

## Is it worth learning differential geometric methods for modelling and control of mechanical systems?\*

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### Abstract

Evidence is presented to indicate that the answer is, “Yes, sometimes.”

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**AMS Subject Classifications.** 70E15, 70F25, 70Q05, 53B20, 93B05, 93B29

### 1. Introduction

Before I get to the actual writing of this paper, let me say a few words about the manner and spirit in which it is written. Normally when writing technical articles I try to be rigorous and precise, giving clear definitions, stating clear theorems, etc. However, I believe that this should not be the tone of this article, given the intent of the special issue for which it is being written. Therefore, I make no claims of the article being precise, rigorous, or even remotely self-contained. Readers looking for a summary of the differential geometric method in mechanics will not find it here. Many such summaries have been written, and many of them are good. Moreover, many books have been written on the topic of differential geometric modelling of mechanical systems, and many of these are good. A reader who has an inkling that they want to learn the differential geometric method will already have ventured into this part of the literature, so it seems pointless to simply reproduce it here.

Instead I aim to give the reader some insights into *why* I think differential geometric modelling is important. In doing so I will (1) make free use of intuitive “definitions” of concepts and (2) make free use of terminology and notation that I have not defined, but is hopefully either clear or for which the absence of a precise definition is inobtrusive. My feeling is that, as soon as I step onto the slippery slope where I disallow myself to do either of these two things, then I will end up writing a paper that I have already written before. And again, I think this is pointless.

One of my hopes for this article is that it will show that there are real advantages to be had from the differential geometric techniques of modelling and control of mechanical systems. Consistent with this, I will also make the point that there are *disadvantages* to not using differential geometric techniques. Moreover, I will illustrate this latter point through particular instances. Therefore, the article will be, in a very few places I hope, somewhat

provocative. In keeping with this, I have decided to write the article in a less formal and more personal style.

That all out of the way, let me get to the actual substance of the paper. Much of the ten or so years I have been doing research has been dedicated to control theory for mechanical systems. My work in this area has been of a pretty mathematical flavour, but with occasional and, I think, substantial diversions into specific problems that are intended to illustrate the value of what I do in practice. The most common feedback I get from applied researchers not familiar with the mathematics I use is (1) “I do not understand your notation” and (2) “I am not sure you are getting any mileage from all of the mathematics.” In this paper I will attempt to address these complaints.

First let me address the notation matter since I feel that it often has little substance as a criticism. In saying, “I do not understand your notation,” I often get the feeling that the complainant feels that I am simply using obtuse terminology in order to show off and make my work appear to have some substance. In this, there is the implicit suggestion that the notation is *just* that: notation. But this is not the case. Behind the notation are real concepts and ideas. To not use the notation I use, and instead use notation more familiar to whomever might not like mine, would be to exactly step away from those concepts and ideas that I actually use to form the contribution of the work. I think that the contribution would simply not exist were it not for the fact that the problem is formulated using the ideas that I actually use, and that includes using the attendant notation. So please, dear reader, do not make the mistake of believing that notation different from yours is simply notation; it is possible that behind the notation lies an idea. And maybe it is this idea, not just its representation by some particular notation, that is preventing you from understanding what is being presented.

The other complaint—that of whether differential geometric methods actually contribute anything in practice—has real substance, and it is to this matter that this paper is primarily addressed. There is wonderful elegance and clarity in the differential geometric formulation of mechanics. Much of the work done in the area of overlap between differential geometry and mechanics—indeed, almost all of it until, maybe, the past fifteen years—has been in understanding the structure of the mathematical models of the mechanics. While such research is certainly justified by its outcome, to an engineer looking for solutions to specific problems, it is not clear that the differential geometric formulation contributes to these ends. Thus any skepticism you might feel towards differential geometric techniques is justified, at least until it is clearly shown by people like me that these techniques contribute towards the solution of problems that are of interest to you. What I will try to do in this paper is indicate some of the kinds of problems where differential geometric ideas have made a clear and decisive contribution. I will argue that this contribution might arise in two ways.

1. It is sometimes the case that, merely by formulating a problem in differential geometric language, the right questions reveal themselves. This is an often overlooked contribution of precise mathematical formulations in general. In solving a problem it is often the question that is the real problem. Sometimes, once one has the right question, the problem either (a) solves itself or (b) reveals itself to be, in a precise way, impossible. In either event, knowing the right question is much more than half the battle.
2. Sometimes the question one is asking is perfectly clear, and it is the solution that is the

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real problem. In some such cases (but definitely not all!) differential geometric methods are just the thing, and can lead one to unexpectedly simple solutions.

I should make some effort to position what I say in this paper relative to the growing volume of literature on differential geometric methods for control of mechanical systems. It is impossible to give an accurate overview of the entirety of this research area, and I will not attempt to do so. Instead I will identify research directions that have achieved a somewhat polished state. Probably the first serious progress in uniting geometric mechanics and geometric control theory was made along Hamiltonian lines. The port-controlled Hamiltonian approach is representative of this, and a huge amount of work has been done here. We refer to the papers of [Dalsmo and van der Schaft \[1998\]](#), [van der Schaft \[2000\]](#) as representative of this work. A book that allies this Hamiltonian approach with topics of interest to readers of this journal is that of [Stramigioli \[2001\]](#).

Another significant body of research, this connected firmly with the origins of geometric mechanics, is that undertaken by researchers including John Baillieul, Anthony Bloch, Peter Crouch, P. S. Krishnaprasad, Naomi Leonard, Jerry Marsden, and Dmitry Zenkov. An account of some of this work can be found in the recent book of [Bloch \[2003\]](#).

The approach to mechanics and control taken in this paper is the one developed in detail in my book [\[Bullo and Lewis 2004\]](#) coauthored with Francesco Bullo from the University of California, Santa Barbara. In this book we develop from “first principles” an approach to the modelling of mechanical systems using Riemannian and affine differential geometry. After this we outline some of the control problems that can be addressed using the differential geometric modelling techniques. We do not attempt to fairly summarise the content of this book. One of the aims, instead, is to try to give the reader an idea of whether reading this book is worth their effort. For some readers I believe it *will* be worth the effort, but for others it may not be.

