Lie symmetries of difference equations *)

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The discrete heat equation is worked out to illustrate the search of symmetries of difference equations. Special attention it is paid to the Lie structure of these symmetries, as well as to their dependence on the derivative's discretization. The case of q-symmetries for discrete equations in a q-lattice is briefly considered at the end.

1 Introduction

As it is well known, Lie point symmetries were introduced by Lie for solving differential equations, providing one of the most efficient methods for obtaining exact analytical solutions of partial differential equations [1]. The interest for discrete systems in the last years has led to extend the Lie method to the case of discrete equations [2–4].

A general difference equation, involving one scalar function $u(x)$ of p independent variables $x = (x_1, x_2, \dots, x_p)$ evaluated at a finite number of points on a lattice will be written in the form

$$E\left(x,\ T^a u(x),\ T^{b_i}\Delta_{x_i} u(x),\ T^{c_{ij}}\Delta_{x_i}\Delta_{x_j} u(x), \dots\right) = 0, \tag{1}$$

where the shift operators $T^a, T^{b_i},\ T^{c_{ij}}$ are defined by

$$T^a u(x) := \left\{T_{x_1}^{a_1} T_{x_2}^{a_2} \cdots T_{x_p}^{a_p}\, u(x)\right\}_{a_i=m_i}^{n_i}, a = (a_1, a_2, \dots, a_p), \quad i = 1, 2, \dots, p,$$

with $a_i,\ m_i,\ n_i,\ (m_i \le n_i)$, fixed integers,

$$T^{a_i} u(x) = u(x_1, x_2, \dots, x_{i-1}, x_i + a_i\sigma_i, x_{i+1}, \dots, x_p),$$

and σ_i is the positive lattice spacing in the uniform lattice of the variable x_i ($i = 1, \dots, p$). The other shift operators $T^{b_i},\ T^{c_{ij}}$ are defined in a similar way. The difference operators Δ_{x_i} are defined so that in the continuous limit they turn into partial derivatives.

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In the following we will make use of the approach presented in [5], based on the formalism of continuous evolutionary vector fields [1]. The infinitesimal symmetry vectors in evolutionary form for the difference equation of order N given in (1) take the general expression

$$X_e \equiv Q\partial u = \left(\sum_i \xi_i(x, T^a u, \sigma_x, \sigma_t) T^b \Delta_{x_i} u - \phi(x, T^c u, \sigma_x, \sigma_t)\right) \partial u, \tag{2}$$

where $\xi_i(x, T^a u, \sigma_x, \sigma_t)$ and $\phi(x, T^c u, \sigma_x, \sigma_t)$ are operator valued functions which in the continuous limit become the functions $\xi_i(x,u)$ and $\phi(x,u)$, respectively, giving rise to Lie point symmetries.

The vector fields X_e generate the symmetry algebra of the discrete equation (1), whose elements transform solutions $u(x)$ of the equation into solutions $\tilde{u}(x)$. The N-th prolongation of X_e must verify the invariance condition

$$\mathrm{pr}^N X_e E|_{E=0} = 0. \tag{3}$$

The prolongation $\mathrm{pr}^N X_e$ is

$$\mathrm{pr}^N X_e = \sum_a T^a Q \partial_{T^a u} + \sum_{b_i} T^{b_i} Q^{x_i} \partial_{T^{b_i}\Delta_{x_i} u} + \sum_{c_{ij}} T^{c_{ij}} Q^{x_i x_j} \partial_{T^{c_{ij}}\Delta_{x_i}\Delta_{x_j} u} + \cdots \tag{4}$$

where summations in (4) are over all the sites present in (1). The symbols Q^{x_i}, $Q^{x_i x_j}$, ... are total variations of Q, i.e., $Q^{x_i} = \Delta^T_{x_i} Q$, $Q^{x_i x_j} = \Delta^T_{x_i}\Delta^T_{x_j} Q$, $\cdots$, defined by

$$\begin{aligned}
&\Delta^T_x f(x, u(x), \Delta_x u(x), \dots) = \\
&\qquad = \frac{1}{\sigma}[f(x+\sigma, u(x+\sigma), (\Delta_x u)(x+\sigma), \dots) - f(x, u(x), \Delta_x u(x), \dots)],
\end{aligned}$$

while the partial variation Δ_{x_i} is

$$\Delta_x f(x, u(x), \Delta_x u(x), \dots) = \frac{1}{\sigma}[f(x+\sigma, u(x), (\Delta_x u)(x), \dots) - f(x, u(x), \Delta_x u(x), \dots)]. \tag{5}$$

