J. Phys. A: Math. Gen. 22 (1989) 4743–4767. Printed in the UK

Group theoretical analysis of a rotating shallow liquid in a rigid container

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Received 10 November 1988

Abstract. A Lie group analysis is undertaken for a nonlinear system which models the motion of a rotating shallow liquid in a rigid basin. The Lie algebra of the symmetry group is presented for elliptic and circular paraboloidal basins. In the elliptic case, the symmetry algebra is a six-dimensional real Lie algebra. In the circular case, the symmetry algebra is nine dimensional. Finite group transformations are constructed which, in the circular paraboloidal case, deliver a theorem concerning the time evolution of a key moment of inertia during the motion. In the elliptic paraboloidal case, a result concerning the motion of the centre of gravity of the liquid is retrieved. The investigation ends with symmetry reduction of the original system and the generation of group-invariant solutions which correspond to various initial data.

Resumé. On donne l'analyse d'un système d'équations différentielles non linéaires décrivant le comportement d'un liquide dans un bassin rigide. On trouve le groupe de symétrie qui est de dimension six si le basin a la forme d'un paraboloïde elliptique et de dimension neuf pour un paraboloïde de révolution. Le groupe est utilisé pour étudier le comportement temporel global du système. Les sous-groupes du groupe de symétrie sont utilisés pour obtenir des solutions exactes invariantes.

1. Introduction

The purpose of this paper is to undertake a group theoretical analysis of a nonlinear system of partial differential equations, describing the motion of a shallow ideal liquid in a rigid basin, rotating with the Earth. The liquid is subject to the force of gravity and the Earth's rotation manifests itself only through Coriolis force.

Exact solutions of nonlinear physical problems in three dimensions are hard to come by. Exceptions are such completely integrable equations as the Kadomtsev-Petviashvili equation [1], the Davey-Stewartson equation [2] and a few others, integrable by inverse scattering techniques [3-5]. These equations describe waves propagating in infinite bodies of fluid, such as oceans, channels or straits [6] (of infinite length). No integrable nonlinear equations are known for waves in a bounded region such as a basin, lake or enclosed sea.

For equations that do not belong to the integrable class, it is of considerable interest to obtain exact analytic solutions. Such solutions often elucidate qualitative features of the model and of the physical situation itself. Exact solutions, be they physically interesting or not, provide a standard against which numerical solutions can be tested. Moreover, exact solutions, whenever they exist, may help in clarifying the dependence of the model on the parameters of the problem (choice of initial conditions, shapes of boundaries, etc).

Probably the most interesting nonlinear non-integrable system, from the fluid dynamics point of view, is the nonlinear shallow water system, obtained as an approximation to the $(3+1)$-dimensional Euler equations [7, 8]. When the system is confined to a finite basin, the shallow water approximation leads to a system of nonlinear partial differential equations, studied by Ball [9–11] and more recently analysed by Thacker [12] and Kirwan and Liu [13].

Most of the applications have concerned tidal or long-wave oscillations in oceans [13, 14]. Solutions of the shallow water system can be classified in terms of the Goldsbrough expansion [15], in which the velocity field and the depth of the fluid above the bottom are expanded in powers of the horizontal coordinates (with time-dependent coefficients). The lowest mode in this expansion is the 'displacement' (motion of the centre of gravity of the fluid), in which the horizontal velocity field depends on time alone. The next modes are 'deformations', (involving uniform rotation, expansion and distortion of the liquid), in which the horizontal velocity field is linear in the horizontal coordinates. 'Higher modes' will thus be characterised by higher powers of the coordinates present in the velocity fields.

Ball [9] presented two theorems dealing with transformations of solutions of the shallow water system. He showed that when applied to a trivial solution, i.e. a horizontal surface and zero velocity field, these transformations give rise to displacement modes and deformations, respectively.

In this paper, we use group theory to provide a natural basis for Ball's theorems, to generalise his result and to obtain new explicit solutions.

In § 2, we indicate briefly the derivation of the shallow water system from the Euler equations for a basin of an arbitrary shape. In § 3, we obtain the Lie algebra of the symmetry group of local point transformations leaving the equations invariant. To obtain an explicit form of the corresponding vector fields, we specify the boundary to be an elliptic or circular paraboloid. The finite group transformations providing, *inter alia*, Ball's theorems are obtained in § 4. In § 5, we concentrate on the circular paraboloid. We determine all subgroups of the symmetry group, having orbits of codimension 1 in the space of independent variables and use some of them to reduce the shallow water system to ordinary differential equations, which must then be solved. The elliptic paraboloidal basin is treated in a similar manner in § 6.

2. The governing equations

In the present context, the relevant hydrodynamic equations consist of the continuity equation for an incompressible fluid

$$\nabla \boldsymbol{v} = u_x + v_y + w_z = 0 \qquad (2.1)$$

together with the Euler equations of motion for an inviscid incompressible fluid, subject to the force of gravity and contained in a basin rotating with the Earth:

$$\rho[u_t + uu_x + vu_y + wu_z] - fv + p_x = 0 \tag{2.2a}$$

$$\rho[v_t + uv_x + vv_y + wv_z] + fu + p_y = 0 \tag{2.2b}$$

$$\rho[w_t + uw_x + vw_y + ww_z] + \rho g + p_z = 0. \tag{2.2c}$$

In the usual notation u, v and w are components of the velocity $\boldsymbol{v}$, ρ is the fluid density, assumed to be constant, p is the fluid pressure relative to that of the atmosphere and the subscripts denote partial derivatives. The constant f is the Coriolis parameter (assumed to be constant over the dimensions of the basin) and g is the gravitational constant. As is common in oceanographic contexts, the centrifugal contribution is neglected [16].

In what follows, we shall be concerned with the motion of a shallow rotating liquid contained in a rigid basin. The geometric configuration is as in Ball [9] and is shown in figure 1. Here

$$z = Z(x, y) \tag{2.3}$$

is the equation of the basin surface underlying the liquid, while

$$z = \eta(x, y, t) = Z + h(x, y, t) \tag{2.4}$$

denotes the free surface. We must now adjoin appropriate boundary conditions to the system (2.1), (2.2). The initial conditions are not specified *a priori* in the present approach.

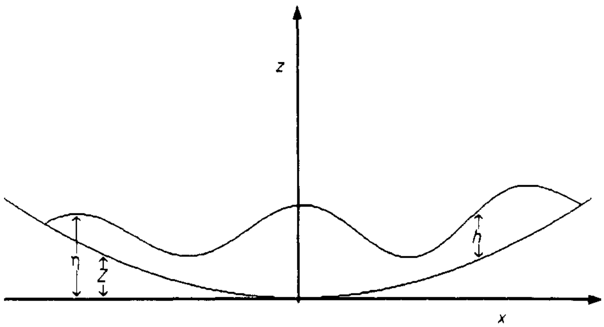

Figure 1. Cross section of the basin described by the function $z = Z(x, y)$; $h(x, y, t)$ is the vertical distance from the free surface to the basin and $\eta(x, y, t) = Z + h$.

The usual boundary conditions consist of the kinematic condition on the free surface, that is

$$\eta_t + u\eta_x + v\eta_y = w \qquad \text{for} \qquad z = \eta(x, y, t) \tag{2.5a}$$

together with the assumption

$$p = 0 \qquad \text{for} \qquad z = \eta(x, y, t) \tag{2.5b}$$

(that the pressure be atmospheric on the free surface). The boundary condition on the basin surface is

$$uZ_x + vZ_y = w \qquad \text{for} \qquad z = Z(x, y). \tag{2.6}$$

The shallow water approximation is now derived in a manner similar to that adopted, in the absence of the Coriolis terms, by Keller [17]. Thus, dimensionless variables, denoted by a bar, are introduced according to

$$\bar{x}=x/k \qquad \bar{y}=y/k \qquad \bar{z}=z/d \qquad \bar{t}=t\sqrt{gd}/k$$
$$\bar{u}=(gd)^{-1/2}u \qquad \bar{v}=(gd)^{-1/2}v \qquad \bar{w}=(d\sqrt{gd}/k)^{-1}w \tag{2.7}$$
$$\bar{p}=p/\rho gd \qquad \bar{\eta}=\eta/d \qquad \bar{Z}=Z/d \qquad \bar{f}=fk/\rho\sqrt{gd}$$

where d represents a typical depth and k a typical length in the horizontal direction.

The 'shallow water' parameter

$$\sigma=d^2/k^2 \tag{2.8}$$

is introduced and will be assumed to be small. Rewriting the governing system (2.1), (2.2), (2.5) and (2.6) in dimensionless variables and dropping the bars, we obtain

$$\begin{aligned}
&u_x+v_y+w_z=0 && \eta_t+u\eta_x+v\eta_y=w \quad \text{for } z=\eta\\
&u_t+uu_x+vu_y-fv+p_x+wu_z=0 && p=0 \quad \text{for } z=\eta\\
&v_t+uv_x+vv_y+fu+p_y+wv_z=0 && uZ_x+vZ_y=w \quad \text{for } z=Z\\
&\sigma[w_t+uw_x+vw_y+ww_z]+1+p_z=0. &&
\end{aligned} \tag{2.9}$$

Power series expansions for the quantities $q=\{u, v, w, \eta, p\}$ in terms of σ are now introduced in the form

$$q=\sum_{n=0} q^{(n)}\sigma^n \tag{2.10}$$

and are inserted into the system (2.9). The terms of order σ^0 yield

$$u_x^{(0)}+v_y^{(0)}+w_z^{(0)}=0 \tag{2.11a}$$
$$u_t^{(0)}+u^{(0)}u_x^{(0)}+v^{(0)}u_y^{(0)}-fv^{(0)}+p_x^{(0)}=0 \tag{2.11b}$$
$$v_t^{(0)}+u^{(0)}v_x^{(0)}+v^{(0)}v_y^{(0)}+fu^{(0)}+p_y^{(0)}=0 \tag{2.11c}$$
$$p_z^{(0)}+1=0 \tag{2.11d}$$
$$\eta_t^{(0)}+u^{(0)}\eta_x^{(0)}+v^{(0)}\eta_y^{(0)}=w^{(0)} \qquad \text{for } z=\eta^{(0)} \tag{2.12a}$$
$$p^{(0)}=0 \qquad \text{for } z=\eta^{(0)} \tag{2.12b}$$
$$u^{(0)}Z_x+v^{(0)}Z_y=w^{(0)} \qquad \text{for } z=Z. \tag{2.12c}$$