## 2. Modelling mechanical systems using affine differential geometry

In this section I present a very quick overview of Chapter 4 of [\[Bullo and Lewis 2004\]](#). The intention is to introduce the major mathematical players in the affine differential geometric formulation of mechanics. A novice reader expecting to read this section and walk away understanding affine differential geometry and its use in mechanics will be disappointed. The fact of the matter is that, if you are starting with only scant knowledge of differential geometry, there is simply no way one can appreciate a geometric formulation of mechanics in an afternoon. It will take a significant investment of effort and, more importantly, a probable abandoning of certain preconceptions learned during the course of a more classical mechanics education. Thus, what I attempt to do in this section is alert the reader to the “brute facts” of a geometric formulation of mechanics. These are the things that they will have to come to grips with in order to merely get off the ground.

**2.1. The configuration manifold.** One of the main reasons why differential geometry is an effective tool for formulating mechanics is that mechanical systems evolve naturally on a configuration space that is a manifold. That is to say, the set of configurations of a mechanical system is in natural 1–1 correspondence with a set that has the structure of a differentiable manifold; this manifold is called the configuration space and typically denoted

by  $Q$ . This idea is stated clearly by [Bullo and Lewis \[2004\]](#).

At this point I could define what a manifold is, but this seems pointless since this definition can be found in many places, including in [\[Bullo and Lewis 2004\]](#). Moreover, the reader will probably have a very good idea, or at least some intuition, about what a manifold is. Instead, therefore, let me indicate why the notion of a manifold is important in robotics. The reader is surely familiar with the fact that the configurations of a rigid body fixed at a point in some inertial frame are in 1–1 correspondence with the set  $SO(3)$  of  $3 \times 3$  orthogonal matrices with determinant 1. What is the best way to view  $SO(3)$ ? Some people choose to parameterise  $SO(3)$  with Euler angles. This makes it appear as though  $SO(3)$  is some subset of 3-dimensional Euclidean space, although clearly it is not. Other representations of  $SO(3)$  are used (e.g., quaternion representations) to attempt to get a handle on  $SO(3)$ . But from my point of view,  $SO(3)$  is a manifold and that is that. (Actually,  $SO(3)$  is a very special manifold, since it is also a Lie group, and this Lie group structure is extremely important in robotics. However, I elect not to get into this aspect of rigid body modelling here, but refer to [\[Bullo and Lewis 2004, Murray, Li, and Sastry 1994\]](#).) A manifold is, essentially a set that can locally be parameterised by Euclidean space (e.g., by Euler angles), and different parameterisations are required to be compatible with one another. A specific choice of parameterisation is sometimes very helpful in working on a concrete example. And the existence of parameterisations in general is sometimes useful in that it allows you to believe, at least temporarily, that you are working with familiar calculus in Euclidean space (typically by saying, “Let  $(q^1, \dots, q^n)$  be coordinates for  $Q$ ”). However, the important thing about manifolds is that, *when used properly*, they force a coordinate independent restriction on the problem formulation. Thus one is forced to deal with the real structure of the problem and not one tacked on, *ad hoc*, by a choice of coordinates.

Let me try to explain this in a very concrete context. In [Figure 1](#) I have depicted two simple systems, systems that are frequently used as benchmark-type systems to implement algorithms and so on. In the top left is a simple mobile robot. Let me suppose the three wheels are actuated, so this is really just a fully actuated planar rigid body. The objective is to steer the body to the  $\times$  with a prescribed orientation. In the bottom right in [Figure 1](#) I depict the classical pendulum/cart system where the objective is to balance the pendulum in the upright configuration by application of a force to the cart. One approach to modelling the mobile robot problem is to declare the coordinates describing the configurations of the system to be  $(x, y, \theta)$ , and then proceed as if the configuration space is  $\mathbb{R}^3$ . Similarly, for the pendulum, one could take coordinates  $(x, \theta)$  and proceed as if the configuration space is  $\mathbb{R}^2$ . In practice this is often accentuated by writing the coordinates as  $[x \ y \ \theta]^T$  and  $[x \ \theta]^T$  to drive home the fact these are “coordinate vectors.” But the configuration spaces are not  $\mathbb{R}^3$  or  $\mathbb{R}^2$  since, in each case,  $\theta = 0$  and  $\theta = 2\pi$  describe the same configuration. The configuration spaces are  $SE(2)$  (which is naturally diffeomorphic to  $\mathbb{R}^2 \times S^1$ ) and  $\mathbb{R} \times S^1$ , where  $S^1$  is the unit circle. (Other representations of the configuration spaces are possible, but they will be equivalent to what I have given in the precise way that they are diffeomorphic.) A very good question is, “Is it important to make this distinction?” Apparently it is, and for the following reason. I have read or reviewed tens of papers and attended tens of talks where it is claimed that a continuous globally stabilising feedback law has been designed for the mobile robot system or the pendulum/cart system (or something like them). But this is

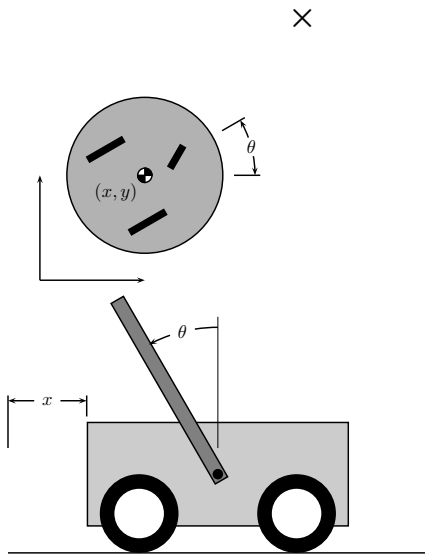


FIGURE 1. These systems are not globally stabilisable using continuous feedback!

impossible because of the topology of the configuration manifold!<sup>1</sup> That is to say, merely by believing that the configuration manifold is Euclidean space many researchers are led to claim something impossible. I once watched an hour long talk where a researcher described a research programme, involving several people, the entire aim of which was to design a global smooth stabilising feedback law for the pendulum/cart system. I was pleased to note, at least, that the programme had not yet been successful!

Returning to the question asked in the title of the paper, a partial answer is this. By formulating mechanical problems in terms of differential geometry, one must recognise that the configuration manifold is a manifold. Doing this properly often forces a recognition of important features of the problem structure.