The solutions of (3) gives the symmetries of equation (1) when using the difference operator (5). The determining equations for ξ_i and ϕ are obtained by considering linearly independent expressions in the discrete derivatives $T^a \Delta_{x_i} u$, $T^b \Delta_{x_i x_j} u$, ... The Lie commutators of the vector fields X_e are obtained by commuting their first prolongations and projecting onto the symmetry algebra $\mathcal{G}$.

Since we will restrict here to linear equations we can assume that the evolutionary vectors (2) have the form $X_e = (\hat{X} u)\partial_u$, where

$$\hat{X} = \sum_i \xi_i(x, T^a, \sigma_x, \sigma_t)\Delta_{x_i} - \phi(x, T^a, \sigma_x, \sigma_t).$$

The operators $\hat{X}$, in general, may span only a subalgebra of the whole Lie symmetry algebra [5].

In Ref. 6 the symmetries were obtained using the above mentioned difference operator

$$\Delta_{x_i} \equiv \Delta^+_{x_i} = \frac{T_{x_i} - 1}{\sigma_i}, \tag{6}$$

which, when $\sigma_i \to 0$, goes over into the standard right derivative with respect to x_i. Since other definitions of the difference operator can be introduced [7], one would like to show that the algebraic structure of the symmetries is independent on the choice undertaken (a similar problem has been dealt in Ref. 8.) In this work we will also use the left derivative

$$\Delta^-_{x_i} = \frac{1 - T^{-1}_{x_i}}{\sigma_i}, \tag{7}$$

and the symmetric derivative (which goes into the derivative with respect to x_i up to terms of order σ_i^2)

$$\Delta^s_{x_i} = \frac{T_{x_i} - T^{-1}_{x_i}}{2\sigma_i}. \tag{8}$$

In the following we will see that by an appropriate definition of the Leibniz rule we can construct Lie symmetries, in principle, for any difference operator. In Section 2 we will introduce this procedure on the example of the discrete heat equation. We shall study separately the cases of the discrete derivatives (6)–(8), as well as that of a q-derivative. In this way we will get different representations of the same Lie algebra. We conclude with some remarks and comments.

2 Discrete heat equation

Let us consider the second order difference equation

$$(\Delta_t - \Delta_{xx})u(x) = 0,$$

as a discretization of the heat equation. Since it is linear, we can consider an evolutionary vector field of the form

$$X_e \equiv Q\partial_u = (\tau\Delta_t + \xi\Delta_x u + fu)\partial_u, \tag{9}$$

where τ, ξ and f are (operator valued) functions of x, t, T_x, T_t, σ_x and σ_t. The determining equation is

$$\Delta^T_t Q - \Delta^T_{xx} Q|_{\Delta_{xx}u = \Delta_t u} = 0,$$

whose explicit expression is

$$\Delta_t(\xi\Delta_x u) + \Delta_t(\tau\Delta_t u) + \Delta_t(fu) - \\ - [\Delta_{xx}(\xi\Delta_x u) + \Delta_{xx}(\tau\Delta_t u) + \Delta_{xx}(fu)]|_{\Delta_{xx}u=\Delta_t u} = 0. \quad (10)$$

When expression (10) is developed one needs to apply a Leibniz rule and, hence, the results will depend from the definition of the corresponding discrete derivative. We propose a Leibniz rule having the form

$$\Delta_x (f(x)g(x)) = f(x)\Delta_x g(x) + D_x(f(x))g(x), \quad (11)$$

where $D_x(f(x)) = [\Delta_x, f(x)]$ is a function of x, T_x and σ_x (similarly for $D_t(f(t))$).

