

# **Math 439 Course Notes**

## **Lagrangian Mechanics, Dynamics, and Control**

Andrew D. Lewis

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## Preface

These notes deal primarily with the subject of Lagrangian mechanics. Matters related to mechanics are the dynamics and control of mechanical systems. While dynamics of Lagrangian systems is a generally well-founded field, control for Lagrangian systems has less of a history. In consequence, the control theory we discuss here is quite elementary, and does not really touch upon some of the really challenging aspects of the subject. However, it is hoped that it will serve to give a flavour of the subject so that people can see if the area is one which they'd like to pursue.

Our presentation begins in Chapter 1 with a very general axiomatic treatment of basic Newtonian mechanics. In this chapter we will arrive at some conclusions you may already know about from your previous experience, but we will also very likely touch upon some things which you had not previously dealt with, and certainly the presentation is more general and abstract than in a first-time dynamics course. While none of the material in this chapter is technically hard, the abstraction may be off-putting to some. The hope, however, is that at the end of the day, the generality will bring into focus and demystify some basic facts about the dynamics of particles and rigid bodies. As far as we know, this is the first thoroughly Galilean treatment of rigid body dynamics, although Galilean particle mechanics is well-understood.

Lagrangian mechanics is introduced in Chapter 2. When instigating a treatment of Lagrangian mechanics at a not quite introductory level, one has a difficult choice to make; does one use differentiable manifolds or not? The choice made here runs down the middle of the usual, “No, it is far too much machinery,” and, “Yes, the unity of the differential geometric approach is exquisite.” The basic concepts associated with differential geometry are introduced in a rather pragmatic manner. The approach would not be one recommended in a course on the subject, but here serves to motivate the need for using the generality, while providing some idea of the concepts involved. Fortunately, at this level, not overly many concepts are needed; mainly the notion of a coordinate chart, the notion of a vector field, and the notion of a one-form. After the necessary differential geometric introductions are made, it is very easy to talk about basic mechanics. Indeed, it is possible that the extra time needed to understand the differential geometry is more than made up for when one gets to looking at the basic concepts of Lagrangian mechanics. All of the principal players in Lagrangian mechanics are simple differential geometric objects. Special attention is given to that class of Lagrangian systems referred to as “simple.” These systems are the ones most commonly encountered in physical applications, and so are deserving of special treatment. What's more, they possess an enormous amount of structure, although this is barely touched upon here. Also in Chapter 2 we talk about forces and constraints. To talk about control for Lagrangian systems, we must have at hand the notion of a force. We give special attention to the notion of a dissipative force, as this is often the predominant effect which is unmodelled in a purely Lagrangian system. Constraints are also prevalent in many application areas, and so demand attention. Unfortunately, the handling of constraints in the literature is often excessively complicated. We try to make things as simple as possible, as the ideas indeed are not all that complicated. While we do not intend these notes to be a detailed description of Hamiltonian mechanics, we do briefly discuss the link between

Lagrangian Hamiltonian mechanics in Section 2.9. The final topic of discussion in Chapter 2 is the matter of symmetries. We give a Noetherian treatment.

Once one uses the material of Chapter 2 to obtain equations of motion, one would like to be able to say something about how solutions to the equations behave. This is the subject of Chapter 3. After discussing the matter of existence of solutions to the Euler-Lagrange equations (a matter which deserves some discussion), we talk about the simplest part of Lagrangian dynamics, dynamics near equilibria. The notion of a linear Lagrangian system and a linearisation of a nonlinear system are presented, and the stability properties of linear Lagrangian systems are explored. The behaviour is nongeneric, and so deserves a treatment distinct from that of general linear systems. When one understands linear systems, it is then possible to discuss stability for nonlinear equilibria. The subtle relationship between the stability of the linearisation and the stability of the nonlinear system is the topic of Section 3.2. While a general discussion the dynamics of Lagrangian systems with forces is not realistic, the important class of systems with dissipative forces admits a useful discussion; it is given in Section 3.5. The dynamics of a rigid body is singled out for detailed attention in Section 3.6. General remarks about simple mechanical systems with no potential energy are also given. These systems are important as they are extremely structure, yet also very challenging. Very little is really known about the dynamics of systems with constraints. In Section 3.8 we make a few simple remarks on such systems.

In Chapter 4 we deliver our abbreviated discussion of control theory in a Lagrangian setting. After some generalities, we talk about “robotic control systems,” a generalisation of the kind of system one might find on a shop floor, doing simple tasks. For systems of this type, intuitive control is possible, since all degrees of freedom are actuated. For underactuated systems, a first step towards control is to look at equilibrium points and linearise. In Section 4.4 we look at the special control structure of linearised Lagrangian systems, paying special attention to the controllability of the linearisation. For systems where linearisations fail to capture the salient features of the control system, one is forced to look at nonlinear control. This is quite challenging, and we give a terse introduction, and pointers to the literature, in Section 4.5.

Please pass on comments and errors, no matter how trivial. Thank you.

Andrew D. Lewis  
andrew@mast.queensu.ca  
420 Jeffery  
x32395

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# Chapter 1

## Newtonian mechanics in Galilean spacetimes

One hears the term relativity typically in relation to Einstein and his two theories of relativity, the special and the general theories. While the Einstein’s general theory of relativity certainly supplants Newtonian mechanics as an accurate model of the macroscopic world, it is still the case that Newtonian mechanics is sufficiently descriptive, and easier to use, than Einstein’s theory. Newtonian mechanics also comes with its form of relativity, and in this chapter we will investigate how it binds together the spacetime of the Newtonian world. We will see how the consequences of this affect the dynamics of a Newtonian system. On the road to these lofty objectives, we will recover many of the more prosaic elements of dynamics that often form the totality of the subject at the undergraduate level.

### 1.1 Galilean spacetime

Mechanics as envisioned first by Galileo Galilei (1564–1642) and Isaac Newton (1643–1727), and later by Leonhard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813), Pierre-Simon Laplace (1749–1827), etc., take place in a Galilean spacetime. By this we mean that when talking about Newtonian mechanics we should have in mind a particular model for physical space in which our objects are moving, and means to measure how long an event takes. Some of what we say in this section may be found in the first chapter of [Arnol’d 1989] and in the paper [Artz 1981]. The presentation here might seem a bit pretentious, but the idea is to emphasise that Newtonian mechanics is a axio-deductive system, with all the advantages and disadvantages therein.

**1.1.1 Affine spaces** In this section we introduce a concept that bears some resemblance to that of a vector space, but is different in a way that is perhaps a bit subtle. An affine space may be thought of as a vector space “without an origin.” Thus it makes sense only to consider the “difference” of two elements of an affine space as being a vector. The elements themselves are not to be regarded as vectors. For a more thorough discussion of affine spaces and affine geometry we refer the reader to the relevant sections of [Berger 1987].

**1.1.1 Definition** Let  $V$  be a  $\mathbb{R}$ -vector space. An *affine space* modelled on  $V$  is a set  $A$  and a map  $\phi: V \times A \rightarrow A$  with the properties

AS1. for every  $x, y \in A$  there exists  $v \in V$  so that  $y = \phi(v, x)$ ,

AS2.  $\phi(v, x) = x$  for every  $x \in A$  implies that  $v = 0$ ,

AS3.  $\phi(0, x) = x$ , and

AS4.  $\phi(u + v, x) = \phi(u, \phi(v, x))$ . □

We shall now cease to use the map  $\phi$  and instead use the more suggestive notation  $\phi(v, x) = v + x$ . By properties [AS1](#) and [AS2](#), if  $x, y \in A$  then there exists a unique  $v \in V$  such that  $v + x = y$ . In this case we shall denote  $v = y - x$ . Note that the minus sign is simply notation; we have not really defined “subtraction” in  $A$ ! The idea is that to any two points in  $A$  we may assign a unique vector in  $V$  and we notationally write this as the difference between the two elements. All this leads to the following result.

**1.1.2 Proposition** *Let  $A$  be a  $\mathbb{R}$ -affine space modelled on  $V$ . For fixed  $x \in A$  define vector addition on  $A$  by*

$$y_1 + y_2 = ((y_1 - x) + (y_2 - x)) + x$$

*(note  $y_1 - x, y_2 - x \in V$ ) and scalar multiplication on  $A$  by*

$$ay = (a(y - x)) + x$$

*(note that  $y - x \in V$ ). These operations make  $A$  a  $\mathbb{R}$ -vector space and  $y \mapsto y - x$  is an isomorphism of this  $\mathbb{R}$ -vector space with  $V$ .*

This result is easily proved once all the symbols are properly understood (see Exercise [E1.1](#)). The gist of the matter is that for *fixed*  $x \in A$  we can make  $A$  a  $\mathbb{R}$ -vector space in a *natural* way, but this *does* depend on the choice of  $x$ . One can think of  $x$  as being the “origin” of this vector space. Let us denote this vector space by  $A_x$  to emphasise its dependence on  $x$ .

A subset  $B$  of a  $\mathbb{R}$ -affine space  $A$  modelled on  $V$  is an **affine subspace** if there is a subspace  $U$  of  $V$  with the property that  $y - x \in U$  for every  $x, y \in B$ . That is to say,  $B$  is an affine subspace if all of its points “differ” by some subspace of  $V$ . In this case  $B$  is itself a  $\mathbb{R}$ -affine space modelled on  $U$ . The following result further characterises affine subspaces. Its proof is a simple exercise in using the definitions and we leave it to the reader (see Exercise [E1.2](#)).

**1.1.3 Proposition** *Let  $A$  be a  $\mathbb{R}$ -affine space modelled on the  $\mathbb{R}$ -vector space  $V$  and let  $B \subset A$ . The following are equivalent:*

- (i)  $B$  is an affine subspace of  $A$ ;*
- (ii) there exists a subspace  $U$  of  $V$  so that for some fixed  $x \in B$ ,  $B = \{u + x \mid u \in U\}$ ;*
- (iii) if  $x \in B$  then  $\{y - x \mid y \in B\} \subset V$  is a subspace.*

**1.1.4 Example** A  $\mathbb{R}$ -vector space  $V$  is a  $\mathbb{R}$ -affine space modelled on itself. To emphasise the difference between  $V$  the  $\mathbb{R}$ -affine space and  $V$  the  $\mathbb{R}$ -vector space we denote points in the former by  $x, y$  and points in the latter by  $u, v$ . We *define*  $v + x$  (the affine sum) to be  $v + x$  (the vector space sum). If  $x, y \in V$  then  $y - x$  (the affine difference) is simply given by  $y - x$  (the vector space difference). Figure [1.1](#) tells the story. The essential, and perhaps hard to grasp, point is that  $u$  and  $v$  are not to be regarded as vectors, but simply as points.

An affine subspace of the affine space  $V$  is of the form  $x + U$  (affine sum) for some  $x \in V$  and a subspace  $U$  of  $V$ . Thus an affine subspace is a “translated” subspace of  $V$ . Note that in this example this means that affine subspaces do not have to contain  $0 \in V$ —affine spaces have no origin.  $\square$

Maps between vector spaces that preserve the vector space structure are called linear maps. There is a similar class of maps between affine spaces. If  $A$  and  $B$  are  $\mathbb{R}$ -affine spaces modelled on  $V$  and  $U$ , respectively, a map  $f: A \rightarrow B$  is a  **$\mathbb{R}$ -affine map** if for each  $x \in A$ ,  $f$  is a  $\mathbb{R}$ -linear map between the  $\mathbb{R}$ -vector spaces  $A_x$  and  $B_{f(x)}$ .

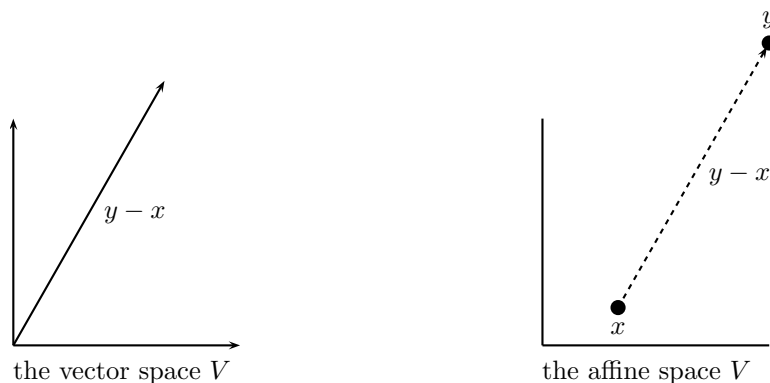


Figure 1.1 A vector space can be thought of as an affine space

1.1.5 Example (1.1.4 cont'd) Let  $V$  and  $U$  be  $\mathbb{R}$ -vector spaces that we regard as  $\mathbb{R}$ -affine spaces. We claim that every  $\mathbb{R}$ -affine map is of the form  $f: x \mapsto Ax + y_0$  where  $A$  is a  $\mathbb{R}$ -linear map and  $y_0 \in U$  is fixed.

First let us show that a map of this form is a  $\mathbb{R}$ -affine map. Let  $x_1, x_2 \in A_x$  for some  $x \in V$ . Then we compute

$$\begin{aligned} f(x_1 + x_2) &= f((x_1 - x) + (x_2 - x) + x) \\ &= f(x_1 + x_2 - x) \\ &= A(x_1 + x_2 - x) + y_0, \end{aligned}$$

and

$$\begin{aligned} f(x_1) + f(x_2) &= (((Ax_1 + y_0) - (Ax + y_0)) + ((Ax_2 + y_0) - (Ax + y_0))) + Ax + y_0 \\ &= A(x_1 + x_2 - x) + y_0 \end{aligned}$$

showing that  $f(x_1 + x_2) = f(x_1) + f(x_2)$ . The above computations will look incorrect unless you realise that the  $+$ -sign is being employed in two different ways. That is, when we write  $f(x_1 + x_2)$  and  $f(x_1) + f(x_2)$ , addition is in  $V_x$  and  $U_{f(x)}$ , respectively. Similarly one show that  $f(ax_1) = af(x_1)$  which demonstrates that  $f$  in a  $\mathbb{R}$ -affine map.

Now we show that any  $\mathbb{R}$ -affine map must have the form given for  $f$ . Let  $0 \in V$  be the zero vector. For  $x_1, x_2 \in V_0$  we have

$$f(x_1 + x_2) = f((x_1 - 0) + (x_2 - 0) + 0) = f(x_1 + x_2),$$

where the  $+$ -sign on the far left is addition in  $V_0$  and on the far right is addition in  $V$ . Because  $f: V_0 \rightarrow U_{f(0)}$  is  $\mathbb{R}$ -linear, we also have

$$f(x_1 + x_2) = f(x_1) + f(x_2) = (f(x_1) - f(0)) + (f(x_2) - f(0)) + f(0) = f(x_1) + f(x_2) - f(0).$$

Again, on the far left the  $+$ -sign is for  $U_{f(0)}$  and on the far right is for  $U$ . Thus we have shown that, for regular vector addition in  $V$  and  $U$  we must have

$$f(x_1 + x_2) = f(x_1) + f(x_2) - f(0). \quad (1.1)$$

Similarly, using linearity of  $f: V_0 \rightarrow U_{f(0)}$  under scalar multiplication we get

$$f(ax_1) = a(f(x_1) - f(0)) + f(0), \quad (1.2)$$

for  $a \in \mathbb{R}$  and  $x_1 \in V_x$ . Here, vector addition is in  $V$  and  $U$ . Together (1.1) and (1.2) imply that the map  $V \ni x \mapsto f(x) - f(0) \in U$  is  $\mathbb{R}$ -linear. This means that there exists  $A \in L(V; U)$  so that  $f(x) - f(0) = Ax$ . After taking  $y_0 = f(0)$  our claim now follows.  $\square$

If  $A$  and  $B$  are  $\mathbb{R}$ -affine spaces modelled on  $\mathbb{R}$ -vector spaces  $V$  and  $U$ , respectively, then we may define a  $\mathbb{R}$ -linear map  $f_V: V \rightarrow U$  as follows. Given  $x_0 \in A$  let  $A_{x_0}$  and  $B_{f(x_0)}$  be the corresponding vector spaces as described in Proposition 1.1.2. Recall that  $A_{x_0}$  is isomorphic to  $V$  with the isomorphism  $x \mapsto x - x_0$  and  $B_{f(x_0)}$  is isomorphic to  $U$  with the isomorphism  $y \mapsto y - f(x_0)$ . Let us denote these isomorphisms by  $g_{x_0}: A_{x_0} \rightarrow V$  and  $g_{f(x_0)}: B_{f(x_0)} \rightarrow U$ , respectively. We then define

$$f_V(v) = g_{f(x_0)} \circ f \circ g_{x_0}^{-1}(v). \quad (1.3)$$

It only remains to check that this definition does not depend on  $x_0$  (see Exercise E1.5).

**1.1.6 Example** (Example 1.1.4 cont'd) Recall that if  $V$  is a  $\mathbb{R}$ -vector space, then it is an  $\mathbb{R}$ -affine space modelled on itself (Example 1.1.4). Also recall that if  $U$  is another  $\mathbb{R}$ -vector space that we also think of as a  $\mathbb{R}$ -affine space, then an affine map from  $V$  to  $U$  looks like  $f(x) = Ax + y_0$  for a  $\mathbb{R}$  linear map  $A$  and for some  $y_0 \in U$  (Example 1.1.5).

Let's see what  $f_V$  looks like in such a case. Well, we can certainly guess what it should be! But let's work through the definition to see how it works. Pick some  $x_0 \in V$  so that

$$g_{x_0}(x) = x - x_0, \quad g_{f(x_0)}(y) = y - f(x_0) = y - Ax_0 - y_0.$$

We then see that

$$g_{x_0}^{-1}(v) = v + x_0.$$

Now apply the definition (1.3):

$$\begin{aligned} f_V(v) &= g_{f(x_0)} \circ f \circ g_{x_0}^{-1}(v) \\ &= g_{f(x_0)} \circ f(v + x_0) \\ &= g_{f(x_0)}(A(v + x_0) + y_0) \\ &= A(v + x_0) + y_0 - Ax_0 - y_0 \\ &= Av. \end{aligned}$$

Therefore we have laboriously derived what can be the only possible answer:  $f_V = A!$   $\square$

Finally, let us talk briefly about convexity, referring to [Berger 1987] for more details. We shall really only refer to this material once (see Lemma 1.5.2), so this material can be skimmed liberally if one is so inclined. A subset  $C$  of an affine space  $A$  is **convex** if for any two points  $x, y \in C$  the set

$$\ell_{x,y} = \{t(y - x) + x \mid t \in [0, 1]\}$$

is contained in  $C$ . This simply means that a set is convex if the line connecting any two points in the set remains within the set. For a given, not necessarily convex, subset  $S$  of  $A$  we define

$$\text{co}(S) = \bigcap_C \{C \text{ is a convex set containing } S\}$$

to be the **convex hull** of  $S$ . Thus  $\text{co}(S)$  is the smallest convex set containing  $S$ . For example, the convex hull of a set of two distinct points  $S = \{x, y\}$  will be the line  $\ell_{x,y}$ , and

the convex hull of three non-collinear points  $S = \{x, y, z\}$  will be the triangle with the points as vertices.

The following characterisation of a convex set will be useful. We refer to Appendix B for the definition of relative interior.

**1.1.7 Proposition** *Let  $A$  be an affine space modelled on  $V$  and let  $C \subsetneq A$  be a convex set. If  $x \in A$  is not in the relative interior of  $C$  then there exists  $\lambda \in V^*$  so that  $C \subset V_\lambda + x$  where*

$$V_\lambda = \{v \in V \mid \lambda(v) > 0\}.$$

The idea is simply that a convex set can be separated from its complement by a hyperplane as shown in Figure 1.2. The vector  $\lambda \in V^*$  can be thought of as being “orthogonal” to the

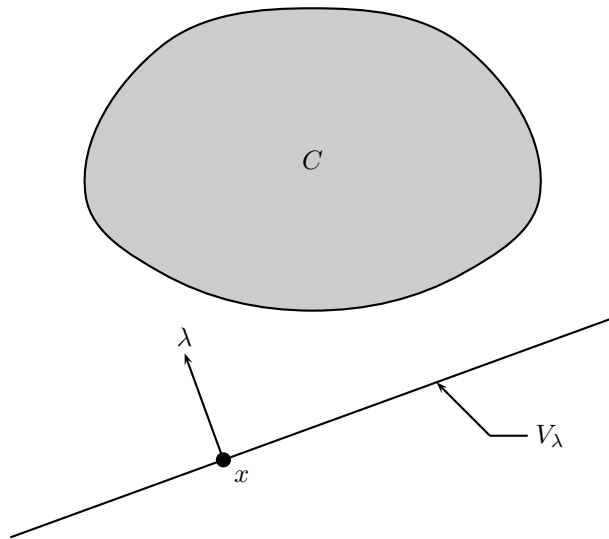


Figure 1.2 A hyperplane separating a convex set from its complement

hyperplane  $V_\lambda$ .

**1.1.2 Time and distance** We begin by giving the basic definition of a Galilean spacetime, and by providing meaning to intuitive notions of time and distance.

**1.1.8 Definition** A *Galilean spacetime* is a quadruple  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  where

GSp1.  $V$  is a 4-dimensional vector space,

GSp2.  $\tau: V \rightarrow \mathbb{R}$  is a surjective linear map called the *time map*,

GSp3.  $g$  is an inner product on  $\ker(\tau)$ , and

GSp4.  $\mathcal{E}$  is an affine space modelled on  $V$ . □

Points in  $\mathcal{E}$  are called *events*—thus  $\mathcal{E}$  is a model for the spatio-temporal world of Newtonian mechanics. With the time map we may measure the time between two events  $x_1, x_2 \in \mathcal{E}$  as  $\tau(x_2 - x_1)$  (noting that  $x_1 - x_2 \in V$ ). Note, however, that it does *not* make sense to talk about the “time” of a particular event  $x \in \mathcal{E}$ , at least not in the way you are perhaps

tempted to do. If  $x_1$  and  $x_2$  are events for which  $\tau(x_2 - x_1) = 0$  then we say  $x_1$  and  $x_2$  are *simultaneous*.

Using the lemma, we may define the *distance* between simultaneous events  $x_1, x_2 \in \mathcal{E}$  to be  $\sqrt{g(x_2 - x_1, x_2 - x_1)}$ . Note that this method for defining distance does not allow us to measure distance between events that are *not* simultaneous. In particular, it does not make sense to talk about two non-simultaneous events as occurring in the same place (i.e., as separated by zero distance). The picture one should have in mind for a Galilean spacetime is of it being a union of simultaneous events, nicely stacked together as depicted in Figure 1.3. That one cannot measure distance between non-simultaneous events reflects there being no

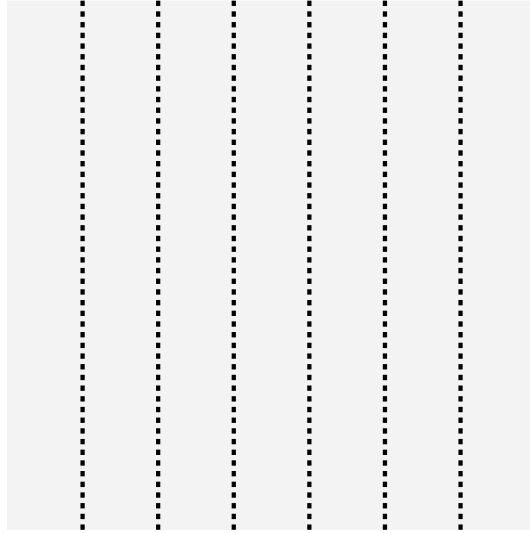


Figure 1.3 Vertical dashed lines represent simultaneous events

natural direction transverse to the stratification by simultaneous events.

Also associated with simultaneity is the *collection* of simultaneous events. For a given Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  we denote by

$$I_{\mathcal{G}} = \{S \subset \mathcal{E} \mid S \text{ is a collection of simultaneous events}\}$$

the collection of all simultaneous events. We shall frequently denote a point in  $I_{\mathcal{G}}$  by  $s$ , but keep in mind that when we do this,  $s$  is actually a collection of simultaneous events. We will denote by  $\pi_{\mathcal{G}}: \mathcal{E} \rightarrow I_{\mathcal{G}}$  the map that assigns to  $x \in \mathcal{E}$  the set of points simultaneous with  $x$ . Therefore, if  $s_0 = \pi_{\mathcal{G}}(x_0)$  then the set

$$\pi_{\mathcal{G}}^{-1}(s_0) = \{x \in \mathcal{E} \mid \pi_{\mathcal{G}}(x) = s_0\}$$

is simply a collection of simultaneous events. Given some  $s \in I_{\mathcal{G}}$ , we denote by  $\mathcal{E}(s)$  those events  $x$  for which  $\pi_{\mathcal{G}}(x) = s$ .

**1.1.9 Lemma** *For each  $s \in I_{\mathcal{G}}$ ,  $\mathcal{E}(s)$  is a 3-dimensional affine space modelled on  $\ker(\tau)$ .*

*Proof* The affine action of  $\ker(\tau)$  on  $\mathcal{E}(s)$  is that obtained by restricting the affine action of  $V$  on  $\mathcal{E}$ . So first we must show this restriction to be well-defined. That is, given  $v \in \ker(\tau)$  we need to show that  $v+x \in \mathcal{E}(s)$  for every  $x \in \mathcal{E}(s)$ . If  $x \in \mathcal{E}(s)$  then  $\tau((v+x)-x) = \tau(v) = 0$  which means that  $v+x \in \mathcal{E}(s)$  as claimed. The only non-trivial part of proving the restriction

defines an affine structure is showing that the action satisfies part [AS1](#) of the definition of an affine space. However, this follows since, thought of as a  $\mathbb{R}$ -vector space (with some  $x_0 \in \mathcal{E}(s)$  as origin),  $\mathcal{E}(s)$  is a 3-dimensional subspace of  $\mathcal{E}$ . Indeed, it is the kernel of the linear map  $x \mapsto \tau(x - x_0)$  that has rank 1. ■

Just as a single set of simultaneous events is an affine space, so too is the set of all simultaneous events.

**1.1.10 Lemma**  $I_{\mathcal{G}}$  is a 1-dimensional affine space modelled on  $\mathbb{R}$ .

*Proof* The affine action of  $\mathbb{R}$  on  $I_{\mathcal{G}}$  is defined as follows. For  $t \in \mathbb{R}$  and  $s_1 \in I_{\mathcal{G}}$ , we define  $t + s_1$  to be  $s_2 = \pi_{\mathcal{G}}(x_2)$  where  $\tau(x_2 - x_1) = t$  for some  $x_1 \in \mathcal{E}(s_1)$  and  $x_2 \in \mathcal{E}(s_2)$ . We need to show that this definition is well-defined, i.e., does not depend on the choices made for  $x_1$  and  $x_2$ . So take  $x'_1 \in \mathcal{E}(s_1)$  and  $x'_2 \in \mathcal{E}(s_2)$ . Since  $x'_1 \in \mathcal{E}(s_1)$  we have  $x'_1 - x_1 = v_1 \in \ker(\tau)$  and similarly  $x'_2 - x_2 = v_2 \in \ker(\tau)$ . Therefore

$$\tau(x'_2 - x'_1) = \tau((v_2 + x_2) - (v_1 + x_1)) = \tau((v_2 - v_1) + (x_2 - x_1)) = \tau(x_2 - x_1),$$

where we have used associativity of affine addition. Therefore, the condition that  $\tau(x_2 - x_1) = t$  does not depend on the choice of  $x_1$  and  $x_2$ . ■

One should think of  $I_{\mathcal{G}}$  as being the set of “times” for a Galilean spacetime, but it is an affine space, reflecting the fact that we do not have a distinguished origin for time (see [Figure 1.4](#)). Following [Artz \[1981\]](#), we call  $I_{\mathcal{G}}$  the **set of instants** in the Galilean spacetime

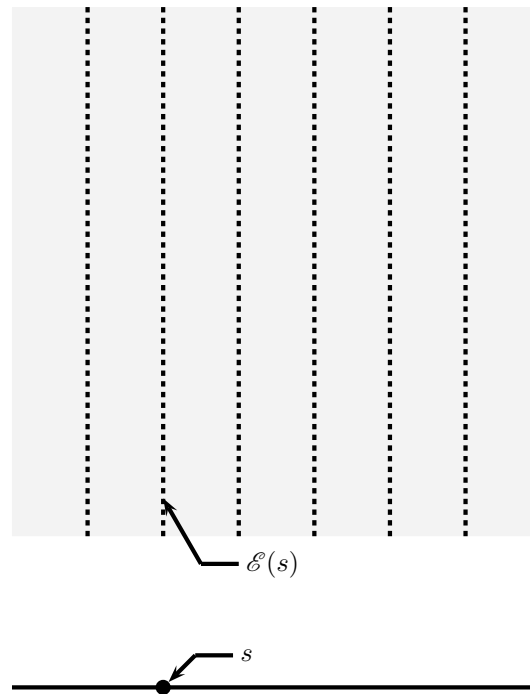


Figure 1.4 The set of instants  $I_{\mathcal{G}}$

$\mathcal{G}$ , the idea being that each of the sets  $\mathcal{E}(s)$  of simultaneous events defines an instant. The Galilean structure also allows for the use of the set

$$V_{\mathcal{G}} = \{v \in V \mid \tau(v) = 1\}.$$

The interpretation of this set is, as we shall see, that of a Galilean invariant velocity. Let us postpone this until later, and for now merely observe the following.

**1.1.11 Lemma**  $V_{\mathcal{G}}$  is a 3-dimensional affine space modelled on  $\ker(\tau)$ .

*Proof* Since  $\tau$  is surjective,  $V_{\mathcal{G}}$  is nonempty. We claim that if  $u_0 \in V_{\mathcal{G}}$  then

$$V_{\mathcal{G}} = \{u_0 + v \mid v \in \ker(\tau)\}.$$

Indeed, let  $u \in V_{\mathcal{G}}$ . Then  $\tau(u - u_0) = \tau(u) - \tau(u_0) = 1 - 1 = 0$ . Therefore  $u - u_0 \in \ker(\tau)$  so that

$$V_{\mathcal{G}} \subset \{u_0 + v \mid v \in \ker(\tau)\}. \quad (1.4)$$

Conversely, if  $u \in \{u_0 + v \mid v \in \ker(\tau)\}$  then there exists  $v \in \ker(\tau)$  so that  $u = u_0 + v$ . Thus  $\tau(u) = \tau(u_0 + v) = \tau(u_0) = 1$ , proving the opposite inclusion.

With this in mind, we define the affine action of  $\ker(\tau)$  on  $V_{\mathcal{G}}$  by  $v + u = v + u$ , i.e., the natural addition in  $V$ . That this is well-defined follows from the equality (1.4). ■

To summarise, given a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ , there are the following objects that one may associated with it:

1. the 3-dimensional vector space  $\ker(\tau)$  that, as we shall see, is where angular velocities and acceleration naturally live;
2. the 1-dimensional affine space  $I_{\mathcal{G}}$  of instants;
3. for each  $s \in I_{\mathcal{G}}$ , the 3-dimensional affine space  $\mathcal{E}(s)$  of events simultaneous with  $\mathcal{E}$ ;
4. the 3-dimensional affine space  $V_{\mathcal{G}}$  of ‘‘Galilean velocities.’’

We shall be encountering these objects continually throughout our development of mechanics in Galilean spacetimes.

When one think of Galilean spacetime, one often has in mind a particular example.

**1.1.12 Example** We let  $\mathcal{E} = \mathbb{R}^3 \times \mathbb{R} \simeq \mathbb{R}^4$  which is an affine space modelled on  $V = \mathbb{R}^4$  in the natural way (see Example 1.1.4). The time map we use is given by  $\tau_{\text{can}}(v^1, v^2, v^3, v^4) = v^4$ . Thus

$$\ker(\tau_{\text{can}}) = \{(v^1, v^2, v^3, v^4) \in V \mid v^4 = 0\}$$

is naturally identified with  $\mathbb{R}^3$ , and we choose for  $g$  the standard inner product on  $\mathbb{R}^3$  that we denote by  $g_{\text{can}}$ . We shall call this particular Galilean spacetime the **standard Galilean spacetime**.

(Notice that we write the coordinates  $(v^1, v^2, v^3, v^4)$  with *superscripts*. This will doubtless cause some annoyance, but as we shall see in Section 2.1, there is some rhyme and reason behind this.)

Given two events  $((x^1, x^2, x^3), s)$  and  $((y^1, y^2, y^3), t)$  one readily verifies that the time between these events is  $t - s$ . The distance between simultaneous events  $((x^1, x^2, x^3), t)$  and  $((y^1, y^2, y^3), t)$  is then

$$\sqrt{(y^1 - x^1)^2 + (y^2 - x^2)^2 + (y^3 - x^3)^2} = \|\mathbf{y} - \mathbf{x}\|$$

where  $\|\cdot\|$  is thus the standard norm on  $\mathbb{R}^3$ .

For an event  $x = ((x_0^1, x_0^2, x_0^3), t)$ , the set of events simultaneous with  $x$  is

$$\mathcal{E}(t) = \{((x^1, x^2, x^3), t) \mid x^i = x_0^i, i = 1, 2, 3\}.$$



The instant associated with  $x$  is naturally identified with  $t \in \mathbb{R}$ , and this gives us a simple identification of  $I_{\mathcal{G}}$  with  $\mathbb{R}$ . We also see that

$$V_{\mathcal{G}} = \{ (v^1, v^2, v^3, v^4) \in V \mid v^4 = 1 \},$$

and so we clearly have  $V_{\mathcal{G}} = (0, 0, 0, 1) + \ker(\tau_{\text{can}})$ .  $\square$

**1.1.3 Observers** An observer is to be thought of intuitively as someone who is present at each instant, and whose world behaves as according to the laws of motion (about which, more later). Such an observer should be moving at a uniform velocity. Note that in a Galilean spacetime, the notion of “stationary” makes no sense. We can be precise about an observer as follows. An **observer** in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  is a 1-dimensional affine subspace  $\mathcal{O}$  of  $\mathcal{E}$  with the property that  $\mathcal{O} \not\subseteq \mathcal{E}(s)$  for any  $s \in I_{\mathcal{G}}$ . That is, the affine subspace  $\mathcal{O}$  should not consist wholly of simultaneous events. There are some immediate implications of this definition.

