Frobenius's Theorem via Spencer theory

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Abstract

We prove Frobenius's Theorem for codistributions using Spencer theory. Specifically, we formulate the theorem as a partial differential equation and apply the conditions for formal integrability of Goldschmidt [1967].

1. Introduction

Frobenius's Theorem for codistributions (i.e., subbundles of the cotangent bundle) states that the codistribution is integrable if and only if the exterior derivative of every one-form taking values in the codistribution lies in the algebraic ideal generated by the codistribution. The standard proofs of this theorem are actually not too difficult, and can be found in many basic texts on differential geometry. Here we will give a difficult proof, valid only in the analytic category (the result actually holds in the C^∞ category). Clearly then, the result we prove is not of significant interest. However, the proof perhaps is. We prove the theorem using Spencer's theory as manifested for linear partial differential equations by Goldschmidt [1967]. This turns out to be a not quite trivial application of the theory.

2. The result

Let X be a manifold of pure dimension n and let $\mathcal{F}$ be a codistribution on X, i.e., $\mathcal{F}$ is a subbundle of $\mathsf{T}^*\mathsf{X}$. To keep things simple, let us suppose that $\mathcal{F}$ is indeed a subbundle, i.e., that its rank is locally constant. When this is not the case, things get complicated [Freeman 1984, Malgrange 1976, 1977].

2.1 Definition: (Integrable codistribution) A codistribution $\mathcal{F}$ on X is ***integrable*** if and only if, for each $x_0 \in \mathsf{X}$, there exists a coordinate chart $(\mathcal{U}, \phi)$ about x_0 such that, in the coordinates $(x^1, \dots, x^n)$ of the chart, we have

$$\mathcal{F}_x = \operatorname{span}_{\mathbb{R}}\{\mathrm{d}x^1(x), \dots, \mathrm{d}x^m(x)\}$$

for each $x \in \mathcal{U}$. •

To state Frobenius's Theorem we need some notation.

2.2 Definition: (Algebraic ideal) Let V be a $\mathbb{R}$-vector space and let F^* be a subspace of V^*. The ***algebraic ideal*** of F^* is the subspace $\mathsf{I}(\mathsf{F}^*)$ of $\bigwedge(\mathsf{V}^*)$ generated by elements of the form $\alpha \wedge \Omega$ for $\alpha \in \mathsf{F}^*$ and $\Omega \in \bigwedge(\mathsf{V}^*)$. For $k \in \mathbb{Z}_{\geq 0}$ we denote $\mathsf{I}_k(\mathsf{F}^*) = \mathsf{I}(\mathsf{F}^*) \cap \bigwedge_k(\mathsf{V}^*)$. •

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Extending the definition to the geometric setting we define

$$\mathsf{I}(\mathcal{F}) = \cup_{x\in\mathsf{X}}\mathsf{I}(\mathcal{F}_x), \qquad \mathsf{I}_k(\mathcal{F}) = \cup_{x\in\mathsf{X}}\mathsf{I}_k(\mathcal{F}_x).$$

We can now state Frobenius's Theorem.

2.3 Theorem: (Frobenius's Theorem) *Suppose* X *is an analytic manifold and* $\mathcal{F}$ *is an analytic codistribution. Then* $\mathcal{F}$ *is integrable if and only if* $\boldsymbol{d}\alpha \in \Gamma(\mathsf{I}_2(\mathcal{F}))$ *for every* $\alpha \in \Gamma(\mathcal{F})$. •

As we mentioned above, the theorem is actually true for C^∞-codistributions. Our proof using the formal integrability of partial differential equations relies on analyticity, however.

3. The proof of the result

We now prove Frobenius's Theorem. As we shall see, the "only if" part of the proof is straightforward. For the "if" part of the proof we shall first reduce the task to a linear partial differential equation. Then we apply Spencer theory.

3.1. The easy part of the proof. It is immediate that if $\mathcal{F}$ is integrable then $\boldsymbol{d}\alpha \in \Gamma(\mathsf{I}_2(\mathcal{F}))$ for every $\alpha \in \Gamma(\mathcal{F})$. Indeed, let $x_0 \in \mathsf{X}$ and let $(\mathcal{U}, \phi)$ be a chart about x_0 for which

$$\mathcal{F}_x = \operatorname{span}_{\mathbb{R}}\{\mathrm{d}x^1(x), \dots, \mathrm{d}x^m(x)\}$$

for each $x \in \mathcal{U}$. If $\alpha \in \Gamma(\mathcal{F})$ we have

$$\alpha|\mathcal{U} = \alpha_1 \mathrm{d}x^1 + \cdots + \alpha_m \mathrm{d}x^m$$

for analytic functions $\alpha_j \colon \mathcal{U} \to \mathbb{R}$. We then have

$$\boldsymbol{d}\alpha|\mathcal{U} = \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{\partial \alpha_j}{\partial x^k} \mathrm{d}x^k \wedge \mathrm{d}x^j,$$

which is clearly a two-form in $\Gamma(\mathsf{I}_2(\mathcal{F}))$.

3.2. Reduction of the hard part of the proof to a partial differential equation. We now embark on the proof of the fact that if $\boldsymbol{d}\alpha \in \Gamma(\mathsf{I}_2(\mathcal{F}))$ for every $\alpha \in \Gamma(\mathcal{F})$ then $\mathcal{F}$ is integrable.

We first reduce this to a partial differential equation.

3.1 Lemma: *A codistribution* $\mathcal{F}$ *on* X *is integrable if and only if, for each* $x_0 \in \mathsf{X}$ *and for each* $\alpha_{x_0} \in \mathcal{F}_{x_0}$*, there exists a section* α *of* $\mathcal{F}$ *such that* $\boldsymbol{d}\alpha = 0$ *in a neighbourhood of* x_0 *and* $\alpha(x_0) = \alpha_{x_0}$.