**2.2. The tangent bundle to the configuration manifold.** The configuration manifold describes the set of configurations of a mechanical system. The tangent bundle of the configuration manifold describes the set of configurations and velocities of a mechanical system. Intuitively, the tangent space at a point  $q \in Q$ , denoted by  $T_qQ$ , is the set of all possible velocities possible at this configuration. From this intuitive description I hope it is

<sup>1</sup>Most simply, the salient feature of  $S^1$  is that any continuous  $\mathbb{R}$ -valued function on  $S^1$  that changes sign once must change sign at least twice. Less simply, the Euler characteristic of  $S^1$  is zero.

evident that  $T_qQ$  is a vector space of the same dimension as  $Q$ . The tangent bundle, denoted by  $TQ$ , is the union of all tangent spaces. The tangent bundle itself is a differentiable manifold, and inherits from  $Q$  a differentiable structure in a natural way. Coordinates for  $TQ$  are often denoted by  $((q^1, \dots, q^n), (v^1, \dots, v^n))$ . Thus  $TQ$  has twice the dimension of  $Q$ . The natural differentiable structure for  $TQ$  works out very nicely since it demands that, if one has two compatible sets of coordinates  $(q^1, \dots, q^n)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  for  $Q$ , then the corresponding velocities obey the very Chain Rule-like formula

$$\tilde{v}^i = \frac{\partial \tilde{q}^i}{\partial q^j} v^j, \quad i \in \{1, \dots, n\}.$$

(Here I use the summation convention commonly employed in differential geometry where summation is implied over indices that are repeated.) The way one should think of the tangent bundle is as the state space for the system. That is to say, the dynamics of the system are uniquely determined by initial conditions in the tangent bundle, this by virtue of the equations of motion being second-order (as we shall soon see, in case you have forgotten).

There is one caveat I would like to make concerning the tangent bundle. The representation of tangent bundle coordinates as  $((q^1, \dots, q^n), (v^1, \dots, v^n))$  would seem to imply that there is some sort of “product structure” to the tangent bundle. That is to say, it appears that the tangent bundle is the Cartesian product of the configuration manifold and the set of velocities. *This is not the case.* It is true that  $TQ$  is *locally* a product, but even this local product representation is not natural; it arises from a specific choice of coordinates and different coordinates give different local product representations. Thus, while the coordinates for  $TQ$  seem to validate the notation of representing points in  $TQ$  as  $(q, v)$ , this representation is, in a very precise way, incorrect. It certainly is the case that one would like to be able to indicate the configuration at which a velocity lives. A good way to do this is by writing a point in  $TQ$  as  $v_q$ , thus declaring by the notation that this is a tangent vector in the tangent space at  $q \in Q$ .

**2.3. External forces.** In a geometric setting for mechanics forces take values in the cotangent bundle. Let me quickly describe the cotangent bundle. The cotangent space at  $q \in Q$  is the dual space of  $T_qQ$  (i.e., the set of  $\mathbb{R}$ -valued linear maps on  $T_qQ$ ) and is denoted by  $T_q^*Q$ . The cotangent bundle is the union of all the cotangent spaces and is denoted by  $T^*Q$ . Often authors feel compelled to attempt to give some intuitive idea of what cotangent vectors are, typically because it is felt that cotangent vectors are somehow more mysterious than tangent vectors. My own view is that the definition is perfectly adequate, and one just has to accept it and learn that various physical quantities, for example forces, naturally are represented as elements of the cotangent bundle.

But why should forces take values in the cotangent bundle? Well, what does a force do? This is actually a fairly involved metaphysical question, but let me sidestep the metaphysics and just answer the question: A force does work on a system as it moves. Precisely, the situation is this. Suppose that a mechanical system undergoes a motion described by a curve  $t \mapsto \gamma(t)$  in  $Q$ . As it undergoes this motion, a force  $t \mapsto F(t)$  is applied to the system (being a little vague about just where  $F$  lives for the moment). The work done is then

$$W = \int_I F(t) \cdot \gamma'(t) dt,$$

where  $I$  is the time interval on which the motion is defined, i.e., the work is the integral of the product of force with velocity. I have just written the product  $F(t) \cdot \gamma'(t)$  as if it means something, but let me consider what  $F$  must be in order for this expression to have meaning. The expression  $F(t) \cdot \gamma'(t)$  should live in  $\mathbb{R}$  (work is a scalar quantity) and should be linear in  $\gamma'(t)$ . Thus  $F(t)$  can live no place other than  $\mathbb{T}_{\gamma(t)}^*\mathbb{Q}$ . This is the explanation I give about why forces are cotangent bundle-valued, and my undergraduate students seem to buy it. Note that the “ $\cdot$ ” in the expression  $F(t) \cdot \gamma'(t)$  is not the “dot product.” It represents an element of a dual space acting on a vector.

Forces in mechanics, at least when one is considering the control theory of mechanical systems, come in two flavours: external forces coming from the environment (potential forces, friction and dissipative forces, etc.) and control forces. Let me denote the total uncontrolled external force by  $F$ , noting that  $F$  may be a function of position, velocity, or time. The total control force I write as a linear combination of forces  $F^1, \dots, F^m$ . Thus the control force is

$$\sum_{a=1}^m u^a F^a,$$

where I ask that the forces  $F^a$  be functions only of position, but allow the controls  $u^a$  to be functions of position, velocity, or time, as necessitated by the control scheme used. The  $u^a$ 's are the controls.

**2.4. The kinetic energy metric.** The key to the affine differential geometric approach to mechanics is the fact that the kinetic energy of the system defines a Riemannian metric on the configuration space. Researchers in robotics will know this Riemannian metric as the “inertia matrix,” the “mass matrix,” or something similar. In the classical way of looking at things, the kinetic energy metric is a symmetric positive-definite matrix-valued function of configuration. In the geometric way of looking at things, the kinetic energy metric is a smooth assignment of an inner product to each tangent space of the configuration space. In differential geometry such an assignment is called a Riemannian metric. In the particular case where the assignment comes from the kinetic energy for a mechanical system we call the corresponding Riemannian metric the kinetic energy metric.