**1.1.13 Proposition** *If  $\mathcal{O}$  is an observer in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  then for each  $s \in I_{\mathcal{G}}$  there exists a unique point  $x \in \mathcal{O} \cap \mathcal{E}(s)$ .*

*Proof* It suffices to prove the proposition for the canonical Galilean spacetime. (The reason for this is that, as we shall see in Section 1.2.4, a “coordinate system” has the property that it preserves simultaneous events.) We may also suppose that  $(\mathbf{0}, 0) \in \mathcal{O}$ . With these simplifications, the observer is then a 1-dimensional subspace passing through the origin in  $\mathbb{R}^3 \times \mathbb{R}$ . What’s more, since  $\mathcal{O}$  is not contained in a set of simultaneous events, there exists a point of the form  $(\mathbf{x}, t)$  in  $\mathcal{O}$  where  $t \neq 0$ . Since  $\mathcal{O}$  is a subspace, this means that all points  $(a\mathbf{x}, at)$  must also be in  $\mathcal{O}$  for any  $a \in \mathbb{R}$ . This shows that  $\mathcal{O} \cap \mathcal{E}(s)$  is nonempty for every  $s \in I_{\mathcal{G}}$ . That  $\mathcal{O} \cap \mathcal{E}(s)$  contains only one point follows since 1-dimensionality of  $\mathcal{O}$  ensures that the vector  $(\mathbf{x}, t)$  is a basis for  $\mathcal{O}$ . Therefore any two distinct points  $(a_1\mathbf{x}, a_1t)$  and  $(a_2\mathbf{x}, a_2t)$  in  $\mathcal{O}$  will not be simultaneous.  $\blacksquare$

We shall denote by  $\mathcal{O}_s$  the unique point in the intersection  $\mathcal{O} \cap \mathcal{E}(s)$ .

This means that an observer, as we have defined it, does indeed have the property of sitting at a place, and only one place, at each instant of time (see Figure 1.5). However, the observer should also somehow have the property of having a uniform velocity. Let us see how this plays out with our definition. Given an observer  $\mathcal{O}$  in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ , let  $U \subset V$  be the 1-dimensional subspace upon which  $\mathcal{O}$  is modelled. There then exists a unique vector  $v_{\mathcal{O}} \in U$  with the property that  $\tau(v_{\mathcal{O}}) = 1$ . We call  $v_{\mathcal{O}}$  the **Galilean velocity** of the observer  $\mathcal{O}$ . Again, it makes no sense to say that an observer is stationary, and this is why we must use the Galilean velocity.

An observer  $\mathcal{O}$  in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  with its Galilean velocity  $v_{\mathcal{O}}$  enables us to resolve *other* Galilean velocities into regular velocities. More generally, it allows us to resolve vectors in  $v \in V$  into a spatial component to go along with their temporal component  $\tau(v)$ . This is done by defining a linear map  $P_{\mathcal{O}}: V \rightarrow \ker(\tau)$  by

$$P_{\mathcal{O}}(v) = v - (\tau(v))v_{\mathcal{O}}.$$

(Note that  $\tau(v - (\tau(v))v_{\mathcal{O}}) = \tau(v) - \tau(v)\tau(v_{\mathcal{O}}) = 0$  so  $P_{\mathcal{O}}(v)$  is indeed in  $\ker(\tau)$ .) Following Artz [1981], we call  $P_{\mathcal{O}}$  the  **$\mathcal{O}$ -spatial projection**. For Galilean velocities, i.e., when  $v \in V_{\mathcal{G}} \subset V$ ,  $P_{\mathcal{O}}(v)$  can be thought of as the velocity of  $v$  relative to the observer’s Galilean velocity  $v_{\mathcal{O}}$ . The following trivial result says just this.

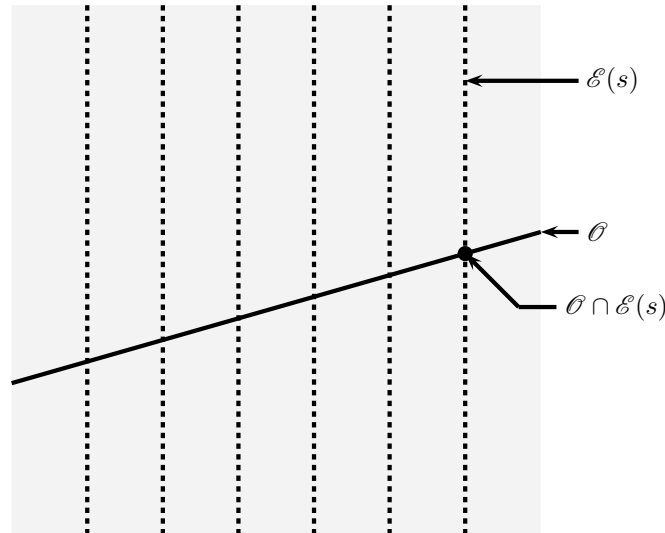


Figure 1.5 The idea of an observer

**1.1.14 Lemma** *If  $\mathcal{O}$  is an observer in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  and if  $v \in V_{\mathcal{G}}$ , then  $v = v_{\mathcal{O}} + P_{\mathcal{O}}(v)$ .*

*Proof* This follows since  $\tau(v) = 1$  when  $v \in V_{\mathcal{G}}$ . ■

The following is a very simple example of an observer in the canonical Galilean spacetime, and represents the observer one unthinkingly chooses in this case.

**1.1.15 Example** We let  $\mathcal{G}_{\text{can}} = (\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}^4, g_{\text{can}}, \tau_{\text{can}})$  be the canonical Galilean spacetime. The **canonical observer** is defined by

$$\mathcal{O}_{\text{can}} = \{(\mathbf{0}, t) \mid t \in \mathbb{R}\}.$$

Thus the canonical observer sits at the origin in each set of simultaneous events. □

**1.1.4 Planar and linear spacetimes** When dealing with systems that move in a plane or a line, things simplify to an enormous extent. But how does one talk of planar or linear systems in the context of Galilean spacetimes? The idea is quite simple.

**1.1.16 Definition** Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime. A subset  $\mathcal{F}$  of  $\mathcal{E}$  is a **sub-spacetime** if there exists a nontrivial subspace  $U$  of  $V$  with the property that

Gsub1.  $\mathcal{F}$  is an affine subspace of  $\mathcal{E}$  modelled on  $U$  and

Gsub2.  $\tau|U: U \rightarrow \mathbb{R}$  is surjective.

The **dimension** of the sub-spacetime  $\mathcal{F}$  is the dimension of the subspace  $U$ . □

Let us denote  $U_{\tau} = U \cap \ker(\tau)$ . The idea then is simply that we obtain a “new” spacetime  $\mathcal{H} = (\mathcal{F}, U, g|U_{\tau}, \tau|U)$  of smaller dimension. We shall often refer to  $\mathcal{H}$  so defined as the sub-spacetime interchangeably with  $\mathcal{F}$ . The idea of condition Gsub2 is that the time map should still be well defined. If we were to choose  $\mathcal{F}$  so that its model subspace  $U$  were a subset of  $\ker(\tau)$  then we would lose our notion of time.

1.1.17 **Examples** If  $(\mathcal{E} = \mathbb{R}^3 \times \mathbb{R}, V = \mathbb{R}^4, g_{\text{can}}, \tau_{\text{can}})$  is the standard Galilean spacetime, then we may choose “natural” planar and linear sub-spacetimes as follows.

1. For the planar sub-spacetime, take

$$\mathcal{F}_3 = \{((x, y, z), t) \in \mathbb{R}^3 \times \mathbb{R} \mid z = 0\},$$

and

$$U_3 = \{(u, v, w, s) \in \mathbb{R}^4 \mid w = 0\}.$$

Therefore,  $\mathcal{F}_3$  looks like  $\mathbb{R}^2 \times \mathbb{R}$  and we may use coordinates  $((x, y), t)$  as coordinates. Similarly  $U_3$  looks like  $\mathbb{R}^3$  and we may use  $(u, v, s)$  as coordinates. With these coordinates we have

$$\ker(\tau_{\text{can}}|_{U_3}) = \{(u, v, s) \in \mathbb{R}^3 \mid s = 0\},$$

so that  $g_{\text{can}}$  restricted to  $\ker(\tau_{\text{can}})$  is the standard inner product on  $\mathbb{R}^2$  with coordinates  $(u, v)$ . One then checks that with the affine structure as defined in Example 1.1.12,  $\mathcal{H}_3 = (\mathcal{F}_3, U_3, g_{\text{can}}|_{U_3}, \tau_{\text{can}}|_{U_3})$  is a 3-dimensional Galilean sub-spacetime of the canonical Galilean spacetime.

2. For the linear sub-spacetime we define

$$\mathcal{F}_2 = \{((x, y, z), t) \in \mathbb{R}^3 \times \mathbb{R} \mid y = z = 0\},$$

and

$$U_2 = \{(u, v, w, s) \in \mathbb{R}^4 \mid v = w = 0\}.$$

Then, following what we did in the planar case, we use coordinates  $(x, t)$  for  $\mathcal{F}_2$  and  $(u, s)$  for  $U_2$ . The inner product for the sub-spacetime is then the standard inner product on  $\mathbb{R}$  with coordinate  $u$ . In this case one checks that  $\mathcal{H}_2 = (\mathcal{F}_2, U_2, g_{\text{can}}|_{U_2}, \tau_{\text{can}}|_{U_2})$  is a 2-dimensional Galilean sub-spacetime of the canonical Galilean spacetime.

The 3-dimensional sub-spacetime of 1 we call the **canonical 3-dimensional Galilean sub-spacetime** and the 2-dimensional sub-spacetime of 2 we call the **canonical 2-dimensional Galilean sub-spacetime**.  $\square$

The canonical 3 and 2-dimensional Galilean sub-spacetimes are essentially the only ones we need consider, in the sense of the following result. We pull a *lassez-Bourbaki*, and use the notion of a coordinate system before it is introduced. You may wish to refer back to this result after reading Section 1.2.4.

1.1.18 **Proposition** *If  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  is a Galilean spacetime and  $\mathcal{F}$  is a  $k$ -dimensional sub-spacetime,  $k \in \{2, 3\}$ , modelled on the subspace  $U$  of  $V$ , then there exists a coordinate system  $\phi$  with the property that*

- (i)  $\phi(\mathcal{F}) = \mathcal{F}_k$ ,
- (ii)  $\phi_V(U) = U_k$ ,
- (iii)  $\tau \circ \phi_V^{-1} = \tau_{\text{can}}|_{U_k}$ , and
- (iv)  $g(u, v) = g_{\text{can}}(\phi_V(u), \phi_V(v))$  for  $u, v \in U$ .

*Proof*  $\mathcal{F}$  is an affine subspace of  $\mathcal{E}$  and  $U$  is a subspace of  $V$ . Since  $U \not\subset \ker(\tau)$ , we must have  $\dim(U \cap \ker(\tau_{\text{can}})) = k - 1$ . Choose a basis  $\mathcal{B} = \{v_1, \dots, v_k, v_{k+1}\}$  for  $V$  with the properties

- 1.  $\{v_1, \dots, v_{k-1}\}$  is a  $g$ -orthonormal basis for  $U$ ,

2.  $\{v_1, \dots, v_k\}$  is a  $g$ -orthonormal basis for  $V$ , and
3.  $\tau(v_{k+1}) = 1$ .

This is possible since  $U \notin \ker(\tau)$ . We may define an isomorphism from  $V$  to  $\mathbb{R}^4$  using the basis we have constructed. That is, we define  $i_{\mathcal{B}}: V \rightarrow \mathbb{R}^4$  by

$$i_{\mathcal{B}}(a^1 v_1 + a^2 v_2 + a^3 v_3 + a^4 v_4) = (a^1, a^2, a^3, a^4).$$

Now choose  $x \in \mathcal{F}$  and let  $\mathcal{E}_x$  be the vector space as in Proposition 1.1.2. The isomorphism from  $\mathcal{E}_x$  to  $V$  let us denote by  $g_x: \mathcal{E} \rightarrow V$ . Now define a coordinate system  $\phi$  by  $\phi = i_{\mathcal{B}} \circ g_x$ . By virtue of the properties of the basis  $\mathcal{B}$ , it follows that  $\phi$  has the properties as stated in the proposition. ■

Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime with  $\mathcal{H} = (\mathcal{F}, U, g|_{U\tau}, \tau|_U)$  a subspacetime. An observer  $\mathcal{O}$  for  $\mathcal{G}$  is  $\mathcal{H}$ -**compatible** if  $\mathcal{O} \subset \mathcal{F} \subset \mathcal{E}$ .

## 1.2 Galilean mappings and the Galilean transformation group

It is useful to talk about mappings between Galilean spacetimes that preserve the structure of the spacetime, i.e., preserve notions of simultaneity, distance, and time lapse. It turns out that the collection of such mappings possesses a great deal of structure. One important aspect of this structure is that of a group, so you may wish to recall the definition of a group.

**1.2.1 Definition** A **group** is a set  $G$  with a map from  $G \times G$  to  $G$ , denoted  $(g, h) \mapsto gh$ , satisfying,

G1.  $g_1(g_2g_3) = (g_1g_2)g_3$  (**associativity**),

G2. there exists  $e \in G$  so that  $eg = ge = g$  for all  $g \in G$  (**identity element**), and

G3. for each  $g \in G$  there exists  $g^{-1} \in G$  so that  $g^{-1}g = gg^{-1} = e$  (**inverse**).

If  $gh = hg$  for every  $g, h \in G$  we say  $G$  is **Abelian**.

A subset  $H$  of a group  $G$  is a **subgroup** if  $h_1h_2 \in H$  for every  $h_1, h_2 \in H$ . □

You will recall, or easily check, that the set of invertible  $n \times n$  matrices forms a group where the group operation is matrix multiplication. We denote this group by  $GL(n; \mathbb{R})$ , meaning the **general linear group**. The subset  $O(n)$  of  $GL(n; \mathbb{R})$  defined by

$$O(n) = \{ \mathbf{A} \in GL(n; \mathbb{R}) \mid \mathbf{A}\mathbf{A}^t = \mathbf{I}_n \},$$

is a subgroup of  $GL(n; \mathbb{R})$  (see Exercise E1.7), and

$$SO(n) = \{ \mathbf{A} \in O(n) \mid \det \mathbf{A} = 1 \}$$

is a subgroup of  $O(n)$  (see Exercise E1.8). (I am using  $\mathbf{A}^t$  to denote the transpose of  $\mathbf{A}$ .)

**1.2.1 Galilean mappings** We will encounter various flavours of maps between Galilean spacetimes. Of special importance are maps from the canonical Galilean spacetime to itself, and these are given special attention in Section 1.2.2. Also important are maps from a given Galilean spacetime into the canonical Galilean spacetime, and these are investigated in Section 1.2.4. But such maps all have common properties that are best illustrated in a general context as follows.

1.2.2 Definition A **Galilean map** between Galilean spacetimes  $\mathcal{G}_1 = (\mathcal{E}_1, V_1, g_1, \tau_1)$  and  $\mathcal{G}_2 = (\mathcal{E}_2, V_2, g_2, \tau_2)$  is a map  $\psi: \mathcal{E}_1 \rightarrow \mathcal{E}_2$  with the following properties:

GM1.  $\psi$  is an affine map;

GM2.  $\tau_2(\psi(x_1) - \psi(x_2)) = \tau_1(x_1 - x_2)$  for  $x_1, x_2 \in \mathcal{E}_1$ ;

GM3.  $g_2(\psi(x_1) - \psi(x_2), \psi(x_1) - \psi(x_2)) = g_1(x_1 - x_2, x_1 - x_2)$  for simultaneous events  $x_1, x_2 \in \mathcal{E}_1$ .  $\square$

Let us turn now to discussing the special cases of Galilean maps.

**1.2.2 The Galilean transformation group** A Galilean map  $\phi: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$  from the standard Galilean spacetime to itself is called a **Galilean transformation**. It is not immediately apparent from the definition of a Galilean map, but a Galilean transformation is invertible. In fact, we can be quite specific about the structure of a Galilean transformation.

The following result shows that the set of Galilean transformations forms a group under composition.

1.2.3 Proposition If  $\phi: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$  is a Galilean transformation, then  $\phi$  may be written in matrix form as

$$\phi: \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r} \\ \sigma \end{pmatrix} \quad (1.5)$$

where  $\mathbf{R} \in O(3)$ ,  $\sigma \in \mathbb{R}$ , and  $\mathbf{r}, \mathbf{v} \in \mathbb{R}^3$ . In particular, the set of Galilean transformations is a 10-dimensional group that we call the **Galilean transformation group** and denote by  $Gal$ .

*Proof* We first find the form of a Galilean transformation. First of all, since  $\phi$  is an affine map, it has the form  $\phi(\mathbf{x}, t) = A(\mathbf{x}, t) + (\mathbf{r}, \sigma)$  where  $A: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$  is  $\mathbb{R}$ -linear and where  $(\mathbf{r}, \sigma) \in \mathbb{R}^3 \times \mathbb{R}$  (see Example 1.1.5). Let us write  $A(\mathbf{x}, t) = (A_{11}\mathbf{x} + A_{12}t, A_{21}\mathbf{x} + A_{22}t)$  where  $A_{11} \in L(\mathbb{R}^3; \mathbb{R}^3)$ ,  $A_{12} \in L(\mathbb{R}; \mathbb{R}^3)$ ,  $A_{21} \in L(\mathbb{R}^3; \mathbb{R})$ , and  $A_{22} \in L(\mathbb{R}; \mathbb{R})$ . By GM3,  $A_{11}$  is an orthogonal linear transformation of  $\mathbb{R}^3$ . GM2 implies that

$$A_{22}(t_2 - t_1) + A_{21}(\mathbf{x}_2 - \mathbf{x}_1) = t_2 - t_1, \quad t_1, t_2 \in \mathbb{R}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3.$$

Thus, taking  $\mathbf{x}_1 = \mathbf{x}_2$ , we see that  $A_{22} = 1$ . This in turn requires that  $A_{21} = 0$ . Gathering this information together shows that a Galilean transformation has the form given by (1.5).

To prove the last assertion of the proposition let us first show that the inverse of a Galilean transformation exists, and is itself a Galilean transformation. To see this, one need only check that the inverse of the Galilean transformation in (1.5) is given by

$$\phi^{-1}: \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{R}^{-1}(\sigma\mathbf{v} - \mathbf{r}) \\ -\sigma \end{pmatrix}.$$

If  $\phi_1$  and  $\phi_2$  are Galilean transformations given by

$$\phi_1 \circ \phi_2: \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R}_1 & \mathbf{v}_1 \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r}_1 \\ \sigma_1 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R}_2 & \mathbf{v}_2 \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r}_2 \\ \sigma_2 \end{pmatrix},$$

we readily verify that  $\phi_1 \circ \phi_2$  is given by

$$\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{v}_1 + \mathbf{R}_1\mathbf{v}_2 \\ \mathbf{0} & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r}_1 + \mathbf{R}_1\mathbf{r}_2 + \sigma_2\mathbf{v}_1 \\ \sigma_1 + \sigma_2 \end{pmatrix}.$$

This shows that the Galilean transformations form a group. We may regard this group as a set to be  $\mathbb{R}^3 \times O(3) \times \mathbb{R}^3 \times \mathbb{R}$  with the correspondence mapping the Galilean transformation (1.5) to  $(\mathbf{v}, \mathbf{R}, \sigma, \mathbf{r})$ . Since the rotations in 3-dimensions are 3-dimensional, the result follows. ■

**1.2.4 Remark** In the proof we assert that  $\dim(O(3)) = 3$ . In what sense does one interpret “dim” in this expression? It is certainly *not* the case that  $O(3)$  is a vector space. But on the other hand, we intuitively believe that there are 3 independent rotations in  $\mathbb{R}^3$  (one about each axis), and so the set of rotations should have dimension 3. This is all true, but the fact of the matter is that to make the notion of “dimension” clear in this case requires that one know about “Lie groups,” and these are just slightly out of reach. We will approach a better understanding of these matters in Section 2.1 □

Note that we may consider  $Gal$  to be a subgroup of the 100-dimensional matrix group  $GL(10; \mathbb{R})$ . Indeed, one may readily verify that the subgroup of  $GL(10; \mathbb{R})$  consisting of those matrices of the form

$$\begin{bmatrix} 1 & \sigma & 0 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{0}^t & 0 & \mathbf{0}^t & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{r} & \mathbf{0}_{3 \times 3} & \mathbf{0} \\ 0 & 0 & \mathbf{0}^t & 1 & \mathbf{0}^t & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{3 \times 3} & \mathbf{0} & \mathbf{R} & \mathbf{v} \\ 0 & 0 & \mathbf{0}^t & 0 & \mathbf{0}^t & 1 \end{bmatrix}$$

is a subgroup that is isomorphic to  $Gal$  under matrix composition: the isomorphism maps the above  $10 \times 10$  matrix to the Galilean transformation given in (1.5).

Thus a Galilean transformation may be written as a composition of one of three basic classes of transformations:

1. *A shift of origin:*

$$\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r} \\ \sigma \end{pmatrix}$$

for  $(\mathbf{r}, \sigma) \in \mathbb{R}^3 \times \mathbb{R}$ .

2. *A rotation of reference frame:*

$$\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix}$$

for  $\mathbf{R} \in O(3)$ .

3. *A uniformly moving frame:*

$$\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix}$$

for  $\mathbf{v} \in \mathbb{R}^3$ .

The names we have given these fundamental transformations are suggestive. A shift of origin should be thought of as moving the origin to a new position, and resetting the clock, but maintaining the same orientation in space. A rotation of reference frame means the origin stays in the same place, and uses the same clock, but rotates their “point-of-view.” The final basic transformation, a uniformly moving frame, means the origin maintains its orientation and uses the same clock, but now moves at a constant velocity relative to the previous origin.

**1.2.3 Subgroups of the Galilean transformation group** In the previous section we saw that elements of the Galilean transformation group  $Gal$  are compositions of temporal and spatial translations, spatial rotations, and constant velocity coordinate changes. In this section we concern ourselves with a more detailed study of certain subgroups of  $Gal$ .

Of particular interest in applications is the subgroup of  $Gal$  consisting of those Galilean transformations that do not change the velocity. Let us also for the moment restrict attention to Galilean transformations that leave the clock unchanged. Galilean transformations with these two properties have the form

$$\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r} \\ 0 \end{pmatrix}$$

for  $\mathbf{r} \in \mathbb{R}^3$  and  $\mathbf{R} \in O(3)$ . Since  $t$  is fixed, we may as well regard such transformations as taking place in  $\mathbb{R}^3$ .<sup>1</sup> These Galilean transformations form a subgroup under composition, and we call it the **Euclidean group** that we denote by  $E(3)$ . One may readily verify that this group may be regarded as the subgroup of  $GL(4; \mathbb{R})$  consisting of those matrices of the form

$$\begin{bmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}^t & 1 \end{bmatrix}. \quad (1.6)$$

The Euclidean group is distinguished by its being precisely the isometry group of  $\mathbb{R}^3$  (we refer to [Berger 1987] for details about the isometry group of  $\mathbb{R}^n$ ).

**1.2.5 Proposition** *A map  $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is an isometry (i.e.,  $\|\phi(\mathbf{x}) - \phi(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$  for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ ) if and only if  $\phi \in E(3)$ .*

*Proof* Let  $g_{\text{can}}$  denote the standard inner product on  $\mathbb{R}^3$  so that  $\|\mathbf{x}\| = \sqrt{g_{\text{can}}(\mathbf{x}, \mathbf{x})}$ . First suppose that  $\phi$  is an isometry that fixes  $\mathbf{0} \in \mathbb{R}^3$ . Recall that the norm on an inner product space satisfies the parallelogram law:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$

(see Exercise E1.9). Using this equality, and the fact that  $\phi$  is an isometry fixing  $\mathbf{0}$ , we compute

$$\begin{aligned} \|\phi(\mathbf{x}) + \phi(\mathbf{y})\|^2 &= 2\|\phi(\mathbf{x})\|^2 + 2\|\phi(\mathbf{y})\|^2 - \|\phi(\mathbf{x}) - \phi(\mathbf{y})\|^2 \\ &= 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2. \end{aligned} \quad (1.7)$$

It is a straightforward computation to show that

$$g_{\text{can}}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} (\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2)$$

for every  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ . In particular, using (1.7) and the fact that  $\phi$  is an isometry fixing  $\mathbf{0}$ , we compute

$$\begin{aligned} g_{\text{can}}(\phi(\mathbf{x}), \phi(\mathbf{y})) &= \frac{1}{2} (\|\phi(\mathbf{x}) + \phi(\mathbf{y})\|^2 - \|\phi(\mathbf{x})\|^2 - \|\phi(\mathbf{y})\|^2) \\ &= \frac{1}{2} (\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2) = g_{\text{can}}(\mathbf{x}, \mathbf{y}). \end{aligned}$$

We now claim that this implies that  $\phi$  is linear. Indeed, let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  be the standard orthonormal basis for  $\mathbb{R}^3$  and let  $(x^1, x^2, x^3)$  be the components of  $\mathbf{x} \in \mathbb{R}^3$  in this basis (thus

<sup>1</sup>One should think of this copy of  $\mathbb{R}^3$  as being a collection of simultaneous events.



$x^i = g_{\text{can}}(\mathbf{x}, \mathbf{e}_i)$ ,  $i = 1, 2, 3$ ). Since  $g_{\text{can}}(\phi(\mathbf{e}_i), \phi(\mathbf{e}_j)) = g_{\text{can}}(\mathbf{e}_i, \mathbf{e}_j)$ ,  $i, j = 1, 2, 3$ , the vectors  $\{\phi(\mathbf{e}_1), \phi(\mathbf{e}_2), \phi(\mathbf{e}_3)\}$  form an orthonormal basis for  $\mathbb{R}^3$ . The components of  $\phi(\mathbf{x})$  in this basis are given by  $\{g_{\text{can}}(\phi(\mathbf{x}), \phi(\mathbf{e}_i)) \mid i = 1, 2, 3\}$ . But since  $\phi$  preserves  $g_{\text{can}}$ , this means that the components of  $\phi(\mathbf{x})$  are precisely  $(x^1, x^2, x^3)$ . That is,

$$\phi \left( \sum_{i=1}^3 x^i \mathbf{e}_i \right) = \sum_{i=1}^3 x^i \phi(\mathbf{e}_i).$$

Thus  $\phi$  is linear. This shows that  $\phi \in O(3)$ .

Now suppose that  $\phi$  fixes not  $\mathbf{0}$ , but some other point  $\mathbf{x}_0 \in \mathbb{R}^3$ . Let  $T_{\mathbf{x}_0}$  be translation by  $\mathbf{x}_0$ :  $T_{\mathbf{x}_0}(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$ . Then we have  $T_{\mathbf{x}_0} \circ \phi \circ T_{\mathbf{x}_0}^{-1}(\mathbf{0}) = \mathbf{0}$ . Since  $T_{\mathbf{x}_0} \in E(3)$ , and since  $E(3)$  is a group, this implies that  $T_{\mathbf{x}_0} \circ \phi \circ T_{\mathbf{x}_0}^{-1} \in O(3)$ . In particular,  $\phi \in E(3)$ .

Finally, suppose that  $\phi$  maps  $\mathbf{x}_1$  to  $\mathbf{x}_2$ . In this case, letting  $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$ , we have  $T_{\mathbf{x}_0} \circ \phi(\mathbf{x}_1) = \mathbf{x}_1$  and so  $T_{\mathbf{x}_0} \circ \phi \in E(3)$ . Therefore  $\phi \in E(3)$ .

To show that  $\phi \in E(3)$  is an isometry is straightforward. ■

Of particular interest are those elements of the Euclidean group for which  $\mathbf{R} \in SO(3) \subset O(3)$ . This is a subgroup of  $E(3)$  (since  $SO(3)$  is a subgroup of  $O(3)$ ) that is called the **special Euclidean group** and denoted by  $SE(3)$ . We refer the reader to [Murray, Li, and Sastry 1994, Chapter 2] for an in depth discussion of  $SE(3)$  beyond what we say here.

The Euclidean group possesses a distinguished subgroup consisting of all translations. Let us denote by  $T_{\mathbf{r}}$  translation by  $\mathbf{r}$ :

$$T_{\mathbf{r}}(\mathbf{x}) = \mathbf{x} + \mathbf{r}.$$

The set of all such elements of  $SE(3)$  forms a subgroup that is clearly isomorphic to the additive group  $\mathbb{R}^3$ .

For sub-spacetimes one can also talk about their transformation groups. Let us look at the elements of the Galilean group that leave invariant the sub-spacetime  $\mathcal{F}_3$  of the canonical Galilean spacetime. Thus we consider a Galilean transformation

$$\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r} \\ \sigma \end{pmatrix} \quad (1.8)$$

for  $\mathbf{R} \in SO(3)$  (the case when  $\mathbf{R} \in O(3) \setminus SO(3)$  is done similarly),  $\mathbf{v}, \mathbf{r} \in \mathbb{R}^3$ , and  $\sigma \in \mathbb{R}$ . Points in  $\mathcal{F}_3$  have the form  $((x, y, 0), t)$ , and one readily checks that in order for the Galilean transformation (1.8) to map a point in  $\mathcal{F}_3$  to another point in  $\mathcal{F}_3$  we must have

$$\mathbf{R} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix}$$

for some  $\theta, u, v, \eta, \xi \in \mathbb{R}$ . In particular, if we are in the case of purely spatial transformations, i.e., when  $\mathbf{v} = \mathbf{0}$  and  $\sigma = 0$ , then a Galilean transformation mapping  $\mathcal{F}_3$  into  $\mathcal{F}_3$  is defined by a vector in  $\mathbb{R}^2$  and a  $2 \times 2$  rotation matrix. The set of all such transformations is a subgroup of *Gal*, and we denote this subgroup by  $SE(2)$ . Just as we showed that  $SE(3)$  is the set of orientation-preserving isometries of  $\mathbb{R}^3$ , one shows that  $SE(2)$  is the set of orientation preserving isometries of  $\mathbb{R}^2$ . These are simply a rotation in  $\mathbb{R}^2$  followed by a translation.



In a similar manner, one shows that the Galilean transformation (1.8) maps points in  $\mathcal{F}_2$  to other points in  $\mathcal{F}_2$  when

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix},$$

for some  $u, \xi \in \mathbb{R}$ . Again, when  $\mathbf{v} = 0$ , the resulting subgroup of  $Gal$  is denoted  $SE(1)$ , the group of orientation preserving isometries of  $\mathbb{R}^1$ . In this case, this simply amounts to translations of  $\mathbb{R}^1$ .

**1.2.4 Coordinate systems** To a rational person it seems odd that we have thus far disallowed one to talk about the distance between events that are not simultaneous. Indeed, from Example 1.1.12 it would seem that this should be possible. Well, such a discussion is possible, but one needs to introduce additional structure. For now we use the notion of a Galilean map to provide a notion of reference. To wit, a **coordinate system** for a Galilean spacetime  $\mathcal{G}$  is a Galilean map  $\phi: \mathcal{E} \rightarrow \mathbb{R}^3 \times \mathbb{R}$  into the standard Galilean spacetime. Once we have chosen a coordinate system, we may talk about the “time” of an event (not just the relative time between two events), and we may talk about the distance between two non-simultaneous events. Indeed, for  $x \in \mathcal{E}$  we define the  **$\phi$ -time** of  $x$  by  $\tau_{\text{can}}(\phi(x))$ . Also, given two events  $x_1, x_2 \in \mathcal{E}$  we define the  **$\phi$ -distance** between these events by  $\|\text{pr}_1(\phi(x_2) - \phi(x_1))\|$  where  $\text{pr}_1: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$  is projection onto the first factor. The idea here is that a coordinate system  $\phi$  establishes a distinguished point  $\phi^{-1}(\mathbf{0}, 0) \in \mathcal{E}$ , called the **origin** of the coordinate system, from which times and distances may be measured. But be aware that this *does* require the additional structure of a coordinate system!

Associated with a coordinate system are various induced maps. Just like the coordinate system itself make  $\mathcal{E}$  “look like” the canonical Galilean spacetime  $\mathbb{R}^3 \times \mathbb{R}$ , the induced maps make other objects associated with  $\mathcal{G}$  look like their canonical counterparts.

1. There is an induced vector space isomorphism  $\phi_V: V \rightarrow \mathbb{R}^4$  as described by (1.3).
2. If we restrict  $\phi_V$  to  $\ker(\tau)$  we may define an isomorphism  $\phi_\tau: \ker(\tau) \rightarrow \mathbb{R}^3$  by

$$\phi_V(v) = (\phi_\tau(v), 0), \quad v \in \ker(\tau). \quad (1.9)$$

This definition makes sense by virtue of the property GM3.

3. A coordinate system  $\phi$  induces a map  $\phi_{I_{\mathcal{G}}}: I_{\mathcal{G}} \rightarrow \mathbb{R}$  by

$$\phi(x) = (\mathbf{x}, \phi_{I_{\mathcal{G}}}(\pi_{\mathcal{G}}(x)))$$

which is possible for *some*  $\mathbf{x} \in \mathbb{R}^3$ . Note that this defines  $\phi_{I_{\mathcal{G}}}(\pi_{\mathcal{G}}(x)) \in \mathbb{R}$ . One can readily determine that this definition only depends on  $s = \pi_{\mathcal{G}}(x)$  and not on a particular choice of  $x \in \mathcal{E}(s)$ .

4. For a fixed  $s_0 \in I_{\mathcal{G}}$  and a coordinate system  $\phi$  for  $\mathcal{E}$  we define a map  $\phi_{s_0}: \mathcal{E}(s_0) \rightarrow \mathbb{R}^3$  by writing

$$\phi(x) = (\phi_{s_0}(x), \sigma), \quad x \in \mathcal{E}(s_0),$$

which is possible for *some*  $\sigma \in \mathbb{R}$  due to the property GM3 of Galilean maps.