Using the general rule (11) for an arbitrary discrete derivative we obtain from (10), equating to zero the coefficients of $\Delta_{xt}u$, $\Delta_t u$, $\Delta_x u$ and u, respectively, the following set of determining equations

$$\begin{aligned} D_x(\tau) &= 0, \\ D_t(\tau) - 2D_x(\xi) &= 0, \\ D_t(\xi) - D_{xx}(\xi) - 2D_x(f) &= 0, \\ D_t(f) - D_{xx}(f) &= 0, \end{aligned} \quad (12)$$

where $D_{xx}(f) = D_x(D_x(f))$. Next, starting from (12) we will study separately the cases for $\Delta^{\pm}$ and Δ^s.

2.1 Symmetries for right (left) discrete derivatives

Choosing as in Ref. [5,6] the derivative Δ^+ and, consequently, the Leibniz rule

$$\Delta^+(fg) = f\Delta^+ g + \Delta^+(f)Tg,$$

we get from (12),

$$\begin{aligned} \Delta_x^+\tau &= 0, \\ (\Delta_t^+\tau)T_t - 2(\Delta_x^+\xi)T_x &= 0, \\ (\Delta_t^+\xi)T_t - (\Delta_{xx}^+\xi)T_x^2 - 2(\Delta_x^+ f)T_x &= 0, \\ (\Delta_t^+ f)T_t - (\Delta_{xx}^+ f)T_x^2 &= 0. \end{aligned} \quad (13)$$

The solution of (13) gives

$$\begin{aligned} &\tau = t^{(2)}\tau_2 + t\tau_1 + \tau_0, \\ &\xi = \tfrac{1}{2}x(\tau_1 + 2t\tau_2)T_t T_x^{-1} + t\xi_1 + \xi_0, \\ &f = \tfrac{1}{4}x^{(2)}\tau_2 T_t^2 T_x^{-2} + \tfrac{1}{2}t\tau_2 T_t + \tfrac{1}{2}x\xi_1 T_t T_x^{-1} + \gamma, \end{aligned} \quad (14)$$

where τ_0, τ_1, τ_2, ξ_0, ξ_1 and γ are arbitrary functions of T_x, T_t, and the spacings σ_x and σ_t. The notation $x^{(n)}$, $t^{(n)}$ is for Pochhammer symbols; for instance

$$x^{(n)} = x(x-\sigma_x)\dots(x-(n-1)\sigma_x).$$

By a suitable choice of the functions τ_i, ξ_i, and γ we get the following symmetries [9]

$$\begin{aligned}
P_0 &= (\Delta_t u)\partial_u, \\
P_1 &= (\Delta_x u)\partial_u, \\
W &= u\partial_u, \\
B &= (2tT_t^{-1}\Delta_x u + xT_x^{-1}u)\partial_u, \\
D &= (2tT_t^{-1}\Delta_t u + xT_x^{-1}\Delta_x u + \tfrac{1}{2}u)\partial_u, \\
K &= (t^2T_t^{-2}\Delta_t u - \sigma_t tT_t^{-2}\Delta_t u + txT_t^{-1}T_x^{-1}\Delta_x u \\
&\quad + \tfrac{1}{4}x^2T_x^{-2}u - \tfrac{1}{4}\sigma_x xT_x^{-2}u + \tfrac{1}{2}tT_t^{-1}u)\partial_u.
\end{aligned} \tag{15}$$

Let us note that the above discrete symmetries have a well defined limit when $\sigma_x, \sigma_t \to 0$, which leads to the symmetries of the continuous heat equation. Also, it can be checked that with this choice, the symmetries (15) close into a 6-dimensional Lie algebra isomorphic to the symmetry algebra of the continuous heat equation, for any value of σ_x, σ_t.

A second choice for the discrete derivative is Δ^-. The Leibniz rule becomes

$$\Delta^-(fg) = f\Delta^- g + \Delta^-(f)T^{-1}g. \tag{16}$$

It gives the same results (14) and (15) provided we make the substitution $T \to T^{-1}$.

