\gamma^*(\mathcal{E}_2)$. Using the fact that the left square commutes, we deduce that $\alpha^*(\mathcal{E})=(\alpha')^*\circ\gamma^*(\mathcal{E}_2)$, but $\alpha^*(\mathcal{E})=(\alpha')^*(\xi_2)$, so $\gamma^*(\mathcal{E}_2)=\xi_2$ that is $\gamma^*(K_2|_{X_{2,o}})=\xi_2[\dim X_2]$. We consider the following cartesian diagram.

$$\begin{array}{ccc}
Y_2 & \xrightarrow{\alpha''} & Y \\
\gamma\downarrow & & i\downarrow \\
X_2 & \xrightarrow{\rho''} & \overline{Y}
\end{array}$$

From the proper base change theorem applied to the above diagram, we deduce that the canonical base change morphism $(\rho''_!K_2)|_Y\to\alpha''_*(\xi_2[\dim X_2])$ is an isomorphism. Since $\alpha''_* : \mathcal{S}h(Y_2) \to \mathcal{S}h(Y)$ is an exact functor (because α'' is a finite morphism), we have $\alpha''_*(\xi_2[\dim X_2]) = (\alpha_*\xi_2)[\dim X_2]$ [3]. Since $\dim X_2 = \dim Y$ (see 5.1.30 (i)) , we deduce (1).

It remains to check the axioms (i), (iii) and (iv) of 4.1.1. The axiom (i) follows from the fact that $\mathcal{H}^i_x(\rho''_! K_2)$ is the hypercohomology $\mathbb{H}^i(\rho''^{-1}(x), K_2|_{\rho''^{-1}(x)})$ which, is equal to 0 when $i < -\dim Y$ since the complex $K_2 = K_2^\bullet$ satisfies $K_2^r = 0$ if $r < -\dim Y$. Now it remains to prove the following inequalities for $i > -\dim Y$,

(a) $\dim\left(Supp\,(\mathcal{H}^i(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K))\right) < -i$,

(b) $\dim\left(Supp\,(\mathcal{H}^i(D_{\mathcal{G}} \circ \mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K))\right) < -i$.

For the same reasons as those evoked in the proof of 5.1.19, the proof of (b) is entirely similar to that of (a), we thus prove only (a).

The proof of (a) is quite similar to that of 5.1.19 (i); the difference is that in 5.1.19, it is enough to use that K_2 is a perverse sheaf while in the proof of (a) we need to use that K_2 is an intersection cohomology complex.

As in the proof of 5.1.19, for any $x \in X_3$, we have a stratification $\rho''^{-1}(x) = \coprod_{\alpha\in A} \rho''^{-1}(x)_\alpha$; we denote by $\alpha_o \in A$, the index corresponding to the open stratum C of $\overline{C}$. To prove (a), we are reduced to prove that the inequality 5.1.20 is strict for any $\alpha \in A$ and $i > -\dim Y$. Let $i > -\dim Y$ and let $\alpha \in A$. When $\alpha \neq \alpha_o$, following [Lus84], we are reduced to prove 5.1.21 with strict inequalities instead of large inequalities. But then, see 5.1.19, it is enough to prove that the inequality 5.1.22, where $X_3^{i,\alpha}$ is replaced by $\{x \in X_3 |\ \dim \rho''^{-1}(x)_\alpha > \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha)\}$, is a strict inequality, which proof is entirely similar to that of 5.1.22 (we only need to replace large inequalities by strict inequalities). Following [Lus84], the case $\alpha = \alpha_o$ is reduced to proving that the inequality 5.1.21 is strict, which proof is entirely similar to that of [Lus84, Proposition 4.5]. □

Notation 5.1.35. We denote by $\mathcal{G}_\sigma$ the closed subset of $\mathcal{G}$ consisting of the elements of $\mathcal{G}$ whose semi-simple part is G-conjugate to σ.

Remark 5.1.36. The variety $\mathcal{G}_\sigma$ is a finite union of G-orbits; this follows from the finiteness of nilpotent orbits. When $\sigma \in z(\mathcal{G})$, note that $\mathcal{G}_\sigma = \sigma + \mathcal{G}_{nil}$.

5.1.37. Let $(P, L, \Sigma, \mathcal{E})$ be as in 5.1.9 such that $\Sigma = \sigma + C$ for some $\sigma \in z(\mathcal{L})$ and $\mathcal{E} = \overline{\mathbb{Q}}_\ell \boxtimes \zeta$ with $\zeta \in ls_L(C)$. Then by 5.1.25, the G-equivariant complex

[3] Note that in the left hand side, $(\alpha'')_*$ is a functor $\mathcal{D}_c^b(Y_2) \to \mathcal{D}_c^b(Y)$ while in the right hand side, it is a functor $\mathcal{S}h(Y_2) \to \mathcal{S}h(Y)$.

$\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}K(\Sigma,\mathcal{E})$ is semi-simple; moreover by 4.2.13 each direct summand is G-equivariant. On the other hand we have $X_3 \subset \mathcal{G}_\sigma$ (see 5.1.17 for the definition of X_3) and so X_3 is a finite union of G-orbits. Hence from 4.2.14, we deduce that the complex $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}K(\Sigma,\mathcal{E})$ is a finite direct sum of orbital perverse sheaves on $\mathcal{G}$.

5.1.38. Let $(P, L, \Sigma, \mathcal{E})$ be as in 5.1.37 such that $\sigma \in z(\mathcal{L})_{reg}$, i.e. $C^o_G(\sigma) = L$. Then we have a diagram as in 5.1.31. Let $x = \sigma + u \in \Sigma$, we then have $\Sigma = \mathcal{O}^L_x$ and $Y = \mathcal{O}^G_x$. First note that since $C^o_G(x) = C^o_L(x)$, we may regard $A_L(x)$ as a normal subgroup of $A_G(x)$. Let ξ_2 be the local system on Y_2 such that $(\alpha')^*(\xi_2) \simeq \alpha^*(\mathcal{E})$ and let χ be the irreducible character of $A_L(x)$ associated to $\mathcal{E}$ as in 4.2.15, then the G-equivariant local system $(\alpha'')_*\xi_2$ corresponds to the character $\mathrm{Ind}^{A_G(x)}_{A_L(x)}(\chi)$ of $A_G(x)$, where $\mathrm{Ind}^{A_G(x)}_{A_L(x)}$ is the usual induction of characters as in [Ser78, 3.3]. Moreover if ξ is the L-equivariant local system on $\mathcal{O}^G_x$ corresponding to a character χ' of $A_G(x)$, then the restriction ξ^L of ξ to $\mathcal{O}^L_x$ is the L-equivariant local system corresponding to the restriction of χ' to $A_L(x)$.

Lemma 5.1.39. *If $C_G(\sigma)$ is connected, then the complex $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}K(\mathcal{O}^L_x, \mathcal{E})$ is isomorphic to the complex $K(\mathcal{O}^G_x, \xi)$ where ξ is the unique irreducible G-equivariant local system on $\mathcal{O}^G_x$ such that $\xi|_{\mathcal{O}^L_x} = \mathcal{E}$.*

Proof: If $C_G(\sigma)$ is connected, then we have $A_G(x) \simeq A_L(x)$. □

Since our assumption on p can not ensure that for any Levi subgroup L of G, the centralizers of the semi-simple elements of $\mathcal{L}$ are all connected, we will need to use the following result.

Lemma 5.1.40. *Let $(\mathcal{O}, \xi)$ be an orbital pair of $\mathcal{G}$, let $x \in \mathcal{O}$ and let $L = C^o_G(x_s)$. Then the complex $K(\mathcal{O}, \xi)$ is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}K(\mathcal{O}^L_x, \xi^L)$ where ξ^L is the restriction of ξ to $\mathcal{O}^L_x = x_s + \mathcal{O}^L_{x_n}$ and where $\mathcal{P}$ is the Lie algebra of a parabolic subgroup of G having L as a Levi subgroup.*

Proof: The lemma follows from 5.1.38 and the fact that if χ is an irreducible character of $A_G(x)$ and χ' denotes its restriction to $A_L(x)$, then the scalar product

$$\left(\mathrm{Ind}^{A_G(x)}_{A_L(x)}(\chi'), \chi\right)_{A_G(x)} = \left(\chi', \chi'\right)_{A_L(x)}$$

is non-zero. □

5.1.41 Restriction of $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma,\mathcal{E})$ to $\mathcal{G}_\sigma$ with $\sigma \in z(\mathcal{G})$

Let $(P, L, \Sigma, \mathcal{E})$ be as in 5.1.9. In this subsection we assume that $\mathcal{Z} = z(\mathcal{L})$ and that $\mathcal{E} = \xi \boxtimes \zeta$ where $\zeta \in ls_L(C)$ and ξ is a one-dimensional local system on $z(\mathcal{L})$. We fix an element $\sigma \in z(\mathcal{L})$. Note that $\mathcal{E}|_{\sigma+C} = \overline{\mathbb{Q}}_\ell \boxtimes \zeta$. We start with the following lemma.

Lemma 5.1.42. *We have* $IC(\overline{\Sigma},\mathcal{E})|_{\sigma+\overline{C}} = IC(\overline{\sigma+C},\overline{\mathbb{Q}}_\ell \boxtimes \zeta)$.

Proof: Since $z(\mathcal{L})$ is smooth, we deduce from 4.1.6 that $\mathrm{IC}(\overline{\Sigma},\mathcal{E}) \simeq \xi \boxtimes \mathrm{IC}(\overline{C},\zeta)$. Hence if $i : \{\sigma\} \times \overline{C} \to z(\mathcal{L}) \times \overline{C}$ is given by the inclusions, then $\mathrm{IC}(\overline{\Sigma},\mathcal{E})|_{\sigma+\overline{C}} \simeq i^*\big(\mathrm{IC}(\overline{\Sigma},\mathcal{E})\big) \simeq \overline{\mathbb{Q}}_\ell \boxtimes \mathrm{IC}(\overline{C},\zeta)$. Applying again 4.1.6, we deduce that $\mathrm{IC}(\overline{\Sigma},\mathcal{E})|_{\sigma+\overline{C}} \simeq \mathrm{IC}(\overline{\sigma+C},\overline{\mathbb{Q}}_\ell \boxtimes \zeta)$. □

Proposition 5.1.43. *Assume that $\sigma \in z(\mathcal{G})$. Then the support of the complex $ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\sigma + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ is contained in $\mathcal{G}_\sigma$ and we have a canonical isomorphism in $\mathcal{D}_c^b(\mathcal{G}_\sigma)$,*

$$\left(ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma,\mathcal{E})\right)|_{\mathcal{G}_\sigma} \xrightarrow{\sim} \left(ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\sigma + C,\overline{\mathbb{Q}}_\ell \boxtimes \zeta)\right)[dim\, z(\mathcal{L})]$$

where we identified $ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\sigma + C,\overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ with its restriction to $\mathcal{G}_\sigma$.

Proof: Note that the triple $(P, L, \sigma+C, \overline{\mathbb{Q}}_\ell\boxtimes\zeta)$ is as in 5.1.9 if we put $\mathcal{Z} = \{\sigma\}$. Hence as in 5.1.11, we have a diagram

5.1.44.

$$\sigma + \overline{C} \xleftarrow{\rho_\sigma} X_1^\sigma \xrightarrow{\rho'_\sigma} X_2^\sigma \xrightarrow{\rho''_\sigma} \mathcal{G}$$

where $(X_1^\sigma, X_2^\sigma, \rho_\sigma, \rho'_\sigma, \rho''_\sigma)$ is defined in terms of $P, L, \sigma+C$ as $(X_1, X_2, \rho, \rho', \rho'')$ is defined in terms of P, L, Σ. Let $K_2^\sigma \in \mathcal{M}(X_2^\sigma)$ be the analogue of K_2; we have $(\rho''_\sigma)_!(K_2^\sigma) = \mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\sigma + C,\overline{\mathbb{Q}}_\ell \boxtimes \zeta)$. Let $f''_\sigma : X_2^\sigma \to \mathcal{G}_\sigma$ be the morphism [4] given by $x \mapsto \rho''_\sigma(x)$. Then we have $\left(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\sigma + C,\overline{\mathbb{Q}}_\ell \boxtimes \zeta)\right)|_{\mathcal{G}_\sigma} = (f''_\sigma)_! K_2^\sigma$.

The following diagram commutes.

5.1.45.

$$\begin{array}{ccccccc}
\sigma + \overline{C} & \xleftarrow{\rho_\sigma} & X_1^\sigma & \xrightarrow{\rho'_\sigma} & X_2^\sigma & \xrightarrow{f''_\sigma} & \mathcal{G}_\sigma \\
{\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow \\
\overline{\Sigma} & \xleftarrow{\rho} & X_1 & \xrightarrow{\rho'} & X_2 & \xrightarrow{\rho''} & \mathcal{G}
\end{array}$$

[4] f''_σ is well-defined since the image of ρ''_σ is contained in $\mathcal{G}_\sigma$.

where i denotes the inclusions.

From 5.1.42 we have $i^*\left(\mathrm{IC}(\overline{\Sigma},\mathcal{E})[\dim\Sigma]\right) = \mathrm{IC}(\overline{\sigma+C},\overline{\mathbb{Q}}_\ell\boxtimes\zeta)[\dim C+\dim z(\mathcal{L})]$, hence since the left square and the middle square of 5.1.45 commute, we deduce that $i^*(K_2)=K_2^{\sigma}[\dim z(\mathcal{L})]$. Since $\sigma\in z(\mathcal{G})$, the right square of 5.1.45 is cartesian. Hence from the proper base change theorem, the canonical base change morphism $(\rho''_!K_2)|_{\mathcal{G}_\sigma}\to\left((f''_\sigma)_!K_2^{\sigma}\right)[\dim z(\mathcal{L})]$ is an isomorphism. □

Remark 5.1.46. The proposition 5.1.43 generalizes as follows: let σ be not necessarily in $z(\mathcal{G})$, then we have a canonical isomorphism (up to a shift)

$$\left(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\Sigma,\mathcal{E})\right)|_{\mathcal{G}_\sigma}\xrightarrow{\sim}\left(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\mathcal{G}_\sigma\cap\Sigma,\mathcal{E}|_{\mathcal{G}_\sigma\cap\Sigma})\right)|_{\mathcal{G}_\sigma}.$$

Note that the right hand side of the above isomorphism might not be the same as that of 5.1.43 (this happens when $\mathcal{G}_\sigma\cap\overline{\Sigma}\neq\sigma+\overline{C}$).

The complexes $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\sigma+C,\overline{\mathbb{Q}}_\ell\boxtimes\zeta)$, with $\sigma\notin z(\mathcal{G})$, will appear to be more important than the complexes $\left(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\Sigma,\mathcal{E})\right)|_{\mathcal{G}_\sigma}$, so we preferred to state 5.1.43 rather than its above generalization which will not be used.

Lemma 5.1.47. *The complex* $\left(ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\Sigma,\mathcal{E})\right)|_{\mathcal{G}_\sigma}$ *is a semi-simple perverse sheaf on* $\mathcal{G}_\sigma$ *up to a shift by* $dim\, z(\mathcal{L})$.

Proof: This follows from 5.1.43 and 5.1.37. □

5.1.48. The isomorphism $\mathcal{G}\simeq z(\mathcal{G})\oplus\overline{\mathcal{G}}$ we have fixed just before 5.1.1 gives rise to a Lie algebra isomorphism $\mathcal{L}\simeq z(\mathcal{G})\oplus(\mathcal{L}/z(\mathcal{G}))$. Let $\hat{P}=P/Z_G^o$, $\hat{L}=L/Z_G^o$, $\hat{\mathcal{L}}=\mathcal{L}/z(\mathcal{G})$, $\hat{\mathcal{P}}=\mathcal{P}/z(\mathcal{G})$ and $\hat{\Sigma}=\Sigma/z(\mathcal{G})$. Then $\mathcal{L}\simeq z(\mathcal{G})\oplus\hat{\mathcal{L}}$ and $\Sigma\simeq z(\mathcal{G})\times\hat{\Sigma}$.

Lemma 5.1.49. *Assume that we have a decomposition* $\mathcal{E}\simeq\xi\boxtimes\hat{\mathcal{E}}$ *with* $\xi\in ls(z(\mathcal{G}))$ *and* $\hat{\mathcal{E}}\in ls_{\hat{L}}(\hat{\Sigma})$. *Then we have* $ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\Sigma,\mathcal{E})\simeq\xi[dim\, z(\mathcal{G})]\boxtimes ind_{\hat{\mathcal{L}}\subset\hat{\mathcal{P}}}^{\overline{\mathcal{G}}}K(\hat{\Sigma},\hat{\mathcal{E}})\in\mathcal{M}(z(\mathcal{G})\oplus\overline{\mathcal{G}})$.

Proof: Let V_1, V_2, π, π' and π'' be defined in terms of (G,P,L) as in 5.1.1 and let $\hat{V}_1$, $\hat{V}_2$, $\hat{\pi}$, $\hat{\pi}'$ and $\hat{\pi}''$ be defined in terms of $(\overline{G},\hat{P},\hat{L})$. Then the lemma follows from the fact that we have the following decompositions $V_1=z(\mathcal{G})\times\hat{V}_1$, $V_2=z(\mathcal{G})\times\hat{V}_2$, and $\pi=Id_{z(\mathcal{G})}\times\hat{\pi}$, $\pi'=Id_{z(\mathcal{G})}\times\hat{\pi}'$ and $\pi''=Id_{z(\mathcal{G})}\times\hat{\pi}''$. □

Similarly we can prove the following lemma.

Lemma 5.1.50. *Assume that $\sigma \in z(\mathcal{G})$. Let $(\mathcal{O}, \eta)$ be a nilpotent pair of $\mathcal{G}$ such that the complex $K(\mathcal{O}, \eta)$ is a direct summand of $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(C, \zeta)$, then the complex $K(\sigma + \mathcal{O}, \overline{\mathbb{Q}}_\ell \boxtimes \eta)$ is a direct summand of $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\sigma + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$.*

5.1.51 Introducing Frobenius

Let $(P, L, \Sigma, \mathcal{E})$ be as in 5.1.9 such that $\mathcal{Z} \cap z(\mathcal{L})_{reg} \neq \emptyset$. We keep the notation of 5.1.9 and 5.1.26. We assume that $(L, \Sigma, \mathcal{E})$ is F-stable and let $\phi_{\mathcal{E}} : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ be an isomorphism.

The construction of the complex $\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ has the following advantage: the morphisms α, α' and α'' of 5.1.31 are F-stable, and so $\phi_{\mathcal{E}}$ induces a canonical isomorphism $\phi_{\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})} : F^*\big(\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})\big) \xrightarrow{\sim} \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ while when P is not F-stable, there is no such a direct way to define an isomorphism $F^*\big(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})\big) \xrightarrow{\sim} \mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})$.

In this subsection, our interest is to follow the action of the Frobenius in the construction of the complex $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})$.

Let $\tilde{P} = F(P)$ (note that L is also a Levi subgroup of $\tilde{P}$). As in 5.1.9, we have a diagram

$$\overline{\Sigma} \xleftarrow{\tilde{\rho}} \tilde{X}_1 \xrightarrow{\tilde{\rho}'} \tilde{X}_2 \xrightarrow{\tilde{\rho}''} \mathcal{G}$$

with

$$\tilde{X}_1 := \{(x, g) \in \mathcal{G} \times G |\ \mathrm{Ad}(g^{-1})x \in \overline{\Sigma} + \mathcal{U}_{\tilde{P}}\}$$

$$\tilde{X}_2 := \{(x, g) \in \mathcal{G} \times (G/\tilde{P}) |\ \mathrm{Ad}(g^{-1})x \in \overline{\Sigma} + \mathcal{U}_{\tilde{P}}\}$$

and where $\tilde{\rho}$, $\tilde{\rho}'$ and $\tilde{\rho}''$ are the analogue for $\tilde{P}$ of ρ, ρ' and ρ''. We denote by $\tilde{K}_2 \in \mathcal{M}(\tilde{X}_2)$ the analogue for $\tilde{P}$ of the complex K_2 (see 5.1.9). Let $F_1 : X_1 \to \tilde{X}_1$, $(x, g) \mapsto (F(x), F(g))$, $F_2 : X_2 \to \tilde{X}_2$, $(x, g) \mapsto (F(x), F(g)\tilde{P})$. We have the following cartesian diagram.

$$\begin{array}{ccccccc} \overline{\Sigma} & \xleftarrow{\rho} & X_1 & \xrightarrow{\rho'} & X_2 & \xrightarrow{\rho''} & \mathcal{G} \\ {\scriptstyle F}\downarrow & & {\scriptstyle F_1}\downarrow & & {\scriptstyle F_2}\downarrow & & {\scriptstyle F}\downarrow \\ \overline{\Sigma} & \xleftarrow{\tilde{\rho}} & \tilde{X}_1 & \xrightarrow{\tilde{\rho}'} & \tilde{X}_2 & \xrightarrow{\tilde{\rho}''} & \mathcal{G} \end{array}$$

Then we can check from the commutativity of the above diagram that $\phi_{\mathcal{E}}$ induces a canonical isomorphism $\phi_2 : F_2^*(\tilde{K}_2) \to K_2$ and so from the proper base change theorem applied to the right square of the above diagram, we deduce that the base change morphism $f : F^*((\tilde{\rho}'')_! \tilde{K}_2) \to (\rho'')_! K_2$ is an isomorphism. We have the following proposition (see [Lus85b, 8.2.4]).

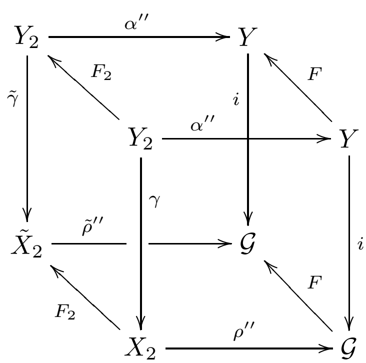
Proposition 5.1.52. *We keep the above notation. Then the above canonical isomorphism* $f : F^*((\tilde{\rho}'')_! \tilde{K}_2) \to (\rho''_!)K_2$ *makes the following diagram commutative*

$$\begin{array}{ccc} F^*((\tilde{\rho}'')_! \tilde{K}_2) & \xrightarrow{f} & (\rho'')_! K_2 \\ {\scriptstyle F^*(\tilde{g})}\downarrow & & {\scriptstyle g}\downarrow \\ F^*(ind_{\Sigma}^{\mathcal{G}}(\mathcal{E})) & \xrightarrow{\phi_{ind_{\Sigma}^{\mathcal{G}}(\mathcal{E})}} & ind_{\Sigma}^{\mathcal{G}}(\mathcal{E}) \end{array}$$

where g and $\tilde{g}$ are the canonical isomorphisms given by 5.1.33.

Proof: Since all complexes occurring in the diagram 5.1.52 are perverse extensions of local systems on Y, it is enough to check the commutativity of the restriction to Y of this diagram. For that, we consider the following cartesian diagram.

5.1.53.



where i is the notation for the natural inclusion and γ, $\tilde{\gamma}$ are given in 5.1.27.

By definition, f is the composition of the base change morphism $F^*(\tilde{\rho}'')_! \tilde{K}_2 \xrightarrow{\sim} (\rho'')_! F_2^* \tilde{K}_2$ with $(\rho'')_!(\phi_2) : (\rho'')_! F_2^* \tilde{K}_2 \xrightarrow{\sim} (\rho'')_! K_2$. We have the following diagram (the composition of the top arrows is the restriction of f to Y).

5.1.54.

$$\begin{array}{ccccc} i^* F^*(\tilde{\rho}'')_! \tilde{K}_2 & \xrightarrow{\sim} & i^*(\rho'')_! F_2^* \tilde{K}_2 & \xrightarrow{i^*(\rho'')_!(\phi_2)} & i^*(\rho'')_! K_2 \\ \downarrow & & \downarrow & & \downarrow \\ F^*(\alpha'')_* \tilde{\gamma}^* \tilde{K}_2 & \xrightarrow{\sim} & (\alpha'')_* \gamma^* F_2^* \tilde{K}_2 & \xrightarrow{(\alpha'')_*(\phi_2)} & (\alpha'')_* \gamma^* K_2 \end{array}$$

where we denote (by abuse of notation) ϕ_2 for $\gamma^*(\phi_2) : \gamma^*(F_2^*\tilde{K}_2) \to \gamma^*(K_2)$ and where the unlabeled arrows are the canonical base change morphisms (which in our case are isomorphisms because the diagram 5.1.53 is cartesian). The right square of the diagram 5.1.54 is commutative because it is given by the base change morphism of functors $i^*(\rho'')_! \to (\alpha'')_*\gamma^*$. The commutativity of the left square follows from the definition of base change morphisms.

To simplify the notation we write ξ_2 instead of $\xi_2[\dim X_2]$. As we saw in the proof of 5.1.33, we have $\tilde{\gamma}^*(\tilde{K}_2) = \xi_2 = \gamma^*(K_2)$. Since $F_2^* \circ \tilde{\gamma}^* = \gamma^* \circ F_2^*$, the isomorphism $\phi_2 = \gamma^*(F_2^*\tilde{K}_2) \to \gamma^*(K_2)$ is thus an isomorphism $F_2^*(\xi_2) \to \xi_2$ (from 5.1.34, we see that this isomorphism is actually the canonical isomorphism induced from $\phi_{\mathcal{E}}$ using the diagram 5.1.31) and an isomorphism $F_2^*\tilde{\gamma}^*\tilde{K}_2 \to \gamma^*K_2$, that is

$$\begin{array}{ccc} F_2^*\tilde{\gamma}^*\tilde{K}_2 & \xrightarrow{\phi_2} & \gamma^*K_2 \\ \| \downarrow & & \downarrow \| \\ F_2^*\xi_2 & \xrightarrow{\phi_2} & \xi_2 \end{array}$$

By applying the functor $(\alpha'')_*$ we get

$$\begin{array}{ccc} (\alpha'')_*F_2^*\tilde{\gamma}^*\tilde{K}_2 & \xrightarrow{(\alpha'')_*(\phi_2)} & (\alpha'')_*\gamma^*K_2 \\ \| \downarrow & & \downarrow \| \\ (\alpha'')_*F_2^*\xi_2 & \xrightarrow{(\alpha'')_*(\phi_2)} & (\alpha'')_*\xi_2 \end{array}$$

We deduce the following commutative diagram.

5.1.55.

$$\begin{array}{ccccc} F^*(\alpha'')_*\tilde{\gamma}^*\tilde{K}_2 & \xrightarrow{\sim} & (\alpha'')_*F_2^*\tilde{\gamma}^*\tilde{K}_2 & \xrightarrow{(\alpha'')_*(\phi_2)} & (\alpha'')_*\gamma^*K_2 \\ \| \downarrow & & \| \downarrow & & \downarrow \| \\ F^*(\alpha'')_*\xi_2 & \xrightarrow{\sim} & (\alpha'')_*F_2^*\xi_2 & \xrightarrow{(\alpha'')_*(\phi_2)} & (\alpha'')_*\xi_2 \end{array}$$

where the unlabeled arrows are the canonical base change morphisms. Using $\gamma^* \circ F_2^* = F_2^* \circ \tilde{\gamma}^*$, we see that the top arrows of the diagram 5.1.55 are the corresponding bottom ones of the diagram 5.1.54. Moreover the composition of the bottom arrows of the diagram 5.1.55 is the restriction of $\phi_{\mathrm{ind}_\Sigma^\mathcal{G}(\mathcal{E})}$ to Y since the isomorphism $\phi_2 : F_2^*(\xi_2) \to \xi_2$ is the canonical isomorphism induced from $\phi_{\mathcal{E}}$ using the diagram 5.1.31. Now if we glue the diagram 5.1.55 together with the diagram 5.1.54 in the obvious way and if we permute i^* with F^* in the left hand side of the diagram 5.1.54, we see that the right and left vertical arrows of the resulting diagram are respectively the restrictions of g and $F^*(\tilde{g})$ to Y. □

5.1.56 Admissible Complexes (or Character Sheaves) on $\mathcal{G}$

Notation 5.1.57. Consider the non-trivial additive character $\Psi : \mathbb{F}_q^+ \to \overline{\mathbb{Q}}_\ell^\times$ of 3.1.9. We denote by $\mathbb{A}^1$ the affine line over k. Let $h : \mathbb{A}^1 \to \mathbb{A}^1$ be the Artin-Shreier covering defined by $h(t) = t^q - t$. Then, since h is a Galois covering of $\mathbb{A}^1$ with Galois group $\mathbb{F}_q$, the sheaf $h_*(\overline{\mathbb{Q}}_\ell)$ is a local system on $\mathbb{A}^1$ on which $\mathbb{F}_q$ acts. We denote by $\mathcal{L}_\Psi$ the subsheaf of $h_*(\overline{\mathbb{Q}}_\ell)$ on which $\mathbb{F}_q$ acts as Ψ^{-1}. There exists an isomorphism $\phi_{\mathcal{L}_\Psi} : F^*(\mathcal{L}_\Psi) \xrightarrow{\sim} \mathcal{L}_\Psi$ such that for any integer $i \geq 1$, we have $\mathbf{X}_{\mathcal{L}_\Psi, \phi_{\mathcal{L}_\Psi}^{(i)}} = \Psi \circ Tr_{\mathbb{F}_{q^i}/\mathbb{F}_q} : \mathbb{F}_{q^i} \to \overline{\mathbb{Q}}_\ell^\times$ (see [Kat80, 3.5.4]).

Definition 5.1.58. *[Lus87] Let $K \in \mathcal{M}_G(\mathcal{G})$ be irreducible.*

(a) If G is semi-simple, we say that K is a cuspidal admissible complex *if its support is a closure of a single nilpotent orbit in $\mathcal{G}$ (i.e if K is of the form $K(\mathcal{O}, \xi)$ for some nilpotent pair $(\mathcal{O}, \xi)$ of $\mathcal{G}$) and if for any proper Levi decomposition $P = LU_P$ in G (with corresponding Lie algebras decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$), we have $(\pi_\mathcal{P})_!(K|_\mathcal{P}) = 0$ (where 0 denotes the zero object in $\mathcal{D}_c^b(X)$).*

(b) If G is reductive, we say that K is a cuspidal admissible complex *if it corresponds, under the identification $\mathcal{G} \simeq z(\mathcal{G}) \oplus \overline{\mathcal{G}}$, to a complex on $z(\mathcal{G}) \oplus \overline{\mathcal{G}}$ of the form $K_1 \boxtimes K_2$ where $K_2 \in \mathcal{M}_{\overline{G}}(\overline{\mathcal{G}})$ is cuspidal and where $K_1 \in \mathcal{M}(z(\mathcal{G}))$ is of the form $m^*(\mathcal{L}_\Psi)[dim\, z(\mathcal{G})]$ with $m : z(\mathcal{G}) \to k$ a linear form and $\mathcal{L}_\Psi$ the one-dimensional local system on $\mathbb{A}^1$ defined in 5.1.57.*

By 4.1.6, we see that any cuspidal admissible complex $K \in \mathcal{M}_G(\mathcal{G})$ is of the form $K\big(z(\mathcal{G}) + C, m^*(\mathcal{L}_\Psi) \boxtimes \zeta\big)$ for some nilpotent pair (C, ζ) of $\mathcal{G}$ and some linear form $m : z(\mathcal{G}) \to k$.

Definition 5.1.59. *Let (C, ζ) be a nilpotent pair of $\mathcal{G}$ and let $\Sigma = z(\mathcal{G}) + C$ and $\mathcal{E} = m^*(\mathcal{L}_\Psi) \boxtimes \zeta$ where m is a linear form on $z(\mathcal{G})$. If the complex $K(\Sigma, \mathcal{E})$ is a cuspidal admissible complex, then we say that*

- *the pair $(\Sigma, \mathcal{E})$ of $\mathcal{G}$ is a* cuspidal admissible pair *(or $\mathcal{E}$ is a cuspidal local system on Σ),*
- *the pair (C, ζ) is a* cuspidal nilpotent pair *of $\mathcal{G}$ (or ζ is a cuspidal local system on C).*

Remark 5.1.60. In the group case, the varieties Σ supporting a cuspidal local system (see [Lus84, 2.1]) are inverse images under the map $G \to \overline{G}$ of isolated conjugacy classes (see [Lus84, 2.6, 2.7]). Note that in the case of Lie algebras, the "isolated" orbits in $\overline{\mathcal{G}}$ are the nilpotent orbits; this follows from the fact that the connected component of the centralizers in G of the semi-simple elements of $\mathcal{G}$ are Levi subgroups of G. Hence the definition of the varieties Σ supporting a cuspidal pair of $\mathcal{G}$ is consistent with that for groups.

Definition 5.1.61. *We say that a triple* $(L, \Sigma, \mathcal{E})$ *is a* cuspidal datum *of* $\mathcal{G}$ *if* L *is a Levi subgroup of* G *and* $(\Sigma, \mathcal{E})$ *is a cuspidal admissible pair of Lie*(L). *We say that a cuspidal datum* $(L, \Sigma, \mathcal{E})$ *of* $\mathcal{G}$ *is* F*-stable if* L, Σ *and* $\mathcal{E}$ *are all* F*-stable.*

Definition 5.1.62. *Let* $K \in \mathcal{M}_G(\mathcal{G})$ *be irreducible. Then we say that* K *is an* admissible complex *(or a* character sheaf*) on* $\mathcal{G}$ *if it is a direct summand of* $ind_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ *for some cuspidal datum* $(L, \Sigma, \mathcal{E})$ *of* $\mathcal{G}$.

Notation 5.1.63. We denote by $\mathcal{A}(\mathcal{G})$ the set of admissible complexes on $\mathcal{G}$.

Remark 5.1.64. Let $a_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ be given by $x \mapsto -x$, then $a_{\mathcal{G}}^*$ permutes the admissible complexes on $\mathcal{G}$ and maps the cuspidals onto the cuspidals.

We have the following proposition.

Proposition 5.1.65. *[Lus87, 3 (a)] Let* $P = LU_P$ *be a Levi decomposition in* G *with corresponding Lie algebra decomposition* $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$. *If* $K \in \mathcal{A}(\mathcal{L})$, *then* $ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K)$ *is a direct sum of finitely many admissible complexes.*

Proof: We verify easily that the proof of [Lus87, 3 (a)] remains valid under our assumption on p. □

Remark 5.1.66. Let (L, C, m, ζ) be such that $(L, z(\mathcal{L}) + C, m^*(\mathcal{L}_\Psi) \boxtimes \zeta)$ is a cuspidal datum of $\mathcal{G}$. We identify $\mathcal{L}$ with $z(\mathcal{G}) \oplus \hat{\mathcal{L}}$ as in 5.1.48. Let $z(\hat{\mathcal{L}}) = z(\mathcal{L})/z(\mathcal{G})$ and let $\hat{C}$ be the image of C in $\hat{\mathcal{L}}$. Then the local system $\mathcal{E}$ decomposes as $(m_{z(\mathcal{G})})^*\mathcal{L}_\Psi \boxtimes (m_{z(\hat{\mathcal{L}})})^*\mathcal{L}_\Psi \boxtimes \zeta \in ls(z(\mathcal{G}) \times z(\hat{\mathcal{L}}) \times \hat{C})$ where $m_{z(\mathcal{G})}$ and $m_{z(\hat{\mathcal{L}})}$ are the restrictions of m respectively to $z(\mathcal{G})$ and $z(\hat{\mathcal{L}})$. Indeed, if $s : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the morphism given by the addition on $\mathbb{A}^1$, then $m = s \circ (m_{z(\mathcal{G})} \times m_{z(\hat{\mathcal{L}})})$. Hence $m^*(\mathcal{L}_\Psi) = (m_{z(\mathcal{G})} \times m_{z(\hat{\mathcal{L}})})^* s^*(\mathcal{L}_\Psi)$. From $s^*(\mathcal{L}_\Psi) = \mathcal{L}_\Psi \boxtimes \mathcal{L}_\Psi$, we deduce that $m^*(\mathcal{L}_\Psi) = (m_{z(\mathcal{G})})^*(\mathcal{L}_\Psi) \boxtimes (m_{z(\hat{\mathcal{L}})})^*(\mathcal{L}_\Psi)$.

Lemma 5.1.67. *With the notation of 5.1.66, let* $(L, \Sigma, \mathcal{E}) = \big(L, z(\mathcal{L}) + C, m^*(\mathcal{L}_\Psi) \boxtimes \zeta\big)$ *and let* $A \in \mathcal{A}(\mathcal{G})$ *be a direct summand of* $ind_{\Sigma}^{\mathcal{G}}(\mathcal{E})$. *Then* A *is of the form* $(m_{z(\mathcal{G})})^*(\mathcal{L}_\Psi)[\dim z(\mathcal{G})] \boxtimes \overline{A}$ *with* $\overline{A} \in \mathcal{A}(\overline{\mathcal{G}})$.

Proof: Follows from 5.1.49, 5.1.33 and 5.1.66. □

Definition 5.1.68. *Let $K \in \mathcal{M}_G(\mathcal{G})$, then we say that K is a* Lusztig complex *over $\mathcal{G}$ if it is of the form $ind_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ for some cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{G}$.*

Definition 5.1.69. *A* cuspidal orbital pair *of $\mathcal{G}$ is an orbital pair of $\mathcal{G}$ of the form $(\sigma + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ where $\sigma \in z(\mathcal{G})$ and (C, ζ) is a cuspidal nilpotent pair of $\mathcal{G}$. An orbital perverse sheaf on $\mathcal{G}$ is said to be* cuspidal *if it is of the form $K(\mathcal{O}, \xi)$ for some cuspidal orbital pair $(\mathcal{O}, \xi)$ of $\mathcal{G}$.*

Remark 5.1.70. Note that when G is semi-simple, the cuspidal admissible pairs of $\mathcal{G}$ and the cuspidal orbital pairs of $\mathcal{G}$ are all nilpotent orbital; thus in the case where G is semi-simple, we use simply the terminology "cuspidal pair". Similarly for complexes, when G is semi-simple, we use the terminology "cuspidal complex" instead of "cuspidal admissible complex" or "cuspidal orbital perverse sheaf".

Remark 5.1.71. Let K be an orbital perverse sheaf on $\mathcal{G}$. Note that K is cuspidal if and only if for any proper Levi decomposition $P = LU_P$ in G with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$, we have $(\pi_{\mathcal{P}})_!(K|_{\mathcal{P}}) = 0$.

5.1.72 Orbital Perverse Sheaves: The Fundamental Theorem

5.1.73. A *cuspidal datum* of G is a triple $(L, \Sigma', \mathcal{E}')$ where L is a Levi subgroup of G and where $(\Sigma', \mathcal{E}')$ is a cuspidal pair of L in the sense of [Lus84, Definition 2.4]. Let $(L, \Sigma', \mathcal{E}')$ be a cuspidal datum of G such that $\Sigma' = Z_L^o C'$ (with C' a unipotent class of L) and $\mathcal{E}' = \overline{\mathbb{Q}}_\ell \boxtimes \zeta'$ with $\zeta' \in ls_L(C')$; such a pair (C', ζ') will be called a *cuspidal unipotent pair* of L. As in [Lus85b, 7.1.7, 8.1.1], we construct a semi-simple perverse sheaf $\mathrm{ind}_{\Sigma'}^G(\mathcal{E}') \in \mathcal{M}_G(G)$, and for any parabolic subgroup P of G having L as a Levi subgroup we construct a complex $\mathrm{ind}_{L \subset P}^G K(\Sigma', \mathcal{E}')$; by [Lus84, Proposition 4.5], these two complexes are canonically isomorphic. These constructions are completely similar to what we have done in the Lie algebra case. From [Lus85b, 7], the pair (C, ζ) of G is a cuspidal unipotent pair of G if and only if for any Levi decomposition PU_P in G, we have $(\pi_P)_!\big(K(C, \zeta)|_P\big) = 0$ where $\pi_P : P \to L$ is the canonical morphism. Note that the Lie algebra version of this assertion is exactly the definition of cuspidal local systems on nilpotent orbits.

Definition 5.1.74. *(i) By a* unipotent complex *over G, we shall mean a perverse sheaf K of the form* $\left(ind_{\Sigma'}^{G}(\mathcal{E}')|_{G_{uni}}\right)[-dim Z_L]$ *extended by zero on $G - G_{uni}$ for some cuspidal datum $(L, \Sigma', \mathcal{E}')$ of G as above. If $K = K(C', \zeta')$ for some cuspidal unipotent pair (C', ζ') of G, then K is called a* cuspidal unipotent complex.

(ii) By a nilpotent complex *over $\mathcal{G}$, we shall mean a perverse sheaf K of the form $ind_{\Sigma}^{\mathcal{G}}(\mathcal{E})|_{\mathcal{G}_{nil}}$ shifted by $-dim z(\mathcal{L})$ and extended by zero on $\mathcal{G} - \mathcal{G}_{nil}$ for some cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{G}$ such that $\mathcal{E}$ is of $\mathcal{G}$ of the form $\overline{\mathbb{Q}}_\ell \boxtimes \zeta$ on $\Sigma = z(\mathcal{L}) + C$ where $\mathcal{L} = Lie(L)$. If $K = K(C, \zeta)$ for some nilpotent cuspidal pair (C, ζ) of $\mathcal{G}$, K is called a* cuspidal nilpotent complex.