Proof: "Only if": Let $(\mathcal{U}, \phi)$ be a chart about x_0 for which

$$\mathcal{F}_x = \operatorname{span}_{\mathbb{R}}\{\mathrm{d}x^1(x), \dots, \mathrm{d}x^m(x)\}$$

for each $x \in \mathcal{U}$. For $\alpha_{x_0} \in \mathcal{F}_{x_0}$ we can write

$$\alpha_{x_0} = a_1 \mathrm{d}x^1(x_0) + \cdots + a_m \mathrm{d}x^m(x_0)$$

for $a_1, \dots, a_m \in \mathbb{R}$. Then define

$$\alpha(x) = a_1 \mathrm{d}x^1(x) + \cdots + a_m \mathrm{d}x^m(x),$$

noting that α is a section of $\mathcal{F}$, $\alpha(x_0) = \alpha_{x_0}$, and $\boldsymbol{d}\alpha = 0$.

"If": Let $\{\alpha^1_{x_0}, \dots, \alpha^m_{x_0}\}$ be a basis for $\mathcal{F}_{x_0}$ and let $\alpha^1, \dots, \alpha^m$ be corresponding sections of $\mathcal{F}$ in a neighbourhood of x_0 such that $\alpha^j(x_0) = \alpha^j_{x_0}$ and $\boldsymbol{d}\alpha^j = 0$ for $j \in \{1, \dots, m\}$. By the Poincaré Lemma let $f^1, \dots, f^m$ be analytic functions such that $\boldsymbol{d}f^j = \alpha^j$ in a neighbourhood $\mathcal{V}$ of x_0, $j \in \{1, \dots, m\}$. Now define $\Phi\colon \mathcal{V} \to \mathbb{R}^m$ by $\Phi(x) = (f^1(x), \dots, f^m(x))$. Since $\{\boldsymbol{d}f^1(x_0), \dots, \boldsymbol{d}f^m(x_0)\}$ are linearly independent, the Regular Value Theorem ensures the existence of a coordinate chart $(\mathcal{U}, \phi)$ for X about x_0 with the following properties:

1. ϕ takes values in $\mathbb{R}^m \times \mathbb{R}^{n-m}$;
2. the local representative of Φ has the form $(\boldsymbol{x}_1, \boldsymbol{x}_2) \mapsto \boldsymbol{x}_1$.

One can readily verify that this chart has the property required in the definition of an integrable codistribution. ■

Thus the thing we need to prove is this:

> *If $\mathcal{F}$ is a codistribution having the property that $\boldsymbol{d}\beta \in \Gamma(\mathsf{I}_2(\mathcal{F}))$ for every $\beta \in \Gamma(\mathcal{F})$, then, for every $x_0 \in \mathsf{X}$ and for every $\alpha_{x_0} \in \mathcal{F}_{x_0}$ there exists $\alpha \in \Gamma(\mathcal{F})$ such that $\alpha(x_0) = \alpha_{x_0}$ and $\boldsymbol{d}\alpha = 0$ in a neighbourhood of x_0.*

This statement is easily converted into a linear partial differential equation. We let $\pi\colon \mathcal{F} \to \mathsf{X}$ be the vector bundle projection so that $\mathsf{J}_k\pi$, $k \in \mathbb{Z}_{\geq 0}$, are the associated jet bundles. The canonical projections are denoted by $\pi_k\colon \mathsf{J}_k\pi \to \mathsf{X}$ and $\pi^{k+l}_k\colon \mathsf{J}_{k+l}\pi \to \mathsf{J}_k\pi$, $k, l \in \mathbb{Z}_{\geq 0}$.

Let $\mathscr{D}_{\mathcal{F}}\colon \Gamma(\mathcal{F}) \to \Gamma(\bigwedge_2(\mathsf{T}^*\mathsf{M}))$ be the first-order differential operator $\mathscr{D}_{\mathcal{F}}(\alpha) = \boldsymbol{d}\alpha$. We let $\Phi_{\mathcal{F}}\colon \mathsf{J}_1\pi \to \bigwedge_2(\mathsf{T}^*\mathsf{X})$ be the corresponding vector bundle map satisfying $\Phi_{\mathcal{F}}(j_1\alpha(x)) = \boldsymbol{d}\alpha(x)$. We then define $\mathsf{R}_{\mathcal{F}} = \ker(\Phi_{\mathcal{F}})$ which is a first-order linear partial differential equation. Lemma 3.1 says that proving the following theorem is enough to prove the "hard part" of Theorem 2.3.

3.2 THEOREM: (The hard part of Frobenius's Theorem) *Let X be an analytic manifold and let $\mathcal{F}$ be an analytic codistribution. Suppose that $\boldsymbol{d}\alpha \in \Gamma(\mathsf{I}_2(\mathcal{F}))$ for every $\alpha \in \Gamma(\mathcal{F})$. Then, for every $p_0 \in \mathsf{R}_{\mathcal{F}}$ there exists an analytic solution α of $\mathsf{R}_{\mathcal{F}}$ defined in a neighbourhood of $x_0 = \pi_1(p_0)$ such that $j_1\alpha(x_0) = p_0$.*

The proof of this theorem will occupy us for the remainder of this note.

3.3. The algebraic part of the hard part of the proof. If V is a $\mathbb{R}$-vector space, we denote by $\otimes^k_{j=1}\mathsf{V}^*$, $S_k(\mathsf{V}^*)$, and $\bigwedge_k(\mathsf{V}^*)$ the vector spaces of $(0,k)$-tensors, symmetric $(0,k)$-tensors, and skew-symmetric $(0,k)$-tensors, respectively, on V. By $\bigwedge(\mathsf{V}^*)$ we denote the set of all skew-symmetric tensors.