The notation used in [Bullo and Lewis 2004] for the kinetic energy metric is  $\mathbb{G}$ . Thus the inner product on the tangent space at  $q \in \mathbb{Q}$  is denoted by  $\mathbb{G}(q)$  (in the geometric formulation, the kinetic energy metric is a genuine map, it being a section of a certain tensor bundle), and the inner product of two tangent vectors  $u_q, v_q \in \mathbb{T}_q\mathbb{Q}$  is denoted by  $\mathbb{G}(q)(u_q, v_q)$ , or simply by  $\mathbb{G}(u_q, v_q)$  since the point  $q \in \mathbb{Q}$  is explicit in the notation already. Therefore, the kinetic energy is the function on  $\mathbb{T}\mathbb{Q}$  given by

$$\text{KE}(v_q) = \frac{1}{2}\mathbb{G}(v_q, v_q).$$

Once one has at hand a physical model for the system, including useful representations of the configuration space and its tangent bundle, the actual task of writing the kinetic energy metric is actually quite simple, at least in principle. Of course, in practice there are genuine issues involved with determining the inertial properties of one's system, etc., but there is not much one can say about these matters, within or without a geometric formulation of mechanics.

There is a useful bit of notation associated with a Riemannian metric  $\mathbb{G}$ . Since I have found that this notation can be off-putting to many readers, let me devote a few moments to an apology for it. Given a Riemannian metric  $\mathbb{G}$  on  $\mathbb{Q}$  there are naturally defined maps  $\mathbb{G}^\flat: \mathbb{T}\mathbb{Q} \rightarrow \mathbb{T}^*\mathbb{Q}$  and  $\mathbb{G}^\sharp: \mathbb{T}^*\mathbb{Q} \rightarrow \mathbb{T}\mathbb{Q}$  defined by

$$\begin{aligned} \mathbb{G}^\flat(v_q) \cdot u_q &= \mathbb{G}(u_q, v_q), & u_q, v_q &\in \mathbb{T}_q\mathbb{Q}, \\ \alpha_q \cdot \mathbb{G}^\sharp(\beta_q) &= \mathbb{G}^{-1}(\alpha_q, \beta_q), & \alpha_q, \beta_q &\in \mathbb{T}_q^*\mathbb{Q}, \end{aligned}$$

where by  $\mathbb{G}^{-1}$  I denote the inner product on  $\mathbb{T}_q^*\mathbb{Q}$  induced by that on  $\mathbb{T}_q\mathbb{Q}$ . If  $\mathbb{G}_{ij}$ ,  $i, j \in \{1, \dots, n\}$ , are the components of  $\mathbb{G}$  in a set of coordinates, then  $\mathbb{G}^\flat$  is simply multiplication by this matrix of components. If  $\mathbb{G}^{ij}$ ,  $i, j \in \{1, \dots, n\}$ , are the components of the inverse matrix with components  $\mathbb{G}_{ij}$ , then  $\mathbb{G}^\sharp$  is multiplication by this inverse matrix. The reason for the funny notation  $\flat$  and  $\sharp$  is that, in terms of indices,  $\mathbb{G}^\flat$  takes an object whose indices are superscripts (a tangent vector) and converts them to an object whose indices are subscripts (a cotangent vector). Thus  $\mathbb{G}^\flat$  “flattens” the indices. In a like manner,  $\mathbb{G}^\sharp$  raises or “sharpens” the indices.

**2.5. The Levi-Civita connection.** The components of the description of a mechanical system I have described thus far, the configuration manifold, its tangent bundle, forces, and the kinetic energy metric, will be well known to anyone in robotics, although not always necessarily in just the way that I think of these things. However, the Levi-Civita connection that I will describe now is, if you are not familiar with it, really something new. While it can be related to some things that may be familiar, it *is* precisely none of these things. The Levi-Civita affine connection is what it is, and, for those unfamiliar with it, this will take a little hard swallowing.

Since I am making the affine connection formulation of mechanics the centrepiece of this article, I ought to say what an affine connection is. I suppose the reader to be familiar with the basic concept of a vector field on a manifold and some of the notation surrounding the differential geometry of dealing with vector fields. Let me work with a manifold  $\mathbb{Q}$  (which may or not be the configuration manifold for a mechanical system). An affine connection<sup>2</sup> on  $\mathbb{Q}$  is a mapping that assigns to two vector fields  $X$  and  $Y$  a third vector field denoted  $\nabla_X Y$ , and this assignment has the following properties:

1. the map  $(X, Y) \mapsto \nabla_X Y$  is  $\mathbb{R}$ -bilinear;
2. for vector fields  $X$  and  $Y$  and a function  $f$  we have  $\nabla_f X Y = f \nabla_X Y$ ;
3. for vector fields  $X$  and  $Y$  and a function  $f$  we have  $\nabla_X(fY) = f \nabla_X Y + (\mathcal{L}_X f)Y$ , where  $\mathcal{L}_X$  is the Lie derivative with respect to  $X$ .

The vector field  $\nabla_X Y$  is called the covariant derivative of  $Y$  with respect to  $X$ . So this is the definition, and how very un insightful it is! However, the notion of an affine connection comes up very naturally in mechanics, and I will explain this as we go along. Let me first say a few general things about affine connections.

1. An affine connection is extra structure on a manifold. That is to say, manifolds do not come equipped with a nice natural affine connection that you can go ahead and

<sup>2</sup>What I call an “affine connection” is called a “linear connection” by some authors, and these authors have something else (something that is related, however) in mind when they say “affine connection.” However, the use of “affine connection” as I use it is fairly widespread.

start using. You have to introduce your own. Thankfully, in mechanics, there are affine connections that come from the mechanical problem data. We shall see how these arise.

2. An affine connection  $\nabla$  is described in coordinates  $(q^1, \dots, q^n)$  by writing

$$\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^k} = \Gamma_{jk}^i \frac{\partial}{\partial q^i},$$

which is always possible for  $n^3$  functions  $\Gamma_{jk}^i$ ,  $i, j, k \in \{1, \dots, n\}$ . These are called the Christoffel symbols for the affine connection and they clearly depend on a choice of coordinates. Using these, the covariant derivative of two general vectors fields is easily written:

$$\nabla_X Y = \left( \frac{\partial Y^i}{\partial q^j} X^j + \Gamma_{jk}^i X^j Y^k \right) \frac{\partial}{\partial q^i}.$$

3. Associated with every Riemannian metric  $\mathbb{G}$  is a unique affine connection, called the Levi-Civita connection and denoted (by me) as  $\overset{\circ}{\nabla}$ . There is a nice intrinsic characterisation of  $\overset{\circ}{\nabla}$  and for this I refer the reader to [Bullo and Lewis 2004]. Here let me give a very hands-on, but very grimy, characterisation by giving the formula for its Christoffel symbols:

$$\overset{\circ}{\Gamma}_{jk}^i = \frac{1}{2} \mathbb{G}^{il} \left( \frac{\partial \mathbb{G}_{lj}}{\partial q^k} + \frac{\partial \mathbb{G}_{lk}}{\partial q^j} - \frac{\partial \mathbb{G}_{jk}}{\partial q^l} \right).$$

We shall see in Section 2.7 where this affine connection arises from.