We now make a physical assumption, namely that $u^{(0)}$ and $v^{(0)}$ are independent of z,

$$u^{(0)}=u^{(0)}(x, y, t) \qquad v^{(0)}=v^{(0)}(x, y, t). \tag{2.13}$$

Integrating (2.11a) subject to (2.13) and (2.12c), we obtain:

$$w^{(0)}=-(u_x^{(0)}+v_y^{(0)})z+(u^{(0)}Z)_x+(v^{(0)}Z)_y. \tag{2.14}$$

From (2.11d) and (2.12b), we obtain the dimensionless hydrostatic approximation for the pressure

$$p^{(0)}(x, y, z, t)=-z+\eta^{(0)}(x, y, t). \tag{2.15}$$

Substituting (2.14) into (2.12*a*), we obtain an equation independent of z which we rewrite as

$$\eta_t^{(0)}+[u^{(0)}(\eta^{(0)}-Z)]_x+[v^{(0)}(\eta^{(0)}-Z)]_y=0. \tag{2.16}$$

In agreement with (2.4), we put

$$\eta^{(0)}(x,y,t)=Z(x,y)+h^{(0)}(x,y,t) \tag{2.17}$$

and, dropping the superscript (0), we rewrite the remaining first-order equations (2.11*b*, *c*) and (2.16) as

$$u_t+uu_x+vu_y+(Z+h)_x-fv=0 \tag{2.18a}$$

$$v_t+uv_x+vv_y+(Z+h)_y+fu=0 \tag{2.18b}$$

$$h_t+(uh)_x+(vh)_y=0. \tag{2.18c}$$

Using two-dimensional vector notation, we put $\boldsymbol{q}=u\boldsymbol{i}+v\boldsymbol{j}$, where $(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$ is the usual orthonormal basis of Euclidean 3-space and rewrite (2.18) as

$$h_t+\operatorname{div}(h\boldsymbol{q})=0 \qquad \frac{\partial \boldsymbol{q}}{\partial t}+(\boldsymbol{q},\nabla\boldsymbol{q})+\nabla(Z+h)+f\boldsymbol{k}\times\boldsymbol{q}=\boldsymbol{0}. \tag{2.19}$$

The system (2.18) (or (2.19)) is the zero-order shallow water system with the inclusion of Coriolis terms, to which we devote the rest of this paper. Notice that the boundary conditions (2.5) and (2.6) have been satisfied to the considered order and that the shape of the basin (2.3) is incorporated in equations (2.18*a*, *b*) explicitly via the function $Z(x, y)$.

Below we shall also make use of the shallow water equations (2.18) written in cylindrical coordinates. These are recorded below for convenience. Thus, putting

$$\begin{gathered} x=r\cos\theta \qquad y=r\sin\theta \\ v_1\equiv v_r=u\cos\theta+v\sin\theta \qquad v_2\equiv v_\theta=-u\sin\theta+v\cos\theta \end{gathered} \tag{2.20}$$

we can rewrite the system (2.18) as:

$$\frac{\partial h}{\partial t}+v_1\frac{\partial h}{\partial r}+\frac{h}{r}v_1+h\frac{\partial v_1}{\partial r}+\frac{1}{r}v_2\frac{\partial h}{\partial \theta}+\frac{h}{r}\frac{\partial v_2}{\partial \theta}=0 \tag{2.21a}$$

$$\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial r}+\frac{\partial}{\partial r}(Z+h)-fv_2-\frac{1}{r}v_2^2+\frac{v_2}{r}\frac{\partial v_1}{\partial \theta}=0 \tag{2.21b}$$

$$\frac{\partial v_2}{\partial t}+v_1\frac{\partial v_2}{\partial r}+fv_1+\frac{1}{r}v_2v_1+\frac{v_2}{r}\frac{\partial v_2}{\partial \theta}+\frac{1}{r}\frac{\partial}{\partial \theta}(Z+h)=0. \tag{2.21c}$$

3. The Lie algebra of the symmetry group

We are looking for the Lie group of local point transformations leaving the shallow water system (2.18) invariant and hence transforming solutions amongst each other.

Such transformations will have the form

$$x_i'=\Lambda_i(x,w,g) \qquad w_a'=\Omega_a(x,w,g) \tag{3.1}$$

where

$$x=(x_1,x_2,x_3)=(x,y,t) \qquad w=(w_1,w_2,w_3)=(u,v,h) \tag{3.2}$$

are the independent and dependent variables, respectively, and g denotes the set of group parameters. The functions Λ_i and Ω_a are such that $w'(x')$ is a solution whenever $w(x)$ is one, and the transformation is defined.

Instead of constructing the group transformations directly, we follow a standard procedure, due to Lie and explained in all books on the subject (see e.g. Olver [18]). It consists of constructing the corresponding Lie algebra realised by vector fields of the form

$$X = \eta_1\partial_x + \eta_2\partial_y + \eta_3\partial_t + \phi_1\partial_u + \phi_2\partial_v + \phi_3\partial_h \tag{3.3}$$

where η_i and w_a are functions of the independent and dependent variables (3.2). These functions are determined from the requirement that the first prolongation $\mathrm{pr}^{(1)}X$ of the vector field X should annihilate the equations (2.18) on the solution set. We have used a specifically written MACSYMA program [19] that greatly facilitates the task of implementing the algorithm for finding the symmetry algebra. The program provides us with a partially solved set of determining equations. Here they consist of first-order linear partial differential equations for the functions η_i and ϕ_a in (3.3).

Dropping all details, we record that for a basin of the general form

$$z = Z(x, y) \tag{3.4}$$

we find the following expressions for the coefficients in (3.3):

$$\begin{aligned}
\eta_1 &= (\tfrac{1}{2}\dot\alpha + c_1)x + (\tfrac{1}{2}f\alpha + c_2)y + \beta \\
\eta_2 &= -(\tfrac{1}{2}f\alpha + c_2)x + (\tfrac{1}{2}\dot\alpha + c_1)y + \gamma \\
\eta_3 &= \alpha \\
\phi_1 &= (-\tfrac{1}{2}\dot\alpha + c_1)u + (\tfrac{1}{2}f\alpha + c_2)v + \tfrac{1}{2}\ddot\alpha x + \tfrac{1}{2}f\dot\alpha y + \dot\beta \\
\phi_2 &= -(\tfrac{1}{2}f\alpha + c_2)u + (-\tfrac{1}{2}\dot\alpha + c_1)v - \tfrac{1}{2}f\dot\alpha x + \tfrac{1}{2}\ddot\alpha y + \dot\gamma \\
\phi_3 &= 2(-\tfrac{1}{2}\dot\alpha + c_1)h.
\end{aligned} \tag{3.5}$$

In (3.5), c_1 and c_2 are constants whereas $\beta = \beta(t)$, $\gamma = \gamma(t)$ and $\eta = \alpha(t)$ are functions of time, subject to the constraints

$$\psi_x = 0 \qquad \psi_y = 0 \tag{3.6}$$

where

$$\begin{aligned}
\psi \equiv\; & Z_x[(\tfrac{1}{2}\dot\alpha + c_1)x + (\tfrac{1}{2}f\alpha + c_2)y + \beta] + Z_y[-(\tfrac{1}{2}f\alpha + c_2)x + (\tfrac{1}{2}\dot\alpha + c_1)y + \gamma] + Z(\dot\alpha - 2c_1) \\
& + \tfrac{1}{4}[f^2\dot\alpha + \dddot\alpha](x^2 + y^2) + (\ddot\beta - \dot\alpha f)x + (\ddot\gamma + \dot\beta f)y.
\end{aligned} \tag{3.7}$$

The constraints (3.6) are actually very stringent and depend crucially on the shape of the boundary given by $Z(x, y)$.

To proceed further, we assume that the basin is an elliptic (or circular) paraboloid. Thus, we have

$$Z(x, y) = Ax^2 + By^2 \qquad A > 0,\ B > 0 \tag{3.8}$$

in (3.4). In passing, we mention that the case $A < 0$, $B < 0$ is also of interest and corresponds, for example, to a partially submerged mountain in a sea. Most, but not all, of the results that follow are also valid in this case.

The constraints (3.6) then greatly simplify and can be written as

$$\begin{aligned}
&(A - B)(f\alpha + 2c_2) = 0 \qquad && \dddot\alpha + (8A + f^2)\dot\alpha = 0 \qquad && \dddot\alpha + (8B + f^2)\dot\alpha = 0 \\
&\ddot\beta - f\dot\gamma + 2A\beta = 0 && \ddot\gamma + f\dot\beta + 2B\gamma = 0. &&
\end{aligned} \tag{3.9}$$

Moreover, these constraints can be completely solved. For $A \neq B$, α is a constant, $\alpha = -2c_2/f$. For $A = B$, the second and third equations coincide and can be solved for

$\alpha(t)$, which then depends on three integration constants. The last two equations yield β and γ, depending altogether on four constants.