We suppose that V is a finite-dimensional $\mathbb{R}$-vector space with V^* its dual space. We let E and F be subspaces of V with the property that $\mathsf{V} = \mathsf{E} \oplus \mathsf{F}$. We denote the corresponding decomposition of V^* as $\mathsf{V}^* = \mathsf{E}^* \oplus \mathsf{F}^*$. The decomposition also induces direct sum decompositions of the vector spaces $\otimes^k_{j=1}\mathsf{V}^*$, $S_k(\mathsf{V}^*)$, and $\bigwedge_k(\mathsf{V}^*)$. For example, we have

$$\textstyle\bigwedge_2(\mathsf{V}^*) = \bigwedge_2(\mathsf{E}^*) \oplus \bigwedge_2(\mathsf{F}^*) \oplus (\mathsf{E}^* \otimes \mathsf{F}^*)$$

and

$$S_2(\mathsf{V}^*) = S_2(\mathsf{E}^*) \oplus S_2(\mathsf{F}^*) \oplus (\mathsf{E}^* \otimes \mathsf{F}^*).$$

(One can think of these decompositions as they manifest themselves for skew-symmetric (resp. symmetric) matrices. If one writes such a matrix in four blocks, one has two skew-symmetric (resp. symmetric) diagonal blocks, and the two off-diagonal blocks differ by a transpose, so it suffices to determine only one of these.) Using these decompositions we have natural inclusions of, for example, $S_k(\mathsf{F}^*)$ and $\bigwedge_k(\mathsf{F}^*)$ in $S_k(\mathsf{V}^*)$ and $\bigwedge_k(\mathsf{V}^*)$, respectively. Moreover, the images of these inclusions have natural complements, so there are also natural projections onto the subspaces. We shall take all of these inclusions and projections for granted in our discussions to follow. We shall also use the decomposition $\mathsf{V} = \mathsf{E} \oplus \mathsf{F}$, and the induced decompositions of the tensor algebra, to identify quotients with complements. We will do all of this without explicit indication.

Let $\mathrm{Alt}\colon \otimes_{j=1}^k \mathsf{V}^* \to \bigwedge_k(\mathsf{V}^*)$ be the projection defined by

$$\mathrm{Alt}(A)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) A(v_{\sigma(1)}, \dots, v_{\sigma(k)}),$$

where S_k denotes the permutation group on k symbols. Let $\sigma\colon \mathsf{V}^* \otimes \mathsf{F}^* \to \bigwedge_2(\mathsf{V}^*)$ be the restriction of $2\mathrm{Alt}$ to $\mathsf{V}^* \otimes \mathsf{F}^*$. Explicitly, writing elements of V with respect to their decomposition in $\mathsf{E} \oplus \mathsf{F}$,

$$\sigma(\alpha \otimes \beta)(u_1 \oplus u_2, v_1 \oplus v_2) = \alpha(u_1 \oplus u_2)\beta(v_2) - \alpha(v_1 \oplus v_2)\beta(u_2),$$

for $\alpha \in \mathsf{V}^*$ and $\beta \in \mathsf{F}^*$. Now define $\sigma_1\colon S_2(\mathsf{V}^*) \otimes \mathsf{F}^* \to \mathsf{V}^* \otimes \bigwedge_2(\mathsf{V}^*)$ by $\sigma_1 = \mathrm{id}_{\mathsf{V}^*} \otimes \sigma$. Explicitly,

$$\begin{aligned}\sigma_1(B \otimes \beta)&(u_1 \oplus u_2, v_1 \oplus v_2, w_1 \oplus w_2) \\ &= B(u_1 \oplus u_2, v_1 \oplus v_2)\beta(w_2) - B(u_1 \oplus u_2, w_1 \oplus w_2)\beta(v_2),\end{aligned}$$

for $B \in S_2(\mathsf{V}^*)$ and $\beta \in \mathsf{F}^*$. The map σ_1 is readily verified to be the first prolongation of σ.

We shall be interested in the kernels and cokernels of the maps σ and σ_1. For σ we have the following result.

3.3 Lemma: *We have* $\ker(\sigma) = S_2(\mathsf{F}^*)$ *and* $\mathrm{coker}(\sigma) = \bigwedge_2(\mathsf{E}^*)$.

Proof: For the assertion concerning $\ker(\sigma)$ consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & S_2(\mathsf{V}^*) \cap (\mathsf{V}^* \otimes \mathsf{F}^*) & \longrightarrow & \mathsf{V}^* \otimes \mathsf{F}^* & \xrightarrow{\sigma} & \bigwedge_2(\mathsf{V}^*) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & S_2(\mathsf{V}^*) & \longrightarrow & \mathsf{V}^* \otimes \mathsf{V}^* & \xrightarrow{2\mathrm{Alt}} & \bigwedge_2(\mathsf{V}^*) & \longrightarrow & 0
\end{array}$$

The bottom row is exact (it is essentially the decomposition of a $(0,2)$-tensor into its symmetric and skew-symmetric parts). One can then easily show that the top row is exact since $\sigma = 2\mathrm{Alt}|\mathsf{V}^* \otimes \mathsf{F}^*$. It remains to show that $S_2(\mathsf{V}^*) \cap (\mathsf{V}^* \otimes \mathsf{F}^*) = S_2(\mathsf{F}^*)$. Since

$\mathsf{V}^* \otimes \mathsf{F}^* = (\mathsf{E}^* \otimes \mathsf{F}^*) \oplus (\mathsf{F}^* \otimes \mathsf{F}^*)$ it follows that $S_2(\mathsf{F}^*) \subset S_2(\mathsf{V}^*) \cap (\mathsf{V}^* \otimes \mathsf{F}^*)$. Now let $A \in S_2(\mathsf{V}^*) \cap (\mathsf{V}^* \otimes \mathsf{F}^*)$. Then

$$\begin{aligned} A(u_1 \oplus u_2, v_1 \oplus v_2) &= A(u_1 \oplus u_2, 0 \oplus v_2) = A(0 \oplus v_2, u_1 \oplus u_2) \\ &= A(0 \oplus v_2, 0 \oplus u_2) = A(0 \oplus u_2, 0 \oplus v_2), \end{aligned} \tag{3.1}$$

since $A \in \mathsf{V}^* \otimes \mathsf{F}^*$ and since A is symmetric. Thus $A \in S_2(\mathsf{F}^*)$. This gives $\ker(\sigma) = S_2(\mathsf{F}^*)$.