5. The coordinate system  $\phi$  induces a map  $\phi_{V_{\mathcal{G}}}: V_{\mathcal{G}} \rightarrow (0, 0, 0, 1) + \ker(\tau_{\text{can}})$  by

$$\phi_{V_{\mathcal{G}}}(u) = (0, 0, 0, 1) + \phi_\tau(u - u_0)$$

where  $u_0 \in V_{\mathcal{G}}$  is defined by  $\phi_V(u_0) = (0, 0, 0, 1)$ .

Just as with the various spaces  $\ker(\tau)$ ,  $\mathcal{E}(s)$ ,  $I_{\mathcal{G}}$ , and  $V_{\mathcal{G}}$  we can associate with a Galilean spacetime  $\mathcal{G}$ , we will often encounter the maps  $\phi_{\tau}$ ,  $\phi_x$ ,  $\phi_{I_{\mathcal{G}}}$ , and  $\phi_{V_{\mathcal{G}}}$  as we proceed with our discussion.

We shall often be in a position where we have a representation of an object in a specific coordinate system, and it is not clear what this object represents, if anything, in terms of the Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ . To understand what something “really” is, one way to proceed is to investigate how it alters if one chooses a different coordinate system. The picture one should have in mind is shown in Figure 1.6. As is evident from

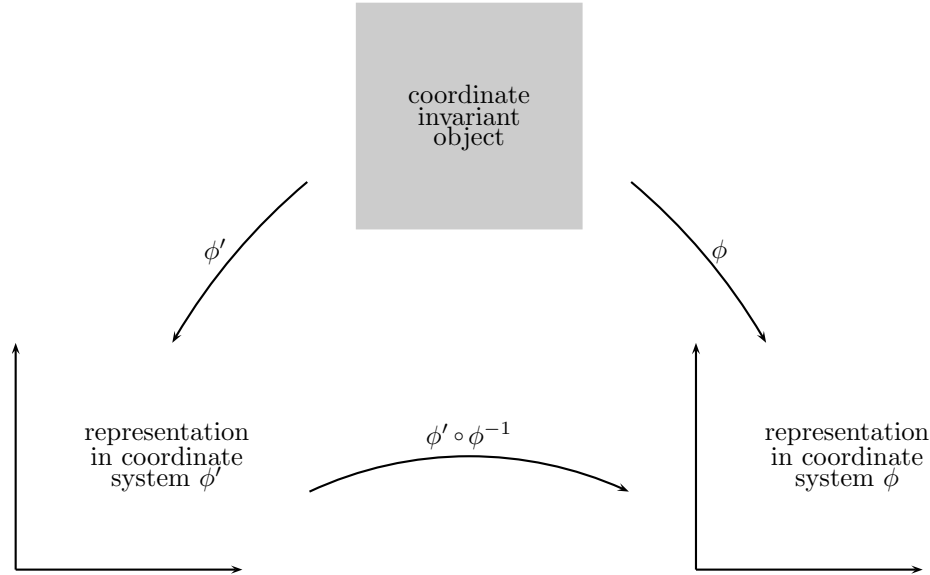


Figure 1.6 Changing coordinate systems

the figure, if we have coordinate systems  $\phi: \mathcal{E} \rightarrow \mathbb{R}^3 \times \mathbb{R}$  and  $\phi': \mathcal{E} \rightarrow \mathbb{R}^3 \times \mathbb{R}$ , the map  $\phi' \circ \phi^{-1}: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$  will tell us what we want to know about how things alter as we change coordinate systems. The advantage of looking at this map is that it is a map between canonical spacetimes and we understand the canonical spacetime well. Note that the map  $\phi' \circ \phi^{-1}$  has associated with it maps  $\phi'_V \circ \phi_V^{-1}$ ,  $\phi'_x \circ \phi_x^{-1}$ , and  $\phi'_\tau \circ \phi_\tau^{-1}$  telling us, respectively, how elements in  $V$ ,  $\mathcal{E}(\pi_{\mathcal{G}}(x))$ , and  $\ker(\tau)$  transform under changing coordinate systems.

The following result records what these maps look like.

**1.2.6 Proposition** *If  $\phi$  and  $\phi'$  are two coordinate systems for a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  then  $\phi' \circ \phi^{-1} \in \text{Gal}$ . Let  $\mathbf{x}_0 \in \mathcal{E}$ ,  $s_0 = \pi_{\mathcal{G}}(\mathbf{x}_0)$ , and denote  $(\mathbf{x}_0, t_0) = \phi(\mathbf{x}_0)$  and  $(\mathbf{x}'_0, t'_0) = \phi'(\mathbf{x}_0)$ . If  $\phi' \circ \phi^{-1}$  is given by*

$$\phi' \circ \phi^{-1}: \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r} \\ \sigma \end{pmatrix},$$

then

(i)  $\phi'_V \circ \phi_V^{-1}$  satisfies

$$\phi'_V \circ \phi_V^{-1}: \begin{pmatrix} \mathbf{u} \\ s \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ s \end{pmatrix},$$

(ii)  $\phi'_{s_0} \circ \phi_{s_0}^{-1}$  satisfies

$$\phi'_{s_0} \circ \phi_{s_0}^{-1}: \mathbf{x} \mapsto \mathbf{R}\mathbf{u} + \mathbf{r} + t_0\mathbf{v},$$

(iii)  $\phi'_\tau \circ \phi_\tau^{-1}$  satisfies

$$\phi'_\tau \circ \phi_\tau^{-1}: \mathbf{u} \mapsto \mathbf{R}\mathbf{u},$$

(iv)  $\phi'_{I_{\mathcal{G}}} \circ \phi_{I_{\mathcal{G}}}^{-1}$  satisfies

$$\phi'_{I_{\mathcal{G}}} \circ \phi_{I_{\mathcal{G}}}^{-1}: t \mapsto t + \sigma,$$

and

(v)  $\phi'_{V_{\mathcal{G}}} \circ \phi_{V_{\mathcal{G}}}^{-1}$  satisfies

$$\phi'_{V_{\mathcal{G}}} \circ \phi_{V_{\mathcal{G}}}^{-1}: (\mathbf{u}, 1) \mapsto (\mathbf{R}\mathbf{u} + \mathbf{v}, 1)$$

Furthermore, let  $\mathbf{A}: \ker(\tau) \rightarrow \ker(\tau)$  be a linear map, with  $\mathbf{A} = \phi_\tau \circ \mathbf{A} \circ \phi_\tau^{-1}$  its representation in the coordinate system  $\phi$ , and  $\mathbf{A}' = \phi'_\tau \circ \mathbf{A} \circ \phi_\tau^{-1}$  its representation in the coordinate system  $\phi'$ . Then  $\mathbf{A}' = \mathbf{R}\mathbf{A}\mathbf{R}^{-1}$ .

*Proof* (i) The map  $\phi'_V \circ \phi_V^{-1}$  is the linear part of the affine map  $\phi' \circ \phi^{-1}$ . That is

$$\phi'_V \circ \phi_V^{-1}: \begin{pmatrix} \mathbf{u} \\ s \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ s \end{pmatrix}.$$

(ii)  $\phi'_{s_0} \circ \phi_{s_0}^{-1}$  is the restriction of  $\phi' \circ \phi^{-1}$  to  $\mathbb{R}^3 \times \{t_0\}$ . Thus

$$\phi'_{s_0} \circ \phi_{s_0}^{-1}: \begin{pmatrix} \mathbf{x} \\ t_0 \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t_0 \end{pmatrix} + \begin{pmatrix} \mathbf{r} \\ t'_0 - t_0 \end{pmatrix},$$

since we are given that  $\phi'(x_0) = (\mathbf{x}'_0, t'_0)$ . Since the  $t$ -component is fixed in this formula,  $\phi'_{s_0} \circ \phi_{s_0}^{-1}$  is determined by the  $\mathbf{x}$ -component, and this is as stated in the proposition.

(iii) By the definition of  $\phi_\tau$ , the map  $\phi'_\tau \circ \phi_\tau^{-1}$  from  $\ker(\tau_{\text{can}})$  to  $\ker(\tau_{\text{can}})$  is as stated, given the form of  $\phi'_V \circ \phi_V^{-1}$  from (i).

(iv) For  $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$  the set of simultaneous events is identified with  $t$ . Then  $\phi'_{I_{\mathcal{G}}} \circ \phi_{I_{\mathcal{G}}}^{-1}(t)$  is the collection of events simultaneous with  $\phi' \circ \phi^{-1}(\mathbf{x}, t)$ . Since  $\phi' \circ \phi^{-1}(\mathbf{x}, t) = (\mathbf{x}', t + \sigma)$  for some appropriately chosen  $\mathbf{x}'$ , this part of the proposition follows.

(v) Let  $u_0 \in V$  be defined by  $\phi(u_0) = (0, 0, 0, 1)$ . From (i) we have  $\phi'(u_0) = (\mathbf{v}, 1)$ . Therefore for  $\mathbf{u} \in \ker(\tau)$  we have

$$\phi'_{V_{\mathcal{G}}} \circ \phi_{V_{\mathcal{G}}}^{-1}((0, 0, 0, 1) + \mathbf{u}) = \phi_V \circ \phi_V^{-1}((0, 0, 0, 1) + \mathbf{u}) = (\mathbf{v}, 1) + (\mathbf{R}\mathbf{u}, 0),$$

as claimed.

The final assertion of the proposition follows from (iii) and the change of basis formula for linear maps. ■

**1.2.5 Coordinate systems and observers** We let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime with  $\mathcal{O}$  an observer. A coordinate system  $\phi$  is  $\mathcal{O}$ -**adapted** if (1)  $\phi_V(v_{\mathcal{O}}) = (\mathbf{0}, 1)$  and (2)  $\phi(\mathcal{O} \cap \mathcal{E}(s_0)) = (\mathbf{0}, 0)$  for some  $s_0 \in I_{\mathcal{G}}$ . We denote the collection of all  $\mathcal{O}$ -adapted coordinate systems by  $\text{Coor}(\mathcal{O})$ . The idea is that the coordinate system is  $\mathcal{O}$ -adapted when the velocity of the observer is zero in that coordinate system.

The definition we give hides an extra simplification, namely the following.

**1.2.7 Proposition** *If  $\mathcal{O}$  is an observer in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  and if  $\phi \in \text{Coor}(\mathcal{O})$ , then for each  $\mathbf{x} \in \mathcal{E}$ ,  $\phi(\mathcal{O}_{\pi_{\mathcal{G}}(\mathbf{x})}) = (\mathbf{0}, t)$  for some  $t \in \mathbb{R}$ .*

*Proof* By definition there exists  $x_0 \in \mathcal{E}$  so that  $\phi(\mathcal{O}_{\pi_{\mathcal{G}}(x_0)}) = (\mathbf{0}, 0)$ . This means that  $\phi(\mathcal{O})$  will be a 1-dimensional subspace through the origin in  $\mathbb{R}^3 \times \mathbb{R}$ . Therefore, since  $\mathcal{O}$  is an observer, we can write a basis for the subspace  $\phi(\mathcal{O})$  as  $(\mathbf{v}, 1)$  for some  $\mathbf{v} \in \mathbb{R}^3$ . However,  $(\mathbf{v}, 1)$  must be  $\phi_V(v_{\mathcal{O}})$  so that  $\mathbf{v} = \mathbf{0}$ . Therefore

$$\phi(\mathcal{O}) = \{(\mathbf{0}, t) \mid t \in \mathbb{R}\},$$

which is what we were to show. ■

The interpretation of this result is straightforward. If at *some* time an observer sits at the origin in coordinate system that is adapted to the observer, then the observer must remain at the origin in that coordinate system for *all* time since an adapted coordinate system has the property that it renders the observer stationary.

Let us see how two  $\mathcal{O}$ -adapted coordinate systems differ from one another.

**1.2.8 Proposition** *Let  $\mathcal{O}$  be an observer in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, \mathbb{V}, \mathbf{g}, \tau)$  with  $\phi, \phi' \in \text{Coord}(\mathcal{O})$ . We then have*

$$\phi' \circ \phi^{-1}: \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \sigma \end{pmatrix}$$

for some  $\mathbf{R} \in O(3)$  and  $\sigma \in \mathbb{R}$ .

*Proof* We will generally have

$$\phi' \circ \phi^{-1}: \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{r} \\ \sigma \end{pmatrix}$$

for  $\mathbf{R} \in O(3)$ ,  $\mathbf{v}, \mathbf{r} \in \mathbb{R}^3$ , and  $\sigma \in \mathbb{R}$  so that

$$\phi'_V \circ \phi_V^{-1}: \begin{pmatrix} \mathbf{u} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ t \end{pmatrix}.$$

Let  $\mathbf{v}_{\mathcal{O}} = \phi_V(v_{\mathcal{O}})$  and  $\mathbf{v}'_{\mathcal{O}} = \phi'_V(v_{\mathcal{O}})$ . Since  $\phi, \phi' \in \text{Coord}(\mathcal{O})$  we have  $\mathbf{v}_{\mathcal{O}} = \mathbf{v}'_{\mathcal{O}} = (\mathbf{0}, 1)$ . Therefore

$$\phi'_V \circ \phi_V^{-1}(\mathbf{v}_{\mathcal{O}}) = (\mathbf{R}\mathbf{0} + \mathbf{v}, 1) = (\mathbf{0}, 1),$$

from which we deduce that  $\mathbf{v} = \mathbf{0}$ .

To show that  $\mathbf{r} = \mathbf{0}$  we proceed as follows. Since  $\phi \in \text{Coord}(\mathcal{O})$  let  $x_0 \in \mathcal{O}$  be the point for which  $\phi(x_0) = (\mathbf{0}, 0)$ . Since  $\phi' \in \text{Coord}(\mathcal{O})$  we also have  $\phi'(x_0) = (\mathbf{0}, t_0)$  for some  $t_0 \in \mathbb{R}$  by Proposition 1.2.7. Therefore

$$\phi' \circ \phi^{-1}(\mathbf{0}, 0) = (\mathbf{R}\mathbf{0} + \mathbf{r}, \sigma) = (\mathbf{0}, t_0),$$

so that in particular,  $\mathbf{r} = \mathbf{0}$  as desired. ■

This is perfectly plausible, of course. If an observer is stationary with respect to two different coordinate systems, then they should have zero velocity with respect to one another. Also, if observers sit at the spatial origin in two coordinate systems, then the spatial origins of the two coordinate systems should “agree.”

In Section 1.2.4 we saw how a coordinate system  $\phi$  induced a variety of maps that are used to represent the various objects associated with a Galilean spacetime (e.g.,  $\ker(\tau)$ ,  $I_{\mathcal{G}}$ ,

$V_{\mathcal{G}}$ ) in the given coordinate system. If we additionally have an observer  $\mathcal{O}$  along with the coordinate system  $\phi$ , there is a map  $\phi_{\mathcal{O}}: V_{\mathcal{G}} \rightarrow \mathbb{R}^3$  defined by

$$\phi_{\mathcal{O}}(v) = \phi_{\tau} \circ P_{\mathcal{O}}(v). \quad (1.10)$$

Referring to Lemma 1.1.14 we see that the interpretation of  $\phi_{\mathcal{O}}$  is that it is the coordinate representation of the relative velocity with respect to the observer's Galilean velocity.

The following coordinate representation of this map will be useful.

**1.2.9 Proposition** *If  $\mathcal{O}$  is an observer in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  and  $\phi$  is a coordinate system, then the map  $\phi_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1}$  satisfies*

$$\phi_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1} : (\mathbf{u}, 1) \mapsto \mathbf{u} - \mathbf{v}_{\mathcal{O}}$$

where  $\mathbf{v}_{\mathcal{O}} \in \mathbb{R}^3$  is defined by  $\phi_{V_{\mathcal{G}}}(\mathbf{v}_{\mathcal{O}}) = (\mathbf{v}_{\mathcal{O}}, 1)$ . In particular  $\phi_{\mathcal{O}}$  is invertible.

*Proof* Let  $\mathbf{u} \in \mathbb{R}^3$  and let  $v = \phi_{V_{\mathcal{G}}}^{-1}(\mathbf{u}, 1) \in V_{\mathcal{G}}$ . We have

$$\begin{aligned} \phi_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1}(\mathbf{u}, 1) &= \phi_{\tau} \circ P_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1}(\mathbf{u}, 1) \\ &= \phi_{\tau} \circ P_{\mathcal{O}}(v) \\ &= \phi_{\tau}(v - (\tau(v))v_{\mathcal{O}}) \\ &= \phi_{\tau} \circ \phi_{\tau}^{-1}((\mathbf{u}, 1) - (\mathbf{v}_{\mathcal{O}}, 1)) \\ &= \mathbf{u} - \mathbf{v}_{\mathcal{O}}, \end{aligned}$$

as claimed. ■

If  $\phi$  is additionally a coordinate system that is  $\mathcal{O}$ -adapted, then  $\phi_{\mathcal{O}}$  has a particularly simple form.

**1.2.10 Corollary** *If  $\mathcal{O}$  is an observer in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  and  $\phi \in \text{Coord}(\mathcal{O})$ , then the map  $\phi_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1}$  satisfies*

$$\phi_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1} : (\mathbf{u}, 1) \mapsto \mathbf{u}.$$

## 1.3 Particle mechanics

In order to get some feeling for the significance of Galilean spacetimes and Galilean transformations in mechanics, let us investigate the dynamics of particles, or point masses. We start with kinematics, following [Artz 1981].

**1.3.1 World lines** Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime. A **world line** in  $\mathcal{G}$  is a continuous map  $c: I_{\mathcal{G}} \rightarrow \mathcal{E}$  with the property that  $c(s) \in \mathcal{E}(s)$ . The idea is that a world line assigns to each instant an event that occurs at that instant. What one should think of is a world line being the spatio-temporal history of something experiencing the spacetime  $\mathcal{G}$  (see Figure 1.7). A world line is **differentiable** at  $s_0 \in I_{\mathcal{G}}$  if the limit

$$\lim_{t \rightarrow 0} \frac{c(t + s_0) - c(s_0)}{t}$$

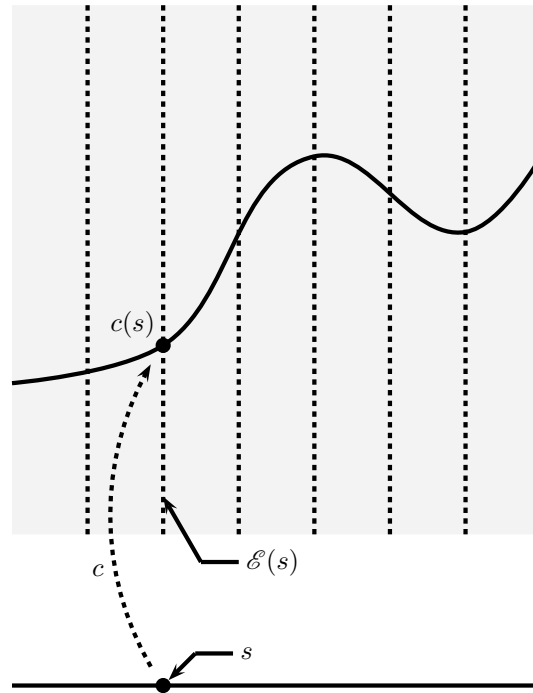


Figure 1.7 The idea of a world line

exists. Note that if the limit exists, it is an element of  $V$  since  $c(t + s_0) - c(s_0) \in V$  and  $t \in \mathbb{R}$ . Thus, we denote this limit by  $c'(s_0) \in V$  called the **velocity** of the world line at the instant  $s_0$ . Similarly, for a differentiable world line, if the limit

$$\lim_{t \rightarrow 0} \frac{c'(t + s_0) - c'(s_0)}{t}$$

exists, we denote it by  $c''(s_0)$  which is the **acceleration** of the world line at the instant  $s_0$ .

The following properties of velocity and acceleration are useful.

**1.3.1 Lemma** *If  $c: I_{\mathcal{G}} \rightarrow \mathcal{E}$  is a world line, then  $c'(s_0) \in V_{\mathcal{G}}$  and  $c''(s_0) \in \ker(\tau)$  for each  $s_0 \in I_{\mathcal{G}}$  where the velocity and acceleration exist.*

*Proof* By the definition of a world line we have

$$\tau(c'(s_0)) = \lim_{t \rightarrow 0} \frac{\tau(c(t + s_0) - c(s_0))}{t} = \lim_{t \rightarrow 0} \frac{t}{t} = 1,$$

and

$$\tau(c''(s_0)) = \lim_{t \rightarrow 0} \frac{\tau(c'(t + s_0) - c'(s_0))}{t} = \lim_{t \rightarrow 0} \frac{1 - 1}{t} = 0.$$

This completes the proof. ■

A world line has a simple form when represented in a coordinate system.

**1.3.2 Proposition** *Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime and  $c: I_{\mathcal{G}} \rightarrow \mathcal{E}$  a world line. If  $\phi$  is a coordinate system for  $\mathcal{E}$  then  $\phi \circ c \circ \phi_{I_{\mathcal{G}}}^{-1}$  has the form  $t \mapsto (\mathbf{x}(t), t)$  for some curve  $t \mapsto \mathbf{x}(t) \in \mathbb{R}^3$ . Thus  $\phi \circ c \circ \phi_{I_{\mathcal{G}}}^{-1}$  is a world line in the standard Galilean spacetime.*

*Proof* Since the map  $\phi_g$  maps instants in  $I_g$  to instants in the canonical Galilean spacetime, and since  $\phi$  maps simultaneous events in  $\mathcal{E}$  to simultaneous events in the canonical Galilean spacetime, the result follows. ■

One readily ascertains that the velocity  $\phi_V \circ c' \circ \phi_{I_g}^{-1}$  and acceleration  $\phi_V \circ c'' \circ \phi_{I_g}^{-1}$  have the form

$$t \mapsto (\dot{\mathbf{x}}(t), 1), \quad t \mapsto (\ddot{\mathbf{x}}(t), 0)$$

in the coordinate system  $\phi$ .

**1.3.2 Interpretation of Newton's Laws for particle motion** One of the basic postulates of Newtonian mechanics is that there are coordinate systems in which the “laws of nature” hold. Just what constitutes the laws of nature is something that one cannot get into too deeply without subjecting oneself to some debate. The essential point is that they are obtained by observing the physical world, and deriving principles from these observations. The laws of nature we will deal with include, for example, *Newton's Laws of Motion*.<sup>2</sup>

NL1. *First Law:* Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

NL2. *Second Law:* The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

NL3. *Third Law:* To every reaction there is always opposed an equal reaction: or, the mutual action of two bodies upon each other are always equal, and directed to contrary parts.

A coordinate system in which the laws of nature are valid is called an *inertial coordinate system*.

**1.3.3 Assumption** An inertial coordinate system exists. □

Inertial coordinate systems do not in fact exist. However, we assume that they do for the purpose of carrying out Newtonian mechanics. Furthermore, we often make choices of coordinate systems as inertial that have varying degrees of badness. For example, the infamous “lab frame” is often used as an inertial coordinate system, where one says that choosing as origin a spot on your workbench, and resetting your clock, defines an inertial coordinate system. This is a demonstrably bad choice of inertial coordinate system (see Exercise E1.14). Despite this lack of existence of actual exact inertial frames, we proceed as if they do exist. And the fact of the matter is that the frames we typically are tempted to use are often good enough for their intended modest purpose.

Now we can talk about particles in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ . We define a *particle* to be a real number  $m > 0$  that is the *mass* of the particle. We suppose the particle resides at  $x_0 \in \mathcal{E}(s_0)$ . An important postulate of Newtonian mechanics the *determinacy principle* that states that the world line of a particle is uniquely determined by its initial position  $x_0 \in \mathcal{E}$  and velocity  $v_0 \in V_g$  in spacetime. Thus, to determine the time-evolution of a particle we need to specify its acceleration. This renders Newton's Second Law to be the following: for a world line  $c$  of a particle of mass  $m$  there exists a map  $F: \mathcal{E} \times V_g \rightarrow \ker(\tau)$  so that at each point  $c(s)$  along the world line  $c$  of the particle we have

$$mc''(s) = F(c(s), c'(s)), \quad c(s_0) = x_0, \quad c'(s_0) = v_0. \quad (1.11)$$

<sup>2</sup>The statements we give are from [Newton 1934]. Newton wrote in Latin, however.

Thus the mass is the “constant of proportionality” and  $F$  is the “impressed force” in Newton’s Second Law. Note that this equation is somewhat remarkable as it is an observer independent formulation of the dynamics of a particle.

Now let us see how this relates to the usual observer *dependent* dynamics one is presented with. Let  $\mathcal{O}$  be an observer with  $\phi$  a coordinate system that is  $\mathcal{O}$ -adapted. Let us see how to represent the force  $F$  in a coordinate system. We recall that the coordinate system  $\phi$  induces a linear map  $\phi_V$  from  $V$  to  $\mathbb{R}^4$ . The observer also induces a linear map  $P_{\mathcal{O}}$  from  $V$  to  $\ker(\tau)$  and so a map  $\phi_{\mathcal{O}}$  from  $V_{\mathcal{G}}$  to  $\mathbb{R}^3$  as in (1.10). We then define  $F_{\phi, \mathcal{O}}: \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$F_{\phi, \mathcal{O}}(t, \mathbf{x}, \mathbf{v}) = \phi_{\tau}(F(\phi^{-1}(\mathbf{x}, t), \phi_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1}(\mathbf{v}, 1))).$$

Thus  $F_{\phi, \mathcal{O}}$  is the representation of the impressed force in the coordinate system  $\phi$ . By Corollary 1.2.10 the initial value problem (1.11) is represented in coordinates by

$$m\ddot{\mathbf{x}}(t) = F_{\phi, \mathcal{O}}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(t_0) = \mathbf{v}_0,$$

where  $t_0 = \phi_{I_{\mathcal{G}}}(s_0)$ . It is *these* equations that one typically deals with in Newtonian dynamics. What we have done is correctly deduce how these are derived from the observer independent equations (1.11) when one has an observer and a coordinate system adapted to this observer. Note that if the coordinate system is *not*  $\mathcal{O}$ -adapted, then one must replace  $\dot{\mathbf{x}}(t)$  with  $\dot{\mathbf{x}}(t) - \mathbf{v}_{\mathcal{O}}$ , where  $\mathbf{v}_{\mathcal{O}}$  is defined by  $\phi_{V_{\mathcal{G}}}(v_{\mathcal{O}}) = (\mathbf{v}_{\mathcal{O}}, 1)$  (cf. Proposition 1.2.9).

Now let us suppose that we are interested in the dynamics of  $N$  particles with masses  $m_1, \dots, m_N$ . Each particle will have associated with it its world line  $c_1, \dots, c_N$ . When one has a collection of particles, one typically wishes to allow that the forces exerted on a particle depend on the positions and velocities of the other particles. One often also asks that the force exerted by particle  $i$  on particle  $j$  at the instant  $s_0$  depends only on the position and velocity of particle  $i$  at that same instant.<sup>3</sup> Therefore, in order to make sure that the forces have the proper property, we introduce the following notation:

$$\tilde{\mathcal{E}}^N = \{ (x_1, \dots, x_N) \in \mathcal{E}^N \mid x_1, \dots, x_N \text{ are simultaneous} \}.$$

We then define the force exerted on particle  $i$  to be a map  $F_I: \tilde{\mathcal{E}}^N \times V^N \rightarrow \ker(\tau)$ , so that the world lines for the entire collection of  $N$  particles are governed by the  $N$  equations

$$\begin{aligned} c_1''(s) &= F_1(c_1(s), \dots, c_N(s), c_1'(s), \dots, c_N'(s)), & c_1(s_0) &= x_{10}, \quad c_1'(s_0) = v_{10} \\ &\vdots \\ c_N''(s) &= F_N(c_1(s), \dots, c_N(s), c_1'(s), \dots, c_N'(s)), & c_N(s_0) &= x_{N0}, \quad c_N'(s_0) = v_{N0}. \end{aligned}$$

Now let us represent these equations with respect to an observer and in a coordinate system adapted to that observer. Define a map  $\phi_N: \tilde{\mathcal{E}}^N \rightarrow \mathbb{R} \times (\mathbb{R}^3)^N$  by

$$\phi_N(x_1, \dots, x_N) = (\phi_{I_{\mathcal{G}}}(s), \phi_{s_0}(x_1), \dots, \phi_{s_0}(x_N)),$$

where  $x_1, \dots, x_N \in \mathcal{E}(s_0)$ . We can then define  $F_{i, \phi, \mathcal{O}}: \mathbb{R} \times (\mathbb{R}^3)^N \times (\mathbb{R}^3)^N \rightarrow \mathbb{R}^3$  by

$$\begin{aligned} F_{i, \phi, \mathcal{O}}(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{v}_1, \dots, \mathbf{v}_N) &= \\ &= \phi_{\tau}(F_i(\phi_N^{-1}(t, \mathbf{x}_1, \dots, \mathbf{x}_N), \phi_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1}(\mathbf{v}_1, 1), \dots, \phi_{\mathcal{O}} \circ \phi_{V_{\mathcal{G}}}^{-1}(\mathbf{v}_N, 1))), \end{aligned}$$

---

<sup>3</sup>It is not altogether uncommon for this not to be the case, but let us simplify our presentation by only dealing with this situation.



for  $i = 1, \dots, N$ . This results in the differential equations

$$\begin{aligned} m_1 \ddot{\mathbf{x}}_1(t) &= F_{1,\phi,\mathcal{E}}(t, \mathbf{x}_1(t), \dots, \mathbf{x}_N(t), \dot{\mathbf{x}}_1(t), \dots, \dot{\mathbf{x}}_N(t)), & \mathbf{x}_1(t_0) &= \mathbf{x}_{10}, \dot{\mathbf{x}}_1(0) = \mathbf{v}_{10}, \\ & \vdots \\ m_N \ddot{\mathbf{x}}_N(t) &= F_{N,\phi,\mathcal{E}}(t, \mathbf{x}_1(t), \dots, \mathbf{x}_N(t), \dot{\mathbf{x}}_1(t), \dots, \dot{\mathbf{x}}_N(t)), & \mathbf{x}_N(t_0) &= \mathbf{x}_{N0}, \dot{\mathbf{x}}_N(0) = \mathbf{v}_{N0}, \end{aligned}$$

where  $t_0 = \phi_{I_{\mathcal{G}}}(s_0)$ . Note that these are  $3N$  coupled second-order differential equations. Typically, any sort of understanding these equations are beyond current knowledge, except for a few simple examples. We will look at a few techniques for investigating the dynamics of such systems in Chapter 3.

For “closed” systems, i.e., those in which the effects of external sources are neglected, things may be made to simplify somewhat. Indeed, the ***Galilean relativity principle*** states that for a closed system in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  the governing physical laws are invariant under Galilean mappings of  $\mathcal{E}$  to itself. For multi-particle systems  $m_1, \dots, m_N$  this means that

$$\begin{aligned} m_i c_i'' &= F_i((c_1(t), \dots, c_N(t)), (c_1'(t), \dots, c_N'(t))) \implies \\ & m(\psi \circ c_i)'' = F_i((\psi \circ c_1(t), \dots, \psi \circ c_N(t)), (\psi_V \circ c_1'(t), \dots, \psi_V \circ c_N'(t))) \end{aligned}$$

for  $i = 1, \dots, N$  and for Galilean maps  $\psi: \mathcal{E} \rightarrow \mathcal{E}$ . Again, since a coordinate system maps world line to world lines, it suffices to investigate the implications of this in a fixed coordinate system  $\phi: \mathcal{E} \rightarrow \mathbb{R}^3 \times \mathbb{R}$ . Some simple cases are given in the exercises at the end of the chapter.

It is easy to imagine that the laws of nature should be invariant under shifts of origin, or rotations of reference frame. That the laws of nature should be invariant under a change to a uniformly moving frame is also consistent with common experience. After all, we on the earth are moving roughly in a uniformly moving frame with respect to, say, the sun. One begins to see a problem with Galilean spacetime when the relative velocity of inertial frames becomes large. Indeed, in such case, Newton’s Laws are seen to break down. The modification of Newtonian mechanics that is currently held to best repair this problem is the theory of general relativity.

## 1.4 Rigid motions in Galilean spacetimes

Many of the examples we will deal with will consist of connected rigid bodies. Unlike particles, rigid bodies possess rotational inertia as well as translational inertia. Thus, to consider rigid body motion, one needs to understand angular velocity and related notions. In this section we deal solely with kinematic issues, saving dynamical properties of rigid bodies for Sections 1.5 and 1.6. We should remark that our treatment of rigid body dynamics is somewhat different from the usual treatment since we allow arbitrary observers. While this to some extent complicates the treatment, at the end of the day it makes all the dependencies clear, whereas this is not altogether obvious in the usual treatment. Some of the issues here are more fully explained in [Lewis 2000a]. Since there are a lot of formulas in this section, and we want to draw attention to some of these, we shall box the equations defining the most essential objects.