Below, we present the symmetry algebras in an appropriate basis. We introduce the notation

$$\omega = R_1 + R_2 = [2(\sqrt{A}+\sqrt{B})^2 + f^2]^{1/2} \qquad f_0 = R_1 - R_2 = [2(\sqrt{A}-\sqrt{B})^2 + f^2]^{1/2} \tag{3.10a}$$

and notice that for a circular basin, we have $A = B$ and hence

$$\omega = R_1 + R_2 = (8A + f^2)^{1/2} \qquad f_0 = R_1 - R_2 = f. \tag{3.10b}$$

For $A \neq B$, the symmetry algebra is a six-dimensional real Lie algebra, for which we choose a basis to be

$$\begin{aligned}
&T = \partial_t \qquad D = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h \\
&Y_1 = \cos R_1 t\partial_x - \frac{1}{f}\left(R_1 - \sqrt{\frac{A}{B}}R_2\right)\sin R_1 t\partial_y - R_1 \sin R_1 t\partial_u \\
&\qquad - \frac{R_1}{f}\left(R_1 - \sqrt{\frac{A}{B}}R_2\right)\cos R_1 t\partial_v \\
&Y_2 = \sin R_1 t\partial_x + \frac{1}{f}\left(R_1 - \sqrt{\frac{A}{B}}R_2\right)\cos R_1 t\partial_y + R_1 \cos R_1 t\partial_u \\
&\qquad - \frac{R_1}{f}\left(R_1 - \sqrt{\frac{A}{B}}R_2\right)\sin R_1 t\partial_v \\
&Y_3 = \cos R_2 t\partial_x - \frac{1}{f}\left(R_2 - \sqrt{\frac{A}{B}}R_1\right)\sin R_2 t\partial_y - R_2 \sin R_2 t\partial_u \\
&\qquad - \frac{R_2}{f}\left(R_2 - \sqrt{\frac{A}{B}}R_1\right)\cos R_2 t\partial_v \\
&Y_4 = \sin R_2 t\partial_x + \frac{1}{f}\left(R_2 - \sqrt{\frac{A}{B}}R_1\right)\cos R_2 t\partial_y + R_2 \cos R_2 t\partial_u \\
&\qquad - \frac{R_2}{f}\left(R_2 - \sqrt{\frac{A}{B}}R_1\right)\sin R_2 t\partial_v .
\end{aligned} \tag{3.11a}$$

We shall call the Lie algebra (3.11*a*) for the elliptic paraboloid basin L_E. This is a solvable Lie algebra; its nilradical (maximal nilpotent ideal) is generated by $\{Y_1, Y_2, Y_3, Y_4\}$ and is Abelian. The operator T corresponds to time translations, D to dilations. We shall see in § 4 that the subgroup of the symmetry group generated by the Abelian subalgebra $\{Y_i\}$ corresponds to a translation of the space coordinates (x, y), depending periodically on time and compensated for by an appropriate translation of the velocities u and v.

For a circular basin, we have $A = B$ and the symmetry algebra is larger, namely nine dimensional . In addition to the six basis elements (3.11*a*) of L_E, we obtain three more symmetry operators, namely

$$\begin{aligned}
&R = y\partial_x - x\partial_y + v\partial_u - u\partial_v \\
&K_1 = \tfrac{1}{2}\cos\omega t[x\partial_x + y\partial_y - u\partial_u - v\partial_v + f(y\partial_u - x\partial_v) - 2h\partial_h] \\
&\qquad + \frac{1}{2\omega}\sin\omega t[f(y\partial_x - x\partial_y + v\partial_u - u\partial_v) - \omega^2(x\partial_u + y\partial_v) + 2\partial_t] \\
&K_2 = -\tfrac{1}{2}\sin\omega t[x\partial_x + y\partial_y - u\partial_u - v\partial_v + f(y\partial_u - x\partial_v) - 2h\partial_h] \\
&\qquad + \frac{1}{2\omega}\cos\omega t[f(y\partial_x - x\partial_y + v\partial_u - u\partial_v) - \omega^2(x\partial_u + y\partial_v) + 2\partial_t].
\end{aligned} \tag{3.11b}$$

Furthermore, for $B = A$, the expressions for $Y_1, \ldots, Y_4$ simplify, since R_1 and R_2 are now given by (3.10*b*) and we have

$$\frac{1}{f}\left(R_1 - \sqrt{\frac{A}{B}}\, R_2\right) = -\frac{1}{f}\left(R_2 - \sqrt{\frac{A}{B}}\, R_1\right) = 1. \tag{3.12}$$

The additional operator R corresponds to simultaneous rotations in coordinate and velocity space. The transformations corresponding to K_1 and K_2 are somewhat harder to interpret. They correspond to a transformation to a moving frame and we shall return to them in § 4.

In order to identify the Lie algebra L_C corresponding to the circular basin, we perform a slight change of basis, replacing the time translation T by the linear combination

$$L_3 = \frac{1}{\omega}\left(T + \tfrac{1}{2} f R\right). \tag{3.13}$$

The commutation relations for the Lie algebra L_C in this basis are given in table 1. The basis is a canonical one, so chosen that we can immediately read off the Levi decomposition (see Levi [20], Jacobson [21]). We have

$$L_C = S \oplus R_0 \tag{3.14a}$$

where

$$S \sim \{K_1, K_2, L_3\} \qquad R_0 \sim \{D, R, Y_1, Y_2, Y_3, Y_4\} \tag{3.14b}$$

are the simple Lie algebra sl(2, $\boldsymbol{R}$) and the maximal solvable ideal (radical), respectively. The nilradical N of L_C is Abelian and is generated by $\{Y_1, Y_2, Y_3, Y_4\}$.

The action of the Lie group corresponding to the algebra $\{K_1, K_2, L_3, R\}$ on the Abelian ideal $\{Y_1, Y_2, Y_3, Y_4\}$ is such that the expression $C = Y_1^2 + Y_2^2 - Y_3^2 - Y_4^2$ is invariant, whereas $e^{\lambda D}$ will scale C; that is, we have

$$[K_1, C] = [K_2, C] = [L_3, C] = [R, C] = 0 \qquad [DC] = -2C. \tag{3.15}$$

Table 1. Commutation table for the symmetry algebra L of the circular basin

	L_3	K_1	K_2	D	R	Y_1	Y_2	Y_3	Y_4
L_3	0	K_2	$-K_1$	0	0	$-\frac{1}{2}Y_2$	$\frac{1}{2}Y_1$	$-\frac{1}{2}Y_4$	$\frac{1}{2}Y_3$
K_1	$-K_2$	0	$-L_3$	0	0	$-\frac{1}{2}Y_3$	$\frac{1}{2}Y_4$	$\frac{1}{2}Y_1$	$\frac{1}{2}Y_2$
K_2	K_1	L_3	0	0	0	$\frac{1}{2}Y_4$	$\frac{1}{2}Y_3$	$\frac{1}{2}Y_2$	$\frac{1}{2}Y_1$
D	0	0	0	0	0	$-Y_1$	$-Y_2$	$-Y_3$	$-Y_4$
R	0	0	0	0	0	Y_2	$-Y_1$	$-Y_4$	Y_3
Y_1	$\frac{1}{2}Y_2$	$\frac{1}{2}Y_3$	$-\frac{1}{2}Y_4$	Y_1	$-Y_2$	0	0	0	0
Y_2	$-\frac{1}{2}Y_1$	$-\frac{1}{2}Y_4$	$-\frac{1}{2}Y_3$	Y_2	Y_1	0	0	0	0
Y_3	$\frac{1}{2}Y_4$	$\frac{1}{2}Y_1$	$-\frac{1}{2}Y_2$	Y_3	Y_4	0	0	0	0
Y_4	$-\frac{1}{2}Y_3$	$-\frac{1}{2}Y_2$	$-\frac{1}{2}Y_1$	Y_4	$-Y_3$	0	0	0	0

Thus, C effectively provides an invariant indefinite metric on $N=\{Y_1, Y_2, Y_3, Y_4\}$ and an element $\alpha_i Y_i \in N$ can have positive, negative or zero length with respect to this metric. The one-parameter subgroups $\exp\lambda(R+2L_3)$ and $\exp\lambda(R-2L_3)$ provide independent rotations in the $\{Y_3, Y_4\}$ and $\{Y_1, Y_2\}$ spaces. Analysing the action of $\{K_1, K_2, L_3, R, D\}$ on N further it is easy to see that any one-dimensional subspace of N is conjugate to $\{Y_1\}$, $\{Y_3\}$ or $\{Y_1+Y_3\}$, depending on whether its general element has $C>0$, $C<0$ or $C=0$. Similarly, any two-dimensional subspace of N is completely characterised by its signature. This will be important in § 5.

4. The group transformations

In order to obtain the group transformations (3.1) leaving the shallow water equations invariant, we must integrate the obtained vector fields (3.11). The general element of the Lie algebra has the form (3.3). The corresponding one-parameter subgroup of group transformations is obtained by solving the system of initial value problems

$$\frac{\mathrm{d}x_i'}{\mathrm{d}\lambda}=\eta_i(x', w') \qquad \frac{\mathrm{d}w_a'}{\mathrm{d}\lambda}=\phi_a(x', w') \qquad x_i'|_{\lambda=0}=x_i \qquad w_a'|_{\lambda=0}=w_a \tag{4.1}$$

where $\lambda \in g$ (see (3.1)).

Let us first consider the Lie algebra L_{E} with basis (3.11a). The corresponding group transformations leave invariant the shallow water equations for both the elliptic and circular paraboloid basins.

We first obtain the transformation corresponding to the general element

$$Y=\sum_{i=1}^{4}\tilde{p}_i Y_i \tag{4.2}$$

or the nilradical N, then compose it with the transformation corresponding to the element

$$F=\tilde{\lambda}D+\tilde{\tau}T \tag{4.3}$$

of the factor algebra L_{E}/N.

The result is that if $u(x,y,t)$, $v(x,y,t)$ and $h(x,y,t)$ satisfy the system (2.18) with Z given by (3.8), then so do

$$\begin{aligned} u'(x', y', t') &= \mathrm{e}^{\lambda}[u(x,y,t)+\dot{X}(t)] \\ v'(x', y', t') &= \mathrm{e}^{\lambda}[v(x,y,t)+\dot{Y}(t)] \\ h'(x', y', t) &= \mathrm{e}^{2\lambda}h(x,y,t) \end{aligned} \tag{4.4a}$$

where

$$x=\mathrm{e}^{-\lambda}x'-X(t'-t_0) \qquad y=\mathrm{e}^{-\lambda}y'-Y(t'-t_0) \qquad t=t'-t_0 \tag{4.4b}$$

with

$$\begin{aligned} X(t) &= p_1\cos R_1t+p_2\sin R_1t+p_3\cos R_2t+p_4\sin R_2t \\ Y(t) &= \frac{1}{f}\left(R_1-\sqrt{\frac{A}{B}}R_2\right)(-p_1\sin R_1t+p_2\cos R_1t) \\ &\quad +\frac{1}{f}\left(R_2-\sqrt{\frac{A}{B}}R_1\right)(-p_3\sin R_2t+p_4\cos R_4t). \end{aligned} \tag{4.4c}$$

Notice that in (4.4), R_1, R_2, f, A and B are parameters of the problem (R_1 and R_2 are given in terms of A, B and f in (3.10)). On the other hand, λ, t_0, p_1, p_2, p_3 and p_4 are group parameters, that is, real numbers that we can choose arbitrarily.