Now we prove that $\operatorname{coker}(\sigma) = \bigwedge_2(\mathsf{E}^*)$. First we note that the sequence

$$\mathsf{V}^* \otimes \mathsf{V}^* \xrightarrow{2\mathrm{Alt}} \textstyle\bigwedge_2(\mathsf{V}^*) \longrightarrow 0$$

is exact since 2Alt is, up to a factor of 2, the natural projection. Thus 2Alt is surjective and so $\operatorname{coker}(2\mathrm{Alt}) = 0$. Now we note that $\mathsf{V}^* \otimes \mathsf{V}^* = (\mathsf{V}^* \otimes \mathsf{E}^*) \oplus (\mathsf{V}^* \oplus \mathsf{F}^*)$. Therefore, to find $\operatorname{coker}(\sigma)$ we need only find a complement to $\operatorname{image}(\sigma)$ in $\operatorname{image}(2\mathrm{Alt})$ and then take the direct sum of this complement with $\operatorname{coker}(2\mathrm{Alt})$ (the latter being trivial in this case). One can readily ascertain that

$$2\mathrm{Alt}(\mathsf{V}^* \otimes \mathsf{E}^*) = \textstyle\bigwedge_2(\mathsf{E}^*) \oplus (\mathsf{E}^* \otimes \mathsf{F}^*), \quad 2\mathrm{Alt}(\mathsf{V}^* \otimes \mathsf{F}^*) = \textstyle\bigwedge_2(\mathsf{F}^*) \oplus (\mathsf{E}^* \otimes \mathsf{F}^*).$$

Therefore, $\operatorname{image}(2\mathrm{Alt}) = \operatorname{image}(\sigma) \oplus \bigwedge_2(\mathsf{E}^*)$. Thus $\operatorname{coker}(\sigma) = \bigwedge_2(\mathsf{E}^*) \oplus \operatorname{coker}(2\mathrm{Alt}) = \bigwedge_2(\mathsf{E}^*)$. ∎

To state the analogous result for σ_1 we need some notation.

1. For a subspace Λ of V^*, let $\mathsf{I}(\Lambda)$ be the ideal generated in $\bigwedge(\mathsf{V}^*)$ by Λ and let $\mathsf{I}_2(\Lambda) = \mathsf{I}(\Lambda) \cap \bigwedge_2(\mathsf{V}^*)$.
2. Define $\tau_1 \colon \mathsf{V}^* \otimes \bigwedge_2(\mathsf{V}^*) \to \bigwedge_3(\mathsf{V}^*)$ by

$$\tau_1(B)(u, v, w) = B(u, v, w) + B(w, u, v) + B(v, w, u).$$

Now we have the following result.

3.4 LEMMA: *We have* $\ker(\sigma_1) = S_3(\mathsf{F}^*)$ *and* $\operatorname{image}(\sigma_1) = (\mathsf{V}^* \otimes \mathsf{I}_2(\mathsf{F}^*)) \cap \ker(\tau_1)$. *In particular, if*

$$\tau \colon \mathsf{V}^* \otimes \textstyle\bigwedge_2(\mathsf{V}^*) \to \operatorname{coker}(\sigma_1), \quad \tau_2 \colon \mathsf{V}^* \otimes \textstyle\bigwedge_2(\mathsf{V}^*) \to \mathsf{V}^* \otimes \textstyle\bigwedge_2(\mathsf{E}^*)$$

are the canonical projections, then $\ker(\tau) = \ker(\tau_1) \cap \ker(\tau_2)$.

Proof: Define $\bar{\sigma}_1 \colon S_2(\mathsf{V}^*) \otimes \mathsf{V}^* \to \mathsf{V}^* \otimes \bigwedge_2(\mathsf{V}^*)$ by

$$\bar{\sigma}_1(B \otimes \beta)(u, v, w) = B(u, v)\beta(w) - B(u, w)\beta(v)$$

for $B \in S_2(\mathsf{V}^*)$ and $\beta \in \mathsf{V}^*$. Note that $\sigma_1 = \bar{\sigma}_1 | S_2(\mathsf{V}^*) \otimes \mathsf{F}^*$. Now consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_3(\mathsf{V}^*) \cap (S_2(\mathsf{V}^*) \otimes \mathsf{F}^*) & \longrightarrow & S_2(\mathsf{V}^*) \otimes \mathsf{F}^* & \xrightarrow{\sigma_1} & \mathsf{V}^* \otimes \bigwedge_2(\mathsf{V}^*) \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & S_3(\mathsf{V}^*) & \longrightarrow & S_2(\mathsf{V}^*) \otimes \mathsf{V}^* & \xrightarrow{\bar{\sigma}_1} & \mathsf{V}^* \otimes \bigwedge_2(\mathsf{V}^*)
\end{array}$$

We claim that the bottom row is exact. Indeed, by definition of $\bar{\sigma}_1$,

$$\ker(\bar{\sigma}_1) = \{B \in S_2(\mathsf{V}^*) \otimes \mathsf{V}^* \mid \ B(u,v,w) = B(u,w,v),\ u,v,w \in \mathsf{V}\}.$$

Thus $\ker(\bar{\sigma}_1)$ are those elements of $\otimes_{j=1}^3 \mathsf{V}^*$ that are symmetric in the first two and last two entries. Thus $\ker(\bar{\sigma}_1) = S_3(\mathsf{V}^*)$, which is exactness of the bottom row. Now it is easy to show that the top row is also exact (this is the same argument as in the first part of the Lemma 3.3). The form for $\ker(\sigma_1)$ asserted in the lemma will follow if we can show that

$$S_3(\mathsf{V}^*) \cap (S_2(\mathsf{V}^*) \otimes \mathsf{F}^*) = S_3(\mathsf{F}^*).$$

This can be shown along the lines of (3.1), using symmetry of A and the fact that $A \in S_2(\mathsf{V}^*) \otimes \mathsf{F}^*$.