Remark 5.1.75. By 5.1.33 and by 5.1.41 applied to $\sigma = 0$, we see that if $(L, \Sigma, \mathcal{E}_1) = \big(L, z(\mathcal{L}) + C, (m_1)^*\mathcal{L}_\Psi \boxtimes \zeta\big)$ and $(L, \Sigma, \mathcal{E}_2) = \big(L, z(\mathcal{L}) + C, (m_2)^*\mathcal{L}_\Psi \boxtimes \zeta\big)$ are two cuspidal data of $\mathcal{G}$ where $\mathcal{L} = \text{Lie}(L)$, then $\text{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}_1)|_{\mathcal{G}_{nil}}$ and $\text{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}_2)|_{\mathcal{G}_{nil}}$ are isomorphic; there are actually both isomorphic to $\text{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(C, \zeta)[\dim z(\mathcal{L})]$ where $\mathcal{P}$ is the Lie algebra of any parabolic subgroup containing L. Hence the complexes $\left(\text{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})|_{\mathcal{G}_{nil}}\right)[-\dim z(\mathcal{L})]$ extended by zero on $\mathcal{G} - \mathcal{G}_{nil}$, where $(L, \Sigma, \mathcal{E})$ runs over the cuspidal data of $\mathcal{G}$, are all nilpotent complexes. We have a similar result for groups, see [Lus85b].

Notation 5.1.76. We denote by $Nil(\mathcal{G})$ the set of nilpotent complexes over $\mathcal{G}$ and by $Uni(G)$ the set of unipotent complexes over G.

Notation 5.1.77. We denote by $\mathcal{D}_c^b(G)_{uni}$ the full subcategory of $\mathcal{D}_c^b(G)$ of unipotently supported complexes and by $\mathcal{D}_c^b(\mathcal{G})_{nil}$ the full subcategory of $\mathcal{D}_c^b(\mathcal{G})$ of nilpotently supported complexes. If f is a G-equivariant isomorphism $G_{uni} \to \mathcal{G}_{nil}$, we denote by $(f_o)^* : \mathcal{D}_c^b(\mathcal{G})_{nil} \to \mathcal{D}_c^b(G)_{uni}$ the functor induced by $f^* : \mathcal{D}_c^b(\mathcal{G}_{nil}) \to \mathcal{D}_c^b(G_{uni})$.

We have the following proposition.

Proposition 5.1.78. *Let $f : G_{uni} \to \mathcal{G}_{nil}$ be a G-equivariant isomorphism. Then the functor $(f_o)^* : \mathcal{D}_c^b(\mathcal{G})_{nil} \to \mathcal{D}_c^b(G)_{uni}$ induces a bijection $Nil(\mathcal{G}) \to Uni(G)$ mapping cuspidals onto cuspidals. More precisely, if $(L, z(\mathcal{L}) + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ is a cuspidal datum of $\mathcal{G}$ with $\mathcal{L} = Lie(L)$, then $(f_o)^*$ maps the nilpotent complex over $\mathcal{G}$ induced by $(L, z(\mathcal{L}) + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ onto the unipotent complex over G induced by the cuspidal datum $\big(L, Z_L^o.\left(f^{-1}(C)\right), \overline{\mathbb{Q}}_\ell \boxtimes f^*(\zeta)\big)$.*

Proof: From the characterization in 5.1.73 of cuspidal unipotent complexes and cuspidal nilpotent complexes, it is clear from 2.7.6, that, for any nilpotent pair (C, ζ) of $\mathcal{G}$, the complex $(f_o)^*(K(C, \zeta)) = K\big(f^{-1}(C), f^*\zeta\big)$ is cuspidal if and only if $K(C,\zeta)$ is cuspidal. Using again 2.7.6, we also have that the functor $(f_o)^*$ commutes with the parabolic induction of equivariant perverse sheaves. Hence if we write the elements of $Nil(\mathcal{G})$ and the elements of $Uni(G)$ in terms of parabolic induction (see 5.1.75), we get the required result. □

Remark 5.1.79. By 5.1.78, a Levi subgroup of G supports a cuspidal unipotent pair if and only if its Lie algebra supports a cuspidal nilpotent pair.

Theorem 5.1.80. *Let $(\mathcal{O},\xi)$ be a nilpotent pair of $\mathcal{G}$. Then there exists a unique (up to G-conjugacy) triple (L,C,ζ) such that L is a Levi subgroup of G and (C,ζ) is a cuspidal nilpotent pair of $\mathcal{L}=Lie(L)$, and such that the complex $K(\mathcal{O},\zeta)$ is a direct summand of $ind^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}K(C,\zeta)$ where $\mathcal{P}$ is the Lie algebra of a parabolic subgroup of G having L as a Levi subgroup.*

Proof: This follows from its group analogue [Lus84, Section 6] and 5.1.78. □

We have the following *fundamental theorem* for orbital complexes.

Theorem 5.1.81. *Let $(\mathcal{O},\mathcal{E})$ be any orbital pair of $\mathcal{G}$. Then there exists a unique (up to G-conjugacy) triple $(L,\mathcal{O}^L,\mathcal{E}^L)$ such that L is a Levi subgroup of G and $(\mathcal{O}^L,\mathcal{E}^L)$ is a cuspidal orbital pair of $\mathcal{L}=Lie(L)$, and such that the complex $K(\mathcal{O},\mathcal{E})$ is a direct summand of $ind^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}K(\mathcal{O}^L,\mathcal{E}^L)$ where $\mathcal{P}$ is the Lie algebra of a parabolic subgroup of G having L as a Levi subgroup.*

Proof: Let $x\in\mathcal{O}$ and let $M=C^o_G(x_s)$. We denote by $\mathcal{M}$ the Lie algebra of M and by $\mathcal{Q}$ the Lie algebra of a parabolic subgroup of G having M as a Levi subgroup. Let $\mathcal{E}_n$ be an irreducible M-equivariant local system on $\mathcal{O}^M_{x_n}$ such that $K(\mathcal{O},\mathcal{E})$ is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}K(\mathcal{O}^M_x,\overline{\mathbb{Q}}_\ell\boxtimes\mathcal{E}_n)$, see 5.1.40. By 5.1.80, there exists a cuspidal datum $\big(L,z(\mathcal{L})+C,\overline{\mathbb{Q}}_\ell\boxtimes\zeta\big)$ of $\mathcal{M}$ such that the complex $K(\mathcal{O}^M_{x_n},\mathcal{E}_n)$ is a direct summand of $\mathrm{ind}^{\mathcal{M}}_{\mathcal{L}\subset\mathcal{P}^M}K(C,\zeta)$ where $\mathcal{P}^M$ is the Lie algebra of a parabolic subgroup of M having L as a Levi subgroup. Hence by 5.1.50, the complex $K(\mathcal{O}^M_x,\overline{\mathbb{Q}}_\ell\boxtimes\mathcal{E}_n)$ is a direct summand of $\mathrm{ind}^{\mathcal{M}}_{\mathcal{L}\subset\mathcal{P}^M}K(x_s+C,\overline{\mathbb{Q}}_\ell\boxtimes\zeta)$. Hence it follows from the transitivity property of parabolic induction that the complex $K(\mathcal{O},\mathcal{E})$ is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}K(x_s+C,\overline{\mathbb{Q}}_\ell\boxtimes\zeta)$.

Let us now prove the unicity up to G-conjugacy. Assume that $(L',\sigma+C',\overline{\mathbb{Q}}_\ell\boxtimes\zeta')$ is another triple such that $(\sigma+C',\overline{\mathbb{Q}}_\ell\boxtimes\zeta')$ is a cuspidal orbital pair on $\mathcal{L}'=\mathrm{Lie}(L')$ and such that $K(\mathcal{O},\mathcal{E})$ is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}'\subset\mathcal{P}'}K(\sigma+C',\overline{\mathbb{Q}}_\ell\boxtimes\zeta')$. Then there exists $g\in G$ such that $\mathrm{Ad}(g)\sigma=x_s$. Hence $(\mathrm{Ad}(g)C',\mathrm{Ad}(g^{-1})^*\zeta')$ is a nilpotent cuspidal pair of $\mathrm{Ad}(g)\mathcal{L}'\subset\mathcal{M}$. Hence we may assume that $\sigma=x_s$ and that L' is a Levi subgroup of M. From the transitivity of induction, the orbital perverse sheaf $K(\mathcal{O},\mathcal{E})$ is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}\big(\mathrm{ind}^{\mathcal{M}}_{\mathcal{L}'\subset\mathcal{P}'^M} K(x_s + C', \overline{\mathbb{Q}}_\ell \boxtimes \zeta')\big)$. Hence there exists an M-equivariant irreducible local system $\mathcal{E}'_n$ on $\mathcal{O}^M_{x_n}$ such that $K(\mathcal{O}, \mathcal{E})$ is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}} K(\mathcal{O}^M_x, \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{E}'_n)$; note that if $C_G(x_s)$ is connected, by 5.1.39 we must have $\mathcal{E}'_n \simeq \mathcal{E}_n$. Now if we interpret the local systems $\mathcal{E}$, $\mathcal{E}_n$ and $\mathcal{E}'_n$ in terms of characters as in 5.1.38, then we see from a theorem of Clifford [Isa94, 6.2], that the local systems $\mathcal{E}_n$ and $\mathcal{E}'_n$ must be G-conjugate. As a consequence, we get that the complexes $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} K(C, \zeta)$ and $\mathrm{ind}^{\mathcal{G}}_{\mathcal{L}'\subset\mathcal{P}'} K(C', \zeta')$ have a common direct summand. Hence from 5.1.80 we get that the triples (L, C, ζ) and (L', C', ζ') are G-conjugate. Since L and L' are two Levi subgroups of M, from the classification of cuspidal data [Lus84] (see also [DLM97, 1.2]), we get that (L, C, ζ) and (L', C', ζ') are M-conjugate from which we deduce that $(L, x_s + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ and $(L', x_s + C', \overline{\mathbb{Q}}_\ell \boxtimes \zeta')$ are M-conjugate. □

Recall that, by 4.2.5 and 4.2.10, the group Z_G/Z^o_G acts on any G-equivariant perverse sheaf on $\mathcal{G}$.

Proposition 5.1.82. *Assume that G is semi-simple and let $\chi : Z_G \to \overline{\mathbb{Q}}_\ell^\times$ be a character of Z_G. Then there exists at most one cuspidal complex on $\mathcal{G}$ on which Z_G acts by χ.*

Proof: The group version of 5.1.82 is known from [Lus84]. Hence the above proposition follows from 5.1.78. □

Proposition 5.1.83. *Assume that G is semi-simple and let $(\mathcal{O}, \xi)$ be a cuspidal pair of $\mathcal{G}$. Then the cuspidal complex $K(\mathcal{O}, \xi)$ is clean, that is its restriction to $\overline{\mathcal{O}} - \mathcal{O}$ is zero.*

Proof: Follows from its group version [Lus86a, 23.1(a)] and from 5.1.78. □

5.2 Deligne-Fourier Transforms and Admissible Complexes

After recalling the definition and the properties of Deligne-Fourier transforms, we expound the main result of [Lus87] and we verify that its proof works for p acceptable; recall that in [Lus87] the characteristic is assumed to be large.

Notation 5.2.1. Let μ denote the non-degenerate G-invariant bilinear form on $\mathcal{G}$ fixed in 3.1.9. If $\mathcal{H}$ and $\mathcal{H}'$ are two Lie subalgebras of $\mathcal{G}$, we denote by $\mu_{\mathcal{H}\times\mathcal{H}'}$ the restriction of μ to $\mathcal{H} \times \mathcal{H}'$; if $\mathcal{H} = \mathcal{H}'$ we write simply $\mu_{\mathcal{H}}$. If L is a Levi subgroup of G with $\mathcal{L} = \mathrm{Lie}(L)$, and if $\sigma \in z(\mathcal{L})$, then we denote by $m_\sigma : z(\mathcal{L}) \to k$ the k-linear form given by $z \mapsto \mu(z, \sigma)$. In order to simplify the notation and since the context should be always clear, we omit the Levi subgroup from the notation m_σ. In view of 5.0.14 (iv), 5.1.79 and 2.5.16, any k-linear form on $z(\mathcal{L})$ is of the form m_σ for some $\sigma \in z(\mathcal{L})$.

5.2.2. Let $\mathcal{H}$ be a Lie subalgebra of $\mathcal{G}$ such that $\mu_{\mathcal{H}}$ is non-degenerate. The Deligne-Fourier transform $\mathcal{F}^{\mathcal{H}} : \mathcal{D}_c^b(\mathcal{H}) \to \mathcal{D}_c^b(\mathcal{H})$ with respect to $(\mu_{\mathcal{H}}, \Psi)$, where $\Psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell$ is the non-trivial additive character fixed in 5.1.57, is defined as follows (see [Bry86, 9.1] [KL85, 2.1]).

$$\mathcal{F}^{\mathcal{H}}(K) = (pr_2)_!\big((pr_1)^* K \otimes (\mu_{\mathcal{H}})^* \mathcal{L}_\Psi\big)[\dim \mathcal{H}]$$

where $pr_1, pr_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ are the two projections and $\mathcal{L}_\Psi$ is the one-dimensional local system on $\mathbb{A}^1$ defined in 5.1.57.

By abuse of notation, we use the symbol $\mathcal{F}^{\mathcal{H}}$ to denote both the Deligne-Fourier transform and the Fourier transform of functions (see 5.2.2 and 3.1.9); this abuse of notation is justified by the following statement (see [Bry86, 9.2]) which relates Deligne-Fourier transforms of complexes with Fourier transforms of functions.

5.2.3. *Let $\mathcal{H}$ be as in 5.2.2. Assume that $\mathcal{H}$ is F-stable. Let $K \in \mathcal{D}_c^b(\mathcal{H})$ be F-stable and let $\phi : F^*(K) \xrightarrow{\sim} K$ be an isomorphism. Then ϕ induces a canonical isomorphism $\mathcal{F}(\phi) : F^*(\mathcal{F}^{\mathcal{H}} K) \xrightarrow{\sim} \mathcal{F}^{\mathcal{H}} K$ such that*

$$X_{\mathcal{F}^{\mathcal{H}}(K), \mathcal{F}(\phi)} = (-1)^{dim\, \mathcal{H}} |\mathcal{H}^F|^{\frac{1}{2}} \mathcal{F}^{\mathcal{H}}(X_{K,\phi}).$$

The proof of 5.2.3 involves the Grothendieck trace formula applied to the F-equivariant complex $\big(\mathcal{F}^{\mathcal{H}}(K), \mathcal{F}(\phi)\big)$ where $\mathcal{F}(\phi)$ is the isomorphism induced by ϕ and the isomorphism $\phi_{\mathcal{L}_\Psi} : F^*(\mathcal{L}_\Psi) \xrightarrow{\sim} \mathcal{L}_\Psi$ of 5.1.57.

5.2.4. Let $\mathcal{H}$ be a Lie subalgebra of $\mathcal{G}$. Following [Bry86, 6], we define the *convolution product* on $\mathcal{D}_c^b(\mathcal{H})$ as follows.

For $K, K' \in \mathcal{D}_c^b(\mathcal{H})$,

$$K * K' := s_!(K \boxtimes K')$$

where $s : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is given by the addition on $\mathcal{H}$.

The following result (see [Bry86, 9.3, 9.6]) is the geometric version of 3.1.10 (ii), (iv), (v).

Lemma 5.2.5. *Let $\mathcal{H}$ be as in 5.2.2. Let $K, K' \in \mathcal{D}_c^b(\mathcal{H})$ and $\mathcal{F} = \mathcal{F}^{\mathcal{H}}$.*

(i) Let $a : \mathcal{H} \to \mathcal{H}$, $x \mapsto -x$, we have an isomorphism

$$\mathcal{F} \circ \mathcal{F}(K) \simeq a^*(K)(-dim\mathcal{H})$$

such that if $\mathcal{H}$ is F-stable and K is an F-equivariant perverse sheaf , then the above isomorphism is an isomorphism of F-equivariant perverse sheaves,

(ii) We have an isomorphism $\mathcal{F}(K * K') \simeq (\mathcal{F}K \otimes \mathcal{F}K')[dim\mathcal{H}]$. Moreover if $\mathcal{H}$ is F-stable and K, K' are F-equivariant perverse sheaves, the above isomorphism is an isomorphism of F-equivariant perverse sheaves.

(iii) We have an isomorphism $\mathcal{F}(K \otimes K') \simeq (\mathcal{F}K * \mathcal{F}K')[dim\mathcal{H}]$. Moreover if $\mathcal{H}$ is F-stable and K, K' are F-equivariant perverse sheaves, the above isomorphism is an isomorphism of F-equivariant perverse sheaves.

Remark 5.2.6. The assertion (iii) of 5.2.5 is not in [Bry86], however it can be easily deduced from (i) and (ii) as this done in the proof of [Leh96, Lemma 4.2] with functions.

We have the following important result (see [Bry86, Corollary 9.11]).

Theorem 5.2.7. *If $K \in \mathcal{D}_c^b(\mathcal{G})$ is a perverse sheaf on $\mathcal{G}$, then $\mathcal{F}^{\mathcal{G}}(K)$ is also a perverse sheaf on $\mathcal{G}$. The functor $\mathcal{F}^{\mathcal{G}} : \mathcal{M}_G(\mathcal{G}) \to \mathcal{M}_G(\mathcal{G})$ is an equivalence of categories. In particular, it permutes the simple G-equivariant perverse sheaves on $\mathcal{G}$.*

The following result which is the geometric version of 3.1.11(i).

Theorem 5.2.8. *[Hen01, Theorem 4.3] [Wal01, II.8(2)] Let $P = LU_P$ be a Levi decomposition in G and $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be its corresponding Lie algebra decomposition. Let $A \in \mathcal{M}_L(\mathcal{L})$, then we have an isomorphism*

$$\mathcal{F}^{\mathcal{G}}(ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} A) \simeq ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(\mathcal{F}^{\mathcal{L}} A)(-dim\mathcal{U}_P).$$

If P, L are F-stable and A is an F-equivariant perverse sheaf, then $\mathcal{F}^{\mathcal{G}}$ $(ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} A)$ and $ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(\mathcal{F}^{\mathcal{L}} A)$ are naturally F-equivariant and the above isomorphism is an isomorphism of F-equivariant perverse sheaves.

We now state the main result of [Lus87].

Theorem 5.2.9. *[Lus87, Theorem 5] Let $A \in \mathcal{M}_G(\mathcal{G})$.*

(a) The complex A is admissible if and only if it is the Deligne-Fourier transform of some orbital perverse sheaf on $\mathcal{G}$.

(b) If G is semi-simple and A is cuspidal, then $\mathcal{F}^{\mathcal{G}}(A) \simeq A$.

The following result is used implicitly in the proof of [Lus87, Theorem 5].

Theorem 5.2.10. *Let* $(L,\Sigma,\mathcal{E})=\big(L,z(\mathcal{L})+C,(m_{-\sigma})^*\mathcal{L}_\Psi\boxtimes\zeta\big)$ *be a cuspidal datum of* $\mathcal{G}$ *where* $\mathcal{L}=Lie(L)$ *and* $\sigma\in z(\mathcal{L})$, *and let* P *be a parabolic subgroup of* G *having* L *as a Levi subgroup. Let* $\mathcal{P}=\mathcal{L}\oplus\mathcal{U}_P$ *be the Lie algebra decomposition corresponding to* $P=LU_P$. *Then, there is an isomorphism*

$$\mathcal{F}^{\mathcal{G}}\left(ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\Sigma,\mathcal{E})\right)\simeq ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K(\sigma+C,\overline{\mathbb{Q}}_\ell\boxtimes\zeta).$$

5.2.11. We are going to prove the implications 5.2.9(b) $\Rightarrow$ 5.2.10 $\Rightarrow$ 5.2.9(a). The implication 5.2.10 $\Rightarrow$ 5.2.9(b) being trivial, this will prove that 5.2.10 is equivalent to 5.2.9(b). The proof of those implications in [Lus87] is very dense and does not show the problems related to the characteristic, moreover Lusztig comes down to problems on functions while thanks to 5.2.8, it is possible to work directly with perverse sheaves.

- We first prove the implication 5.2.9(b) $\Rightarrow$ 5.2.10. Let $(L,\Sigma,\mathcal{E})$ be as in 5.2.10. Thanks to 5.2.8, we are reduced to show that $\mathcal{F}^{\mathcal{L}}\left(K(\Sigma,\mathcal{E})\right)\simeq K(\sigma+C,\overline{\mathbb{Q}}_\ell\boxtimes\zeta)$. Hence we may assume that G supports a cuspidal admissible pair and that $L=G$. To simplify the notation, we put $m=m_{-\sigma}$. Let $K=K(\Sigma,\mathcal{E})$, then $K=m^*\mathcal{L}_\Psi[\dim z(\mathcal{G})]\boxtimes K^{\overline{\mathcal{G}}}(C,\zeta)\in\mathcal{M}(z(\mathcal{G})\oplus\overline{\mathcal{G}})$.

By 2.5.16, the bilinear forms $\mu_{\overline{\mathcal{G}}}$ and $\mu_{z(\mathcal{G})}$ are non-degenerate; hence the Deligne-Fourier transforms $\mathcal{F}^{\overline{\mathcal{G}}}:\mathcal{D}_c^b(\overline{\mathcal{G}})\to\mathcal{D}_c^b(\overline{\mathcal{G}})$ and $\mathcal{F}^{z(\mathcal{G})}:\mathcal{D}_c^b(z(\mathcal{G}))\to\mathcal{D}_c^b(z(\mathcal{G}))$ are well-defined. We have

5.2.12.
$$\mathcal{F}^{\mathcal{G}}(K)=\mathcal{F}^{z(\mathcal{G})}(m^*\mathcal{L}_\Psi[\dim z(\mathcal{G})])\boxtimes\mathcal{F}^{\overline{\mathcal{G}}}(K^{\overline{\mathcal{G}}}(C,\zeta)).$$

Indeed, let $s:\mathbb{A}_1\times\mathbb{A}_1\to\mathbb{A}_1$ be given by the addition on k, we have $s^*(\mathcal{L}_\Psi)=\mathcal{L}_\Psi\boxtimes\mathcal{L}_\Psi$. On the other hand, the form μ decomposes as $\mu=s\circ(\mu_{z(\mathcal{G})}\times\mu_{\overline{\mathcal{G}}})$ since by 2.5.16, the subspaces $\overline{\mathcal{G}}$ and $z(\mathcal{G})$ of $\mathcal{G}$ are orthogonal with respect to μ. We deduce that $\mu^*\mathcal{L}_\Psi=(\mu_{z(\mathcal{G})})^*\mathcal{L}_\Psi\boxtimes(\mu_{\overline{\mathcal{G}}})^*\mathcal{L}_\Psi$ from which 5.2.12 follows.

In view of 5.2.12 and 5.2.9 (b), and since $K^{\mathcal{G}}(\sigma+C,\overline{\mathbb{Q}}_\ell\boxtimes\zeta)=K^{z(\mathcal{G})}(\sigma,\overline{\mathbb{Q}}_\ell)\boxtimes K^{\overline{\mathcal{G}}}(C,\zeta)\in\mathcal{M}(z(\mathcal{G})\boxtimes\overline{\mathcal{G}})$, note that $K^{z(\mathcal{G})}(\sigma,\overline{\mathbb{Q}}_\ell)$ is the constant sheaf on $\{\sigma\}$ extended by zero on $z(\mathcal{G})-\{\sigma\}$, it remains to see that

5.2.13.
$$\mathcal{F}^{z(\mathcal{G})}(m^*\mathcal{L}_\Psi[\dim z(\mathcal{G})])\simeq K^{z(\mathcal{G})}(\sigma,\overline{\mathbb{Q}}_\ell).$$

We prove that $m^*\mathcal{L}_\Psi[\dim z(\mathcal{G})] \simeq \mathcal{F}^{z(\mathcal{G})}\big(K^{z(\mathcal{G})}(-\sigma,\overline{\mathbb{Q}}_\ell)\big)$; then 5.2.13 will be obtained by applying $\mathcal{F}^{z(\mathcal{G})}$ to this isomorphism. Let $pr_1, pr_2 : z(\mathcal{G})\times z(\mathcal{G}) \to z(\mathcal{G})$ be the two projections. Note that $(pr_1)^*K^{z(\mathcal{G})}(-\sigma,\overline{\mathbb{Q}}_\ell)$ is the extension of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\{-\sigma\}\times z(\mathcal{G})$ by zero on $\big(z(\mathcal{G})\times z(\mathcal{G})\big)-\big(\{-\sigma\}\times z(\mathcal{G})\big)$. Hence the complex $(pr_1)^*\big(K^{z(\mathcal{G})}(-\sigma,\overline{\mathbb{Q}}_\ell)\big)\otimes(\mu_{z(\mathcal{G})})^*\mathcal{L}_\Psi$ is the extension of $\overline{\mathbb{Q}}_\ell\boxtimes m^*\mathcal{L}_\Psi \in \mathcal{S}h\big(\{-\sigma\}\times z(\mathcal{G})\big)$ by zero on $\big(z(\mathcal{G})\times z(\mathcal{G})\big)-\big(\{-\sigma\}\times z(\mathcal{G})\big)$, from which we deduce that $(pr_2)_!\,\big((pr_1)^*\big(K^{z(\mathcal{G})}(-\sigma,\overline{\mathbb{Q}}_\ell)\big)\otimes(\mu_{z(\mathcal{G})})^*\mathcal{L}_\Psi\big) = m^*\mathcal{L}_\Psi$.

We have proved the implication 5.2.9(b) $\Rightarrow$ 5.2.10.

• Let us now prove the implication 5.2.10 $\Rightarrow$ 5.2.9(a). From 5.2.10, 5.2.7 and 5.1.37, we see that the Deligne-Fourier transform of an admissible complex of $\mathcal{G}$ is an orbital perverse sheaf on $\mathcal{G}$. Hence by 5.2.5 (i), we deduce that any admissible complex on $\mathcal{G}$ is the Deligne-Fourier transform of some orbital perverse sheaf on $\mathcal{G}$. Conversely, if K is an orbital perverse sheaf on $\mathcal{G}$, then by 5.1.81, it is the direct summand of the parabolic induction of some cuspidal orbital perverse sheaf. From 5.2.8, it is thus enough to see that the Deligne-Fourier transform of a cuspidal orbital perverse sheaf is a cuspidal admissible complex. From 5.2.10, it is clear that a cuspidal orbital perverse sheaf is the Deligne-Fourier transform of a cuspidal admissible complex, hence using 5.2.5(i) together with 5.1.64, we see that the Deligne-Fourier transform of a cuspidal orbital perverse sheaf is an admissible complex. □

Since by 5.1.64 the functor $a_{\mathcal{G}}^*$ permutes the non-cuspidal admissible complexes on $\mathcal{G}$, we have in fact proved in the last paragraph that:

5.2.14. *If 5.2.10 holds whenever $L \subsetneq G$, and if K is a non-cuspidal orbital perverse sheaf on $\mathcal{G}$, then $\mathcal{F}^{\mathcal{G}}(K)$ is admissible non-cuspidal.*

Proof of 5.2.9, 5.2.10:

From 5.2.11, we are reduced to proving the assertion 5.2.9(b). We now sketch Lusztig's proof of 5.2.9(b). We assume G semi-simple and let K be a cuspidal complex on $\mathcal{G}$. We first assume that the following result is true.

5.2.15. *The complex $\mathcal{F}^{\mathcal{G}}(K)$ is nilpotently supported.*

5.2.16. We now prove 5.2.9(b) by induction on $\dim G$. We thus assume that 5.2.9(b) is true for any simple algebraic group of dimension $< \dim G$. Hence 5.2.14 holds for any $L \subsetneq G$ (see 5.2.11). Since $\mathcal{F}^{\mathcal{G}}(K)$ is a G-equivariant simple nilpotently supported perverse sheaf on $\mathcal{G}$, it is orbital by 4.2.14. Hence $\mathcal{F}^{\mathcal{G}}(K)$

must be cuspidal. Indeed, if not, then by 5.2.14, the complex $\mathcal{F}^{\mathcal{G}}\big(\mathcal{F}^{\mathcal{G}}(K)\big) \simeq a_{\mathcal{G}}^*(K)$ is admissible non-cuspidal which is in contradiction with 5.1.64. Since the Deligne-Fourier transforms preserve the action of Z_G on K, by 5.1.82, we must have $\mathcal{F}^{\mathcal{G}}(K) \simeq K$.

5.2.17. It remains to prove 5.2.15. We proceed by induction on $\dim G$ in order to apply 5.2.14. We consider the notation of 4.4.13 with $H = G$ and $X = \mathcal{G}$. We denote by I_o the subset of I consisting of the cuspidal pairs of $\mathcal{G}$. Since G is semi-simple, the cuspidal pairs are nilpotent and so I_o is finite. We assume that q is large enough so that $I_o^F = I_o$. Note that by 5.1.83, for $\iota_o \in I_o$, the nilpotently supported functions $\mathcal{Y}_{\iota_o}$ and $\mathcal{X}_{\iota_o}$ are proportional. Using a G-equivariant homeomorphism $G_{uni} \rightarrow \mathcal{G}_{nil}$ defined over $\mathbb{F}_q$, we can transfer [Lus86a, 24.4(d)] to the Lie algebra setting and we thus deduce that

$$\big(\mathcal{X}_{\iota_o}, \mathcal{Y}_{\iota}\big) = 0 \qquad \text{for } \iota_o \in I_o, \iota \in I^F - I_o, \tag{*}$$

where the non-degenerate bilinear form $(,)$ on $\mathcal{C}(\mathcal{G}^F)$ is defined by $(f, f') = \sum_{x \in \mathcal{G}^F} f(x)f'(x)$. Applying again [Lus86a, 24.4(d)], we get that the functions $\mathcal{X}_{\iota}|_{\mathcal{G}_{nil}^F}$ with $\iota \in I^F - I_o$ are linear combinations of the functions $\{\mathcal{Y}_{\iota} |\, \iota \in I^F - I_o\}$, hence by (*), we deduce that the space spanned by the functions $\{\mathcal{X}_{\iota_o} |\, \iota_o \in I_o\}$ is the orthogonal complement of the space spanned by $\{\mathcal{X}_{\iota} |\, \iota \in I^F - I_o\}$. Now by applying the induction hypothesis we deduce from 5.2.16 that 5.2.9(b) is true for any group of dimension $< \dim G$, hence by 5.2.14, we get that the complexes $\mathcal{F}^{\mathcal{G}}(K_{\iota})$ with $\iota \in I^F - I_o$ are admissible non-cuspidal. We deduce from [Lus86a, 24.4(d)], that the functions $\mathcal{F}^{\mathcal{G}}(\mathcal{X}_{\iota})|_{\mathcal{G}_{nil}^F}$ with $\iota \in I^F - I_o$, are linear combinations of the functions $\mathcal{Y}_{\iota}$ with $\iota \in I^F - I_o$. Since $(,)$ is preserved by $\mathcal{F}^{\mathcal{G}}$ up to a scalar, it follows from (*) that the space spanned by $\{\mathcal{F}^{\mathcal{G}}(\mathcal{X}_{\iota_o}) |\, \iota_o \in I_o\}$ is the orthogonal complement of the space spanned by $\{\mathcal{X}_{\iota} |\, \iota \in I^F - I_o\}$, from which we deduce that the functions $\{\mathcal{F}^{\mathcal{G}}(\mathcal{X}_{\iota_o}) |\, \iota_o \in I_o\}$ and $\{\mathcal{X}_{\iota_o} |\, \iota_o \in I_o\}$ span the same subspace of $\mathcal{C}(\mathcal{G}^F)$. Hence the Fourier transforms of the functions $\mathcal{X}_{\iota_o}$ with $\iota_o \in I_o$ are nilpotently supported. The last assertion being true for any Frobenius F^n with $n > 1$, we deduce from 4.4.9, that $\mathcal{F}^{\mathcal{G}}(K)$ is nilpotently supported for any cuspidal complex K on $\mathcal{G}$. □

Corollary 5.2.18. *Let A be an admissible complex and let $(L, \Sigma, \mathcal{E})$ and $(L', \Sigma', \mathcal{E}')$ be two cuspidal data of $\mathcal{G}$ such that A is a direct summand of both $ind_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ and $ind_{\Sigma'}^{\mathcal{G}}(\mathcal{E}')$. Then $(L, \Sigma, \mathcal{E})$ and $(L', \Sigma', \mathcal{E}')$ are G-conjugate.*

Proof: Put $(L, \Sigma, \mathcal{E}) = \big(L, z(\mathcal{L}) + C, (m_\sigma)^*\mathcal{L}_\Psi \boxtimes \zeta\big)$ and $(L', \Sigma', \mathcal{E}') = \big(L', z(\mathcal{L}') + C', (m_{\sigma'})^*\mathcal{L}_\Psi \boxtimes \zeta'\big)$. We have to show that $(L, \sigma + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ and

$(L', \sigma' + C', \overline{\mathbb{Q}}_\ell \boxtimes \zeta')$ are G-conjugate. But taking the Deligne-Fourier transform of A, $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}) \simeq \mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})$ and $\mathrm{ind}_{\Sigma'}^{\mathcal{G}}(\mathcal{E}') \simeq \mathrm{ind}_{\mathcal{L}'\subset\mathcal{P}'}^{\mathcal{G}} K(\Sigma', \mathcal{E}')$, we see that this is a consequence of 5.2.10 together with 5.1.81. □

Corollary 5.2.19. *Let $(L, \Sigma, \mathcal{E})$ be as in 5.2.10. We assume that $M = C_G(\sigma)$ is connected (recall that this is always true if p is not a torsion prime for G). Let Q be a parabolic subgroup of G having M as a Levi subgroup and let $\mathcal{Q} = \mathcal{M} \oplus \mathcal{U}_Q$ be the Lie algebra decomposition corresponding to $Q = MU_Q$.*

(i) The functor $\mathrm{ind}_{\mathcal{M}\subset\mathcal{Q}}^{\mathcal{G}}$ induces a bijection between the admissible complexes which are direct summands of $\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})$ and the admissible complexes which are direct summands of $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})$.

(ii) Let A be an admissible complex on $\mathcal{G}$ which is a direct summand of $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})$ and let B be an admissible complex on $\mathcal{M}$ such that $A = \mathrm{ind}_{\mathcal{M}\subset\mathcal{Q}}^{\mathcal{G}}(B)$. Then, if $\mathcal{F}^{\mathcal{G}}(A) = K(\mathcal{O}_x^G, \xi)$ with $x_s = \sigma$, we have $\mathcal{F}^{\mathcal{M}}(B) = K(\mathcal{O}_x^M, \xi|_{\mathcal{O}_x^M})$; in particular such an admissible complex B is unique (up to isomorphism) and so, by (i), must be a direct summand of $\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})$.

(iii) Assume that $(L, \Sigma, \mathcal{E})$ is F-stable, then the bijection of (i) induces a bijection between the two subsets consisting of the F-stable objects.

Proof: Let B be an admissible complex which is a direct summand of $\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})$. By 5.2.9(a), the complex B is of the form $\mathcal{F}^{\mathcal{M}}(K(\mathcal{O}, \xi))$ for some orbital pair $(\mathcal{O}, \xi)$ of $\mathcal{M}$ and by 5.2.10, we have $\mathcal{O} = \sigma + \mathcal{O}_n$ where $\mathcal{O}_n$ is the orbit formed by the nilpotent elements of $\mathcal{O}$. Hence we deduce from 5.2.8, 5.1.39 and 5.2.9(a) that $\mathrm{ind}_{\mathcal{M}\subset\mathcal{Q}}^{\mathcal{G}}(B)$ is an admissible complex on $\mathcal{G}$ and that the map induced by $\mathrm{ind}_{\mathcal{M}\subset\mathcal{Q}}^{\mathcal{G}}$ from the set of admissible complexes which are direct summand of $\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})$ to the set of admissible complex on $\mathcal{G}$ is injective. Since $\mathrm{ind}_{\mathcal{M}\subset\mathcal{Q}}^{\mathcal{G}}\big(\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})\big) \simeq \mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})$, we get the assertion (i). The assertion (ii) is a straightforward consequence of 5.2.9(a), 5.2.8, 5.1.39. Now, if σ and B are F-stable, then the complex $K(\mathcal{O}, \xi)$ is also F-stable and so, from 5.1.33 applied to $(M, \sigma + \mathcal{O}_n, \xi)$ and from the remark at the beginning of 5.1.51, we see that the complex $\mathrm{ind}_{\mathcal{M}\subset\mathcal{Q}}^{\mathcal{G}}\big(K(\mathcal{O}, \xi)\big)$ is also F-stable. Applying 5.2.8, we get that $\mathrm{ind}_{\mathcal{M}\subset\mathcal{Q}}^{\mathcal{G}}(B)$ is F-stable. Hence the map of (i) induces a well-defined map on the F-stable objects; this map is surjective by (ii). □

Remark 5.2.20. With the notation and assumption of 5.2.19, we see that the endomorphism algebra of $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})$ is canonically isomorphic to that of $\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})$.

5.2.21. We use the notation of 4.4.13 for $H = G$ and $X = \mathcal{G}$. Then by 5.2.9(a), the set $I(\mathcal{G}) := I$ of 4.4.13 parametrizes the isomorphic classes of the admissible complexes on $\mathcal{G}$ and $I(\mathcal{G})^F := I^F$ parametrizes the isomorphic classes of the F-stable admissible complexes. For $\iota \in I(\mathcal{G})$, we denote by A_ι the admissible complex $\mathcal{F}^{\mathcal{G}}(K_\iota)$ and for $\iota \in I(\mathcal{G})^F$, we choose an isomorphism $\phi_{A_\iota} : F^*(A_\iota) \xrightarrow{\sim} A_\iota$. Then we have the following proposition which is the Lie algebra analogue of [Lus86a, 25.2].

Proposition 5.2.22. *The set* $\{\mathbf{X}_{A_\iota,\phi_{A_\iota}} \mid \iota \in I(\mathcal{G})^F\}$ *is a basis of* $\mathcal{C}(\mathcal{G}^F)$.

Proof: This follows from 4.4.13, 5.2.3 and the easy fact that $\mathcal{F}^{\mathcal{G}}$ transforms a basis of $\mathcal{C}(\mathcal{G}^F)$ into a basis of $\mathcal{C}(\mathcal{G}^F)$. □

5.3 Endomorphism Algebra of Lusztig Complexes

Let $(L, \Sigma, \mathcal{E}) = (L, z(\mathcal{L}) + C, m^*\mathcal{L}_\Psi \boxtimes \zeta)$ with $\mathcal{L} = \mathrm{Lie}(L)$ be a cuspidal datum of $\mathcal{G}$. Let $\sigma \in z(\mathcal{L})$ be such that $m = m_\sigma$ where m_σ is as in 5.2.1. We use the notation of 5.1.26 relatively to $(L, \Sigma, \mathcal{E})$.

Let

$$N_G(\mathcal{E}) := \{n \in N_G(L) \mid \mathrm{Ad}(n)\Sigma = \Sigma,\ \mathrm{Ad}(n)^*(\mathcal{E}) \simeq \mathcal{E}\}$$

and let $\mathcal{W}_G(\mathcal{E})$ be the finite group $N_G(\mathcal{E})/L$.

5.3.1. Following [Lus84] and [Lus85b, 10.2], we are going to describe the endomorphism algebra $\mathcal{A} := \mathrm{End}(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}))$ in terms of $\mathcal{W}_G(\mathcal{E})$. Let $w \in \mathcal{W}_G(\mathcal{E})$ and let $\delta_w : Y_2 \xrightarrow{\sim} Y_2$ be the isomorphism defined by $\delta_w(x, gL) = (x, g\dot{w}^{-1}L)$ where $\dot{w}$ denotes a representative of w in $N_G(\mathcal{E})$; the map δ_w does not depend on the choice of the representative $\dot{w}$ of w. We have the following cartesian diagram.

$$\begin{array}{ccccccc}
\Sigma & \xleftarrow{\alpha} & Y_1 & \xrightarrow{\alpha'} & Y_2 & \xrightarrow{\alpha''} & Y \\
{\scriptstyle \mathrm{Ad}(\dot{w})}\downarrow & & {\scriptstyle f_{\dot{w}}}\downarrow & & {\scriptstyle \delta_w}\downarrow & & \| \downarrow \\
\Sigma & \xleftarrow{\alpha} & Y_1 & \xrightarrow{\alpha'} & Y_2 & \xrightarrow{\alpha''} & Y
\end{array}$$

where $f_{\dot{w}}(x, g) = (x, g\dot{w}^{-1})$. From the above diagram we see that any isomorphism $\mathrm{Ad}(\dot{w})^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ induces a canonical isomorphism $\delta_w^*(\xi_2) \xrightarrow{\sim} \xi_2$. Conversely, since $\alpha : Y_1 \to \Sigma_{reg}$ is a trivial principal G-bundle if we let G act on Y_1 by left translation on both coordinates and on Σ_{reg} trivially, the functor $\alpha^* : \mathcal{S}h(\Sigma_{reg}) \to \mathcal{S}h_G(Y_1)$ is an equivalence of categories and so any isomorphism $\delta_w^*(\xi_2) \simeq \xi_2$ defines a unique isomorphism $\mathrm{Ad}(\dot{w})^*(\mathcal{E}) \simeq \mathcal{E}$. Since the local system ξ_2 is irreducible, the $\overline{\mathbb{Q}}_\ell$-vector space $\mathcal{A}_w$ of all homomorphisms $\delta_w^*(\xi_2) \to \xi_2$ is one-dimensional.

For each $w \in \mathcal{W}_G(\mathcal{E})$, we choose a non-zero element θ_w of $\mathcal{A}_w$. Note that for $w, w' \in \mathcal{W}_G(\mathcal{E})$, we have $\delta_w \circ \delta_{w'} = \delta_{ww'}$. Hence for any $w, w' \in \mathcal{W}_G(\mathcal{E})$, we have $\theta_{w'} \circ \delta_{w'}^*(\theta_w) \in \mathcal{A}_{ww'}$. We thus have a well-defined product on $\bigoplus_{w \in \mathcal{W}_G(\mathcal{E})} \mathcal{A}_w$ given by $\theta_w.\theta_{w'} := \theta_{w'} \circ \delta_{w'}^*(\theta_w)$. This makes $\bigoplus_{w \in \mathcal{W}_G(\mathcal{E})} \mathcal{A}_w$ into a $\overline{\mathbb{Q}}_\ell$-algebra.