**1.4.1 Isometries** We let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime. For  $s \in I_{\mathcal{G}}$ , a map  $\psi: \mathcal{E}(s) \rightarrow \mathcal{E}(s)$  is an ***isometry*** if, in the usual manner,

$$g(\psi(x_2) - \psi(x_1), \psi(x_2) - \psi(x_1)) = g(x_2 - x_1, x_2 - x_1)$$

for every  $x_1, x_2 \in \mathcal{E}(s)$ . We denote by  $\text{Isom}(\mathcal{G})$  the collection of all isometries of  $\mathcal{E}(s)$  for all  $s \in I_{\mathcal{G}}$ . We call the projection  $\Pi_{\mathcal{G}}: \text{Isom}(\mathcal{G}) \rightarrow I_{\mathcal{G}}$  the **isometry bundle** of  $\mathcal{G}$ . Since for fixed  $s \in I_{\mathcal{G}}$  the set  $\mathcal{E}(s)$  is “a lot like” (i.e., exactly like in a coordinate system)  $\mathbb{R}^3$ , we expect there to be a relationship between isometries  $\mathcal{E}(s)$  and the characterisation of  $E(3)$  as the isometries of  $\mathbb{R}^3$ . In preparation for such a description, we let  $O(\ker(\tau))$  denote the  $g$ -orthogonal linear transformations of  $\ker(\tau)$ . Corresponding to this, for reasons we shall see shortly, is the collection  $\mathfrak{o}\ker(\tau)$  of linear transformations of  $\ker(\tau)$  which are  $g$ -skew-symmetric. That is, a linear transformation  $A: \ker(\tau) \rightarrow \ker(\tau)$  is in  $\mathfrak{o}\ker(\tau)$  if

$$g(A(u_1), u_2) = -g(u_1, A(u_2))$$

for all  $u_1, u_2 \in \ker(\tau)$ . It is possible to establish an isomorphism from  $\ker(\tau)$  to  $\mathfrak{o}\ker(\tau)$  as follows. Let  $\{v_1, v_2, v_3\}$  be an orthonormal basis for  $\ker(\tau)$ . Thus any vector  $u \in \ker(\tau)$  can be written as

$$u = u^1 v_1 + u^2 v_2 + u^3 v_3$$

for  $u^1, u^2, u^3 \in \mathbb{R}$ . Now, to such a  $u \in \ker(\tau)$  we assign the element of  $\mathfrak{o}\ker(\tau)$  whose matrix representation with respect to the basis  $\{v_1, v_2, v_3\}$  is

$$\begin{bmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{bmatrix}.$$

Let us denote this matrix by  $\hat{u}$ , and it is easy to check that the map  $\hat{\cdot}: \ker(\tau) \rightarrow \mathfrak{o}\ker(\tau)$  defined in this manner does not depend on the choice of orthonormal basis, and is an isomorphism. We can also define the cross-product on  $\ker(\tau)$  by using an orthonormal basis  $\{v_1, v_2, v_3\}$ . Thus if  $x, y \in \ker(\tau)$  are written

$$x = x^1 v_1 + x^2 v_2 + x^3 v_3, \quad y = y^1 v_1 + y^2 v_2 + y^3 v_3,$$

we define  $x \times y$  to be the vector whose components are the usual cross-product of the vectors  $(x^1, x^2, x^3)$  and  $(y^1, y^2, y^3)$ :

$$(x^1, x^2, x^3) \times (y^1, y^2, y^3) = (x^2 y^3 - x^3 y^2, x^3 y^1 - x^1 y^3, x^1 y^2 - x^2 y^1).$$

One then readily verifies that for  $u \in \ker(\tau)$ , the matrix  $\hat{u}$  is the unique  $g$ -skew-symmetric linear transformation with the property that  $\hat{u}(v) = u \times v$  for each  $v \in \ker(\tau)$ .

Now, if we have an observer  $\mathcal{O} \subset \mathcal{E}$  then the isometry bundle can be more easily understood. We then let  $O(\ker(\tau)) \ltimes \ker(\tau)$  be the “semi-direct product” of the groups  $O(\ker(\tau))$  and  $\ker(\tau)$ . Thus  $O(\ker(\tau)) \ltimes \ker(\tau)$  as a set is simply  $O(\ker(\tau)) \times \ker(\tau)$ , and is a group with the product defined by

$$(R_1, u_1) \cdot (R_2, u_2) = (R_1 R_2, u_1 + R_1 u_2).$$

The following result gives a concrete description of  $\text{Isom}(\mathcal{G})$  in the presence of an observer.

**1.4.1 Proposition** *Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime with  $\mathcal{O} \subset \mathcal{E}$  an observer. There exists a bijection  $\Phi_{\mathcal{O}}: (O(\ker(\tau)) \ltimes \ker(\tau)) \times I_{\mathcal{G}} \rightarrow \text{Isom}(\mathcal{G})$  with the properties*

- (i)  $\Phi_{\mathcal{O}}((R, u), s)$  is an isometry of  $\mathcal{E}(s)$  and
- (ii)  $\Phi_{\mathcal{O}}$  commutes with the product structure for fixed  $s \in I_{\mathcal{G}}$ , i.e.,

$$\Phi_{\mathcal{O}}(R_1 R_2, u_1 + R_1 u_2, s) = \Phi_{\mathcal{O}}((R_1, u_2), s) \circ \Phi_{\mathcal{O}}((R_2, u_2), s).$$

*Proof* We define  $\Phi_{\mathcal{O}}(R, u)$  by

$$\Phi_{\mathcal{O}}(R, u)(x) = (R(x - \mathcal{O}_s) + u) + \mathcal{O}_s,$$

it is then easy to check that  $\Phi_{\mathcal{O}}$  has the properties set out in the proposition.  $\blacksquare$

The observer  $\mathcal{O}$  establishes an “origin”  $\mathcal{O}_s$  in the affine space  $\mathcal{E}(s)$ , thus rendering it a vector space as in Proposition 1.1.2. We can thus consider rotations about this origin. This makes it possible to describe a general isometry as a rotation about this origin plus a translation, and it is this which gives us the correspondence  $\Phi_{\mathcal{O}}$ .

**1.4.2 Rigid motions** We define a *rigid motion* of a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  to be a pair  $\Sigma = (\Psi, \nu)$  where  $\Psi: I_{\mathcal{G}} \rightarrow \text{Isom}(\mathcal{G})$  is a smooth map having the property that  $\Psi(s)$  is an isometry of  $\mathcal{E}(s)$ , and  $\nu \in V_{\mathcal{G}}$ . Corresponding to a rigid motion  $\Sigma$  we define a map  $T_{\Sigma}: \mathcal{E} \rightarrow I_{\mathcal{G}} \times \mathcal{E}$  by

$$T_{\Sigma}(s, x) = \Psi(s)((s - \pi_{\mathcal{G}}(x))\nu + x),$$

and we call  $T_{\Sigma}$  the *motion* corresponding to the rigid motion  $\Sigma$ . That  $\nu \in V_{\mathcal{G}}$  ensures that  $\tau(\nu) = 1$  so that  $T_{\Sigma}(s, x) \in \mathcal{E}(s)$ . The idea is that if an event  $x \in \mathcal{E}$  is subjected to the rigid motion  $\Sigma$ , then at the instant  $s$  it is located at  $T_{\Sigma}(s, x) \in \mathcal{E}$  (see Figure 1.8). The

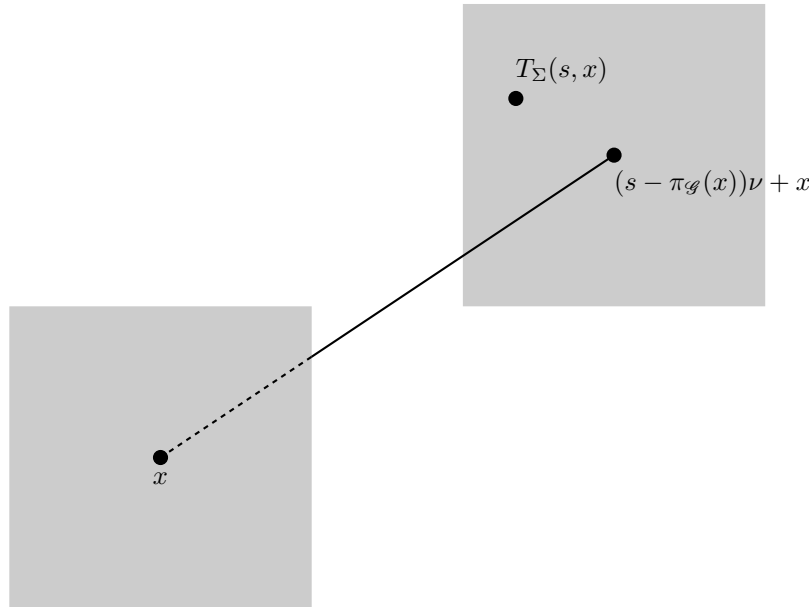


Figure 1.8 The motion of a rigid motion

Galilean velocity  $\nu$  defines the manner in which the motion is transferred from one instant to the next. Note that it is not necessarily the case that  $T_{\Sigma}(\pi_{\mathcal{G}}(x), x) = x$ . This reflects the fact that there is no reason to expect  $\Psi(\pi_{\mathcal{G}}(x))$  to be the identity isometry on  $\mathcal{E}(\pi_{\mathcal{G}}(x))$ .

**1.4.2 Remarks 1.** There are other ways one could define a rigid motion. For example, one could define a rigid motion to be a triple  $(R, r, \mathcal{O})$  where  $R: I_{\mathcal{G}} \rightarrow O(\ker(\tau))$  and  $r: I_{\mathcal{G}} \rightarrow \ker(\tau)$  are smooth maps, and  $\mathcal{O}$  is an observer. One then could define an isometry  $\Psi(s)$

of  $\mathcal{E}(s)$  by  $\Psi(s)(x) = (R(s)(x - \mathcal{O}_s) + r(s)) + \mathcal{O}_s$  where  $\mathcal{O}_s$  is, as usual, the unique point in  $\mathcal{O} \cap \mathcal{E}(s)$ . One could also define the motion of this rigid motion by

$$(s, x) \mapsto \Psi(s)((s - \pi_{\mathcal{G}}(x))v_{\mathcal{O}} + x).$$

As we shall shortly see, this notion of a rigid motion is contained in the definition we have given. The disadvantage of the alternate definition is that it unnecessarily provides a distinguished origin  $\mathcal{O}_s$  for each instant.

2. It is evident that the most interesting thing about a rigid motion, at least as concerns the motion of a rigid body, is the motion it defines. However, it is readily shown (Exercise E1.15) that a motion uniquely determines the rigid motion which gives rise to it. That is, if  $T_{\Sigma_1} = T_{\Sigma_2}$  then  $\Sigma_1 = \Sigma_2$ .  $\square$

To make the notion of a rigid motion more concrete, we shall fix an observer  $\mathcal{O}$ . Following Proposition 1.4.1, we define smooth maps  $R_{\Psi}: I_{\mathcal{G}} \rightarrow O(\ker(\tau))$  and  $r_{\Psi, \mathcal{O}}: I_{\mathcal{G}} \rightarrow \ker(\tau)$  by asking that  $\Psi(s) = \Phi_{\mathcal{O}}(R_{\Psi}(s), r_{\Psi, \mathcal{O}}(s))$ . We may then write

$$\Psi(s)(x) = (R_{\Psi}(s)(x - \mathcal{O}_s) + r_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s, \quad x \in \mathcal{E}(s). \quad (1.12)$$

The following result makes sensible our notation that does not include the dependence of  $R_{\Psi}$  on  $\mathcal{O}$ .

**1.4.3 Proposition** *Let  $\Sigma = (\Psi, \nu)$  be a rigid motion in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$ . The maps  $R_{\Psi}$  and  $r_{\Psi, \mathcal{O}}$  defined by (1.12) are uniquely determined by  $\Psi$  and the observer  $\mathcal{O}$ . If  $\tilde{\mathcal{O}}$  is another observer and  $\tilde{R}_{\Psi}$  and  $\tilde{r}_{\Psi, \tilde{\mathcal{O}}}$  are defined by*

$$\Psi(s)(x) = (\tilde{R}_{\Psi}(s)(x - \tilde{\mathcal{O}}_s) + \tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s)) + \tilde{\mathcal{O}}_s, \quad x \in \mathcal{E}(s),$$

then

$$\begin{aligned} \tilde{R}_{\Psi}(s) &= R_{\Psi}(s) \\ \tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s) &= r_{\Psi, \mathcal{O}}(s) + (R_{\Psi}(s) - \text{id}_{\tau})(\tilde{\mathcal{O}}_s - \mathcal{O}_s), \end{aligned}$$

where  $\text{id}_{\tau}$  denotes the identity map on  $\ker(\tau)$ .

*Proof* Let us show that for a given observer  $\mathcal{O}$ ,  $R_{\Psi}$  and  $r_{\Psi, \mathcal{O}}$  are uniquely determined. If  $\tilde{R}_{\Psi}$  and  $\tilde{r}_{\Psi, \mathcal{O}}$  satisfy

$$\Psi(s)(x) = (\tilde{R}_{\Psi}(s)(x - \mathcal{O}_s) + \tilde{r}_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s, \quad x \in \mathcal{E}(s),$$

then we have

$$(\tilde{R}_{\Psi}(s)(x - \mathcal{O}_s) - R_{\Psi}(s)(x - \mathcal{O}_s)) + (\tilde{r}_{\Psi, \mathcal{O}}(s) - r_{\Psi, \mathcal{O}}(s)) = 0$$

for every  $x \in \mathcal{E}(s)$ . In particular, choosing  $x = \mathcal{O}_s$  shows that  $\tilde{r}_{\Psi, \mathcal{O}}(s) = r_{\Psi, \mathcal{O}}(s)$  from which we then ascertain that  $\tilde{R}_{\Psi}(s) = R_{\Psi}(s)$ .

Now let us choose a different observer  $\tilde{\mathcal{O}}$  and let us denote  $\tilde{\mathcal{O}}_s$  to be the unique point in  $\tilde{\mathcal{O}} \cap \mathcal{E}(s)$ . We then have a unique  $\tilde{R}_{\Psi}(s) \in O(\ker(\tau))$  and  $\tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s) \in \ker(\tau)$  so that

$$\Psi(s)(x) = (\tilde{R}_{\Psi}(s)(x - \tilde{\mathcal{O}}_s) + \tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s)) + \tilde{\mathcal{O}}_s, \quad x \in \mathcal{E}(s).$$

This means that

$$(\tilde{R}_\Psi(s)(x - \tilde{\mathcal{O}}_s) - R_\Psi(s)(x - \mathcal{O}_s)) + (\tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s) - r_{\Psi, \mathcal{O}}(s)) + (\tilde{\mathcal{O}}_s - \mathcal{O}_s) = 0$$

for every  $x \in \mathcal{E}(s)$ . Letting  $x = \tilde{\mathcal{O}}_s$  we get

$$R_\Psi(s)(\tilde{\mathcal{O}}_s - \mathcal{O}_s) = \tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s) - r_{\Psi, \mathcal{O}}(s) + (\tilde{\mathcal{O}}_s - \mathcal{O}_s)$$

and letting  $x = \mathcal{O}_s$  we get

$$\tilde{R}_\Psi(s)(\tilde{\mathcal{O}}_s - \mathcal{O}_s) = \tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s) - r_{\Psi, \mathcal{O}}(s) + (\tilde{\mathcal{O}}_s - \mathcal{O}_s).$$

From this we deduce that

$$\tilde{R}_\Psi(s)(\tilde{\mathcal{O}}_s - \mathcal{O}_s) = R_\Psi(s)(\tilde{\mathcal{O}}_s - \mathcal{O}_s).$$

Therefore

$$\tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s) = r_{\Psi, \mathcal{O}}(s) + (R_\Psi(s) - \text{id}_\tau)(\tilde{\mathcal{O}}_s - \mathcal{O}_s) \quad (1.13)$$

and so we compute

$$\begin{aligned} (\tilde{R}_\Psi(s)(x - \tilde{\mathcal{O}}_s) + \tilde{r}_{\Psi, \tilde{\mathcal{O}}}(s)) + \tilde{\mathcal{O}}_s &= (\tilde{R}_\Psi(s)(x - \mathcal{O}_s) + \tilde{R}_\Psi(s)(\mathcal{O}_s - \tilde{\mathcal{O}}_s) + \\ &\quad r_{\Psi, \mathcal{O}}(s) + (R_\Psi(s) - \text{id}_\tau)(\tilde{\mathcal{O}}_s - \mathcal{O}_s)) + \tilde{\mathcal{O}}_s \\ &= (\tilde{R}_\Psi(x - \mathcal{O}_s) + r_{\Psi, \mathcal{O}}(s) + (\mathcal{O}_s - \tilde{\mathcal{O}}_s)) + \tilde{\mathcal{O}}_s \\ &= (\tilde{R}_\Psi(x - \mathcal{O}_s) + r_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s. \end{aligned}$$

From the first part of the proposition it now follows that  $\tilde{R}_\Psi = R_\Psi$ . The desired expression for  $\tilde{r}_{\Psi, \tilde{\mathcal{O}}}$  is already given in (1.13).  $\blacksquare$

Note that our definitions of  $R_\Psi$  and  $r_{\Psi, \mathcal{O}}$  make no reference to the Galilean velocity  $\nu$  that comes with a rigid motion  $\Sigma = (\Psi, \nu)$ . However, this dependence on  $\nu$  will show up when we write the motion  $T_\Sigma$  in terms of  $R_\Psi$  and  $r_{\Psi, \mathcal{O}}$ . That is, we have

$$T_\Sigma(s, x) = (R_\Psi(s)(x - ((\pi_{\mathcal{G}}(x) - s)\nu + \mathcal{O}_s)) + r_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s.$$

It will be convenient when we write the motion to use the fact that

$$\mathcal{O}_s = (s - \pi_{\mathcal{G}}(x))v_{\mathcal{O}} + \mathcal{O}_{\pi_{\mathcal{G}}(x)}.$$

In this case we have

$$T_\Sigma(s, x) = (R_\Psi(s)(x - ((\pi_{\mathcal{G}}(x) - s)(\nu - v_{\mathcal{O}}) + \mathcal{O}_{\pi_{\mathcal{G}}(x)})) + r_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s.$$

A rigid motion  $\Sigma = (\Psi, \nu)$  distinguishes a class of observers  $\mathcal{O}$  with the property that  $v_{\mathcal{O}} = \nu$ . Let us denote this class of observer by  $\text{Ob}(\Sigma)$ . If  $\mathcal{O} \in \text{Ob}(\Sigma)$  then  $\mathcal{O}_{s_2} = (s_2 - s_1)\nu + \mathcal{O}_{s_1}$ . In this case, the expression for the motion thus simplifies to

$$T_\Sigma(s, x) = (R_\Psi(s)(x - \mathcal{O}_{\pi_{\mathcal{G}}(x)}) + r_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s.$$

Let us look at a simple example.

**1.4.4 Example** We consider perhaps the simplest rigid motion: rotation about a fixed line. We work with the canonical Galilean spacetime  $(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}^4, g_{\text{can}}, \tau_{\text{can}})$  and we fix  $\boldsymbol{\omega}_0 \in \mathbb{R}^3$ . We define  $\nu = (\mathbf{0}, 1)$  and for  $t \in \mathbb{R}$  we define an isometry  $\Psi(t)$  of  $\mathbb{R}^3$  by

$$\Psi(t)(\mathbf{x}) = \exp(t\hat{\boldsymbol{\omega}}_0)\mathbf{x} \quad (1.14)$$

where  $\exp$  is here the matrix exponential. This gives us our rigid motion  $\Sigma = (\Psi, \nu)$ . If we choose the canonical observer  $\mathcal{O}_{\text{can}}$  then  $\mathbf{R}(t) \in O(3)$  and  $\mathbf{r}(t) \in \mathbb{R}^3$  are defined by  $\Phi_{\mathcal{O}_{\text{can}}}((\mathbf{R}, \mathbf{r}), t) = \Psi(t)$ . Therefore  $\mathbf{R}(t) = \exp(t\hat{\boldsymbol{\omega}}_0)$  and  $\mathbf{r}(t) = \mathbf{0}$ . The definition of  $T_\Sigma$  gives

$$T_\Sigma(t, \mathbf{x}) = (\exp(t\hat{\boldsymbol{\omega}}_0)\mathbf{x}, t).$$

Thus the motion is simply a rotation about  $\mathbf{0}$ .

Let us now show that this is a rotation about the vector  $\boldsymbol{\omega}_0$ . We claim that all points on the line

$$\ell_{\boldsymbol{\omega}_0} = \{a\boldsymbol{\omega}_0 \mid a \in \mathbb{R}\}.$$

are fixed by the rigid motion  $\Sigma$ . Since this particular rigid motion is linear in the given coordinate system, this will follow if we can show that  $\boldsymbol{\omega}_0$  is fixed by the rigid motion. We compute

$$\mathbf{R}(t)\boldsymbol{\omega}_0 = \exp(t\hat{\boldsymbol{\omega}}_0)\boldsymbol{\omega}_0 = \boldsymbol{\omega}_0$$

where we have used Exercise [E1.11\(c\)](#). Therefore  $\Psi(t)$  satisfies

$$\Psi(t)(\boldsymbol{\omega}_0) = \boldsymbol{\omega}_0,$$

which shows that  $\Psi(t)$  fixes  $\ell_{\boldsymbol{\omega}_0}$  as claimed. It therefore follows that  $\Psi(t)$  is indeed a rotation about the vector  $\boldsymbol{\omega}_0$ . We will have to wait until we actually talk about angular velocity to see that this is a uniform rotation, i.e., one with constant speed.  $\square$

**1.4.3 Rigid motions and relative motion** It is convenient in our discussion of spatial and angular velocities to regard these as contributing to the description of the same thing in two different coordinate systems. Let us fix  $s_0 \in I_{\mathcal{G}}$  and we think of  $\mathcal{E}(s_0)$  as something which gets moved around with the rigid motion  $\Sigma = (\Psi, \nu)$ . Thus one could think of fixing some origin  $O$  in  $\mathcal{E}(s_0)$  and attaching to it an orthonormal frame  $\{v_1, v_2, v_3\}$ . As we subject  $\mathcal{E}(s_0)$  to the motion  $T_\Sigma$ , the origin and the frame will be moved in a corresponding manner. A planar picture is given in [Figure 1.9](#). What we wish to do is relate how something looks in the moving coordinate systems to how it looks in an inertial coordinate system.

**1.4.4 Spatial velocities** In this section we introduce concepts of velocity associated with a rigid motion and an observer. The notions of velocity we introduce will have the advantage of being Galilean invariant quantities in a sense that we will make precise. We also talk about velocity in *two* ways, one as seen from an inertial frame, and the other as seen in a frame moving with the motion. Except for the fact that we are working in a Galilean spacetime, much of what we say here may be found in [\[Murray, Li, and Sastry 1994\]](#).

We retain the notation from the previous sections, in particular that of the maps  $R_\Psi$  and  $r_{\Psi, \mathcal{O}}$  that are associated to a rigid motion  $\Sigma = (\Psi, \nu)$  and an observer  $\mathcal{O}$ . We do not assume that  $\mathcal{O} \in \text{Ob}(\Sigma)$ . The following result provides us with the natural interpretation of spatial velocity, both angular and linear.

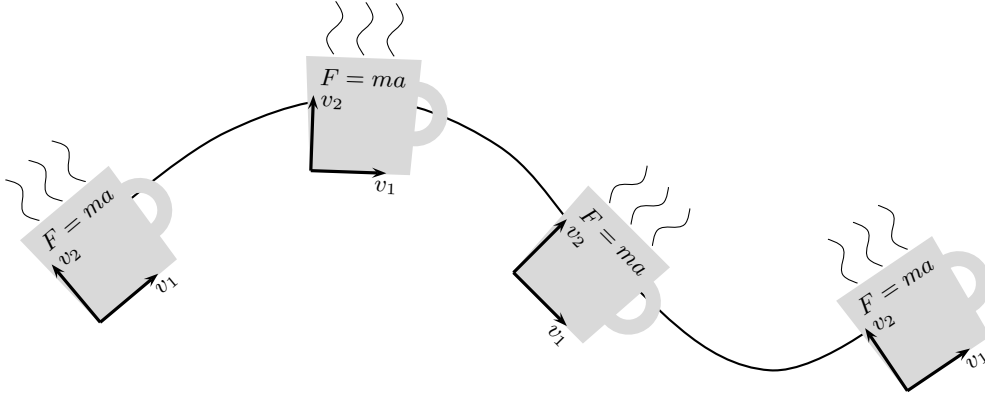


Figure 1.9 Rigid motion in the plane

**1.4.5 Proposition (Spatial velocities)** *Let  $\Sigma = (\Psi, \nu)$  be a rigid motion in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, \mathbb{V}, g, \tau)$  and let  $\mathcal{O}$  be an observer in  $\mathcal{G}$ . Let  $\mathcal{O}_s$  denote the unique point in  $\mathcal{O} \cap \mathcal{E}(s)$ . Fix  $s_0 \in I_{\mathcal{G}}$  and suppose that curves  $s \mapsto \xi(s) \in \mathcal{E}(s_0)$  and  $s \mapsto x(s) \in \mathcal{E}$  are related by  $x(s) = T_{\Sigma}(s, \xi(s))$ .*

*Then there exists smooth maps  $\omega_{\Psi}: I_{\mathcal{G}} \rightarrow \ker(\tau)$  and  $v_{\Psi, \mathcal{O}}: I_{\mathcal{G}} \rightarrow \ker(\tau)$  so that*

$$P_{\mathcal{O}}(x'(s)) = R_{\Psi}(s)(\xi'(s) + (\nu - v_{\mathcal{O}})) + \omega_{\Psi}(s) \times (x(s) - \mathcal{O}_s) + v_{\Psi, \mathcal{O}}(s).$$

*Furthermore,  $\omega_{\Psi}$  and  $v_{\Psi, \mathcal{O}}$  may be chosen in such a manner that  $\omega_{\Psi}$  does not depend on the observer  $\mathcal{O}$ , and neither  $\omega_{\Psi}$  nor  $v_{\Psi, \mathcal{O}}$  depend on  $\nu$ .*

*Proof* We differentiate the equality

$$x(s) = T_{\Sigma}(s, \xi(s)) = (R_{\Psi}(s)(\xi(s) - ((s_0 - s)(\nu - v_{\mathcal{O}}) + \mathcal{O}_{s_0})) + r_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s$$

with respect to  $s$ :

$$\begin{aligned} x'(s) &= R_{\Psi}(s)(\xi'(s) + (\nu - v_{\mathcal{O}})) + R'_{\Psi}(s)(\xi(s) - ((s_0 - s)(\nu - v_{\mathcal{O}}) + \mathcal{O}_{s_0})) + \\ &\quad r'_{\Psi, \mathcal{O}}(s) + \mathcal{O}'_s \\ &= R_{\Psi}(s)(\xi'(s) + (\nu - v_{\mathcal{O}})) + R'_{\Psi}(s) \circ R_{\Psi}^{-1}(s)((x(s) - \mathcal{O}_s) - r_{\Psi, \mathcal{O}}(s)) + \\ &\quad r'_{\Psi, \mathcal{O}}(s) + \mathcal{O}'_s. \end{aligned} \quad (1.15)$$

We note that since  $R_{\Psi}(s)$  is orthogonal,  $R_{\Psi}(s) \circ R_{\Psi}^t(s) = \text{id}_{\tau}$ . Differentiating with respect to  $s$  gives

$$\begin{aligned} R_{\Psi}(s) \circ R'_{\Psi}(s) + R'_{\Psi}(s) \circ R_{\Psi}^t(s) &= 0 \\ \implies (R'_{\Psi}(s) \circ R_{\Psi}^t(s))^t &= -R'_{\Psi}(s) \circ R_{\Psi}^t(s), \end{aligned}$$

which means that  $R'_{\Psi}(s) \circ R_{\Psi}^{-1}(s)$  is skew-symmetric with respect to  $g$ . Thus there exists a vector  $\omega_{\Psi}(s) \in \ker(\tau)$  so that  $R'_{\Psi}(s) \circ R_{\Psi}^{-1}(s) = \hat{\omega}_{\Psi}(s)$ .

The following simple lemma records some useful facts.

**1 Lemma** *For each  $s \in I_{\mathcal{G}}$ ,  $P_{\mathcal{O}}(\mathcal{O}'_s) = 0$  and  $P_{\mathcal{O}} \circ R_{\Psi}(x)(\xi'(s) + (\nu - v_{\mathcal{O}})) = R_{\Psi}(s)(\xi'(s) + (\nu - v_{\mathcal{O}}))$ .*

*Proof* For the first assertion we compute

$$\begin{aligned}\mathcal{O}'_s &= \lim_{t \rightarrow 0} \frac{\mathcal{O}_{t+s} - \mathcal{O}_s}{t} \\ &= \lim_{t \rightarrow 0} \frac{(tv_\mathcal{O} + \mathcal{O}_s) - \mathcal{O}_s}{t} \\ &= \lim_{t \rightarrow 0} \frac{tv_\mathcal{O}}{t} = v_\mathcal{O}.\end{aligned}$$

By definition of  $P_\mathcal{O}$  our assertion follows.

For the second assertion, we claim that if  $A \in \mathfrak{gl}(\ker(\tau))$  then

$$P_\mathcal{O} \circ A(u) = A \circ P_\mathcal{O}(u) = A(u)$$

for every  $u \in \ker(\tau)$ . Indeed, since  $u \in \ker(\tau)$  we have

$$P_\mathcal{O} \circ A(u) = A(u) - \tau(A(u))v_\mathcal{O} = A(u),$$

and

$$A \circ P_\mathcal{O}(u) = A((u) - \tau(u)v_\mathcal{O}) = A(u),$$

as claimed. The statement in the lemma follows directly from our more general claim.  $\blacktriangledown$

Using the lemma, applying  $P_\mathcal{O}$  to (1.15), existence of  $\omega_\Psi$  and  $v_{\Psi,\mathcal{O}}$  follows if we now define

$$v_{\Psi,\mathcal{O}} = r'_{\Psi,\mathcal{O}}(s) + r_{\Psi,\mathcal{O}}(s) \times \omega_\Psi(s).$$

The final assertion of the proposition follows since our definition of  $\omega_\Psi$  depends only upon  $R_\Psi$  which is itself independent of  $\mathcal{O}$ , and  $v_{\Psi,\mathcal{O}}$  is evidently independent of  $\nu$ .  $\blacksquare$

Summarising the previous result, given a rigid motion  $\Sigma = (\Psi, \nu)$  and an observer  $\mathcal{O}$ , we define the **spatial angular velocity** for  $\Sigma$  to be the unique map  $\omega_\Psi: I_\mathcal{G} \rightarrow \ker(\tau)$  satisfying

$$\hat{\omega}_\Psi(s) = R'_\Psi(s) \circ R_\Psi^{-1}(s).$$

The **spatial linear velocity** for the rigid motion  $\Sigma$  with respect to the observer  $\mathcal{O}$  is given by

$$v_{\Psi,\mathcal{O}} = r'_{\Psi,\mathcal{O}}(s) + r_{\Psi,\mathcal{O}}(s) \times \omega_\Psi(s).$$

Clearly  $\omega_\Psi$  is an observer independent quantity, and  $v_{\Psi,\mathcal{O}}$  depends on a choice of observer. Note that the first term in the expression for  $P_\mathcal{O}(x'(s))$  in the proposition simplifies if  $\mathcal{O} \in \text{Ob}(\Sigma)$ . Furthermore, this term exactly vanishes if  $\xi'(s)$  is also independent of  $s$ , e.g., in the case when one is following the motion of a single point as it gets transported by the rigid motion.

Let us make some comments concerning these definitions.

**1.4.6 Remarks 1.** The definition we give of spatial linear velocity seems weird. Why not just go with  $r'_{\Psi,\mathcal{O}}(s)$ ? The answer will come when we define body velocities. Given the definition as it is, it becomes not entirely clear what it means. An interpretation is given in [Murray, Li, and Sastry 1994].



2. The term  $R_{\Psi}(s)(\xi'(s) + (\nu - v_{\mathcal{O}}))$  in the proposition is to be thought of as the relative velocity measured in the frame of the body. For example, if  $R_{\Psi}(s) = \text{id}_{\tau}$  for all  $s \in I_{\mathcal{G}}$ , and if  $\mathcal{O} \in \text{Ob}(\Sigma)$ , then we simply obtain

$$P_{\mathcal{O}}(x'(s)) = \xi'(s) + r_{\Psi, \mathcal{O}}(s),$$

which is the usual addition of velocities for rectilinear motion.

3. When  $r_{\Psi, \mathcal{O}}(s) = 0$  for all  $s \in I_{\mathcal{G}}$ , then we have rotational motion with respect to the observer  $\mathcal{O}$ . If we further suppose that  $\xi(s)$  is independent of  $s$  (as for example if we are observing the motion of a point in the body itself) and that  $\mathcal{O} \in \text{Ob}(\Sigma)$ , then  $\xi'(s) = \omega_{\Psi}(s) \times (x(s) - \mathcal{O}_s)$ , which may resemble a familiar formula.  $\square$

It is about time to resume our example.

**1.4.7 Example** (Example 1.4.4 cont'd) We again take the canonical Galilean spacetime  $(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}^4, g_{\text{can}}, \tau_{\text{can}})$  and the rigid motion  $\Sigma = (\Psi, \nu)$  defined by (1.14). We compute the spatial angular velocity to be, by definition,

$$\hat{\omega}(t) = \mathbf{R}'(t)\mathbf{R}^{-1}(t) = \exp(t\hat{\omega}_0)\hat{\omega}_0 \exp(-t\hat{\omega}_0) = \hat{\omega}_0$$

where we have used Exercise E1.10. This justifies our original claim that  $\Psi$  is a uniform rotation about the vector  $\omega_0$ . Note also that the spatial linear velocity  $\mathbf{v}(t)$  vanishes since  $\mathbf{r}(t) = \mathbf{0}$ .

Note that  $\mathcal{O}_{\text{can}} \in \text{Ob}(\Sigma)$  and that  $\mathbf{r}'(t) = \mathbf{0}$  for each  $t \in \mathbb{R}$ . Thus if we fix a point  $\mathbf{x}$  in the set  $\mathcal{E}(0)$  of simultaneous events, Proposition 1.4.5 gives its velocity at time  $t$  as

$$\mathbf{x}'(t) = \omega(t) \times \mathbf{x}(t) = \omega(t) \times (\exp(t\hat{\omega}_0)\mathbf{x}) = \omega_0 \times (\exp(t\hat{\omega}_0)\mathbf{x}).$$

Note then that if  $\mathbf{x} = a\omega_0$  for some  $a \in \mathbb{R}$ , then the velocity of this point at time  $t$  is

$$\dot{\mathbf{x}}(t) = \omega_0 \times (\exp(t\hat{\omega}_0)\omega_0) = \omega_0 \times \omega_0 = \mathbf{0},$$

by Exercise E1.11(c). This again verifies our previous conclusion that  $\Psi$  fixes points on the line  $\ell_{\omega_0}$ . See Figure 1.10. Note that rotations obey the “right-hand rule:” If you point your right thumb in the direction of the spatial angular velocity, the rotation will be in the direction that your fingers curl.  $\square$

**1.4.5 Body velocities** Now let us turn to defining body velocities. We proceed in very much the same manner as in Proposition 1.4.5.