For the assertion about $\mathrm{image}(\sigma_1)$, we first claim that the sequence

$$S_2(\mathsf{V}^*) \otimes \mathsf{V}^* \xrightarrow{\bar{\sigma}_1} \mathsf{V}^* \otimes \textstyle\bigwedge_2(\mathsf{V}^*) \xrightarrow{\tau_1} \textstyle\bigwedge_3(\mathsf{V}^*) \longrightarrow 0 \tag{3.2}$$

is exact, i.e., that $\mathrm{coker}(\bar{\sigma}_1) = \bigwedge_3(\mathsf{V}^*)$. One can easily verify that $\tau_1 \circ \bar{\sigma}_1 = 0$. This gives $\mathrm{image}(\bar{\sigma}_1) \subset \ker(\tau_1)$. Also, τ_1 is clearly surjective since if $\Omega \in \bigwedge_3(\mathsf{V}^*)$ then $\Omega = \tau_1(\frac{1}{3}\Omega)$. The exactness of the sequence above will now follow if we can show that $\dim(\mathrm{image}(\bar{\sigma}_1)) = \dim(\ker(\tau_1))$. This identity is readily verified from the following formulae:

$$\begin{aligned}\dim(\mathrm{image}(\bar{\sigma}_1)) &= \dim(S_2(\mathsf{V}^*) \otimes \mathsf{V}^*) - \dim(\ker(\bar{\sigma}_1)),\\ \dim(\ker(\tau_1)) &= \dim(\mathsf{V}^* \otimes \textstyle\bigwedge_2(\mathsf{V}^*)) - \dim(\mathrm{image}(\tau_1)),\end{aligned}$$

along with the facts that $\ker(\bar{\sigma}_1) = S_3(\mathsf{V}^*)$ and $\mathrm{image}(\tau_1) = \bigwedge_3(\mathsf{V}^*)$.

Next we claim that the sequence

$$\mathsf{V}^* \otimes \mathsf{I}_2(\mathsf{F}^*) \longrightarrow \mathsf{V}^* \otimes \textstyle\bigwedge_2(\mathsf{V}^*) \xrightarrow{\tau_2} \mathsf{V}^* \otimes \textstyle\bigwedge_2(\mathsf{E}^*) \longrightarrow 0 \tag{3.3}$$

is exact. Let us first give the explicit formula for τ_2:

$$\tau_2(B)(u_1 \oplus u_2, v_1 \oplus v_2, w_1 \oplus w_2) = B(u_1 \oplus u_2, v_1 \oplus 0, w_1 \oplus 0).$$

If $B \in \mathsf{V}^* \otimes \mathsf{I}_2(\mathsf{F}^*)$ then $\tau_2(B) = 0$ by definition of $\mathsf{I}_2(\mathsf{F}^*)$. Thus $\mathsf{V}^* \otimes \mathsf{I}_2(\mathsf{F}^*) \subset \ker(\tau_2)$. We next claim that τ_2 is surjective. Indeed, let $B \in \mathsf{V}^* \otimes \bigwedge_2(\mathsf{E}^*)$ and note that $\tau_2(B) = B$. Now the exactness of (3.3) will follow if we can show that

$$\dim(\mathsf{V}^* \otimes \mathsf{I}_2(\mathsf{F}^*)) = \dim(\ker(\tau_2)).$$

However, this follows by a direct computation, noting that

$$\mathsf{I}_2(\mathsf{F}^*) = \textstyle\bigwedge_2(\mathsf{F}) \oplus (\mathsf{E}^* \oplus \mathsf{F}^*).$$

We claim that $\mathrm{image}(\sigma_1) = (\mathsf{V}^* \otimes \mathsf{I}_2(\mathsf{F}^*)) \cap \mathrm{image}(\bar{\sigma}_1)$. Exactness of the sequences (3.2) and (3.3) ensures that this is equivalent to the assertion that $\mathrm{image}(\sigma_1) = \ker(\tau_1) \cap \ker(\tau_2)$. One can check directly, using the definitions of τ_1 and τ_2, that if $B \in \mathrm{image}(\sigma_1)$ then $\tau_1(B) = 0$ and $\tau_2(B) = 0$. Thus $\mathrm{image}(\sigma_1) \subset \ker(\tau_1) \cap \ker(\tau_2)$. For the converse inclusion, let $B \in \ker(\tau_1) \cap \ker(\tau_2)$. Then, since $B \in \ker(\tau_1) = \text{image}(\bar{\sigma}_1)$, there exists $A \in S_2(\mathsf{V}^*) \otimes \mathsf{F}^*$ such that

$$B(u_1 \oplus u_2, v_1 \oplus v_2, w_1 \oplus w_2) = A(u_1 \oplus u_2, v_1 \oplus v_2, w_2) - A(u_1 \oplus u_2, w_1 \oplus w_2, v_2).$$

Define $A' \in S_2(\mathsf{V}^*) \otimes \mathsf{F}^*$ by

$$A'(u_1 \oplus u_2, v_1 \oplus v_2, w_2) = A(u_1 \oplus u_2, 0 \oplus v_2, w_2).$$

Then

$$\begin{aligned}\sigma_1(A')(u_1 \oplus u_2, v_1 \oplus v_2, w_1 \oplus w_2) &= A(u_1 \oplus u_2, 0 \oplus v_2, w_2) - A(u_1 \oplus u_2, 0 \oplus w_2, v_2)\\ &= B(u_1 \oplus u_2, 0 \oplus v_2, 0 \oplus w_2)\\ &= B(u_1 \oplus u_2, v_1 \oplus v_2, w_1 \oplus w_2),\end{aligned}$$

using the fact that $B \in \ker(\tau_2)$. Thus $\ker(\tau_1) \cap \ker(\tau_2) \subset \text{image}(\sigma_1)$.

The final assertion has already been proved during the course of the preceding argument. ■

Let us define $\mathsf{G} = S_2(\mathsf{F}^*) \subset \mathsf{V}^* \otimes \mathsf{F}^*$ and $\mathsf{G}_1 = S_3(\mathsf{F}^*) \subset S_2(\mathsf{V}^*) \otimes \mathsf{F}^*$. We note that, by definition of σ_1, G_1 is the first prolongation of G. The following lemma gives an extremely important property of the subspace G.