Using $\alpha''_* \circ \delta_w^* = \alpha''_*$ we identify $\mathcal{A}_w$ with a subspace of $\mathcal{A}$. Then as in [Lus84, Proposition 3.5], we show that $\bigoplus_{w \in \mathcal{W}_G(\mathcal{E})} \mathcal{A}_w = \mathcal{A}$ as $\overline{\mathbb{Q}}_\ell$-algebras. If $\phi : \mathrm{Ad}(\dot{w})^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is an isomorphism, we denote by $\theta_w(\phi)$ the element of $\mathcal{A}_w$ induced by ϕ. From the previous discussion, the map $\mathrm{Hom}(\mathrm{Ad}(\dot{w})^*\mathcal{E}, \mathcal{E}) \to \mathcal{A}_w$, $\phi \mapsto \theta_w(\phi)$ is an isomorphism of $\overline{\mathbb{Q}}_\ell$-vector spaces.

5.3.2. We fix $w \in \mathcal{W}_G(\mathcal{E})$ together with a representative $\dot{w}$ of w in $N_G(\mathcal{E})$. Let P be a parabolic subgroup of G having L as a Levi subgroup and let $P^w = \dot{w}P\dot{w}^{-1}$. Let $\mathcal{P}$ and $\mathcal{P}^w$ be the respective Lie algebras of P and P^w. Let $(X_1^w, X_2^w, \rho_w, \rho'_w, \rho''_w)$ be defined in terms of (P^w, L, Σ) as $(X_1, X_2, \rho, \rho', \rho'')$ is defined in terms of (P, L, Σ), and let $K_2^w \in \mathcal{M}(X_2^w)$ be the analogue of K_2, see 5.1.9. We have the following cartesian diagram.

5.3.3.

$$\begin{array}{ccccccc}
\overline{\Sigma} & \xleftarrow{\rho} & X_1 & \xrightarrow{\rho'} & X_2 & \xrightarrow{\rho''} & \overline{Y} \\
{\scriptstyle \mathrm{Ad}(\dot{w})}\big\downarrow & & {\scriptstyle \tilde{f}_{\dot{w}}}\big\downarrow & & {\scriptstyle \tilde{\delta}_w}\big\downarrow & & \|\big\downarrow \\
\overline{\Sigma} & \xleftarrow{\rho_w} & X_1^w & \xrightarrow{\rho'_w} & X_2^w & \xrightarrow{\rho''_w} & \overline{Y}
\end{array}$$

where $\tilde{f}_{\dot{w}} : (x, g) \mapsto (x, g\dot{w}^{-1})$ extends the map $f_{\dot{w}} : Y_1 \to Y_1$ and where $\tilde{\delta}_w : (x, gP) \mapsto (x, gP^w)$. Let $\phi : \mathrm{Ad}(\dot{w})^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ be an isomorphism; it induces an isomorphism $\mathrm{Ad}(\dot{w})^*K(\Sigma, \mathcal{E}) \simeq K(\Sigma, \mathcal{E})$ which by 5.3.3 induces a canonical isomorphism $h_w(\phi) : (\rho''_w)_!K_2^w \xrightarrow{\sim} (\rho'')_!K_2$ such that the following diagram commutes.

5.3.4.

$$\begin{array}{ccc}
(\rho''_w)_!K_2^w & \xrightarrow{h_w(\phi)} & (\rho'')_!K_2 \\
{\scriptstyle g_w}\big\downarrow & & {\scriptstyle g}\big\downarrow \\
\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}) & \xrightarrow{\theta_w(\phi)} & \mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})
\end{array}$$

where the vertical maps are the canonical isomorphisms given by 5.1.33.

Proposition 5.3.5. *Assume that $C_G(\sigma)$ is connected and let $M = C_G(\sigma)$. We have $\mathcal{W}_G(\mathcal{E}) = \mathcal{W}_M(\mathcal{E}) = \mathcal{W}_M(\overline{\mathbb{Q}}_\ell \boxtimes \zeta)$.*

Proof: We show that the inclusion $\mathcal{W}_M(\mathcal{E}) \subset \mathcal{W}_G(\mathcal{E})$ is an equality. Let $n \in N_G(\mathcal{E})$; it satisfies $\mathrm{Ad}(n)^* m^*(\mathcal{L}_\Psi) = m^*(\mathcal{L}_\Psi)$. But the map $m \circ \mathrm{Ad}(n)$ is the linear form on $z(\mathcal{L})$ given by $t \mapsto \mu(t, \mathrm{Ad}(n^{-1})\sigma)$. Hence from the fact that the character Ψ is non-trivial and the fact that the restriction of μ to $z(\mathcal{L})$ is non-degenerate (see 2.5.16), we deduce that $\sigma = \mathrm{Ad}(n^{-1})\sigma$, that is $n \in C_G(\sigma)$. We thus proved that $\mathcal{W}_G(\mathcal{E}) = \mathcal{W}_M(\mathcal{E})$. The equality $\mathcal{W}_M(\mathcal{E}) = \mathcal{W}_M(\overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ is obvious. □

Proposition 5.3.6. *We use the notation and assumption of 5.3.5 and 5.3.1. Then we have,*

(i) $\mathcal{W}_G(\mathcal{E}) = W_M(L)$. Hence by [Lus84, 9.2(a)], the group $\mathcal{W}_G(\mathcal{E})$ is a Coxeter group.

(ii) The natural morphism $End(ind_\Sigma^{\mathcal{M}}(\mathcal{E})) \to End\left(ind_\Sigma^{\mathcal{M}}(\mathcal{E})|_{\mathcal{M}_{nil}}\right)$ is an isomorphism.

(iii) We can choose the $\theta_w \in \mathcal{A}_w \subset \mathcal{A}$ such that for any $w, w' \in \mathcal{W}_G(\mathcal{E})$, we have $\theta_w.\theta_{w'} = \theta_{ww'}$; that is the map $w \mapsto \theta_w$ gives rise to an isomorphism between the group algebra $\overline{\mathbb{Q}}_\ell[\mathcal{W}_G(\mathcal{E})]$ of $\mathcal{W}_G(\mathcal{E})$ and $End(ind_\Sigma^{\mathcal{G}}(\mathcal{E}))$.

Proof: Since, by 5.3.5, we have $\mathcal{W}_G(\mathcal{E}) = \mathcal{W}_M(\mathcal{E})$, and, by 5.2.20, we have $\mathrm{End}\left(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})\right) \simeq \mathrm{End}\left(\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})\right)$, we are reduced to prove the proposition in the case where $\sigma \in z(\mathcal{G})$. We thus assume that $\sigma \in z(\mathcal{G})$, i.e $M = G$. We now reduce the proofs of (i) and (ii) to the case where $\sigma = 0$, i.e $\mathcal{E} = \overline{\mathbb{Q}}_\ell \boxtimes \zeta$. Since by 5.3.5 we have $\mathcal{W}_G(\mathcal{E}) = \mathcal{W}_G(\overline{\mathbb{Q}}_\ell \boxtimes \zeta)$, to prove (i) it is thus enough to prove it for $\sigma = 0$. By 5.1.66 we have a decomposition $\mathcal{E} = (m_{z(\mathcal{G})})^* \mathcal{L}_\Psi \boxtimes \left(m_{z(\hat{\mathcal{L}})}\right)^* \mathcal{L}_\Psi \boxtimes \zeta$. But since $\sigma \in z(\mathcal{G})$ and since by 2.5.16, the space $z(\mathcal{G})$ is orthogonal to $\overline{\mathcal{G}}$ with respect to μ, we have $\left(m_{z(\hat{\mathcal{L}})}\right)^* \mathcal{L}_\Psi = \overline{\mathbb{Q}}_\ell$. We deduce from 5.1.49 that $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}) \simeq (m_{z(\mathcal{G})})^* \mathcal{L}_\Psi[\dim z(\mathcal{L})] \boxtimes \mathrm{ind}_{\hat{\Sigma}}^{\overline{\mathcal{G}}}(\hat{\mathcal{E}})$ where $\hat{\mathcal{E}} = \overline{\mathbb{Q}}_\ell \boxtimes \zeta$. Hence the natural morphism $\mathrm{End}\big(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})\big) \to \mathrm{End}\big(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})|_{\mathcal{G}_{nil}}\big)$ is an isomorphism if and only if the morphism $\mathrm{End}\big(\mathrm{ind}_{\hat{\Sigma}}^{\overline{\mathcal{G}}}(\hat{\mathcal{E}})\big) \to \mathrm{End}\big(\mathrm{ind}_{\hat{\Sigma}}^{\overline{\mathcal{G}}}(\hat{\mathcal{E}})|_{\mathcal{G}_{nil}}\big)$ is an isomorphism. We thus have reduced the proof of (ii) to the case where $\sigma = 0$.

We now prove that (i), (ii) and (iii) hold for $\sigma = 0$; we will prove afterwards that (iii) holds for any $\sigma \in z(\mathcal{G})$. Let $f : G_{uni} \to \mathcal{G}_{nil}$ be a G-equivariant isomorphism. Let $(C^L, \zeta^L) := (f^{-1}(C), f^*\zeta)$ and let $(\Sigma^L, \mathcal{E}^L) := (Z_L^o.C^L, \overline{\mathbb{Q}}_\ell \boxtimes \zeta^L)$; then $(L, \Sigma^L, \mathcal{E}^L)$ is a cuspidal datum of G. Moreover we have

$$N_G(\mathcal{E}) = N_G(\mathcal{E}^L) := \{n \in N_G(L) | n\Sigma^L n^{-1} = \Sigma^L, (\mathrm{Int}_n)^*(\mathcal{E}^L) \simeq \mathcal{E}^L\} \qquad (*)$$

and from [Lus84, Theorem 9.2 (b)] we have $\mathcal{W}_G(\mathcal{E}^L) := N_G(\mathcal{E}^L)/L = W_G(L)$, hence we deduce (i). Put $\mathcal{W} = \mathcal{W}_G(\mathcal{E}^L) = \mathcal{W}_G(\mathcal{E})$.

Let us prove the assertion (ii) for $\sigma = 0$. By 5.1.78, the complex $\mathrm{ind}_{\Sigma^L}^G(\mathcal{E}^L)|_{G_{uni}}$ is isomorphic to $f^*\big(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})|_{\mathcal{G}_{nil}}\big)$, and by [Lus84, 6.8.2, 9.2],

$$\dim \mathrm{End}\big(\mathrm{ind}_{\Sigma^L}^G(\mathcal{E}^L)|_{G_{uni}}\big) = |\mathcal{W}|.$$

Hence

$$\dim \mathrm{End}\big(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})|_{\mathcal{G}_{uni}}\big) = \dim \mathrm{End}\big(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})\big). \qquad (1)$$

As in the group case (see [Lus84, 6.8.3]), we show that the restriction to $\mathcal{G}_{nil}$ of any irreducible direct summand of $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})$ is non-zero, hence we get that the map in (ii) is injective which is thus bijective by (1).

From [Lus84, 6.8.2, 9.2], we have an isomorphism

$$\overline{\mathbb{Q}}_\ell[\mathcal{W}] \xrightarrow{\sim} \mathrm{End}\big(\mathrm{ind}_{\Sigma^L}^G(\mathcal{E}^L)|_{G_{uni}}\big)$$

of $\overline{\mathbb{Q}}_\ell$-algebras, and so via $\mathrm{ind}_{\Sigma^L}^G(\mathcal{E}^L)|_{G_{uni}} \simeq f^*\big(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})|_{\mathcal{G}_{nil}}\big)$, we get that $\mathrm{End}\big(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})|_{\mathcal{G}_{nil}}\big) \simeq \overline{\mathbb{Q}}_\ell[\mathcal{W}]$. Thus we can choose the $\theta_w|_{\mathcal{G}_{nil}}$ such that $(\theta_w|_{\mathcal{G}_{nil}}).(\theta_v|_{\mathcal{G}_{nil}}) = \theta_{wv}|_{\mathcal{G}_{nil}}$ for any $w, v \in \mathcal{W}$. The assertion (iii) for $\sigma = 0$ follows thus from (ii).

Assume now that $\sigma \in z(\mathcal{G})$. By 5.1.75, we have

$$\mathrm{End}\left(\mathrm{ind}_\Sigma^{\mathcal{G}}(\overline{\mathbb{Q}}_\ell \boxtimes \zeta)|_{\mathcal{G}_{nil}}\right) = \mathrm{End}\left(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})|_{\mathcal{G}_{nil}}\right).$$

From the previous discussion we also have $\mathrm{End}\left(\mathrm{ind}_\Sigma^{\mathcal{G}}(\overline{\mathbb{Q}}_\ell \boxtimes \zeta)|_{\mathcal{G}_{nil}}\right) = \overline{\mathbb{Q}}_\ell[\mathcal{W}]$, hence by (ii) we have $\mathrm{End}\left(\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})\right) = \overline{\mathbb{Q}}_\ell[\mathcal{W}]$. □

The following result is a consequence of 5.3.6(ii).

Proposition 5.3.7. *Assume that $\sigma \in z(\mathcal{G})$. Then the restriction to $\mathcal{G}_{nil}$ of any simple direct summand of $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})$ is a simple perverse sheaf on $\mathcal{G}_{nil}$ (up to a shift by $\dim z(\mathcal{L})$).*

5.4 Geometrical Induction: Definition

The geometrical induction for invariant functions will be defined using a formula (see 5.4.7) expressing the characteristic functions of the F-equivariant admissible complexes in terms of characteristic functions of F-equivariant Lusztig complexes. To establish this formula, we follow [Lus85b, 10].

5.4.1 Preliminaries

5.4.2. Let $(L, \Sigma, \mathcal{E})$ be an F-stable cuspidal datum of $\mathcal{G}$ and let $\phi : F^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ be an isomorphism. For any $w \in \mathcal{W}_G(\mathcal{E})$, we choose arbitrarily a non-zero element $\theta_w \in \mathcal{A}_w$ where $\mathcal{A}_w$ is the one-dimensional vector space defined in 5.3.1. We fix an element w of $\mathcal{W}_G(\mathcal{E})$ together with a representative $\dot{w}$ of w in $N_G(\mathcal{E})$. By the Lang-Steinberg theorem there is an element $z \in G$ such that $z^{-1}F(z) = \dot{w}^{-1}$. Let $L_w := zLz^{-1}$ and let $\mathcal{L}_w$ be its Lie algebra. Then L_w and $\Sigma_w := \mathrm{Ad}(z)\Sigma$ are both F-stable. Let $\mathcal{E}_w$ be the local system $\mathrm{Ad}(z^{-1})^*(\mathcal{E})$. We are going to define an isomorphism $\phi_w : F^*(\mathcal{E}_w) \xrightarrow{\sim} \mathcal{E}_w$ in terms of ϕ. By 5.3.1, the automorphism θ_w defines an isomorphism $\mathcal{E} \simeq \mathrm{Ad}(\dot{w})^*(\mathcal{E})$ which leads to an isomorphism

$$F^*\mathrm{Ad}(z^{-1})^*(\mathcal{E}) \simeq F^*\mathrm{Ad}(z^{-1})^*\mathrm{Ad}(\dot{w})^*(\mathcal{E}). \qquad (*)$$

Since we have $\mathrm{Ad}(\dot{w}) \circ \mathrm{Ad}(z^{-1}) \circ F = F \circ \mathrm{Ad}(z^{-1})$, the isomorphism (*) gives rise to an isomorphism $h : F^*\mathrm{Ad}(z^{-1})^*(\mathcal{E}) \simeq \mathrm{Ad}(z^{-1})^*F^*(\mathcal{E})$. Then the isomorphism $\phi_w : F^*(\mathcal{E}_w) \simeq \mathcal{E}_w$ is $\mathrm{Ad}(z^{-1})^*(\phi) \circ h$.

We denote by $\phi^{\mathcal{G}} : F^*\big(\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})\big) \xrightarrow{\sim} \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ the natural isomorphism induced by ϕ and by $\phi_w^{\mathcal{G}} : F^*\big(\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w)\big) \xrightarrow{\sim} \mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w)$ the natural isomorphism induced by ϕ_w. As in [Lus85b, 10.6], there is a natural isomorphism $j : \mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w) \xrightarrow{\sim} \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ such that the following diagram commutes.

$$\begin{array}{ccc} F^*\left(\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w)\right) & \xrightarrow{F^*(j)} & F^*\left(\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})\right) \\ \downarrow{\scriptstyle \phi_w^{\mathcal{G}}} & & \downarrow{\scriptstyle \theta_w \circ \phi^{\mathcal{G}}} \\ \mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w) & \xrightarrow{\quad j \quad} & \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}) \end{array}$$

As a consequence we get that

$$\mathbf{X}_{\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}), \theta_w \circ \phi^{\mathcal{G}}} = \mathbf{X}_{\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w), \phi_w^{\mathcal{G}}}.$$

5.4.3. Let $(L, \Sigma, \mathcal{E})$ be a cuspidal datum of $\mathcal{G}$, let $K^{\mathcal{G}} = \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ and let $\mathcal{A} = \mathrm{End}(K^{\mathcal{G}})$. If A is a simple direct summand of $K^{\mathcal{G}}$, we denote by V_A the abelian group $\mathrm{Hom}(A, K^{\mathcal{G}})$. Then V_A is endowed with a structure of $\mathcal{A}$-module defined by $\mathcal{A} \times V_A \to V_A$, $(a, f) \mapsto a \circ f$; since A is a simple perverse sheaf, the $\mathcal{A}$-module V_A is irreducible. We have a natural isomorphism

$$\bigoplus_A (V_A \otimes A) \xrightarrow{\sim} K^{\mathcal{G}}$$

where A runs over the set of simple components of $K^{\mathcal{G}}$ (up to isomorphism). For any $x \in \mathcal{G}$ and any integer i, it gives rise to an isomorphism

$$\bigoplus_A (V_A \otimes \mathcal{H}^i_x A) \xrightarrow{\sim} \mathcal{H}^i_x K^{\mathcal{G}} \qquad (*)$$

under which an element $v \otimes a \in V_A \otimes \mathcal{H}^i_x A$ corresponds to $v^i_x(a)$ where $v^i_x : \mathcal{H}^i_x A \to \mathcal{H}^i_x K^{\mathcal{G}}$ is the morphism induced by $v : A \to K^{\mathcal{G}}$.

Assume now that the datum $(L, \Sigma, \mathcal{E})$ is F-stable and let ϕ be an isomorphism $F^*(\mathcal{E}) \simeq \mathcal{E}$. The complex $K^{\mathcal{G}}$ is thus F-stable and we denote by $\phi^{\mathcal{G}} : F^*(K^{\mathcal{G}}) \xrightarrow{\sim} K^{\mathcal{G}}$ the isomorphism induced by ϕ. Let A be an F-stable simple direct summand of $K^{\mathcal{G}}$ together with an isomorphism $\phi_A : F^*(A) \xrightarrow{\sim} A$. This defines a linear map $\sigma_A : V_A \to V_A$, $v \mapsto \phi^{\mathcal{G}} \circ F^*(v) \circ \phi_A^{-1}$ such that for any $x \in \mathcal{G}^F$ and any integer i, the isomorphism $\sigma_A \otimes (\phi_A)^i_x : V_A \otimes \mathcal{H}^i_x A \xrightarrow{\sim} V_A \otimes \mathcal{H}^i_x A$ corresponds under (*) to $(\phi^{\mathcal{G}})^i_x : \mathcal{H}^i_x K^{\mathcal{G}} \xrightarrow{\sim} \mathcal{H}^i_x K^{\mathcal{G}}$. On the other hand, if B is a simple component of $K^{\mathcal{G}}$ which is not F-stable, then $(\phi^{\mathcal{G}})^i_x$ maps $V_B \otimes \mathcal{H}^i_x B \hookrightarrow \mathcal{H}^i_x K^{\mathcal{G}}$ onto a different direct summand. It follows that

5.4.4.

$$\mathbf{X}_{K^{\mathcal{G}}, \phi^{\mathcal{G}}} = \sum_A \mathrm{Tr}(\sigma_A, V_A) \mathbf{X}_{A, \phi_A}$$

where A runs over the set of F-stable simple components of $K^{\mathcal{G}}$ (up to isomorphism). If for $w \in \mathcal{W}_G(\mathcal{E})$, we replace $\phi^{\mathcal{G}}$ by $\theta_w \circ \phi^{\mathcal{G}}$ with θ_w as in 5.4.2 and we keep ϕ_A unchanged, then the formula 5.4.4 becomes

5.4.5.

$$\mathbf{X}_{K^{\mathcal{G}}, \theta_w \circ \phi^{\mathcal{G}}} = \sum_A \mathrm{Tr}(\theta_w \circ \sigma_A, V_A) \mathbf{X}_{A, \phi_A}.$$

Following [Lus86a, 10.4] we deduce that

5.4.6.

$$\mathbf{X}_{A, \phi_A} = |\mathcal{W}_G(\mathcal{E})|^{-1} \sum_{w \in \mathcal{W}_G(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_A)^{-1}, V_A) \mathbf{X}_{K^{\mathcal{G}}, \theta_w \circ \phi^{\mathcal{G}}}$$

for any F-equivariant complex (A, ϕ_A) with A a simple direct summand of $K^{\mathcal{G}}$.

We use the notation of 5.4.2; by 5.4.2 and 5.4.6 we get that

5.4.7.

$$\mathbf{X}_{A,\phi_A} = |\mathcal{W}_G(\mathcal{E})|^{-1} \sum_{w \in \mathcal{W}_G(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_A)^{-1}, V_A) \mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma_w}(\mathcal{E}_w), \phi^{\mathcal{G}}_w}$$

for any F-equivariant admissible complex (A, ϕ_A) with A a simple direct summand of $K^{\mathcal{G}}$.

5.4.8. Let A be an F-stable admissible complex on $\mathcal{G}$. Now we prove that there exists a unique (up to G-conjugacy) F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{G}$ such that A is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E})$; this will show that we have a formula like 5.4.7 for any F-equivariant admissible complex (A, ϕ_A) on $\mathcal{G}$.

By 5.2.18, we need only to prove the existence of such an F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$. Let $(L, \Sigma, \mathcal{E})$ be a cuspidal datum of $\mathcal{G}$ such that A is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E})$. Since A is F-stable, it is also a direct summand of $F^*\big(\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E})\big) \simeq \mathrm{ind}^{\mathcal{G}}_{F^{-1}(\Sigma)}\big(F^*(\mathcal{E})\big)$. By 5.2.18, the cuspidal data $(L, \Sigma, \mathcal{E})$ and $(F^{-1}(L), F^{-1}(\Sigma), F^*(\mathcal{E}))$ are G-conjugate i.e. there exists $g \in G$ such that $F^{-1}(L) = gLg^{-1}$, $F^{-1}(\Sigma) = \mathrm{Ad}(g)\Sigma$ and $F^*(\mathcal{E}) = \mathrm{Ad}(g^{-1})^*(\mathcal{E})$. By Lang-Steinberg theorem, there exists $g_1 \in G$ such that $F(g) = g_1^{-1}F(g_1)$. Put

$$(L_1, \Sigma_1, \mathcal{E}_1) := (g_1 L g_1^{-1}, \mathrm{Ad}(g_1)\Sigma, \mathrm{Ad}(g_1^{-1})^*(\mathcal{E})).$$

Then the cuspidal datum $(L_1, \Sigma_1, \mathcal{E}_1)$ is F-stable and A is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\Sigma_1}(\mathcal{E}_1)$.

Remark 5.4.9. Let A be an F-stable admissible complex on $\mathcal{G}$. Assume that $\mathcal{F}^{\mathcal{G}}(A)$ is supported by the Zariski closure of an F-stable G-orbit of the form $\sigma + \mathcal{O}$ with $\sigma \in z(\mathcal{G})$ and $\mathcal{O}$ a nilpotent orbit of $\mathcal{G}$. Then there exists a unique (up to G^F-conjugacy) F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ such that L is G-split and such that A is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E})$.

Indeed, by 5.4.8, there exists an F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{G}$ such that A is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E})$. Since $\mathcal{F}^{\mathcal{G}}(A)$ is supported by $\sigma + \overline{\mathcal{O}}$ with $\sigma \in z(\mathcal{G})$, the datum $(L, \Sigma, \mathcal{E})$ is of the form $\big(L, z(\mathcal{L}) + C, (m_{-\sigma})^* \mathcal{L}_\Psi \boxtimes \zeta\big)$ for some cuspidal nilpotent pair (C, ζ) of $\mathcal{L} = \mathrm{Lie}(L)$. Since L supports a cuspidal pair, any two parabolic subgroup of G having L as a Levi subgroup are $N_G(L)$-conjugate (see [DLM97, 1.1(i)]). As a consequence there is a unique (up to G^F-conjugacy) F-stable G-split Levi subgroup L_o of G which is G-conjugate to L. Since $\mathcal{W}_G(\mathcal{E}) = W_G(L)$, see 5.3.6(i), it is thus possible, as in 5.4.2, to construct an F-stable cuspidal datum $(L_w, \Sigma_w, \mathcal{E}_w)$ of $\mathcal{G}$, for some $w \in \mathcal{W}_G(\mathcal{E})$, such that $L_o = L_w$.

5.4.10 Geometrical Induction

5.4.11. Let M be an F-stable Levi subgroup of G and let $\mathcal{M}$ be the Lie algebra of M. We are now in position to define the geometrical induction $R_{\mathcal{M}}^{\mathcal{G}} : \mathcal{C}(\mathcal{M}^F) \to \mathcal{C}(\mathcal{G}^F)$.

We define the geometrical induction $R_{\mathcal{M}}^{\mathcal{G}} : \mathcal{C}(\mathcal{M}^F) \to \mathcal{C}(\mathcal{G}^F)$ using a basis $\{\mathbf{X}_{A_\iota,\phi_\iota} | \iota \in I(\mathcal{M})^F\}$ of $\mathcal{C}(\mathcal{M}^F)$ as in 5.2.22.

Let $\iota \in I(\mathcal{M})^F$ and let $(L, \Sigma, \mathcal{E})$ be an F-stable cuspidal datum of $\mathcal{M}$ such that A_ι is a direct summand of $\mathrm{ind}_{\Sigma}^{\mathcal{M}}(\mathcal{E})$.

Let $\phi : F^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ be an isomorphism. For $w \in \mathcal{W}_M(\mathcal{E})$, let θ_w be a non-zero element of $\mathcal{A}_w \subset \mathrm{End}(\mathrm{ind}_{\Sigma}^{\mathcal{M}}(\mathcal{E}))$ where $\mathcal{A}_w$ is as in 5.3.1. As in 5.4.7 we have

5.4.12.

$$\mathbf{X}_{A_\iota,\phi_\iota} = |\mathcal{W}_M(\mathcal{E})|^{-1} \sum_{w \in \mathcal{W}_M(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_{A_\iota})^{-1}, V_{A_\iota}) \mathbf{X}_{\mathrm{ind}_{\Sigma_w}^{\mathcal{M}}(\mathcal{E}_w),\phi_w^{\mathcal{M}}}.$$

Then we define $R_{\mathcal{M}}^{\mathcal{G}}(\mathbf{X}_{A_\iota,\phi_\iota})$ by

5.4.13.

$$R_{\mathcal{M}}^{\mathcal{G}}(\mathbf{X}_{A_\iota,\phi_\iota}) = |\mathcal{W}_M(\mathcal{E})|^{-1} \sum_{w \in \mathcal{W}_M(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_{A_\iota})^{-1}, V_{A_\iota}) \mathbf{X}_{\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w),\phi_w^{\mathcal{G}}}.$$

We will prove after the following remark that 5.4.13 does not depend on the choice of the F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$.

Remark 5.4.14. (i) Note that the definition of $R_{\mathcal{M}}^{\mathcal{G}} : \mathcal{C}(\mathcal{M}^F) \to \mathcal{C}(\mathcal{G}^F)$ does not depend on the choice of the isomorphisms ϕ_ι with $\iota \in I(\mathcal{M})^F$. Indeed, let ${R'}_{\mathcal{M}}^{\mathcal{G}}$ be the induction defined on another basis $\{\mathbf{X}_{A_\iota,\phi'_\iota} | \iota \in I(\mathcal{M})^F\}$ and let $\iota \in I(\mathcal{M})^F$. Since A_ι is a simple perverse sheaf, there exists a constant $c \in \overline{\mathbb{Q}}_\ell^\times$ such that $\phi_\iota = c\phi'_\iota$. Let $\sigma'_{A_\iota} : V_{A_\iota} \to V_{A_\iota}$ be defined in terms of $\phi^{\mathcal{M}}$, ϕ'_ι as σ_{A_ι} is defined in terms of $\phi^{\mathcal{M}}$, ϕ_ι. We thus have $\sigma_{A_\iota} = c^{-1}\sigma'_{A_\iota}$. Hence for any $w \in \mathcal{W}_M(\mathcal{E})$, we have $(\theta_w \circ \sigma_{A_\iota})^{-1} = c(\theta_w \circ \sigma'_{A_\iota})^{-1}$ and so from 5.4.13, we get that $R_{\mathcal{M}}^{\mathcal{G}}(\mathbf{X}_{A_\iota,\phi_\iota}) = c{R'}_{\mathcal{M}}^{\mathcal{G}}(\mathbf{X}_{A_\iota,\phi'_\iota})$. But since $\mathbf{X}_{A_\iota,\phi_\iota} = c\mathbf{X}_{A_\iota,\phi'_\iota}$, this proves that $R_{\mathcal{M}}^{\mathcal{G}}(\mathbf{X}_{A_\iota,\phi_\iota}) = {R'}_{\mathcal{M}}^{\mathcal{G}}(\mathbf{X}_{A_\iota,\phi'_\iota})$. It is also clear that the induction $R_{\mathcal{M}}^{\mathcal{G}}$ does not depend on the choice of the isomorphisms $\phi : F^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$. Finally it is clearly independent on the choice of the isomorphisms $\theta_w \in \mathcal{A}_w$ since if we denote by $\theta_w^{\mathcal{G}}$ the canonical endomorphism of $\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ induced by θ_w (recall that θ_w defines a unique isomorphism $\mathrm{Ad}(\dot{w})^*\mathcal{E} \simeq \mathcal{E}$ which induces a canonical isomorphism $\theta_w^{\mathcal{G}} : \mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}) \simeq \mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})$), then $\mathbf{X}_{\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w),\phi_w^{\mathcal{G}}} = \mathbf{X}_{\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}),\theta_w^{\mathcal{G}}\circ\phi^{\mathcal{G}}}$ by 5.4.2.

(ii) If $(M, \Sigma, \mathcal{E})$ is an F-stable cuspidal datum of $\mathcal{G}$ together with an isomorphism $\phi : F^*(\mathcal{E}) \simeq \mathcal{E}$, then

$$R_{\mathcal{M}}^{\mathcal{G}}\big(\mathbf{X}_{K(\Sigma,\mathcal{E}),\phi}\big) = \mathbf{X}_{\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}),\phi^{\mathcal{G}}}.$$

(iii) Note that unlike Deligne-Lusztig induction, the definition of geometrical induction does not involve any parabolic subgroup of G.

We have the following lemma.

Lemma 5.4.15. *We use the notation of 5.4.11. Assume that $X_{\mathcal{M}}^{\mathcal{G}} : \mathcal{C}(\mathcal{M}^F) \to \mathcal{C}(\mathcal{G}^F)$ is a $\overline{\mathbb{Q}}_\ell$-linear map such that for any F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{M}$ and any isomorphism $\phi : F^*(\mathcal{E}) \simeq \mathcal{E}$, we have $X_{\mathcal{M}}^{\mathcal{G}}\big(\mathbf{X}_{ind_\Sigma^{\mathcal{M}}(\mathcal{E}),\phi^{\mathcal{M}}}\big) = \mathbf{X}_{ind_\Sigma^{\mathcal{G}}(\mathcal{E}),\phi^{\mathcal{G}}}$. Then $X_{\mathcal{M}}^{\mathcal{G}} = R_{\mathcal{M}}^{\mathcal{G}}$.*

The following result will show the transitivity of geometrical induction, and, together with 5.4.15, it will show the independence of the formula 5.4.13 from the F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$.

Proposition 5.4.16. *For any F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{M}$ and any isomorphism $\phi : F^*(\mathcal{E}) \simeq \mathcal{E}$, we have $R_{\mathcal{M}}^{\mathcal{G}}\big(\mathbf{X}_{ind_\Sigma^{\mathcal{M}}(\mathcal{E}),\phi^{\mathcal{M}}}\big) = \mathbf{X}_{ind_\Sigma^{\mathcal{G}}(\mathcal{E}),\phi^{\mathcal{G}}}$.*

Proof: Thanks to 5.4.15, it is enough to show the existence of a $\overline{\mathbb{Q}}_\ell$-linear map $X_{\mathcal{M}}^{\mathcal{G}} : \mathcal{C}(\mathcal{M}^F) \to \mathcal{C}(\mathcal{G}^F)$ such that for any F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{M}$ and any isomorphism $\phi : F^*(\mathcal{E}) \simeq \mathcal{E}$, we have $X_{\mathcal{M}}^{\mathcal{G}}\big(\mathbf{X}_{\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E}),\phi^{\mathcal{M}}}\big) = \mathbf{X}_{\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}),\phi^{\mathcal{G}}}$. We define $X_{\mathcal{M}}^{\mathcal{G}} : \mathcal{C}(\mathcal{M}^F) \longrightarrow \mathcal{C}(\mathcal{G}^F)$ on each element of the basis $\{\mathbf{X}_{A_\iota,\phi_\iota} | \iota \in I(\mathcal{M})^F\}$ using the formula 5.4.13 with the following additional condition concerning the choice of the F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$: for $\iota \in I(\mathcal{M})^F$, we assume that the F-stable cuspidal data $a_\iota = (L, \Sigma, \mathcal{E})$ such that $A_\iota \hookrightarrow \mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})$ is chosen such that if A_μ, with $\mu \in I(\mathcal{M})^F$, is a direct summand of $\mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})$, then $a_\mu = a_\iota$. Now, let $\iota \in I(\mathcal{M})^F$ and put $a_\iota = (L, \Sigma, \mathcal{E})$. Put $K^{\mathcal{M}} = \mathrm{ind}_\Sigma^{\mathcal{M}}(\mathcal{E})$ and $K^{\mathcal{G}} = \mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E})$. We want to show that for any $w' \in \mathcal{W}_M(\mathcal{E})$, we have $X_{\mathcal{M}}^{\mathcal{G}}\big(\mathbf{X}_{K^{\mathcal{M}},\theta_{w'}\circ\phi^{\mathcal{M}}}\big) = \mathbf{X}_{K^{\mathcal{G}},\theta_{w'}^{\mathcal{G}}\circ\phi^{\mathcal{G}}}$ where $\theta_{w'}^{\mathcal{G}}$ denotes the canonical endomorphism of $K^{\mathcal{G}}$ induced by the endomorphism $\theta_{w'} \in \mathcal{A}_w$. Let $w' \in \mathcal{W}_M(\mathcal{E})$, from 5.4.5, we have

$$X_{\mathcal{M}}^{\mathcal{G}}\big(\mathbf{X}_{K^{\mathcal{M}},\theta_{w'}\circ\phi^{\mathcal{M}}}\big) = \sum_\iota \mathrm{Tr}(\theta_{w'} \circ \sigma_{A_\iota}, V_{A_\iota}) X_{\mathcal{M}}^{\mathcal{G}}(\mathbf{X}_{A_\iota,\phi_\iota})$$

where ι runs over the set $Z := \{\iota \in I(\mathcal{M})^F | A_\iota \hookrightarrow K^{\mathcal{M}}\}$. From the definition of $X_{\mathcal{M}}^{\mathcal{G}}$, we have

5.4.17. $X^{\mathcal{G}}_{\mathcal{M}}\big(\mathbf{X}_{K^{\mathcal{M}},\theta_{w'}\circ\phi^{\mathcal{M}}}\big) = |\mathcal{W}_M(\mathcal{E})|^{-1}\times$

$$\sum_{w\in\mathcal{W}_M(\mathcal{E})}\left(\sum_{\iota\in Z}\mathrm{Tr}(\theta_{w'}\circ\sigma_{A_\iota}, V_{A_\iota})\mathrm{Tr}\big((\theta_w\circ\sigma_{A_\iota})^{-1}, V_{A_\iota}\big)\mathbf{X}_{K^{\mathcal{G}},\theta^{\mathcal{G}}_w\circ\phi^{\mathcal{G}}}\right).$$

Let $\mathcal{A}$ be the endomorphims algebra of $K^{\mathcal{M}}$. Define $\rho:\mathcal{A}\longrightarrow\mathcal{A}$ by $\rho(\theta)=\phi^{\mathcal{M}}\circ F^*(\theta)\circ\big(\phi^{\mathcal{M}}\big)^{-1}$. This is an automorphism of $\overline{\mathbb{Q}}_\ell$-algebras and for any $v\in\mathcal{W}_M(\mathcal{E})$, we have $\rho(\mathcal{A}_v)=\mathcal{A}_{F^{-1}(v)}$. For any $\iota\in Z$, the linear map $\sigma_{A_\iota}:V_{A_\iota}\to V_{A_\iota}$ is $\mathcal{A}$-semi-linear, t hat is $\sigma_{A_\iota}(\theta v)=\rho(\theta)\sigma_{A_\iota}(v)$. Hence from [Lus85b, (10.3.2)], the term

$$\sum_{\iota\in Z}\mathrm{Tr}(\theta_{w'}\circ\sigma_{A_\iota}, V_{A_\iota})\mathrm{Tr}\big((\theta_w\circ\sigma_{A_\iota})^{-1}, V_{A_\iota}\big) \qquad (*)$$

is equal to the trace of the linear map $\mathcal{A}\to\mathcal{A}, \theta\mapsto\theta_w^{-1}\rho^{-1}(\theta)\theta_{w'}$. Recall that the set $\{\theta_v|v\in\mathcal{W}_M(\mathcal{E})\}$ is a basis of $\mathcal{A}$ such that for any $v,v'\in\mathcal{W}_M(\mathcal{E})$, we have $\theta_v\theta_{v'}\in\mathcal{A}_{vv'}$, hence for any $v\in\mathcal{W}_M(\mathcal{E})$, we have

$$\theta_w^{-1}\rho^{-1}(\theta_v)\theta_{w'}=\varepsilon(v)\theta_{w^{-1}F(v)w'}$$

for some scalar $\varepsilon(v)$. The term (*) is thus equal to

$$\sum_{\{v\in\mathcal{W}_M(\mathcal{E})|F(v)^{-1}wv=w'\}}\varepsilon(v).$$

The formula 5.4.17 becomes

$$X^{\mathcal{G}}_{\mathcal{M}}\big(\mathbf{X}_{K^{\mathcal{M}},\theta_{w'}\circ\phi^{\mathcal{M}}}\big)=|\mathcal{W}_M(\mathcal{E})|^{-1}\sum_{\substack{w\in\mathcal{W}_M(\mathcal{E})\\ w\underset{F}{\sim}w'}}\left(\sum_{\substack{v\in\mathcal{W}_M(\mathcal{E})\\ F(v)^{-1}wv=w'}}\varepsilon(v)\mathbf{X}_{K^{\mathcal{G}},\theta^{\mathcal{G}}_w\circ\phi^{\mathcal{G}}}\right)$$

where for two elements $v,v'\in\mathcal{W}_M(\mathcal{E})$, the expression $v\underset{F}{\sim}v'$ means: v and v' are F^{-1}-conjugate. Hence to prove the equality $X^{\mathcal{G}}_{\mathcal{M}}\big(\mathbf{X}_{K^{\mathcal{M}},\theta_{w'}\circ\phi^{\mathcal{M}}}\big)=\mathbf{X}_{K^{\mathcal{G}},\theta^{\mathcal{G}}_{w'}\circ\phi^{\mathcal{G}}}$, it remains to show that if $v\in\mathcal{W}_M(\mathcal{E})$ satisfies $F(v)^{-1}wv=w'$, then $\varepsilon(v)\mathbf{X}_{K^{\mathcal{G}},\theta^{\mathcal{G}}_w\circ\phi^{\mathcal{G}}}=\mathbf{X}_{K^{\mathcal{G}},\theta^{\mathcal{G}}_{w'}\circ\phi^{\mathcal{G}}}$.