2.2 Symmetries for symmetric discrete derivatives

Next, let us consider the case of the symmetric derivative Δ^s (8). The commutator

$$[\Delta_x^s, x] = \tfrac{1}{2}(T_x + T_x^{-1})$$

can always be rewritten by introducing a function $\beta_x^s = \beta^s(T_x) = 2(T_x + T_x^{-1})^{-1}$ as

$$[\Delta_x, x\beta_x] = 1.$$

This fact will help us in the computation of general commutators, so

$$[\Delta_x^s, f(x)\beta_x^s] = (\Delta_x^s f(x))\, T_x\beta_x^s + \left(T_x^{-1}f(x) - f(x)\right)\beta_x^s\Delta_x^s. \tag{17}$$

From relation (17) we get the explicit Leibniz rule

$$\begin{aligned}
&\Delta_x^s\left(f(x)g(x)\right) = \\
&\quad = f(x)\Delta_x^s g(x) + \left[\frac{1}{\sigma_x}\left((T_x^{-1}-1)f(x)\right)\left(T_x - (\beta_x^s)^{-1}\right) + \left(\Delta_x^s f(x)\right)T_x\right] g(x).
\end{aligned}$$

This formula allows us to write explicitly the determining equations (12). Their solution is given by [9]

$$\begin{aligned}
\tau^s &= t^{(2)}\tau_2 + t\tau_1 + \tau_0,\\
\xi^s &= \tfrac{1}{2}x\left(2t\tau_2 + \tau_1 + \sigma_t T_t^{-1}\beta_t^s\tau_2\right)(\beta_t^s)^{-1}\beta_x^s + t\xi_1 + \xi_0,\\
f^s &= \tfrac{1}{4}x^{(2)}\tau_2(\beta_x^s)^2(\beta_t^s)^{-2} + \tfrac{1}{2}x\xi_1\beta_x^s(\beta_t^s)^{-1}\\
&\quad + \tfrac{1}{4}x\sigma_x\tau_2 T_x^{-1}(\beta_x^s)^3(\beta_t^s)^{-2} + \tfrac{1}{2}t\tau_2(\beta_t^s)^{-1} + f_0,
\end{aligned}$$

where τ_2, τ_1, τ_0, ξ_1, ξ_0 and f_0 are arbitrary functions of T_x, T_t, σ_x and σ_t. Now, from these solutions and (9) we obtain, with a suitable choice of τ_2, τ_1, τ_0, ξ_1, ξ_0 and f_0, the following symmetries

$$\begin{aligned}
P_0^s &= (\Delta_t^s u)\partial_u,\\
P_1^s &= (\Delta_x^s u)\partial_u,\\
W^s &= u\partial_u,\\
B^s &= (2t\beta_t^s\Delta_x^s u + x\beta_x^s u)\partial_u,\\
D^s &= (2t\beta_t^s\Delta_t^s u + x\beta_x^s\Delta_x^s u + \tfrac{1}{2}u)\partial_u,\\
K^s &= ((t^2(\beta_t^s)^2 - t{\sigma_t}^2(\beta_t^s)^3\Delta_t^s)\Delta_t^s u + tx\beta_t^s\beta_x^s\Delta_x^s u\\
&\quad - \tfrac{1}{4}x{\sigma_x}^2(\beta_x^s)^3\Delta_x^s u + \tfrac{1}{4}x^2(\beta_x^s)^2 u + \tfrac{1}{2}t\beta_t^s u)\partial_u.
\end{aligned} \tag{18}$$

These symmetries close the same 6-dimensional Lie algebra generated by the operators (15), and have a well defined continuous limit.