4. Associated with an affine connection  $\nabla$  are special curves on  $\mathbb{Q}$  called the geodesics of  $\nabla$ . These admit a nice intrinsic description too, but let me simply give the differential equation in coordinates that these curves must satisfy:

$$\ddot{q}^i(t) + \Gamma_{jk}^i \dot{q}^j(t) \dot{q}^k(t) = 0, \quad i \in \{1, \dots, n\}. \quad (2.1)$$

In coordinate-free language this equation is written as  $\nabla_{\gamma'(t)} \gamma'(t) = 0$ . Thus  $\nabla_{\gamma'(t)} \gamma'(t)$  is the slick way of writing what is on the left-hand side of (2.1), where  $t \mapsto \gamma(t)$  is the curve that in coordinates has the form  $t \mapsto (q^1(t), \dots, q^n(t))$ .

It is through geodesics that one can most easily begin to get at what an affine connection “means.” So let me say a few words about this.

- (a) If  $\nabla = \overset{\circ}{\nabla}$  is the Levi-Civita connection, then geodesics are those curves which locally minimise length.
- (b) Note that the coordinate expression in (2.1) for  $\nabla_{\gamma'(t)} \gamma'(t)$  is a sum of “ $\ddot{q}^i$ ” and “ $\gamma_{jk}^i \dot{q}^j(t) \dot{q}^k(t)$ .” *It is an act of unspeakable violence to break these terms apart.* These terms always belong together, and together they mean “ $\nabla_{\gamma'(t)} \gamma'(t)$ .” Separately they mean absolutely, positively nothing.<sup>3</sup> One consequence of this is that the expression “ $\nabla_{\gamma'(t)} \gamma'(t)$ ” gives a way of understanding the acceleration along the curve  $\gamma$ . But note that this expression of acceleration is not intrinsic, it depends on a choice of affine connection. Some lengths are taken in Section 4.3.5 of [Bullo and Lewis 2004] to describe this.

<sup>3</sup>Okay, I am telling a fib here for emphasis. The expression  $\ddot{q}^i$  makes sense as a local coordinate for the second jet bundle. But I do not wish to go in that direction, so let me stick to my expression of the unspeakable violence of rendering these terms asunder, since you cannot be led wrong in accepting it.

The primary message of the preceding discussion is this:

*Associated in a unique way with the kinetic energy metric  $\mathbb{G}$  for a mechanical connection is an affine connection called the Levi-Civita connection. The importance of this will be made clear in Section 2.7.*

**2.6. Nonholonomic constraints.** The final ingredient in my description of mechanics before I write the equations of motion is nonholonomic constraints. Sometimes it seems to me as if there are as many techniques for deriving the equations of motion in the presence of nonholonomic constraints as there are people who have thought about doing this. These methods are, at least the correct ones, all distinguished by one simple fact: *they are all the same!* That is to say, Method X for writing equations of motion for a system subject to nonholonomic constraints is correct if and only if it agrees with Newton’s and Euler’s equations in the intersection where Method X and Newton’s and Euler’s equations apply. Many authors claim mystical advantages of their correct technique over other (necessarily equivalent) correct techniques. The mystical advantages too often are described in some metaphysical form, and Newton and Euler do not even get mentioned. My take on this is: If your technique is equivalent to Newton and Euler, then you are correct and so is everyone else whose method is equivalent to Newton and Euler. There can be real issues in practice, such as coming up with a technique that is easy to implement. However, there are no “in principle” advantages of any one correct method over any other correct method.

That all being said, let me just say here what I mean when I say “nonholonomic constraint,” and how these constraints get accounted for in the equations of motion. A nonholonomic constraint on a configuration manifold  $\mathbb{Q}$  is a distribution  $\mathcal{D}$  on  $\mathbb{Q}$ . That is, we select a subspace  $\mathcal{D}_q$  of each tangent space  $\mathbb{T}_q \mathbb{Q}$  that describes the set of velocities admissible at that configuration. Other generalisations are possible, for example, to time-dependent or nonlinear constraints. But this complication adds little to the conceptual picture.

The way in which nonholonomic constraints get incorporated into the equations of motion is by adding a force, called the constraint force, to the equations of motion. A constraint force is, by definition, one that does no work on admissible velocities. That is, at a configuration  $q \in \mathbb{Q}$ , a force  $\lambda_q \in \mathbb{T}_q^* \mathbb{Q}$  is a constraint force if  $\lambda_q \cdot v_q = 0$  for every  $v_q \in \mathcal{D}_q$ . The idea, as we shall see clearly in the next section, is that a constraint force is introduced into the problem as an unknown to be solved for, a little like, but not exactly like, a Lagrange multiplier.

**2.7. The equations of motion.** As I suggested in the preceding section, the equations of motion for a mechanical system are given by Newton’s and Euler’s equations that are based on simple laws of force and moment balance. The precise variant of Newton’s and Euler’s equations one wishes to use—for example, the standard Euler–Lagrange equations, the Gibbs–Appell equations, Gauss’s Principle of Least Action, the Principle of Virtual Work—is subject to taste. For my approach, the key is the following result.

**2.1 THEOREM:** (General equations of motion) *Consider a mechanical system with configuration manifold  $\mathbb{Q}$ , kinetic energy metric  $\mathbb{G}$ , external force  $F$ , control force  $\sum_{a=1}^m u^a F^a$ , and nonholonomic constraint  $\mathcal{D}$ . For a curve  $t \mapsto \gamma(t)$  the following are equivalent:*

- (i) *the force/moment balance equations of Newton/Euler hold along  $\gamma$ ;*

(ii)  $\gamma$  satisfies the equations

$$\mathbb{G}^b(\overset{\circ}{\nabla}_{\gamma'(t)}\gamma'(t)) = F(t, \gamma'(t)) + \sum_{a=1}^m u^a(t)F^a(\gamma(t)) + \lambda(t),$$

$$\gamma'(t) \in \mathcal{D}_{\gamma(t)},$$

where  $t \mapsto \lambda(t)$  is a constraint force along  $\gamma$ .