Assume that $v\in\mathcal{W}_M(\mathcal{E})$ is such that $F(v)^{-1}wv=w'$. We have

$$\theta_w^{-1}\rho^{-1}(\theta_v)\theta_{w'}=\varepsilon(v)\theta_v$$

from which we deduce that $\theta_{F(v)}\theta_{w'}\rho(\theta_{F(v)})^{-1}=\varepsilon(v)\theta_w$, that is,

$$\theta_{F(v)}\theta_{w'}\big(\phi^{\mathcal{M}}\circ F^*(\theta_{F(v)}^{-1})\circ(\phi^{\mathcal{M}})^{-1}\big)=\varepsilon(v)\theta_w.$$

We thus have

$$\theta^{\mathcal{G}}_{F(v)}\theta^{\mathcal{G}}_{w'}\big(\phi^{\mathcal{G}}\circ F^*(\theta^{\mathcal{G}}_{F(v)})^{-1}\circ(\phi^{\mathcal{G}})^{-1}\big)=\varepsilon(v)\theta^{\mathcal{G}}_{w}.$$

By composing this equality with $\phi^{\mathcal{G}}$ on the right we get that

$$\theta^{\mathcal{G}}_{F(v)}\circ\theta^{\mathcal{G}}_{w'}\circ\phi^{\mathcal{G}}\circ F^*(\theta^{\mathcal{G}}_{F(v)})^{-1}=\varepsilon(v)\theta^{\mathcal{G}}_{w}\circ\phi^{\mathcal{G}}$$

from which we see that $\mathbf{X}_{K^{\mathcal{G}},\theta^{\mathcal{G}}_{w'}\circ\phi^{\mathcal{G}}}=\varepsilon(v)\mathbf{X}_{K^{\mathcal{G}},\theta^{\mathcal{G}}_{w}\circ\phi^{\mathcal{G}}}$. □

Corollary 5.4.18. *The geometrical induction is transitive, that is for any inclusion $L\subset M$ of F-stable Levi subgroups of G, we have $R^{\mathcal{G}}_{\mathcal{M}}\circ R^{\mathcal{M}}_{\mathcal{L}}=R^{\mathcal{G}}_{\mathcal{L}}$ where $\mathcal{L}=Lie(L)$ and $\mathcal{M}=Lie(M)$.*

Proof: This is a straightforward consequence of 5.4.16. □

5.5 Deligne-Lusztig Induction and Geometrical Induction

Let L be an F-stable Levi subgroup of G and let (C,ζ) be an F-stable cuspidal nilpotent pair of $\mathcal{L}=\mathrm{Lie}(L)$. Let P be a parabolic subgroup of G having L as a Levi subgroup and let $\mathcal{P}=\mathrm{Lie}(P)$. For any $\sigma\in z(\mathcal{L})$ we denote by $\mathcal{E}_\sigma$ the local system $(m_\sigma)^*\mathcal{L}_\Psi\boxtimes\zeta$ on $\Sigma=z(\mathcal{L})+C$ and by K_σ the complex $K(\Sigma,\mathcal{E}_\sigma)$; then $\mathcal{E}_o=\overline{\mathbb{Q}}_\ell\boxtimes\zeta$ and $K_o=K(\Sigma,\mathcal{E}_o)$. We fix an isomorphism $\phi:F^*(\zeta)\xrightarrow{\sim}\zeta$. Let $K^{\mathcal{G}}_\sigma:=\mathrm{ind}^{\mathcal{G}}_\Sigma(\mathcal{E}_\sigma)$ and $K^{\mathcal{G}}_o:=\mathrm{ind}^{\mathcal{G}}_\Sigma(\mathcal{E}_o)$. We denote by $\phi^{\mathcal{G}}_o:F^*(K^{\mathcal{G}}_o)\xrightarrow{\sim}K^{\mathcal{G}}_o$ the canonical isomorphism induced by $1\boxtimes\phi:F^*(\mathcal{E}_o)\xrightarrow{\sim}\mathcal{E}_o$.

5.5.1 Generalized Green Functions

Definition 5.5.2. *We define the generalized Green function $\mathcal{Q}^{\mathcal{G}}_{\mathcal{L},C,\zeta,\phi}\in\mathcal{C}(\mathcal{G}^F)_{nil}$ as the characteristic function of $(K^{\mathcal{G}}_o|_{\mathcal{G}_{nil}},\phi^{\mathcal{G}}_o|_{\mathcal{G}_{nil}})$ extended by zero on $\mathcal{G}^F-\mathcal{G}^F_{nil}$, .*

The following proposition is the Lie algebra version of [Lus85b, 8.3.2].

Proposition 5.5.3. *Let $\sigma\in z(\mathcal{L})^F$, and let $\phi_\sigma:F^*(\mathcal{E}_\sigma)\xrightarrow{\sim}\mathcal{E}_\sigma$ be an isomorphism extending ϕ, then if $\phi^{\mathcal{G}}_\sigma:F^*(K^{\mathcal{G}}_\sigma)\xrightarrow{\sim}K^{\mathcal{G}}_\sigma$ is the isomorphism induced by ϕ_σ, we have*

$$\mathcal{Q}^{\mathcal{G}}_{\mathcal{L},C,\zeta,\phi}(u)=\mathbf{X}_{K^{\mathcal{G}}_\sigma,\phi^{\mathcal{G}}_\sigma}(u)$$

for any $u\in\mathcal{G}^F_{nil}$.

Proof: There is a gap in the proof of [Lus85b, 8.3.2]. However a complete proof of [Lus85b, 8.3.2] can be found in [Lus04] where the result [Lus85b, 8.3.2] has been generalized to the case where the reductive group G is not necessarily connected. Here we adapt Lusztig's argument to the Lie algebra case.

Let $h_P : \big(K_\sigma^{\mathcal{G}}\big)|_{\mathcal{G}_{nil}} \xrightarrow{\sim} \big(K_o^{\mathcal{G}}\big)|_{\mathcal{G}_{nil}}$ be the isomorphism as in 5.1.75. Since L supports a cuspidal pair, by [DLM97, 1.1(i)], the parabolic subgroups $F(P)$ and P are conjugate in $N_G(L)$. Let $w \in W_G(L)$ be such that $F(P) = P^w := \dot{w}P\dot{w}^{-1}$ with $\dot{w} \in N_G(L)$ a representative of w. By 5.1.52, we have the following commutative diagram.

5.5.4.

$$\begin{CD} F^*\big(\big(K_\sigma^{\mathcal{G}}\big)|_{\mathcal{G}_{nil}}\big) @>{F^*(h_{P^w})}>> F^*\big(\big(K_o^{\mathcal{G}}\big)|_{\mathcal{G}_{nil}}\big) \\ @V{\phi_\sigma^{\mathcal{G}}|_{\mathcal{G}_{nil}}}VV @VV{\phi^{\mathcal{G}}|_{\mathcal{G}_{nil}}}V \\ \big(K_\sigma^{\mathcal{G}}\big)|_{\mathcal{G}_{nil}} @>{h_P}>> \big(K_o^{\mathcal{G}}\big)|_{\mathcal{G}_{nil}} \end{CD}$$

We thus have to show that $h_P = h_{P^w}$, i.e. that the following diagram commutes.

5.5.5.

$$\begin{array}{ccccc} & & \big(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(C,\zeta)\big)[\dim z(\mathcal{L})] & & \\ & \overset{\sim}{\swarrow} & & \overset{\sim}{\searrow} & \\ \big(K_\sigma^{\mathcal{G}}\big)|_{\mathcal{G}_{nil}} & & & & \big(K_o^{\mathcal{G}}\big)|_{\mathcal{G}_{nil}} \\ & \underset{\sim}{\searrow} & & \underset{\sim}{\swarrow} & \\ & & \big(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}^w}^{\mathcal{G}} K(C,\zeta)\big)[\dim z(\mathcal{L})] & & \end{array}$$

where the arrows are the restriction to the nilpotent set of the canonical isomorphisms given by 5.1.33.

We thus come down to the following problem:

Let $v \in W_G(L)$ and let $\dot{v}$ be a representative of v in $N_G(L)$. From 5.3.6(i), the element $\dot{v}$ normalizes Σ. Let $\sigma' = \mathrm{Ad}(\dot{v}^{-1})\sigma$ and $\zeta' = \mathrm{Ad}(\dot{v})^*\zeta$. Put

$\mathcal{E}'_{\sigma'} = (m_{\sigma'})^* \mathcal{L}_\Psi \boxtimes \zeta'$ and $K'_{\sigma'} = K(\Sigma, \mathcal{E}'_{\sigma'})$; note that $\mathrm{Ad}(\dot{v})^* \mathcal{E}_\sigma = \mathcal{E}'_{\sigma'}$. The complexes $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}'_{\sigma'})$ and $\mathrm{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}_\sigma)$ are then isomorphic and so we get an isomorphism

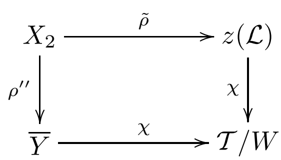
5.5.6.
$$\Phi : \mathrm{ind}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}(K_\sigma) \xrightarrow{\sim} \mathrm{ind}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}}(K'_{\sigma'}).$$

Restricting Φ to the nilpotent set, we get an isomorphism

$$\Phi_n : \mathrm{ind}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}} K(C, \zeta) \xrightarrow{\sim} \mathrm{ind}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}} K(C, \zeta')$$

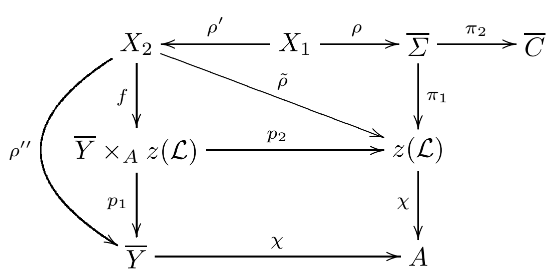
which depends on v. We want to prove that the isomorphism Φ_n does not depend on $\sigma \in z(\mathcal{L})$.

Let T be an F-stable maximal torus of L and let $\mathcal{T}$ be its Lie algebra; put $W = W_G(T)$. We use the notation of 5.1.11 and 5.1.31 relatively to (P, L, Σ). Let $\pi_1 : \overline{\Sigma} \to z(\mathcal{L})$ be the first projection and $\pi_2 : \overline{\Sigma} \to \overline{C}$ be the second projection. The morphism $X_1 \xrightarrow{\rho} \overline{\Sigma} \xrightarrow{\pi_1} z(\mathcal{L})$ factorizes through a morphism $\tilde{\rho} : X_2 \to z(\mathcal{L})$. We have the following commutative diagram.



where χ denotes the Steinberg map that maps $x \in \mathcal{G}$ onto $\mathcal{O}^G_{x_s} \cap \mathcal{T}$. Assume that $z = \mathrm{Ad}(g^{-1})z'$ with $z, z' \in z(\mathcal{L})$ and $g \in G$. Put $L' = g^{-1}Lg$. Then L and L' are two Levi subgroups of $C_G(z)^o$ which support a cuspidal pair, hence from [DLM97, 1.1(ii)] we have $hLh^{-1} = g^{-1}Lg$ for some $h \in C_G(z)^o$, i.e. $gh \in N_G(L)$. We thus proved that two elements of $z(\mathcal{L})$ which are G-conjugate are conjugate in $N_G(L)$. As a consequence we get that $\chi\big(z(\mathcal{L})\big) \simeq z(\mathcal{L})/W_G(L) \hookrightarrow \mathcal{T}/W$. Let $A = z(\mathcal{L})/W_G(L)$. We have the following commutative diagram.

5.5.7.



Put $K = K_\sigma$, $K' = K'_{\sigma'}$, $\vartheta = (m_\sigma)^* \mathcal{L}_\Psi[\dim z(\mathcal{L})]$, $\vartheta' = (m_{\sigma'})^* \mathcal{L}_\Psi[\dim z(\mathcal{L})]$, $\mathcal{K} = \mathrm{IC}(\overline{C}, \zeta)[\dim C]$ and $\mathcal{K}' = \mathrm{IC}(\overline{C}, \zeta')[\dim C]$. We have $K = (\pi_1)^* \vartheta \otimes (\pi_2)^* \mathcal{K}$ and $K' = (\pi_1)^* \vartheta' \otimes (\pi_2)^* \mathcal{K}'$. Let $\vartheta_2, \vartheta'_2, \mathcal{K}_2, \mathcal{K}'_2 \in \mathcal{D}^b_c(X_2)$ be such that

$$(\rho')^* \vartheta_2[\dim P] = \rho^* \pi_1^* \vartheta[\dim G + \dim U_P],$$

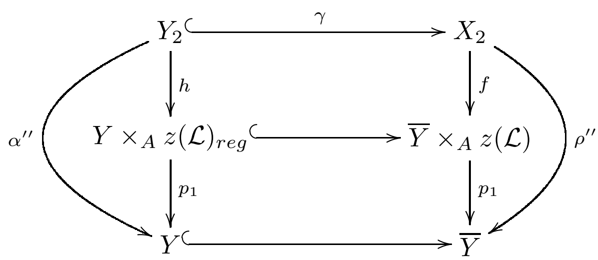
$$(\rho')^* \vartheta'_2[\dim P] = \rho^* \vartheta' \pi_1^*[\dim G + \dim U_P],$$

$$(\rho')^* \mathcal{K}_2[\dim P] = \rho^* \pi_2^* \mathcal{K}[\dim G + \dim U_P],$$

$$(\rho')^* \mathcal{K}'_2[\dim P] = \rho^* \pi_2^* \mathcal{K}'[\dim G + \dim U_P].$$

Put $K_2 = \vartheta_2 \otimes \mathcal{K}_2$ and $K'_2 = \vartheta'_2 \otimes \mathcal{K}'_2$. Since $(\rho')^* K_2[\dim P] = \rho^* K[\dim G + \dim U_P]$ and $(\rho')^* K'_2[\dim P] = \rho^* K'[\dim G + \dim U_P]$, we have $(\rho'')_! K_2 = \mathrm{ind}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}} K$ and $(\rho'')_! K'_2 = \mathrm{ind}^{\mathcal{G}}_{\mathcal{L} \subset \mathcal{P}} K'$. We have the following cartesian diagram.

5.5.8.



where γ is given by $\gamma(x, gL) = (x, gP)$, see 5.1.27. Let $V = Y \times_A z(\mathcal{L})_{reg}$, then $\dim V = \dim Y = \dim Y_2$. The morphism α'' being finite, the morphism h is also finite and so $h(Y_2) = V$. Since the morphism ρ'' is proper, the morphism f is also proper, in particular it is closed; we thus have $f(X_2) = \overline{Y} \times_A z(\mathcal{L}) = \overline{V}$. Let ξ_2 and ξ'_2 be the local systems on Y_2 such that $\gamma^*(K_2) = \xi_2[\dim X_2]$ and $\gamma^*(K'_2) = \xi'_2[\dim X_2]$. We now prove that $f_!(K_2) = \mathrm{IC}\big(\overline{V}, h_*(\xi_2)\big)[\dim V]$. From 5.5.8, we have $\mathcal{H}^{-\dim V}(f_! K_2)|_V \simeq h_*(\xi_2)$. Since K_2 is an intersection cohomology complex on X_2, we also have $\mathcal{H}^i(f_! K_2) = 0$ if $i < -\dim V$. It remains to check that

(i) $\dim \big(Supp(\mathcal{H}^i(f_! K_2))\big) < -i$ if $i > \dim V$,

(ii) $\dim \big(Supp(\mathcal{H}^i(D_{f(X_2)} f_! K_2))\big) < -i$ if $i > \dim V$.

Clearly we have $\dim \big(X_2 \times_{f(X_2)} X_2\big) \leq \dim \big(X_2 \times_{\overline{Y}} X_2\big)$. From 5.1.15, we deduce that

$$\dim \big(X_2 \times_{f(X_2)} X_2\big) \leq \dim G - \dim L + \dim \Sigma.$$

Hence, we prove (i) as we proved (a) in 5.1.33. Since f is proper, the functor $f_!$ commutes with the Verdier dual, hence the proof of (ii) is completely similar to that of (i).

Similarly, we prove that $f_!(K_2') = \mathrm{IC}\big(\overline{V}, h_*(\xi_2')\big)[\dim V]$.

Let $W_G(L)$ act on $\overline{Y} \times_A z(\mathcal{L})$ by Ad on the second coordinate; if $w \in W_G(L)$, we denote by f_w the corresponding automorphism of $\overline{Y} \times_A z(\mathcal{L})$. The set V is then $W_G(L)$-stable and we have a canonical isomorphism $(f_v)^*(h_*\xi_2) \simeq h_*\xi_2'$. Hence from the properties of intersection cohomology complexes, this isomorphism extends to a unique isomorphism

$$\Phi' : (f_v)^*(f_! K_2) \xrightarrow{\sim} f_! K_2'.$$

Then the isomorphism $(p_1)_!(\Phi')$ is nothing but the isomorphism Φ : $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K) \xrightarrow{\sim} \mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K')$ of 5.5.6; we thus have to show that the restriction of $(p_1)_!(\Phi')$ to the nilpotent set does not depend on σ.

Let $d = \dim G - \dim L$. From 5.5.7, note that

$$\big(f^*(p_2)^*[d]\vartheta\big) \otimes \mathcal{K}_2 = K_2 \quad \text{and} \quad \big(f^*(p_2)^*[d]\vartheta'\big) \otimes \mathcal{K}_2' = K_2'.$$

Put $\tilde{\vartheta} = (p_2)^*[d]\vartheta$ and $\tilde{\vartheta}' = (p_2)^*[d]\vartheta'$. Since the morphism p_2 is $W_G(L)$-invariant, we get that $(f_v)^*\tilde{\vartheta} = \tilde{\vartheta}'$. On the other hand, from the projection formula we have

$$f_!(K_2) \simeq \tilde{\vartheta} \otimes f_!\big(\mathcal{K}_2\big) \quad \text{and} \quad f_!(K_2') \simeq \tilde{\vartheta}' \otimes f_!\big(\mathcal{K}_2'\big).$$

We deduce that

$$\Phi' : \tilde{\vartheta}' \otimes (f_v)^*\big(f_!\mathcal{K}_2\big) \xrightarrow{\sim} \tilde{\vartheta}' \otimes f_!\mathcal{K}_2'.$$

We have $\Phi' = \mathrm{Id}_{\tilde{\vartheta}'} \otimes \alpha$ for some $\alpha : (f_v)^*\big(f_!\mathcal{K}_2\big) \xrightarrow{\sim} f_!\mathcal{K}_2'$ which does not depend on σ. Put $\mathcal{N} = p_1^{-1}(\overline{Y}_{nil})$, then $\Phi'|_{\mathcal{N}} = (\mathrm{Id}_{\tilde{\vartheta}'})|_{\mathcal{N}} \otimes (\alpha|_{\mathcal{N}})$. But $p_2^{-1}(0) = \mathcal{N}$, hence $(\mathrm{Id}_{\tilde{\vartheta}'})|_{\mathcal{N}} = (p_2)^*[d]\big(\mathrm{Id}_{\vartheta_n'}\big)$ where ϑ_n' is the constant sheaf on $\{0\}$ shifted by $\dim z(\mathcal{L})$. Hence $\Phi'|_{\mathcal{N}}$ does not depend on σ and so we get that $(p_1)_!(\Phi'|_{\mathcal{N}}) = \Phi_n$ does not depend on σ. □

5.5.9 The Character Formula

Let $\sigma \in z(\mathcal{L})^F$. Assume that $\mathcal{E}_\sigma$ is F-stable and let $\phi_\sigma : F^*(\mathcal{E}_\sigma) \xrightarrow{\sim} \mathcal{E}_\sigma$ be an isomorphism. We denote by $\phi_\sigma^{\mathcal{G}} : F^*(K_\sigma^{\mathcal{G}}) \xrightarrow{\sim} K_\sigma^{\mathcal{G}}$ the isomorphism induced by ϕ_σ. We now give a formula which expresses the values of the function $\mathbf{X}_{K_\sigma^{\mathcal{G}}, \phi_\sigma^{\mathcal{G}}}$ in terms of the values of some generalized Green functions. Let $x \in \mathcal{G}^F$ and assume that there exists $g \in G^F$ such that $\mathrm{Ad}(g^{-1})x_s \in z(\mathcal{L})$. Put $L_g = gLg^{-1}$ and $\mathcal{L}_g = \mathrm{Lie}(L_g) = \mathrm{Ad}(g)(\mathcal{L})$. We have $x_s \in z(\mathcal{L}_g)$ and so L_g is a Levi subgroup of $C_G^o(x_s)$. Let $C_g = \mathrm{Ad}(g)C$ and let (ζ_g, ϕ_g) be the inverse image of the F-equivariant sheaf $(\mathcal{E}_\sigma, \phi_\sigma)$ by $C_g \to \Sigma$, $v \mapsto \mathrm{Ad}(g^{-1})(x_s + v)$. Note that the irreducible local system ζ_g is isomorphic to $\mathrm{Ad}(g^{-1})^*\zeta$. Then the following formula is the Lie algebra version of [Lus85b, 8.5].

$$\mathbf{X}_{K^{\mathcal{G}}_{\sigma},\phi^{\mathcal{G}}_{\sigma}}(x) = |C^o_G(x_s)^F|^{-1} \sum_{\{g\in G^F \mid \mathrm{Ad}(g^{-1})x_s\in z(\mathcal{L})\}} \mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{\mathcal{L}_g,C_g,\zeta_g,\phi_g}(x_n).$$

Remark 5.5.10. The proof of the above formula is entirely similar to that of [Lus85b, 8.5], in particular it uses 5.5.3. We do not prove it here, however we will prove a similar formula for another kind of complexes (see next chapter).

5.5.11 Generalized Green Functions and Two-Variable Green Functions

5.5.12. Let $f : G_{uni} \xrightarrow{\sim} \mathcal{G}_{nil}$ be a G-equivariant isomorphism defined over $\mathbb{F}_q$, i.e. which commutes with F. Put $(C^L, \zeta^L, \phi^L) = (f^{-1}(C), f^*(\zeta), f^*(\phi))$ and denote by K^G_o the complex on G induced by the cuspidal datum $(L, Z^o_L C^L, \overline{\mathbb{Q}}_\ell \boxtimes \zeta^L)$. Let $\phi^G_o : F^*(K^G_o) \xrightarrow{\sim} K^G_o$ be the isomorphism induced by $1 \boxtimes \phi : F^*(\overline{\mathbb{Q}}_\ell \boxtimes \zeta^L) \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell \boxtimes \zeta^L$. Let us show that for any $v \in \mathcal{G}^F_{nil}$,

$$\mathbf{X}_{K^{\mathcal{G}}_o,\phi^{\mathcal{G}}_o}(v) = \mathbf{X}_{K^G_o,\phi^G_o}(f^{-1}(v)). \qquad (*)$$

To prove this, by 5.4.2, 5.3.6(i) and 5.4.9, we may assume that P is F-stable and prove, for any $v \in \mathcal{G}^F_{nil}$ and any $w \in W_G(L)$, that

$$\mathbf{X}_{K^{\mathcal{G}}_o,\theta^{\mathcal{G}}_w\circ\phi^{\mathcal{G}}_o}(v) = \mathbf{X}_{K^G_o,\theta^G_w\circ\phi^G_o}(f^{-1}(v)) \qquad (1)$$

where $\theta^{\mathcal{G}}_w$ and θ^G_w are as in 5.3.6(iii). We denote by $\{A_\iota | \iota \in I^F_o\}$ the set of admissible complexes (up to isomorphism) which are direct summand of $K^{\mathcal{G}}_o$. By 5.3.7, the restriction of $A_\iota[-\dim z(\mathcal{L})]$ to $\mathcal{G}_{nil}$ is a simple perverse sheaf K_ι supported by the Zariski closure of a nilpotent orbit. For $\iota \in I^F_o$, let $\phi_\iota : F^*(K_\iota) \xrightarrow{\sim} K_\iota$ be an isomorphism and let $\mathcal{X}^{\mathcal{G}}_\iota$ be the characteristic function of (K_ι, ϕ_ι). We then denote by $\mathcal{X}^G_\iota$ the characteristic function of $(f^*(K_\iota), f^*(\phi_\iota))$. By 5.3.6, the irreducible characters of $W_G(L)$ are in bijection with I_o, and the F-stable ones are in bijection with I^F_o. For $\iota \in I^F_o$, let χ_ι be the F-stable irreducible character corresponding to ι and let $\tilde{\chi}_\iota$ be the "preferred" extension [Lus86a, 24] of χ_ι to the semi-direct product $W_G(L) \rtimes \langle F \rangle$. Then with a specific choice of ϕ and $\{\phi_\iota | \iota \in I^F_o\}$ (see [Lus92, section 5] or [DLM97, 1.4]), the formula 5.4.5 becomes

$$\mathbf{X}_{K^{\mathcal{G}}_o,\theta^{\mathcal{G}}_w\circ\phi^{\mathcal{G}}_o}(v) = \sum_{\iota\in I^F_o} \tilde{\chi}_\iota(wF)\mathcal{X}^{\mathcal{G}}_\iota(v) \quad \text{ for } v \in \mathcal{G}^F_{nil}.$$

Since P is F-stable, the F-equivariant complex $\left(K^G_o|_{G_{uni}}, \phi^G_o|_{G_{uni}}\right)$ is the inverse image by f^* of $\left(K^{\mathcal{G}}_o|_{\mathcal{G}_{nil}}, \phi^{\mathcal{G}}_o|_{\mathcal{G}_{nil}}\right)$, and so the isomorphisms $f^*(\phi)$ and

$\{f^*(\phi_\iota)|\,\iota\in I_o^F\}$ satisfies the same properties as ϕ and $\{\phi_\iota|\,\iota\in I_o^F\}$. We thus have

$$\mathbf{X}_{K_o^G,\theta_w^G\circ\phi_o^G}(u)=\sum_{\iota\in I_o^F}\tilde{\chi}_\iota(wF)\mathcal{X}_\iota^G(u),\qquad \text{for } u\in G_{uni}^F.$$

Actually to establish this formula, we also use the fact that $f^*(K_\iota)$, which (up to isomorphism) does not depend on the choice of f, is the direct summand of $K_o^G|_{G_{uni}}$ corresponding to the irreducible character χ_ι of $W_G(L)$. This fact follows from the explicit computation of the generalized Springer correspondence [Lus84][LS85] both in the group case and in the Lie algebra case. Recall that the generalized Springer correspondence is the map $\mathrm{Irr}(W_G(L))\longrightarrow\{K_\iota|\iota\in I_o\}$. We thus proved (1).

5.5.13. Let $\omega:\mathcal{G}_{nil}\xrightarrow{\sim}G_{uni}$ be a G-equivariant isomorphism (defined over $\mathbb{F}_q$) and let $\mathcal{Q}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}$ be the two-variable Green function as in 3.2.11. Assume that q is large enough so that [Lus90, 1.14] is available. Then from 5.5.12(*) and [Lus90, 1.14] we have

$$\mathcal{Q}_{\mathcal{L},C,\zeta,\phi}^{\mathcal{G}}(x)=\sum_{v\in\mathcal{L}_{nil}^F}\mathcal{Q}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(x,v)\mathbf{X}_{K_o,1\boxtimes\phi}(v)$$

for any $x\in\mathcal{G}^F$. If L is a maximal torus, then we can drop the assumption on q by [Sho95].

5.5.14 Geometrical Induction and Deligne-Lusztig Induction

Assume that q is large enough so that the formula in 5.5.13 holds. Let $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}$ be the Deligne-Lusztig induction relatively to $\omega:\mathcal{G}_{nil}\xrightarrow{\sim}G_{uni}$. Recall that $R_{\mathcal{L}}^{\mathcal{G}}$ denotes the geometrical induction (whose definition does not depend on a G-equivariant homeomorphism $\mathcal{G}_{nil}\to G_{uni}$). Then we have the following result.

Proposition 5.5.15. *Let $\sigma\in z(\mathcal{L})^F$ and let $\phi_\sigma:F^*(K_\sigma)\xrightarrow{\sim}K_\sigma$ be an isomorphism. Then*

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K_\sigma,\phi_\sigma}\big)=R_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K_\sigma,\phi_\sigma}\big).$$

Proof: Let $x\in\mathcal{G}^F$, we have

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K_\sigma,\phi_\sigma}\big)(x) = |L^F|^{-1} \sum_{y\in\mathcal{L}^F} S_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(x,y)\mathbf{X}_{K_\sigma,\phi_\sigma}(y).$$

Since the complex K_σ is supported by $\overline{\Sigma}$, we have

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K_\sigma,\phi_\sigma}\big)(x) = |L^F|^{-1} \sum_{(t,v)\in z(\mathcal{L})^F\times\overline{C}^F} S_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(x,t+v)\mathbf{X}_{K_\sigma,\phi_\sigma}(t+v).$$

But for $(t,v)\in z(\mathcal{L})^F\times\overline{C}^F$, we have

$$S_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(x,t+v) = \sum_{h\in G^F\,|\,\mathrm{Ad}(h)t=x_s} |C_L^o(t)^F||C_G^o(t)^F|^{-1}\mathcal{Q}_{C_{\mathcal{L}}(t)}^{C_{\mathcal{G}}(t)}(\mathrm{Ad}(h^{-1})x_n,v).$$

Hence, we get that $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K_\sigma,\phi_\sigma}\big)(x) =$

$$\sum_{(t,v)\in z(\mathcal{L})^F\times\overline{C}^F}\ \sum_{\substack{h\in G^F\\ \mathrm{Ad}(h)t=x_s}} |C_G^o(t)^F|^{-1}\mathcal{Q}_{\mathcal{L}}^{C_{\mathcal{G}}(t)}(\mathrm{Ad}(h^{-1})x_n,v)\mathbf{X}_{K_\sigma,\phi_\sigma}(t+v).$$

By interchanging the sums we have

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K_\sigma,\phi_\sigma}\big)(x) = |C_G^o(x_s)^F|^{-1}\times$$

$$\sum_{\substack{h\in G^F\\ \mathrm{Ad}(h^{-1})x_s\in z(\mathcal{L})}}\ \sum_{v\in\overline{C}^F} \mathcal{Q}_{\mathcal{L}}^{C_{\mathcal{G}}(\mathrm{Ad}(h^{-1}x_s))}(\mathrm{Ad}(h^{-1})x_n,v)\mathbf{X}_{K_\sigma,\phi_\sigma}(\mathrm{Ad}(h^{-1})x_s+v).$$

Using the notation of 5.5.9, we may re-write this formula as follows.

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K_\sigma,\phi_\sigma}\big)(x) = |C_G^o(x_s)^F|^{-1} \sum_{\substack{h\in G^F\\ \mathrm{Ad}(h^{-1})x_s\in z(\mathcal{L})}}\ \sum_{v\in\overline{C_h}^F} \mathcal{Q}_{\mathcal{L}}^{C_{\mathcal{G}}(x_s)}(x_n,v)\mathbf{X}_{K_h,\phi_h}(v)$$

where $K_h = K(C_h,\zeta_h)$. Hence the proposition is a consequence of 5.5.9 and 5.5.12. □

Theorem 5.5.16. *The geometrical induction coincides with Deligne-Lusztig induction.*

Proof: This is a straightforward consequence of 5.5.15, 5.4.15 and 3.2.22. □

Remark 5.5.17. The theorem shows the independence of $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}$ from the choice of the G-equivariant isomorphism ω.

6

Deligne-Lusztig Induction and Fourier Transforms

Throughout this chapter, unless specified, we assume that the prime p is acceptable for G and that q is large enough such that the geometrical induction coincides with Deligne-Lusztig induction. Fourier transforms considered will be with respect to (μ, Ψ) as in 5.2. The goal of the chapter is to discuss the commutation formula conjectured in 3.2.30. We reduce this conjecture to the case where the function f of 3.2.30 is the characteristic function of a cuspidal nilpotently supported F-equivariant orbital perverse sheaf. We then prove the conjecture in almost all cases under a stronger assumption on p.

6.1 Frobenius Action on the Parabolic Induction of Cuspidal Orbital Perverse Sheaves

Throughout this section we fix a Levi subgroup L of G, a parabolic subgroup P of G having L as a Levi subgroup and we denote by $\mathcal{P}$, $\mathcal{L}$ the respective Lie algebras of P and L. We assume that $\mathcal{L}$ supports a cuspidal nilpotent pair (C, ζ). When the variety $z(\mathcal{L})$ will be used as a set parametrazing the cuspidal orbital pairs of $\mathcal{L}$ of the form $(\sigma + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$, $\sigma \in z(\mathcal{L})$, it will be denoted by S. For any $\sigma \in z(\mathcal{L})$, we put $(L, \Sigma, \mathcal{E}_{1,\sigma}) = (L, z(\mathcal{L}) + C, (m_\sigma)^* \mathcal{L}_\Psi \boxtimes \zeta)$ where m_σ is as in 5.2.1, $K_{1,\sigma} = K(\Sigma, \mathcal{E}_{1,\sigma})$, $\mathcal{E}_{2,\sigma} = \overline{\mathbb{Q}}_\ell \boxtimes \zeta \in ls(\sigma + C)$ and $K_{2,\sigma} = K(\sigma + C, \mathcal{E}_{2,\sigma})$.

In the following, the group L acts on $S \times \mathcal{L}$ by Ad on the second coordinate and trivially on the first coordinate, and G acts on $S \times \mathcal{G}$ by Ad on $\mathcal{G}$ and trivially on S. Following [Wal01, Chapter II], we define a functor $\mathrm{ind}_{S \times \mathcal{L}, \mathcal{P}}^{S \times \mathcal{G}}$: $\mathcal{M}_L(S \times \mathcal{L}) \rightarrow \mathcal{D}_c^b(S \times \mathcal{G})$ and two L-equivariant simple perverse sheaves K_1 and K_2 on $S \times \mathcal{L}$ such that, for any $\sigma \in z(\mathcal{L})$,

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• the restrictions of K_1 and K_2 to $\{\sigma\}\times\mathcal{L}\simeq\mathcal{L}$ are respectively $K_{1,\sigma}[\dim S]$ and $K_{2,\sigma}[\dim S]$,

• the complexes $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K_1)$ and $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K_2)$ are G-equivariant **simple** perverse sheaves on $S\times\mathcal{G}$ and their restrictions to $\{\sigma\}\times\mathcal{G}\simeq\mathcal{G}$ are respectively $\left(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K_{1,\sigma}\right)[\dim S]$ and $\left(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K_{2,\sigma}\right)[\dim S]$.

6.1.1 The Functor $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}:\mathcal{M}_L(S\times\mathcal{L})\rightarrow\mathcal{D}_c^b(S\times\mathcal{G})$

Define

$$V_{S,1}:=\{(s,x,h)\in S\times\mathcal{G}\times G|\mathrm{Ad}(h^{-1})x\in\mathcal{P}\},$$

$$V_{S,2}:=\{(s,x,hP)\in S\times\mathcal{G}\times(G/P)|\mathrm{Ad}(h^{-1})x\in\mathcal{P}\}.$$

We have the following diagram

$$S\times\mathcal{L}\xleftarrow{\ \pi_S\ }V_{S,1}\xrightarrow{\ \pi'_S\ }V_{S,2}\xrightarrow{\ \pi''_S\ }S\times\mathcal{G}$$

where $\pi_S=\mathrm{Id}_S\times\pi$, $\pi'_S=\mathrm{Id}_S\times\pi'$ and $\pi''_S=\mathrm{Id}_S\times\pi''$ with π, π' and π'' as in 5.1.1.

Let K be an L-equivariant perverse sheaf on $S\times\mathcal{L}$. The morphism π_S is smooth with connected fibers of dimension $m=\dim G+\dim U_P$ and is P-equivariant if we let P acts on $V_{S,1}$ by $p.(s,x,g)=(s,x,gp^{-1})$ and on $S\times\mathcal{L}$ by $p.(s,x)=(s,\mathrm{Ad}(\pi_P(p))x)$. Hence the complex $(\pi_S)^*K[m]$ is P-equivariant. Since the morphism π'_S is a locally trivial principal P-bundle, we deduce that there exists a unique perverse sheaf $\tilde{K}$ on $V_{S,2}$ such that $(\pi'_S)^*\tilde{K}[\dim P]=(\pi_S)^*K[m]$. Define

$$\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K):=(\pi''_S)_!\tilde{K}.$$

Let G act on $S\times\mathcal{L}$ trivially and let G act on $V_{S,1}$ and $V_{S,2}$ by the adjoint action on the second coordinate, by left translation on the third coordinate and trivially on S. Then if the complex $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K)$ is a perverse sheaf, it is naturally G-equivariant.

6.1.2 The Complexes $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}K(\mathcal{Z}\times C,\mathcal{E})$

Let $\mathcal{Z}$ be a smooth irreducible closed subvariety of $S\times z(\mathcal{L})$; we identify the variety $\mathcal{Z}\times C$ with a subvariety of $S\times\mathcal{L}$ via the morphism $((s,z),v)\mapsto(s,z+v)$.

We assume that the fibers of the morphism $\phi : \mathcal{Z} \to S$ given by the projection on the first coordinate are smooth, irreducible and are all of dimension $\dim \mathcal{Z} - \dim S$; if $s \in S$, we denote by $\mathcal{Z}_s$ the set $\{z \in z(\mathcal{L}) | (s,z) \in \mathcal{Z}\} \simeq \phi^{-1}(s)$. Let ξ be a local system on $\mathcal{Z}$ and let $\mathcal{E} = \xi \boxtimes \zeta \in ls_L(\mathcal{Z} \times C)$. We denote by $K(\mathcal{Z} \times C, \mathcal{E})$ the complex $K^{S\times\mathcal{L}}(\mathcal{Z} \times C, \mathcal{E})$; recall that $K^{S\times\mathcal{L}}(\mathcal{Z} \times C, \mathcal{E})$ denotes the extension by zero on $(S \times \mathcal{L}) - (\overline{\mathcal{Z} \times C})$ of the complex $\mathrm{IC}(\overline{\mathcal{Z} \times C}, \mathcal{E})[\dim(\mathcal{Z} \times C)]$.

The complexes K_1 and K_2 mentioned at the beginning of 6.1 will be of the form $K(\mathcal{Z} \times C, \mathcal{E})$ for some $\mathcal{Z}$ and $\mathcal{E}$ as above. Before defining K_1 and K_2, we study, as we did with the complexes $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\Sigma, \mathcal{E})$ in 5.1.9 and 5.1.26, the general properties of the complexes $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}} K(\mathcal{Z} \times C, \mathcal{E})$; most of these properties will be deduced from the results of 5.1.9 and 5.1.26.

For $s \in S$, let $j_{s,\mathcal{G}} : \mathcal{G} \to S \times \mathcal{G}$, $x \mapsto (s,x)$ and let $\xi_s \in ls(\mathcal{Z}_s)$ be the inverse image of ξ by $\mathcal{Z}_s \to \mathcal{Z}$, $z \mapsto (s,z)$. Since $\mathcal{Z}$ and $\mathcal{Z}_s$ are smooth and irreducible, we verify as in 5.1.42 and 5.1.43 that

6.1.3. $(j_{s,\mathcal{L}})^* \left(K(\mathcal{Z} \times C, \mathcal{E})\right) = K(\mathcal{Z}_s + C, \xi_s \boxtimes \zeta)[dim\, S]$, *and*

$$(j_{s,\mathcal{G}})^* \left(ind_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}} K(\mathcal{Z} \times C, \mathcal{E})\right) = \left(ind_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K(\mathcal{Z}_s + C, \xi_s \boxtimes \zeta)\right)[dim\, S]$$

for any $s \in S$.

We also have:

6.1.4. *The complex* $ind_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}} \left(K(\mathcal{Z} \times C, \mathcal{E})\right)$ *is a* G*-equivariant perverse sheaf on* $S \times \mathcal{G}$.

Proof: Let K be the complex $K(\mathcal{Z} \times C, \mathcal{E})$ and $K^{S\times\mathcal{G}}$ be the complex $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}} \left(K(\mathcal{Z} \times C, \mathcal{E})\right)$. We show that for any $i \in \mathbb{Z}$,
(i) $\dim \left(Supp\,(\mathcal{H}^i K^{S\times\mathcal{G}})\right) \leq -i$ and ,
(ii) $\dim \left(Supp\,(\mathcal{H}^i D_{S\times\mathcal{G}} K^{S\times\mathcal{G}})\right) \leq -i$.

Let us prove (i). Let $i \in \mathbb{Z}$; we have

$$Supp\,(\mathcal{H}^i K^{S\times\mathcal{G}}) = \{(s,x) \in S \times \mathcal{G} | \mathcal{H}^i_{(s,x)} K^{S\times\mathcal{G}} \neq 0\}.$$

For $s \in S$, we denote by K_s the complex $K(\mathcal{Z}_s + C, \xi_s \boxtimes \zeta)$. Let $pr_S : Supp\,(\mathcal{H}^i K^{S\times\mathcal{G}}) \to S$ be the projection on the first coordinate. It follows from 6.1.3 that

$$\dim\left(pr_S^{-1}(s)\right) = \dim Supp\left(\mathcal{H}^{i+\dim S}(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K_s)\right).$$

The complex $\operatorname{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K_s)$ being a perverse sheaf by 5.1.19, we have

$$\dim Supp\left(\mathcal{H}^{i+\dim S}(\operatorname{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K_s)\right) \leq -i - \dim S.$$

Hence we deduce that

$$\dim Supp\,(\mathcal{H}^i K^{S\times\mathcal{G}}) \leq -i.$$

The Verdier dual operator commutes with the functor $\operatorname{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}$; the proof of (ii) is thus completely similar to that of (i). □

Remark 6.1.5. Note that 6.1.4 has nothing to do with the fact that the nilpotent pair (C,ζ) is cuspidal.

6.1.6. Define

$$X_{S,1} = \{(s,x,g) \in S\times\mathcal{G}\times G|\operatorname{Ad}(g^{-1})x \in \mathcal{Z}_s + \overline{C} + \mathcal{U}_P\},$$

$$X_{S,2} = \{(s,x,gP) \in S\times\mathcal{G}\times(G/P)|\operatorname{Ad}(g^{-1})x \in \mathcal{Z}_s + \overline{C} + \mathcal{U}_P\}.$$

We have the following commutative diagram

$$\begin{array}{ccccccc}
\overline{\mathcal{Z}\times C} & \xleftarrow{\rho_S} & X_{S,1} & \xrightarrow{\rho'_S} & X_{S,2} & \xrightarrow{\rho''_S} & \mathcal{G} \\
\downarrow & & \downarrow & & \downarrow & & \| \\
S\times\mathcal{L} & \xleftarrow{\pi_S} & V_{S,1} & \xrightarrow{\pi'_S} & V_{S,2} & \xrightarrow{\pi''_S} & \mathcal{G}
\end{array}$$

where ρ_S, ρ'_S and ρ''_S are given respectively by the restrictions of π_S, π'_S and π''_S. As in 5.1.10, we show that the varieties $X_{S,1}$ and $X_{S,2}$ are irreducible and respectively closed in $V_{S,1}$ and $V_{S,2}$.