**1.4.8 Proposition** (Body velocities) *Let  $\Sigma = (\Psi, \nu)$  be a rigid motion in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, g, \tau)$  and let  $\mathcal{O}$  be an observer in  $\mathcal{G}$ . Let  $\mathcal{O}_s$  denote the unique point in  $\mathcal{O} \cap \mathcal{E}(s)$ . Fix  $s_0 \in I_{\mathcal{G}}$  and suppose that curves  $s \mapsto \xi(s) \in \mathcal{E}(s_0)$  and  $s \mapsto \mathbf{x}(s) \in \mathcal{E}$  are related by  $\mathbf{x}(s) = \mathbf{T}_{\Sigma}(s, \xi(s))$ .*

*Then there exists smooth maps  $\Omega_{\Psi}: I_{\mathcal{G}} \rightarrow \ker(\tau)$  and  $V_{\Psi, \mathcal{O}}: I_{\mathcal{G}} \rightarrow \ker(\tau)$  so that*

$$\xi'(s) + (\nu - v_{\mathcal{O}}) = \mathbf{R}_{\Psi}^{-1}(s)(P_{\mathcal{O}}(\mathbf{x}'(s))) - \Omega_{\Psi}(s) \times (\xi(s) - ((s_0 - s)(\nu - v_{\mathcal{O}}) + \mathcal{O}_{s_0})) - V_{\Psi, \mathcal{O}}(s). \quad (1.16)$$

*Furthermore,  $\Omega_{\Psi}$  and  $V_{\Psi, \mathcal{O}}$  may be chosen in such a manner that  $\Omega_{\Psi}$  does not depend on the observer  $\mathcal{O}$ , and neither  $\Omega_{\Psi}$  nor  $V_{\Psi, \mathcal{O}}$  depend on  $\nu$ .*

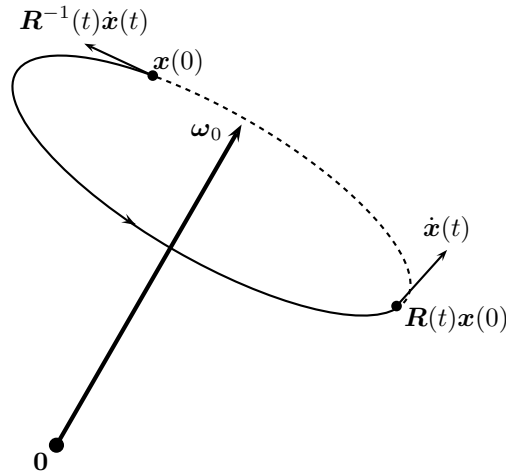


Figure 1.10 Simple rotational motion

*Proof* Differentiating the equality

$$x(s) = (R_{\Psi}(s)(\xi(s) - ((s_0 - s)(\nu - v_{\mathcal{O}}) + \mathcal{O}_{s_0})) + r_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s$$

and solving for  $\xi'(s) + (\nu - v_{\mathcal{O}})$  gives

$$\begin{aligned} \xi'(s) + (\nu - v_{\mathcal{O}}) &= R_{\Psi}^{-1}(s)((x'(s) - \mathcal{O}'_s) - r'_{\Psi, \mathcal{O}}(s)) - \\ &\quad R_{\Psi}^{-1}(s) \circ R'_{\Psi}(s)(\xi(s) - ((s_0 - s)(\nu - v_{\mathcal{O}}) + \mathcal{O}_{s_0})). \end{aligned}$$

If we follow along the same lines as the proof of Proposition 1.4.5 and differentiate the equality  $R_{\Psi}^t(s) \circ R_{\Psi}(s) = \text{id}_{\tau}$  with respect to  $s$ , we can show that  $R_{\Psi}^{-1}(s) \circ R'_{\Psi}(s)$  is skew-symmetric with respect to  $g$ . Therefore there exists  $\Omega_{\Psi}(s) \in \ker(\tau)$  so that  $R_{\Psi}^{-1}(s) \circ R'_{\Psi}(s) = \hat{\Omega}_{\Psi}(s)$ .

By Lemma 1 in the proof of Proposition 1.4.5, we have

$$P_{\mathcal{O}} \circ R_{\Psi}^{-1}((x'(s) - \mathcal{O}'_s) - r'_{\Psi, \mathcal{O}}(s)) = R_{\Psi}^{-1}(P_{\mathcal{O}}(x'(s))) - R_{\Psi}^{-1}(r'_{\Psi, \mathcal{O}}(s))$$

and

$$P_{\mathcal{O}}(\xi'(s) + (\nu - v_{\mathcal{O}})) = \xi'(s) + (\nu - v_{\mathcal{O}}).$$

Thus the result follows if we take

$$V_{\Psi, \mathcal{O}}(s) = R_{\Psi}^{-1}(s)(r'_{\Psi, \mathcal{O}}(s)).$$

This concludes the proof. ■

We use this result to define the body velocities for the rigid motion  $\Sigma = (\Psi, \nu)$  with respect to the observer  $\mathcal{O}$ . Thus we define the **body angular velocity** to be the map  $\Omega_{\Psi}: I_{\mathcal{G}} \rightarrow \ker(\tau)$  with the property that

$$\boxed{\hat{\Omega}_{\Psi}(s) = R_{\Psi}^{-1}(s) \circ R'_{\Psi}(s).}$$

The **body linear velocity** for the rigid motion  $\Psi$  with respect to the observer  $\mathcal{O}$  is given by

$$\boxed{V_{\Psi, \mathcal{O}}(s) = R_{\Psi}^{-1}(s)(r'_{\Psi, \mathcal{O}}(s)).}$$

Again, we reiterate that  $\Omega_{\Psi}$  is observer independent. Note that the relation (1.16) simplifies, as did the similar expression for spatial velocity, when  $\mathcal{O} \in \text{Ob}(\Sigma)$ .

- 1.4.9 **Remarks 1.** The interpretation of the body linear velocity here is somewhat easier to digest than is its spatial counterpart: it is merely the velocity of the origin of the moving frame.
2. If  $\xi(s)$  is constant, i.e., the point is stationary with respect to the moving frame, and if  $\mathcal{O} \in \text{Ob}(\Sigma)$  then we have

$$R_{\Psi}^{-1}(s)(P_{\mathcal{O}}(x'(s))) = \Omega_{\Psi}(s) \times (\xi(s) - \mathcal{O}_{s_0}) + V_{\Psi, \mathcal{O}}(s).$$

In particular, for rotational motion where  $V_{\Psi, \mathcal{O}}(s) = 0$ , we have a ready interpretation of the body angular velocity.  $\square$

Let us look at our simple example of rotation about a fixed line.

1.4.10 **Example** (Example 1.4.4 cont'd) The body angular velocity in this example is given by

$$\hat{\Omega}(t) = \mathbf{R}^{-1}(t)\mathbf{R}'(t) = \exp(-t\hat{\omega}_0)\exp(t\hat{\omega}_0)\hat{\omega}_0 = \hat{\omega}_0,$$

where we have used Exercise E1.11(c). Note that  $\Omega(t)$  can be thought of as the angular velocity at the time  $t$  as it would appear to an observer who is moving along with the rigid motion. This is generally the interpretation of body angular velocity, and in this case it is just quite simple. It just happens in this case that because the motion is so simple the body and spatial angular velocities coincide. One also readily checks that the body linear velocity  $\mathbf{V}(t)$  is identically zero.  $\square$

Let us now observe the relationship between the spatial and body velocities. The following result is readily verified by direct calculation using the definitions of spatial and body velocities.

1.4.11 **Proposition** *Let  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$  be a Galilean spacetime with  $\Sigma = (\Psi, \nu)$  a rigid motion in  $\mathcal{G}$  and  $\mathcal{O}$  an observer. Let  $\omega_{\Psi}$  and  $v_{\Psi, \mathcal{O}}$  be the spatial angular and linear velocity, and let  $\Omega_{\Psi}$  and  $V_{\Psi, \mathcal{O}}$  be the body angular and linear velocity. We then have*

$$\begin{bmatrix} \omega_{\Psi, \mathcal{O}}(s) \\ v_{\Psi}(s) \end{bmatrix} = \begin{bmatrix} R_{\Psi}(s) & 0 \\ \hat{r}_{\Psi, \mathcal{O}}(s) \circ R_{\Psi}(s) & R_{\Psi}(s) \end{bmatrix} \begin{bmatrix} \Omega_{\Psi, \mathcal{O}}(s) \\ V_{\Psi}(s) \end{bmatrix}.$$

Let us look at this in another way, just for fun. Motivated by (1.6), let us represent elements of the group  $O(\ker(\tau)) \times \ker(\tau)$  by matrices like

$$\begin{bmatrix} R & r \\ 0^t & 1 \end{bmatrix},$$

for  $R \in O(\ker(\tau))$  and  $r \in \ker(\tau)$ . Thus, given a rigid motion  $\Sigma = (\Psi, \nu)$  and an observer  $\mathcal{O}$  with  $R_{\Psi}(s)$  and  $r_{\Psi, \mathcal{O}}(s)$  curves in  $O(\ker(\tau))$  and  $\ker(\tau)$  defined in the usual manner, we denote

$$\mathbf{G}_{\Psi, \mathcal{O}}(s) = \begin{bmatrix} R_{\Psi}(s) & r_{\Psi, \mathcal{O}}(s) \\ 0^t & 1 \end{bmatrix}.$$

One then verifies (Exercise E1.17) that

$$\mathbf{G}'_{\Psi, \mathcal{O}}(s)\mathbf{G}_{\Psi, \mathcal{O}}^{-1}(s) = \begin{bmatrix} \hat{\omega}_{\Psi}(s) & v_{\Psi, \mathcal{O}}(s) \\ 0^t & 0 \end{bmatrix}, \quad \mathbf{G}_{\Psi, \mathcal{O}}^{-1}(s)\mathbf{G}'_{\Psi, \mathcal{O}}(s) = \begin{bmatrix} \hat{\Omega}_{\Psi}(s) & V_{\Psi, \mathcal{O}}(s) \\ 0^t & 0 \end{bmatrix}. \quad (1.17)$$

We therefore also see that

$$\begin{bmatrix} \hat{\omega}_\Psi(s) & v_{\Psi,\sigma}(s) \\ 0^t & 0 \end{bmatrix} = \mathbf{G}_{\Psi,\sigma}(s) \begin{bmatrix} \hat{\Omega}_\Psi(s) & V_{\Psi,\sigma}(s) \\ 0^t & 0 \end{bmatrix} \mathbf{G}_{\Psi,\sigma}^{-1}(s). \quad (1.18)$$

The implications of this formalism run all through [Murray, Li, and Sastry 1994], and form part of the analysis of Section 4.4 of [Abraham and Marsden 1978].

**1.4.12 Example** (Example 1.4.4 cont'd) We can also check that Proposition 1.4.11 holds for our example of uniform rotation about  $\omega_0$ . In this case it amounts to verifying that

$$\begin{bmatrix} \omega(t) \\ \mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} \omega_0 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \exp(t\hat{\omega}_0)\omega_0 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{R}(t) & \mathbf{0}_{3 \times 3} \\ \hat{\mathbf{r}}(t)\mathbf{R}(t) & \mathbf{0}_{3 \times 3} \end{bmatrix} \begin{bmatrix} \Omega(t) \\ \mathbf{V}(t) \end{bmatrix},$$

since  $\mathbf{r}(t) = \mathbf{0}$  and using Exercise E1.11(c).  $\square$

**1.4.6 Planar rigid motions** Very often one wishes to restrict consideration to motions that take place in a plane. Although one can consider these cases within the context of the general notion of a rigid motion, things simplify so significantly for planar rigid motions that it is worth going to the effort of treating them separately.

Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime and let  $\mathcal{F}$  be a 3-dimensional Galilean subspacetime of  $\mathcal{E}$  modelled on a subspace  $U$  of  $V$ . Recall that  $U_\tau = U \cap \ker(\tau)$ . Let us denote  $\mathcal{H} = (\mathcal{F}, U, g|_{U_\tau}, \tau|_U)$ . A rigid motion  $\Sigma = (\Psi, \nu)$  is  $\mathcal{H}$ -compatible if  $T_\Sigma(s, x) \in \mathcal{F}$  for every  $(s, x) \in I_\mathcal{G} \times \mathcal{F}$ . Thus the motion of an  $\mathcal{H}$ -compatible rigid motion will leave  $\mathcal{F}$  invariant. The following result gives the description of an  $\mathcal{H}$ -compatible rigid motion relative to an  $\mathcal{H}$ -compatible observer.

**1.4.13 Proposition** *Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime with  $\mathcal{F}$  a 3-dimensional sub-spacetime modelled on a subspace  $U$  of  $V$ . Let  $\mathcal{O}$  be an  $\mathcal{H}$ -compatible observer. For an  $\mathcal{H}$ -compatible rigid motion  $\Sigma = (\Psi, \nu)$  define  $\Psi(s) = \Phi_\mathcal{O}(R_\Psi(s), r_{\Psi,\sigma}(s))$ . Then  $r_{\Psi,\sigma}(s) \in U_\tau$  and  $R_\Psi(s)$  leaves  $U_\tau$  invariant for each  $s \in I_\mathcal{G}$ .*

*Proof* Fix  $s \in I_\mathcal{G}$  and denote  $\mathcal{F}(s) = \mathcal{E}(s) \cap \mathcal{F}$ . By definition of  $\Phi_\mathcal{O}$ , we have

$$\Psi(s)(x) = (R_\Psi(s)(x - \mathcal{O}_s) + r_{\Psi,\sigma}(s)) + \mathcal{O}_s.$$

Since  $\Sigma$  and  $\mathcal{O}$  are  $\mathcal{H}$ -compatible, if  $x \in \mathcal{F}(s)$  then  $R_\Psi(s)(x - \mathcal{O}_s) + r_{\Psi,\sigma}(s) \in U_\tau$ . In particular, choosing  $x = \mathcal{O}_s$  shows that  $r_{\Psi,\sigma}(s) \in U_\tau$ . Therefore, for each  $x \in \mathcal{F}(s)$  we have

$$R_\Psi(s)(x - \mathcal{O}_s) + r_{\Psi,\sigma}(s) \in U_\tau \implies R_\Psi(s)(x - \mathcal{O}_s) \in U_\tau,$$

which implies that  $R_\Psi(s)$  leaves  $U_\tau$  invariant since every vector in  $U_\tau$  can be written as  $x - \mathcal{O}_s$  for some  $x \in \mathcal{F}(s)$ .  $\blacksquare$

Let us investigate the upshot of this for the spatial and body velocities. Recall (see Appendix A) that since  $(\ker(\tau), g)$  is an inner product space and  $U_\tau$  is a codimension 1 subspace of  $U$ , there exists a unique positively oriented unit vector which is  $g$ -orthogonal to  $U_\tau$ . Let us denote this unit vector by  $u_\tau$ .

**1.4.14 Proposition** *Let  $\mathcal{H} = (\mathcal{F}, U, g|_{U_\tau}, \tau|_U)$  be a 3-dimensional sub-spacetime of a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ . Let  $\Sigma = (\Psi, \nu)$  be an  $\mathcal{H}$ -compatible rigid motion*

and let  $\mathcal{O}$  be an  $\mathcal{H}$ -compatible observer. Let  $\omega_\Psi$  and  $v_{\Psi,\mathcal{O}}$  be the spatial angular and linear velocities, and let  $\Omega_\Psi$  and  $V_{\Psi,\mathcal{O}}$  be the body angular and linear velocities.

Then  $v_{\Psi,\mathcal{O}}(s), V_{\Psi,\mathcal{O}}(s) \in U_\tau$  for each  $s \in I_{\mathcal{G}}$ , and for each  $s \in I_{\mathcal{G}}$  there exists  $\bar{\omega}_\Psi(s), \bar{\Omega}_\Psi(s) \in \mathbb{R}$  so that

$$\omega_\Psi(s) = \bar{\omega}_\Psi(s)u_\tau, \quad \Omega_\Psi(s) = \bar{\Omega}_\Psi(s)u_\tau.$$

*Proof* Let us first deal with the angular velocities. Since  $R_\Psi(s)$  is orthogonal and leaves  $U_\tau$  invariant, then  $R_\Psi(s)(u_\tau) = u_\tau$ . Therefore  $R'_\Psi(s)(u_\tau) = 0$  or  $\hat{\omega}_\Psi(s) \circ R_\Psi^{-1}(s)(u_\tau) = 0$ . This in turn implies that  $\hat{\omega}_\Psi(s)(u_\tau) = 0$  which implies, by Exercise E1.11(c), that  $\omega_\Psi(s) = \bar{\omega}_\Psi(s)u_\tau$  for some  $\bar{\omega}_\Psi(s) \in \mathbb{R}$ , as claimed. A similar argument holds for the body angular velocity.

For the spatial linear velocity we have

$$v_{\Psi,\mathcal{O}}(s) = r'_{\Psi,\mathcal{O}}(s) + r_{\Psi,\mathcal{O}}(s) \times \omega_\Psi(s).$$

By Proposition 1.4.13,  $r'_{\Psi,\mathcal{O}}(s) \in U_\tau$ . Properties of the cross-product, and the above proved characterisation of  $\omega_\Psi$ , imply that  $r_{\Psi,\mathcal{O}}(s) \times \omega_\Psi(s)$  is orthogonal to  $u_\tau$ . But this means that  $r_{\Psi,\mathcal{O}}(s) \times \omega_\Psi(s) \in U_\tau$ . For the body linear velocity,  $V_\Psi(s) = R_\Psi^{-1}(s)(r'_{\Psi,\mathcal{O}}(s))$ , the lemma follows since  $R_\Psi^{-1}(s)(u_\tau) = u_\tau$ . ■

The above result merely formalises the usual notion that for a planar motion the angular velocity is represented by a real number rather than by a vector.

## 1.5 Rigid bodies

In the previous section we have on occasion been talking about rigid bodies as if we knew what they were, and without relation to an actual rigid body. Now let us actually say what we mean by a rigid body. We will spend some considerable effort obtaining the inertial characteristics of a rigid body as they are defined by its inertia tensor. Though these are well-known, they seem to be “lore,” and it is instructive to use our general setting to give proofs.

**1.5.1 Definitions** To discuss in a precise way the inertial properties of a rigid body one actually should use some very simple measure theory. We present the necessary background in Appendix D. However, if one wishes to exclude such degenerate rigid bodies as point masses and lines with no thickness, one may escape without the measure theory. For those who wish to avoid Appendix D, whenever a  $\mu$  is encountered in the sequel, you may replace it with  $\rho dV$  where  $\rho: \mathcal{B} \rightarrow \mathbb{R}$  is the “mass density” and where  $dV$  is the standard volume element. But know that in doing this, you are not allowed to think of a point mass as being a rigid body without invoking absurdities.

Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime. A **rigid body** is a pair  $(\mathcal{B}, \mu)$  where  $\mathcal{B} \subset \mathcal{E}(s_0)$  is a compact subset of simultaneous events, and  $\mu$  is a mass distribution (see Appendix D) on  $\mathcal{E}(s_0)$  with support equal to  $\mathcal{B}$ . Our definition thus allows such degenerate rigid bodies as point masses, and bodies whose mass distribution is contained in a line in  $\mathcal{E}(s_0)$ . We denote

$$\mu(\mathcal{B}) = \int d\mu$$

as the **mass** of the body. If  $\mathcal{H} = (\mathcal{F}, U, g|_{U_\tau}, \tau|_U)$  is a sub-spacetime of  $\mathcal{G}$ , a **planar rigid body** is  $\mathcal{H}$ -compatible if  $\mathcal{B} \subset \mathcal{F}(s_0) = \mathcal{E}(s_0) \cap \mathcal{F}$ . In actuality one can talk about planar

motions for rigid bodies not contained in  $\mathcal{F}$ . However, one loses no generality in making this restriction, and it is very often made as a matter of course when planar motions are discussed.

Here are some typical rigid bodies, forgoing for the moment any discussion of their inertial properties.

**1.5.1 Examples 1.** A particle of mass  $m$  is a special case of a rigid body. In this case we suppose the particle to be situated at the point  $x_0 \in \mathcal{E}(s_0)$ .

2. A slightly more interesting case consists of a rigid body that is contained in a line segment

$$\ell_{y_1, y_2} = \{ \xi(y_2 - y_1) / \|y_2 - y_1\| + y_1 \mid \xi \in [0, \|y_2 - y_1\|] \},$$

for some  $y_1, y_2 \in \mathcal{E}(s_0)$ . An example of such a situation would consist of two particles constrained so as to be equidistant at all times. Also, an infinitely thin rod will fall into this category.

3. The above two cases are degenerate examples of rigid bodies. The typical nondegenerate situation is when  $\mathcal{B}$  occupies a subset of  $\mathcal{E}(s_0)$  that cannot be contained in any line.  $\square$

The **centre of mass** of the body is the point

$$x_c = \frac{1}{\mu(\mathcal{B})} \left( \int (x - x_0) d\mu \right) + x_0,$$

for some  $x_0 \in \mathcal{E}(s_0)$ . Note that the integrand is in  $\ker(\tau)$  and so too will be the integral. The following lemma gives some properties of this definition.

**1.5.2 Lemma** *Let  $(\mathcal{B}, \mu)$  be a rigid body in a Galilean spacetime with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . The following statements hold:*

(i) *the expression*

$$x_c = \frac{1}{\mu(\mathcal{B})} \left( \int (x - x_0) d\mu \right) + x_0$$

*is independent of choice of  $x_0 \in \mathcal{E}(s_0)$ ;*

(ii)  *$x_c$  is the unique point in  $\mathcal{E}(s_0)$  with the property that*

$$\int (x - x_c) d\mu = 0;$$

(iii)  *$x_c$  is in the interior of the convex hull of  $\mathcal{B}$  (relative to the topology induced from  $\mathcal{E}(s_0)$ ).*

**Proof** (i) To check that the definition of  $x_c$  is independent of  $x_0 \in \mathcal{E}(s_0)$  we let  $\tilde{x}_0 \in \mathcal{E}(s_0)$  and compute

$$\begin{aligned} \frac{1}{\mu(\mathcal{B})} \left( \int (x - \tilde{x}_0) d\mu \right) + \tilde{x}_0 &= \frac{1}{\mu(\mathcal{B})} \left( \int (x - x_0) d\mu \right) + \\ &\quad \frac{1}{\mu(\mathcal{B})} \left( \int (x_0 - \tilde{x}_0) d\mu \right) + (\tilde{x}_0 - x_0) + x_0 \\ &= \frac{1}{\mu(\mathcal{B})} \left( \int (x - x_0) d\mu \right) + (x_0 - \tilde{x}_0) + (\tilde{x}_0 - x_0) + x_0 \\ &= \frac{1}{\mu(\mathcal{B})} \left( \int (x - x_0) d\mu \right) + x_0. \end{aligned}$$

(ii) By definition of  $x_c$  and by part (i) we have

$$x_c = \frac{1}{\mu(\mathcal{B})} \left( \int (x - x_c) \, d\mu \right) + x_c,$$

from which it follows that

$$\int (x - x_c) \, d\mu = \mu(\mathcal{B})(x_c - x_c) = 0.$$

Now suppose that  $\tilde{x}_c \in \mathcal{E}(s_0)$  is an arbitrary point with the property that

$$\int (x - \tilde{x}_c) \, d\mu = 0.$$

Then, by (i),

$$x_c = \frac{1}{\mu(\mathcal{B})} \left( \int (x - \tilde{x}_c) \, d\mu \right) + \tilde{x}_c,$$

from which we conclude that  $x_c = \tilde{x}_c$ .

(iii) If  $x_c$  is on the relative boundary of  $\text{co}(\mathcal{B})$  or not in  $\mathcal{B}$  at all, then there exists a hyperplane  $P$  in  $\mathcal{E}(s_0)$  passing through  $x_c$  so that there are points in  $\mathcal{B}$  which lie on one side of  $P$ , but there are no points in  $\mathcal{B}$  on the opposite side. In other words, as in Proposition 1.1.7, there exists  $\lambda \in \ker(\tau)^*$  so that the set

$$\{x \in \mathcal{B} \mid \lambda(x - x_c) > 0\}$$

is nonempty, but the set

$$\{x \in \mathcal{B} \mid \lambda(x - x_c) < 0\}$$

is empty. But this would imply that

$$\int \lambda(x - x_c) \, d\mu > 0,$$

contradicting (ii). ■

Let us look at our examples of rigid bodies to see how one might determine their centres of mass.

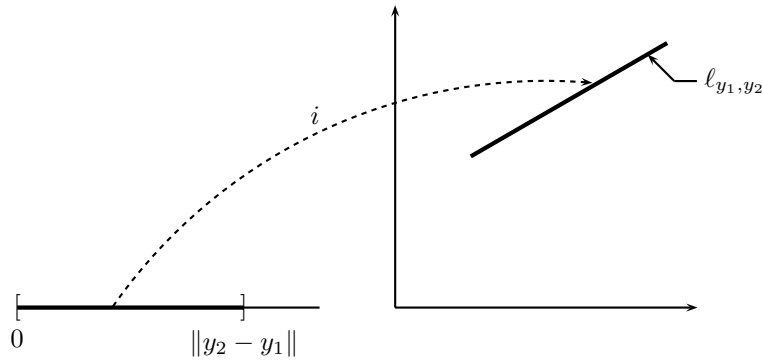
**1.5.3 Examples 1.** *Example 1.5.1-1 cont'd:* In this case we had a particle of mass  $m$  sitting at  $x_0 \in \mathcal{E}(s_0)$ . Since the support of  $\mu$  must be  $\{x_0\}$ , the centre of mass is

$$x_c = \mu(\mathcal{B})^{-1} \int (x - x_0) \, d\mu = m^{-1}m(x_0 - x_0) + x_0 = x_0.$$

Therefore, unsurprisingly the centre of mass for a point mass is that point at which the mass is located.

2. *Example 1.5.1-2 cont'd:* Here we have a rigid body  $(\mathcal{B}, \mu)$  so that  $\mathcal{B}$  is contained in a line segment in  $\mathcal{E}(s_0)$ . Let us better understand the mass distribution in this case. Define a map  $i: [0, \|y_2 - y_1\|] \rightarrow \mathcal{E}(s_0)$  by

$$i(\xi) = \frac{\xi}{\|y_2 - y_1\|} (y_2 - y_1) + y_1.$$

Figure 1.11 Coordinates for  $\ell_{y_1, y_2}$ 

Thus  $i$  takes a point on the interval in  $\mathbb{R}$  whose length is the same as  $\ell_{y_1, y_2}$ , and maps this point to the corresponding point on the rigid body (see Figure 1.11). With  $i$  so defined, we define a mass distribution  $d\mu_{\ell_{y_1, y_2}}$  on  $[0, \|y_2 - y_1\|]$  by

$$\int f \circ i \, d\mu_{\ell_{y_1, y_2}} = \int f \, d\mu.$$

With this notation, the centre of mass  $x_c$  can be written as

$$x_c = \mu(\mathcal{B})^{-1} \int \left( \frac{\xi}{\|y_2 - y_1\|} (y_2 - y_1) + (y_1 - x_0) \right) d\mu_{\ell_{y_1, y_2}} + x_0.$$

3. *Example 1.5.1–3 cont'd:* For the case where the rigid body does not have its mass distribution supported in a line, there is not much one can say about its centre of mass other than reiterate its definition.  $\square$

**1.5.2 The inertia tensor** The properties of a rigid body are characterised by three things: (1) its mass, (2) its centre of mass, and (3) its inertia tensor. It remains to define the latter. Let  $x_0 \in \mathcal{E}(s_0)$ . We define the ***inertia tensor about  $x_0$***  of a rigid body  $(\mathcal{B}, \mu)$  to be the linear map  $\mathbb{I}_{x_0}: \ker(\tau) \rightarrow \ker(\tau)$  defined by

$$\mathbb{I}_{x_0}(u) = \int (x - x_0) \times (u \times (x - x_0)) \, d\mu.$$

We denote the inertia tensor about the centre of mass of  $(\mathcal{B}, \mu)$  by  $\mathbb{I}_c$ .

Let us record some of the basic properties of the inertia tensor.

**1.5.4 Proposition** *The inertia tensor  $\mathbb{I}_{x_0}$  of a rigid body  $(\mathcal{B}, \mu)$  is symmetric with respect to the inner product  $g$ .*

*Proof* Using the vector identity

$$g(u, v \times w) = g(w, u \times v) \tag{1.19}$$



we compute

$$\begin{aligned}
 g(\mathbb{I}_{x_0}(u_1), u_2) &= \int g((x - x_0) \times (u_1 \times (x - x_0)), u_2) \, d\mu \\
 &= \int g(u_1 \times (x - x_0), u_2 \times (x - x_0)) \, d\mu \\
 &= \int g(u_1, (x - x_0) \times (u_2 \times (x - x_0))) \, d\mu \\
 &= g(u_1, \mathbb{I}_{x_0}(u_2)),
 \end{aligned}$$

which is what we wished to show. ■

It is often useful to be able to compute the inertia tensor about a general point by first computing it about its centre of mass. The following result records how this is done.

**1.5.5 Proposition**  $\mathbb{I}_{x_0}(u) = \mathbb{I}_c(u) + \mu(\mathcal{B})(x_c - x_0) \times (u \times (x_c - x_0))$ .

*Proof* We compute

$$\begin{aligned}
 \mathbb{I}_{x_0}(u) &= \int (x - x_0) \times (u \times (x - x_0)) \, d\mu \\
 &= \int ((x - x_c) + (x_c - x_0)) \times (u \times ((x - x_c) + (x_c - x_0))) \, d\mu \\
 &= \int (x - x_c) \times (u \times (x - x_c)) \, d\mu + \int (x_c - x_0) \times (u \times (x_c - x_0)) \, d\mu + \\
 &\quad \int (x - x_c) \times (u \times (x_c - x_0)) \, d\mu + \int (x_c - x_0) \times (u \times (x - x_c)) \, d\mu. \quad (1.20)
 \end{aligned}$$

It then follows from Lemma 1.5.2(ii) that the last two terms in (1.20) vanish, and from this the result follows. ■

**1.5.3 Eigenvalues of the inertia tensor** Since  $\mathbb{I}_{x_0}$  is symmetric, its eigenvalues are certainly real. Furthermore, they are nonnegative. The following result demonstrates this, as well as other eigenvalue related assertions.

**1.5.6 Proposition** *Let  $(\mathcal{B}, \mu)$  be a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$  and let  $x_0 \in \mathcal{E}(s_0)$ . Let  $\mathbb{I}_{x_0}$  denote the inertia tensor of  $(\mathcal{B}, \mu)$  about  $x_0$ . The following statements hold:*

- (i) *the eigenvalues of the inertia tensor  $\mathbb{I}_{x_0}$  of a rigid body  $(\mathcal{B}, \mu)$  are never negative;*
- (ii) *if  $\mathbb{I}_{x_0}$  has a zero eigenvalue, then the other two eigenvalues are equal;*
- (iii) *if  $\mathbb{I}_{x_0}$  has two zero eigenvalues, then  $\mathbb{I}_{x_0} = 0$ .*

*Proof* (i) Since  $\mathbb{I}_{x_0}$  is symmetric, its eigenvalues will be nonnegative if and only if the quadratic form  $u \mapsto g(\mathbb{I}_{x_0}(u), u)$  is positive semi-definite. For  $u \in \ker(\tau)$  we compute

$$\begin{aligned}
 \mathbb{I}_{x_0}(u, u) &= \int g(u, (x - x_0) \times (u \times (x - x_0))) \, d\mu \\
 &= \int g(u \times (x - x_0), u \times (x - x_0)) \, d\mu.
 \end{aligned}$$

Since the integrand is nonnegative, so too will be the integral.

(ii) Let  $I_1$  be the zero eigenvalue with  $v_1$  a unit eigenvector. We claim that the support of the mass distribution  $\mu$  must be contained in the line

$$\ell_{v_1} = \{sv_1 + x_0 \mid s \in \mathbb{R}\}.$$

To see that this must be so, suppose that the support of  $\mu$  is not contained in  $\ell_{v_1}$ . Then there exists a Borel set  $S \subset \mathcal{E}(s_0) \setminus \ell_{v_1}$  so that  $\mu(S) > 0$ . This would imply that

$$\begin{aligned} \mathbb{I}_{x_0}(v_1, v_1) &= \int g(v_1 \times (x - x_0), v_1 \times (x - x_0)) \, d\mu \\ &\geq \int_S g(v_1 \times (x - x_0), v_1 \times (x - x_0)) \, d\mu. \end{aligned}$$

Since  $S \cap \ell_{v_1} = \emptyset$  it follows that for all points  $x \in S$ , the vector  $x - x_0$  is not collinear with  $v_1$ . Therefore

$$g(v_1 \times (x - x_0), v_1 \times (x - x_0)) > 0$$

for all  $x \in S$ , and this would imply that  $\mathbb{I}_{x_0}(v_1, v_1) > 0$ . But this contradicts  $v_1$  being an eigenvector with zero eigenvalue, and so the support of  $\mathcal{B}$  must be contained in the line  $\ell_{v_1}$ .

To see that this implies that the eigenvalues  $I_2$  and  $I_3$  are equal, we shall show that any vector that is  $g$ -orthogonal to  $v_1$  is an eigenvector for  $\mathbb{I}_{x_0}$ . First write

$$x - x_0 = f^1(x)v_1 + f^2(x)v_2 + f^3(x)v_3$$

for functions  $f^i: \mathcal{E}(s_0) \rightarrow \mathbb{R}$ ,  $i = 1, 2, 3$ . Since the support of  $\mu$  is contained in the line  $\ell_{v_1}$  we have

$$\int (x - x_0) \times (u \times (x - x_0)) \, d\mu = v_1 \times (u \times v_1) \int (f^1(x))^2 \, d\mu$$

for all  $u \in \ker(\tau)$ . Now recall the property of the cross-product that  $v_1 \times (u \times v_1) = u$  provided  $u$  is orthogonal to  $v_1$  and that  $v_1$  has unit length. Therefore we see that for any  $u$  that is orthogonal to  $v_1$  we have

$$\mathbb{I}_{x_0}(u) = \left( \int (f^1(x))^2 \, d\mu \right) u,$$

meaning that all such vectors  $u$  are eigenvectors with the same eigenvalue, which is what we wished to show.