For unconstrained systems, i.e., where the nonholonomic constraint is  $\mathcal{D} = \text{TQ}$ , this immediately gives the following corollary.

**2.2 COROLLARY:** (Equations of motion for unconstrained systems) *Consider an unconstrained mechanical system with configuration manifold  $\mathbf{Q}$ , kinetic energy metric  $\mathbb{G}$ , external force  $F$ , and control force  $\sum_{a=1}^m u^a F^a$ . For a curve  $t \mapsto \gamma(t)$  the following are equivalent:*

- (i) the force/moment balance equations of Newton/Euler hold along  $\gamma$ ;
- (ii)  $\gamma$  satisfies the equations

$$\mathbb{G}^b(\overset{\circ}{\nabla}_{\gamma'(t)}\gamma'(t)) = F(t, \gamma'(t)) + \sum_{a=1}^m u^a(t)F^a(\gamma(t)).$$

It is perhaps worth comparing these equations with the equations of motion in the usual form seen in the robotics and mechanics literature. The usual form of the equations is something like

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + N(t, \mathbf{q}, \dot{\mathbf{q}}) = \mathbf{B}\mathbf{u}.$$

In components the equations of Corollary 2.2 are

$$\mathbb{G}_{il}\ddot{q}^l + \mathbb{G}_{ij}\overset{\circ}{\Gamma}_{jk}^l \dot{q}^j \dot{q}^k = F_i(t, q, \dot{q}) + \sum_{a=1}^m u^a F_i^a(q).$$

Thus we can directly compare the terms in the two equations as per Table 1. Note that I

**TABLE 1.** Comparison of terms in “classical,” “geometric component,” and “intrinsic” formulation of equations of motion

Classical form	Geometric coordinate form	Intrinsic form
$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$	$\mathbb{G}_{il}\ddot{q}^l + \mathbb{G}_{ij}\overset{\circ}{\Gamma}_{jk}^l \dot{q}^j \dot{q}^k$	$\overset{\circ}{\nabla}_{\gamma'(t)}\gamma'(t)$
$N(t, \mathbf{q}, \dot{\mathbf{q}})$	$-F_i(t, q, \dot{q})$	$-F(t, \gamma'(t))$
$\mathbf{B}\mathbf{u}$	$\sum_{a=1}^m u^a F_i^a(q)$	$\sum_{a=1}^m u^a F^a(\gamma(t))$

do not separate  $M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ , and neither should you, given my comments in item 4b in Section 2.5.

For systems with nonholonomic constraints the usual way to proceed is to eliminate the constraint force in some way. In terms of the affine connection picture I am painting, the magical thing is that, after elimination of the constraint force, one still gets equations involving an affine connection. I will not explain the way this works here since the most

direct explanation involves manipulations with affine connections that are one step beyond elementary, and I do not want to get into too much intricate stuff. The elimination of the constraint force also results in a modification of the external and control forces. Let me just summarise descriptively the final tagline and refer to [Bullo and Lewis 2004] for a thorough discussion of how this works.

*For a mechanical system with configuration manifold  $\mathbf{Q}$ , kinetic energy metric  $\mathbb{G}$ , external force  $F$ , control force  $\sum_{a=1}^m u^a F^a$ , and nonholonomic constraint  $\mathcal{D}$ , there exists (1) an affine connection  $\overset{\circ}{\nabla}$  depending on  $\mathbb{G}$  and  $\mathcal{D}$ , (2) a  $\text{TQ}$ -valued map  $Y$  depending on  $F$ ,  $\mathbb{G}$ , and  $\mathcal{D}$ , and (3) vector fields  $Y_1, \dots, Y_m$ , depending on  $F^1, \dots, F^m$ ,  $\mathbb{G}$ , and  $\mathcal{D}$ , such that, for a curve  $t \mapsto \gamma(t)$  the following are equivalent:*

- (i) the force/moment balance equations of Newton/Euler hold along  $\gamma$ ;
- (ii)  $\gamma$  satisfies the equations

$$\overset{\circ}{\nabla}_{\gamma'(t)}\gamma'(t) = Y(t, \gamma'(t)) + \sum_{a=1}^m u^a(t)Y_a(\gamma(t)), \quad (2.2)$$

$$\gamma'(0) \in \mathcal{D}_{\gamma(0)}.$$

Therefore, the point is that the equations of motion are given by the equations (2.2), and a fundamental rôle in the equations is played by an affine connection.

### 3. Some problems whose solution is facilitated by using differential geometry

The bottom line of the preceding section is equations (2.2), which include the equations of motion for a large class of mechanical systems. These equations are very nice since they are so compact, general, and revealing of the geometric structure of the problem. However, the fact of the matter is that, if this is the end of what we can do with the affine connection formalism, then serious questions can be raised about whether there is a payoff in learning the mathematics necessary to understand the formulae. Thankfully, there are problems beyond the mere formulation of the equations of motion that are effectively addressed within the differential geometric framework. In this section I investigate some of these. Since everything I will talk about here has already appeared in the literature, and since a detailed presentation of the results would necessitate reproducing huge chunks of the papers in question, I instead *describe* the results in the papers in question rather than reproduce them. I think that this approach is more in keeping with the objectives of the paper.

**3.1. Controllability.** One of the fundamental problems in the control of mechanical systems is, “Can a system be steered from Point A to Point B?” This is a controllability question. For systems that are fully actuated, i.e., when the control forces  $F^1, \dots, F^m$  span  $\text{T}_q^*\mathbf{Q}$  for each  $q \in \mathbf{Q}$ , these controllability questions are trivial since one can always apply a force to make the system go wherever you want. However, for underactuated systems, controllability questions are generally extremely difficult. Moreover, if the system

is underactuated *and* if there are no uncontrolled external forces (e.g., no potential forces, so ruling out the multitude of pendulum systems that are often considered), then controllability becomes a very difficult problem. The approach I take here is to give a brief general discussion of the sort of controllability problems I will consider, and then look at an example where I outline the controllability that can be deduced based on some more or less recent work that is heavily steeped in techniques of differential geometry.