Define

$$X_{S,1,o} = \{(s,x,g) \in S\times\mathcal{G}\times G|\operatorname{Ad}(g^{-1})x \in \mathcal{Z}_s + C + \mathcal{U}_P\},$$

$$X_{S,2,o} = \{(s,x,gP) \in S\times\mathcal{G}\times(G/P)|\operatorname{Ad}(g^{-1})x \in \mathcal{Z}_s + C + \mathcal{U}_P\}.$$

As in 5.1.9 we prove that the varieties $X_{S,1,o}$ and $X_{S,2,o}$ are respectively smooth open subsets of $X_{S,1}$ and $X_{S,2}$, and we construct a G-equivariant local system $\tilde{\mathcal{E}}$ on $X_{S,2,o}$ such that

$$(\rho''_S)_!\left(\mathrm{IC}(\overline{X_{S,2,o}},\tilde{\mathcal{E}})[\dim X_{S,2,o}]\right) = \mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}} K(\mathcal{Z}\times C,\mathcal{E}).$$

We have the following proposition.

Proposition 6.1.7. *The varieties $X_{S,2}$ and $\rho''_S(X_{S,2})$ are both of dimension $dim\,G - dim\,L + dim\,(\mathcal{Z}\times C)$.*

Proof: The fact that the variety $X_{S,2}$ is of dimension $\dim G - \dim L + \dim(\mathcal{Z}\times C)$ is clear. Let $f : \rho''_S(X_{S,2}) \to S$, $(s,x)\mapsto s$. We have

$$f^{-1}(s) \simeq \{x\in\mathcal{G}|\,\exists g\in G, \mathrm{Ad}(g^{-1})x\in\mathcal{Z}_s+\overline{C}+\mathcal{U}_P\}$$

for any $s\in S$. Moreover for any $s\in S$, we have $\dim\mathcal{Z}_s = \dim\mathcal{Z} - \dim S$. Hence we deduce from 5.1.18 that the fibers of f are all of dimension $\dim G - \dim L + \dim(\mathcal{Z}\times C) - \dim S$. As a consequence we get that dim $(\rho''_S(X_{S,2})) = \dim G - \dim L + \dim(\mathcal{Z}\times C)$. □

6.1.8. Recall that $z(\mathcal{L})_{reg}$ denotes the set of L-regular elements in $\mathcal{G}$. Let

$$\mathcal{Z}_{reg} = \{(s,z)\in\mathcal{Z}|z\in z(\mathcal{L})_{reg}\}$$

and for $s\in S$, let $(\mathcal{Z}_s)_{reg} = \mathcal{Z}_s\cap z(\mathcal{L})_{reg}$. We assume from now and until the end of this section that $\mathcal{Z}_{reg}\neq\emptyset$. Define

$$Y_{S,1} = \{(s,x,g)\in S\times\mathcal{G}\times G|\,\mathrm{Ad}(g^{-1})x\in(\mathcal{Z}_s)_{reg}+C\},$$

$$Y_{S,2} = \{(s,x,gL)\in S\times\mathcal{G}\times(G/L)|\,\mathrm{Ad}(g^{-1})x\in(\mathcal{Z}_s)_{reg}+C\},$$

$$Y_S = \{(s,x)\in S\times\mathcal{G}|\,\exists g\in G, \mathrm{Ad}(g^{-1})x\in(\mathcal{Z}_s)_{reg}+C\}.$$

We have a diagram

6.1.9. $$\mathcal{Z}\times C \xleftarrow{\alpha_S} Y_{S,1} \xrightarrow{\alpha'_S} Y_{S,2} \xrightarrow{\alpha''_S} Y_S$$

where $\alpha'_S(s,x,g) = (s,x,gL)$, $\alpha''_S(s,x,gL) = (s,x)$ and where $\alpha_S(s,x,g) = ((s,t),v)$ if $\mathrm{Ad}(g^{-1})x = t+v$ with $t\in(\mathcal{Z}_s)_{reg}$ and $v\in C$.

The morphism α''_S is a Galois covering with Galois group, the normalizer of $\mathcal{Z}\times C$ in $W_G(L)$ where $N_G(L)$ acts on $S\times\mathcal{L}$ by Ad on $\mathcal{L}$ and trivially on S.

We have the following proposition.

Proposition 6.1.10. *The map $\gamma_S : Y_{S,2}\to(\rho''_S)^{-1}(Y_S)$ defined by $(s,x,gL)\mapsto(s,x,gP)$ is an isomorphism.*

Proof: We verify as in the proof of 5.1.27 that the image of γ_S is a variety and that γ_S induces in isomorphism onto its image; the proof of the surjectivity of γ_S reduces easily to 5.1.27. □

Proposition 6.1.11. *The variety Y_S is a smooth irreducible locally closed subvariety of $S \times \mathcal{G}$ of dimension $\dim G - \dim L + \dim(\mathcal{Z} \times C)$.*

Proof: The proof is completely similar to that of 5.1.28. □

From 6.1.7 and 6.1.11 we deduce the follwing fact.

Corollary 6.1.12. *We have $\overline{Y_S} = \rho''_S(X_{S,2})$.*

Now let ξ_2 be the irreducible local system on $Y_{S,2}$ such that $(\alpha'_S)^*\xi_2 = (\alpha_S)^*\mathcal{E}$. Define

$$\mathrm{ind}_{\mathcal{Z}\times C}^{S\times\mathcal{G}}(\mathcal{E}) := K(Y_S, (\alpha''_S)_*\xi_2)$$

where $K(Y_S, (\alpha''_S)_*\xi_2) = K^{S\times\mathcal{G}}(Y_S, (\alpha''_S)_*\xi_2)$. Since α''_S is a Galois covering, the local system $(\alpha''_S)_*\xi_2$ is semi-simple and so the complex $\mathrm{ind}_{\mathcal{Z}\times C}^{S\times\mathcal{G}}(\mathcal{E})$ is semi-simple.

From 6.1.12, note that the supports of the perverse sheaves $\mathrm{ind}_{\mathcal{Z}\times C}^{S\times\mathcal{G}}(\mathcal{E})$ and $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}K(\mathcal{Z}\times C,\mathcal{E})$ are both contained in the closed subvariety $\overline{Y_S}$ of $S\times\mathcal{G}$; moreover from 6.1.10, we show easily, as in the proof of 5.1.33, that the sheaf $\mathcal{H}^{-\dim Y_S}\left(\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}K(\mathcal{Z}\times C,\mathcal{E})\right)|_{Y_S}$ is canonically isomorphic to the local system $(\alpha''_S)_*\xi_2$. We have the following lemma.

Lemma 6.1.13. *Assume that $(\mathcal{Z}_s)_{reg} \neq \emptyset$ for any $s \in S$, then the complexes $ind_{\mathcal{Z}\times C}^{S\times\mathcal{G}}(\mathcal{E})$ and $ind_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}K(\mathcal{Z}\times C,\mathcal{E})$ are canonically isomorphic.*

Proof: Let $K^{S\times\mathcal{G}} = \mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}K(\mathcal{Z}\times C,\mathcal{E})$. The sheaf $\mathcal{H}^{-\dim Y_S}\left(K^{S\times\mathcal{G}}|_{Y_S}\right)$ is canonically isomorphic to the local system $(\alpha''_S)_*\xi_2$, moreover we have $\mathcal{H}^iK^{S\times\mathcal{G}} = 0$ if $i < -\dim Y_S$ since $K^{S\times\mathcal{G}} = (\rho''_S)_!\left(\mathrm{IC}(\overline{X_{S,2,o}},\tilde{\mathcal{E}})[\dim X_{S,2}]\right)$ and $\dim Y_S = \dim X_{S,2}$. It remains to see that for any $i > -\dim Y_S$,

(i) $\dim Supp(\mathcal{H}^iK^{S\times\mathcal{G}}) < -i$,

(ii) $\dim Supp(\mathcal{H}^iD_{S\times\mathcal{G}}K^{S\times\mathcal{G}}) < -i$.

Let $i > -\dim Y_S$. We use the notation of the proof of 6.1.4. Let $s \in S$; from 6.1.11, we have $\dim Y_S = \dim G - \dim L + \dim\mathcal{Z} + \dim C$, hence

$$i + \dim S > -(\dim G - \dim L + \dim\mathcal{Z}_s + \dim C). \tag{1}$$

Since $(\mathcal{Z}_s)_{reg} \neq \emptyset$, from 5.1.33, we get that $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K_s)$ is an intersection cohomology complex with support of dimension $\dim G - \dim L + \dim \mathcal{Z}_s + \dim C$, and so we deduce from (1) that

$$\dim Supp\left(\mathcal{H}^{i+\dim S}(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K_s)\right) < -i - \dim S.$$

We have proved that $\dim\left(pr_S^{-1}(s)\right) < -i$ for any $s \in S$, hence we deduce that

$$\dim Supp\left(\mathcal{H}^i K^{S\times\mathcal{G}}\right) < -i.$$

Since the Verdier dual operator commutes with the functor $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}$, the proof of (ii) is completely similar to that of (i). □

Remark 6.1.14. The assumption "$(\mathcal{Z}_s)_{reg} \neq \emptyset$ for any $s \in S$" in 6.1.13 is not necessary. Under the assumption 6.1.8, we can prove as in 5.1.33 that the two complexes $\mathrm{ind}_{\mathcal{Z}\times C}^{S\times\mathcal{G}}(\mathcal{E})$ and $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}} K(\mathcal{Z}\times C, \mathcal{E})$ are isomorphic; however we will not use this more general result.

6.1.15 The Complexes K_1 and K_2

Define

- $\mathcal{Z}_1 := S \times z(\mathcal{L})$,
- $\mathcal{Z}_2 := \{(z,z) | z \in z(\mathcal{L})\}$,
- $\mathcal{E}_1 := (\mu_{z(\mathcal{L})})^* \mathcal{L}_\Psi \boxtimes \zeta \in ls_L(\mathcal{Z}_1 \times C)$,
- $\mathcal{E}_2 := \overline{\mathbb{Q}}_\ell \boxtimes \zeta \in ls_L(\mathcal{Z}_2 \times C)$,
- $K_1 := K(\mathcal{Z}_1 \times C, \mathcal{E}_1) \in \mathcal{M}_L(S \times \mathcal{L})$,
- $K_2 := K(\mathcal{Z}_2 \times C, \mathcal{E}_2) \in \mathcal{M}_L(S \times \mathcal{L})$.

The complexes K_1 and K_2 are both irreducible and from 6.1.3, we have $(j_{s,\mathcal{L}})^* K_1 = K_{1,s}[\dim S]$ and $(j_{s,\mathcal{L}})^* K_2 = K_{2,s}[\dim S]$ for any $s \in S$.

Proposition 6.1.16. *[Wal01, page 43] Let $i \in \{1,2\}$. The G-equivariant perverse sheaf $ind_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K_i)$ is isomorphic to $ind_{\mathcal{Z}_i\times C}^{S\times\mathcal{G}}(\mathcal{E}_i)$; moreover it is a simple perverse sheaf.*

We outline the proof of 6.1.16 (see [Wal01]).

The fact that the complex $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K_1)$ is isomorphic to $\mathrm{ind}_{\mathcal{Z}_i\times C}^{S\times\mathcal{G}}(\mathcal{E}_1)$ follows from 6.1.13. Note that we can not use 6.1.13 to prove 6.1.16 with $i = 2$. Following Waldspurger [Wal01], we extended the definition of the Deligne-Fourier transform $\mathcal{F}^{\mathcal{G}}$ into a transformation $\mathcal{F}^{S\times\mathcal{G}} : \mathcal{M}(S\times\mathcal{G}) \to \mathcal{M}(S\times\mathcal{G})$ which transforms $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K_2)$ into $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K_1)$, see next chapter. The only element w of the Galois group of α''_S (with $\mathcal{Z}=\mathcal{Z}_1$) such that $w^*(\mathcal{E}_1)\simeq\mathcal{E}_1$ is the neutral element, hence the perverse sheaf $\mathrm{ind}_{\mathcal{Z}_1\times C}^{S\times\mathcal{G}}(\mathcal{E}_1)\simeq\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K_1)$ is simple. As a consequence we get that $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}(K_2)$ is also a simple perverse sheaf; it is thus an intersection cohomology complex. It follows from the remark just before 6.1.13 that it is isomorphic to $\mathrm{ind}_{\mathcal{Z}_2\times C}^{S\times\mathcal{G}}(\mathcal{E}_2)$.

Remark 6.1.17. The proof of 6.1.16 outlined above works if the pair (C,ζ) is cuspidal and if p is large enough so that the Fourier transforms exist; as noticed in 6.1.14, we can prove the same result without using the fact that (C,ζ) is cuspidal and with a better condition on p.

6.1.18. Assume that L, C and ζ are all F-stable. Then the complexes K_1 and K_2 are both F-stable; let $\phi_1 : F^*(K_1)\xrightarrow{\sim}K_1$ and $\phi_2 : F^*(K_2)\xrightarrow{\sim}K_2$ be two isomorphisms. Note that ϕ_1 and ϕ_2 induce two isomorphisms $\phi_1^{S\times\mathcal{G}}$: $F^*\left(\mathrm{ind}_{\mathcal{Z}_1\times C}^{S\times\mathcal{G}}(\mathcal{E}_1)\right)\xrightarrow{\sim}\mathrm{ind}_{\mathcal{Z}_1\times C}^{S\times\mathcal{G}}(\mathcal{E}_1)$ and $\phi_2^{S\times\mathcal{G}}$: $F^*\left(\mathrm{ind}_{\mathcal{Z}_2\times C}^{S\times\mathcal{G}}(\mathcal{E}_2)\right)\xrightarrow{\sim}$ $\mathrm{ind}_{\mathcal{Z}_2\times C}^{S\times\mathcal{G}}(\mathcal{E}_2)$. Let $\sigma\in z(\mathcal{L})^F$. From 6.1.3 and 6.1.16 we deduce isomorphisms

$$(j_{\sigma,\mathcal{G}})^*\left(\phi_1^{S\times\mathcal{G}}\right)[-r] : F^*\left(\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}_{1,\sigma})\right)\xrightarrow{\sim}\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}_{1,\sigma}),\ \text{and}$$

$$(j_{\sigma,\mathcal{G}})^*\left(\phi_2^{S\times\mathcal{G}}\right)[-r] : F^*\left(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K_{2,\sigma})\right)\xrightarrow{\sim}\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K_{2,\sigma})$$

where $r=\dim S$. Put $\psi_{\sigma,1}^{\mathcal{G}}=(j_{\sigma,\mathcal{G}})^*\left(\phi_1^{S\times\mathcal{G}}\right)[-r]$, $\psi_{\sigma,2}^{\mathcal{G}}=(j_{\sigma,\mathcal{G}})^*\left(\phi_2^{S\times\mathcal{G}}\right)[-r]$; we will prove that the characteristic functions of the F-equivariant complexes $(\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}_{1,\sigma}),\psi_{1,\sigma}^{\mathcal{G}})$ and $(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K_{2,\sigma},\psi_{2,\sigma}^{\mathcal{G}})$ are respectively the Deligne-Lusztig induction of $\mathbf{X}_{K_{1,\sigma},\psi_{\sigma,1}}$ and of $\mathbf{X}_{K_{2,\sigma},\psi_{\sigma,2}}$ where $\psi_{\sigma,1}=(j_{\sigma,\mathcal{L}})^*(\phi_1)[-r]$ and $\psi_{\sigma,2}=(j_{\sigma,\mathcal{L}})^*(\phi_2)[-r]$. The proof will involve two ingredients, the Lie algebra version of [Lus90, Theorem 1.14] (see 5.5.13), and the character formula for the characteristic functions of the F-equivariant complexes $\left(\mathrm{ind}_{\mathcal{Z}_1\times C}^{S\times\mathcal{G}}(\mathcal{E}_1),\phi_1^{S\times\mathcal{G}}\right)$ and $\left(\mathrm{ind}_{\mathcal{Z}_2\times C}^{S\times\mathcal{G}}(\mathcal{E}_2),\phi_2^{S\times\mathcal{G}}\right)$.

6.1.19 The Character Formula

From now we denote respectively by $K_1^{S\times\mathcal{G}}$ and by $K_2^{S\times\mathcal{G}}$ the complexes $\mathrm{ind}_{\mathcal{Z}_1\times C}^{S\times\mathcal{G}}(\mathcal{E}_1)$ and $\mathrm{ind}_{\mathcal{Z}_2\times C}^{S\times\mathcal{G}}(\mathcal{E}_2)$ and we assume that the datum (L,C,ζ) is F-stable. We fix two isomorphisms $\phi_1 : F^*(K_1)\xrightarrow{\sim}K_1$ and $\phi_2 : F^*(K_2)\xrightarrow{\sim}K_2$, and we denote by $\phi_1^{S\times\mathcal{G}} : F^*\left(K_1^{S\times\mathcal{G}}\right)\xrightarrow{\sim}K_1^{S\times\mathcal{G}}$ and $\phi_2^{S\times\mathcal{G}} : F^*\left(K_2^{S\times\mathcal{G}}\right)\xrightarrow{\sim}K_2^{S\times\mathcal{G}}$ the two isomorphisms induced respectively by ϕ_1 and ϕ_2. By analogy with 5.5.9, we give an expression of the values of the characteristic functions of the F-equivariant complexes $(K_1^{S\times\mathcal{G}}, \phi_1^{S\times\mathcal{G}})$ and $(K_2^{S\times\mathcal{G}}, \phi_2^{S\times\mathcal{G}})$ in terms of the values of some generalized Green functions.

We fix $s \in S^F$ and $\sigma, u \in \mathcal{G}^F$ with σ semi-simple and u nilpotent such that $[\sigma, u] = 0$.

The character formula

Let $i \in \{1, 2\}$. Assume that there is $x \in G^F$ such that $(s, \mathrm{Ad}(x^{-1})\sigma) \in \mathcal{Z}_i$. Then put $L_x = xLx^{-1}$ and $\mathcal{L}_x = \mathrm{Lie}(L_x)$. We have $\sigma \in z(\mathcal{L}_x)$ and so L_x is a Levi subgroup of $C_G^o(\sigma)$. Let $C_x = \mathrm{Ad}(x)C$ and let $(\mathcal{E}_{i,x}, \phi_{i,x})$ be the inverse image of the F-equivariant sheaf $(\mathcal{E}_i, \phi_i)$ by $C_x \to \mathcal{Z}_i \times C$, $v \mapsto \big((s, \mathrm{Ad}(x^{-1})\sigma), \mathrm{Ad}(x^{-1})v\big)$. Note that the irreducible L_x-equivariant local system $\mathcal{E}_{i,x}$ is isomorphic to $\mathrm{Ad}(x^{-1})^*\zeta \in ls(C_x)$; we thus denote $\mathcal{E}_{i,x}$ by ζ_x.

We are going to prove the following theorem.

Theorem 6.1.20 (Character formula). *With the above notation we have*

(i) $$\mathbf{X}_{K_1^{S\times\mathcal{G}}, \phi_1^{S\times\mathcal{G}}}(s, \sigma+u) = |C_G^o(\sigma)^F|^{-1} \sum_{\{x\in G^F \mid Ad(x^{-1})\sigma\in z(\mathcal{L})\}} \mathcal{Q}^{C_{\mathcal{G}}(\sigma)}_{\mathcal{L}_x, C_x, \zeta_x, \phi_{1,x}}(u).$$

(ii) $$\mathbf{X}_{K_2^{S\times\mathcal{G}}, \phi_2^{S\times\mathcal{G}}}(s, \sigma+u) = |C_G^o(\sigma)^F|^{-1} \sum_{\{x\in G^F \mid Ad(x^{-1})\sigma = s\}} \mathcal{Q}^{C_{\mathcal{G}}(\sigma)}_{\mathcal{L}_x, C_x, \zeta_x, \phi_{2,x}}(u).$$

Remark 6.1.21. If $\sigma = 0$, the formula 6.1.20(ii) is a result of Waldspurger [Wal01].

Proof of Theorem 6.1.20(ii)

The proof of 6.1.20(ii) is an adaptation of the proof of [Lus85b, Theorem 8.5]. We start with the following intermediate result whose proof is entirely similar to that of [Lus85b, Lemma 8.6].

Lemma 6.1.22. *There exists an open subset* $\mathbf{U}$ *of* $C_{\mathcal{G}}(\sigma)$ *containing* 0 *such that:*

(a) for any element g of $C_G^o(\sigma)$, we have $Ad(g)(\mathbf{U}) = \mathbf{U}$,

(b) if $x \in C_{\mathcal{G}}(\sigma)$, we have $x \in \mathbf{U}$ if and only if $x_s \in \mathbf{U}$,

(c) $F(\mathbf{U}) = \mathbf{U}$,

(d) if $x \in \mathbf{U}$*,* $g \in G$ *such that* $Ad(g^{-1})(\sigma + x) \in z(\mathcal{L}) + \overline{C} + \mathcal{U}_P$*, then* $Ad(g^{-1})x_s \in z(\mathcal{L}) + \mathcal{U}_P$ *and* $Ad(g^{-1})\sigma \in z(\mathcal{L}) + \mathcal{U}_P$*,*

(e) if $x \in \mathbf{U}$*,* $g \in G$ *such that* $Ad(g^{-1})(\sigma+x) \in z(\mathcal{L})+C$*, then* $Ad(g^{-1})x_s \in z(\mathcal{L})$ *and* $Ad(g^{-1})\sigma \in z(\mathcal{L})$*.*

We fix once for all an open subset $\mathbf{U}$ of $C_{\mathcal{G}}(\sigma)$ as in 6.1.22; since $0 \in \mathbf{U}$, it follows from 6.1.22(b) that $C_{\mathcal{G}}(\sigma)_{nil} \subset \mathbf{U}$.

We put $\mathcal{Z} = \mathcal{Z}_2$ and we use the notation of 6.1.2 relatively to $\mathcal{Z}$. Then we have

$$X_{S,2} = \{(t, x, gP) \in S \times \mathcal{G} \times (G/P) |\, \mathrm{Ad}(g^{-1})x \in t + \overline{C} + \mathcal{U}_P\},$$

and $Y_{S,2} = \{(t, x, gL) \in S_{reg} \times \mathcal{G} \times (G/L) |\, \mathrm{Ad}(g^{-1})x \in t + C\}$.

Define

$$X^{\mathbf{U}}_{S,2} := (\rho''_S)^{-1}\,(S \times (\sigma + \mathbf{U})) = \{(t, x, gP) \in X_{S,2} |\, x \in \sigma + \mathbf{U}\}.$$

Let

$$\Delta := \{g \in G | \mathrm{Ad}(g^{-1})\sigma \in z(\mathcal{L})\}, \qquad \Gamma = C^o_G(\sigma)\backslash\Delta/L,$$
$$\hat{\Delta} := \{g \in G | \mathrm{Ad}(g^{-1})\sigma \in z(\mathcal{L}) + \mathcal{U}_P\}, \quad \hat{\Gamma} = C^o_G(\sigma)\backslash\hat{\Delta}/P.$$

We assume that the set Δ (and therefore $\hat{\Delta}$) is non-empty.

The canonical map $\Gamma \to \hat{\Gamma}$ is a bijection. Indeed, let $x \in \hat{\Delta}$, we have $\mathrm{Ad}(x^{-1})\sigma = z + v$ for some $z \in z(\mathcal{L})$ and $v \in \mathcal{U}_P$. Since $z + v$ is semi-simple, by 2.7.1, there exists an element $u \in U_P$ such that $z + v = \mathrm{Ad}(u)z$, i.e. $xu \in \Delta$; we thus proved the surjectivity of $\Gamma \to \hat{\Gamma}$. Assume now that $x, y \in \Delta$ and $x \in C^o_G(\sigma)yP$ i.e. $x = gyul$ with $g \in C^o_G(\sigma)$, $l \in L$ and $u \in U_P$. Since $\mathrm{Ad}(y^{-1})\sigma \in z(\mathcal{L})$, we have $\mathrm{Ad}(u^{-1}y^{-1})\sigma = \mathrm{Ad}(y^{-1})\sigma + \mathcal{U}_P$ and so we deduce that $\mathrm{Ad}(x^{-1})\sigma = \mathrm{Ad}(l^{-1}u^{-1}y^{-1}g^{-1})\sigma \in \mathrm{Ad}(u^{-1}y^{-1})\sigma + \mathcal{U}_P$. Hence we deduce that $\mathrm{Ad}(x^{-1})\sigma = \mathrm{Ad}(y^{-1})\sigma$ i.e. $xy^{-1} \in C_G(\sigma)$. But $xy^{-1} = gyuly^{-1} \in C_G(\sigma)$ if and only if $yuy^{-1} \in C_G(\sigma)$. Since yuy^{-1} is unipotent, we have $yuy^{-1} \in C^o_G(\sigma)$. We thus proved that $x \in C^o_G(\sigma)yL$ and so the injectivity of $\Gamma \to \hat{\Gamma}$.

It is also easy to verify that the set Γ (therefore $\hat{\Gamma}$) is finite. For $\hat{\mathcal{O}} \in \hat{\Gamma}$, define

$$X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}} := \{(t, x, gP) \in X^{\mathbf{U}}_{S,2} |\, g \in \hat{\mathcal{O}}\}.$$

6.1.23. *The sets* $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ *with* $\hat{\mathcal{O}} \in \hat{\Delta}$, *are open and closed in* $X^{\mathbf{U}}_{S,2}$ *and we have* $X^{\mathbf{U}}_{S,2} = \coprod_{\hat{\mathcal{O}}\in\hat{\Gamma}} X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$.

Proof of 6.1.23: Let $(t, \sigma+z, gP) \in X^{\mathbf{U}}_{S,2}$; then $z \in \mathbf{U}$ and $\mathrm{Ad}(g^{-1})(\sigma+z) \in t+\overline{C}+\mathcal{U}_P$. From 6.1.22(d) we get that $\mathrm{Ad}(g^{-1})\sigma \in z(\mathcal{L})+\mathcal{U}_P$, hence $g \in \hat{\Delta}$ and so we deduce that $(t, \sigma+z, gP) \in X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ with $\hat{\mathcal{O}} = C^o_G(\sigma)gP$. Now let $\hat{\mathcal{O}} \in \hat{\Gamma}$; we can view $\hat{\mathcal{O}}$ as a closed $C^o_G(\sigma)$-orbit of G/P. The set $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ is thus closed in $X^{\mathbf{U}}_{S,2}$ since it is the inverse image of $\hat{\mathcal{O}}$ by the morphism $X^{\mathbf{U}}_{S,2} \to G/P$ given by the projection on the third coordinate. From the fact that $X^{\mathbf{U}}_{S,2}$ is the (finite) disjoint union of the $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$, we deduce that the $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ are also open in $X^{\mathbf{U}}_{S,2}$. □

From now, if $\mathcal{O} \in \Gamma$, we denote by $\hat{\mathcal{O}}$ the element of $\hat{\Gamma}$ corresponding to $\mathcal{O}$.

6.1.24. For $\mathcal{O} \in \Gamma$, we fix an element $x_{\mathcal{O}}$ of $\mathcal{O}$ such that $F(x_{\mathcal{O}}) = x_{F(\mathcal{O})}$; note that $\mathrm{Ad}(x_{\mathcal{O}}^{-1})\sigma \in z(\mathcal{L})$, hence $x_{\mathcal{O}}Lx_{\mathcal{O}}^{-1}$ is a Levi subgroup of $C^o_G(\sigma)$. Let $\mathcal{O} \in \Gamma$, define

- $P_{\mathcal{O}} = (x_{\mathcal{O}}Px_{\mathcal{O}}^{-1}) \cap C^o_G(\sigma)$,
- $L_{\mathcal{O}} = x_{\mathcal{O}}Lx_{\mathcal{O}}^{-1}$.

We denote by $\mathcal{P}_{\mathcal{O}}$ and by $\mathcal{L}_{\mathcal{O}}$ the respective Lie algebras of $P_{\mathcal{O}}$ and $L_{\mathcal{O}}$. Put $S_{\mathcal{O}} = \mathrm{Ad}(x_{\mathcal{O}})S$, $\mathcal{Z}_{\mathcal{O}} = \{(t,t)\,|\, t \in z(\mathcal{L}_{\mathcal{O}})\}$ and $C_{\mathcal{O}} = \mathrm{Ad}(x_{\mathcal{O}})C$.

We denote by $(S_{\mathcal{O}})_{\sigma-reg}$ the subset of $S_{\mathcal{O}}$ consisting of the $L_{\mathcal{O}}$-regular elements in $C_{\mathcal{G}}(\sigma)$. Define

$$Y_{\sigma,S_{\mathcal{O}},2} = \{(t, x, gL_{\mathcal{O}}) \in (S_{\mathcal{O}})_{\sigma-reg} \times C_{\mathcal{G}}(\sigma) \times (C^o_G(\sigma)/L_{\mathcal{O}}) \,|\, \mathrm{Ad}(g^{-1})x \in t + C_{\mathcal{O}}\},$$

$$X_{\sigma,S_{\mathcal{O}},2} = \{(t, x, gP_{\mathcal{O}}) \in S_{\mathcal{O}} \times C_{\mathcal{G}}(\sigma) \times (C^o_G(\sigma)/P_{\mathcal{O}}) \,|\, \mathrm{Ad}(g^{-1})x \in t + \overline{C_{\mathcal{O}}} + \mathcal{U}_{P_{\mathcal{O}}}\},$$

$$Y_{\sigma,S_{\mathcal{O}}} = \{(t, x) \in (S_{\mathcal{O}})_{\sigma-reg} \times C_{\mathcal{G}}(\sigma) \,|\, \exists g \in C^o_G(\sigma), \mathrm{Ad}(g^{-1})x \in t + C_{\mathcal{O}}\}.$$

Let $\alpha''_{\sigma,S_{\mathcal{O}}} : Y_{\sigma,S_{\mathcal{O}},2} \to Y_{\sigma,S_{\mathcal{O}}}$ and $\rho''_{\sigma,S_{\mathcal{O}}} : X_{\sigma,S_{\mathcal{O}},2} \to \overline{Y_{\sigma,S_{\mathcal{O}}}}$ be given by the projection on the first and second coordinates. Note that $X_{\sigma,S_{\mathcal{O}},2}, Y_{\sigma,S_{\mathcal{O}},2}$, $Y_{\sigma,S_{\mathcal{O}}}, \alpha''_{\sigma,S_{\mathcal{O}}}, \rho''_{\sigma,S_{\mathcal{O}}}$ are defined in terms of $C^o_G(\sigma)$, $P_{\mathcal{O}}$, $L_{\mathcal{O}}$, $\mathcal{Z}_{\mathcal{O}}$, $C_{\mathcal{O}}$ as $X_{S,2}$, $Y_{S,2}$, Y_S, α''_S, ρ''_S are defined in terms of $G, P, L, \mathcal{Z}, C$.

6.1.25. For $\mathcal{O} \in \Gamma$, define

$$X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2} = (\rho''_{\sigma,S_{\mathcal{O}}})^{-1}\,(S_{\mathcal{O}} \times \mathbf{U}) \subset X_{\sigma,S_{\mathcal{O}},2}.$$

Since $\mathbf{U}$ is open in $C_{\mathcal{G}}(\sigma)$ we have following assertion.

6.1.26. *$X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}$ is an open subset of $X_{\sigma,S_{\mathcal{O}},2}$ for any $\mathcal{O} \in \Gamma$.*

6.1.27. We denote by $f''_{\mathcal{O}}$ the morphism $X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2} \to X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ given by

$$(t, x, gP_{\mathcal{O}}) \mapsto \big(\mathrm{Ad}(x_{\mathcal{O}}^{-1})(\sigma + t), \sigma + x, gx_{\mathcal{O}}P\big)$$

We have the following result.

6.1.28. *The map $f''_{\mathcal{O}}$ is well-defined and is an isomorphism.*

Proof: We first verify that the map $f''_{\mathcal{O}}$ is well-defined. Let $(t, x, gP_{\mathcal{O}}) \in X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}$, then we have $\mathrm{Ad}(g^{-1})x \in t + \overline{C_{\mathcal{O}}} + \mathcal{U}_{P_{\mathcal{O}}}$ and so we get that

$$\mathrm{Ad}(x_{\mathcal{O}}^{-1}g^{-1})x \in \mathrm{Ad}(x_{\mathcal{O}}^{-1})t + \overline{C} + \mathcal{U}_P.$$

We deduce that $\mathrm{Ad}(x_{\mathcal{O}}^{-1}g^{-1})(\sigma + x) \in \mathrm{Ad}(x_{\mathcal{O}}^{-1})\sigma + \mathrm{Ad}(x_{\mathcal{O}}^{-1})t + \overline{C} + \mathcal{U}_P$ and so that
$\big(\mathrm{Ad}(x_{\mathcal{O}}^{-1})(\sigma + t), \sigma + x, gx_{\mathcal{O}}P\big) \in X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$. We thus proved that $f''_{\mathcal{O}}$ is well-defined.

The fact that $f''_{\mathcal{O}}$ is injective is clear. We prove now the surjectivity of $f''_{\mathcal{O}}$. Let $(r, \sigma + x, tP) \in X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ and write $t = gx_{\mathcal{O}}p$ with $g \in C^o_G(\sigma)$ and $p \in P$. Let $h \in S_{\mathcal{O}}$ be defined by $h = -\sigma + \mathrm{Ad}(x_{\mathcal{O}})r$. We verify that $(h, x, gP_{\mathcal{O}}) \in X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}$, i.e. that $\mathrm{Ad}(g^{-1})x \in h + \overline{C_{\mathcal{O}}} + \mathcal{U}_{P_{\mathcal{O}}}$. We have $\mathrm{Ad}(t^{-1})(\sigma + x) \in r + \overline{C} + \mathcal{U}_P$ and so $\mathrm{Ad}(g^{-1})(\sigma + x) \in \mathrm{Ad}(x_{\mathcal{O}})r + \overline{C_{\mathcal{O}}} + \mathbf{U}_{x_{\mathcal{O}}Px_{\mathcal{O}}^{-1}}$. Since $\mathrm{Ad}(g^{-1})x \in C_{\mathcal{G}}(\sigma)$, we get that $\mathrm{Ad}(g^{-1})(\sigma + x) \in \mathrm{Ad}(x_{\mathcal{O}})r + \overline{C_{\mathcal{O}}} + \mathcal{U}_{P_{\mathcal{O}}}$, hence $\mathrm{Ad}(g^{-1})x \in -\sigma + \mathrm{Ad}(x_{\mathcal{O}})r + \overline{C_{\mathcal{O}}} + \mathcal{U}_{P_{\mathcal{O}}}$. We thus proved the surjectivity of $f''_{\mathcal{O}}$. □

We denote by $\tilde{f}_{\mathcal{O}}$ the morphism $S_{\mathcal{O}} \times C_{\mathcal{G}}(\sigma) \to S \times \mathcal{G}$ given by $(t, x) \mapsto (\mathrm{Ad}(x_{\mathcal{O}}^{-1})(\sigma + t), \sigma + x)$. We have the following commutative diagram.

$$\begin{array}{ccc}
X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2} & \xrightarrow{\rho''_{\sigma,S_{\mathcal{O}}}} & \overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U}) \\
\downarrow f''_{\mathcal{O}} & & \downarrow \tilde{f}_{\mathcal{O}} \\
X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}} & \xrightarrow{\rho''_S} & \overline{Y_S} \cap (S \times (\sigma + \mathbf{U}))
\end{array}$$

and we have the following assertions.

6.1.29. *(i) The morphisms* $X^{\mathbf{U}}_{S,2} \to \overline{Y_S} \cap (S \times (\sigma + \mathbf{U}))$ *and* $X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2} \to \overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})$ *given respectively by restriction of* ρ''_S *and* $\rho''_{\sigma,S_{\mathcal{O}}}$ *are surjective and proper.*

(ii) The set $Y_S \cap (S \times (\sigma + \mathbf{U}))$ *is open in* $\overline{Y_S} \cap (S \times (\sigma + \mathbf{U}))$ *and the set* $Y_{\sigma,S_{\mathcal{O}}} \cap (S_{\mathcal{O}} \times \mathbf{U})$ *is an open dense irreducible subset of* $\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})$*.*

(iii) The morphism $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}} \to \overline{Y_S} \cap (S \times (\sigma + \mathbf{U}))$ *given by restriction of* ρ''_S *is proper with image* $\tilde{f}_{\mathcal{O}}\left(\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right)$ *and we have*

$$\overline{Y_S} \cap (S \times (\sigma + \mathbf{U})) = \coprod_{\mathcal{O} \in \Gamma} \tilde{f}_{\mathcal{O}}\left(\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right).$$

(iv) The variety $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ *is irreducible.*

Proof: The morphisms of (i) are obtained by base change respectively from $\rho''_S : X_{S,2} \to \overline{Y_S}$ and $\rho''_{\sigma,S_{\mathcal{O}}} : X_{\sigma,S_{\mathcal{O}},2} \to \overline{Y_{\sigma,S_{\mathcal{O}}}}$ which are proper morphisms, hence they are proper; the fact they are surjective is clear. Let us see (ii). The fact that $Y_S \cap (S \times (\sigma + \mathbf{U}))$ and $Y_{\sigma,S_{\mathcal{O}}} \cap (S_{\mathcal{O}} \times \mathbf{U})$ are respectively open in $\overline{Y_S} \cap (S \times (\sigma + \mathbf{U}))$ and in $\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})$ is clear. The set $S_{\mathcal{O}} \times \mathbf{U}$ is an open subset of $S_{\mathcal{O}} \times C_{\mathcal{G}}(\sigma)$, hence the set $\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})$ is open in $\overline{Y_{\sigma,S_{\mathcal{O}}}}$. Moreover it is non-empty since $\mathbf{U} \supset C_{\mathcal{G}}(\sigma)_{nil}$ and $\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times C_{\mathcal{G}}(\sigma)_{nil}) \neq \emptyset$. The variety $Y_{\sigma,S_{\mathcal{O}}}$ being open in $\overline{Y_{\sigma,S_{\mathcal{O}}}}$ and $\overline{Y_{\sigma,S_{\mathcal{O}}}}$ being irreducible, we deduce that $Y_{\sigma,S_{\mathcal{O}}} \cap (S_{\mathcal{O}} \times \mathbf{U})$ is a non-empty open subset of $\overline{Y_{\sigma,S_{\mathcal{O}}}}$ and so is irreducible and dense in $\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})$. We now prove (iii). By 6.1.23, the set $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ is closed in $X^{\mathbf{U}}_{S,2}$, hence from (i), we get that the morphism of (iii) is also proper. The fact that $\rho''_S\left(X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}\right) = \tilde{f}_{\mathcal{O}}\left(\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right)$ is a straightforward consequence of 6.1.28 and (i). Let us prove (iv). From 6.1.6, we know that the variety $X_{\sigma,S_{\mathcal{O}},2}$ is irreducible and from 6.1.26, we know that $X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}$ is an open subset of $X_{\sigma,S_{\mathcal{O}},2}$. Hence we deduce that the variety $X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}$ is irreducible and so from 6.1.28, it follows that $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ is also irreducible. □

6.1.30. Define

$$Y^{\mathbf{U}}_{S,2} := (\alpha''_S)^{-1}\big(S \times (\sigma + \mathbf{U})\big) = \{(t, x, gL) \in Y_{S,2} | \, x \in \sigma + \mathbf{U}\},$$

and for $\mathcal{O} \in \Gamma$, define $Y^{\mathbf{U}}_{S,2,\mathcal{O}} := \{(t, x, gL) \in Y^{\mathbf{U}}_{S,2} | \, g \in \mathcal{O}\}$, $Y^{\mathbf{U}}_{S,\mathcal{O}} := \alpha''_S(Y^{\mathbf{U}}_{S,2,\mathcal{O}})$, and

$$Y^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}:=(\alpha''_{\sigma,S_{\mathcal{O}}})^{-1}(S_{\mathcal{O}}\times\mathbf{U})\subset Y_{\sigma,S_{\mathcal{O}},2}.$$

6.1.31. *The sets* $Y^{\mathbf{U}}_{S,2,\mathcal{O}}$ *and* $Y^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}$ *are non-empty.*

Proof: From 6.1.29(ii), we have $Y_{\sigma,S_{\mathcal{O}}}\cap(S_{\mathcal{O}}\times\mathbf{U})\neq\emptyset$ from which we see that $Y^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}\neq\emptyset$. Let $v\in C_{\mathcal{O}}$, then the intersection $z(\mathcal{L}_{\mathcal{O}})\cap(\mathbf{U}-v)$ is open dense in $z(\mathcal{L}_{\mathcal{O}})$; indeed it is non-empty since we have $0\in z(\mathcal{L}_{\mathcal{O}})\cap(\mathbf{U}-v)$ (recall that $\mathbf{U}\supset C_{\mathcal{G}}(\sigma)_{nil}$). The intersection $z(\mathcal{L}_{\mathcal{O}})\cap(-\sigma+\mathrm{Ad}(x_{\mathcal{O}})(z(\mathcal{L})_{reg}+C)-v)$ is also a dense open subset of $z(\mathcal{L}_{\mathcal{O}})$. Indeed, $-\sigma+\mathrm{Ad}(x_{\mathcal{O}})(z(\mathcal{L})_{reg}+C)-v$ is an open subset of $z(\mathcal{L}_{\mathcal{O}})+C_{\mathcal{O}}-v$. But $z(\mathcal{L}_{\mathcal{O}})+C_{\mathcal{O}}-v$ is irreducible and contains $z(\mathcal{L}_{\mathcal{O}})$ as an open subset, hence $-\sigma+\mathrm{Ad}(x_{\mathcal{O}})(z(\mathcal{L})_{reg}+C)-v$ intersects $z(\mathcal{L}_{\mathcal{O}})$. As a consequence we have $z(\mathcal{L}_{\mathcal{O}})\cap(\mathbf{U}-v)\cap(-\sigma+\mathrm{Ad}(x_{\mathcal{O}})(z(\mathcal{L})_{reg}+C)-v)\neq\emptyset$. Hence there exists $t\in z(\mathcal{L}_{\mathcal{O}})$, $h\in z(\mathcal{L})_{reg}$ such that $(h,\sigma+t+v,x_{\mathcal{O}}L)\in Y^{\mathbf{U}}_{S,2,\mathcal{O}}$. □

We have the following assertions.