3.5 Lemma: *The subspace $\mathsf{G} \subset \mathsf{V}^* \oplus \mathsf{F}^*$ is involutive.*

Proof: Let $\{v_1, \dots, v_n\}$ be a basis for V with the property that $\{v^1, \dots, v^m\}$ forms a basis for F^*. We claim that this basis is quasi-regular which will show involutivity. We have

$$\begin{aligned}\mathsf{G} &= \ker(\sigma) = S_2(\mathsf{F}^*) \simeq \mathbb{R}_2[x_1, \dots, x_m]\\ \mathsf{G}_{v_1} &= \{B \in \mathsf{G} \mid \ v_1 \lrcorner B = 0\} \simeq \mathbb{R}_2[x_2, \dots, x_m]\\ \mathsf{G}_{v_1,v_2} &= \{B \in \mathsf{G} \mid \ v_1 \lrcorner B = v_2 \lrcorner B = 0\} \simeq \mathbb{R}_2[x_3, \dots, x_m]\\ &\ \vdots\\ \mathsf{G}_{v_1,\dots,v_{m-1}} &= \{B \in \mathsf{G} \mid \ v_1 \lrcorner B = \dots = v_{m-1} \lrcorner B = 0\} \simeq \mathbb{R}_2[x_m],\end{aligned}$$

where $\mathbb{R}_2[\xi_1, \dots, \xi_k]$ denotes the homogeneous polynomials of degree 2 in indeterminates $\xi_1, \dots, \xi_k$. Note that $\mathsf{G}_{v_1,\dots,v_k} = \{0\}$ for $k \geq m$. We have

$$\begin{aligned}\dim(\mathsf{G}_{v_1}) &= \tfrac{1}{2}m(m-1)\\ \dim(\mathsf{G}_{v_1,v_2}) &= \tfrac{1}{2}(m-1)(m-2)\\ &\ \vdots\\ \dim(\mathsf{G}_{v_1,\dots,v_{m-1}}) &= \tfrac{1}{2}2 \cdot 1.\end{aligned}$$

Therefore,

$$\dim(\mathsf{G}) + \sum_{j=1}^{m-1} \dim(\mathsf{G}_{v_1,\dots,v_j}) = \sum_{j=1}^{m} \tfrac{1}{2}j(j+1) = \tfrac{1}{6}m(m+1)(m+2) = \dim(\mathsf{G}_1),$$

giving involutivity, as desired. ■

3.4. The geometric part of the hard part of the proof. Let $\tau\colon \mathsf{F}\to\mathsf{X}$ be a vector bundle and let ∇ be a connection in F. The $\mathsf{T}^*\mathsf{X}\otimes\mathsf{F}$-valued first-order differential operator $\xi\mapsto\nabla\xi$ gives a vector bundle map $\Phi_\nabla\colon \mathsf{J}_1\tau\to\mathsf{T}^*\mathsf{X}\otimes\mathsf{F}$ defined by $\Phi_\nabla(j_1\xi(x))=\nabla\xi(x)$. One can readily check that the symbol of Φ_∇ is $\sigma(\Phi_\nabla)=\mathrm{id}_{\mathsf{T}^*\mathsf{X}\otimes\mathsf{F}}$. Thus Φ_∇ defines a splitting on the left of the exact sequence associated with the jet bundle $\mathsf{J}_1\tau$:

$$0 \longrightarrow \mathsf{T}^*\mathsf{X}\otimes\mathsf{F} \underset{\Phi_\nabla}{\overset{\longrightarrow}{\longleftarrow}} \mathsf{J}_1\tau \longrightarrow \mathsf{F} \longrightarrow 0 \tag{3.4}$$

We shall use this fact below when checking the compatibility conditions for the potential shaping partial differential equation.

Now we let X be a manifold and focus on the case when ∇ is an affine connection on X; the reader will understand that ∇ now means something different than in the constructions immediately above. We first recall [Nelson 1967] that there is a formula relating the exterior derivative and the covariant differential.

3.6 Lemma: *If Ω is a differential k-form on X then $\boldsymbol{d}\Omega=(k+1)\mathrm{Alt}(\nabla\Omega)$.*

The constant in the lemma relating $\boldsymbol{d}$ and $\mathrm{Alt}\circ\nabla$ will vary depending on one's convention for the definitions of $\boldsymbol{d}$ and Alt.

Next we suppose that we have a subbundle Λ of $\mathsf{T}^*\mathsf{X}$. Given an affine connection ∇ on X one can construct from it an affine connection $\overset{\wedge}{\nabla}$ that restricts to the subbundle $\mathrm{coann}(\Lambda)$ that is annihilated by Λ. To see how this can be done, we refer to [Lewis 1998]. It is easy to see that $\overset{\wedge}{\nabla}$ also restricts to the subbundle Λ: for a $\mathrm{coann}(\Lambda)$-valued vector field X and a Λ-valued one-form λ we have

$$\langle\lambda;X\rangle=0 \quad\Longrightarrow\quad \langle\overset{\wedge}{\nabla}_Y\lambda;X\rangle+\langle\lambda;\overset{\wedge}{\nabla}_YX\rangle=\langle\overset{\wedge}{\nabla}_Y\lambda;X\rangle=0$$

for every vector field Y.

We now have the following result.

3.7 Lemma: *If Λ is a regular codistribution, if ∇ is an affine connection restricting to Λ, and if $\Omega\in\Gamma(\mathsf{I}_k(\Lambda))$, then $\nabla\Omega\in\Gamma(\mathsf{T}^*\mathsf{X}\otimes\mathsf{I}_k(\Lambda))$.*

Proof: Let $\mathcal{U}$ be an open subset of X such that Λ is generated by $\{\lambda^1,\dots,\lambda^m\}$. Then $\Omega\in\Gamma(\mathsf{I}_k(\Lambda))$ if and only if

$$\Omega=\sum_{j=1}^m\Omega^j\wedge\lambda^j$$

for some $(k-1)$-forms $\Omega^1,\dots,\Omega^m$. Then, for a vector field X,

$$\nabla_X\Omega=\sum_{j=1}^m\nabla_X\Omega^j\wedge\lambda^j+\sum_{j=1}^m\Omega^j\wedge\nabla_X\lambda^j$$

Clearly $\nabla_X\Omega^j\wedge\lambda^j\in\Gamma(\mathsf{I}_k(\Lambda))$, $j\in\{1,\dots,m\}$. Since ∇ restricts to Λ it also follows that $\Omega^j\wedge\nabla_X\lambda^j\in\Gamma(\mathsf{I}_k(\Lambda))$, $j\in\{1,\dots,m\}$. ■

3.5. The rest of the hard part of the proof. The strategy for the remainder of the proof is to prove the formal integrability of the partial differential equation $\mathsf{R}_{\mathcal{F}}$. It then follows from a general result of Malgrange [1972a, 1972b] that analytic solutions exist as stated. (This also follows from the Cartan–Kähler Theorem for partial differential equations.)