Equation (4.4) provides a generalisation and group theoretical justification of a theorem due to Ball [10]. Indeed, $X(t)$ and $Y(t)$ of (4.4c) satisfy equations (3.9) for β and γ. Ball interpreted these X and Y to be the coordinates of the centre of gravity of a given volume of liquid.

$$X=\int\!\!\int hx\,\mathrm{d}S\left(\int\!\!\int h\,\mathrm{d}S\right)^{-1}\qquad Y=\int\!\!\int hy\,\mathrm{d}S\left(\int\!\!\int h\,\mathrm{d}S\right)^{-1}. \tag{4.5}$$

Then from the fact that u and u' are solutions of the shallow water equations, follows Ball's statement that the form of the system govering the motion of the liquid relative to its centre of gravity is identical to that of the original shallow water system.

The symmetry group can in particular be used to generate a non-trivial exact solution from a solution in which the liquid is orginally at rest, that is

$$u(x,y,t)=0\qquad v(x,y,t)=0\qquad h(x,y,t)=-(Ax^2+By^2)+h_0. \tag{4.6}$$

The solution we obtain from (4.4) is

$$\begin{aligned}&u'(x',y',t')=\mathrm{e}^{\lambda}\dot{X}(t'-t_0)\qquad v'(x',y',t')=\mathrm{e}^{\lambda}\dot{Y}(t'-t_0)\\&h'(x',y',t')=\mathrm{e}^{2\lambda}\{h_0-A[\mathrm{e}^{-\lambda}x'-X(t'-t_0)]^2-B[\mathrm{e}^{-\lambda}y'-Y(t'-t_0)]^2\}\end{aligned} \tag{4.7}$$

with X and Y as in (4.4c). This is a displacement-type solution, described in detail by Ball [9].

Let us now restrict ourselves to the case of a circular paraboloid, that is $A=B$, and consider the group transformations corresponding to the additional elements R, K_1 and K_2 of the symmetry algebra (3.11b).

For convenience, we introduce polar coordinates

$$x=r\cos\theta\qquad y=r\sin\theta \tag{4.8}$$

and also the radial and tangential components of the velocity:

$$v_1=\cos\theta u+\sin\theta v\qquad v_2=-\sin\theta u+\cos\theta v. \tag{4.9}$$

As was mentioned in § 3, the vector fields L_3, K_1 and K_2 of (3.11b) and (3.13) generate an sl(2, $\boldsymbol{R}$) Lie algebra. A general element of the corresponding Lie group can be written as

$$G=\mathrm{e}^{L_3\beta}\mathrm{e}^{K_1\lambda}\mathrm{e}^{L_3\alpha} \tag{4.10}$$

where β, λ, α are group parameters (β and α are angles, i.e. $0\leqslant\alpha,\ \beta<2\pi$, λ is a 'boost' parameter, i.e. $0\leqslant\lambda<\infty$). To obtain the transformation (4.10) in our case, it is hence sufficient to integrate separately the vector fields L_3 and K_1, and then to compose the results.

Let us start with the 'rotation' L_3, which is actually a combination of a physical rotation R and a time translation T (3.13).

Integrating as in (4.1), we obtain

$$\begin{aligned}&u'(x',y',t')=\cos\frac{f\alpha}{2\omega}\,u(x,y,t)+\sin\frac{f\alpha}{2\omega}\,v(x,y,t)\\&v'(x',y',t')=-\sin\frac{f\alpha}{2\omega}\,u(x,y,t)+\cos\frac{f\alpha}{2\omega}\,v(x,y,t)\\&h'(x',y',t')=h(x,y,t)\end{aligned} \tag{4.11a}$$

where on the right-hand side of (4.11), we have

$$x=\cos\frac{f\alpha}{2\omega}x'-\sin\frac{f\alpha}{2\omega}y' \qquad y=\sin\frac{f\alpha}{2\omega}x'+\cos\frac{f\alpha}{2\omega}y' \qquad t=t'-\frac{\alpha}{\omega}. \tag{4.11b}$$

In polar coordinates, this transformation simplifies to

$$v_1'(r',\theta',t')=v_1(r,\theta,t) \qquad v_2'(r',\theta',t')=v_2(r,\theta,t) \qquad h'(r',\theta',t')=h(r,\theta,t)$$
$$r=r' \qquad \theta=\theta'+\frac{f\alpha}{2\omega} \qquad t=t'-\frac{\alpha}{\omega}. \tag{4.12}$$

Let us now find the group transformation corresponding to the vector field K_1 of (3.11*b*). A straightforward, though somewhat lengthy, computation yields the following result in polar coordinates:

$$h'(r',\theta',t')=h(r,\theta,t)[\cosh\lambda+\sinh\lambda\cos\omega t']^{-1}$$
$$v_1'(r',\theta',t')=v_1(r,\theta,t)[\cosh\lambda+\sinh\lambda\cos\omega t']^{-1/2}$$
$$-\frac{\omega'}{2}\sinh\lambda\sin\omega t'[\cosh\lambda+\sinh\lambda\cos\omega t']^{-1}$$
$$v_2'(r',\theta',t')=v_2(r,\theta,t)[\cosh\lambda+\sinh\lambda\cos\omega t']^{-1/2}$$
$$+\frac{fr'}{2}(1-\cosh\lambda-\sinh\lambda\cos\omega t')(\cosh\lambda+\sinh\lambda\cos\omega t')^{-1} \tag{4.13}$$
$$t=\frac{2}{\omega}\tan^{-1}\left[e^{-\lambda}\tan\frac{\omega t'}{2}\right] \qquad r=r'[\cosh\lambda+\sinh\lambda\cos\omega t']^{-1/2}$$
$$\theta=\theta'+\tfrac{1}{2}ft'-\frac{f}{\omega}\tan^{-1}\left[e^{-\lambda}\tan\frac{\omega t'}{2}\right].$$

The most general group transformation leaving the shallow water equations for a circular paraboloidal basin invariant is obtained as follows. First perform a rotation (4.12) through the angle β, then a boost (4.13) and a further rotation (4.12) through a different angle α. Then compose the obtained transformation with the remaining transformation (4.4).

As a special case, we obtain a result similar to one obtained by Ball [9, 10]. Consider a four-dimensional subalgebra of the symmetry algebra, the general element of which is

$$X=aK_1+bK_2+cL_3+pT \qquad T=\omega L_3-\tfrac{1}{2}fR. \tag{4.14}$$

Integrating (4.14), we obtain a transformation that can be written as

$$t'=\phi^{-1}[\lambda+\phi(t)] \qquad \rho'=\rho\left(\frac{I}{I_0}\right)^{1/2} \qquad \frac{\mathrm{d}t'}{\mathrm{d}t}=\frac{I}{I_0} \qquad \theta'=\theta+\int_t^{t'}\left(\frac{I}{I(s)}-\frac{f}{2}\right)\mathrm{d}s$$
$$h'=h\frac{I_0}{I} \qquad v_1'=v_1\left(\frac{I_0}{I}\right)^{1/2}+\frac{1}{2I}\rho'[\dot I-\dot I_0] \qquad v_2'=v_2\left(\frac{I_0}{I}\right)^{1/2}+\tfrac{1}{2}f\rho'\left[\frac{I_0}{I}-1\right]. \tag{4.15}$$

We have put

$$\phi(t)=\int_0^t \frac{\mathrm{d}s}{I(s)} \qquad I\equiv I(t')=\frac{1}{\omega}[a\sin\omega t'+b\cos\omega t'+c+p\omega] \qquad I_0=I(t) \tag{4.16}$$

where $I(t')$ satisfies the equation

$$\ddot{I}(t')+\omega^2 I(t')=\omega(c+\omega p)=4K \qquad J=fp/2$$

and by ϕ^{-1}, we denote the inverse of the function ϕ.

Formulae (4.15) should be compared with Ball's [10] formulae (25)-(30). Their form is the same; however our $I_0(t)$ is a function of time, his I_0 is constant. Our functions $h'(\rho', \theta', t')$, $v_1'(\rho', \theta', t')$, and $v_2'(\rho', \theta', t')$ satisfy exactly the same equations as h, v_1 and v_2, whereas Ball obtains modified equations in which the parameters f and A of the problem are changed. Therefore his transformations do not in general correspond to a symmetry group of the problem.

The quantity $I(t)$ in (4.16) is interpreted in [10] as the moment of inertia of the liquid about a verical axis through the origin

$$I=\int\int h\rho^2\,\mathrm{d}S. \tag{4.17}$$

the constants K and J are then the 'absolute energy' of the system and its total absolute angular momentum about a vertical axis through the origin

$$K=E+\tfrac{1}{2}fJ \qquad J=\int\int hr(\rho+\tfrac{1}{2}f\rho)\,\mathrm{d}S \tag{4.18}$$

and E is the total energy of the liquid

$$E=\int\int \tfrac{1}{2}h(v_1^2+v_2^2+h+2A\rho^2)\,\mathrm{d}S. \tag{4.19}$$

5. Group-invariant solutions for the circular paraboloid

Having found the symmetry group of the shallow water equations (2.18) for the case of an elliptic or circular paraboloid basin (3.8), we now wish to use this group to reduce the system to one in fewer independent variables, and ultimately to obtain particular solutions. The method to be applied is called symmetry reduction and it leads to 'group-invariant solutions', that is to solutions invariant under a subgroup of the symmetry group of the equation. We are interested in subgroups $G_0 \subset G$ that will reduce the system (2.18) to a system of ordinary differential equations (ODEs). The condition for this to be so is that the generic orbits of the group G_0, when acting on the space $\{x, y, t, u, v, h\}$ have dimension 4 and that their projection onto the space $\{x, y, t\}$ have dimension 1. This already implies that G_0 is at least two dimensional. Moreover, in view of the structure of the Lie algebras L_E and L_C, G_0 must be precisely two dimensional.

The method consists of several steps, to be performed separately for the elliptic and circular cases.

1. Find all conjugacy classes of two-dimensional subalgebras of the appropriate Lie algebra $L_0 \subset L$. Conjugacy is to be considered under the corresponding symmetry group G. Choose a representative of each conjugacy class.

2. Find the invariants of the action of the Lie group $G_0 = \exp L_0$ when acting on the space of independent and dependent variables, for each subgroup in the list.