6.1.32. *(i) The map* $\gamma_S:Y_{S,2}\to X_{S,2}$ *given by* $(t,x,gL)\mapsto(t,x,gP)$ *induces an isomorphism* $Y^{\mathbf{U}}_{S,2}\to(\rho''_S)^{-1}\left(Y_S\cap(S\times(\sigma+\mathbf{U}))\right)$.

(ii) The map $\gamma_{\sigma,S_{\mathcal{O}}}:Y_{\sigma,S_{\mathcal{O}},2}\to X_{\sigma,S_{\mathcal{O}},2}$ *given by* $(t,x,gL_{\mathcal{O}})\mapsto(t,x,gP_{\mathcal{O}})$ *induces an isomorphism* $Y^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}\to(\rho''_{\sigma,S_{\mathcal{O}}})^{-1}(Y_{\sigma,S_{\mathcal{O}}}\cap(S_{\mathcal{O}}\times\mathbf{U}))$.

Proof: This follows from 6.1.10. □

We have the following result.

6.1.33. *The sets* $Y^{\mathbf{U}}_{S,2,\mathcal{O}}$ *with* $\mathcal{O}\in\Gamma$, *are open and closed in* $Y^{\mathbf{U}}_{S,2}$; *they form a finite partition of* $Y^{\mathbf{U}}_{S,2}$.

Proof: By 6.1.31, the sets $Y^{\mathbf{U}}_{S,2,\mathcal{O}}$ are non-empty. Let $(t,x,gL)\in Y^{\mathbf{U}}_{S,2}$; we have $\mathrm{Ad}(g^{-1})x\in t+C$. From 6.1.22 we deduce that $g\in\Delta$ and so that $(t,x,gL)\in Y^{\mathbf{U}}_{S,2,\mathcal{O}}$ with $\mathcal{O}=C^o_G(\sigma)gL$. We thus proved that $Y^{\mathbf{U}}_{S,2}$ is the disjoint union of the $Y^{\mathbf{U}}_{S,2,\mathcal{O}}$ with $\mathcal{O}\in\Gamma$; this union is finite since Γ is finite. Let $\mathcal{O}\in\Gamma$; the isomorphism $Y^{\mathbf{U}}_{S,2}\to(\rho''_S)^{-1}\big(Y_S\cap(S\times(\sigma+\mathbf{U}))\big)$ given by $(t,x,gL)\mapsto(t,x,gP)$ induces an isomorphism from $Y^{\mathbf{U}}_{S,2,\mathcal{O}}$ onto $(\rho''_S)^{-1}\big(Y_S(S\times(\sigma+\mathbf{U}))\big)\cap X_{S,2,\hat{\mathcal{O}}}$. But from 6.1.23, the set $X_{S,2,\hat{\mathcal{O}}}$ is open and closed in $X_{S,2}$, hence $(\rho''_S)^{-1}\big(Y_S\cap(\times(\sigma+\mathbf{U}))\big)\cap X_{S,2,\hat{\mathcal{O}}}$ is open and closed in $(\rho''_S)^{-1}\big(Y_S\cap(S\times(\sigma+\mathbf{U}))\big)$. As a consequence, we get that $Y^{\mathbf{U}}_{S,2,\mathcal{O}}$ is closed and open in $Y^{\mathbf{U}}_{S,2}$. □

We have the following commutative diagram.

$$\begin{array}{ccc} Y^{\mathbf{U}}_{S,2,\mathcal{O}} & \xrightarrow{\alpha''_S} & Y^{\mathbf{U}}_{S,\mathcal{O}} \\ \downarrow{\scriptstyle \gamma_S} & & \downarrow \\ X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}} & \xrightarrow{\rho''_S} & \overline{Y_S} \cap \big(S \times (\sigma + \mathbf{U})\big) \end{array}$$

6.1.34. *(i) We have* $\gamma_S\left(Y^{\mathbf{U}}_{S,2,\mathcal{O}}\right) = (\rho''_S)^{-1}(Y^{\mathbf{U}}_{S,\mathcal{O}}) \cap X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$.

(ii) The variety $Y^{\mathbf{U}}_{S,\mathcal{O}}$ *is open dense in* $\tilde{f}_{\mathcal{O}}\left(\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right)$.

Proof: Let us prove (i). Since $\gamma_S : Y^{\mathbf{U}}_{S,2} \to (\rho''_S)^{-1}\big(Y_S \cap (S \times (\sigma + \mathbf{U}))\big)$ is an isomorphism (see 6.1.32(i)), we have

$$\gamma_S(Y^{\mathbf{U}}_{S,2,\mathcal{O}}) = (\rho''_S)^{-1}\big(Y_S \cap (S \times (\sigma + \mathbf{U}))\big) \cap X_{S,2,\hat{\mathcal{O}}}.$$

Hence from the fact that $\gamma_S\left(Y^{\mathbf{U}}_{S,2,\mathcal{O}}\right) \subset (\rho''_S)^{-1}(Y^{\mathbf{U}}_{S,\mathcal{O}}) \cap X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$ and $(\rho''_S)^{-1}$ $(Y^{\mathbf{U}}_{S,\mathcal{O}}) \cap X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}} \subset (\rho''_S)^{-1}\big(Y_S \cap (S \times (\sigma + \mathbf{U}))\big) \cap X_{S,2,\hat{\mathcal{O}}}$, we deduce that $\gamma_S\left(Y^{\mathbf{U}}_{S,2,\mathcal{O}}\right) = (\rho''_S)^{-1}(Y^{\mathbf{U}}_{S,\mathcal{O}}) \cap X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$.

Let us now prove (ii). The morphism γ_S maps $Y_{S,2}$ onto an open subset of $X_{S,2}$, hence $\gamma_S(Y_{S,2}) \cap X^{\mathbf{U}}_{S,2}$ is open in $X^{\mathbf{U}}_{S,2}$. Since $\gamma_S(Y_{S,2}) \cap X^{\mathbf{U}}_{S,2} = \gamma_S(Y^{\mathbf{U}}_{S,2})$, the isomorphism γ_S induces an isomorphism from $Y^{\mathbf{U}}_{S,2}$ onto an open subset of $X^{\mathbf{U}}_{S,2}$. By 6.1.33, the set $Y^{\mathbf{U}}_{S,2,\mathcal{O}}$ is open in $Y^{\mathbf{U}}_{S,2}$, hence we deduce that $\gamma_S(Y^{\mathbf{U}}_{S,2,\mathcal{O}})$ is open in $X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}}$. From (i) we see that $\gamma_S(Y^{\mathbf{U}}_{S,2,\mathcal{O}})$ is a union of fibers, hence from the fact that the morphism $\rho''_S : X_{S,2,\hat{\mathcal{O}}} \to \overline{Y_S} \cap (S \times (\sigma + \mathbf{U}))$ is proper (and so closed), it follows that $\rho''_S\left(\gamma_S(Y^{\mathbf{U}}_{S,2,\mathcal{O}})\right)$ is open in $\rho''_S(X_{S,2,\hat{\mathcal{O}}})$. We thus proved that $\alpha''_S(Y^{\mathbf{U}}_{S,2,\mathcal{O}}) = Y^{\mathbf{U}}_{S,\mathcal{O}}$ is open in $\rho''_S(X_{S,2,\hat{\mathcal{O}}}) = \tilde{f}_{\mathcal{O}}(\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U}))$; the last equality comes from 6.1.29(iii). □

6.1.35. We now describe the irreducible components of the variety $Y_S \cap (S \times (\sigma + \mathbf{U}))$.

6.1.36. *The irreducible components of* $Y_S \cap (S \times (\sigma + \mathbf{U}))$ *are disjoint. They are in bijection with the set* Γ*. The irreducible component corresponding to* $\mathcal{O} \in \Gamma$ *is* $Y^{\mathbf{U}}_{S,\mathcal{O}}$.

Proof: The map $Y^{\mathbf{U}}_{S,2} \to Y_S \cap (S \times (\sigma + \mathbf{U}))$ obtained from α''_S by base change is a finite surjective morphism, hence it follows from 6.1.33 that the varieties

$Y^{\mathbf{U}}_{S,\mathcal{O}}$ with $\mathcal{O} \in \Gamma$ cover $Y_S \cap (S \times (\sigma + \mathbf{U}))$ and are closed in $Y_S \cap (S \times (\sigma + \mathbf{U}))$. From 6.1.29(ii), the variety $\tilde{f}_{\mathcal{O}}\left(\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right)$ is irreducible, hence from 6.1.34, we deduce that the varieties $Y^{\mathbf{U}}_{S,\mathcal{O}}$ with $\mathcal{O} \in \Gamma$ are irreducible. The fact that the union of the $Y^{\mathbf{U}}_{S,\mathcal{O}}$ with $\mathcal{O} \in \Gamma$ is disjoint is clear from the definition of $Y^{\mathbf{U}}_{S,\mathcal{O}}$. □

6.1.37. We now define for each $\mathcal{O} \in \Gamma$ an open subset $V_{\mathcal{O}}$ of $Y_S \cap (S \times (\sigma + \mathbf{U}))$. For $\mathcal{O} \in \Gamma$ define

$$V_{\mathcal{O}} := Y^{\mathbf{U}}_{S,\mathcal{O}} \cap \tilde{f}_{\mathcal{O}}\left(Y_{\sigma,S_{\mathcal{O}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right).$$

For $\mathcal{O} \in \Gamma$, the set $\tilde{f}_{\mathcal{O}}\left(Y_{\sigma,S_{\mathcal{O}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right)$ is open in $\tilde{f}_{\mathcal{O}}\left(\overline{Y_{\sigma,S_{\mathcal{O}}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right)$, hence from 6.1.34(ii), we see that the set $Y^{\mathbf{U}}_{S,\mathcal{O}} \cap \tilde{f}_{\mathcal{O}}\left(Y_{\sigma,S_{\mathcal{O}}} \cap (S_{\mathcal{O}} \times \mathbf{U})\right)$ is open dense in $Y^{\mathbf{U}}_{S,\mathcal{O}}$. Since $Y^{\mathbf{U}}_{S,\mathcal{O}}$ is open in $Y_S \cap (S \times (\sigma + \mathbf{U}))$, we get that the set $V_{\mathcal{O}}$ is open in $Y_S \cap (S \times (\sigma + \mathbf{U}))$. Note also that $V_{\mathcal{O}}$ is isomorphic to an open subset of $Y_{\sigma,S_{\mathcal{O}}}$ which is known to be smooth (see 6.1.11), hence $V_{\mathcal{O}}$ is also smooth. This also shows that $V_{\mathcal{O}}$ is of dimension $\dim(C^o_G(\sigma)) - \dim L_{\mathcal{O}} + \dim(\mathcal{Z}_{\mathcal{O}} \times C_{\mathcal{O}})$ i.e. $\dim V_{\mathcal{O}} = \dim(C^o_G(\sigma)) - \dim L + \dim(\mathcal{Z} \times C)$. Moreover note that $F(V_{\mathcal{O}}) = V_{F(\mathcal{O})}$. We thus have the following assertion.

6.1.38. *The set $V = \coprod_{\mathcal{O} \in \Gamma} V_{\mathcal{O}}$ is an F-stable open dense smooth equidimensional subset of $Y_S \cap (S \times (\sigma + \mathbf{U}))$. The subsets $V_{\mathcal{O}}$ are open and closed in V (in particular they are the irreducible components of V) and are of dimension equal to $\dim(C^o_G(\sigma)) - \dim L + \dim(\mathcal{Z} \times C)$.*

6.1.39. We are now in position to prove the assertion (ii) of the theorem 6.1.20.

For $\mathcal{O} \in \Gamma$, define

$$Y_{\sigma,S_{\mathcal{O}},1} = \{(t,x,g) \in (S_{\mathcal{O}})_{\sigma-reg} \times C_{\mathcal{G}}(\sigma) \times C^o_G(\sigma) |\, \mathrm{Ad}(g^{-1})x \in t + C_{\mathcal{O}}\},$$

and let $\alpha'_{\sigma,S_{\mathcal{O}}} : Y_{\sigma,S_{\mathcal{O}},1} \to Y_{\sigma,S_{\mathcal{O}},2}$ be given by $(t,x,g) \mapsto (t,x,gL_{\mathcal{O}})$ and $\alpha_{\sigma,S_{\mathcal{O}}} : Y_{\sigma,S_{\mathcal{O}},1} \to \mathcal{Z}_{\mathcal{O}} \times C_{\mathcal{O}}$ be given by $(t,x,g) \mapsto ((t,t),v)$ where $v \in C_{\mathcal{O}}$ is such that $\mathrm{Ad}(g^{-1})x = t + v$.

We denote by $Y_{S,2}|V$ the inverse image of V by α''_S and by $Y_{S,1}|V$ the inverse image of $Y_{S,2}|V$ by α'_S. We denote by $Y_{S,2,\mathcal{O}}|V_{\mathcal{O}}$ the inverse image of $V_{\mathcal{O}}$ by $\alpha''_{S,2}$; note that this is an open subset of $Y^{\mathbf{U}}_{S,2,\mathcal{O}}$. Put $W_{\mathcal{O}} = \tilde{f}^{-1}_{\mathcal{O}}(V_{\mathcal{O}})$. We denote by $Y_{\sigma,S_{\mathcal{O}},2}|W_{\mathcal{O}}$ the inverse image of $W_{\mathcal{O}}$ by $\alpha''_{\sigma,S_{\mathcal{O}}}$, and by $Y_{\sigma,S_{\mathcal{O}},1}|W_{\mathcal{O}}$ the inverse image of $Y_{\sigma,S_{\mathcal{O}},2}|W_{\mathcal{O}}$ by $\alpha'_{\sigma,S_{\mathcal{O}}}$. Put $W = \coprod_{\mathcal{O} \in \Gamma} W_{\mathcal{O}}$, we have the following commutative diagram.

6.1.40.
$$\begin{array}{ccc}
\coprod_{\mathcal{O}\in\Gamma}(\mathcal{Z}_{\mathcal{O}}\times C_{\mathcal{O}}) & \xrightarrow{\coprod h_{\mathcal{O}}} & \mathcal{Z}\times C \\
\Big\uparrow{\scriptstyle \coprod \alpha_{\sigma,S_{\mathcal{O}}}} & & \Big\uparrow{\scriptstyle \alpha_S} \\
\coprod_{\mathcal{O}\in\Gamma} Y_{\sigma,S_{\mathcal{O}},1} & & Y_{S,1} \\
\Big\uparrow & & \Big\uparrow \\
\coprod_{\mathcal{O}\in\Gamma}(Y_{\sigma,S_{\mathcal{O}},1}|W_{\mathcal{O}}) & \xrightarrow{\coprod h'_{\mathcal{O}}} & Y_{S,1}|V \\
\Big\downarrow{\scriptstyle \coprod \alpha'_{\sigma,S_{\mathcal{O}}}} & & \Big\downarrow{\scriptstyle \alpha'_S} \\
\coprod_{\mathcal{O}\in\Gamma}(Y_{\sigma,S_{\mathcal{O}},2}|W_{\mathcal{O}}) & \xrightarrow{\coprod h''_{\mathcal{O}}} & Y_{S,2}|V=\coprod_{\mathcal{O}\in\Gamma}(Y_{S,2,\mathcal{O}}|V_{\mathcal{O}}) \\
\Big\downarrow{\scriptstyle \coprod \alpha''_{\sigma,S_{\mathcal{O}}}} & & \Big\downarrow{\scriptstyle \alpha''_S} \\
W=\coprod_{\mathcal{O}\in\Gamma} W_{\mathcal{O}} & \xrightarrow{\tilde{f}} & V=\coprod_{\mathcal{O}\in\Gamma} V_{\mathcal{O}}
\end{array}$$

where

$h_{\mathcal{O}}\,((t,t),v) = \big(\mathrm{Ad}(x_{\mathcal{O}}^{-1})(t+\sigma), \mathrm{Ad}(x_{\mathcal{O}}^{-1})(t+\sigma), \mathrm{Ad}(x_{\mathcal{O}}^{-1})v\big)$ if $((t,t),v)\in \mathcal{Z}_{\mathcal{O}}\times C_{\mathcal{O}}$,

$h'_{\mathcal{O}}(t,x,g) = (\mathrm{Ad}(x_{\mathcal{O}}^{-1})(t+\sigma), \sigma+x, gx_{\mathcal{O}})$ if $(t,x,g)\in (Y_{\sigma,S_{\mathcal{O}},1}|W_{\mathcal{O}})$,

$h''_{\mathcal{O}}(t,x,gL_{\mathcal{O}}) = (\mathrm{Ad}(x_{\mathcal{O}}^{-1})(t+\sigma), \sigma+x, gx_{\mathcal{O}}L)$ if $(t,x,gL_{\mathcal{O}})\in (Y_{\sigma,S_{\mathcal{O}},2}|W_{\mathcal{O}})$ and

where $\tilde{f}=\coprod_{\mathcal{O}\in\Gamma}\tilde{f}_{\mathcal{O}}$.

Remark 6.1.41. (i) Note that the map $Y_{\sigma,S_{\mathcal{O}},2}\to Y_{S,2,\mathcal{O}}$ given by $(t,X,gL_{\mathcal{O}})\mapsto (\mathrm{Ad}(x_{\mathcal{O}}^{-1}(\sigma+t), \sigma+X, gx_{\mathcal{O}}L)$ is not well-defined since if t is an $L_{\mathcal{O}}$-regular element in $C_{\mathcal{G}}(\sigma)$, the element $\mathrm{Ad}(x_{\mathcal{O}}^{-1})(\sigma+t)$ might not be L-regular in $\mathcal{G}$. However from our definition of $V_{\mathcal{O}}$, we can verify easily that its restriction to $(\alpha''_{\sigma,S_{\mathcal{O}}})^{-1}(W_{\mathcal{O}})$ which gives $h''_{\mathcal{O}}$, is well-defined; similarly for $h'_{\mathcal{O}}$. Moreover the maps $h''_{\mathcal{O}}:(Y_{\sigma,S_{\mathcal{O}},2}|W_{\mathcal{O}})\to(Y_{S,2,\mathcal{O}}|V_{\mathcal{O}})$ with $\mathcal{O}\in\Gamma$ are isomorphisms; the map $\coprod h''_{\mathcal{O}}$ is thus an isomorphism.

(ii) The bottom square of 6.1.40 is cartesian.

We put $\mathcal{E} = \mathcal{E}_2$; recall that $\mathcal{E}_2$ is the local system on $\mathcal{Z} \times C = \mathcal{Z}_2 \times C$ defined in 6.1.15. For $\mathcal{O} \in \Gamma$, let $\mathcal{E}_\mathcal{O}$ be the irreducible $L_\mathcal{O}$-equivariant local system on $\mathcal{Z}_\mathcal{O} \times C_\mathcal{O}$ defined by $\mathcal{E}_\mathcal{O} = (h_\mathcal{O})^* \mathcal{E}$; note that for any $\mathcal{O} \in \Gamma$, the restriction of the local system $(\coprod h_\mathcal{O})^* \mathcal{E}$ to $\mathcal{Z}_\mathcal{O} \times C_\mathcal{O}$ is $\mathcal{E}_\mathcal{O}$ and so $(\coprod h_\mathcal{O})^* \mathcal{E}$ is the direct sum of the local systems $\mathcal{E}_\mathcal{O}$ extended by zero outside $\mathcal{Z}_\mathcal{O} \times C_\mathcal{O}$. Let ξ_2 denote the local system on $Y_{S,2}$ such that $(\alpha'_S)^* \xi_2 = (\alpha_S)^* \mathcal{E}$ (see 6.1.9) and for $\mathcal{O} \in \Gamma$, let $\xi_{\mathcal{O},2}$ be the local system on $Y_{\sigma, S_\mathcal{O}, 2}$ such that $(\alpha'_{\sigma, S_\mathcal{O}})^* (\xi_{\mathcal{O},2}) = (\alpha_{\sigma, S_\mathcal{O}})^* (\mathcal{E}_\mathcal{O})$. For $\mathcal{O} \in \Gamma$, we denote by $(\alpha''_{\sigma, S_\mathcal{O}})_* (\xi_{\mathcal{O},2})|_W$ the local system on $\coprod_{\mathcal{O} \in \Gamma} W_\mathcal{O}$ whose restriction to $W_\mathcal{O}$ is $(\alpha''_{\sigma, S_\mathcal{O}})_* (\xi_{\mathcal{O},2})$ and whose restriction to $W_{\mathcal{O}'}$ with $\mathcal{O}' \neq \mathcal{O}$ is zero.

Now the local system $((\alpha''_S)_* \xi_2)\,|_V$ is isomorphic to the local system on V induced from $\mathcal{E}$ using the right vertical diagram of 6.1.40, and the local system $\bigoplus_{\mathcal{O} \in \Gamma} \big((\alpha''_{\sigma, S_\mathcal{O}})_* (\xi_{\mathcal{O},2})|_W \big)$ is isomorphic to the local system on W induced from $(\coprod h_\mathcal{O})^* \mathcal{E}$ using the left vertical diagram of 6.1.40.

Hence from 6.1.41 and the fact that 6.1.40 is commutative, we deduce that there exists a canonical isomorphism

6.1.42.
$$\tilde{f}^* \big(((\alpha''_S)_* \xi_2)\,|_V \big) \simeq \bigoplus_{\mathcal{O} \in \Gamma} \big((\alpha''_{\sigma, S_\mathcal{O}})_* (\xi_{\mathcal{O},2})|_W \big).$$

Since all the maps of 6.1.40 are defined over $\mathbb{F}_q$, this isomorphism is compatible with the two isomorphisms $F^* \left(\tilde{f}^* \left(((\alpha''_S)_* \xi_2)|_V \right) \right) \xrightarrow{\sim} \tilde{f}^* (((\alpha''_S)_* \xi_2)|_V)$ and

$$F^* \left(\bigoplus_{\mathcal{O} \in \Gamma} \big((\alpha''_{\sigma, S_\mathcal{O}})_* (\xi_{\mathcal{O},2})|_W \big) \right) \xrightarrow{\sim} \bigoplus_{\mathcal{O} \in \Gamma} \big((\alpha''_{\sigma, S_\mathcal{O}})_* (\xi_{\mathcal{O},2})|_W \big)$$

induced respectively by $\phi = \phi_2$ and $(\coprod h_\mathcal{O})^*(\phi) : F^* ((\coprod h_\mathcal{O})^* \mathcal{E}) \simeq (\coprod h_\mathcal{O})^* \mathcal{E}$. We put $K_\mathcal{O}^{S_\mathcal{O} \times C_\mathcal{G}(\sigma)} = \mathrm{ind}_{\mathcal{Z}_\mathcal{O} \times C_\mathcal{O}}^{S_\mathcal{O} \times C_\mathcal{G}(\sigma)} (\mathcal{E}_\mathcal{O})$ and $K^{S \times \mathcal{G}} = K_2^{S \times \mathcal{G}}$; then the isomorphism 6.1.42 can be regarded as an isomorphism

6.1.43.
$$\tilde{f}^* \left(K^{S \times \mathcal{G}}|_V \right) [-\delta] \simeq \bigoplus_{\mathcal{O} \in \Gamma} \left(K_\mathcal{O}^{S_\mathcal{O} \times C_\mathcal{G}(\sigma)}|_W \right)$$

where $\delta = \dim Y_S - \dim Y_{\sigma, S_\mathcal{O}} = \dim G - \dim C_G^o(\sigma)$.

We now show the following assertion.

6.1.44. The isomorphism 6.1.43 is the restriction to W of an isomorphism

$$\tilde{f}^* \left(K^{S \times \mathcal{G}}|_{\overline{Y_S} \cap (S \times (\sigma + \mathbf{U}))} \right) [-\delta] \simeq \bigoplus_{\mathcal{O} \in \Gamma} \left(K_\mathcal{O}^{S_\mathcal{O} \times C_\mathcal{G}(\sigma)}|_{\coprod_{\mathcal{O} \in \Gamma} \overline{Y_{\sigma, S_\mathcal{O}}} \cap (S_\mathcal{O} \times \mathbf{U})} \right)$$

where we still denote by $\tilde{f}$ the isomorphism $\coprod_{\mathcal{O}\in\Gamma}\left(\overline{Y_{\sigma,S_{\mathcal{O}}}}\cap(S_{\mathcal{O}}\times\mathbf{U})\right)\xrightarrow{\simeq}\overline{Y_S}\cap(S\times(\sigma+\mathbf{U}))$ given by $(t,x)\mapsto\tilde{f}_{\mathcal{O}}(t,x)$ if $(t,x)\in\overline{Y_{\sigma,S_{\mathcal{O}}}}\cap(S_{\mathcal{O}}\times\mathbf{U})$.

Proof of 6.1.44: We put $K=K_2$ and we regard K as a perverse sheaf on $\overline{\mathcal{Z}\times C}$. Let $\tilde{K}$ be the perverse sheaf on $X_{S,2}$ such that

6.1.45.
$$(\rho'_S)^*\tilde{K}[\dim P]\simeq(\rho_S)^*K[\dim G+\dim U_P].$$

For $\mathcal{O}\in\Gamma$, define

$$X_{\sigma,S_{\mathcal{O}},1}=\{(t,x,g)\in S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)\times C_G^o(\sigma)|\operatorname{Ad}(g^{-1})x\in t+\overline{C_{\mathcal{O}}}+\mathcal{U}_{P_{\mathcal{O}}}\},$$

and let $\rho_{\sigma,S_{\mathcal{O}}}:X_{\sigma,S_{\mathcal{O}},1}\to\mathcal{Z}_{\mathcal{O}}\times\overline{C_{\mathcal{O}}}$ be given by $(t,x,g)\mapsto((t,t),v)$ where $v\in\overline{C_{\mathcal{O}}}$ is such that $\operatorname{Ad}(g^{-1})x\in t+v+\mathcal{U}_{P_{\mathcal{O}}}$, and $\rho_{\sigma,S_{\mathcal{O}}}:X_{\sigma,S_{\mathcal{O}},1}\to X_{\sigma,S_{\mathcal{O}},1}$ given by $(t,x,g)\mapsto(t,x,gP_{\mathcal{O}})$.

Put $K_{\mathcal{O}}=\operatorname{IC}(\overline{\mathcal{Z}_{\mathcal{O}}\times C_{\mathcal{O}}},\mathcal{E}_{\mathcal{O}})[\dim(\mathcal{Z}_{\mathcal{O}}\times C_{\mathcal{O}})]$ and let $\tilde{K}_{\mathcal{O}}$ be the perverse sheaf on $X_{\sigma,S_{\mathcal{O}},2}$ such that

$$(\rho'_{\sigma,S_{\mathcal{O}}})^*\tilde{K}_{\mathcal{O}}[\dim P_{\mathcal{O}}]\simeq(\rho_{\sigma,S_{\mathcal{O}}})^*K_{\mathcal{O}}[\dim\,(C_G(\sigma))+\dim U_{P_{\mathcal{O}}}].$$

Since the morphism $X_{S,2}\to\overline{Y_S}\cap(S\times(\sigma+\mathbf{U}))$ given by $(t,x,gP)\mapsto(t,x)$ is obtained by base change from the proper morphism ρ''_S, we find from the proper base change theorem an isomorphism

6.1.46.
$$(\rho''_S)_!(\tilde{K}|_{X^{\mathbf{U}}_{S,2}})\simeq((\rho''_S)_!\tilde{K})|_{\overline{Y_S}\cap(S\times(\sigma+\mathbf{U}))}.$$

Similarly by applying the proper base change theorem to $\rho''_{\sigma,S_{\mathcal{O}}}:X_{\sigma,S_{\mathcal{O}},2}\to\overline{Y_{\sigma,S_{\mathcal{O}}}}$, we get a canonical isomorphism

6.1.47.
$$(\rho''_{\sigma,S_{\mathcal{O}}})_!\big(\tilde{K}_{\mathcal{O}}|_{X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}}\big)\simeq\big((\rho''_{\sigma,S_{\mathcal{O}}})_!\tilde{K}_{\mathcal{O}}\big)|_{\overline{Y_{\sigma,S_{\mathcal{O}}}}\cap(S_{\mathcal{O}}\times\mathbf{U})}.$$

On the other hand, we have the following commutative diagram.

6.1.48.

$$\begin{array}{ccc}
\coprod_{\mathcal{O}\in\Gamma}(\overline{\mathcal{Z}_{\mathcal{O}}\times C_{\mathcal{O}}}) & \xrightarrow{\coprod f_{\mathcal{O}}} & \mathcal{Z}\times\overline{C} \\
\Big\uparrow{\scriptstyle \coprod \rho_{\sigma,S_{\mathcal{O}}}} & & \Big\uparrow{\scriptstyle \rho_S} \\
\coprod_{\mathcal{O}\in\Gamma} X_{\sigma,S_{\mathcal{O}},1} & \xrightarrow{\coprod f'_{\mathcal{O}}} & X_{S,1} \\
\Big\downarrow{\scriptstyle \coprod \rho'_{\sigma,S_{\mathcal{O}}}} & & \Big\downarrow{\scriptstyle \rho'_S} \\
\coprod_{\mathcal{O}\in\Gamma} X_{\sigma,S_{\mathcal{O}},2} & \xrightarrow{\coprod f''_{\mathcal{O}}} & X_{S,2}
\end{array}$$

where $f''_{\mathcal{O}}(t,x,gP_{\mathcal{O}}) = (\mathrm{Ad}(x_{\mathcal{O}}^{-1})(\sigma+t), \sigma+x, gx_{\mathcal{O}}P)$,

$f'_{\mathcal{O}}(t,x,g) = (\mathrm{Ad}(x_{\mathcal{O}}^{-1})(\sigma+t), \sigma+x, gx_{\mathcal{O}})$ and

$f_{\mathcal{O}}((t,t),v) = \big(\mathrm{Ad}(x_{\mathcal{O}}^{-1})(\sigma+t), \mathrm{Ad}(x_{\mathcal{O}}^{-1})(\sigma+t), \mathrm{Ad}(x_{\mathcal{O}}^{-1})v\big)$.

The inverse image of K by $\coprod f_{\mathcal{O}}$ is the complex K' given by $K'|_{\overline{\mathcal{Z}_{\mathcal{O}}\times C_{\mathcal{O}}}} = K_{\mathcal{O}}$ for any $\mathcal{O}\in\Gamma$. If $\tilde{K}'$ denotes the complex on $\coprod_{\mathcal{O}\in\Gamma} X_{\sigma,S_{\mathcal{O}},2}$ given by $\tilde{K}'|_{X_{\sigma,S_{\mathcal{O}},2}} = \tilde{K}_{\mathcal{O}}$, then we have

$$(\coprod \rho'_{\sigma,S_{\mathcal{O}}})^*\tilde{K}'[\dim P_{\mathcal{O}}] \simeq (\coprod \rho'_{\sigma,S_{\mathcal{O}}})^* K'[\dim\,(C_G(\sigma)) + \dim U_{P_{\mathcal{O}}}].$$

Hence from 6.1.45 and the commutativity of 6.1.48, we deduce that

$$(\coprod f''_{\mathcal{O}})^*\tilde{K}[-\delta] \simeq \tilde{K}'$$

and so we get that

6.1.49.

$$(\coprod f''_{\mathcal{O}})^*\left(\tilde{K}|_{X^{\mathbf{U}}_{S,2}}\right)[-\delta] \simeq \tilde{K}'|_{\coprod X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}}.$$

We now consider the following cartesian diagram.

$$\begin{array}{ccc}
\coprod_{\mathcal{O}\in\Gamma} X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2} & \xrightarrow{\coprod f''_{\mathcal{O}}} & X^{\mathbf{U}}_{S,2} = \coprod_{\hat{\mathcal{O}}\in\hat{\Gamma}} X^{\mathbf{U}}_{S,2,\hat{\mathcal{O}}} \\
\Big\downarrow{\scriptstyle \coprod \rho''_{\sigma,S_{\mathcal{O}}}} & & \Big\downarrow{\scriptstyle \rho''_S} \\
\coprod_{\mathcal{O}\in\Gamma} \overline{Y_{\sigma,S_{\mathcal{O}}}}\cap(S_{\mathcal{O}}\times\mathbf{U}) & \xrightarrow{\tilde{f}} & \overline{Y_S}\cap(S\times(\sigma+\mathbf{U}))
\end{array}$$

from which we find together with 6.1.49 an isomorphism

$$\tilde{f}^*\left((\rho''_S)_!\left(\tilde{K}|_{X^{\mathbf{U}}_{S,2}}\right)\right)[-\delta] \simeq (\coprod \rho''_{\sigma,S_{\mathcal{O}}})_!\left(\tilde{K}'|_{\coprod X^{\mathbf{U}}_{\sigma,S_{\mathcal{O}},2}}\right).$$

Combined with the isomorphisms 6.1.46 and 6.1.47, we get the isomorphism 6.1.44 extending 6.1.43. □

As a consequence, it follows from the properties of intersection cohomology complexes that the isomorphism of 6.1.44 is the unique one extending 6.1.43, and that it is an isomorphism of F-equivariant complexes if we regard the two complexes of 6.1.44 as the F-equivariant complexes induced from the F-equivariant sheaf $(\mathcal{E},\phi)$. Put $\phi^{S\times\mathcal{G}} = \phi_2^{S\times\mathcal{G}}$ and for $\mathcal{O}\in\Gamma$, let $\phi^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}$: $F^*\big(K_{F(\mathcal{O})}^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}\big) \xrightarrow{\sim} K_{\mathcal{O}}^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}$ be the isomorphism induced by $(\mathcal{E},\phi)$.

The isomorphism 6.1.44 gives rise to an isomorphism of stalks

$$\mathcal{H}^{i-\delta}_{(s,\sigma+v)}K^{S\times\mathcal{G}} \simeq \bigoplus_{\mathcal{O}\in\Gamma}\mathcal{H}^i_{(\mathrm{Ad}(x_{\mathcal{O}}^{-1})s-\sigma,v)}K_{\mathcal{O}}^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}$$

where $u\in C_{\mathcal{G}}(\sigma)^F_{nil}$ and $s\in S^F$ are as in 6.1.20. And since the complexes $K_{\mathcal{O}}^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}$ are supported by $\overline{Y_{\sigma,S_{\mathcal{O}}}}$, it follows that $\mathcal{H}^i_{(\mathrm{Ad}(x_{\mathcal{O}}^{-1})s-\sigma,v)}K_{\mathcal{O}}^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}$ is zero unless $\mathrm{Ad}(x_{\mathcal{O}})s-\sigma=0$, i.e $\mathrm{Ad}(x_{\mathcal{O}}^{-1})\sigma=s$. Hence taking the characteristic functions in 6.1.44 with respect to $\phi^{S\times\mathcal{G}}=\phi_2^{S\times\mathcal{G}}$ and $\phi^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}$ ($\mathcal{O}\in\Gamma$), we get that

$$\mathbf{X}_{K^{S\times\mathcal{G}},\phi^{S\times\mathcal{G}}}(s,\sigma+u) = \sum_{\{\mathcal{O}\in\Gamma\,|\,F(\mathcal{O})=\mathcal{O},\mathrm{Ad}(x_{\mathcal{O}}^{-1})\sigma=s\}} \mathbf{X}_{K_{\mathcal{O}}^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)},\phi^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}}(0,u).$$

But as in [Wal01, page 44, (10)], we prove that

$$\mathbf{X}_{K_{\mathcal{O}}^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)},\phi^{S_{\mathcal{O}}\times C_{\mathcal{G}}(\sigma)}}(0,u) = \mathcal{Q}^{C_{\mathcal{G}}(\sigma)}_{\mathcal{L}_{\mathcal{O}},C_{\mathcal{O}},\zeta_{\mathcal{O}},\phi_{\mathcal{O}}}(u)$$

where $(\zeta_{\mathcal{O}},\phi_{\mathcal{O}})$ is the inverse image by $C_{\mathcal{O}}\hookrightarrow\mathcal{Z}_{\mathcal{O}}\times C_{\mathcal{O}}$, $v\mapsto((0,0),v)$ of the F-equivariant sheaf $((h_{\mathcal{O}})^*\mathcal{E},(h_{\mathcal{O}})^*(\phi))$ with $h_{\mathcal{O}}$ as in 6.1.40 i.e. $(\zeta_{\mathcal{O}},\phi_{\mathcal{O}})$ is the inverse image of $(\mathcal{E},\phi)$ by $C_{\mathcal{O}}\to\mathcal{Z}\times C$, $v\mapsto((\mathrm{Ad}(x_{\mathcal{O}}^{-1})\sigma,\mathrm{Ad}(x_{\mathcal{O}}^{-1})\sigma),$ $\mathrm{Ad}(x_{\mathcal{O}}^{-1})v)$.

Hence from the fact that $|\mathcal{O}^F|=|C_G^o(\sigma)^F|$ if $\mathcal{O}\in\Gamma$ and $F(\mathcal{O})=\mathcal{O}$, we deduce that

$$\mathbf{X}_{K^{S\times\mathcal{G}},\phi^{S\times\mathcal{G}}}(s,\sigma+u) = |C_G^o(\sigma)^F|^{-1}\sum_{\{x\in G^F\,|\,\mathrm{Ad}(x^{-1})\sigma=s\}}\mathcal{Q}^{C_{\mathcal{G}}(\sigma)}_{\mathcal{L}_x,C_x,\zeta_x,\phi_x}(u)$$

where $L_x, C_x, \zeta_x, \phi_x=\phi_{2,x}$ are as in 6.1.20(ii). □

Proof of Theorem 6.1.20(i)

Let $(Y_{S,1}, Y_{S,2}, Y_S, \alpha_S, \alpha'_S, \alpha''_S)$ and $(Y_1, Y_2, Y, \alpha, \alpha', \alpha'')$ be respectively as in 6.1.8 (with $\mathcal{Z} = \mathcal{Z}_1$), and as in 5.1.31 (with $\Sigma = z(\mathcal{L}) + C$). Note that we have $\mathcal{Z}_1 \times C \simeq S \times \Sigma$. We have the following cartesian diagram.

6.1.50.

$$\begin{array}{ccccccc} S\times\Sigma & \xleftarrow{\alpha_S} & Y_{S,1} & \xrightarrow{\alpha'_S} & Y_{S,2} & \xrightarrow{\alpha''_S} & Y_S \\ \uparrow{\scriptstyle j_{s,\mathcal{L}}} & & \uparrow{\scriptstyle j'_s} & & \uparrow{\scriptstyle j''_s} & & \uparrow{\scriptstyle j_{s,\mathcal{G}}} \\ \Sigma & \xleftarrow{\alpha} & Y_1 & \xrightarrow{\alpha'} & Y_2 & \xrightarrow{\alpha''} & Y \end{array}$$

where $j_{s,\mathcal{G}}(x) = (s, x)$, $j'_s(x, g) = (s, x, g)$, $j''_s(x, gL) = (s, x, gL)$.

From 6.1.50 we find a canonical isomorphism

6.1.51.

$$(j_{s,\mathcal{G}})^*\big((\alpha''_S)_*\xi_{S,2}\big) \simeq (\alpha'')_*\xi_{s,2}$$

where $\xi_{S,2}$ is the unique local system on $Y_{S,2}$ such that $(\alpha'_S)^*(\xi_{S,2}) = (\alpha_S)^*\mathcal{E}_1$ and where $\xi_{s,2}$ is the unique local system on Y_2 such that $(\alpha')^*\xi_{s,2} \simeq \alpha^*(\mathcal{E}_{1,s})$; recall that $\mathcal{E}_{1,s} := (m_s)^*\mathcal{L}_\Psi \boxtimes \zeta$.