Following [Pommaret 1978, Corollary 2.4.9] we have the following result which was proved by Goldschmidt [1967]. We denote by $\rho_1(\mathsf{R}_{\mathcal{F}}) \subset \mathsf{J}_2\pi$ the first prolongation of $\mathsf{R}_{\mathcal{F}}$.

3.8 Theorem: *The partial differential equation $\mathsf{R}_{\mathcal{F}}$ is formally integrable in a neighbourhood of x_0 if*

(i) it has an involutive symbol at every point in that neighbourhood, if

(ii) the first prolongation of the symbol is a vector bundle, and if

(iii) $\rho_1(\mathsf{R}_{\mathcal{F}})$ projects surjectively onto $\mathsf{R}_{\mathcal{F}}$ in that neighbourhood.

In the remainder of the proof we will let $\mathcal{E}$ be a subbundle of $\mathsf{T}^*\mathsf{X}$ which is complementary to $\mathcal{F}$: $\mathsf{T}^*\mathsf{X} = \mathcal{E} \oplus \mathcal{F}$. This is valid in a neighbourhood $\mathcal{U}_1$ of x_0 since x_0 is a regular point for $\mathcal{F}$ (recall that we are assuming that $\mathcal{F}$ has locally constant rank). Note that this gives a corresponding direct sum decomposition $\mathsf{TX} = \mathrm{coann}(\mathcal{F}) \oplus \mathrm{coann}(\mathcal{E})$. We shall use this decomposition below without explicit reference. As we did in Section 3.3, we shall also suppose that this decomposition gives rise to decompositions of the tensor algebra, and we shall use these decompositions to give explicit inclusions and projections from and onto various subspaces of tensors.

Let us first determine the symbol $\mathsf{G}(\mathsf{R}_{\mathcal{F}})$ for $\mathsf{R}_{\mathcal{F}}$.

3.9 Lemma: *We have $\mathsf{G}(\mathsf{R}_{\mathcal{F}}) = S_2(\mathcal{F})$.*

Proof: Note that $\mathsf{G}(\mathsf{R}_{\mathcal{F}}) = \ker(\sigma(\Phi_{\mathcal{F}}))$, where $\sigma(\Phi_{\mathcal{F}})$ is the symbol of $\Phi_{\mathcal{F}}$. A direct computation (in coordinates, for example) using the definition of $\Phi_{\mathcal{F}}$ gives

$$\sigma(\Phi_{\mathcal{F}})(\alpha \otimes \beta)(u_1 \oplus u_2, v_1 \oplus v_2) = \alpha(u_1 \oplus u_2)\beta(v_2) - \alpha(v_1 \oplus v_2)\beta(u_2)$$

for $\alpha \in \mathsf{T}^*_x\mathsf{X}$, $\beta \in \mathcal{F}_x$, $u_1, v_1 \in \mathrm{coann}(\mathcal{F}_x)$, and $u_2, v_2 \in \mathrm{coann}(\mathcal{E}_x)$. By Lemma 3.3 our claim about $\mathsf{G}(\mathsf{R}_{\mathcal{F}})$ follows. ■

From the preceding lemma and from Lemma 3.5 we know that $\mathsf{G}(\mathsf{R}_{\mathcal{F}})$ is involutive.

Let us give the first prolongation of $\mathsf{G}(\mathsf{R}_{\mathcal{F}})$, which we denote by $\rho_1(\mathsf{G}(\mathsf{R}_{\mathcal{F}}))$.

3.10 Lemma: *We have $\rho_1(\mathsf{G}(\mathsf{R}_{\mathcal{F}})) = S_3(\mathcal{F})$.*

Proof: We make the following observations:

1. $\mathsf{G}(\mathsf{R}_{\mathcal{F}})_x$ is the kernel of the map σ used in Section 3.3 if we take $\mathsf{V}^* = \mathsf{T}^*_x\mathsf{X}$ and $\mathsf{F}^* = \mathcal{F}_x$;
2. the map σ_1 in Section 3.3 is the first prolongation of σ.

An application of Lemma 3.4 gives the result. ■

We can then see that the first prolongation of $\mathsf{G}(\mathsf{R}_{\mathcal{F}})$ is a vector bundle on the open subset $\mathcal{U}_2$ of X on which $\mathcal{F}$ is a vector bundle. Clearly $x_0 \in \mathrm{int}(\mathcal{U}_2)$.

We have now to verify only the third of the hypotheses of Theorem 3.8.

Let $\mathsf{K} = \mathrm{coker}(\sigma_1(\Phi_{\mathcal{F}}))$ and denote by τ the canonical projection from $\mathsf{T}^*\mathsf{X} \otimes \bigwedge_2(\mathsf{T}^*\mathsf{X})$ to K. We let $\chi\colon \bigwedge_2(\mathsf{T}^*\mathsf{X}) \to \mathsf{X}$ be the canonical projection.