(iii) It follows from our above arguments that if the eigenvalues  $I_1$  and  $I_2$  are zero, then the support of  $\mu$  must lie in the intersection of the lines  $\ell_{v_1}$  and  $\ell_{v_2}$ , and this intersection is a single point, that must therefore be  $x_0$ . From this and the definition of  $\mathbb{I}_{x_0}$  it follows that  $\mathbb{I}_{x_0} = 0$ . ■

Note that in proving the result we have proven the following corollary.

**1.5.7 Corollary** *Let  $(\mathcal{B}, \mu)$  be a rigid body with inertia tensor  $\mathbb{I}_{x_0}$ . The following statements hold:*

- (i)  $\mathbb{I}_{x_0}$  has a zero eigenvalue if and only if  $\mathcal{B}$  is contained in a line through  $x_0$ ;
- (ii) if  $\mathbb{I}_{x_0}$  has two zero eigenvalues then  $\mathcal{B} = \{x_0\}$ , i.e.,  $\mathcal{B}$  is a particle located at  $x_0$ ;
- (iii) if there is no line through  $x_0$  that contains the support of  $\mu$ , then the inertia tensor is an isomorphism.

In coming to an understanding of the “appearance” of a rigid body, it is most convenient to refer to its inertia tensor  $\mathbb{I}_c$  about its centre of mass. Let  $\{I_1, I_2, I_3\}$  be the eigenvalues of  $\mathbb{I}_c$  that we call the **principal inertias** of  $(\mathcal{B}, \mu)$ . If  $\{v_1, v_2, v_3\}$  are orthonormal eigenvectors associated with these eigenvalues, we call these the **principal axes** of  $(\mathcal{B}, \mu)$ . Related to these is the **inertial ellipsoid** which is the ellipsoid in  $\ker(\tau)$  given by

$$E(\mathcal{B}) = \{x^1 v_1 + x^2 v_2 + x^3 v_3 \in \ker(\tau) \mid I_1(x^1)^2 + I_2(x^2)^2 + I_3(x^3)^2 = 1\},$$

provided that none of the eigenvalues of  $\mathbb{I}_{x_0}$  vanish. If one of the eigenvalues *does* vanish, then by Proposition 1.5.6(ii), the other two eigenvalues are equal. If we suppose that  $I_1 = 0$  and  $I_2 = I_3 = I$  then in the case of a single zero eigenvalue, the inertial ellipsoid is

$$E(\mathcal{B}) = \{x^1 v_1 + x^2 v_2 + x^3 v_3 \in \ker(\tau) \mid x^2 = x^3 = 0, x^1 \in \{-\frac{1}{\sqrt{I}}, \frac{1}{\sqrt{I}}\}\}.$$

In the most degenerate case, when all eigenvalues are zero, we define  $E(\mathcal{B}) = \{0\}$ . These latter two inertia ellipsoids, note, correspond to the cases (i) and (ii) in Corollary 1.5.7.

To relate properties of the eigenvalues of  $\mathbb{I}_c$  with the inertial ellipsoid  $E(\mathcal{B})$ , it is helpful to introduce the notion of an axis of symmetry for a rigid body. We let  $\mathbb{I}_c$  be the inertia tensor about the centre of mass, and denote by  $\{I_1, I_2, I_3\}$  its eigenvalues and  $\{v_1, v_2, v_3\}$  its orthonormal eigenvectors. A vector  $v \in \ker(\tau) \setminus \{0\}$  is an **axis of symmetry** for  $(\mathcal{B}, \mu)$  if for every  $R \in O(\ker(\tau))$  which fixes  $v$  we have  $R(E(\mathcal{B})) = E(\mathcal{B})$ . The following result gives the relationship between axes of symmetry and the eigenvalues of  $\mathbb{I}_c$ .

**1.5.8 Proposition** *Let  $(\mathcal{B}, \mu)$  be a rigid body with inertia tensor  $\mathbb{I}_c$  about its centre of mass. Let  $\{I_1, I_2, I_3\}$  be the eigenvalues of  $\mathbb{I}_c$  with orthonormal eigenvectors  $\{v_1, v_2, v_3\}$ . If  $I_1 = I_2$  then  $v_3$  is an axis of symmetry of  $\mathcal{B}$ .*

*Conversely, if  $v \in \ker(\tau)$  is an axis of symmetry, then  $v$  is an eigenvector for  $\mathbb{I}_c(\mathcal{B})$ . If  $I$  is the eigenvalue for which  $v$  is an eigenvector, then the other two eigenvalues of  $\mathbb{I}_c(\mathcal{B})$  are equal.*

*Proof* Write  $I_1 = I_2 = I$ . We then see that any vector  $v \in \text{span}_{\mathbb{R}}\{v_1, v_2\}$  will have the property that

$$\mathbb{I}_c(v) = Iv.$$

Now let  $R \in O(\ker(\tau))$  fix the vector  $v_3$ . Because  $R$  is orthogonal, if  $v \in \text{span}_{\mathbb{R}}\{v_1, v_2\}$  then  $R(v) \in \text{span}_{\mathbb{R}}\{v_1, v_2\}$ . What's more if

$$v = a^1 v_1 + a^2 v_2,$$

then

$$R(v) = (\cos \theta a^1 + \sin \theta a^2)v_1 + (-\sin \theta a^1 + \cos \theta a^2)v_2 \quad (1.21)$$

for some  $\theta \in \mathbb{R}$  since  $R$  is simply a rotation in the plane spanned by  $v_1, v_2$ . Now let  $u \in E(\mathcal{B})$ . We then write  $u = x^1 v_1 + x^2 v_2 + x^3 v_3$ , and note that

$$I(x^1)^2 + I(x^2)^2 + I_3(x^3)^2 = 1.$$

It is now a straightforward but tedious calculation to verify that  $R(v) \in E(\mathcal{B})$  using (1.21) and the fact that  $R$  fixes  $v_3$ . This shows that  $R(E(\mathcal{B})) = E(\mathcal{B})$ , and so  $v_3$  is an axis of symmetry for  $\mathcal{B}$ .

For the second part of the proposition, note that  $R \in O(\ker(\tau))$  has the property that  $R(E(\mathcal{B})) = E(\mathcal{B})$  if and only if  $R$  maps the principal axes of the ellipsoid  $E(\mathcal{B})$  to principal

axes. Since  $R$  is a rotation about *some* axis, this means that  $R$  fixes a principal axis of  $E(\mathcal{B})$ . Thus if  $v \in \ker(\tau)$  is an axis of symmetry, then  $v$  must lie along a principal axis of the ellipsoid  $E(\mathcal{B})$ . By our definition of  $E(\mathcal{B})$ , this means that  $v$  is an eigenvector of  $\mathbb{I}_c$ . Let  $I$  be the associated eigenvalue, and let  $\{I_1, I_2, I_3 = I\}$  be the collection of all eigenvalues of  $\mathbb{I}_c$  with eigenvectors  $\{v_1, v_2, v_3 = v\}$ . Since  $v$  is an axis of symmetry, any rotation about  $v$  must map principal axes of  $E(\mathcal{B})$  to principal axes. This means that for every  $\theta \in \mathbb{R}$  the vectors

$$v'_1 = \cos \theta v_1 - \sin \theta v_2, \quad v'_2 = \sin \theta v_1 + \cos \theta v_2 \quad (1.22)$$

are eigenvectors for  $\mathbb{I}_c$ . This means that all nonzero vectors in  $\text{span}_{\mathbb{R}}\{v_1, v_2\}$  are eigenvectors for  $\mathbb{I}_c$ . This means that the restriction of  $\mathbb{I}_c$  to  $\text{span}_{\mathbb{R}}\{v_1, v_2\}$  is diagonal in every orthonormal basis for  $\text{span}_{\mathbb{R}}\{v_1, v_2\}$ . Therefore, if  $\{v_1, v_2\}$  are chosen to be orthonormal then  $\{v'_1, v'_2\}$  as defined in (1.22) are also orthonormal. Our conclusions assert the existence of  $I'_1, I'_2 \in \mathbb{R}$  so that

$$\mathbb{I}_c(v'_i) = I_i v'_i, \quad i = 1, 2.$$

But by the definition of  $v'_1$  and  $v'_2$  we also have

$$\begin{aligned} \mathbb{I}_c(v'_1) &= \cos \theta \mathbb{I}_c(v_1) - \sin \theta \mathbb{I}_c(v_2) \\ &= \cos \theta I_1 v_1 - \sin \theta I_2 v_2 \\ &= \cos \theta I_1 (\cos \theta v'_1 + \sin \theta v'_2) - \sin \theta I_2 (-\sin \theta v'_1 + \cos \theta v'_2). \end{aligned}$$

Therefore,

$$I'_1 v'_1 = (\cos^2 \theta I_1 + \sin^2 \theta I_2) v'_1 + \sin \theta \cos \theta (I_1 - I_2) v'_2$$

for every  $\theta \in \mathbb{R}$ . Since  $v'_1$  and  $v'_2$  are orthogonal, this means choosing  $\theta$  so that  $\sin \theta \cos \theta \neq 0$  implies that  $I_1 - I_2 = 0$ . This is what we wished to show. ■

Finally, let us say what happens to the inertia tensor for a planar rigid body. Recall that  $u_\tau$  is denotes the positively-oriented unit vector orthogonal to  $U_\tau$ .

**1.5.9 Proposition** *If  $\mathcal{H} = (\mathcal{F}, \mathbb{U}, \mathbb{g}|_{U_\tau}, \tau|_{\mathbb{U}})$  is a 3-dimensional sub-spacetime of a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, \mathbb{V}, \mathbb{g}, \tau)$ , and if  $(\mathcal{B}, \mu)$  is an  $\mathcal{H}$ -compatible rigid body with  $\mathcal{B} \subset \mathcal{F}(s_0)$ , then  $u_\tau$  is a principal axis for  $(\mathcal{B}, \tau)$ .*

*Proof* Let  $\{v_1, v_2, v_3 = u_\tau\}$  be a positively-oriented basis for  $\ker(\tau)$ . For  $x \in \mathcal{E}(s_0)$  let us write

$$x - x_c = f^1(x)v_1 + f^2(x)v_2 + f^3(x)v_3,$$

thus defining  $f^i(x) \in \mathbb{R}$ ,  $i = 1, 2, 3$ . We therefore compute

$$\begin{aligned} \mathbb{I}_c(u_\tau) &= \int (x - x_c) \times (u_\tau \times (x - x_c)) \, d\mu \\ &= \int (f^1(x)v_1 + f^2(x)v_2 + f^3(x)v_3) \times (u_\tau \times (f^1(x)v_1 + f^2(x)v_2 + f^3(x)v_3)) \, d\mu. \end{aligned}$$

Since  $\mathcal{B} \subset \mathcal{F}(s_0)$ ,  $x_c \in \mathcal{F}(s_0)$  so that we get

$$\mathbb{I}_c(u_\tau) = \int (f^1(x)v_1 + f^2(x)v_2) \times (u_\tau \times (f^1(x)v_1 + f^2(x)v_2)) \, d\mu.$$

Now one recalls that if  $u$  is a unit vector and if  $v$  is a unit vector orthogonal to  $u$  we have  $v \times (u \times v) = u$ . Therefore

$$\mathbb{I}_c(u_\tau) = \left( \int \|f^1(x)v_1 + f^2(x)v_2\|^2 \, d\mu \right) u_\tau.$$

Thus  $u_\tau$  is an eigenvector of  $\mathbb{I}_c$  as asserted. ■

As we shall see, this property of a planar rigid body is essential if the natural motion of a rigid body is indeed to remain in the sub-spacetime  $\mathcal{H}$ . With this in mind, one can more generally consider motions of a general rigid body in a planar sub-spacetime  $\mathcal{H}$  provided that  $u_\tau$  is a principal axis of  $(\mathcal{B}, \mu)$ . But, again, this extra little piece of generality does not add any substance.

**1.5.4 Examples of inertia tensors** In this section we give the three “typical” types of inertia tensors one can encounter. As we shall see, two of the three are somewhat degenerate.

**1.5.10 Examples 1.** *Example 1.5.1–1 cont’d:* Let us compute the inertia tensor of a particle of mass  $m$ , thought of as a rigid body. Thus  $\mathcal{B} = \{x_c\}$  and since the particle is located at its centre of mass by Example 1.5.3–1. Note that the inertia tensor of the particle about its centre of mass is obviously zero. Therefore, using Proposition 1.5.5 we readily compute

$$\mathbb{I}_{x_0}(u) = m(x_c - x_0) \times (u \times (x_c - x_0)).$$

The matrix representation of  $\mathbb{I}_{x_0}$  with respect to an orthonormal basis  $\{v_1, v_2, v_3\}$  for  $\ker(\tau)$  is

$$m \begin{bmatrix} (a^2)^2 + (a^3)^2 & -a^1 a^2 & -a^1 a^3 \\ -a^1 a^2 & (a^1)^2 + (a^3)^2 & -a^2 a^3 \\ -a^1 a^3 & -a^2 a^3 & (a^1)^2 + (a^2)^2 \end{bmatrix} \quad (1.23)$$

where  $(a^1, a^2, a^3)$  are the components of  $(x_c - x_0)$  in the basis  $\{v_1, v_2, v_3\}$ . One may verify that the eigenvalues of this matrix are  $\{\|\mathbf{a}\|^2, \|\mathbf{a}\|^2, 0\}$  with the corresponding orthonormal eigenvectors  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{a}/\|\mathbf{a}\|\}$  where  $\{\mathbf{u}_1, \mathbf{u}_2\}$  are any orthonormal basis for  $\text{span}_{\mathbb{R}}\{\mathbf{a}\}^\perp$ , the orthogonal complement to  $\text{span}_{\mathbb{R}}\{\mathbf{a}\}$ . Thus, for a particle at  $x_c$ , there is no resistance to angular velocities collinear with  $x_c - x_0$ . This is reflected by the zero eigenvalue for  $\mathbb{I}_{x_0}$  with eigenvector  $(x_c - x_0)$ . What’s more, if we choose  $x_0 = x_c$  then  $\mathbb{I}_{x_0} = 0$ , again reflecting the fact that the inertia tensor about the centre of mass is zero. Note that this all agrees with Propositions 1.5.6 and 1.5.8. Also note that  $x_c - x_0$  is an axis of symmetry if  $x_0 \neq x_c$ .

2. *Example 1.5.1–2 cont’d:* Let us consider the slightly less degenerate case where  $\mu$  is a mass distribution whose support, is contained in the line  $\ell_{y_1, y_2}$ . We shall determine the inertia tensor of  $\mathcal{B}$  about its centre of mass. By Example 1.5.3–2, the centre of mass, that we denote by  $x_c$ , can be written as

$$x_c = \mu(\mathcal{B})^{-1} \int \left( \frac{\xi}{\|y_2 - y_1\|} (y_2 - y_1) + (y_1 - x_0) \right) d\mu_{\ell_{y_1, y_2}} + x_0.$$

The reader will recall the definition of the coordinate  $\xi$  from Example 1.5.3–2. Using this relation, a straightforward computation yields

$$\mathbb{I}_c(u) = \frac{(y_2 - y_1) \times (u \times (y_2 - y_1))}{\|y_2 - y_1\|^2} \left[ \int \xi^2 d\mu_{\ell_{y_1, y_2}} - \frac{1}{m} \left( \int \xi d\mu_{\ell_{y_1, y_2}} \right)^2 \right] \quad (1.24)$$

Similar with the single particle, the inertia tensor vanishes when applied to angular velocities collinear with  $y_2 - y_1$ . However, unlike what is seen with the single particle,

the inertia tensor in this example *never* vanishes, regardless of the choice of the point about which the moment is measured.

A special case when  $\mu$  has support in  $\ell_{y_1, y_2}$  occurs when there are particles  $m_1$  and  $m_2$  located at  $y_1$  and  $y_2$ . In this case one may compute the centre of mass to be

$$x_c = \mu(\mathcal{B})^{-1}(m_1(y_1 - x_0) + m_2(y_2 - x_0)) + x_0$$

where  $m = m_1 + m_2$ . It is also an easy computation to determine that in this case

$$\mathbb{I}_c(u) = \frac{m_1 m_2}{m_1 + m_2} (y_2 - y_1) \times (u \times (y_2 - y_1)). \quad (1.25)$$

One may verify that if  $m_1$  and  $m_2$  are given by

$$m_1 = \frac{1}{2}(m + \Delta), \quad m_2 = \frac{1}{2}(m - \Delta), \quad (1.26)$$

where

$$\Delta = \left[ m^2 + \frac{4}{\|y_2 - y_1\|^2} \left( \int \xi \, d\mu_{\ell_{y_1, y_2}} \right) - \frac{4m}{\|y_2 - y_1\|^2} \int \xi^2 \, d\mu_{\ell_{y_1, y_2}} \right]^{1/2},$$

then (1.24) and (1.25) agree. One may show that  $m_1$  and  $m_2$  make sense in (1.26) (see Exercise E1.21). Thus we have the following assertion:

*A rigid body whose mass distribution is supported on a line segment is equivalent to two appropriately chosen particles at the endpoints of the line segment.*

3. *Example 1.5.1–3 cont'd:* The remaining case to consider is when the mass distribution  $\mu$  corresponding to  $\mathcal{B} \subset \mathcal{E}(s_0)$  has support that cannot be contained in a line. Let us first note that  $\mathbb{I}_c$  is an isomorphism in this case by Corollary 1.5.7(iii). Now let  $m$  be the mass of  $\mathcal{B}$  and let  $\{I_1, I_2, I_3\}$  be the three principal moments of inertia with  $\{v_1, v_2, v_3\}$  the corresponding principal axes, and suppose that these vectors are orthonormal. We now construct a “simple” rigid body that is equivalent to a nondegenerate rigid body. Define  $r_i = \sqrt{6I_i/m}$ ,  $i = 1, 2, 3$ . We now define a new rigid body  $\mathcal{B}'$  consisting of six points:

$$\mathcal{B}' = \{r_1 v_1\} \cup \{-r_1 v_1\} \cup \{r_2 v_2\} \cup \{-r_2 v_2\} \cup \{r_3 v_3\} \cup \{-r_3 v_3\}.$$

Now place a mass distribution on  $\mathcal{B}'$  by asking that the mass of each of the points in  $\mathcal{B}'$  be  $\frac{1}{6}m$ . One then readily verifies that  $\mathcal{B}'$  has the same inertial ellipsoid as  $\mathcal{B}$ , and thus the rigid bodies are equivalent.

*A rigid body whose mass distribution cannot be supported on a line segment is equivalent to six appropriately chosen particles, two placed at points along each of the principal axes of the rigid body.*  $\square$

## 1.6 Dynamics of rigid bodies

In this section we get to the heart of the matter: the “Euler equations” for a rigid body. To derive the Euler equations we first define spatial and body momenta for a rigid body. It is comparatively simple within our general formalism to demonstrate that the laws of conservation of momentum are independent of observer. We then invoke these conservation laws to derive the equations that govern the behaviour of a rigid body. Note that this is not a typical situation. That is, one cannot typically just derive the equations of motion for a system knowing that it conserves linear and angular momentum. The reason it will work for us in this section is part of the magic of the dynamics of a rigid body, the principles of which are generalised in Section 4.4 of [Abraham and Marsden 1978].

**1.6.1 Spatial momenta** We let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime with  $\Sigma = (\Psi, \nu)$  a rigid motion,  $\mathcal{O}$  an observer, and  $(\mathcal{B}, \mu)$  a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . We wish to see how to define angular momentum with respect to the point  $x_0$  and so shall suppose that the observer  $\mathcal{O}$  contains the point  $x_0$ . As usual, we denote by  $\mathcal{O}_s$  the unique point in the intersection  $\mathcal{O} \cap \mathcal{E}(s)$  for  $s \in I_{\mathcal{G}}$ .

Let us try to motivate our definition of spatial angular momentum by recalling how it might be defined for a particle of mass  $m$ . If the particle is moving in  $\mathbb{R}^3$  following the curve  $t \mapsto \mathbf{x}(t)$ , then we would define the spatial angular momentum at time  $t$  to be  $m \mathbf{x}(t) \times \dot{\mathbf{x}}(t)$ . Motivated by this, if the particle were following a world line  $s \mapsto x(s)$ , then we might define its spatial angular momentum with respect to the observer  $\mathcal{O}$  to be

$$m(x(s) - \mathcal{O}_s) \times P_{\mathcal{O}}(x'(s)).$$

Note that the angular momentum at the instant  $s$  is measured about the observers origin  $\mathcal{O}_s$ . Now let us see the implications of this for a rigid body. We suppose that we have a rigid motion  $\Sigma = (\Psi, \nu)$ . We let  $s_0 = \pi_{\mathcal{G}}(x_0)$  so that the rigid body is located at the instant  $s_0$ . At the instant  $s$  a point  $x \in \mathcal{E}(s_0)$  will be located at

$$T_{\Sigma}(s, x) = (R_{\Psi}(s)(x - ((s_0 - s)(\nu - v_{\mathcal{O}}) + \mathcal{O}_{s_0})) + r_{\Psi, \mathcal{O}}(s)) + \mathcal{O}_s \in \mathcal{E}(s).$$

By Proposition 1.4.5 its velocity will be given by

$$P_{\mathcal{O}}(T'_{\Sigma}(s, x)) = R_{\Psi}(s)(\nu - v_{\mathcal{O}}) + \omega_{\Psi}(s) \times (T_{\Sigma}(s, x) - \mathcal{O}_s) + v_{\Psi, \mathcal{O}}(s).$$

Note that since we are supposing that  $x_0 \in \mathcal{O}$  it must be the case that  $\mathcal{O}_{s_0} = x_0$ . Motivated by the situation for the point mass, we use as our definition of **spatial angular momentum**,

$$\ell_{\Sigma, \mathcal{O}}(s) = \int (T_{\Sigma}(s, x) - \mathcal{O}_s) \times P_{\mathcal{O}}(T'_{\Sigma}(s, x)) d\mu.$$

Generally, a substitution of the expressions  $T_{\Sigma}(s, x)$  and  $P_{\mathcal{O}}(T'_{\Sigma}(s, x))$  into the formula for spatial angular momentum will produce a somewhat lengthy expression. We do not include it here, although it is in principle straightforward. In any case, one of the advantages to our Galilean presentation is that it *is* straightforward to write down the formula for spatial angular momentum with respect to a general observer.

Now let us talk about the spatial linear momentum. Again, we start by analogy with a particle of mass  $m$  moving along a curve  $t \mapsto \mathbf{x}(t)$  in  $\mathbb{R}^3$ . Its linear momentum is taken to be  $m \dot{\mathbf{x}}(t)$ . For a particle following a world line  $s \mapsto x(s)$ , an observer  $\mathcal{O}$  would determine the linear momentum of the particle to be  $m P_{\mathcal{O}}(x'(s))$ . The key point here is that one measures the linear momentum of the centre of mass. Therefore, for a rigid body  $(\mathcal{B}, \mu)$  undergoing a rigid motion  $\Sigma = (\Psi, \nu)$ , a reasonable definition for the **spatial linear momentum** would be

$$m_{\Sigma, \mathcal{O}}(s) = \mu(\mathcal{B}) P_{\mathcal{O}}(T'_{\Sigma}(s, x_c)).$$

Using Proposition 1.4.5 one can readily write this expression in terms of spatial angular and linear velocities.

When the observer has a special form, then the expressions for spatial momenta may be simplified to the point where they may conveniently be expressed in terms of spatial linear velocity. The following result records the result of the corresponding easily verified computations.



**1.6.1 Proposition** *Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime with  $(\mathcal{B}, \mu)$  a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . Also let  $\Sigma = (\Psi, \nu)$  be a rigid motion with  $\mathcal{O} \in \text{Ob}(\Sigma)$  an observer having the property that  $\mathbf{x}_c \in \mathcal{O}$ . Then we have*

$$\begin{aligned}\ell_{\Sigma, \mathcal{O}}(s) &= \mathbb{I}_c(R_{\Psi}^{-1}(s)\omega_{\Psi}(s)) + \mu(\mathcal{B})\left(r_{\Psi, \mathcal{O}}(s) \times ((\omega_{\Psi}(s) \times r_{\Psi, \mathcal{O}}(s)) + v_{\Psi, \mathcal{O}}(s))\right) \\ m_{\Sigma, \mathcal{O}}(s) &= \mu(\mathcal{B})(\omega_{\Psi}(s) \times r_{\Psi, \mathcal{O}}(s) + v_{\Psi, \mathcal{O}}(s)).\end{aligned}$$

Prove

*Proof* ■

Note that by using Proposition 1.4.5 it is readily verified that

$$\omega_{\Psi}(s) \times r_{\Psi, \mathcal{O}}(s) + v_{\Psi, \mathcal{O}}(s) = r'_{\Psi, \mathcal{O}}(s),$$

so that the expression for the spatial momenta in Proposition 1.6.1 are simply

$$\begin{aligned}\ell_{\Sigma, \mathcal{O}}(s) &= \mathbb{I}_c(R_{\Psi}^{-1}(s)\omega_{\Psi}(s)) + \mu(\mathcal{B})(r_{\Psi, \mathcal{O}}(s) \times r'_{\Psi, \mathcal{O}}(s)) \\ m_{\Sigma, \mathcal{O}}(s) &= \mu(\mathcal{B})r'_{\Psi, \mathcal{O}}(s),\end{aligned}$$

which are the familiar expressions.

Let us look at the spatial angular momentum for our rotational rigid motion.

**1.6.2 Example** (Example 1.4.4 cont'd) For this example, recall that we had a rotation about the origin in the canonical Galilean spacetime which fixed the vector  $\omega_0 \in \mathbb{R}^3$ . As we have already computed, the spatial angular velocity of this rigid motion is  $\omega_0$ . To talk about the angular momentum, one must have a rigid body of some sort. Let us suppose that we have a rigid body  $(\mathcal{B}, \mu)$  and that  $\mathcal{B} \subset \mathcal{E}(0)$ . Thus our rigid body occupies the set of points  $\mathcal{B}$  at time 0. As our base point for measuring the inertia tensor we choose  $\mathbf{x}_0 = \mathbf{0}$ , and we shall also suppose this to be the centre of mass for  $(\mathcal{B}, \mu)$ . We shall also suppose that  $\omega_0$  is in fact an eigenvalue of the inertia tensor  $\mathbb{I}_0$  with eigenvalue  $I$ . We also keep with our previous convention with this example, and use the canonical observer  $\mathcal{O}_{\text{can}}$ .

Now, using Proposition 1.6.1 we compute the spatial angular momentum to be

$$\ell(t) = \mathbb{I}_0(\omega(t)) = \mathbb{I}_0(\omega_0) = I\omega_0.$$

Thus for this simple example the spatial angular momentum is simply a multiple of the angular velocity. This is reminiscent of the situation where linear momentum is a multiple (the mass) of the linear velocity of a particle. Of course, in this example since the spatial linear velocity is zero, so too is the spatial linear momentum zero.  $\square$

To finish off our discussion of spatial momentum, let us look at the spatial momentum in the planar case. To state the result, it is helpful to introduce an extra bit of notation. For a 3-dimensional sub-spacetime  $\mathcal{H} = (\mathcal{F}, U, g|_{U_\tau}, \tau|_U)$  of  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ ,  $U_\tau$  is as usual a 2-dimensional subspace of  $\ker(\tau)$ . Therefore, given  $u_1, u_2 \in U_\tau$  the cross-product  $u_1 \times u_2$  is orthogonal to  $U_\tau$ . Also recall that  $u_\tau$  is the positively-oriented unit vector orthogonal to  $U_\tau$ . We denote by  $f_\tau(u_1, u_2) \in \mathbb{R}$  the number satisfying

$$u_1 \times u_2 = f_\tau(u_1, u_2)u_\tau.$$

With this notation we have the following.



**1.6.3 Proposition** Let  $\mathcal{H} = (\mathcal{F}, U, g|_{U_\tau}, \tau|_U)$  be a 3-dimensional Galilean sub-spacetime of  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ , let  $\Sigma = (\Psi, \mathcal{O})$  be an  $\mathcal{H}$ -compatible rigid motion, let  $\mathcal{O}$  be an  $\mathcal{H}$ -compatible observer, and let  $(\mathcal{B}, \mu)$  be an  $\mathcal{H}$ -compatible rigid body. Then  $m_{\Sigma, \mathcal{O}}(s) \in U_\tau$  for each  $s \in I_{\mathcal{G}}$ , and for each  $s \in I_{\mathcal{G}}$  there exists  $\bar{\ell}_{\Sigma, \mathcal{O}}(s) \in \mathbb{R}$  so that

$$\ell_{\Sigma, \mathcal{O}}(s) = \bar{\ell}_{\Sigma, \mathcal{O}}(s)u_\tau.$$

Further suppose that we have  $\mathcal{O} \in \text{Ob}(\Sigma)$  and  $x_c \in \mathcal{O}$ , and then define  $\bar{\omega}_\Psi(s) \in \mathbb{R}$  by  $\omega_\Psi(s) = \bar{\omega}_\Psi(s)u_\tau$  as in Proposition 1.4.14. Also let  $I$  be the eigenvalue of  $\mathbb{I}_c$  for which  $u_\tau$  is an eigenvector. Then the expression for  $\bar{\ell}_{\Sigma, \mathcal{O}}$  simplifies to

$$\bar{\ell}_{\Sigma, \mathcal{O}}(s) = I\bar{\omega}_\Psi(s) + \mu(\mathcal{B})f_\tau(r_{\Psi, \mathcal{O}}(s), r'_{\Psi, \mathcal{O}}(s)).$$

*Proof* First note the following facts:

1.  $T_\Sigma(s, x) - \mathcal{O}_s \in U_\tau$  for each  $s \in I_{\mathcal{G}}$  and  $x \in \mathcal{B}$ ;
2.  $P_\mathcal{O}(T'_\Sigma(s, x)) \in U_\tau$ .

The first fact holds since  $\mathcal{O}$ ,  $(\mathcal{B}, \mu)$ , and  $\Sigma$  are  $\mathcal{H}$ -compatible. The second holds since  $T'_\Sigma(s, x) \in U$  for each  $s \in I_{\mathcal{G}}$  and for each  $x \in \mathcal{B}$ . It now holds that

$$(T_\Sigma(s, x) - \mathcal{O}_s) \times P_\mathcal{O}(T'_\Sigma(s, x))$$

is orthogonal to  $U_\tau$  for each  $s \in I_{\mathcal{G}}$  and each  $x \in \mathcal{B}$ . Therefore, by definition,  $\ell_{\Sigma, \mathcal{O}}(s)$  is orthogonal to  $U_\tau$  for each  $s \in I_{\mathcal{G}}$  which gives that part of the result referring to  $\ell_{\Sigma, \mathcal{O}}$  with respect to a general  $\mathcal{H}$ -compatible observer. The similar statement for the spatial linear momentum follows directly from Proposition 1.6.1 and the fact 2 above.

The second part of the proposition follows directly from the definition of  $f_\tau$ . ■

**1.6.2 Body momenta** Now let us turn to defining the body momentum. Here we take our motivation from Proposition 1.4.11 and define the body angular momentum  $L_{\Sigma, \mathcal{O}}$  and the body linear momentum  $M_{\Sigma, \mathcal{O}}$  by asking that

$$\begin{aligned} \begin{bmatrix} \ell_{\Sigma, \mathcal{O}}(s) \\ m_{\Sigma, \mathcal{O}}(s) \end{bmatrix} &= \begin{bmatrix} R_\Psi(s) & 0 \\ \hat{r}_{\Psi, \mathcal{O}}(s) \circ R_\Psi(s) & R_\Psi(s) \end{bmatrix}^{-t} \begin{bmatrix} L_{\Sigma, \mathcal{O}}(s) \\ M_{\Sigma, \mathcal{O}}(s) \end{bmatrix} \\ &= \begin{bmatrix} R_\Psi(s)(L_{\Sigma, \mathcal{O}}(s)) + r_{\Psi, \mathcal{O}}(s) \times (R_\Psi(s)(M_{\Sigma, \mathcal{O}}(s))) \\ R_\Psi(s)(M_{\Sigma, \mathcal{O}}(s)) \end{bmatrix}. \end{aligned}$$

This reflects the fact that momentum is to be thought of as an element in the dual to the vector space occupied by velocity. We shall not be able to understand all the implications of this here, but again refer to Section 4.4 of [Abraham and Marsden 1978]. Since spatial and body velocities are related by the adjoint representation, the spatial and body momenta should be related by the coadjoint representation. In any event, inverting the above relation gives

$$\boxed{L_{\Sigma, \mathcal{O}}(s) = R_\Psi^{-1}(s)(\ell_{\Sigma, \mathcal{O}}(s) + m_{\Sigma, \mathcal{O}}(s) \times r_{\Psi, \mathcal{O}}(s))}$$

as the **body angular momentum** and

$$\boxed{M_{\Sigma, \mathcal{O}}(s) = R_\Psi^{-1}(s)m_{\Sigma, \mathcal{O}}(s)}$$

as the **body linear momentum** of the body  $(\mathcal{B}, \mu)$  undergoing the motion  $\Psi$  with respect to the observer  $\mathcal{O}$ .

Again, if the observer has a special form, the expressions for body momenta may be conveniently expressed in terms of the body velocities. The following result, easily derived from our definitions for body momenta and Proposition 1.6.1, records the form of the expressions in the case of the simplified observer.

**1.6.4 Proposition** *Let  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$  be a Galilean spacetime with  $(\mathcal{B}, \mu)$  a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . Also let  $\Sigma = (\Psi, \nu)$  be a rigid motion with  $\mathcal{O} \in \text{Ob}(\Sigma)$  an observer having the property that  $\mathbf{x}_c \in \mathcal{O}$ . Then we have*

$$\begin{aligned} \mathbf{L}_{\Sigma, \mathcal{O}}(s) &= \mathbb{I}_c(\Omega_{\Psi}(s)) \\ \mathbf{M}_{\Sigma, \mathcal{O}}(s) &= \mu(\mathcal{B}) \mathbf{V}_{\Psi, \mathcal{O}}(s). \end{aligned}$$

Let us return to our running example.