The controllability problem I consider is purely a local one. Namely, I consider a mechanical system with configuration manifold  $Q$  and whose equations of motion are

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^m u^a(t)Y_a(\gamma(t)), \quad (3.1)$$

recalling from Section 2.7 that this equation models mechanical systems, possibly with nonholonomic constraints, that are subject to no uncontrolled external forces. I let  $q_0 \in Q$  and denote by  $\mathcal{R}_Q(q_0, \leq T)$  the set of configurations reachable from  $q_0$  in time at most  $T$ . The system is small-time locally configuration controllable (STLCC) from  $q_0$  if  $q_0$  is an interior point of  $\mathcal{R}_Q(q_0, \leq T)$ . This definition is a bit pared down from the most general definition, and I refer to [Bullo and Lewis 2004] for the full details. The results I will refer to in this section are from the work of Bullo and Lewis [2003b], Lewis [2000], Lewis and Murray [1997], Lewis and Tyner [2004]. All of the results in these papers rely heavily on the affine connection structure of the system equations (3.1).

As an example of the sort of system whose controllability can be described by the results I refer to, I consider the planar rigid body system depicted in Figure 2. This is a

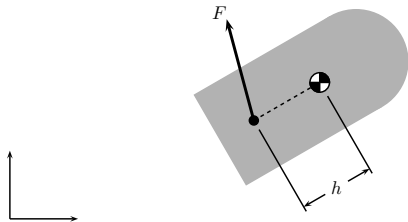


FIGURE 2. A controlled planar body

system moving on a planar surface orthogonal to the direction of the gravitational field, and I assume that there are no friction forces. One might want to think of this system as an idealised model of a hovercraft whose motive force is supplied by a single thrust fan for which the direction and magnitude of the force can be varied. One problem of interest for this system is to steer the system from rest in a configuration  $q_A$  to rest in a configuration  $q_B$ . In order to do this, one would require that  $q_B$  be in the set of configurations reachable from  $q_A$ . Thus the controllability problem is fundamental to this motion planning problem. In [Lewis and Murray 1997] it is shown that this system is, indeed, STLCC from every configuration. The results in [Lewis and Murray 1997] are an affine differential geometric formulation of general accessibility results of Sussmann and Jurdjevic [1972] and local controllability results of Sussmann [1987].

Let me now consider a slight alteration of the system by supposing that the direction of the force is now fixed, say at an angle of  $\frac{\pi}{2}$  as shown in Figure 3. Effectively, this

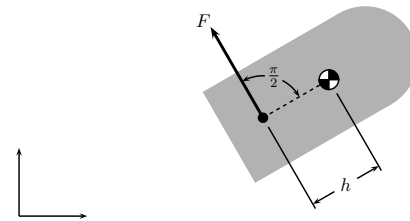


FIGURE 3. A controlled planar body with a single control input

system now has a single-input. It turns out that the controllability of this system cannot be determined using the results of Lewis and Murray [1997]. However, these results were refined in a certain direction in [Bullo and Lewis 2003b] where it is shown that this system, and indeed any single-input system of the form (3.1), is controllable from at most a “thin”<sup>4</sup> set of points.

Finally let me consider another alteration of the model as depicted in Figure 4. For

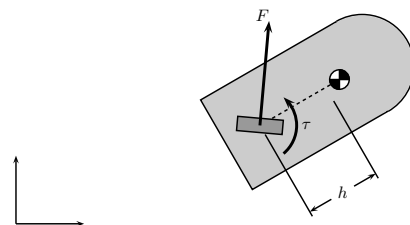


FIGURE 4. A controlled planar body with the fan dynamics modelled

this model, the fan is assumed to have inertia, so this adds an extra degree of freedom to the model. Once again, this system was one whose controllability I was unable to analyse when I first encountered it. However, after some work by Lewis and Tyner [2004] it was shown that this system was only STLCC from the configurations shown in Figure 5 and SE(2)-translations of these configurations.

The point of the presentation in this section might be said to be the following.

1. Underactuated mechanical systems often have subtle local controllability properties. This is clearly evidenced in this section by the difficulty of determining the controllability

<sup>4</sup>Precisely, a set that is locally the set of zeros of a finite collection of analytic functions.

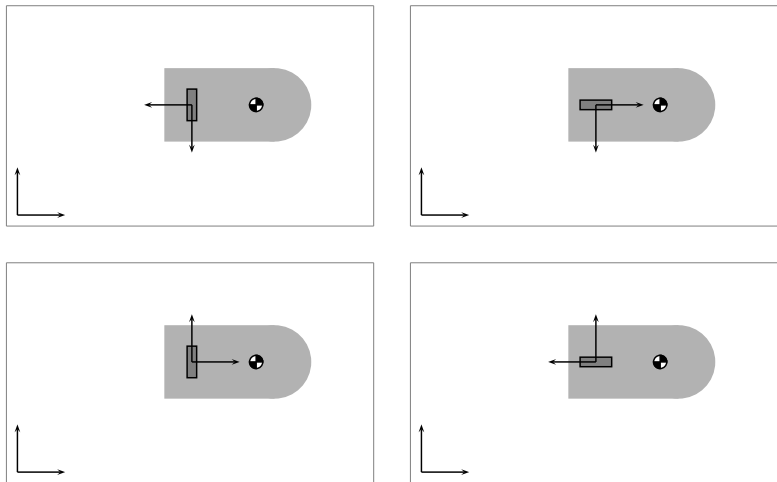


FIGURE 5. The configurations from which the planar body with fan dynamics is controllable

of a simple planar body system with some simple, natural variations in the model.

2. By understanding the geometric structure of the model of the system, particularly the rôle played by the affine connection, it is possible to develop general results that allow the analysis of some nontrivial examples.

Moreover, the controllability results referred to in this section lead naturally to some techniques in motion planning that I now describe.