3. Express the original equations in terms of these invariants. This will provide a system of three ODEs.

4. Solve (if possible) the obtained ODEs.

The shallow water equations (2.18) can, for the boundary (3.8), be written as

$$\begin{aligned} &u_t + uu_x + vu_y + 2Ax + h_x - fv = 0 \\ &v_t + uv_x + vv_y + 2By + h_y + fu = 0 \qquad h_t + (uh)_x + (vh)_y = 0. \end{aligned} \tag{5.1}$$

In addition to the invariance group (4.4) found in § 4, we note that the system (5.1) is invariant under a discrete group, generated by the reflections

$$\begin{aligned} &x' = \varepsilon_x x \qquad y' = \varepsilon_y y \qquad t' = \varepsilon_x \varepsilon_y t \qquad u' = \varepsilon_y u \\ &v' = \varepsilon_x v \qquad h' = h \qquad \varepsilon_x = \pm 1 \qquad \varepsilon_y = \pm 1. \end{aligned} \tag{5.2}$$

We now turn to the case of the circular paraboloidal basin, that is the case with $A = B$ in (3.8). The Lie algebra L_C of § 3 is nine dimensional. The commutation relations are given in table 1 and the subgroup structure is quite complicated. We are only interested in two-dimensional subalgebras. Classifying them using the algorithm developed in [22, 23] and described also in [24], we find that every two-dimensional subalgebra of L_C is conjugate, under the symmetry group G_C of the considered equations, to precisely one of the subalgebras, given in table 2. The classification group G_C includes the discrete transformations (5.2)).

Table 2. Two-dimensional subalgebras of the symmetry algebra for the circular basin. Throughout a, b and ε are real parameters with $a \geqslant 0$, $\varepsilon = \pm 1$ and b unrestricted, unless a restriction is indicated. The algebras A_i are Abelian, B_i non-Abelian. For the B algebras, the commutation relation in the basis given below is $[X, Y] = -Y$.

No	Basis	No	Basis
A_1	$\{R, L_3\}$	A_{14}	$\{Y_1, Y_2 + Y_3\}$
A_2	$\{R, K_1\}$	A_{15}	$\{Y_3, Y_4\}$
A_3	$\{R, K_2 + L_3\}$	A_{16}	$\{Y_3, Y_1 + Y_4\}$
A_4	$\{R, D\}$	A_{17}	$\{Y_1 + Y_4, Y_2 + Y_3\}$
A_5^b	$\{R, K_1 + bD, b \neq 0\}$	A_{18}	$\{K_2 + L_3, Y_1 + Y_3\}$
A_6^a	$\{R, L_3 + aD, a > 0\}$	A_{19}	$\{D + 2K_1, Y_1 - Y_3\}$
A_7	$\{R, K_2 + L_3 + D\}$	A_{20}	$\{R + 2L_3, Y_1\}$
A_8^{ab}	$\{K_1 + bR, D + aR\}$	A_{21}	$\{R - 2L_3, Y_3\}$
A_9^{ab}	$\{L_3 + bR, D + aR\}$	A_{22}^b	$\{K_2 + L_3 + Y_1 + bY_2, Y_1 + Y_3\}$
A_{10}^a	$\{K_2 + L_3, D + aR\}$	A_{23}	$\{K_2 + L_3 + Y_2, Y_1 + Y_3\}$
$A_{11}^{\varepsilon a}$	$\{K_2 + L_3 + \varepsilon R, D + aR\}$	A_{24}	$\{D + 2K_1 + Y_2, Y_1 - Y_3\}$
A_{12}	$\{Y_1, Y_2\}$	A_{25}	$\{R + 2L_3 + Y_2, Y_1\}$
A_{13}	$\{Y_1, Y_3\}$	A_{26}	$\{R - 2L_3 + Y_4, Y_3\}$
B_{27}^{ab}	$\{K_1 + aR + bD, K_2 + L_3\}$	B_{34}^a	$\{(2L_3 + aD - R)/2a, Y_3, a \neq 0\}$
B_{28}	$\{-2K_1, Y_1 - Y_3\}$	B_{35}	$\{K_2 + L_3 + D, Y_1 + Y_3\}$
B_{29}	$\{D, Y_1\}$	B_{36}	$\{(2K_1 + 3D)/2, K_2 + L_3 + Y_1 - Y_3\}$
B_{30}	$\{D, Y_3\}$	B_{37}	$\{(2K_1 - D + 2Y_1)/2, 2K_2 + 2L_3 - Y_2 + Y_4\}$
B_{31}	$\{D, Y_1 + Y_3\}$	B_{38}^a	$\{(-2K_1 + D - Y_1 - aY_2)/2, Y_1 - Y_3\}$
B_{32}^b	$\{[2/(2b-1)](K_1 + bD), Y_1 - Y_3, b \neq \frac{1}{2}\}$	B_{39}	$\{(-2K_1 + D - 2Y_2)/2, Y_1 - Y_3\}$
B_{33}^a	$\{(2L_3 + aD + R)/2a, Y_1, a \neq 0\}$		

In view of the large number of conjugacy classes of subalgebras and subgroups, we shall not make use of all of them. We shall concentrate on the simpler ones among those that involve rotational invariance, i.e. the subalgebras $A_1, \ldots, A_4$ containing R, or those that are contained in the ideal N, i.e. $A_{12}, \ldots, A_{17}$. Some other subalgebras are treated in the context of the elliptic paraboloidal basin.

Let us start with the algebras $A_{12}, \ldots, A_{17}$. The expressions (3.11*a*) simplify when we put $A = B$ and use (3.10*b*).

1. $A_{12} = \{Y_1, Y_2\}$. We find the invariants of the corresponding subgroup by putting

$$Y_i \Phi(x, y, t, u, v, h) = 0 \qquad i = 1, 2 \tag{5.3}$$

where Φ is the general invariant. Solving (5.3), we find the elementary invariants t, h, $\alpha = u - R_1 y$, $\beta = v + R_1 x$. Hence, the reduction to an ODE is obtained by putting

$$u = R_1 y + \alpha(t) \qquad v = -R_1 x + \beta(t) \qquad h = h(t). \tag{5.4}$$

Substituting into (5.1) for $A = B$, we obtain

$$\dot{h} = 0 \qquad \dot{\alpha} + R_2 \beta = 0 \qquad \dot{\beta} - R_2 \alpha = 0. \tag{5.5}$$

These equations can be immediately solved to yield the solution

$$h = h_0 \qquad u = R_1 y + \alpha_0 \cos(R_2 t + \theta_0) \qquad v = -R_1 x + \alpha_0 \sin(R_2 t + \theta_0) \tag{5.6a}$$

where h_0, α_0 and θ_0 are integration constants. In the polar coordinate frame, (5.6*a*) is replaced by

$$h = h_0 \qquad v_1 = \alpha_0 \cos(R_2 t - \theta + \theta_0) \qquad v_2 = -R_1 r + \alpha_0 \sin(R_2 t - \theta + \theta_0). \tag{5.6b}$$

Since we have $h = h_0 =$ constant, this solution corresponds to a layer of fluid, the height of which is h_0 (from the underlying surface). Applying the group transformation (4.13), we obtain a class of solutions, for which the surface has the form

$$h = \frac{h_0}{\cosh \lambda + \sinh \lambda \cos \omega t} \tag{5.7}$$

and the velocities are obtained by inserting v_1 and v_2 of (5.6*b*) into (4.13).

2. $A_{15} = \{Y_3, Y_4\}$. The result in this case is quite similar, namely

$$h = h_0 \qquad u = -R_2 y + \alpha_0 \cos(R_1 t - \theta + \theta_0) \qquad v = R_2 x + \alpha_0 \sin(R_1 t - \theta + \theta_0) \tag{5.8}$$

and applying (4.13), we again obtain (5.7).

3. $A_{13} = \{Y_1, Y_3\}$. Here, we have

$$\begin{aligned} h &= \frac{h_0}{\sin(R_1 + R_2) t} \\ u &= R_1 y - (R_1 + R_2) \frac{\sin R_2 t}{\sin(R_1 + R_2) t} (x \sin R_1 t + y \cos R_1 t) + \alpha(t) \\ v &= -R_1 x + (R_1 + R_2) \frac{\cos R_2 t}{\sin(R_1 + R_2) t} (x \sin R_1 t + y \cos R_1 t) + \beta(t) \end{aligned} \tag{5.9}$$

where α and β satisfy a pair of coupled linear equations, namely

$$\begin{aligned}
&\dot{\alpha}\sin(R_1+R_2)t-\alpha(R_1+R_2)\sin R_1t\sin R_2t\\
&\qquad+\beta[R_2\sin R_1t\cos R_2t-R_1\cos R_1t\sin R_2t]=0\\
&\dot{\beta}\sin(R_1+R_2)t+\alpha[R_1\sin R_1t\cos R_2t-R_2\cos R_1t\sin R_2t]\\
&\qquad+\beta(R_1+R_2)\cos R_1t\cos R_2t=0.
\end{aligned}\tag{5.10}$$

4. $A_{14}=\{Y_1, Y_2+Y_3\}$, $A_{16}=\{Y_3, Y_1+Y_4\}$, $A_{17}=\{Y_1+Y_4, Y_2+Y_3\}$. In all these cases, we obtain

$$h=\frac{h_0}{1+\sin(R_1+R_2)t}\tag{5.11}$$

$$u=[P_1(t)x+Q_1(t)y]+\alpha(t)\qquad v=[P_2(t)x+Q_2(t)y]+\beta(t)\tag{5.12}$$

where P_1, Q_1, P_2 and Q_2 are known trigonometric functions of t with periodically spaced singularities on the real time axis. The functions $\alpha(t)$ and $\beta(t)$ again satisfy a pair of coupled first-order linear differential equations with periodic coefficients.

At this stage, we note that all the solutions obtained so far are what Ball [9] calls 'deformation' solutions (uniform rotation, expansion and distorsion of the liquid) with u and v linear in x and y. The solutions (5.6), . . . , (5.8) are finite; all the other solutions have singularities on the real time axis and are hence non-physical. From the group theoretical point of view, the subalgebras A_{12} and A_{15}, yielding finite solutions, are distinguished by the fact that the subspaces $\{Y_1, Y_2\}$ and $\{Y_3, Y_4\}$ are respectively positive and negative definite in the metric introduced at the end of § 3. The subalgebras A_{14} and A_{16} correspond to vector spaces with degenerate metrics; A_{17} is completely isotropic.