**1.6.5 Example** (Example 1.4.4 cont'd) We carry on with the situation described in Example 1.6.2. Using the definition of body angular momentum we determine it to be

$$\mathbf{L}(t) = \mathbf{R}^{-1}(t)\boldsymbol{\ell}(t) = I \exp(-t\boldsymbol{\omega}_0)\boldsymbol{\omega}_0 = I \boldsymbol{\omega}_0.$$

Thus, as with spatial and body angular velocities, in this case the spatial and body angular momenta are the same.  $\square$

Let us duplicate for body momenta Proposition 1.6.3, describing the spatial momenta for planar rigid motions. The proof of this result follows exactly along the lines of the proof of Proposition 1.6.3

**1.6.6 Proposition** *Let  $\mathcal{H} = (\mathcal{F}, \mathbf{U}, \mathbf{g}|_{\mathbf{U}_\tau}, \tau|_{\mathbf{U}})$  be a 3-dimensional Galilean sub-spacetime of  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$ , let  $\Sigma = (\Psi, \mathcal{O})$  be a  $\mathcal{H}$ -compatible rigid motion, let  $\mathcal{O}$  be an  $\mathcal{H}$ -compatible observer, and let  $(\mathcal{B}, \mu)$  be an  $\mathcal{H}$ -compatible rigid body. Then  $\mathbf{M}_{\Sigma, \mathcal{O}}(s) \in \mathbf{U}_\tau$  for each  $s \in \mathbf{I}_{\mathcal{G}}$ , and for each  $s \in \mathbf{I}_{\mathcal{G}}$  there exists  $\bar{\mathbf{L}}_{\Sigma, \mathcal{O}}(s) \in \mathbb{R}$  so that*

$$\mathbf{L}_{\Sigma, \mathcal{O}}(s) = \bar{\mathbf{L}}_{\Sigma, \mathcal{O}}(s)\mathbf{u}_\tau.$$

*Further suppose that we have  $\mathcal{O} \in \text{Ob}(\Sigma)$  and  $\mathbf{x}_c \in \mathcal{O}$ , and then define  $\bar{\Omega}_{\Psi}(s) \in \mathbb{R}$  by  $\Omega_{\Psi}(s) = \bar{\Omega}_{\Psi}(s)\mathbf{u}_\tau$  as in Proposition 1.4.14. Also let  $I$  be the eigenvalue of  $\mathbb{I}_c$  for which  $\mathbf{u}_\tau$  is an eigenvector. Then the expression for  $\bar{\mathbf{L}}_{\Sigma, \mathcal{O}}$  simplifies to  $\bar{\mathbf{L}}_{\Sigma, \mathcal{O}}(s) = I\bar{\Omega}_{\Psi}(s)$ .*

**1.6.3 Conservation laws** The equations of motion of a free rigid body are derived from the principle, experimentally verified to some degree of accuracy, of the conservation of momentum. In this section we demonstrate that this principle is observer independent. This is rather simple, given the notation we have adopted.

**1.6.7 Proposition** *Let  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$  be a Galilean spacetime with  $(\mathcal{B}, \mu)$  a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . Suppose that  $\Sigma = (\Psi, \nu)$  be a rigid motion and let  $\mathcal{O}$  and  $\tilde{\mathcal{O}}$  be observers. Denote by  $\boldsymbol{\ell}_{\Sigma, \mathcal{O}}$ ,  $\mathbf{m}_{\Sigma, \mathcal{O}}$ ,  $\boldsymbol{\ell}_{\Sigma, \tilde{\mathcal{O}}}$ , and  $\mathbf{m}_{\Sigma, \tilde{\mathcal{O}}}$  the spatial angular and linear momenta for the rigid motion  $\Sigma$  relative to the observers  $\mathcal{O}$  and  $\tilde{\mathcal{O}}$ . If the rigid motion has the property that  $\boldsymbol{\ell}'_{\Sigma, \mathcal{O}} = 0$  and  $\mathbf{m}'_{\Sigma, \mathcal{O}} = 0$ , then it also has the property that  $\boldsymbol{\ell}'_{\Sigma, \tilde{\mathcal{O}}} = 0$  and  $\mathbf{m}'_{\Sigma, \tilde{\mathcal{O}}} = 0$ .*

*Proof* First let us show that  $m'_{\Sigma, \mathcal{O}} = 0$  implies that  $m'_{\Sigma, \tilde{\mathcal{O}}} = 0$ . We write

$$\begin{aligned} m_{\Sigma, \tilde{\mathcal{O}}}(s) &= \mu(\mathcal{B}) P_{\tilde{\mathcal{O}}}(T'_{\Sigma}(s, x_c)) \\ &= m_{\Sigma, \mathcal{O}}(s) + \mu(\mathcal{B}) (P_{\tilde{\mathcal{O}}}(T'_{\Sigma}(s, x_c)) - P_{\mathcal{O}}(T'_{\Sigma}(s, x_c))). \end{aligned}$$

Now we note that for  $v \in V_{\mathcal{G}}$  we have

$$P_{\tilde{\mathcal{O}}}(v) - P_{\mathcal{O}}(v) = (v - \tau(v)v_{\tilde{\mathcal{O}}}) - (v - \tau(v)v_{\mathcal{O}}) = v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}. \quad (1.27)$$

It now follows directly from the definition of  $T_{\Sigma}$  that  $m'_{\Sigma, \tilde{\mathcal{O}}}(s) = m'_{\Sigma, \mathcal{O}}(s)$ .

Now we look at angular momentum. Here we compute, using (1.27),

$$\begin{aligned} \ell_{\Sigma, \tilde{\mathcal{O}}} &= \int (T_{\Sigma}(s, x) - \tilde{\mathcal{O}}_s) \times P_{\tilde{\mathcal{O}}}(T'_{\Sigma}(s, x)) \, d\mu \\ &= \int ((T_{\Sigma}(s, x) - \mathcal{O}_s) + (\mathcal{O}_s - \tilde{\mathcal{O}}_s)) \times (P_{\mathcal{O}}(T'_{\Sigma}(s, x)) + (v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}})) \, d\mu \\ &= \ell_{\Sigma, \mathcal{O}}(s) + \int (\mathcal{O}_s - \tilde{\mathcal{O}}_s) \times P_{\mathcal{O}}(T'_{\Sigma}(s, x)) \, d\mu + \\ &\quad \int (T_{\Sigma}(s, x) - \mathcal{O}_s) \times (v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}) \, d\mu + \int (\mathcal{O}_s - \tilde{\mathcal{O}}_s) \times (v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}) \, d\mu. \end{aligned}$$

Let us investigate the derivative with respect to  $s$  of the last three terms.

*Term 1:* Here we compute

$$\begin{aligned} \mathcal{O}_s - \tilde{\mathcal{O}}_s &= (\mathcal{O}_{s_0} + (s - s_0)v_{\mathcal{O}}) - (\tilde{\mathcal{O}}_{s_0} + (s - s_0)v_{\tilde{\mathcal{O}}}) \\ &= (\mathcal{O}_{s_0} - \tilde{\mathcal{O}}_{s_0}) + (s - s_0)(v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}). \end{aligned} \quad (1.28)$$

Therefore, the derivative of the first term is

$$\int (v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}) \times P_{\mathcal{O}}(T'_{\Sigma}(s, x)) \, d\mu + \int ((\mathcal{O}_{s_0} - \tilde{\mathcal{O}}_{s_0}) + (s - s_0)(v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}})) \times P_{\mathcal{O}}(T''_{\Sigma}(s, x)) \, d\mu.$$

Here we have used the fact that differentiation with respect to  $s$  commutes with  $P_{\mathcal{O}}$ , a fact that is readily checked. We now claim that

$$\int P_{\mathcal{O}}(T''_{\Sigma}(s, x)) \, d\mu = 0.$$

To prove this claim we write

$$T''_{\Sigma}(s, x) = T''_{\Sigma}(s, (x - x_c) + x_c).$$

By direct calculation it is observed that  $x$  appears in the expression  $T''_{\Sigma}(s, x)$  “linearly,” so that an application of Lemma 1.5.2(ii) gives

$$\int P_{\mathcal{O}}(T''_{\Sigma}(s, x)) \, d\mu = P_{\mathcal{O}}(T''_{\Sigma}(s, x_c)) \, d\mu.$$

However, since we are supposing that  $m'_{\Sigma, \mathcal{O}} = 0$ , our claim follows. Thus we have the derivative of the first term as

$$\int (v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}) \times P_{\mathcal{O}}(T'_{\Sigma}(s, x)) \, d\mu \quad (1.29)$$

*Term 2:* We first note that by direct computation and using Proposition 1.4.5 we may ascertain that the derivative with respect to  $s$  of  $T_\Sigma(s, x) - \mathcal{O}_s$  is equal to  $P_{\mathcal{O}}(T'_\Sigma(s, x))$ . Therefore the derivative of the second term is determined easily to be

$$\int P_{\mathcal{O}}(T'_\Sigma(s, x)) \times (v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}) d\mu. \quad (1.30)$$

*Term 3:* Here we use (1.28) to readily check that the derivative of the third term is

$$\int (v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}) \times (v_{\mathcal{O}} - v_{\tilde{\mathcal{O}}}) d\mu = 0 \quad (1.31)$$

Now we combine the expressions (1.29), (1.30), and (1.31) to see that, provided that  $m'_{\Sigma, \mathcal{O}} = 0$ ,  $\ell'_{\Sigma, \mathcal{O}} = \ell'_{\Sigma, \tilde{\mathcal{O}}}$ .  $\blacksquare$

- 1.6.8 **Remarks 1.** Let us call a rigid motion  $\Sigma = (\Psi, \nu)$  *natural* for a rigid body  $(\mathcal{B}, \mu)$  if the spatial momenta  $\ell_{\Sigma, \mathcal{O}}$  and  $m_{\Sigma, \mathcal{O}}$  are constant functions of  $s$  for some observer  $\mathcal{O}$  (and hence for all observers by the proposition).
2. Typically when one deals with rigid body dynamics, an observer  $\mathcal{O}$  is assumed fixed, and then one tacitly considers rigid motions  $\Sigma = (\Psi, \nu)$  for which  $\mathcal{O} \in \text{Ob}(\Sigma)$ . This restricts the changes of observer to effectively changing the point about which one measures spatial angular momentum. Therefore, when we demonstrate here that conservation of momentum is observer independent, we are doing more than is usually done.  $\square$

Returning to our example.

1.6.9 **Example** (Example 1.4.4 cont'd) Let us now show that the rigid motion  $\Sigma$  defined in Example 1.4.4 is natural for the rigid body  $(\mathcal{B}, \mu)$  described in Example 1.6.2. Indeed, by Proposition 1.6.7 all we need to observe is that the spatial momentum is constant, the constant in this case being  $I\omega_0$ . Note that if  $\omega$  were not an eigenvector for the inertia tensor, then the rigid motion  $\Sigma$  would not be natural.  $\square$

**1.6.4 The Euler equations in Galilean spacetimes** Now we use the conservation laws to derive the equations of motion for a rigid body relative to a general observer. When we use an observer with the properties of those in Propositions 1.6.1 and 1.6.4, we recover the usual Euler equations. However, when the observer is not so distinguished, the equations do not decouple in the usual manner to give separate equations that determine the evolution of the body momenta.

1.6.10 **Proposition** *Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime with  $(\mathcal{B}, \mu)$  a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . Suppose that  $\Sigma = (\Psi, \nu)$  be a natural rigid motion for the rigid body  $(\mathcal{B}, \mu)$  and let  $\mathcal{O}$  be an observer with  $\Omega_{\Psi, \mathcal{O}}$  and  $V_{\Psi, \mathcal{O}}$  the body angular and linear velocities, and  $L_{\Sigma, \mathcal{O}}$  and  $M_{\Sigma, \mathcal{O}}$  the body angular and linear momenta. Define  $R_\Psi(s)$  and  $r_{\Psi, \mathcal{O}}(s)$  by  $\Psi(s) = \Phi_{\mathcal{O}}(R_\Psi(s), r_{\Psi, \mathcal{O}}(s))$ .*

*The body momenta, and  $R_\Psi$  and  $r_{\Psi, \mathcal{O}}$  satisfy*

$$\begin{aligned} R'_\Psi(s) &= R_\Psi(s) \circ \hat{\Omega}_\Psi(s) \\ r'_{\Psi, \mathcal{O}}(s) &= R_\Psi(s)(V_{\Psi, \mathcal{O}}(s)) \\ L'_{\Sigma, \mathcal{O}}(s) &= L_{\Sigma, \mathcal{O}}(s) \times \Omega_\Psi(s) + M_{\Sigma, \mathcal{O}}(s) \times V_{\Psi, \mathcal{O}}(s) \\ M'_{\Sigma, \mathcal{O}}(s) &= M_{\Sigma, \mathcal{O}}(s) \times \Omega_\Psi(s). \end{aligned}$$

If the inertia tensor  $\mathbb{I}_{x_0}$  (where  $x_0 = \mathcal{O}_{s_0}$ ) is an isomorphism, then the body velocities  $\Omega_\Psi$  and  $V_{\Psi,\sigma}$  can be expressed in terms of  $R_\Psi$ ,  $r_{\Psi,\sigma}$ ,  $L_{\Sigma,\sigma}$ , and  $M_{\Sigma,\sigma}$  using the definitions for body momenta.

*Proof* The equations for  $R'_\Psi$  and  $r'_{\Psi,\sigma}$  follow directly from the definition of  $R_\Psi$  and  $r_{\Psi,\sigma}$ .

To obtain the equations for  $L'_{\Sigma,\sigma}$  and  $M'_{\Sigma,\sigma}$  we use the expressions giving spatial momenta in terms of body momenta:

$$\begin{aligned}\ell_{\Sigma,\sigma}(s) &= R_\Psi(s)(L_{\Sigma,\sigma}(s)) + r_{\Psi,\sigma}(s) \times (R_\Psi(s)(M_{\Sigma,\sigma}(s))) \\ m_{\Sigma,\sigma}(s) &= R_\Psi(s)(M_{\Sigma,\sigma}(s)).\end{aligned}$$

Differentiating  $m_{\Sigma,\sigma}$  gives

$$0 = R'_\Psi(s)(M_{\Sigma,\sigma}(s)) + R_\Psi(s)(M'_{\Sigma,\sigma}(s)) = R_\Psi(s)(\Omega_\Psi(s) \times M_{\Sigma,\sigma}(s) + M'_{\Sigma,\sigma}(s)),$$

and from this directly follows the equation for  $M'_{\Sigma,\sigma}$ . Similarly, differentiating the expression for  $\ell_{\Sigma,\sigma}$ , and using the fact that spatial linear momentum is conserved, we have

$$\begin{aligned}0 &= R'_\Psi(s)(L_{\Sigma,\sigma}(s)) + R_\Psi(s)(L'_{\Sigma,\sigma}(s)) + r'_{\Psi,\sigma}(s) \times (R_\Psi(s)(M_{\Sigma,\sigma}(s))) \\ &= R_\Psi(s)(L'_{\Sigma,\sigma}(s) + \Omega_\Psi(s) \times L_{\Sigma,\sigma}(s) + (R_\Psi^{-1}(s)(r'_{\Psi,\sigma}(s))) \times M_{\Sigma,\sigma}(s)).\end{aligned}$$

From this follows the equation given for  $L'_{\Sigma,\sigma}$ .

For the final assertion of the proposition, we merely remark that a lengthy calculation does indeed show that the body velocities may be determined as functions of the body momenta and  $R_\Psi$  and  $r_{\Psi,\sigma}$ .  $\blacksquare$

**1.6.11 Remarks 1.** We shall call the equations of Proposition 1.6.10 the *Galilean Euler equations*.

2. Although initially the governing equations of Proposition 1.6.10 look innocuous enough, they are in fact quite formidable if one wishes to work with a general observer and then express the body velocities in terms of the body momenta. We shall see in Section 2.1 that it is possible to make things more palatable by eliminating the complications that derive from the choice of an observer not in  $\text{Ob}(\Sigma)$ . However, if one wishes to use an observer that does not contain the centre of mass of the body, the equations are still somewhat complicated, and in particular, the equations for  $L'_{\Sigma,\sigma}$  and  $M'_{\Sigma,\sigma}$  will not decouple from the equations for  $R_\Psi$  and  $r_{\Psi,\sigma}$ .
3. Even when one makes the simplification of measuring the spatial angular momentum about the centre of mass, the equations of Proposition 1.6.10 will be potentially more complicated than the usual Euler equations, resulting from the fact that  $\mathcal{O}$  may not be in  $\text{Ob}(\Sigma)$ . Thus we see that an observer whose Galilean velocity is not the same as that for the rigid motion will not see motion of the same nature as an observer with the same Galilean velocity. Following Remark 1.6.8–2, the upshot of this is that we obtain equations that are genuinely more general than the usual rigid body equations, although they are still derived from the principle of conservation of spatial momentum.
4. When  $\mathbb{I}_{x_0}$  is not an isomorphism, then the equations of Proposition 1.6.10 will be overdetermined. This is a result of the fact that when the inertia tensor has zero eigenvalues, then  $O(\ker(\tau))$  is too large to describe the rotations of the body, as may be seen from Corollary 1.5.7. For a body that lies in a line through  $x_0$ , an appropriate space to describe the rotational motion of the body is the 2-sphere  $\mathbb{S}^2$ . When the body is the point  $\{x_0\}$ , then there are no rotational degree of freedom, and all equations describing them can be eliminated.  $\square$

As usual, if the observer has certain typical properties, then the equations of motion simplify. The next result follows directly from Propositions 1.6.1 and 1.6.4.

**1.6.12 Proposition** *Let  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$  be a Galilean spacetime with  $(\mathcal{B}, \mu)$  a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . Also let  $\Sigma = (\Psi, \nu)$  be a natural rigid motion for the rigid body  $(\mathcal{B}, \mu)$  with  $\mathcal{O} \in \text{Ob}(\Sigma)$  an observer having the property that  $\mathbf{x}_c \in \mathcal{O}$ . Let  $\Omega_{\Psi, \mathcal{O}}$  and  $\mathbf{V}_{\Psi, \mathcal{O}}$  be the body angular and linear momenta. Define  $\mathbf{R}_{\Psi}(s)$  and  $\mathbf{r}_{\Psi, \mathcal{O}}(s)$  by  $\Psi(s) = \Phi_{\mathcal{O}}(\mathbf{R}_{\Psi}(s), \mathbf{r}_{\Psi, \mathcal{O}}(s))$ .*

*The body velocities, and  $\mathbf{R}_{\Psi}$  and  $\mathbf{r}_{\Psi, \mathcal{O}}$  satisfy*

$$\begin{aligned} \mathbf{R}'_{\Psi}(s) &= \mathbf{R}_{\Psi}(s) \circ \hat{\Omega}_{\Psi}(s) \\ \mathbf{r}'_{\Psi, \mathcal{O}}(s) &= \mathbf{R}_{\Psi}(s)(\mathbf{V}_{\Psi, \mathcal{O}}(s)) \\ \mathbb{I}_c(\Omega'_{\Psi}(s)) &= (\mathbb{I}_c(\Omega_{\Psi}(s))) \times \Omega_{\Psi}(s) \\ \mu(\mathcal{B}) \mathbf{V}'_{\Psi, \mathcal{O}}(s) &= \mu(\mathcal{B}) \mathbf{V}_{\Psi, \mathcal{O}}(s) \times \Omega_{\Psi}(s). \end{aligned}$$

Our running example illustrates the concepts.

**1.6.13 Example** (Example 1.4.4 cont'd) In Example 1.6.9 we saw that the rigid motion of Example 1.4.4 was natural for the rigid body considered in Example 1.6.2. Let us verify that it also satisfies the Galilean Euler equations. Things are made simpler by our considering an observer whose Galilean velocity is the same as that for the rigid motion, and that the observer contains the centre of mass. We thus only need to consider the equations of Proposition 1.6.12. Since the body linear velocity  $\mathbf{V}(t)$  is zero and since  $\mathbf{r}(t) = \mathbf{0}$ , the linear equations are trivially satisfied. To verify that the equations for  $\mathbf{R}(t)$  and  $\Omega(t)$  are also satisfied we need only see that

$$\mathbf{R}'(t) = \exp(t\hat{\omega}_0)\hat{\omega}_0 = \mathbf{R}(t)\hat{\Omega}(t),$$

and that

$$\mathbb{I}_0(\Omega'(t)) = \mathbf{0} = I \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 = (\mathbb{I}_0(\Omega(t))) \times \Omega(t),$$

as required. □

Let us now see how the Galilean Euler equations simplify for planar rigid motions. The proof of the following result follows directly from Proposition 1.6.10, and the characterisations of spatial and body angular momenta for planar rigid motions.

**1.6.14 Proposition** *Let  $\mathcal{H} = (\mathcal{F}, \mathbf{U}, \mathbf{g}|_{\mathbf{U}}, \tau|_{\mathbf{U}})$  be a 3-dimensional Galilean sub-spacetime of  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$ , let  $\Sigma = (\Psi, \mathcal{O})$  be an  $\mathcal{H}$ -compatible rigid motion, let  $\mathcal{O}$  be an  $\mathcal{H}$ -compatible observer, and let  $(\mathcal{B}, \mu)$  be an  $\mathcal{H}$ -compatible rigid body. Define  $\Psi(s) = \Phi_{\mathcal{O}}(\mathbf{R}_{\Psi}(s), \mathbf{r}_{\Psi, \mathcal{O}}(s))$ , define  $\bar{\Omega}_{\Psi}(s)$  as in Proposition 1.4.14, and define  $\bar{\mathbf{L}}_{\Psi}(s)$  as in Proposition 1.6.6.*

*The following equations hold:*

$$\begin{aligned} \mathbf{R}'_{\Psi}(s) &= \bar{\Omega}_{\Psi}(s) \mathbf{R}_{\Psi}(s) \circ \hat{\mathbf{u}}_{\tau} \\ \mathbf{r}'_{\Psi, \mathcal{O}}(s) &= \mathbf{R}_{\Psi}(s)(\mathbf{V}_{\Psi, \mathcal{O}}(s)) \\ \bar{\mathbf{L}}'_{\Sigma, \mathcal{O}}(s) &= \mathbf{f}_{\tau}(\mathbf{M}_{\Sigma, \mathcal{O}}(s), \mathbf{V}_{\Psi, \mathcal{O}}(s)) \\ \mathbf{M}'_{\Sigma, \mathcal{O}}(s) &= \mathbf{M}_{\Sigma, \mathcal{O}}(s) \times \Omega_{\Psi}(s). \end{aligned}$$

Further suppose that we have  $\mathcal{O} \in \text{Ob}(\Sigma)$  and  $\mathbf{x}_c \in \mathcal{O}$ , and then define  $\bar{\Omega}_\Psi(s) \in \mathbb{R}$  by  $\Omega_\Psi(s) = \bar{\Omega}_\Psi(s)\mathbf{u}_\tau$  as in Proposition 1.4.14. Let  $I$  be the eigenvalue of  $\mathbb{I}_c$  for which  $\mathbf{u}_\tau$  is an eigenvector. Then the above equations simplify to:

$$\begin{aligned} \mathbf{R}'_\Psi(s) &= \bar{\Omega}_\Psi(s) \mathbf{R}_\Psi(s) \circ \hat{\mathbf{u}}_\tau \\ \mathbf{r}'_{\Psi,\mathcal{O}}(s) &= \mathbf{R}_\Psi(s) \mathbf{V}_{\Psi,\mathcal{O}}(s) \\ I \bar{\Omega}'_\Psi(s) &= 0 \\ \mu(\mathcal{B}) \mathbf{V}'_{\Psi,\mathcal{O}}(s) &= \mu(\mathcal{B}) \bar{\Omega}_\Psi(s) \mathbf{V}_{\Psi,\mathcal{O}}(s) \times \mathbf{u}_\tau. \end{aligned} \tag{1.32}$$

Of course, these equations now have a very simple solution (see Exercise E1.24).

**1.6.5 Solutions of the Galilean Euler equations** Now that we have seen that the equations which describe the motion of a rigid body can be quite complicated in form, let us show that, as expected, one may determine the solutions of these equations directly from the solutions for observers satisfying the hypotheses of Proposition 1.6.12.

**1.6.15 Proposition** Let  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$  be a Galilean spacetime with  $(\mathcal{B}, \mu)$  a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . Also let  $\Sigma = (\Psi, \nu)$  be a natural rigid motion for the rigid body  $(\mathcal{B}, \mu)$  with  $\tilde{\mathcal{O}}$  a general observer. Let  $\Omega_{\Psi,\tilde{\mathcal{O}}}$  and  $\mathbf{V}_{\Psi,\tilde{\mathcal{O}}}$  be the body angular and linear velocities. Define  $\mathbf{R}_\Psi(s)$  and  $\mathbf{r}_{\Psi,\tilde{\mathcal{O}}}(s)$  by  $\Psi(s) = \Phi_{\tilde{\mathcal{O}}}(\mathbf{R}_\Psi(s), \mathbf{r}_{\Psi,\tilde{\mathcal{O}}}(s))$ . Let  $\mathcal{O} \in \text{Ob}(\Sigma)$  be an observer having the property that  $\mathbf{x}_c \in \mathcal{O}$  with  $\Omega_\Psi$  and  $\mathbf{V}_{\Psi,\mathcal{O}}$  the corresponding body angular and linear velocities (note that the body angular velocities with respect to the two observers are indeed the same).

Then we have

$$\begin{aligned} \mathbf{V}_{\Psi,\tilde{\mathcal{O}}}(s) &= \mathbf{V}_{\Psi,\mathcal{O}}(s_0) + (\text{id}_\tau - \mathbf{R}_\Psi^{-1}(s))(\mathbf{v}_{\tilde{\mathcal{O}}} - \mathbf{v}_\mathcal{O}) + \\ &\quad \Omega_\Psi(s) \times ((\tilde{\mathcal{O}}_{s_0} - \mathcal{O}_{s_0}) + (s - s_0)(\mathbf{v}_{\tilde{\mathcal{O}}} - \mathbf{v}_\mathcal{O})). \end{aligned}$$

In particular, if  $\mathcal{O}$  is also in  $\text{Ob}(\Sigma)$  then

$$\mathbf{V}_{\Psi,\tilde{\mathcal{O}}}(s) = \mathbf{V}_{\Psi,\mathcal{O}}(s_0) + \Omega_\Psi(s) \times (\tilde{\mathcal{O}}_{s_0} - \mathcal{O}_{s_0}).$$

*Proof* Note that  $\mathbf{V}_{\Psi,\mathcal{O}}(s) = \mathbf{V}_{\Psi,\mathcal{O}}(s_0)$  for all  $s \in I_\mathcal{G}$  by Proposition 1.6.12. The second equation of the proposition follows from combining the relations

$$\begin{aligned} \mathbf{r}_{\Psi,\tilde{\mathcal{O}}}(s) &= \mathbf{r}_{\Psi,\mathcal{O}}(s) + (\mathbf{R}_\Psi(s) - \text{id}_\tau)(\tilde{\mathcal{O}}_s - \mathcal{O}_s) \\ \Omega_\Psi(s) &= \mathbf{R}_\Psi^{-1}(s) \mathbf{R}'_\Psi(s) \\ \tilde{\mathcal{O}}_s - \mathcal{O}_s &= (\tilde{\mathcal{O}}_{s_0} - \mathcal{O}_{s_0}) + (s - s_0)(\mathbf{v}_{\tilde{\mathcal{O}}} - \mathbf{v}_\mathcal{O}), \end{aligned}$$

which are derived in Propositions 1.4.3, 1.4.5, and Proposition 1.6.7, respectively, with the definition of  $\mathbf{V}_{\Psi,\tilde{\mathcal{O}}}$ . ■

Let us see how this plays out with our ongoing example.

**1.6.16 Example** (Example 1.4.4 cont'd) We have observed that the rigid motion  $\Sigma$  described in Example 1.4.4 is natural for the rigid body  $(\mathcal{B}, \mu)$  described in Example 1.6.2. For the canonical observer  $\mathcal{O}_{\text{can}}$  this led us to the verification that the body angular momentum does indeed satisfy the Galilean Euler equations.

Now let us introduce a non-canonical observer  $\tilde{\mathcal{O}}$  defined by

$$\tilde{\mathcal{O}} = \{(\mathbf{x}_0 + \mathbf{v}_0 t, t) \mid t \in \mathbb{R}\}.$$



Thus this new observer does not contain the centre of mass (unless  $\mathbf{x}_0 = \mathbf{0}$ , and it has a Galilean velocity  $(\mathbf{v}_0, 1)$  which differs from the rigid body Galilean velocity  $\nu = (\mathbf{0}, 1)$ . Nevertheless we can use Proposition 1.6.15 to obtain the solution to the Galilean Euler equations with respect to this observer *without having to solve the equations*. According to Proposition 1.6.15, if  $\tilde{\boldsymbol{\Omega}}(t)$  and  $\tilde{\mathbf{V}}(t)$  are the body angular velocities with respect to the new observer we have

$$\tilde{\boldsymbol{\Omega}}(t) = \boldsymbol{\Omega}(t) = \boldsymbol{\omega}_0.$$

We also have

$$\begin{aligned} \tilde{\mathbf{V}}(t) &= \mathbf{V}(t) + (\mathbf{I}_3 - \mathbf{R}(t))\mathbf{v}_0 + \boldsymbol{\Omega}(t) \times (\mathbf{x}_0 + t\mathbf{v}_0) \\ &= (\mathbf{I}_3 - \exp(t\hat{\boldsymbol{\omega}}_0))\mathbf{v}_0 + \boldsymbol{\omega}_0 \times (\mathbf{x}_0 + t\mathbf{v}_0). \end{aligned}$$

Thus, even though the Galilean Euler equations may not look so pleasant with respect to a general observer, their solutions may be written down. Note that the body linear velocity is not constant with respect to a general observer.  $\square$

## 1.7 Forces on rigid bodies

The previous sections have been dedicated to producing a description of rigid body motion in the absence of forces. Let us now consider how forces affect a rigid body. Note that the matter of adding a force to a particle did not cause us concern, so let us use this as our basis. Let us consider a mass moving in  $\mathbb{R}^3$  and following a curve  $t \mapsto \mathbf{x}(t)$ . We also suppose that the mass is subjected to a force which at time  $t$  is  $\mathbf{F}(t)$ . We then know that the familiar Newton equations tell us that  $m\ddot{\mathbf{x}}(t) = \mathbf{F}(t)$ . The spatial momentum about the origin, which is conserved for unforced motions, is no longer conserved when forces is present. The linear momentum evolves according to

$$\frac{d}{dt}(m\dot{\mathbf{x}}(t)) = m\ddot{\mathbf{x}}(t) = \mathbf{F}(t),$$

and the angular momentum evolves according to

$$\frac{d}{dt}(m\mathbf{x}(t) \times \dot{\mathbf{x}}(t)) = m\mathbf{x}(t) \times \ddot{\mathbf{x}}(t) = \mathbf{x}(t) \times \mathbf{F}(t).$$

With these calculations in mind, let us proceed to look at a general rigid body.

We let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime with  $(\mathcal{B}, \mu)$  a rigid body with  $\mathcal{B} \subset \mathcal{E}(s_0)$ . We let  $x_c$  denote the centre of mass of  $\mathcal{B}$  and suppose that a force acts on  $\mathcal{B}$ , being applied at the point  $x_c$ , and that a pure moment acts on  $\mathcal{B}$ , again about the point  $x_c$ . The force we regard as a map  $F: I_{\mathcal{G}} \rightarrow \ker(\tau)$  and the moment we similarly regard as a map  $\tau: I_{\mathcal{G}} \rightarrow \ker(\tau)$ . We suppose that the rigid body undergoes the rigid motion  $\Sigma = (\Psi, \nu)$  as a result of the given applied forces, and that  $\mathcal{O} \in \text{Ob}(\Sigma)$  has the property that  $x_c \in \mathcal{O}$ . Using as motivation our above calculations with a particle, we *declare* that the rigid motion  $\Sigma$  should satisfy

$$m'_{\Sigma, \mathcal{O}}(s) = F(s), \quad \ell'_{\Sigma, \mathcal{O}}(s) = \tau(s) + (T_{\Sigma}(s, x_c) - \mathcal{O}_s) \times F(s).$$



## 1.8 The status of the Newtonian world view

Newtonian mechanics is a powerful tool. Let us see what our above development says about how the world looks from a Newtonian point of view.

1. In a Newtonian world, we do not have a distinguished origin in either space or time. That is, one cannot say, “Time starts here,” or, “All distances should be measured relative to this point.”
2. Although there is no “time zero,” all clocks in a Newtonian world move at the same rate. Similarly, although there is no spatial origin, all rulers look the same to all observers.
3. The notion of “stationary” is senseless in the Newtonian world. That is to say, if one is in an inertial frame, there is no way to determine whether you are moving or not moving by performing experiments within your inertial frame. Thus, in Newtonian mechanics one does not say the train is moving at constant velocity and the person at the station is fixed. One merely says that each is an inertial reference frame, and the one can only determine *relative* velocities of one observer with respect to another.
4. The Newtonian world is comprised of objects sitting in a universe (that is a Galilean spacetime) and interacting with one another in a way consistent with the Galilean relativity principle. In particular, determinacy principle says that to see what will happen in the universe, one need only specify initial conditions for the differential equations of Newtonian mechanics, and all else follows, at least in principle. As you might imagine, there are philosophical upshots of this that may or may not be disturbing.

Is the Newtonian world view empirically accurate? That is, is it consistent with all experimental observations? For many years after Newton, his view of the world, or more accurately, that view of the world that follows from his mechanics was *extremely* influential exactly because it was so very successful. The world did indeed behave as if it were governed by the laws of Newton. However, things began to fall apart for Newton in the latter part of the 19th century. A Scottish physicist, James Clerk Maxwell (1831–1879) introduced in 1873 the partial differential equations that describe the interaction of electric and magnetic fields. One of the consequences of his equations is that the speed of light is an observer independent quantity. This throws a wrench into Galilean invariance, because according to the latter, if the speed of light is observer independent, then if observer  $A$  is travelling at the speed of light with respect to observer  $B$ , then the speed of light as seen by observer  $B$  must be *twice* the speed of light; clearly a contradiction! This is repaired by special relativity, a theory attributed to Albert Einstein (1879–1955).<sup>4</sup> However, Newtonian mechanics is still the tool of choice for everyday use, and it is only when we are dealing with objects moving at enormous relative velocities that relativistic effects need to be accounted for.