2.3 Symmetries for q-derivatives

In the following we shall extend the preceding method to the case of q-derivatives and q-symmetries. We deal briefly with a q-discretized heat equation where the q-difference operator is defined by

$$\Delta_x^q = \frac{1}{(q_x-1)x}(T_x - 1),$$

with the help of a q-shift operator

$$T_x = e^{q_x x\partial_x}, \qquad T_x f(x) = f(q_x x).$$

In this case we have the commutator

$$[\Delta_x^q, x] = T_x.$$

The function $\beta_x(T_x)$ satisfying

$$[\Delta_x^q, \beta_x x] = 1,$$

is formally given by

$$\beta_x(T_x) = (q_x - 1)x\partial_x(T_x - 1)^{-1}.$$

Thus, we can perform a change of basic operators $\{x, T_x\} \to \{\tilde{x}, \Delta_x^q\}$, where $\tilde{x} = \beta_x(T_x)x$, so that, formally, we can express any function $f(x, T_x)$ as $f(x, T_x) = \widetilde{f}(\tilde{x}, \Delta_x^q)$. In this way the determining equations (12) take the form

$$\begin{aligned} \widetilde{\tau}_{\tilde{x}} &= 0, \\ \widetilde{\tau}_{\tilde{t}} - 2\widetilde{\xi}_{\tilde{x}} &= 0, \\ \widetilde{\xi}_{\tilde{t}} - \widetilde{\xi}_{\tilde{x}\tilde{x}} - 2\widetilde{f}_{\tilde{x}} &= 0, \\ \widetilde{f}_{\tilde{t}} - \widetilde{f}_{\tilde{x}\tilde{x}} &= 0. \end{aligned}$$

Therefore, we obtain solutions that have similar appearance as the classical symmetries

$$\begin{aligned} P_0^q &= (\Delta_t^q\, u)\partial_u, \\ P_1^q &= (\Delta_x^q\, u)\partial_u, \\ W^q &= u\partial_u, \\ B^q &= (2\beta_t t\Delta_x^q\, u + \beta_x x\, u)\partial_u, \\ D^q &= (2\beta_t t\Delta_t^q\, u + \beta_x x\Delta_x^q\, u + \tfrac{1}{2}\, u)\partial_u, \\ K^q &= (\gamma_t t^2\Delta_t^q\, u + \beta_x\beta_t tx\Delta_x^q\, u + \tfrac{1}{4}\gamma_x x^2\, u + \tfrac{1}{2}\beta_t t\, u)\partial_u, \end{aligned}$$

where $\gamma_x = \left[\dfrac{(q-1)x\partial_x}{T_x - 1}\right]\left[\dfrac{(q-1)(-1+x\partial_x)}{q_x^{-1}T_x - 1}\right]$ (for γ_t replace x by t).

3 Conclusions

The key point in obtaining the explicit determining equations (12) and, consequently, the discrete symmetries is the use of the Leibniz rule defined in (11). Of course, this approach is not the only possibility; in fact, it must be checked whether it works correctly or not in each case. The choice of a commutator in order to define the Leibniz rule implicitly leads us to Lie symmetries, since the natural algebraic structure will be given also in terms of commutators.

Some of the above result deserves some comments. The symmetries associated to the symmetric derivatives (Subsection 2.2) include functions β_t^s (β_x^s) of T_t (T_x) that can only be understood as infinite series expansions. Therefore, not all the symmetries (18) have a local character, in the sense that they are not (finite) polynomials in the operators $T_t^{\pm 1}$, $T_x^{\pm 1}$. Note that although these discrete symmetries give rise to the classical symmetries in the limit $\sigma_x \to 0, \sigma_t \to 0$, one of them, K^s, also includes surprisingly a term in $(\Delta_t)^2$, which vanishes in the continuous limit since it is multiplied by σ_t^2.

Something similar happens with the q-symmetries (Subsection 2.3): they have also a highly non-local character. The origin of this unpleasant feature is that the basic commutator $[\Delta_x^q, \beta_x x] = 1$ needs a non-local function $\beta_x(T_x)$. If we want to investigate local symmetries it is necessary to find a commutator which is free of this problem. For instance, we could take as starting point the q-commutator: $[\Delta_x^q, xT_x^{-1}]_{q_x} = 1$, where $[A, B]_{q_x} = AB - q_x^{-1}BA$. However, in this case we have a q-algebra [10] which is out of the scope of the present article.

Let us insist that the procedure here exposed can be straightforwardly applied to other discretizations such as for the wave equation [11] or even equations including a potential term as long as we stay inside the field of linear equations. Non-linear equations need additional improvements in order to have reasonable determining equations.

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