We now turn to subalgebras containing the rotation R and hence to rotationally invariant solutions. In polar coordinates, we have $R=-\partial/\partial\theta$ and hence any reduction involving R will imply that v_1, v_2 and h are independent of θ. The shallow water equations (2.21) in the cylindrical frame then reduce to

$$\begin{aligned}
&h_t+v_1h_r+\frac{h}{r}v_1+hv_{1,r}=0\\
&v_{1,t}+v_1v_{1,r}+2Ar+h_r-fv_2-\frac{1}{r}v_2^2=0\\
&v_{2,t}+v_1v_{2,r}+fv_1+\frac{1}{r}v_1v_2=0.
\end{aligned}\tag{5.13}$$

Among the algebras $A_1, \ldots, A_7$ leading to further reductions of (5.13), we confine our attention here to $A_1, \ldots, A_4$.

5. $A_1=\{R, L_3\}$. The corresponding subgroup leads to static solutions. We have h, v_1 and v_2 as functions of r alone, satisfying

$$\begin{aligned}
&(hv_1)\ +\frac{1}{r}(hv_1)=0\qquad v_1\left(\dot{v}_2+f+\frac{1}{r}v_2\right)=0\\
&v_1\dot{v}_1+\dot{h}+2Ar-fv_2-\frac{1}{r}v_2^2=0.
\end{aligned}\tag{5.14}$$

For the case of vanishing radial velocity, we obtain

$$v_1 = 0 \qquad v_2 = v_2(r) \qquad h = -Ar^2 + h_0 + \int \left(fv_2 + \frac{1}{r} v_2^2 \right) \mathrm{d}r \tag{5.15}$$

where the angular velocity v_2 is an arbitrary function of the radial distance r. In general, (5.15) represents a higher mode solution [9, 15]. In the special case of constant angular velocity ω_0, we have

$$v_1 = 0 \qquad v_2 = \omega_0 r \qquad h = h_0 + [\tfrac{1}{2}\omega_0(f_0 + \omega_0) - A]r^2. \tag{5.16}$$

This is a deformation mode motion; the surface acquires a paraboloidal shape. For $v_1 \neq 0$, we obtain

$$h = \frac{h_0}{rv_1} \qquad v_2 = -\tfrac{1}{2}fr + \frac{c_1}{r} \tag{5.17a}$$

and the radial velocity satisfies a cubic equation

$$v_1^3 + \left(\tfrac{1}{4}\omega r^2 + c_1^2 \frac{1}{r^2} + c_2 \right) v_1 + \frac{2h_0}{r} = 0 \tag{5.17b}$$

where h_0, c_1 and c_2 are integration constants. Notice that for $r \to 0$, (5.17b) implies $rv_1 \to 0$ and hence $h \to \infty$ for $r \to 0$. The angular velocity v_2 also diverges for $r \to 0$, unless we have $c_1 = 0$. Thus we see that (5.17) represents a higher-mode solution, though clearly a non-physical one: it corresponds to water piling up at the centre of the basin.

6. $A_2 = \{R, K_1\}$. The reduction formulae are

$$h = \frac{\alpha(\xi)}{r} \qquad v_1 = \frac{\omega r}{2} \cot \omega t + \frac{\gamma(\xi)}{r} \qquad v_2 = -\tfrac{1}{2}fr + \frac{\beta(\xi)}{r} \qquad \xi = \frac{1}{r^2} \sin \omega t \tag{5.18}$$

with

$$\alpha\gamma = \frac{c_1}{\xi} \qquad \dot{\beta}\gamma = 0 \qquad (\gamma^2\xi)^{\cdot} + 2(\alpha\xi)^{\cdot} + \beta^2 - \frac{\omega^2}{4\xi^2} = 0. \tag{5.19}$$

For $\gamma = 0$, we obtain

$$v_1 = \tfrac{1}{2}\omega r \cot \omega t \qquad v_2 = -\tfrac{1}{2}fr + \frac{\beta(\xi)}{r} \qquad h = -\frac{1}{2 \sin \omega t} \left(\frac{\omega^2 r^2}{4 \sin \omega t} + h_0 + \int \beta^2(\xi)\, \mathrm{d}\xi \right) \tag{5.20}$$

where $\beta(\xi)$ is an arbitrary function. We again note that this is a higher-mode solution, albeit a non-physical one.

For $\gamma \neq 0$, we have $\beta = \beta_0 =$ constant and obtain

$$h = \alpha(\xi) \frac{1}{r^2} \qquad v_1 = \frac{\omega r}{2} \cot \omega t + \frac{c_1 r}{\sin \omega t \alpha(\xi)} \qquad v_2 = -\tfrac{1}{2}fr + \beta_0 \frac{1}{r} \tag{5.21}$$

where $\alpha(\xi)$ satisfies a cubic equation

$$2(\alpha\xi)^3 + \left(\beta_0^2 \xi + \frac{\omega^2}{4} \frac{1}{\xi} + c_2 \right) \alpha^2 \xi^2 + c_1^2 \xi = 0. \tag{5.22}$$

For $\xi \to 0$, i.e. $\omega t \to k\pi$, (5.22) has two imaginary solutions for $\alpha\xi$ and one real one, behaving as

$$\alpha\xi \to -\frac{\omega^2}{8}\frac{1}{\xi} \qquad \text{i.e.} \qquad \alpha(\xi) \xrightarrow[\omega t \to k\pi]{} -\frac{\omega^2}{8}\frac{r^4}{\sin^2 \omega t}. \tag{5.23}$$

Thus $h(\xi)$ can be singular for $r \to 0$ and for $t \to k\pi/\omega$.

7. $A_3 = \{R, K_2 + L_3\}$. Here we obtain

$$h = \frac{\alpha(\xi)}{r^2} \qquad v_1 = -\frac{\omega}{2} r \tan\frac{\omega t}{2} + \frac{\gamma(\xi)}{r} \qquad v_2 = -\tfrac{1}{2} f r + \frac{\beta(\xi)}{r} \qquad \xi = \frac{1}{r^2}\cos^2\frac{\omega t}{2} \tag{5.24}$$

with α, β and γ satisfying

$$\alpha\gamma = \frac{c_1}{\xi} \qquad \dot{\beta}\gamma = 0 \qquad 2(\alpha\xi)^{\cdot} + (\gamma^2\xi)^{\cdot} + \beta^2 = 0. \tag{5.25}$$

For $\gamma = 0$, we obtain

$$h = \frac{1}{2\cos^2(\omega t/2)}\left(h_0 - \int_0^{\xi} \beta^2 \, d\xi'\right) \qquad v_1 = -\frac{\omega}{2} r \tan\frac{\omega t}{2} \qquad v_2 = -\tfrac{1}{2} f r + \frac{\beta(\xi)}{r} \tag{5.26}$$

where $\beta(\xi)$ is an arbitrary function. We see that the solution is singular; that is, v_1 diverges for $\omega t = (2k+1)\pi$, k = integer.

For $\gamma \neq 0$, we have $\beta = \beta_0$ = constant and

$$h = \frac{\alpha(\xi)}{r^2} \qquad v_1 = -\frac{\omega}{2} r \tan\frac{\omega t}{2} + \frac{c_1 r}{\alpha \cos^2(\omega t/2)} \qquad v_2 = -\tfrac{1}{2} f r + \frac{\beta_0}{r} \tag{5.27}$$

where $\alpha(\xi)$ satisfies a cubic equation

$$2(\alpha\xi)^3 + (\beta_0^2\xi^3 + c_2\xi^2)(\alpha\xi) + c_1^2\xi = 0. \tag{5.28}$$

8. $A_4 = \{R, D\}$. In this case, the reduction is

$$h = r^2\alpha(t) \qquad v_1 = r\gamma(t) \qquad v_2 = r\beta(t) \tag{5.29}$$

where

$$\dot{\alpha} + 4\alpha\gamma = 0 \qquad \dot{\beta} + 2\beta\gamma + f\gamma = 0 \qquad \dot{\gamma} + \gamma^2 - \beta^2 + 2\alpha + 2A - f\beta = 0. \tag{5.30}$$

We proceed only with the case $\alpha \neq 0$ so that $h \neq 0$. Then the system (5.30) yields

$$\ddot{\alpha} = \frac{5}{4}\frac{\dot{\alpha}^2}{\alpha} + 4(2 - c_1^2)\alpha^2 + \omega^2\alpha \tag{5.31}$$

where β, γ are given in terms of α by

$$\beta = -\tfrac{1}{2} f + c_1\sqrt{\alpha} \qquad \gamma = -\frac{1}{4\alpha}\dot{\alpha} \qquad (c_1 = \text{constant}). \tag{5.32}$$

If we now set

$$\Omega = \alpha^{-1/4} \tag{5.33}$$

then (5.31) reduces to

$$\ddot{\Omega}+\frac{\omega^2}{4}\Omega=(c_1^2-2)\Omega^{-3}. \tag{5.34}$$

The latter is a special case of Pinney's equation [25]. If h and $\dot{h}$ are known, for $t=0$, then the initial data

$$\Omega(0)=\omega_0 \qquad \dot{\Omega}(0)=v_0 \tag{5.35}$$

are determined. The initial value problem consisting of the Pinney equation (5.34) subject to the initial values (5.35) is readily solved to yield

$$\Omega=[(\omega_0\cos\mu t+(v_0/\mu)\sin\mu t)^2+(1/\omega_0)^2\sin^2\mu t]^{1/2} \tag{5.36}$$

where $\mu=\omega/2$ and $\omega_0^2=\frac{1}{8}(8-\omega^2-4c_1^2)v_0^{-2}$. Thus, h is given at time $t\geqslant 0$ by

$$h=\frac{r^2}{\Omega^4}=\frac{r^2}{[(\omega_0\cos\mu t+(v_0/\mu)\sin\mu t)^2+(1/\omega_0)^2\sin^2\mu t]^2} \tag{5.37a}$$

while the velocity components v_1 and v_2 are expressed in terms of Ω by

$$v_1=r+\frac{\dot{\Omega}}{\Omega} \qquad v_2=r(-\tfrac{1}{2}f+c_1\Omega^{-2}). \tag{5.37b}$$

This solution corresponds to a layer of water of finite depth, pulsating with frequency $\mu=\omega/2$.

It is of interest to note that recently another solution of the present shallow water system in a circular paraboloidal basin has been derived based on Pinney's equation [26]. It cannot be obtained directly by symmetry reduction from the system (5.13), nor by applying the group transformations of § 4 to a solution obtained by symmetry reduction. It does however have a group theoretical interpretation. The transformations of the symmetry group G of an equation can be applied to an arbitrary solution of the equation. If, on the other hand, we restrict ourselves to particular types of solutions, then in some cases, it is possible to find a larger group $G_p\supset G$ that transforms the particular solutions into more general solutions of the considered equation.