Moreover, since all the maps of 6.1.50 are defined over $\mathbb{F}_q$, this isomorphism is compatible with the two canonical isomorphisms $F^*\big((j_{s,\mathcal{G}})^*\left((\alpha''_S)_*\xi_{S,2}\right)\big) \xrightarrow{\sim} (j_{s,\mathcal{G}})^*\big((\alpha''_S)_*\xi_{S,2}\big)$ and $F^*\big((\alpha'')_*\xi_{s,2}\big) \xrightarrow{\sim} (\alpha'')_*\xi_{s,2}$ induced respectively by ϕ_1 and $\psi_{1,s} := (j_{s,\mathcal{L}})^*(\phi_1) : F^*(\mathcal{E}_{1,s}) \xrightarrow{\sim} \mathcal{E}_{1,s}$. The isomorphism 6.1.51 can be regarded as an isomorphism

$$(j_{s,\mathcal{G}})^*\left(K_1^{S\times\mathcal{G}}|_{Y_S}\right)[-\dim S] \simeq K_{1,s}^{\mathcal{G}}|_Y$$

where $K_{1,s}^{\mathcal{G}} := \operatorname{ind}_\Sigma^{\mathcal{G}}(\mathcal{E}_{1,s})$. From the properties of intersection cohomology complexes, this isomorphism is the restriction to Y of an isomorphism

$$(j_{s,\mathcal{G}})^*(K_1^{S\times\mathcal{G}})[-\dim S] \simeq K_{1,s}^{\mathcal{G}}$$

which is compatible with the isomorphisms $(j_{s,\mathcal{G}})^*(\phi_1^{S\times\mathcal{G}}) : F^*(K_1^{S\times\mathcal{G}}) \xrightarrow{\sim} K_1^{S\times\mathcal{G}}$ and $\psi_{1,s}^{\mathcal{G}} : F^*(K_{1,s}^{\mathcal{G}}) \xrightarrow{\sim} K_{1,s}^{\mathcal{G}}$ where $\psi_{1,s}^{\mathcal{G}}$ is the canonical isomorphism induced by $\psi_{1,s}$. As a consequence we get that

6.1.52.

$$\mathbf{X}_{K_1^{S\times\mathcal{G}},\phi_1^{S\times\mathcal{G}}}(s, z) = \mathbf{X}_{K_{1,s}^{\mathcal{G}},\psi_{1,s}^{\mathcal{G}}}(z)$$

for any $z \in \mathcal{G}^F$. Hence the assertion (ii) of 6.1.20 will follow from

6.1.53.

$$\mathbf{X}_{K^{\mathcal{G}}_{1,s},\psi^{\mathcal{G}}_{1,s}}(\sigma+u)=|C^o_G(\sigma)^F|^{-1}\sum_{\{x\in G^F|\mathrm{Ad}(x^{-1})\sigma\in z(\mathcal{L})\}}\mathcal{Q}^{C_{\mathcal{G}}(\sigma)}_{\mathcal{L}_x,C_x,\zeta_x,\phi_{1,x}}(u).$$

Note that $\phi_{1,x}:F^*(\zeta_x)\simeq\zeta_x$ is the inverse image by $C_x\to\Sigma$, $v\mapsto\mathrm{Ad}(x^{-1})(\sigma+v)$ of $\psi_{1,s}:F^*(\mathcal{E}_{1,s})\simeq\mathcal{E}_{1,s}$. Hence 6.1.53 is 5.5.9. □

End of the proof of theorem 6.1.20

6.1.54 Deligne-Lusztig Induction and Geometrical Induction

We use the notation and assumption of 6.1.19. We denote by $\mathcal{C}(S^F\times\mathcal{L}^F)$ the $\overline{\mathbb{Q}}_\ell$-vector space of L^F-invariant $\overline{\mathbb{Q}}_\ell$-valued functions on $S^F\times\mathcal{L}^F$ and by $\mathcal{C}(S^F\times\mathcal{G}^F)$ the space of G^F-invariant $\overline{\mathbb{Q}}_\ell$-valued functions on $S^F\times\mathcal{G}^F$ where L (resp. G) acts on $S\times\mathcal{L}$ (resp. on $S\times\mathcal{G}$) by Ad on the second coordinate and trivially on the first coordinate. Then we define the Deligne-Lusztig induction $\mathcal{R}^{S\times\mathcal{G}}_{S\times\mathcal{L}}:\mathcal{C}(S^F\times\mathcal{L}^F)\to\mathcal{C}(S^F\times\mathcal{G}^F)$ by

$$\mathcal{R}^{S\times\mathcal{G}}_{S\times\mathcal{L}}(f)(t,x)=|L^F|^{-1}\sum_{y\in\mathcal{L}^F}S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)f(t,y)$$

where $f\in\mathcal{C}(S^F\times\mathcal{L}^F)$ and $(t,x)\in S^F\times\mathcal{G}^F$, and where $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ is the function on $\mathcal{G}^F\times\mathcal{L}^F$ defined in 3.2.17.

Remark 6.1.55. Let $f\in\mathcal{C}(S^F\times\mathcal{L}^F)$ and for $t\in S^F$, let $f_t\in\mathcal{C}(\mathcal{L}^F)$ be given by $f_t(x)=f(t,x)$, then we have $\mathcal{R}^{S\times\mathcal{G}}_{S\times\mathcal{L}}(f)(t,x)=\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}(f_t)(x)$ for any $(t,x)\in S^F\times\mathcal{G}^F$.

We are now in position to state the main result of this section.

Theorem 6.1.56. *With the above notation we have*

(i) $\mathcal{R}^{S\times\mathcal{G}}_{S\times\mathcal{L}}(\mathbf{X}_{K_1,\phi_1})=\mathbf{X}_{K_1^{S\times\mathcal{G}},\phi_1^{S\times\mathcal{G}}}$.

(ii) $\mathcal{R}^{S\times\mathcal{G}}_{S\times\mathcal{L}}(\mathbf{X}_{K_2,\phi_2})=\mathbf{X}_{K_2^{S\times\mathcal{G}},\phi_2^{S\times\mathcal{G}}}$.

Proof: Let $i\in\{1,2\}$ and let $(s,\sigma+u)\in S^F\times\mathcal{G}^F$ be such that σ is semi-simple, u is nilpotent, $[\sigma,u]=0$ and $(s,\mathrm{Ad}(g^{-1})\sigma)\in\mathcal{Z}_i$ for some $g\in G^F$. By 6.1.20 we have

6.1.57.

$$\mathbf{X}_{K_i^{S\times\mathcal{G}},\phi_i^{S\times\mathcal{G}}}(s,\sigma+u)=|C_G^o(\sigma)^F|^{-1}\sum_{\substack{x\in G^F\\(s,\mathrm{Ad}(x^{-1})\sigma)\in\mathcal{Z}_i}}\mathcal{Q}_{\mathcal{L}_x,C_x,\zeta_x,\phi_{i,x}}^{C_{\mathcal{G}}(\sigma)}(u).$$

On the other hand we have

$$\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\left(\mathbf{X}_{K_i,\phi_i}\right)(s,\sigma+u)=|L^F|^{-1}\sum_{x\in\mathcal{L}^F}S_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(\sigma+u,x)\mathbf{X}_{K_i,\phi_i}(s,x).$$

Since the complex K_i is supported by $\mathcal{Z}_i\times\overline{C}$, we get that

$$\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\left(\mathbf{X}_{K_i,\phi_i}\right)(s,\sigma+u)=|L^F|^{-1}\sum_{\substack{(t,v)\in z(\mathcal{L})^F\times\overline{C}^F\\(s,t)\in\mathcal{Z}_i}}S_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(\sigma+u,t+v)\mathbf{X}_{K_i,\phi_i}(s,t+v).$$

But

$$S_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(\sigma+u,t+v)=\sum_{\substack{h\in G^F\\\mathrm{Ad}(h)t=\sigma}}|C_L^o(t)^F||C_G^o(t)^F|^{-1}\mathcal{Q}_{C_{\mathcal{L}}(t)}^{C_{\mathcal{G}}(t)}\left(\mathrm{Ad}(h^{-1})u,v\right).$$

Hence we get that

$$\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\left(\mathbf{X}_{K_i,\phi_i}\right)(s,\sigma+u)=$$

$$\sum_{\substack{(t,v)\in z(\mathcal{L})^F\times\overline{C}^F\\(s,t)\in\mathcal{Z}_i}}\ \sum_{\substack{h\in G^F\\\mathrm{Ad}(h)t=\sigma}}|C_G^o(t)^F|^{-1}\mathcal{Q}_{\mathcal{L}}^{C_{\mathcal{G}}(t)}\left(\mathrm{Ad}(h^{-1})u,v\right)\mathbf{X}_{K_i,\phi_i}(s,t+v).$$

By interchanging the sums we have

$$\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\left(\mathbf{X}_{K_i,\phi_i}\right)(s,\sigma+u)=|C_G^o(\sigma)^F|^{-1}\times$$

$$\sum_{\substack{h\in G^F\\(s,\mathrm{Ad}(h^{-1})\sigma)\in\mathcal{Z}_i}}\ \sum_{v\in\overline{C}^F}\mathcal{Q}_{\mathcal{L}}^{C_{\mathcal{G}}\left(\mathrm{Ad}(h^{-1})\sigma\right)}\left(\mathrm{Ad}(h^{-1})u,v\right)\mathbf{X}_{K_i,\phi_i}\left(s,\mathrm{Ad}(h^{-1})\sigma+v\right).$$

But by definition of $(C_h,\zeta_h,\phi_{i,h})$, see 6.1.20, we have

$$\mathbf{X}_{K_i,\phi_i}\left(s,\mathrm{Ad}(h^{-1})\sigma+v\right)=\mathbf{X}_{K_h,\phi_{i,h}}(\mathrm{Ad}(h)v)$$

where $K_h=\mathrm{IC}(\overline{C_h},\zeta_h)[\dim(\mathcal{Z}\times C)]$ and where we still denote by $\phi_{i,h}$ the canonical isomorphism $F^*(K_h)\xrightarrow{\sim}K_h$ induced by $\phi_{i,h}:F^*(\zeta_h)\xrightarrow{\sim}\zeta_h$. Hence it follows that $\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\left(\mathbf{X}_{K_i,\phi_i}\right)(s,\sigma+u)=|C_G^o(\sigma)^F|^{-1}\times$

$$\sum_{\substack{h\in G^F\\(s,\mathrm{Ad}(h^{-1})\sigma)\in\mathcal{Z}_i}}\ \sum_{v\in\overline{C_h}^F}\mathcal{Q}_{\mathcal{L}_h}^{C_{\mathcal{G}}(\sigma)}(u,v)\mathbf{X}_{K_h,\phi_{i,h}}(v).$$

Hence 6.1.56 follows from 6.1.57 together with 5.5.13. □

6.2 On the Conjecture 3.2.30

6.2.1 Reduction of 3.2.30 to the Case of Nilpotently Supported Cuspidal Functions

We first reduce the conjecture to the case of characteristic functions of F-equivariant cuspidal admissible complexes (or cuspidal orbital perverse sheaves).

Proposition 6.2.2. *The following three assertions are equivalent.*

(1) For any inclusion $L \subset M$ of F-stable Levi subgroups of G with corresponding Lie algebras inclusion $\mathcal{L} \subset \mathcal{M}$, we have:

$$\mathcal{F}^{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{M}} = \epsilon_M \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{M}} \circ \mathcal{F}^{\mathcal{L}}.$$

(2) For any inclusion $L \subset M$ of F-stable Levi subgroups of G with corresponding Lie algebras inclusion $\mathcal{L} \subset \mathcal{M}$, and any F-equivariant cuspidal admissible complex (K, ϕ) on $\mathcal{L}$, we have:

$$\mathcal{F}^{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\big(\mathbf{X}_{K,\phi}\big) = \epsilon_M \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{M}} \circ \mathcal{F}^{\mathcal{L}}\big(\mathbf{X}_{K,\phi}\big).$$

(3) For any inclusion $L \subset M$ of F-stable Levi subgroups of G with corresponding Lie algebras inclusion $\mathcal{L} \subset \mathcal{M}$, and any F-equivariant cuspidal orbital perverse sheaf (K, ϕ) on $\mathcal{L}$, we have:

$$\mathcal{F}^{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\big(\mathbf{X}_{K,\phi}\big) = \epsilon_M \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{M}} \circ \mathcal{F}^{\mathcal{L}}\big(\mathbf{X}_{K,\phi}\big).$$

Proof: The assertions (2) and (3) are particular cases of (1). We assume that (2) holds. Let us prove (1). By 5.2.22, we have to verify that the commutation formula holds for the characteristic functions of the F-equivariant admissible complexes. Let M be an F-stable Levi subgroup of G with Lie algebra $\mathcal{M}$. Let (A, ϕ_A) be an F-equivariant admissible complex on $\mathcal{M}$. By 5.4.12 and 5.5.16, we have a formula

$$\mathbf{X}_{A,\phi_A} = |\mathcal{W}_M(\mathcal{E})|^{-1} \sum_{w \in \mathcal{W}_M(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_A)^{-1}, V_A) \mathcal{R}_{\mathcal{L}_w}^{\mathcal{M}}(\mathbf{X}_{K(\Sigma_w, \mathcal{E}_w), \phi_w}).$$

Let us now apply $\mathcal{R}_{\mathcal{M}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{M}}$ to this formula. By (2) and the transitivity of Deligne-Lusztig induction, we get that $\mathcal{R}_{\mathcal{M}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{M}}\big(\mathbf{X}_{A,\phi_A}\big) = |\mathcal{W}_M(\mathcal{E})|^{-1} \times$

$$\sum_{w \in \mathcal{W}_M(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_A)^{-1}, V_A) \epsilon_M \epsilon_{L_w} \mathcal{R}_{\mathcal{L}_w}^{\mathcal{G}}\big(\mathcal{F}^{\mathcal{L}_w}(\mathbf{X}_{K(\Sigma_w, \mathcal{E}_w), \phi_w})\big).$$

Applying again (2) we finally deduce that

$$\mathcal{R}_{\mathcal{M}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{M}}\big(\mathbf{X}_{A,\phi_A}\big) = \epsilon_G \epsilon_M \mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{M}}^{\mathcal{G}}\big(\mathbf{X}_{A,\phi_A}\big).$$

Let us now prove (3). Let $L \subset M$ be an inclusion of F-stable Levi subgroups of G and let (K, ϕ) be an F-equivariant cuspidal admissible complex on $\mathcal{L}$. By 5.2.10, the Fourier transform of $\mathbf{X}_{K,\phi}$ is a function of the form $\mathbf{X}_{K',\phi'}$ where (K', ϕ') is an F-equivariant orbital perverse sheaf on $\mathcal{L}$. Hence the identity in (2) becomes

$$\mathcal{F}^{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{M}}(\mathbf{X}_{K,\phi}) = \epsilon_M \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\big(\mathbf{X}_{K',\phi'}\big).$$

Applying $\mathcal{F}^{\mathcal{M}}$ to this equality, from 3.1.10(ii), we get that

$$\big(\mathcal{R}_{\mathcal{L}}^{\mathcal{M}}(\mathbf{X}_{K,\phi})\big)^{-} = \epsilon_M \epsilon_L \mathcal{F}^{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\big(\mathbf{X}_{K',\phi'}\big).$$

But $\big(\mathcal{R}_{\mathcal{L}}^{\mathcal{M}}(f)\big)^{-} = \mathcal{R}_{\mathcal{L}}^{\mathcal{M}}(f^{-})$ for any $f \in \mathcal{C}(\mathcal{L}^F)$, hence applying again 3.1.10(ii), we have

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\big(\mathcal{F}^{\mathcal{L}} \circ \mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K,\phi})\big) = \epsilon_M \epsilon_L \mathcal{F}^{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\big(\mathbf{X}_{K',\phi'}\big)$$

that is

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{M}} \circ \mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K',\phi'}) = \epsilon_M \epsilon_L \mathcal{F}^{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\big(\mathbf{X}_{K',\phi'}\big).$$

Since the above equalities are in fact equivalent this prove the equivalence between (2) and (3). □

The following result reduces the proof of 6.2.2(3) to the case of nilpotently supported cuspidal orbital perverse sheaves.

Theorem 6.2.3. *Let (L, C, ζ) be such that L is an F-stable Levi subgroup of G and (C, ζ) is an F-stable nilpotent cuspidal pair of $\mathcal{L} = Lie(L)$. Then there is a constant $c \in \overline{\mathbb{Q}}_\ell^{\times}$ such that for any $\sigma \in z(\mathcal{L})^F$ and any $\psi : F^*(K_{2,\sigma}) \xrightarrow{\sim} K_{2,\sigma}$ where $K_{2,\sigma}$ is as in 6.1, we have*

$$\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathbf{X}_{K_{2,\sigma},\psi}) = c \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K_{2,\sigma},\psi}).$$

Proof: Define the Fourier transform $\mathcal{F}^{S\times\mathcal{G}} : \mathcal{C}(S^F \times \mathcal{G}^F) \to \mathcal{C}(S^F \times \mathcal{G}^F)$ with respect to (μ, Ψ) by

$$\mathcal{F}^{S\times\mathcal{G}}(f)(s,x) = |\mathcal{G}^F|^{-\frac{1}{2}} \sum_{y \in \mathcal{G}^F} \Psi(\mu(y,x)) f(s,y)$$

with $f \in \mathcal{C}(S^F \times \mathcal{G}^F)$, $(s,x) \in S^F \times \mathcal{G}^F$. We also define the Deligne-Fourier transform $\mathcal{F}^{S\times\mathcal{G}} : \mathcal{D}_c^b(S\times\mathcal{G}) \to \mathcal{D}_c^b(S\times\mathcal{G})$ with respect to (μ,Ψ) as follows (see [Wal01]).

Let $p_{12}, p_{13} : S\times\mathcal{G}\times\mathcal{G} \to S\times\mathcal{G}$ be given by $p_{12}(s,x,y) = (s,x)$ and $p_{13}(s,x,y) = (s,y)$ and let $p_{23} : S\times\mathcal{G}\times\mathcal{G} \to \mathcal{G}\times\mathcal{G}$ be given by $p_{23}(s,x,y) = (x,y)$, then for $K \in \mathcal{D}_c^b(S\times\mathcal{G})$, define

$$\mathcal{F}^{S\times\mathcal{G}}(K) = (p_{13})_!\big((p_{12})^*K \otimes (p_{23})^*(\mu^*\mathcal{L}_\Psi)\big)[\dim\mathcal{G}].$$

The Fourier transform $\mathcal{F}^{S\times\mathcal{G}}$ have the following properties.

6.2.4. *(i) The functor $\mathcal{F}^{S\times\mathcal{G}}$ leaves $\mathcal{M}_G(S\times\mathcal{G})$ stable.*

(ii) If (K,ϕ) is an F-equivariant complex on $S\times\mathcal{G}$, then ϕ induces an isomorphism $\mathcal{F}(\phi) : F^*\big(\mathcal{F}^{S\times\mathcal{G}}K\big) \xrightarrow{\sim} \mathcal{F}^{S\times\mathcal{G}}K$ such that

$$\mathbf{X}_{\mathcal{F}^{S\times\mathcal{G}}(K),\mathcal{F}(\phi)} = (-1)^{dim\,\mathcal{G}}|\mathcal{G}^F|^{\frac{1}{2}}\mathcal{F}^{S\times\mathcal{G}}(\mathbf{X}_{K,\phi}).$$

(iii) We have

$$\mathcal{F}^{S\times\mathcal{G}} \circ ind_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}} = ind_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}} \circ \mathcal{F}^{S\times\mathcal{L}}(-dim\mathcal{U}_P).$$

The assertion (i) of 6.2.4 can be found in [Wal01, Page 38]. As in 5.2.3, the proof of (ii) involves the Grothendieck trace formula applied to the F-equivariant complex $\big(\mathcal{F}^{S\times\mathcal{G}}(K), \mathcal{F}(\phi)\big)$ where $\mathcal{F}(\phi)$ is the isomorphism induced by ϕ and by the isomorphism $\phi_{\mathcal{L}_\Psi}$ of 5.1.57. As noticed in [Wal01, Page 40], since the variety S does not play any role in (iii), we refer to 5.2.8.

Let K_1 and K_2 be the two perverse sheaves on $S\times\mathcal{L}$ defined in 6.1.15; we have $\mathcal{F}^{S\times\mathcal{L}}(K_2) \simeq K_1$. Hence from 6.2.4(iii), we get that $\mathcal{F}^{S\times\mathcal{G}}\big(K_2^{S\times\mathcal{G}}\big) \simeq K_1^{S\times\mathcal{G}}$ where $K_1^{S\times\mathcal{G}}$ and $K_2^{S\times\mathcal{G}}$ are as in 6.1.19. Since the pair (C,ζ) is F-stable, the complexes K_1 and K_2 are also F-stable. Let $\phi_2 : F^*(K_2) \xrightarrow{\sim} K_2$ be an isomorphism and let $\phi_1 : F^*(K_1) \xrightarrow{\sim} K_1$ be given by $\phi_1 = \mathcal{F}(\phi_2)$ (see 6.2.4(ii)). As in 6.1.19, we denote by $\phi_1^{S\times\mathcal{G}}$ and $\phi_2^{S\times\mathcal{G}}$ the canonical isomorphisms $F^*\big(K_1^{S\times\mathcal{G}}\big) \xrightarrow{\sim} K_1^{S\times\mathcal{G}}$ and $F^*\big(K_2^{S\times\mathcal{G}}\big) \xrightarrow{\sim} K_2^{S\times\mathcal{G}}$ induced respectively by ϕ_1 and ϕ_2. Since the perverse sheaves $K_1^{S\times\mathcal{G}}$ and $K_2^{S\times\mathcal{G}}$ are simple (see 6.1.16), from the isomorphism $\mathcal{F}^{S\times\mathcal{G}}\big(K_2^{S\times\mathcal{G}}\big) \simeq K_1^{S\times\mathcal{G}}$ we get that there exists a constant $c' \in \overline{\mathbb{Q}}_\ell^\times$ such that $\phi_1^{S\times\mathcal{G}} = c'\mathcal{F}\big(\phi_2^{S\times\mathcal{G}}\big)$ where $\mathcal{F}\big(\phi_2^{S\times\mathcal{G}}\big)$ denotes the isomorphism $F^*\left(\mathcal{F}^{S\times\mathcal{G}}\big(K_2^{S\times\mathcal{G}}\big)\right) \simeq \mathcal{F}^{S\times\mathcal{G}}\big(K_2^{S\times\mathcal{G}}\big)$ induced by $\phi_2^{S\times\mathcal{G}}$ as in 6.2.4(ii). As a consequence we have

$$\mathbf{X}_{K_1^{S\times\mathcal{G}},\phi_1^{S\times\mathcal{G}}} = c'\mathbf{X}_{\mathcal{F}^{S\times\mathcal{G}}(K_2^{S\times\mathcal{G}}),\mathcal{F}(\phi_2^{S\times\mathcal{G}})}.$$

From 6.2.4(ii), it follows that

$$\mathbf{X}_{K_1^{S\times\mathcal{G}},\phi_1^{S\times\mathcal{G}}} = c'(-1)^{\dim\mathcal{G}}|\mathcal{G}^F|^{\frac{1}{2}}\mathcal{F}^{S\times\mathcal{G}}\big(\mathbf{X}_{K_2^{S\times\mathcal{G}},\phi_2^{S\times\mathcal{G}}}\big).$$

Hence from 6.1.56, we get that

$$\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\big(\mathbf{X}_{K_1,\phi_1}\big) = c'(-1)^{\dim\mathcal{G}}|\mathcal{G}^F|^{\frac{1}{2}}\mathcal{F}^{S\times\mathcal{G}}\left(\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\big(\mathbf{X}_{K_2,\phi_2}\big)\right).$$

But $\phi_1 = \mathcal{F}(\phi_2)$ by definition, hence from 6.2.4(ii), it follows that

$$\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\big(\mathcal{F}^{S\times\mathcal{L}}\big(\mathbf{X}_{K_2,\phi_2}\big)\big) = c'|\mathcal{L}^F|^{-\frac{1}{2}}|\mathcal{G}^F|^{\frac{1}{2}}\mathcal{F}^{S\times\mathcal{G}}\left(\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\big(\mathbf{X}_{K_2,\phi_2}\big)\right).$$

Restricting the functions of this equality to $\{\sigma\}\times\mathcal{G}^F$ with $\sigma\in z(\mathcal{L})^F$, we get 6.2.5.

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathcal{F}^{\mathcal{L}}\big(\mathbf{X}_{K_{2,\sigma},\phi_{2,\sigma}}\big)\big) = c'|\mathcal{L}^F|^{-\frac{1}{2}}|\mathcal{G}^F|^{\frac{1}{2}}\mathcal{F}^{\mathcal{G}}\left(\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K_{2,\sigma},\phi_{2,\sigma}}\big)\right)$$

where $\phi_{2,\sigma}: F^*(K_{2,\sigma})\xrightarrow{\sim}K_{2,\sigma}$ is the isomorphism obtained by restricting ϕ_2.

Now if we choose another isomorphism $\psi: F^*(K_{2,\sigma})\xrightarrow{\sim}K_{2,\sigma}$, then it is proportional to $\phi_{2,\sigma}$ since $K_{2,\sigma}$ is a simple perverse sheaf, hence the formula 6.2.5 remains true if we replace $\phi_{2,\sigma}$ by ψ. We thus have proved 6.2.3. □

From the previous discussion we have the following result.

Corollary 6.2.6. *The two following assertions are equivalent.*

(1) For any inclusion $L\subset M$ of F-stable Levi subgroups of G with corresponding Lie algebras inclusion $\mathcal{L}\subset\mathcal{M}$, we have

$$\mathcal{F}^{\mathcal{M}}\circ\mathcal{R}_{\mathcal{L}}^{\mathcal{M}} = \epsilon_M\epsilon_L\mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\circ\mathcal{F}^{\mathcal{L}}.$$

(2) For any inclusion $L\subset M$ of F-stable Levi subgroups of G with corresponding Lie algebras inclusion $\mathcal{L}\subset\mathcal{M}$, and any F-equivariant nilpotently supported cuspidal orbital perverse sheaf (K,ϕ) on $\mathcal{L}$, we have

$$\mathcal{F}^{\mathcal{M}}\circ\mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\big(\mathbf{X}_{K,\phi}\big) = \epsilon_M\epsilon_L\mathcal{R}_{\mathcal{L}}^{\mathcal{M}}\circ\mathcal{F}^{\mathcal{L}}\big(\mathbf{X}_{K,\phi}\big).$$

6.2.7 The Main Results

6.2.8. Let L be an F-stable Levi subgroup of G and let (C,ζ) be an F-stable cuspidal nilpotent pair of $\mathcal{L}$. Let L_o be an F-stable G-split Levi subgroup of G which is G-conjugate to L. The triple (L,C,ζ) is of the form $\big((L_o)_w, (C_o)_w, (\zeta_o)_w\big)$ for some $w \in W_G(L_o)$ and some F-stable cuspidal nilpotent pair (C_o, ζ_o) of $\mathcal{L}_o = \mathrm{Lie}(L_o)$ (see 5.4.2 and 5.4.9). Let $\phi : F^*\big(K(C,\zeta)\big) \xrightarrow{\sim} K(C,\zeta)$ and $\phi_o : F^*\big(K(C_o,\zeta_o)\big) \xrightarrow{\sim} K(C_o,\zeta_o)$ be such that $\phi = (\phi_o)_w$ (see 5.4.2). By 5.2.10, there exist two constants γ and γ_o such that

$$\mathcal{F}^{\mathcal{L}}\big(\mathbf{X}_{K(\Sigma,\mathcal{E}),1\boxtimes\phi}\big) = \gamma \mathbf{X}_{K(C,\zeta),\phi},$$

$$\mathcal{F}^{\mathcal{L}_o}\big(\mathbf{X}_{K(\Sigma_o,\mathcal{E}_o),1\boxtimes\phi_o}\big) = \gamma_o \mathbf{X}_{K(C_o,\zeta_o),\phi_o}$$

where $\Sigma = z(\mathcal{L}) + C$, $\mathcal{E} = \overline{\mathbb{Q}}_\ell \boxtimes \zeta$, $\Sigma_o = z(\mathcal{L}_o) + C_o$, $\mathcal{E}_o = \overline{\mathbb{Q}}_\ell \boxtimes \zeta_o$. Note that the two constants γ and γ_o do not depend on the choice of the isomorphisms ϕ and ϕ_o.

Let $e : W_G(L_o) \to \overline{\mathbb{Q}}_\ell^\times$ be the sign character of $W_G(L_o)$. We have the following result.

Proposition 6.2.9. *With the above notation we have*

$$\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathbf{X}_{K(C,\zeta),\phi}\big) = e(w)\gamma^{-1}\gamma_o \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}\big(\mathbf{X}_{K(C,\zeta),\phi}\big)$$

where $w \in W_G(L_o)$ is such that $L = (L_o)_w$.

Proof:
Put $A_o = K(\Sigma_o, \overline{\mathbb{Q}}_\ell \boxtimes \zeta_o)$, $K_o = K(C_o, \zeta_o)$,

$A = K(\Sigma, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$, $K = K(C,\zeta)$,

and

$f_{A_o} = \mathbf{X}_{A_o,1\boxtimes\phi_o}$, $f_{K_o} = \mathbf{X}_{K_o,\phi_o}$,

$f_A = \mathbf{X}_{A,1\boxtimes\phi}$, $f_K = \mathbf{X}_{K,\phi}$.

We thus have

6.2.10. $\mathcal{F}^{\mathcal{L}_o}(f_{A_o}) = \gamma_o f_{K_o}$ and $\mathcal{F}^{\mathcal{L}}(f_A) = \gamma f_K$.

We have to show that $\mathcal{F}^{\mathcal{G}}\big(\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f_K)\big) = e(w)\gamma^{-1}\gamma_o \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big((\mathcal{F}^{\mathcal{L}}(f_K)\big)$.

As in the proof of 6.2.2, we see that it is equivalent to show that

$$\mathcal{F}^{\mathcal{G}}\big(\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f_A)\big) = e(w)\gamma^{-1}\gamma_o \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big((\mathcal{F}^{\mathcal{L}}(f_A)\big).$$

From 6.2.10 and 4.4.7, we have an isomorphism of F-equivariant complexes

$$\delta : \big(\mathcal{F}^{\mathcal{L}_o}(A_o), \mathcal{F}(1 \boxtimes \phi_o)\big) \simeq \big(K_o, \gamma_o'\phi_o\big)$$

where $\gamma_o' = (-1)^{\dim \mathcal{L}_o} q^{\frac{\dim \mathcal{L}_o}{2}} \gamma_o$. Let P_o be an F-stable parabolic subgroup of G having L_o as a Levi subgroup and let $\mathcal{P}_o$ be its Lie algebra. Put $\big(A_o^{\mathcal{G}}, \psi_o^{\mathcal{G}}\big) =$

$\big(\mathrm{ind}_{\Sigma_o}^{\mathcal{G}}(\mathcal{E}_o), \mathrm{ind}_{\Sigma_o}^{\mathcal{G}}(1\boxtimes\phi_o)\big)$ and put $\big(K_o^{\mathcal{G}}, \phi_o^{\mathcal{G}}\big) = \big(\mathrm{ind}_{\mathcal{L}_o\subset\mathcal{P}_o}^{\mathcal{G}}(K_o), \mathrm{ind}_{\mathcal{L}_o\subset\mathcal{P}_o}^{\mathcal{G}}(\phi_o)\big)$. Then, from 5.2.8 and 5.1.33, the isomorphism δ induces an isomorphism of F-equivariant complexes

$$\theta_{P_o} : \big(\mathcal{F}^{\mathcal{G}}(A_o^{\mathcal{G}})(\dim U_{P_o}), \mathcal{F}(\psi_o^{\mathcal{G}})\big) \simeq \big(K_o^{\mathcal{G}}, \gamma_o'\phi_o^{\mathcal{G}}\big).$$

Let $\theta_w : A_o^{\mathcal{G}} \xrightarrow{\sim} A_o^{\mathcal{G}}$ be as in 5.3.6(iii) and let $\theta_{w,n} : K_o^{\mathcal{G}} \xrightarrow{\sim} K_o^{\mathcal{G}}$ be the isomorphism induced by the automorphism $\theta_w|_{\mathcal{G}_{nil}}$ of $A_o^{\mathcal{G}}|_{\mathcal{G}_{nil}} = K_o^{\mathcal{G}}|_{\mathcal{G}_{nil}}[\dim z(\mathcal{L})]$. From [Lus92, 5.5], we have $\theta_{w,n}\circ\theta_{P_o} = e(w)\theta_{P_o}\circ\mathcal{F}(\theta_w)$. As a consequence, θ_{P_o} induces an isomorphism

6.2.11.

$$\big(\mathcal{F}^{\mathcal{G}}(A_o^{\mathcal{G}})(\dim U_{P_o}), \mathcal{F}(\theta_w\circ\psi_o^{\mathcal{G}})\big) \simeq \big(K_o^{\mathcal{G}}, e(w)\gamma_o'\theta_{w,n}\circ\phi_o^{\mathcal{G}}\big).$$

Now the characteristic function of $\big(A_o^{\mathcal{G}}, \theta_w\circ\psi_o^{\mathcal{G}}\big)$ is equal to $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f_A)$ in view of 5.4.2 and 5.5.16. The characteristic function of $(K_o^{\mathcal{G}}, \theta_{w,n}\circ\phi_o^{\mathcal{G}})$ is thus equal to $(-1)^{\dim z(\mathcal{L})}\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f_A).\eta_0^{\mathcal{G}}$, and so it is equal to $(-1)^{\dim z(\mathcal{L})}\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big((f_A).\eta_o^{\mathcal{L}}\big) = \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f_K)$ by 3.2.16. Taking the characteristic functions in 6.2.11, we thus get that

$$\mathcal{F}^{\mathcal{G}}\big(\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f_A)\big) = e(w)\gamma_o\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(f_K\big).$$

On the other hand, from the identity $f_K = \gamma^{-1}\mathcal{F}^{\mathcal{L}}\big(f_A\big)$ we deduce that

$$\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(f_K\big) = \gamma^{-1}\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathcal{F}^{\mathcal{L}}\big(f_A\big)\big).$$

Hence

$$\mathcal{F}^{\mathcal{G}}\big(\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f_A)\big) = e(w)\gamma_o\gamma^{-1}\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}\big(\mathcal{F}^{\mathcal{L}}\big(f_A\big)\big).$$

□

Remark 6.2.12. With the notation of 6.2.8, put $A = K(\Sigma,\mathcal{E})$, $K = K(C,\zeta)$, and $f_A = \mathbf{X}_{A,1\boxtimes\phi}$, $f_K = \mathbf{X}_{K,\phi}$. Since A is homogeneous with respect to homotheties, there exists $\nu\in\{1,-1\}$ such that $(f_A)^- = \nu f_A$ where for $f\in\mathcal{C}(\mathcal{L}^F)$, $f^-(x) := f(-x)$. Then we have

$$\mathcal{F}^{\mathcal{L}}\big(f_K\big) = \nu\gamma^{-1}f_A. \qquad (*)$$

Now if we identify $\mathcal{L}$ with $z(\mathcal{L})\oplus\overline{\mathcal{L}}$ and if $f\in\mathcal{C}(\mathcal{L}^F)$ is such that for any $z\in z(\mathcal{L})^F$ and $x\in\overline{\mathcal{L}}^F$, $f(z+x) = f_1(z)f_2(u)$ with $f_1\in\mathcal{C}(z(\mathcal{L})^F)$ and $f_2\in\mathcal{C}(\overline{\mathcal{L}}^F)$, then we show (as we did with complexes in 5.2.12) that $\mathcal{F}^{\mathcal{L}}(f)(z+x) = \mathcal{F}^{z(\mathcal{L})}f_1(z)\mathcal{F}^{\overline{\mathcal{L}}}f_2(x)$ for any $z\in z(\mathcal{L})^F$ and $x\in\overline{\mathcal{L}}^F$. As a consequence, using the decomposition $f_A = (-1)^{\dim z(\mathcal{L})}(\mathrm{Id}_{z(\mathcal{L})^F}\times f_K)\in\mathcal{C}(z(\mathcal{L})^F\times\overline{\mathcal{L}}^F)$ where f_K

is regarded as a function over $\overline{\mathcal{L}}^F$, we see that $\mathcal{F}^{\mathcal{L}} \circ \mathcal{F}^{\mathcal{L}}(f_A) = q^{-\dim z(\mathcal{L})}\gamma^2 f_A$ i.e. that $\gamma^2 = \nu q^{\dim z(\mathcal{L})}$. As a consequence from (*), we get that

$$\mathcal{F}^{\mathcal{L}}(f_K) = \gamma q^{-\dim z(\mathcal{L})} f_A. \tag{1}$$

Definition 6.2.13. *With the notation of 6.2.8, the constant γ is called the* Lusztig constant *attached to the cuspidal datum $(L, \Sigma, \mathcal{E})$ with respect to the Frobenius F. Put $\tilde{\gamma} = \eta_L \sigma_L \gamma$ with $\eta_L = (-1)^{semi\text{-}simple\ \mathbb{F}_q-rank(L)}$ and $\sigma_L = (-1)^{rk_{ss}(L)}$. The constant $\tilde{\gamma}$ is called the* modified Lusztig constant *attached to $(L, \Sigma, \mathcal{E})$ with respect to F.*

Remark 6.2.14. Let $(L, \Sigma, \mathcal{E})$ be an F-stable cuspidal datum of $\mathcal{G}$ such that L is G-split and $\mathcal{E}$ is of the form $\overline{\mathbb{Q}}_\ell \boxtimes \zeta$ on $\Sigma = z(\mathcal{L}) + C$. Denote by γ the Lusztig constant attached to $(L, \Sigma, \mathcal{E})$ with respect to F, and for $w \in W_G(L)$, let γ^w be the Lusztig constant attached to $(L_w, \Sigma_w, \mathcal{E}_w)$ with respect to F. Then the equality $\gamma = \epsilon_G \epsilon_{L_w} e(w)\gamma^w$ is equivalent to $\tilde{\gamma} = \tilde{\gamma}^w$. Now saying that $\tilde{\gamma} = \tilde{\gamma}^w$ for any $w \in W_G(L)$ is equivalent of saying that the modified Lusztig constant attached to $(L, \Sigma, \mathcal{E})$ does not depend on the Frobenius wF on L for any $w \in W_G(L)$.

Remark 6.2.15. Let T be an F-stable maximal torus of G with Lie algebra $\mathcal{T}$. Note that the Lusztig constant attached to $(T, \mathcal{T}, \overline{\mathbb{Q}}_\ell)$ is equal to $(-1)^{rk(G)} q^{\frac{rk(G)}{2}}$ and so does not depend on the $\mathbb{F}_q$-structure on G for which the induced Frobenius endomorphism stabilizes T. As a consequence from 6.2.14, 6.2.9 and 6.2.3 we get that:

$$\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}^{\mathcal{G}}_{\mathcal{T}} = \epsilon_G \epsilon_T \mathcal{R}^{\mathcal{G}}_{\mathcal{T}} \circ \mathcal{F}^{\mathcal{T}}.$$

Moreover, a theorem of T.Shoji [Sho95] says that [Lus90, 1.14] holds without restriction on q if the Levi subgroup considered is a maximal torus. Hence this commutation formula holds without restriction on q.

Theorem 6.2.16. *The following assertions are equivalent.*

(i) The conjecture 3.2.30 is true.

(ii) For any F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{G}$ as in 6.2.14, the modified Lusztig constant attached to $(L, \Sigma, \mathcal{E})$ does not depend on wF for any $w \in W_G(L)$.

Proof: Follows from 6.2.6, 6.2.9 and 6.2.14. □

If G is either $GL_n(k)$ or a simple group of type E_8, F_4 or G_2 (in which case p is acceptable for G if and only if it is good), then the only proper Levi subgroups of G which support a cuspidal pair are the maximal tori, hence by 6.2.16 and 6.2.15 we have:

Corollary 6.2.17. *Assume that G is either $GL_n(k)$ or a simple group of type E_8, F_4 or G_2, and that p is good for G, then the conjecture 3.2.30 is true for any q.*

6.2.18. We can easily reduces 6.2.16(ii) to the case where G is simple. Then using the classification of cuspidal data of simple algebraic groups [Lus84], we see that to prove 6.2.16(ii) it is enough to prove that if G is either semi-simple of type A_n, or simple of type either B_n, C_n or D_n, and if $\mathcal{G}$ supports an F-stable cuspidal pair (C, ζ), then

(*) the modified Lusztig constant attached to (C, ζ) does not depend on the $\mathbb{F}_q$-structure on G for which the induced Frobenius endomorphism stabilizes (C, ζ).

Note that if G is simple of type either B_n or C_n, then the statement (*) holds since in that case the pair (G, F) is unique up to isomorphism. If $p > 3(h_o^G - 1)$ and if G is semi-simple of type A_n, then from the explicit computation of the Lusztig constant [DLM97], the assertion (*) follows [DLM03, 6.12]. If $p > 3(h_o^G - 1)$ and if G is either the simple adjoint group of type D_n or $SO_{2n}(k)$, then the assertion (*) follows from the explicit computation of the Lusztig constant [Wal01, V.8]. Actually, in [Wal01, V.8], the adjoint case $G = G_{ad}$ of type D_n is not explicitely mentioned, but using the canonical central isogeny $f : SO_{2n}(k) \to G_{ad}$, we see that if (C, ζ) is an F-stable cuspidal pair of G_{ad}, then the Lusztig constant attached to (G_{ad}, C, ζ) is equal to that attached to $\big(SO_{2n}(k), f^{-1}(C), f^*(\zeta)\big)$. The remaining case is the case where G is simple simply connected of type D_n and $SO_{2n}(k)$ does not admit a cuspidal pair. As far as I know, this case is still unknown.

Theorem 6.2.19. *Assume that $p > 3(h_o^G - 1)$ and that every simple component of G/Z_G^o of type D_n is either $SO_{2n}(k)$ or the adjoint group of type D_n, then the conjecture 3.2.30 holds.*

Proof: Follows from 6.2.16 and 6.2.18. □

6.2.20 Lusztig Constants: A Formula

In this section, we give a formula for the Lusztig constant attached to an F-stable cuspidal pair of the Lie algebra of a simple algebraic group. Such a preliminary formula has been obtained by Digne-Lehrer-Michel [DLM97] when the nilpotent orbit supporting the cuspidal pair is regular, by Waldspurger [Wal01] for the classical groups $Sp_{2n}(k)$, $SO_n(k)$, and by Kawanaka [Kaw86] for the exceptional groups E_8, F_4 and G_2. Although it has been used by the previously named authors to compute the Lusztig constants, this formula is not explicit enough to verify the required property on Lusztig constants (see 6.2.18(*)).

We assume that $p > 3(h_o^G - 1)$, that G is simple, and that $\mathcal{G}$ admits an F-stable cuspidal pair (C, ζ). We denote by γ^F the Lusztig constant attached to (C, ζ) with respect to F.

6.2.21. We fix an element $u_o \in C^F$. Under our assumption, we can use Dynkin-Kostant-Springer-Steinberg's theory on nilpotent orbits on $\mathcal{G}$. Hence there exists an F-stable $\mathbb{Z}$-grading $\mathcal{G} = \bigoplus_i \mathcal{G}(i)$ of $\mathcal{G}$ i.e. $F(\mathcal{G}(i)) = \mathcal{G}(i)$ and $[\mathcal{G}(i), \mathcal{G}(j)] \subset \mathcal{G}(i+j)$, with the following properties (i)-(vii).