Following [Pommaret 1978, page 69] we have the following commutative and exact diagram:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & S_2(\mathsf{T}^*\mathsf{X})\otimes\mathcal{F} & \xrightarrow{\sigma_1(\Phi_{\mathcal{F}})} & \mathsf{T}^*\mathsf{X}\otimes\bigwedge_2(\mathsf{T}^*\mathsf{X}) & \xrightarrow{\tau} & \mathsf{K} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \rho_1(\mathsf{R}_{\mathcal{F}}) & \longrightarrow & \mathsf{J}_2\pi & \xrightarrow{\rho_1(\Phi_{\mathcal{F}})} & \mathsf{J}_1\chi & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathsf{R}_{\mathcal{F}} & \longrightarrow & \mathsf{J}_1\pi & \xrightarrow{\Phi_{\mathcal{F}}} & \bigwedge_2(\mathsf{T}^*\mathsf{X}) & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

All unmarked arrows are either canonical inclusions or canonical projections. We define a map κ from $\mathsf{R}_{\mathcal{F}}$ to K as follows. Let $p\in\mathsf{R}_{\mathcal{F}}$ and denote $x=\pi_1(p)$. Then $\Phi_{\mathcal{F}}(p)=0_x$. Choose $p'\in\mathsf{J}_2\pi$ such that $\pi^2_1(p')=p$ and define $\omega=\rho_1(\Phi_{\mathcal{F}})(p')\in\mathsf{J}_1\chi$. Then, by commutativity of the diagram, $\chi^1_0(\omega)=\Phi_{\mathcal{F}}(p)=0_x\in\bigwedge_2(\mathsf{T}^*\mathsf{X})$. Therefore, by exactness of the third column, ω is the image of $A\in\mathsf{T}^*_x\mathsf{X}\otimes\bigwedge_2(\mathsf{T}^*_x\mathsf{X})$. It thus makes sense to define $\kappa(p)=\tau(A)$. One can easily show that this definition of κ is independent of the choice of p'. Pommaret [1978, Theorem 2.4.1] shows that p lies in the image of the projection of $\rho_1(\mathsf{R}_{\mathcal{F}})$ to $\mathsf{R}_{\mathcal{F}}$ if and only if $\kappa(p)=0$.

Let us give a way of explicitly constructing κ, as this will be essential in our proof. Let $p\in\mathsf{R}_{\mathcal{F}}$ and $x=\pi_1(p)$. We let ∇ be an affine connection on X and let α be a section of $\mathcal{F}$ such that $j_1\alpha(x)=p$. Thus $\boldsymbol{d}\alpha(x)=0_x$. Then define $p'=j_2\alpha(x)$ so that

$$\rho_1(\Phi_{\mathcal{F}})(p')=j_1\boldsymbol{d}\alpha(x).$$

We next claim that

$$j_1\boldsymbol{d}\alpha(x)=\nabla\boldsymbol{d}\alpha(x),$$

where we identify the fibre of $\mathsf{J}_1\chi$ over $\boldsymbol{d}\alpha(x)\in\bigwedge_2(\mathsf{T}^*\mathsf{X})$ with $\mathsf{T}^*_x\mathsf{X}\otimes\bigwedge_2(\mathsf{T}^*_x\mathsf{X})$. This follows from (3.4), combined with the fact that $\boldsymbol{d}\alpha(x)=0_x$. Then we have

$$\kappa(p)=\tau(j_1\boldsymbol{d}\alpha(x))=\tau(\nabla\boldsymbol{d}\alpha(x)).$$

The point is that this representation of κ is independent of the affine connection ∇.

Let $\tau_1\colon\mathsf{T}^*\mathsf{X}\otimes\bigwedge_2(\mathsf{T}^*\mathsf{X})\to\bigwedge_3(\mathsf{T}^*\mathsf{X})$ be defined by

$$\tau_1(B)(u,v,w)=B(u,v,w)+B(w,u,v)+B(v,w,u),\qquad u,v,w\in\mathsf{T}_x\mathsf{X}.$$

Denote by $\tau_2\colon\mathsf{T}^*\mathsf{X}\otimes\bigwedge_2(\mathsf{T}^*\mathsf{X})\to\mathsf{T}^*\mathsf{X}\otimes\bigwedge_2(\mathcal{E})$ the canonical projection. By Lemma 3.4 we have

$$\ker(\tau)=\ker(\tau_1)\cap\ker(\tau_2). \tag{3.5}$$

Now let ∇ be an arbitrary torsion-free affine connection on X and let $p \in \mathsf{R}_{\mathcal{F}}$ with $x \in \mathsf{X}$. Using Lemma 3.6 and the definition of τ_1 we may easily show that

$$\tau_1(\nabla \boldsymbol{d}\alpha(x)) = \boldsymbol{d} \circ \boldsymbol{d}\alpha(x) = 0, \qquad x \in \mathsf{X}, \tag{3.6}$$

for any section α of $\mathcal{F}$ for which $j_1\alpha(x) = p$.

Now let $p \in \mathsf{R}_{\mathcal{F}}$ and let α have the property that $j_1\alpha(x) = p$ where $x = \pi_1(p)$. Assume that ∇' is an affine connection that restricts to $\mathcal{F}$ in the sense discussed in the run up to Lemma 3.7. (Note that this means that we will generally have to sacrifice ∇' being torsion free.) We then compute

$$\tau_2(\nabla' \boldsymbol{d}\alpha(x)) = 0 \tag{3.7}$$

using Lemma 3.7 and the fact that $\boldsymbol{d}\alpha \in \Gamma(\mathsf{I}_2(\mathcal{F}))$.

Let us now wrap up all of the above arguments. Let $p \in \mathsf{R}_{\mathcal{F}}$ and let α be a section of $\mathcal{F}$ such that $j_1\alpha(x) = p$ for $x = \pi_1(p)$. By (3.6) we have

$$\tau_1(j_1\boldsymbol{d}\alpha(x)) = \tau_1(\nabla \boldsymbol{d}\alpha(x)) = 0, \qquad x \in \mathsf{X}.$$

By (3.7) we have

$$\tau_2(j_1\boldsymbol{d}\alpha(x)) = \tau_2(\nabla' \boldsymbol{d}\alpha(x)) = 0, \qquad x \in \mathcal{U}_2.$$

Therefore, by (3.5) we have $\tau(j_1\boldsymbol{d}\alpha(x)) = 0$, giving $\kappa(p) = 0$.

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