Let us consider the solutions (5.37) from this point of view. We apply a transformation that adds a functions $\mu(t)$ to the depth function h, while leaving the velocities unchanged. Thus, we put

$$h=r^2\alpha(t)+\mu(t) \qquad v_1=r\gamma(t) \qquad v_2=r\beta(t). \tag{5.38}$$

We find that α, β and γ satisfy equations (5.31) and (5.32) as before, whereas μ satisfies

$$\dot{\mu}-2\mu\gamma=0. \tag{5.39}$$

This can be solved and we obtain the new solution

$$h=r^2\Omega^{-4}-\mu_0\Omega^{-2} \qquad v_1=r\dot{\Omega}\Omega^{-1} \qquad v_2=r(-\tfrac{1}{2}f+c_1\Omega^{-2}) \tag{5.40}$$

where μ_0 is a constant and Ω is a solution (5.36) of Pinney's equation (5.34).

The solution (5.40) corresponds to a moving shoreline that is a circle with a radius oscillating in accordance with Pinney's equation. Indeed, putting $h(r,t)=0$, we obtain the shoreline, where the free surface meets the paraboloid, namely

$$r^2=x^2+y^2=\mu_0\Omega^2 \qquad \mu_0>0. \tag{5.41}$$

It is worth mentioning that Pinney's equation also arises in a similar manner in elastodynamics in the context of the large-amplitude radial oscillator of thin-shelled tubes of neo-Hookean material (Shahinpoor and Nowinski [27], Rogers and Ames [28]).

In conclusion, it is noted that symmetry reductions to ordinary differential equations are readily obtained for the other subalgebras involving R, namely A_5^b, A_6^a and A_7 are readily obtained. However, in view of the complexity of the resulting equations, these subgroups are not considered here.

6. Group-invariant solutions for the elliptic paraboloidal basin

We now turn to the Lie algebra L_E of (3.11a), corresponding to the symmetry group G for $A \neq B$. Any two-dimensional subalgebra of L_E is conjugate under the symmetry group of the shallow water equations to precisely one algebra in table 3. Notice that all these algebras also exist for $A = B$, that is the circular paraboloidal basin. For $A = B$, the transformations corresponding to $\{K_1, K_2, L_3\}$ can be used for further simplication. As we have seen above, the action of $\{T, K_1, K_2, L_3\}$ on the space $\{Y_1, \ldots, Y_4\}$ introduces an invariant O(2, 2) metric on this space, which influences the correspondence between subalgebras of tables 2 and 3.

Table 3. Two-dimensional subalgebras of the symmetry algebra L_E for the elliptic basin. The parameters a, b and c are real, $b \geq 0$. The algebras M_i are Abelian, N_i non-Abelian.

No	Basis	No	Basis
M_1	$\{Y_3, Y_4\}$	N_1	$\{D, Y_3\}$
M_2^a	$\{Y_1 + aY_3, Y_4\}$	N_2^{ab}	$\{D, Y_1 + aY_3 + bY_4\}$
M_3^{ab}	$\{Y_1 + aY_4, Y_3 + bY_4\}$ (for $b = 0$, we have $a \geq 0$)		
M_4^{abc}	$\{Y_1 + aY_3 + bY_4, Y_2 - bY_3 + cY_4\}$		
M_5	$\{D, T\}$		

For $A = B$, we obviously have $M_1 = A_{15}$, $M_5 = A_9^{ab}$ ($a = 0$, $b = -f/(2\omega)$) and $N_1 = B_{30}$. The signature of the space M_2 can be $(+, -)$, $(--)$ or $(0-)$ and M_2 reduces to A_{13}, A_{15}, or A_{16} for $a^2 < 1$, $a^2 > 1$, or $a^2 = 1$, respectively. Similarly, M_3 reduces to A_{13}, A_{15}, or A_{16} for $a^2/(1+b^2) < 1$, >1, or $=1$, respectively. For M_4, we can also determine the signature of the corresponding space. The result is that for $A = B$, we have

$$
\begin{array}{llll}
M_4 = A_{12} & \text{if} & 1 - b^2 - c^2 > 0 & (1 - b^2 - a^2)(1 - b^2 - c^2) > b^2(a-c)^2 \\
M_4 = A_{13} & \text{if} & 1 - b^2 - c^2 > 0 & (1 - b^2 - a^2)(1 - b^2 - c^2) < b^2(a-c)^2 \\
 & \text{or} & 1 - b^2 - c^2 < 0 & (1 - b^2 - a^2)(1 - b^2 - c^2) > b^2(a-c)^2 \\
M_4 = A_{14} & \text{if} & 1 - b^2 - c^2 > 0 & (1 - b^2 - a^2)(1 - b^2 - c^2) = b^2(a-c)^2 \\
 & \text{or} & b = 0 \quad c = \pm 1 & a^2 < 0 \\
M_4 = A_{15} & \text{if} & 1 - b^2 - c^2 < 0 & (1 - b^2 - a^2)(1 - b^2 - c^2) > b^2(a-c)^2 \\
M_4 = A_{16} & \text{if} & 1 - b^2 - c^2 < 0 & (1 - b^2 - a^2)(1 - b^2 - c^2) = b^2(a-c)^2 \\
 & \text{or} & b = 0 \quad c = \pm 1 & a^2 > 0 \\
M_4 = A_{17} & \text{if} & b^2 + c^2 = 1 & c = a \\
 & \text{or} & b = 0 \quad c = 1 & a = \pm 1.
\end{array}
\qquad (6.1)
$$

Finally, N_2 for $A=B$ is equivalent to B_{29}, B_{30}, or B_{31}, for $a^2+b^2<1$, $a^2+b^2>1$, or $a^2+b^2=1$, respectively.

Let us now run through the individual subalgebras and perform the corresponding reductions of equations (5.1). To abbreviate formulae, we introduce the following notation:

$$R_0=\frac{1}{f}\left(R_2-\sqrt{\frac{A}{B}}R_1\right) \qquad S_0=\frac{1}{f}\left(R_1-\sqrt{\frac{A}{B}}R_2\right).$$

1. $M_1=\{Y_3, Y_4\}$. The reduction formulae in this case are

$$\begin{aligned} &u=\frac{R_2}{R_0}y+\alpha(t) \qquad v=-R_0R_2x+\beta(t) \qquad h=h(t)\\ &\dot{h}=0 \qquad \dot{\alpha}+\left(\frac{R_2}{R_0}-f\right)\beta=0 \qquad \dot{\beta}+(f-R_0R_2)\alpha=0. \end{aligned} \tag{6.2}$$

Solving these equations, we obtain a constant-layer deformation mode solution;

$$\begin{aligned} &h=h_0 \qquad u=\frac{R_2}{R_0}y+C\cos\sqrt{\Omega}(t-t_0)\\ &v=-R_0R_2x-\frac{f\sqrt{\Omega}}{2B+f^2-R_2^2}C\sin\sqrt{\Omega}\,(t-t_0) \qquad \Omega=\frac{2A}{R_2^2-2A}(R_2^2-2A-f^2). \end{aligned} \tag{6.3}$$

For $A=B$, this solution is equivalent to (5.8). The 'boost' transformation (4.13) is, however, not available in this case.

2. $M_2=\{Y_1+aY_3, Y_4\}$. The result of the reduction in this case is

$$\begin{aligned} &u=\sqrt{\frac{A}{B}}\frac{R_1^2-R_2^2}{R_0}\frac{1}{D(t)}[xR_0\cos R_2t-y\sin R_2t]\sin R_1t+\frac{R_2}{R_0}y+\alpha(t)\\ &v=-(R_1^2-R_2^2)\frac{1}{D(t)}[xR_0\cos R_2t-y\sin R_2t]\cos R_1t-R_2R_0x+\beta(t) \end{aligned} \tag{6.4a}$$

$$h=\frac{h_0}{D(t)} \qquad D(t)=R_0f(a+\cos R_1t\cos R_2t)+S_0f\sin R_1t\sin R_2t. \tag{6.4b}$$

The functions $\alpha(t)$ and $\beta(t)$ satisfy linear equations

$$\begin{aligned} &D\dot{\beta}+[(R_1^2-R_2^2)\cos R_1t\sin R_2t]\beta+\{R_0f[f-S_0R_1]\cos R_1t\cos R_2t\\ &\qquad +S_0\sin R_1t\sin R_2t+aR_0f[f-R_2R_0]\}\alpha=0\\ &D\dot{\alpha}+\left[\sqrt{\frac{A}{B}}(R_1^2-R_2^2)\sin R_1t\cos R_2t\right]\alpha\\ &\qquad +f\sqrt{\frac{A}{B}}[R_2\sin R_1t\sin R_2t+R_1\cos R_1t\cos R_2t+R_1a]\beta=0. \end{aligned} \tag{6.4c}$$

We again obtain a deformation mode solution. Singularities occur for $D(t)=0$. For $|a|>1$, these singularities will not occur for t real; for $a^2\leqslant 1$, $h(t)$, as well as u and v, will have singularities for $t\in\boldsymbol{R}$.

3. $M_3 = \{Y_1 + aY_4,\ Y_3 + bY_4\}$. Conceptually the results in this case are similar to those for M_1 and M_2. We have

$$u = \frac{1}{D(t)}(E_1(t)x + E_2(t)y) + \alpha(t)$$
$$v = \frac{1}{D(t)}(F_1(t)x + F_2(t)y) + \beta(t) \qquad h = \frac{h_0}{D(t)} \tag{6.5a}$$

$$D(t) = R_0 f[\sin R_2 t - b\cos R_2 t][\cos R_1 t + a \sin R_2 t] + [\cos R_2 t + b \sin R_2 t]$$
$$\times [aR_0 f \cos R_2 t - S_0 f \sin R_1 t]. \tag{6.5b}$$

The functions E_1, E_2, F_1 and F_2 are non-singular and may be determined explicitly but are not set down here; they depend on t via $\sin R_i t$ and $\cos R_i t$, $i = 1, 2$. The functions $\alpha(t)$ and $\beta(t)$ satisfy a coupled pair of linear ordinary differential equations. The singularities of h, u and v occur for $D(t) = 0$. There are no singularities for real t if $a^2 > 1 + b^2$; for $a^2 \leqslant b^2 + 1$ real time singularities do occur.