(i) $u_o \in \mathcal{G}(2)$.

(ii) $\mathcal{P} = \oplus_{i \geq 0} \mathcal{G}(i)$ is the Lie algebra of an F-stable parabolic subgroup P of G and $\mathcal{L} = \mathcal{G}(0)$ is the Lie algebra of an F-stable Levi subgroup L of P.

(iii) $\mathcal{G}(2)$ is stable under the adjoint action of L and $\mathcal{O}_{u_o}^L$ is dense in $\mathcal{G}(2)$.

(iv) $\mathcal{U}_P = \oplus_{i>0} \mathcal{G}(i)$.

(v) The group $C_{U_P}(u_o)$ is unipotent and connected, and the group $C_G(u_o)$ is the semi-direct product of $C_L(u_o)$ and $C_{U_P}(u_o)$ as an algebraic group.

(vi) We have $\mathcal{O}_{u_o}^G \cap \big(\oplus_{i \geq 2} \mathcal{G}(i) \big) = \mathcal{O}_{u_o}^P$.

(vii) The pair (C, ζ) being cuspidal, by [Lus84, 2.8] the element u_o is *distinguished* i.e. the map $\mathrm{ad}(u_o) : \mathcal{G}(0) \to \mathcal{G}(2)$ is bijective. Hence we have $\mathcal{G}(i) = \{0\}$ if i is odd i.e. $\mathcal{U}_P = \bigoplus_{i \geq 2} \mathcal{G}(i)$, and from (iii) we deduce that $C_L^o(u_o) = \{0\}$.

6.2.22. We now define the generalized Gelfand-Graev functions following [Kaw85]. Let $H^1(F, A_G(u_o))$ be the group of F-conjugacy classes of $A_G(u_o)$. By setting that $1 \in H^1(F, A_G(u_o))$ corresponds to the G^F-orbit of u_o, we have a well-defined parametrization of the G^F-orbits in C^F by $H^1(F, A_G(u_o))$ (see 2.1.20). From 6.2.21(v), we have $A_G(u_o) \simeq A_L(u_o)$, hence for $z \in H^1(F, A_G(u_o)) \simeq H^1(F, A_L(u_o))$, we can choose an element $u_z \in \mathcal{G}(2)^F$ which is in the G^F-orbit of C^F corresponding to z. Let $\mathcal{U}_P^- = \bigoplus_{i \leq -2} \mathcal{G}(i)$, then for each $z \in H^1(F, A_L(u_o))$, we define a linear additive character $\Psi_z : (\mathcal{U}_P^-)^F \to \overline{\mathbb{Q}}_\ell$ by $\Psi_z(u) = \Psi\big(\mu(u_z, u)\big)$. The corresponding generalized Gelfand-Graev function $\Gamma_z : \mathcal{G}^F \to \overline{\mathbb{Q}}_\ell$ is defined by

$$\Gamma_z(x) = |U_P^F|^{-1} \sum_{\{g \in G^F | \mathrm{Ad}(g)x \in \mathcal{U}_P^-\}} \Psi_z\big(\mathrm{Ad}(g)x\big).$$

The G-equivariant irreducible local system ζ corresponds to a unique F-stable irreducible character (denoted again by ζ) of $A_G(u_o)$ which can be extended to a character of the semi-direct product $A_G(u_o) \rtimes \langle F \rangle$ where $\langle F \rangle$ is the cyclic group generated by the Frobenius F. The restriction to $A_G(u_o).F$ of this extended character is constant on the $A_G(u_o)$-orbits and so leads to a unique function $\tilde{\zeta}$ on $H^1(F, A_G(u_o)) \simeq H^1(F, A_L(u_o))$. We then define a nilpotently supported function $\Gamma_\zeta : \mathcal{G}^F \to \overline{\mathbb{Q}}_\ell$ by

$$\Gamma_\zeta = \sum_{z \in H^1(F, A_L(u_o))} |z| \tilde{\zeta}(z) \Gamma_z.$$

6.2.23. By [Lus92, 7.6], the function Γ_ζ is proportional to the characteristic function of the F-equivariant perverse sheaf $\big(K(C,\zeta), \phi\big)$ for any isomorphism $\phi : F^*(K(C,\zeta)) \xrightarrow{\sim} K(C,\zeta)$. As a consequence we get that

$$\mathcal{F}^{\mathcal{G}}(\Gamma_\zeta) = \gamma^F \Gamma_\zeta.$$

From the classification of the distinguished parabolic subgroups of G, we can verify that the longest element w_o of $W_G(T)$ (with T a maximal torus of L) normalizes L and $\mathrm{Ad}(w_o)$ maps $\mathcal{G}(2)$ onto $\mathcal{G}(-2)$. As a consequence $\mathcal{O}^G_{-u_o} \cap \mathcal{G}(-2) \neq \emptyset$ and any element of $\mathcal{O}^G_{-u_o} \cap \mathcal{G}(-2)$ is distinguished with associated parabolic subgroup $P^- = LU_P^-$. Let $u_o^* \in \mathcal{O}^{G^F}_{-u_o} \cap \mathcal{G}(-2)^F$. From [Lus92, 6.13] we have $\mathcal{F}^{\mathcal{G}}(\Gamma_\zeta)(u_o^*) = \tilde{\zeta}(1)|1||C_G(u_o)^F| q^{-\frac{\dim C_{\mathcal{G}}(u_o)}{2}}$ where by definition $|1| = \sharp\{x^{-1}F(x) | x \in A_L(u_o)\}$. Hence by 6.2.21(v), we deduce that:

$$\gamma^F = \frac{\tilde{\zeta}(1)|1||C_L(u_o)^F|}{\Gamma_\zeta(u_o^*)} q^{-\frac{\dim C_{\mathcal{L}}(u_o)}{2}} q^{\frac{\dim C_{\mathcal{U}_P}(u_o)}{2}}.$$

Hence the computation of γ^F reduces to that of $\Gamma_\zeta(u_o^*)$. For any $z \in H^1\big(F, A_L(u_o)\big)$ we have

$$\Gamma_z(u_o^*) = |U_P^F|^{-1} \sum_{g \in (P^-)^F} \Psi_z\big(\mathrm{Ad}(g)u_o^*\big) = \sum_{g \in L^F} \Psi_z\big(\mathrm{Ad}(g)u_o^*\big).$$

These equalities come from 6.2.21(vi), 6.2.21(iii) where (u_o, P) is replaced by (u_o^*, P^-), and the fact that the restriction of Ψ_z to $\bigoplus_{i<-2} \mathcal{G}(i)$ is trivial. We thus get that

$$\Gamma_\zeta(u_o^*) = \sum_{z \in H^1(F, A_L(u_o))} |z| \tilde{\zeta}(z) \sum_{l \in L^F} \Psi_z\big(\mathrm{Ad}(l)u_o^*\big).$$

Let $\mathcal{L}_L : L \to L$, $t \mapsto t^{-1}F(t)$ be the Lang map. Then we have a surjective map

$$\overline{\mathcal{L}} : \mathcal{L}_L^{-1}(C_L(u_o))/C_L(u_o) \to H^1(F, C_L(u_o)) \simeq H^1(F, A_L(u_o))$$

which maps $tC_L(u_o)$ onto the F-conjugacy class of $t^{-1}F(t)$.

For $z \in H^1(F, A_L(u_o))$, let $l_z \in L$ be such that $l_z^{-1}F(l_z) = \dot{z}$ where $\dot{z} \in C_L(u_o)$ is a representative of z, and $u_z = \mathrm{Ad}(l_z)u_o$. Then we have a well-defined map $\phi_z : L^F \to \overline{\mathcal{L}}^{-1}(z)$ given by $t \mapsto tl_zC_L(u_o)$. This map is clearly surjective and its fibers are all of cardinality

$$a_z = \sharp\{h \in C_L(u_o) | h^{-1}\dot{z}F(h) = \dot{z}\}.$$

For $g \in \mathcal{L}_L^{-1}(C_L(u_o))$ and $x \in (\mathcal{U}_P^-)^F$, define ${}^g\Psi_o(x) := \Psi\big(\mu(\mathrm{Ad}(g)u_o, x)\big) = \Psi\big(\mu(u_o, \mathrm{Ad}(g^{-1}(x))\big) = \Psi_o\big(\mathrm{Ad}(g^{-1})x\big)$. We thus have:

$$\sum_{t\in L^F} \Psi_z\big(\mathrm{Ad}(t)u_o^*\big) = \sum_{t\in L^F} {}^{tl_z}\Psi_o(u_o^*) = a_z \sum_{l\in\overline{\mathcal{L}}^{-1}(z)} {}^l\Psi_o(u_o^*).$$

We finally deduce that:

$$\Gamma_\zeta(u_o^*) = |C_L(u_o)| \sum_{l\in\mathcal{L}_L^{-1}(C_L(u_o))/C_L(u_o)} \tilde{\zeta}\big(\overline{\mathcal{L}}(l)\big){}^l\Psi_o(u_o^*).$$

Indeed we have $a_z|z| = |C_L(u_o)|$ since by 6.2.21(vii), we have $A_L(u_o) = C_L(u_o)$. Note that $\mathcal{L}_L^{-1}(C_L(u_o))/C_L(u_o) = \big(L/C_L(u_o)\big)^F$. We define the quantity

$$\sigma_\zeta := \tilde{\zeta}(1)^{-1} \sum_{l\in\left(L/C_L(u_o)\right)^F} \tilde{\zeta}\big(\overline{\mathcal{L}}(l)\big)\Psi_o^*(\mathrm{Ad}(l)u_o)$$

where Ψ_o^* is the additive character of $\mathcal{G}(2)^F$ defined by $\Psi_o^*(v) = \Psi\big(\mu(u_o^*, v)\big)$. Note that σ_ζ does not depend on the choice of the extension of ζ on $A_G(u_o) \rtimes \langle F\rangle$. Since $|1||C_L(u_o)^F| = |C_L(u_o)|$, we thus have

6.2.24.

$$\gamma^F = \sigma_\zeta^{-1} q^{\frac{d}{2}}$$

where $d = \dim C_{\mathcal{U}_P}(u_o) - \dim C_{\mathcal{L}}(u_o)$.

Remark 6.2.25. From the formula 6.2.24, we see that γ^F is a "generalized character sum" [KP00] associated to the regular prehomogeneous vector space $\big(L, \mathrm{Ad}, \mathcal{G}(2)\big)$.

7

Fourier Transforms of the Characteristic Functions of the Adjoint Orbits

The goal of this chapter is to give a formula which reduces the computation of the values of the Fourier transforms of the characteristic functions of the adjoint orbits of $\mathcal{G}^F$ to the computation of the values of the generalized Green functions and the computation of the Lusztig constants attached to the F-stable cuspidal data of $\mathcal{G}$. As in the previous chapter, we assume, unless specified, that p is acceptable for G and that q is large enough.

7.1 Preliminaries

7.1.1 A Decomposition of $\mathcal{C}(\mathcal{G}^F)$

In this subsection, we give a decomposition of $\mathcal{C}(\mathcal{G}^F)$ which is "conserved" by Fourier transforms and by Deligne-Lusztig induction.

7.1.2. We denote by $J(\mathcal{G})$ the set of G-conjugacy classes (see 5.1.23) of triples of the form (L, C, ζ) with L a Levi subgroup of G and (C, ζ) a cuspidal nilpotent pair on $\mathcal{L}$. Note that the Frobenius map F induces a map $J(\mathcal{G}) \to J(\mathcal{G})$ such that if (L, C, ζ) is a representative of $\mathcal{O} \in J(\mathcal{G})$, then the image of $\mathcal{O}$ is the G-conjugacy class of $(F^{-1}(L), f^{-1}(C), F^*(\zeta))$. We denote by $J(\mathcal{G})^F$ the subset of $J(\mathcal{G})$ of F-stable elements of $J(\mathcal{G})$. Recall (see 5.4.8) that if $\mathcal{O} \in J(\mathcal{G})^F$, it is possible to choose an F-stable representative of $\mathcal{O}$. We also use the notation of 4.4.13 with $X = \mathcal{G}$ and $H = G$, and we put $I(\mathcal{G}) := I$. In particular, for each $\iota = (\mathcal{O}_\iota, \mathcal{E}_\iota) \in I(\mathcal{G})^F$, we have fixed an isomorphism $\phi_\iota : F^*(\mathcal{E}_\iota) \xrightarrow{\sim} \mathcal{E}_\iota$, and we have denoted by $\mathcal{Y}_\iota$ the characteristic function of $(\mathcal{E}_\iota, \phi_\iota)$ extended by zero on $\mathcal{G}^F - \mathcal{O}_\iota^F$ and by $\mathcal{X}_\iota$ the characteristic function of the F-equivariant perverse sheaf $\big(K(\mathcal{O}_\iota, \mathcal{E}_\iota), \phi_\iota\big)$. By 5.1.81, we have a well-defined surjective map $I(\mathcal{G}) \to J(\mathcal{G})$ defined as follows. Let $(\mathcal{O}, \mathcal{E}) \in I(\mathcal{G})$ and

E. Letellier: LNM 1859, pp. 151–158, 2005.

let $(L, \mathcal{O}^L, \mathcal{E}^L)$ be as in 5.1.81. Then to $(\mathcal{O}, \mathcal{E})$ we associate the G-conjugacy class of (L, C, ζ) where (C, ζ) is the unique cuspidal nilpotent pair of $\mathcal{L}$ such that $\mathcal{O}^L = z + C$ with $z \in z(\mathcal{L})$ and $\mathcal{E}^L = \overline{\mathbb{Q}}_\ell \boxtimes \zeta$. This map restricts to a map $h : I(\mathcal{G})^F \to J(\mathcal{G})^F$. Then, for $j \in J(\mathcal{G})^F$, we denote by $\mathcal{C}(\mathcal{G}^F)_j$ the subspace of $\mathcal{C}(\mathcal{G}^F)$ generated by the functions $\{\mathcal{X}_\iota | \, \iota \in I(\mathcal{G})^F, h(\iota) = j\}$. By 4.4.13 we have a decomposition

$$\mathcal{C}(\mathcal{G}^F) = \bigoplus_{j \in J(\mathcal{G})^F} \mathcal{C}(\mathcal{G}^F)_j. \tag{1}$$

Denote by $I_n(\mathcal{G})$ the subset of $I(\mathcal{G})$ consisting of nilpotent orbital pairs of $\mathcal{G}$, and, for $j \in J(\mathcal{G})^F$, denote by $\left(\mathcal{C}(\mathcal{G}^F)_{nil}\right)_j$, the subspace of $\mathcal{C}(\mathcal{G}^F)_{nil}$ generated by $\{\mathcal{X}_\iota | \, \iota \in I_n(\mathcal{G})^F, h(\iota) = j\}$. Then we also have a decomposition

$$\mathcal{C}(\mathcal{G}^F)_{nil} = \bigoplus_{j \in J(\mathcal{G})^F} \left(\mathcal{C}(\mathcal{G}^F)_{nil}\right)_j.$$

7.1.3. Let L be an F-stable Levi subgroup G with $\mathcal{L} := \text{Lie}(L)$. From the classification of cuspidal data [Lus84], the natural map $J(\mathcal{L}) \hookrightarrow J(\mathcal{G})$ is injective (see also [DLM97, 1.2]) and so we may identify $J(\mathcal{L})$ with a subset of $J(\mathcal{G})$. By 6.1.56(ii), 6.1.55 and 6.1.3, for any $j \in J(\mathcal{L})^F$ and any $f \in \mathcal{C}(\mathcal{L}^F)_j$, we have $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}(f) \in \mathcal{C}(\mathcal{G}^F)_j$.

7.1.4. We want to see that the Fourier transforms conserve the decomposition 7.1.2(1). Firstable, if $j = (G, C, \zeta) \in J(\mathcal{G})^F$, then it is clear that $\mathcal{F}^{\mathcal{G}}$ leaves stable the subspace $\mathcal{C}(\mathcal{G}^F)_j$, i.e. the characteristic functions of the F-equivariant cuspidal admissible complexes on $\mathcal{G}$ span the spaces $\mathcal{C}(\mathcal{G}^F)_j$ with $j \in J(\mathcal{G})^F$. Hence by 7.1.3, for each $j \in J(\mathcal{G})^F$, there is a basis of $\mathcal{C}(\mathcal{G}^F)_j$ formed by the characteristic functions of some F-equivariant Lusztig complexes on $\mathcal{G}$ (recall that the characteristic functions of the F-stable Lusztig complexes span $\mathcal{C}(\mathcal{G}^F)$, see 5.2.22 and 5.4.4). Looking now at the Fourier transforms of these bases, we see from 6.2.3, that $\mathcal{F}^{\mathcal{G}}$ leaves the subspaces $\mathcal{C}(\mathcal{G}^F)_j$, with $j \in J(\mathcal{G})^F$, stable.

Proposition 7.1.5. *Let M be an F-stable Levi subgroup of G. Let $j \in J(\mathcal{M})^F$ and let (L, C, ζ) be an F-stable representative of j. Assume that the modified Lusztig constant attached to $(L, \Sigma, \mathcal{E}) = (L, z(\mathcal{L}) + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ does not depend on the Frobenius wF with $w \in W_G(L)$. Then*

$$\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}^{\mathcal{G}}_{\mathcal{M}}(f) = \epsilon_G \epsilon_M \mathcal{R}^{\mathcal{G}}_{\mathcal{M}} \circ \mathcal{F}^{\mathcal{M}}(f)$$

for any $f \in \mathcal{C}(\mathcal{M}^F)_j$.

Proof: By 7.1.4, the subspaces $\mathcal{C}(\mathcal{M}^F)_j$, with $j \in J(\mathcal{M})^F$, have a basis formed by the characteristic functions of F-equivariant admissible complexes on $\mathcal{M}$. Hence it is enough to verify the commutation formula when f is the characteristic function of an F-equivariant admissible complex on $\mathcal{M}$. Hence the proof is similar to that of 6.2.16. □

7.1.6 A Geometric Analogue of 3.2.24

We use the notation of 7.1.1 which includes the notation of 4.4.13. We assume now that $G = GL_n(k)$ or that p is very good for G (note that if G is a simple group, then p is acceptable for G if and only if it is very good for G). Let $\iota = (\mathcal{O}_\iota, \mathcal{E}_\iota) \in I(\mathcal{G})^F$ and let $x \in \mathcal{O}_\iota^F$. We denote by M the Levi subgroup $C_G(x_s)$. Note that $C_G(x_s)$ is connected by 2.6.18, hence the local system $\mathcal{E}_\iota|_{\mathcal{O}_x^M}$ is irreducible by 5.1.39; we denote by $(\mathcal{O}_\iota^M, \mathcal{E}_\iota^M)$ the orbital pair $\left(\mathcal{O}_x^M, \mathcal{E}_\iota|_{\mathcal{O}_x^M}\right) \in I(\mathcal{M})^F$. Let $\phi_\iota^M : F^*(\mathcal{E}_\iota^M) \simeq \mathcal{E}_\iota^M$ be the restriction of $\phi_\iota : F^*(\mathcal{E}_\iota) \simeq \mathcal{E}_\iota$, and let $\mathcal{Y}_\iota^M$ be the characteristic function of the F-equivariant sheaf $(\mathcal{E}_\iota^M, \phi_\iota^M)$ extended by zero on $\mathcal{M}^F - \mathcal{O}_\iota^M$. We also denote by $\mathcal{X}_\iota^M$, the characteristic function of the F-equivariant perverse sheaf $\left(K(\mathcal{O}_\iota^M, \mathcal{E}_\iota^M), \phi_\iota^M\right)$.

Lemma 7.1.7. *We have* $\overline{\mathcal{O}_\iota} = \bigcup_{g\in G} Ad(g)\left(\overline{\mathcal{O}_\iota^M}\right)$.

Proof: Let Q be a parabolic subgroup of G having M as a Levi subgroup. By replacing (P, L, Σ) by $(Q, M, \mathcal{O}_\iota^M)$ in 5.1.26, and by applying 5.1.30(ii) we get that

$$\bigcup_{g\in G} \mathrm{Ad}(g)\left(\overline{\mathcal{O}_\iota^M} + \mathcal{U}_Q\right) = \overline{\mathcal{O}_\iota}.$$

But since x_s is M-regular in $\mathcal{G}$, we get from 2.6.6 that

$$\bigcup_{g\in G} \mathrm{Ad}(g)\left(\overline{\mathcal{O}_\iota^M} + \mathcal{U}_Q\right) = \bigcup_{g\in G} \mathrm{Ad}(g)\left(\overline{\mathcal{O}_\iota^M}\right).$$

□

Proposition 7.1.8. *We have* $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathcal{X}_\iota^M) = \mathcal{X}_\iota$ *and* $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathcal{Y}_\iota^M) = \mathcal{Y}_\iota$.

Proof: Write $\mathcal{Y}_\iota^M = \sum_{\mathcal{O}\in Orb(\mathcal{M}^F)} \lambda_{\mathcal{O}} \xi_{x_{\mathcal{O}}}^M$ where $Orb(\mathcal{M}^F)$ denotes the set of M^F-orbits of $\mathcal{M}^F$ and where for $\mathcal{O} \in Orb(\mathcal{M}^F)$, the symbol $x_{\mathcal{O}}$ denotes an element of $\mathcal{O}$. From 3.2.24 and the fact that $C_G(x_s) = C_M(x_s)$, we have $\mathcal{R}_{\mathcal{M}}^{\mathcal{G}}(\mathcal{Y}_\iota^M) = \sum_{\mathcal{O}\in Orb(\mathcal{M}^F)} \lambda_{\mathcal{O}} \xi_{x_{\mathcal{O}}}^G$; note that $\lambda_{\mathcal{O}} \neq 0$ only if $\mathcal{O} \subset \mathcal{O}_\iota^M$. Since $\mathcal{O}_\iota^M$ is of the form $x_s + C_\iota^M$ for some nilpotent orbit C_ι^M of $\mathcal{M}$, we get that the semi-simple part of $x_{\mathcal{O}}$, for $\mathcal{O}$ such that $\lambda_{\mathcal{O}} \neq 0$, is equal to x_s. As a consequence, the functions $\xi^G_{x_\mathcal{O}} \in \mathcal{C}(\mathcal{G}^F)$ such that $\lambda_\mathcal{O} \neq 0$ are linearly independent. Since the function $\mathcal{Y}^M_\iota$ and $\mathcal{Y}_\iota$ coincide over $(\mathcal{O}^M_\iota)^F$, we deduce that both $\mathcal{Y}_\iota$ and $\mathcal{R}^\mathcal{G}_\mathcal{M}(\mathcal{Y}^M_\iota)$ take the value $\lambda_\mathcal{O}$ at $x_\mathcal{O}$ for any $\mathcal{O} \subset (\mathcal{O}^M_\iota)^F$. Now let $y \in \mathcal{O}^F_\iota$; its semi-simple part is then G-conjugate to x_s. Since $C_G(x_s)$ is connected, it follows that y_s and x_s are actually G^F-conjugate. Hence we get that any G^F-orbit of $\mathcal{G}^F$ supporting $\mathcal{Y}_\iota$ is of the form $\mathcal{O}^{G^F}_{x_\mathcal{O}}$ for some $\mathcal{O} \in Orb(\mathcal{M}^F)$ such that $\mathcal{O} \subset \mathcal{O}^M_\iota$. We thus proved (i).

The proof of (ii) is similar to that of (i) as long as we can see that

(a) the functions $\mathcal{X}_\iota$ and $\mathcal{X}^M_\iota$ coincide over $\overline{\mathcal{O}^M_\iota}^F$,

(b) any rational element of $\overline{\mathcal{O}_\iota}^F$ is G^F-conjugate to an element of $\overline{\mathcal{O}^M_\iota}^F$.

From 4.3.6, the restriction of the complex $K(\mathcal{O}_\iota, \mathcal{E}_\iota)$ to $\overline{\mathcal{O}^M_\iota}$ is nothing but $\mathrm{IC}(\overline{\mathcal{O}^M_\iota}, \mathcal{E}^M_\iota)[\dim \mathcal{O}_\iota]$. We thus get that $\mathcal{X}_\iota(x) = (-1)^{\dim \mathcal{O}_\iota - \dim \mathcal{O}^M_\iota} \mathcal{X}^M_\iota(x)$ for any $x \in \overline{\mathcal{O}^M_\iota}^F$. Hence (a) follows from the fact that the integer $\dim \mathcal{O}_\iota - \dim \mathcal{O}^M_\iota$ is even. From 7.1.7, any element of $\overline{\mathcal{O}_\iota}$ is G-conjugate to an element of $\overline{\mathcal{O}^M_\iota}$. Since centralizers in G of semi-simple elements of $\mathcal{G}$ are connected, any element of $\overline{\mathcal{O}_\iota}^F$ is thus G^F-conjugate to an element of $\overline{\mathcal{O}^M_\iota}^F$. □

Proposition 7.1.9. *For $\mu, \mu' \in I(\mathcal{G})^F$, let $a_{\mu,\mu'} \in \overline{\mathbb{Q}}_\ell$ be such that $\mathcal{X}_\mu = \sum_{\mu' \in I(\mathcal{G})^F} a_{\mu,\mu'} \mathcal{Y}_{\mu'}$. Then $a_{\mu,\mu'} \neq 0$ only if μ and μ' have the same image by $h : I(\mathcal{G})^F \to J(\mathcal{G})^F$, i.e. for any $j \in J(\mathcal{G})^F$, the set $\{\mathcal{Y}_\mu | \, h(\mu) = j\}$ forms a basis of $\mathcal{C}(\mathcal{G}^F)_j$.*

Proof: Using 7.1.8, we see that the proof reduces to the case where μ is a nilpotent pair. But via a G-equivariant isomorphism $G_{uni} \to \mathcal{G}_{nil}$, the nilpotent case follows from its group version [Lus86a, 24.4(d)]. □

Remark 7.1.10. Note that thanks to 7.1.8, the computation of the coefficients $a_{\mu,\mu'}$ in 7.1.9 reduces to the nilpotent case.

7.2 Fourier Transforms of the Characteristic Functions of the Adjoint Orbits

In this section we keep the assumptions and the notation of 7.1.6. We also assume that $\iota \in I(\mathcal{G})^F$ is such that the modified Lusztig constant attached to $(L, z(\mathcal{L}) + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$, with $(L, C, \zeta) \in h(i)^F$, does not depend on the Frobenius wF with $w \in W_G(L)$. We first give a formula for $\mathcal{F}^\mathcal{G}(\mathcal{X}_\iota)$.

By 7.1.5 and 7.1.8, we have

$$\mathcal{F}^{\mathcal{G}}(\mathcal{X}_\iota) = \epsilon_G \epsilon_M \mathcal{R}^{\mathcal{G}}_{\mathcal{M}} \circ \mathcal{F}^{\mathcal{M}}(\mathcal{X}^{\mathcal{M}}_\iota). \tag{1}$$

Note that $\mathcal{O}^M_\iota = x_s + \mathcal{O}^M_{\iota,n}$ and $\mathcal{E}^M_\iota = \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{E}^M_{\iota,n}$ for some nilpotent pair $(\mathcal{O}^M_{\iota,n}, \mathcal{E}^M_{\iota,n})$ of $\mathcal{M}$. Let f_{x_s} be the $\mathbb{F}_q$-linear form $\mathcal{M}^F \to \mathbb{F}_q$ given by $z \mapsto \mu(z, x_s)$. Then there is an isomorphism $\phi^M_{\iota,n} : F^*(\mathcal{E}^M_{\iota,n}) \simeq \mathcal{E}^M_{\iota,n}$ such that $\mathcal{F}^{\mathcal{M}}(\mathcal{X}^M_\iota) = (\Psi \circ f_{x_s}).\mathcal{F}^{\mathcal{M}}(\mathcal{X}^{\mathcal{M}}_{\iota,n})$ where $\mathcal{X}^{\mathcal{M}}_{\iota,n}$ is the characteristic function of $\big(K(\mathcal{O}^M_{\iota,n}, \mathcal{E}^M_{\iota,n}), \phi^M_{\iota,n}\big)$. We assume that L is contained in M and that it is M-split. We choose an isomorphism $\phi : F^*(\zeta) \xrightarrow{\sim} \zeta$. For any $w \in W_M(L)$, we denote by $(L_w, C_w, \zeta_w, \phi_w)$ the datum obtained from (L, C, ζ, ϕ) as in 5.4.2. As usual, we denote by $\mathcal{L}_w$ the Lie algebra of L_w. Then taking the restriction to nilpotent elements of a formula like 5.4.12, we may write

$$\mathcal{X}^M_{\iota,n} = (-1)^{\dim z(\mathcal{L})} |W_M(L)|^{-1} \sum_{w \in W_M(L)} \mathrm{Tr}\big((\theta^M_w \circ \sigma^M_\iota)^{-1}, V^M_\iota\big) \mathcal{Q}^{\mathcal{M}}_{\mathcal{L}_w, C_w, \zeta_w, \phi_w}$$

where θ^M_w is chosen as in 5.3.6(iii). Put $\lambda^M_\iota(w) = \mathrm{Tr}\big((\theta^M_w \circ \sigma^M_\iota)^{-1}, V^M_\iota\big)$. For $w \in W_M(L)$, let K_w be the complex $K(C_w, \zeta_w)$ and let $\mathcal{X}_w$ be the characteristic function of the F-equivariant complex (K_w, ϕ_w). We thus have: $\mathcal{R}^{\mathcal{M}}_{\mathcal{L}_w}(\mathcal{X}_w) = (-1)^{\dim z(\mathcal{L})} \mathcal{Q}^{\mathcal{M}}_{\mathcal{L}_w, C_w, \zeta_w, \phi_w}$ since Deligne-Lusztig induction coincides with geometrical induction. Applying 7.1.5, we thus get that

$$\mathcal{F}^{\mathcal{M}}(\mathcal{Q}^{\mathcal{M}}_{\mathcal{L}_w, C_w, \zeta_w, \phi_w}) = \epsilon_M \epsilon_{L_w} (-1)^{\dim z(\mathcal{L})} \mathcal{R}^{\mathcal{M}}_{\mathcal{L}_w}(\mathcal{F}^{\mathcal{L}_w}(\mathcal{X}_w)).$$

Since $x_s \in z(\mathcal{M})$, we have

$$(\Psi \circ f_{x_s}).\mathcal{R}^{\mathcal{M}}_{\mathcal{L}_w}(\mathcal{F}^{\mathcal{L}_w}(\mathcal{X}_w)) = \mathcal{R}^{\mathcal{M}}_{\mathcal{L}_w}\big((\Psi \circ f^{\mathcal{L}_w}_{x_s}).\mathcal{F}^{\mathcal{L}_w}(\mathcal{X}_w)\big)$$

where $f^{\mathcal{L}_w}_{x_s}$ is the restriction of f_{x_s} to $\mathcal{L}^F_w$. From 6.2.12(1), we have

$$(\Psi \circ f^{\mathcal{L}_w}_{x_s}).\mathcal{F}^{\mathcal{L}_w}(\mathcal{X}_w) = \gamma_w q^{-\dim z(\mathcal{L})} \mathbf{X}_{A_{w,x_s}, \phi_{w,x_s}}$$

where $A_{w,x_s} = K\big(z(\mathcal{L}_w) + C_w, m^*_{x_s}(\mathcal{L}_\Psi) \boxtimes \zeta_w\big)$, γ^w is the Lusztig constant, and where $\phi_{w,x_s} = m^*_{x_s}(\phi_{\mathcal{L}_\Psi}) \boxtimes \phi_w$. From (1), we finally deduce that

$$\mathcal{F}^{\mathcal{G}}(\mathcal{X}_\iota) = q^{-\dim z(\mathcal{L})} |W_M(L)|^{-1} \sum_{w \in W_M(L)} \epsilon_G \epsilon_{L_w} \lambda^M_\iota \gamma^w \mathcal{R}^{\mathcal{G}}_{\mathcal{L}_w}(\mathbf{X}_{A_{w,x_s}, \phi_{w,x_s}}). \tag{2}$$

7.2.1. From now, we choose $\phi : F^*(\zeta) \xrightarrow{\sim} \zeta$ and the isomorphisms $F^*(\mathcal{E}_\mu) \xrightarrow{\sim} \mathcal{E}_\mu$, with $\mu \in I_n(\mathcal{M})^F$ and $h(\mu) = j$, as in 5.5.12; in particular we have $\lambda^M_\iota(w) = \tilde{\chi}_\iota(wF)$ where χ_ι is the F-stable irreducible character of $W_M(L)$ corresponding to the pair $(\mathcal{O}^M_{\iota,n}, \mathcal{E}^M_{\iota,n})$ and where $\tilde{\chi}_\iota$ denotes the "preferred extension" of χ_ι to $W_M(L) \rtimes \langle F \rangle$. Note that the choice of $\phi_{\iota,n} : F^*(\mathcal{E}_{\iota,n}) \xrightarrow{\sim} \mathcal{E}_{\iota,n}$ we just made determine uniquely the isomorphism $\phi_\iota : F^*(\mathcal{E}_\iota) \xrightarrow{\sim} \mathcal{E}_\iota$. Now we see from (2) that the explicit computation of the values of $\mathcal{F}^{\mathcal{G}}(\mathcal{X}_\iota)$ reduces to the computation of the values of the generalized Green functions (by 5.5.9) and to the computation of the Lusztig constants γ^w which are known in many cases (see 6.2.20). From 5.5.12, we see that the problem of computing the values of the generalized Green functions is the same both in the Lie algebra case and in the group case. By 5.5.12, this problem reduces to the problem of computing the values of the functions $\mathcal{X}_\mu$ with μ a nilpotent pair. Lusztig has invented an algorithm [Lus86a] which allows the computation of the coefficient $a_{\mu,\mu'}$ of 7.1.9 in the case where μ, μ' are nilpotent pairs. The problem reduces thus to the computation of the values of the functions $\mathcal{Y}_\mu$ with μ a nilpotent pair. However, if $u \in \mathcal{O}_\mu^F$, the values of the function $\mathcal{Y}_\mu$ can be described (up to a root of unity) in terms of the values of the corresponding function on $H^1(F, A(u))$ which is defined in the second paragraph of 6.2.22.

Concerning the computation of the values of the functions $\mathcal{F}^{\mathcal{G}}(\mathcal{Y}_\iota)$, we see by 7.1.9, that it reduces to the computation of the values of the functions $\mathcal{F}^{\mathcal{G}}(\mathcal{X}_\iota)$ (we already outlined) and the computation of the coefficients $a_{\mu,\mu'}$. But by 7.1.10, the computation of the coefficients $a_{\mu,\mu'}$ reduces to the case where μ, μ' are nilpotent pairs which coefficients can be computed by Lusztig's algorithm.

7.2.2. Concerning the computation of the values of the function $\mathcal{F}^{\mathcal{G}}(\xi_x^G)$ for some $x \in \mathcal{G}^F$: we know (up to some roots of unity) the base change matrix between the functions ξ_y^G, $y \in \mathcal{G}^F$ and the functions $\mathcal{Y}_\mu$, $\mu \in I(\mathcal{G})^F$, where the isomorphisms $\phi_\mu : F^*(\mathcal{E}_\mu) \simeq \mathcal{E}_\mu$ are chosen as ϕ_ι in 7.2.1. Hence the computation of the value of the function $\mathcal{F}^{\mathcal{G}}(\xi_x^G)$ reduces to that of $\mathcal{F}^{\mathcal{G}}(\mathcal{Y}_\mu)$ for $\mu \in I(\mathcal{G}^F)$ such that $\mathcal{O}_\mu = \mathcal{O}_x^G$, which computation is outlined above. However, since in general the functions ξ_x^G do not belongs to a $\mathcal{C}(\mathcal{G}^F)_j$, with $j \in J(\mathcal{G})^F$, one need to assume the assumption of 7.1.5 for any $j \in J(\mathcal{G})^F$ such that, under the decomposition 7.1.2(1), the function ξ_x^G has a non-zero component in $\mathcal{C}(\mathcal{G}^F)_j$. In particular, one knows how to compute the values of $\mathcal{F}^{\mathcal{G}}(\xi_x^G)$ in the following cases:

(a) p, q and G are as in 6.2.19, although the value of the Lusztig constant is in general not known in this case.

(b) G and p are as in 6.2.17 (no assumption on q). Note that in these cases the Lusztig constants are explicitly known [Kaw86].

(c) p and q are as in 6.2.19 and x is a regular element, i.e. $\dim C_G(x) = rk(G)$. Indeed, in that case, the $j \in J(\mathcal{G})^F$ such that under the decomposition

7.1.2(1) the function ξ_x^G has a non-zero component in $\mathcal{C}(\mathcal{G}^F)_j$, are "supported" by a regular nilpotent orbit [DLM97, 1.10]. But, the $j \in J(\mathcal{G})^F$ which are supported by a regular (nilpotent) orbit satisfy the assumption of 7.1.5 as it can be seen from the explicit computation of the Lusztig constant [DLM97, section 2] attached to such j.

(d) p is very good for G (no assumption on q) and x is a semi-simple element of $\mathcal{G}^F$. Indeed, in that case $\xi_x^G \in \mathcal{C}(\mathcal{G}^F)_j$ where j the G-conjugacy class of $(T, \{0\}, \overline{\mathbb{Q}}_\ell)$ with T a maximal torus of G.

7.3 Fourier Transforms of the Characteristic Functions of the Semi-simple Orbits

We now give a more explicit formula for $\mathcal{F}^{\mathcal{G}}(\xi_x^G)$ in the situation of 7.2.2(d). We thus assume that p is very good for G and that $x \in \mathcal{G}^F$ is semi-simple. Here T is an F-stable maximal torus of $M = C_G(x)$ which is M-split and Φ is the root system of M with respect to T.

Lemma 7.3.1. *We have*

$$\xi_0^M = q^{-|\Phi^+|}|W_M(T)|^{-1} \sum_{w \in W_M(T)} \epsilon_M \epsilon_{T_w} \mathcal{Q}_{T_w}^{\mathcal{M}}.$$

Proof: Via the M-equivariant isomorphism $\omega : \mathcal{M}_{nil} \to M_{uni}$, it is equivalent to prove it in the group setting. From [DM91, 12.13] we have

$$\mathrm{Id}_M = |W_M(T)|^{-1} \sum_{w \in W_M(T)} R_{T_w}^M(\mathrm{Id}_{T_w}). \tag{1}$$

As in [DM91], we denote by D_M the dual map. Applying D_M to this formula, we get that

$$D_M(\mathrm{Id}_M) = |W_M(T)|^{-1} \sum_{w \in W_M(T)} \epsilon_M \epsilon_{T_w} R_{T_w}^M\big(D_{T_w}(\mathrm{Id}_{T_w})\big).$$

Let St_M denote the Steinberg character of M^F; it is equal to $D_M(\mathrm{Id}_M)$. Since $St_{T_w} = \mathrm{Id}_{T_w}$, we have

$$St_M = |W_M(T)|^{-1} \sum_{w \in W_M(T)} \epsilon_M \epsilon_{T_w} R_{T_w}^M(\mathrm{Id}_{T_w}). \tag{2}$$

Let η_0^M denote the function on M^F that takes the value 1 on M^F_{uni} and 0 on $M^F - M^F_{uni}$. We have

$$(St_M).\eta_o^M = |W_M(T)|^{-1} R^M_{T_w}(h_1^{T_w})$$

where $h_1^{T_w}(y) = 1$ if $y = 1$ and $h_1^{T_w}(y) = 0$ otherwise. From [DM91, 9.3], we see that $(St_M).\eta_o^M = q^{|\Phi^+|} h_1^M$ whence the result. □

Remark 7.3.2. Applying $\mathcal{F}^{\mathcal{G}}$ to 7.3.1 and using 6.2.15, we get the Lie algebra version of (1) above, from which we deduce the Lie algebra version of (2) above; the Steinberg function on $\mathcal{G}^F$ has been defined in [Spr80].

Theorem 7.3.3. *We have*

$$\mathcal{F}^{\mathcal{G}}(\xi_x^G) = \epsilon_G \epsilon_M q^{-\frac{dim\, M}{2}} |W_M(T)|^{-1} \sum_{w \in W_M(T)} \mathcal{R}^{\mathcal{G}}_{\mathcal{T}_w}\big(\Psi \circ f_x^{\mathcal{T}_w}\big)$$

where $f_x^{\mathcal{T}_w} : \mathcal{T}_w^F \to \overline{\mathbb{Q}}_\ell^\times,\ z \mapsto \Psi\big(\mu(z,x)\big)$.

Proof: Since $C_G(x)$ is connected (because p is very good), the constant sheaf $\overline{\mathbb{Q}}_\ell$ is the unique (up to isomorphism) G-equivariant irreducible local system on $\mathcal{O}_x^G$. Moreover, the orbit $\mathcal{O}_x^G$ is closed in $\mathcal{G}$, hence the complex $K(\mathcal{O}_x^G, \overline{\mathbb{Q}}_\ell)$ is isomorphic to $\overline{\mathbb{Q}}_\ell[\dim \mathcal{O}_x^G]$. Thus, for an appropriate choice of ϕ_ι we may identify $\mathcal{X}_\iota$ of 7.2 with ξ_x^G and $\mathcal{X}_{\iota,n}^M$ with ξ_0^M. Applying the formulas 7.2(2) and 7.3.1 we get 7.3.3. Note that the Lusztig constant γ^w of 7.2(2) is then equal to $(-1)^{rk(G)} q^{\frac{rk(G)}{2}}$. □

Remark 7.3.4. The 7.3.3 is nothing but the generalization of the Kazhdan-Springer formula 3.2.12 to the case where the semi-simple element x is not necessarily regular, although even in the regular case, the formula 7.3.3 is slighty more general since Kazdan's result [Kaz77] is available for p large.

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