**3.2. Trajectory planning using decoupling vector fields.** In this section I shall describe what can be achieved in terms of converting the controllability results used in the preceding section to motion planning results. The idea I describe here was motivated by explorations of Arai, Tanie, and Shiroma [1998], Lynch, Shiroma, Arai, and Tanie [2000]. The initial geometric presentation is due to Bullo and Lynch [2001] and the most polished form of the results appear in [Bullo and Lewis 2003b]. The idea is fairly simple. For a given mechanical system whose equations of motion are given by

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^m u^a(t)Y_a(\gamma(t)), \quad (3.2)$$

one seeks vector fields  $X_1, \dots, X_k$ , called decoupling vector fields, that have the property that their integral curves, and any reparameterisations of these integral curves, can be followed by a trajectory of the system (3.2). One then hopes to find sufficiently many

such vector fields that any motion planning problem can be solved by concatenating their integral curves.

Again, rather than restating the results which are given in the language of affine differential geometry, let me give a problem for which the ideas lead to a solution of the motion planning problem. The system we consider is the snakeboard system introduced by Lewis, Ostrowski, Murray, and Burdick [1994]. The motion planning strategy we present here is described by Bullo and Lewis [2003a].

The snakeboard system is shown in Figure 6. This is an example of a system with

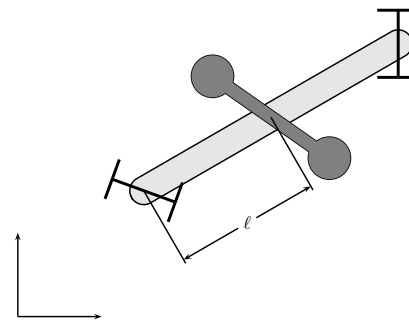


FIGURE 6. The snakeboard

nonholonomic constraints. The idea of the system is that there is a central coupler at each end of which are wheels whose angle relative to the coupler is controllable. Atop the coupler, between the wheels, is an inertial rotor that is also controlled. Thus the system has five degrees of freedom and two controls. The objective is to steer the snakeboard from an initial configuration at rest to a desired configuration at rest. Bullo and Lewis [2003a] show that the snakeboard has two decoupling vector fields and indicate how to use these to explicitly solve the motion planning problem. In Figure 7 we show trace of a solution of the motion planning problem where the snakeboard is steered from the position at the origin to a specified final position. The motion planning controller that generates Figure 7 is very simple, but is directly derived from the controllability results of Bullo and Lewis [2003b].

Other systems where the motion planning problem can be solved are given in Chapter 13 of Bullo and Lewis [2004].

The point of what we say in this section includes the following.

1. The idea of motion planning using decoupling vector fields is an excellent example of one where the right question is suggested by the differential geometric formulation.
2. Controllability is often seen as being a somewhat esoteric undertaking. Very often systems are controllable in a simple way, and in such cases sophisticated controllability techniques are more than what is needed. However, mechanical systems provide a rich class of systems where controllability is often not easy to characterise. Moreover, by understanding the controllability properties of a system, it is possible that one might



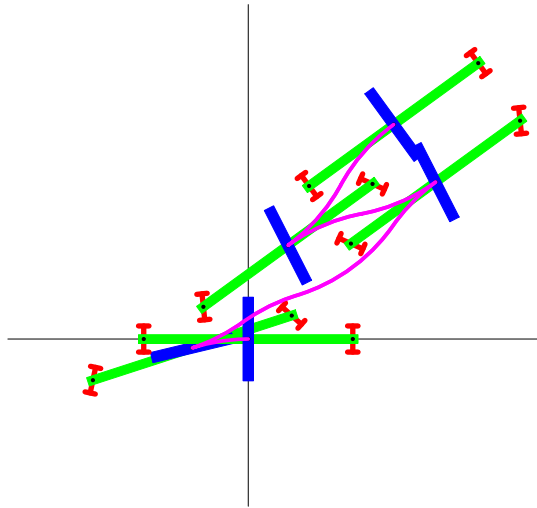


FIGURE 7. A solution of the snakeboard motion planning problem

be able to derive explicit solutions to the motion planning problem.

3. While the affine connection formulation of nonholonomic systems is fairly complicated, the application of this formulation to the solution of the motion planning problem for the snakeboard justifies the complication.

**3.3. Energy shaping.** The last idea we discuss in this section is something that is actually very complicated, and is still the topic of ongoing research. The idea is this. Suppose one is given a mechanical system, let me call it the “open-loop system,” with kinetic energy metric  $\mathbb{G}_{ol}$  and with the uncontrolled external forces being those coming from a potential function  $V_{ol}$ . Using feedback, one wishes to transform the system into a mechanical system, let me call it the “closed-loop system,” with kinetic energy metric  $\mathbb{G}_{cl}$  and external forces being those coming from a potential function  $V_{cl}$ . The value of this idea is that much can be said about the behaviour of a mechanical system by knowing properties of the potential function. In particular, equilibria and their stability are easily determined from the potential. Thus this technique of “energy shaping,” i.e., changing the energy of the system using feedback, is a useful tool for stabilisation. The original idea of potential shaping for fully actuated systems dates to [Takegaki and Arimoto 1981]. The situation in the underactuated case was worked out by van der Schaft [1986]. The possibility of shaping the kinetic energy was considered first in a Lagrangian setting by Bloch, Chang, Leonard, and Marsden [2001], Bloch, Leonard, and Marsden [2000] and in a Hamiltonian setting by

Ortega, Spong, Gómez-Estern, and Blankenstein [2002].

These ideas of energy shaping lead to partial differential equations and, particularly when kinetic energy shaping is considered, these partial differential equations are extremely complicated. It is difficult to even give conditions ensuring that these partial differential equations have solutions. A rather course general result is given by Auckly and Kapitanski [2002], and they make use of some Riemannian geometry in their development. A fairly refined result concerning part of the problem is given by Lewis [2006]. In both cases, the results are too complicated to reproduce here. But the point is this.

1. Sometimes very natural engineering questions can lead to extremely challenging problems that require sophisticated mathematics for their solution.

#### 4. In conclusion

In this paper I have attempted to illustrate that there is value in the differential geometric approach to modelling and control of mechanical systems. In particular, in Section 3 I have attempted to make it clear that this value extends beyond a mere elegant formulation of the equations of motion. What I hope is that an uninitiated reader interested in the differential geometric approach might now attack the daunting literature with something of a sense of purpose, knowing that there are possible rewards awaiting them at the end of their journey.

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