4. $M_4 = \{Y_1 + aY_3 + bY_4,\ Y_2 - bY_3 + cY_4\}$. The expressions for u, v and h again have the form (6.5a) and $\alpha(t)$, $\beta(t)$ again satisfy a system of linear ODEs. The denominator $D(t)$ of (6.5b) is replaced by

$$D(t) = [\cos R_1 t + a\cos R_2 t + b \sin R_2 t] f[S_0 \cos R_1 t + bR_0 \sin R_2 t + cR_0 \cos R_2 t]$$
$$-[\sin R_1 t - b\cos R_2 t + c \sin R_2 t]$$
$$\times f[bR_0 \cos R_2 t - S_0 \sin R_1 t - aR_0 \sin R_2 t]. \tag{6.6}$$

The solutions are non-singular if the equation $D(t) = 0$ has no solutions for real time t. This is the case for

$$\{1 - b^2 - c^2 > 0,\ (1 - b^2 - a^2)(1 - b^2 - c^2) > b^2(a - c)^2\}$$

and

$$\{1 - b^2 - c^2 < 0,\ (1 - b^2 - a^2)(1 - b^2 - c^2) > b^2(a - c)^2\}.$$

5. $M_5 = \{D, T\}$. Here we have

$$u = x\alpha(\xi) \qquad v = x\beta(\xi) \qquad h = x^2\gamma(\xi) \qquad \xi = y/x \tag{6.7}$$

and α, β, γ satisfy the following system of ODEs:

$$(\beta\gamma)^{\cdot} - \xi(\alpha\gamma)^{\cdot} + 3\alpha\gamma = 0$$
$$(\beta - \alpha\xi)\dot{\alpha} - \xi\dot{\gamma} + 2\gamma + \alpha^2 - f\beta + 2A = 0 \tag{6.8}$$
$$(\beta - \alpha\xi)\dot{\beta} + \dot{\gamma} + \alpha\beta + f\alpha + 2B\xi = 0.$$

(The dot denotes differentiation with respect to the argument ξ.)

While we are not able to solve the system (6.8) in its generality, we can obtain a one-parameter family of particular solutions. Indeed,

$$\alpha(\xi) = \alpha_0 \xi \qquad \beta(\xi) = \beta_0 \qquad \gamma(\xi) = \gamma_0 + \gamma_1 \xi^2 \tag{6.9}$$

provides a solution, if the four constants α_0, β_0, γ_0 and γ_1 satisfy the three equations

$$\gamma_0 = -A + \frac{f}{2}\beta_0 - \tfrac{1}{2}\alpha_0\beta_0 \qquad \gamma_1 = -B - \frac{f}{2}\alpha_0 - \tfrac{1}{2}\alpha_0\beta_0$$
$$\alpha_0\beta_0(\alpha_0 + \beta_0) + 2A\alpha_0 + 2B\beta_0 = 0. \tag{6.10}$$

Moreover, we can generate a family of further solutions from this one by the transformation

$$h = \gamma_0 x^2 + \gamma_1 y^2 - h_0 \qquad u = \alpha_0 y \qquad v = \beta_0 x \tag{6.11}$$

(where h_0 is the transformation parameter).

Setting $h = 0$, we get an equation for the shoreline, in this case a constant ellipse

$$\gamma_0 x^2 + \gamma_1 y^2 = h_0 \tag{6.12}$$

where the constants must be so chosen that γ_0, γ_1 and h_0 all have the same sign (if the amount of water and the dimensions of the basin are finite).

6. $N_1 = \{D, Y_3\}$. The reduction by the subgroup corresponding to N_1 yields deformation mode solutions of the form

$$\begin{aligned} u &= \frac{R_2}{R_0} y + (R_0 x \sin R_2 t + y \cos R_2 t)\alpha(t) \\ v &= -R_0 R_2 x + (R_0 x \sin R_2 t + y \cos R_2 t)\beta(t) \\ h &= (R_0 x \sin R_2 t + y \cos R_2 t)^2 \gamma(t) \end{aligned} \tag{6.13}$$

with

$$\begin{aligned} \dot\alpha &= -\left(\frac{R_2}{R_0} - f\right)\beta - 2R_0 \sin R_2 t\gamma - \alpha(R_0 \sin R_2 t\alpha + \cos R_2 t\beta) \\ \dot\beta &= (R_0 R_2 - f)\alpha - 2\cos R_2 t\gamma - \beta(R_0 \sin R_2 t\alpha + \cos R_2 t\beta) \\ \dot\gamma &= -3\gamma(R_0 \sin R_2 t\alpha + \cos R_2 t\beta). \end{aligned} \tag{6.14}$$

Since we did not discuss this reduction for the circular basin in the previous section, we shall now show how the equations simplify for $A = B$. Thus, as a special case, we have

$$A = B \qquad R_0 = -1 \qquad R_1 + R_2 = \omega \qquad R_1 - R_2 = f. \tag{6.15}$$

We put

$$\mu = -\sin R_2 t\alpha + \cos R_2 t\beta \qquad \lambda = \cos R_2 t\alpha + \sin R_2 t\beta. \tag{6.16}$$

Using (6.15) and (6.16), we reduce the ODEs (6.14) to

$$\dot\lambda = (\omega - \lambda)\mu \qquad \dot\mu = -\omega\lambda - 2\gamma - \mu^2 \qquad \dot\gamma = -3\gamma\mu. \tag{6.17}$$

For $\lambda = \omega$, we obtain

$$\gamma = -\tfrac{1}{2}(\dot\mu + \mu^2 + \omega^2) \tag{6.18a}$$

where μ satisfies

$$\ddot\mu + 5\mu\dot\mu + 3\mu\omega^2 = 0. \tag{6.18b}$$

For $\lambda \neq \omega$, we find

$$\lambda = \nu + \omega \qquad \mu = -\dot\nu/\nu \tag{6.19a}$$

where $\nu(t)$ satisfies

$$\ddot\nu = \dot\nu^2/\nu + \omega^2\nu + \omega\nu^2 - 2c_1\nu^4. \tag{6.19b}$$

The solution of the shallow water system in this case reduces to the solving of a single nonlinear ODE, namely (6.18*b*) or (6.19*b*). Moreover (6.19*b*) can be reduced to a first-order equation by putting

$$\nu=\frac{1}{z} \qquad \dot{z}^2=-\omega^2 z^2-2\omega z-\frac{4c_1}{z}+c_2 \tag{6.19c}$$

and the problem is reduced to quadratures. For example, for $c_1=0$, we obtain

$$\nu=\frac{\omega}{-1+\sqrt{1+c_2}\sin\omega(t-t_0)} \qquad \mu=\frac{\omega\sqrt{1+c_2}\cos\omega(t-t_0)}{-1+\sqrt{1+c_2}\sin\omega(t-t_0)}. \tag{6.20}$$

A real non-singular solution is obtained for

$$-1\leqslant c_2<0 \tag{6.21}$$

namely (6.13) with (6.15) and

$$\alpha=\frac{\omega\sqrt{1+c_2}\sin R_1(t-t_0)}{-1+\sqrt{1+c_2}\,sin\,\omega(t-t_0)} \qquad \beta=\frac{\omega\sqrt{1+c_2}\cos R_1(t-t_0)}{-1+\sqrt{1+c_2}\sin\omega(t-t_0)}$$
$$\gamma=\frac{\gamma_0}{[-1+\sqrt{1+c_2}\sin\omega(t-t_0)]^2}. \tag{6.22}$$

Returning to the general case of (6.13) and (6.14), we note that any such solution, once obtained, can be transformed by a transformation in the group G_p (corresponding to this particular solution) to a family of solutions with a shoreline. To see this, we keep u and v as in (6.13), but put

$$h=(R_0x\sin R_2t+y\cos R_2t)^2\gamma(t)-\mu(t). \tag{6.23}$$

Then α, β and γ satisfy the same equations (6.14), whereas $\mu(t)$ satisfies

$$\dot{\mu}+(\alpha R_0\sin R_2t+\beta\cos R_2t)\mu=0. \tag{6.24}$$

Hence, $\mu(t)$ is given explicitly as

$$\mu(t)=\mu_0\exp\left\{-\int(\alpha R_0\sin R_2t+\beta\cos R_2t)\right\}\mathrm{d}t. \tag{6.25}$$

The shoreline is hence a straight line

$$R_0x\sin R_2t+y\cos R_2t=\mu^{-1/2}\gamma^{-1/2}$$

that is rotating in time with frequency R_2 and has a time-varying intercept, governed by the equations (6.24) and (6.14). We note that this exact solution of the shallow water equations is a 'biblical' one: it corresponds, for example, to the waters of the Red Sea being 'divided' along a straight path.

7. $N_2=\{D, Y_1+aY_3+bY_4\}$. The equations in this case are somewhat more complicated and general than for the algebra N_1. We again obtain deformation mode solutions, since we have

$$u=\left(\frac{R}{Q}-P\alpha\right)y+Q\alpha x$$
$$v=\left(\frac{S}{P}+Q\beta\right)x-P\beta y \tag{6.26}$$
$$h=(Qx-Py)^2\gamma$$

where

$$P = \cos R_1 t + a \cos R_2 t + b \sin R_2 t$$
$$Q = -\{S_0 \sin R_1 t + R_0(a \sin R_2 t - b \cos R_2 t)\} \tag{6.27}$$
$$R = \dot{P} \qquad S = \dot{Q}$$

and the dots denote time derivatives. The functions of time α, β and γ satisfy the following ODEs:

$$\dot{\alpha} = -\left(\frac{\dot{P}}{Q} - f\right)\beta - 2Q\gamma + \frac{1}{P}\left(\frac{\dot{P}}{Q}\right)^{\cdot} - \alpha[Q\alpha - P\beta]$$
$$\dot{\beta} = -\left(\frac{\dot{Q}}{P} + f\right)\alpha + 2P\gamma - \frac{1}{Q}\left(\frac{\dot{Q}}{P}\right)^{\cdot} - \beta[Q_\alpha - P\beta] \tag{6.28}$$
$$\dot{\gamma} = -3\gamma[Q\alpha - P\beta].$$

Any solution of this type could again be transformed into a solution with a shoreline that is a straight line undergoing a more complicated time evolution.

Acknowledgments

The research reported in this paper was partially sponsored by research grants from NSERC of Canada, the FCAR du Québec, NATO and the Italian Ministry of Education.

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