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<sup>4</sup>The ideas behind the special theory of relativity predate Einstein’s 1905 paper in the work of Hendrik Antoon Lorentz (1853–1928), and the mathematical ideas were present in a paper of Henri Poincaré (1854–1912) that was published roughly concurrently with Einstein’s paper. Thus special relativity was not a uniquely Einsteinian vision. However, his *general* theory of relativity presented in 1915 was uniquely his.

## Exercises

- E1.1 Prove Proposition 1.1.2.
- E1.2 Prove Proposition 1.1.3.
- E1.3 Let  $U$  and  $V$  be  $\mathbb{R}$ -vector spaces,  $A: U \rightarrow V$  a linear map, and  $v \in V$ . Show that the set of solutions to the equation  $Au = v$  is a  $\mathbb{R}$ -affine space modelled on  $\ker(A)$ . You may suppose that  $v \in \text{image}(A)$ .
- E1.4 Let  $A$  be an affine space modelled on  $V$  and let  $v_1, v_2 \in V$  and  $x \in A$ . Show that  $(v_1 + x) - (v_2 + x) = v_1 - v_2$ .
- E1.5 Show that the definition (1.3) is independent of the choice of  $x_0 \in A$ .
- E1.6 Is the group  $GL(n; \mathbb{R})$  Abelian?
- E1.7 Prove that  $O(n)$  is a subgroup of  $GL(n; \mathbb{R})$ .
- E1.8 Show that  $SO(n)$  a subgroup of  $O(n)$ . Is the set

$$\{\mathbf{A} \in O(n) \mid \det \mathbf{A} = -1\}$$

a subgroup of  $O(n)$ ?

- E1.9 Let  $V$  be a  $\mathbb{R}$  vector space with inner product  $g$  and let  $\|\cdot\|$  be the norm on  $V$  induced by  $g$ . Show that  $\|g\|$  satisfies the parallelogram law:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2), \quad u, v \in V. \quad (\text{E1.1})$$

If you are feeling *very* energetic, try to demonstrate the converse. That is, let  $V$  be a vector space with norm  $\|\cdot\|$  *not* necessarily derived from an inner product. Show that if  $\|\cdot\|$  satisfies (E1.1) then there exists an inner product  $g$  on  $V$  so that  $\|v\| = \sqrt{g(v, v)}$  for every  $v \in V$ .

- E1.10 Show that for every matrix  $A \in \mathfrak{gl}(n; \mathbb{R})$  we have  $e^A A = A e^A$ .

- E1.11 Let  $\omega \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ .

- (a) Show that  $\ker(\hat{\omega}) = \text{span}_{\mathbb{R}} \{\omega\}$ .
- (b) Show that  $\text{image}(\hat{\omega}) = (\text{span}_{\mathbb{R}} \{\omega\})^\perp$ , where  $\perp$  means orthogonal complement.
- (c) Show that  $e^{\hat{\omega}t} \omega = \omega$  for  $\omega \in \mathbb{R}^3$ .

For  $\omega_1, \omega_2 \in \mathbb{R}^3$  show that

- (d)  $\widehat{\omega_1 \times \omega_2} = \hat{\omega}_1 \hat{\omega}_2 - \hat{\omega}_2 \hat{\omega}_1$  and that
- (e)  $\omega_1 \cdot \omega_2 = \text{tr}(\hat{\omega}_1 \hat{\omega}_2)$ .

- E1.12 Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be observers in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ , and suppose that  $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset$ . Show that there exists  $\phi_1 \in \text{Coor}(\mathcal{O}_1)$  and  $\phi_2 \in \text{Coor}(\mathcal{O}_2)$  so that  $\phi_2 \circ \phi_1^{-1}$  satisfy

$$\phi_2 \circ \phi_1^{-1}: \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{v} \\ \mathbf{0}^t & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix}.$$

- E1.13 Suppose that you inhabited a world that was accelerating at a uniform rate with respect to an inertial coordinate system. Also suppose that you were Isaac Newton. What would be the laws of motion you would formulate?

- E1.14 In this exercise, you will show that a “lab frame” is not inertial.

- E1.15 Show that for rigid motions  $\Sigma_1 = (\Psi_1, \nu_1)$  and  $\Sigma_2 = (\Psi_2, \nu_2)$ , the equality  $T_{\Sigma_1} = T_{\Sigma_2}$  implies that  $\Sigma_1 = \Sigma_2$ .

E1.16 Suppose that you inhabited a world that was rotating at a variable angular velocity  $\Omega(t)$  with respect to an inertial reference frame. Show that an object of mass  $m$  in your world, located at a point  $\mathbf{X}$  in its own coordinate system with respect to the point of rotation, and moving with velocity  $\mathbf{V}$  with respect to its own coordinate system, would feel the effects of the following three forces:

1. the *rotational inertial force*:  $m\dot{\Omega} \times \mathbf{X}$ ;
2. the *Coriolis force*:  $2m\Omega \times \mathbf{V}$ ;
3. the *centrifugal force*:  $m\Omega \times (\Omega \times \mathbf{X})$ .

Are you an occupant of such a world (assuming that an inertial frame exists)?

E1.17 Verify the formulae (1.17).

E1.18 Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime inhabited by a single particle of mass  $m$ . If the particle is subjected to an external force  $F: \mathcal{E} \times V_{\mathcal{G}} \rightarrow \ker(\tau)$ , what can one say about the form of  $F$  if the Galilean relativity principle holds? What are the corresponding motions of the particle?

E1.19 Let  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  be a Galilean spacetime inhabited by two particles, one of mass  $m_1$  and the other of mass  $m_2$ .

- (a) If the particle  $m_i$  is acted upon by a force  $F_i: \mathcal{E}^2, V^2 \rightarrow \ker(\tau)$  for  $i \in \{1, 2\}$ , what must be the form of  $F_1$  and  $F_2$  if the Galilean relativity principle is to hold?
- (b) Show that for each set of initial conditions  $(x_1, v_1) \in \mathcal{E} \times V$  for  $m_1$  and  $(x_2, v_2) \in \mathcal{E} \times V$  for  $m_2$ , there exists a 3-dimensional sub-spacetime  $\mathcal{F}$  of  $\mathcal{E}$  so that the world line for each of  $m_1$  and  $m_2$  lie in  $\mathcal{F}$ .
- (c) Show that the total angular of the system is conserved.

E1.20 Let  $(\mathcal{B}, \mu)$  be a rigid body with the property that  $\mathcal{B}$  is contained in a line  $\ell$ , but is not a point. Show that if  $x_0 \notin \ell$  then the inertia tensor  $\mathbb{I}_{x_0}$  is an isomorphism.

E1.21 Show that  $m_1$  and  $m_2$  as defined by (1.26) are positive real numbers.

E1.22 For a planar rigid body, state and prove the Parallel Axis Theorem. Make sure that the statement of the theorem begin, “Let  $(\mathcal{B}, \mu)$  be a planar rigid body in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ ...”

E1.23 In this exercise you will derive the equations of motion for a rigid body which is fixed at a point  $x_0$ , not necessarily the centre of mass.

- (a) In a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ , make sense of the statement that a rigid body  $(\mathcal{B}, \mu)$  is to move so that it is fixed at a point  $x_0 \in \mathcal{B}$ .

Let  $\Sigma = (\Psi, \nu)$  be a rigid motion which fixes the point  $x_0$  (according to your notion of doing so from (a)), and let  $\mathcal{O} \in \text{Ob}(\Sigma)$  be an observer for which  $x_0 \in \mathcal{O}$ .

- (b) Determine the spatial and body angular and linear velocities of the rigid motion with respect to the observer  $\mathcal{O}$ .
- (c) Determine the spatial and body angular and linear momenta for the rigid motion with respect to the observer  $\mathcal{O}$ .
- (d) What are the Galilean Euler equations for the rigid body motion with respect to the observer  $\mathcal{O}$ ?

E1.24 Determine the solution to the equations (1.32) subject to the initial conditions

$$R_{\Psi}(s_0) = R_0, \quad r_{\Psi, \mathcal{O}}(s_0) = r_0, \quad \hat{\Omega}_{\Psi}(s_0) = \Omega_0, \quad V_{\Psi, \mathcal{O}}(s_0) = V_0.$$



# Chapter 2

## Lagrangian mechanics

After understanding some essential concepts in Newtonian mechanics we are in a position to get to Lagrangian mechanics proper. Lagrangian mechanics can be quite abstract, but to be really precise, some of this abstraction is necessary. The starting point for Lagrangian mechanics in the modern framework [Abraham and Marsden 1978, Arnol'd 1989] is with a configuration space that is a “differentiable manifold.” From here one is able to construct the objects of Lagrangian mechanics as defined on a differentiable manifold. While I feel that such a treatment has much to recommend it, it does necessitate something of a buildup, and at this level this buildup may not be appropriate. Thus our approach is to (1) identify the reasons why concepts such as a differentiable manifold might be helpful, (2) initiate a way to redress the issues without having to actually define a differentiable manifold (although we come close), and (3) essentially fall back on the classical “generalised coordinates,” but now hopefully with a better understanding for what they are and what they are not.

After a discussion of coordinate related matters, we discuss Lagrangian mechanics, starting with its formulation as a variational principle. Then we discuss so-called simple mechanical systems. These are not so named because their behaviour is simple, but rather because they are not completely general. Most of the systems one encounters in applications are in fact simple mechanical systems. Next we introduce the concept of a force in the Lagrangian setting. Our treatment will seem upsetting to some, because a force does not naturally appear as a vector, but rather as a “covector.” Constraints are often part and parcel of a description of a mechanical system, and a common class of constraints is presented in Section 2.6. With the basic data for Lagrangian mechanics now at hand, in Section 2.7 we investigate the relationship between Newtonian and Lagrangian mechanics, proving that the two descriptions are equivalent whenever they may both be applied. Although we are primarily interested in Lagrangian mechanics, it would be something of a sham not to mention the Hamiltonian formalism, and this we do in Section 2.9. Connoisseurs of the Hamiltonian formalism will be disappointed by the shallowness of the presentation, but they will already know where to go to get what they want [e.g., Abraham and Marsden 1978, Arnol'd 1989, Guillemin and Sternberg 1984, Libermann and Marle 1987]. Finally, in Section 2.10 we observe the effects that symmetries of the Lagrangian can have on the behaviour of solutions of the Euler-Lagrange equations. This study of symmetry in mechanics is something of an industry these days [Marsden and Ratiu 1999], and we give only the barest outline.

### 2.1 Configuration spaces and coordinates

When one sees a classical treatment of Lagrangian mechanics, the typical starting point is “generalised coordinates.” This is normally satisfactory, and we will eventually get ourselves to this starting point. However, we wish to not *start* there in order that we may emphasise

the critical concept of *coordinate invariance*. The idea with coordinate invariance is that, at least when dealing with general considerations (as opposed to a specific example), one should never deal with things that are not coordinate invariant. In order to see what that might mean, we begin at the beginning.

**2.1.1 Configuration spaces** Suppose that we are given a collection of particles and rigid bodies, perhaps interconnected in some way. We wish to assign to this mechanical entity a set that describes its configuration. That is to say, we wish to find a set whose points are in one-to-one correspondence with the set of configurations of the system. There is generally speaking something of an art to doing this, and we shall demonstrate how this might be done via examples. As we introduce the examples, we will introduce the notation one often encounters when trying to assign a set to the collection of configurations for a system.

**2.1.1 Examples 1.** Consider a mass moving in a plane and constrained to move on a circle of radius  $r$  in that plane.

We need to parse this description with the language we know. First of all, when we say something is moving in a plane, what we really mean is that it is constrained to move in a three-dimensional Galilean sub-spacetime  $\mathcal{F}$  of a big Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ . Let  $U$  be the subspace of  $V$  upon which  $\mathcal{F}$  is modelled. We then choose a coordinate system  $\phi$  for  $\mathcal{E}$  that maps  $\mathcal{F}$  to the canonical 3-dimensional Galilean sub-spacetime  $\mathcal{F}_3$ . If the mass is further constrained to move on a circle, this means that for each collection  $\mathcal{E}(s)$  of simultaneous events there exists a point  $x_s \in \mathcal{E}(s)$  with the property that the particle must lie on the circle

$$\{x_s + ru \mid u \in U, g(u, u) = 1\}.$$

It is not clearly stated, but we may suppose that with respect to some observer, the centre of the circle is stationary. This means that we may select the coordinate system  $\phi$  so that all points  $x_s$  get mapped to  $\mathbf{0} \in \mathbb{R}^3$ . Therefore, we have reduced ourselves to the situation that we would have guessed by the initial description:

*The particle moves on the circle of radius  $r$  centred at  $(0, 0) \in \mathbb{R}^2$ .*

We still need to describe the configuration space. Let us define

$$\mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$$

to be the ***n-dimensional sphere***. We claim that we may take  $\mathbb{S}^1$  as the configuration space for the system. Indeed, the particle moves on the circle of radius  $r$ , and there is certainly a simple correspondence between points on the circle of radius  $r$  and those on the circle of radius 1 (see Figure 2.1). Thus we select  $Q = \mathbb{S}^1$  as the configuration space for the system.

Why not choose the configuration space to be the sphere of radius  $r$  rather than the sphere of radius 1? There is no reason not to do this. However, the configuration space is typically chosen to be some dimensionless abstract object, and the physics of the problem, in this case the exact radius of the circle, are shoved into other aspects of the problem description, as we shall see.

2. Our next mechanical system is depicted in Figure 2.2. We have two rigid bodies, each

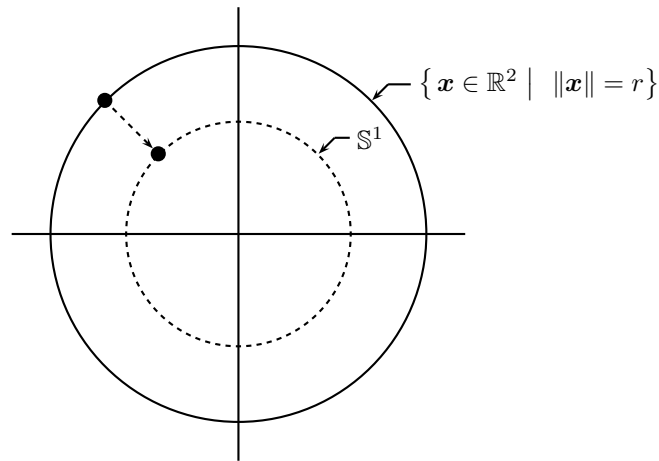


Figure 2.1 Assigning a point on the unit circle to a point on the circle of radius  $r$

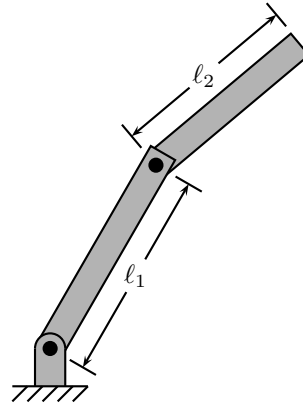


Figure 2.2 A double link pendulum

moving in the same plane, one with a stationary base point, and connected to the base of the other link at its tip.

Let us see how to define a configuration space for this system. The links move in the same plane so there exists a three-dimensional Galilean sub-spacetime  $\mathcal{F}$  of  $\mathcal{E}$  to which the dynamics restricts. We take a coordinate system  $\phi$  that maps  $\mathcal{F}$  to  $\mathcal{F}_3$ . The links then move around in a nice copy of  $\mathbb{R}^2$ . As one of the links has a fixed point, let us choose our coordinate system so that this point is at the origin in  $\mathbb{R}^2$ . Now, to fix the position of the link whose base point is fixed, we need only specify its angle relative to, say, the  $x$ -axis in  $\mathbb{R}^2$ . Note that once we know the position of the link with the fixed base point, we also know the position of one point on the other link since we are told that they are connected at a point. Therefore we need only position the second link with respect to this point whose position we know. But to do this, we again only need specify its angle with respect to, say, the  $x$ -axis. Therefore, the configuration space for this system consists of two copies of  $\mathbb{S}^1$ . If we define

$$\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ times}}$$

to be the  **$n$ -dimensional torus** then the configuration space of our double linked

pendulum is  $\mathbb{T}^2$ .

Note that the datum with respect to which we measure the angles—we arbitrarily choose the  $x$ -axis—will not change the configuration space. However, it *will* change the correspondence between the abstract configuration space and the actual configurations of the system. This will generally be the case. That is to say, my choice of correspondence might differ from yours.

3. Next we consider a rigid body that is fixed at a point. Again, we want to put this in terms of the language of the previous chapter.

As we have a rigid body, this means that the body is  $\mathcal{B} \subset \mathcal{E}(s_0)$  for some  $s_0 \in I_{\mathcal{G}}$ . We are also told that one point in  $\mathcal{B}$  remains fixed throughout the motion. A good interpretation of this is to infer the existence of an observer  $\mathcal{O}$  for  $\mathcal{E}$  so that any motion of the body is described by a rigid motion  $\Sigma = (\Psi, \nu)$  with  $\mathcal{O} \in \text{Ob}(\Sigma)$  and so that  $\Psi(s) = \Phi_{\mathcal{O}}(R_{\Psi}(s), 0)$  for every  $s \in I_{\mathcal{G}}$ . Let  $\phi \in \text{Coor}(\mathcal{O})$  and suppose that

$$\phi_s \circ R_{\Psi}(s) \circ \phi_s^{-1}(\mathbf{0}) = \mathbf{0}.$$

Thus in the coordinate system  $\phi$ , the origin in  $\mathbb{R}^3$  is fixed by the rigid motion. Let us denote  $\mathbf{R}(s) \in O(3)$  by

$$\mathbf{R}(s) = \phi_s \circ R_{\Psi}(s) \circ \phi_s^{-1}.$$

Thus, after time  $s$  of the motion, the rigid body  $\phi_{s_0}(\mathcal{B}) \subset \mathbb{R}^3$  is mapped to the rigid body

$$\{\mathbf{R}(s)(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \phi_{s_0}(\mathcal{B})\}.$$

Thus we may identify each configuration of the body with the matrix  $\mathbf{R}(s) \in SO(3)$ . Therefore, we take  $SO(3)$  as the configuration space for the system.  $\square$

Although we have made long work of defining the configuration spaces in these examples, with a little experience one often “just knows” from the problem description what is the configuration space. But with these examples we want to illustrate how the problem description interacts with our machinations with Galilean spacetimes in Chapter 1. Note that in none of the above examples have we said anything about the system other than its configuration space. We have said nothing about the quantitative geometry of the system (e.g., the radius  $r$  of the circle in Example 2.1.1–1 or the lengths of the links in Example 2.1.1–2). We have also said nothing about the inertial properties of the system or the external forces acting on the system. As a general rule, a good idea with a mechanical problem is to first determine its configuration space. It is with respect to the configuration space that everything else is done, as we shall see.

**2.1.2 Coordinates** In the preceding examples we identified sets that describe the configurations of a system. Note that in none of the cases was the configuration space Euclidean. That is to say, it is not possible to establish a smooth correspondence with the configuration spaces and open subsets of  $\mathbb{R}^n$  for some appropriate  $n$ . This is problematic. To describe the dynamics of these systems, we need to be able to differentiate things, and if we are only able to differentiate on Euclidean spaces, then it is not clear how to proceed to deal with the configuration spaces in the examples. But on the other hand, what’s the problem? Certainly in Example 2.1.1–1 you would just choose an angular coordinate and proceed. Similarly, in Example 2.1.1–2 you’d assign two angular coordinates and proceed. Here, however, you begin to see that you have some choices to make. Should you measure the angle of the second link with respect to the first link, or with respect to the “horizontal”? Moving



on to Example 2.1.1–3 things seem altogether less transparent. What coordinates should you use? The fact of the matter is that things were not so transparent in Example 2.1.1–1, but you got over it because you were familiar with the configuration space. In Example 2.1.1–2 you saw you had a choice to make, but any hesitation about how things might depend on this choice were washed away by your anxiousness to proceed. However, for Example 2.1.1–3 it is not really clear how to begin.

We wish to systematise the above discussion by clearly identifying the process you readily undertook in the easy examples. By understanding these, you at least know what you need to do to start with Example 2.1.1–3. The idea is that we make somewhat precise the idea of choosing coordinates. Our definition here actually destructs under close scrutiny, but it will suffice as long as we are gentle with it.

**2.1.2 Definition** Let  $Q$  be a configuration space for a mechanical system. A **coordinate chart** for  $Q$  is a pair  $(U, \phi)$  where

- (i)  $U$  is a subset of  $Q$  and
- (ii)  $\phi: U \rightarrow \mathbf{U}_\phi$  is a map from  $U$  to an open subset  $\mathbf{U}_\phi$  of  $\mathbb{R}^n$ , for some  $n$ , that is a bijection (i.e., is one-to-one and onto).  $\square$

Let us see if we can make sense of this for the three systems of Example 2.1.1.

**2.1.3 Examples 1.** *Example 2.1.1–1 cont'd:* We wish to choose a coordinate chart  $(U, \phi)$  for  $Q = \mathbb{S}^1$ . We take

$$U = \mathbb{S}^1 \setminus \{(-1, 0)\} \subset \mathbb{S}^1$$

and

$$\phi(x, y) = \text{atan}(x, y).$$

Here  $\text{atan}: \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow (-\pi, \pi]$  is the usual angle measured so that  $\text{atan}(x, 0) = 0$  for  $x > 0$ . Note here that  $\mathbf{U}_\phi = (-\pi, \pi)$  is indeed an open subset of  $\mathbb{R}^1$ .

**2.** *Example 2.1.1–2 cont'd:* Here we choose a coordinate chart for  $Q = \mathbb{T}^2$ . We take

$$\begin{aligned} S_1 &= \{((x_1, y_1), (x_2, y_2)) \in Q \mid x_1 = -1\} \\ S_2 &= \{((x_1, y_1), (x_2, y_2)) \in Q \mid x_2 = -1\}, \end{aligned}$$

and then

$$U = Q \setminus (S_1 \cup S_2), \quad \phi((x_1, y_1), (x_2, y_2)) = (\text{atan}(x_1, y_1), \text{atan}(x_2, y_2)).$$

Note here that  $\mathbf{U}_\phi = (-\pi, \pi) \times (-\pi, \pi)$ .

**3.** *Example 2.1.1–3 cont'd* As expected, choosing a coordinate chart for  $SO(3)$  is not so easy to do. What's more, it is regarded in some circles as a silly thing to do as often there are better ways to handle systems involving  $SO(3)$  than choosing coordinates. Nonetheless, we shall illustrate how one can do that, just so that we might illustrate that it is possible. We shall provide coordinates for a neighbourhood of  $\mathbf{I}_3$  in  $SO(3)$  by using **Euler angles**. Other coordinate charts are possible, and we refer to [Murray, Li, and Sastry 1994] for details.

Let us first be formal, and then we shall give the intuition behind what we do. Let  $\mathbf{R}$  be a matrix in  $SO(3)$  which is “close” to, but not equal to,  $\mathbf{I}_3$ . We shall be clear about how

close shortly. Let us write the components for  $\mathbf{R}$  as  $R_{ij}$ ,  $i, j = 1, 2, 3$  where first index is the row index, and the second the column index. We then define

$$\beta \in (0, \pi), \quad \alpha \in (-\pi, \pi), \quad \gamma \in (-\pi, \pi) \quad (2.1)$$

by

$$\begin{aligned} \beta &= \operatorname{atan}(R_{33}, \sqrt{R_{31}^2 + R_{32}^2}) \\ \alpha &= \operatorname{atan}(R_{13}/\sin \beta, R_{23}/\sin \beta) \\ \gamma &= \operatorname{atan}(-R_{31}/\sin \beta, R_{32}/\sin \beta). \end{aligned} \quad (2.2)$$

This then defines a map  $\phi$  from some subset  $U$  of  $SO(3)$  to  $\mathbb{R}^3$ , thereby defining a coordinate chart. The subset of  $SO(3)$  is given by inverting the relations (2.2) to give the matrix entries, and doing so gives the matrix

$$\begin{bmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{bmatrix}. \quad (2.3)$$

Thus  $U$  is the subset of all matrices in  $SO(3)$  of the form (2.3) where the numbers  $\alpha$ ,  $\beta$ , and  $\gamma$  satisfy the relations (2.1). The map  $\phi: U \rightarrow \mathbb{R}^3$  is given by

$$\begin{aligned} \phi(\mathbf{R}) &= \left( \operatorname{atan}(R_{33}, \sqrt{R_{31}^2 + R_{32}^2}), \operatorname{atan}(R_{13}/\sin \beta, R_{23}/\sin \beta), \right. \\ &\quad \left. \operatorname{atan}(-R_{31}/\sin \beta, R_{32}/\sin \beta) \right). \end{aligned}$$

Note that this map does not include  $\mathbf{I}_3$  in the chart domain  $U$ , and so that is kind of sucky. The fact of the matter is, we will never use coordinates for  $SO(3)$ . It is far more convenient when dealing with  $SO(3)$  to proceed as we did in Chapter 1 with rigid body dynamics.  $\square$

If  $Q$  is a configuration space with  $(U, \phi)$  a coordinate chart, we will often pretend that  $U_\phi = \phi(U) \subset \mathbb{R}^n$  is the configuration manifold. But note that this is just pretend. But when we do this, we will often write a point  $q \in Q$  by its image under  $\phi$  that we might denote  $\phi(q) = (q^1, \dots, q^n)$ . Thus  $(q^1, \dots, q^n)$  are **coordinates** in the coordinate chart  $(U, \phi)$ . We will think of functions on  $Q$  as being functions of the real variables  $(q^1, \dots, q^n)$ . In this way we reduce ourselves to something with which we are familiar. Note that we put *superscripts* on our coordinates! That is,  $q^2$  is the second coordinate, not the square of  $q$ . We do this because tradition dictates that this is what we should do. Other objects we encounter will have subscripts as labels, and one can ascertain the character of something by looking at where its indices are located. But we will get to this in time.

The idea of Definition 2.1.2 is that one establishes an exact correspondence of a subset of one's configuration space with something one can deal with, an open subset of Euclidean space. In doing so, one makes it so that one can deal with these portions of the configuration space in a way with which one is familiar. Everything seems okay, but the problem is that it is generally not possible to find a chart  $(U, \phi)$  where  $U = Q$ . Thus to cover the entire configuration space we may need two or more charts. Generally, these charts may overlap. Thus we may have a situation where a subset of the configuration space may have two or more coordinate charts describing it. These coordinate charts should have some properties relative to one another.

**2.1.4 Definition** Let  $Q$  be a configuration space for a mechanical system and let  $(U_1, \phi_1)$  and  $(U_2, \phi_2)$  be coordinate charts for  $Q$ . The coordinate charts are **compatible** if the map

$$\begin{aligned}\phi_{12}: \phi_1(U_1 \cap U_2) &\rightarrow \phi_2(U_1 \cap U_2) \\ \mathbf{x} &\mapsto \phi_2 \circ \phi_1^{-1}(\mathbf{x})\end{aligned}$$

satisfies

- (i)  $\phi_{12}$  is a bijection,
- (ii)  $\phi_{12}$  is differentiable, and
- (iii)  $\phi_{12}^{-1}$  (that exists by (i)) is differentiable.

If there exists compatible coordinate charts  $(U_1, \phi_1), \dots, (U_k, \phi_k)$  so that  $U_1 \cup \dots \cup U_k = Q$  and so that all maps  $\phi_1, \dots, \phi_k$  take their values in  $\mathbb{R}^n$ , then the **dimension** of  $Q$  is  $n$ . A mechanical system with an  $n$ -dimensional configuration space has  $n$  **degrees of freedom**.  $\square$

In the following, we shall consider configuration spaces with a well-defined dimension, and we will reserve the letter  $n$  to represent the dimension of an arbitrary configuration space. Thus, below, whenever you see an  $n$ , it will be the dimension of the configuration manifold that is lying about at the time.

A picture to have in mind with respect to this definition is provided in Figure 2.3. I will

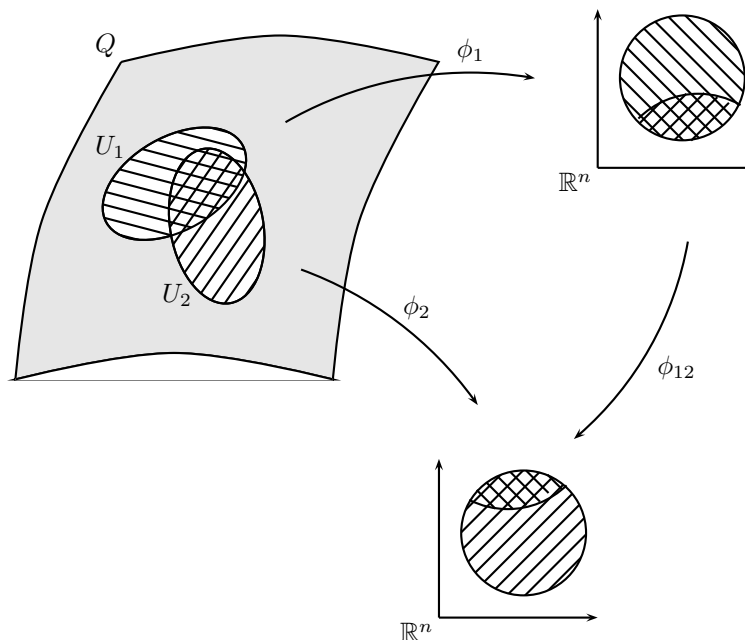


Figure 2.3 Compatibility of coordinate charts

grant that this seems complicated. But you are actually quite familiar with the content of this definition, but the formalism is perhaps intimidating. Perhaps a really simple example will make this clear.

**2.1.5 Example** We take  $Q = \mathbb{R}^2$ . Let us not be too concerned with how this represents a physical system, but rather let us think about  $Q$  as something quite abstract to which we will assign two coordinate charts.

Our first coordinate chart is the “obvious” one. We define a chart  $(U_1, \phi_1)$  by  $U_1 = Q$  and  $\phi(x, y) = (x, y)$ . Note that we are able to choose a chart that covers all of  $Q$ . This is typically not the case.

Now we define  $(U_2, \phi_2)$  by

$$U_2 = \mathbb{R}^2 \setminus \{(x, y) \mid x \leq 0\}, \quad \phi_2(x, y) = (\sqrt{x^2 + y^2}, \text{atan}(x, y)).$$

The coordinate chart is illustrated in Figure 2.4. These coordinates are, of course, the usual

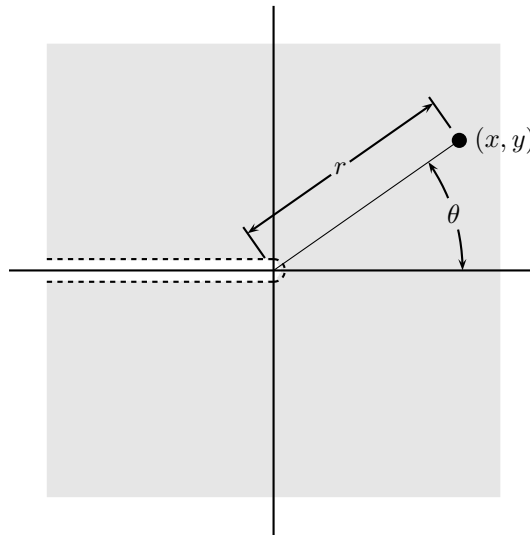


Figure 2.4 Polar coordinates for  $Q = \mathbb{R}^2$

polar coordinates you are all familiar with. We are simply being more careful with the description.

Let us now see whether these two coordinate charts are compatible in the sense of Definition 2.1.4. We have

$$\begin{aligned} \phi_1(U_1 \cap U_2) &= \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\} \\ \phi_2(U_1 \cap U_2) &= \{(r, \theta) \mid r > 0, \theta \in (-\pi, \pi)\}. \end{aligned}$$

Note that we use coordinates  $(x, y)$  for  $\phi_1(U_1 \cap U_2)$  and coordinates  $(r, \theta)$  for  $\phi_2(U_1 \cap U_2)$ . The map  $\phi_{12}$  is then computed to be

$$\phi_{12}(x, y) = (\sqrt{x^2 + y^2}, \text{atan}(x, y)).$$

This map is certainly one-to-one and onto—it establishes a correspondence between points in  $\phi_1(U_1 \cap U_2)$  and points in  $\phi_2(U_1 \cap U_2)$ . Thus  $\phi_{12}$  satisfies condition (i) of Definition 2.1.4. Keeping in mind that the domain of  $\phi_{12}$  is  $\phi_1(U_1 \cap U_2)$ , we also see that  $\phi_{12}$  satisfies condition (ii) of Definition 2.1.4 since its Jacobian is

$$D\phi_{12}(x, y) = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{bmatrix}.$$

This matrix has continuous entries, so  $\phi_{12}$  is differentiable. The inverse of  $\phi_{12}$  is readily determined to be

$$\phi_{12}^{-1}(r, \theta) = (r \cos \theta, r \sin \theta),$$

so that

$$D\phi_{12}^{-1}(r, \theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix}.$$

Since this matrix has continuous entries,  $\phi_{12}^{-1}$  is differentiable, and so  $\phi_{12}$  satisfies condition (iii) of Definition 2.1.4.

All of the above shows that the two charts  $(U_1, \phi_1)$  and  $(U_2, \phi_2)$  are compatible.  $\square$

With Definition 2.1.4 we are converging to the proper object to describe the configuration space of a mechanical system: a differentiable manifold. However, having gotten this close, we are going to back off and generally work in a fixed coordinate chart for  $Q$ , and ignore the fact that it may not cover all of  $Q$ . At various times, however, we will point out the effects of demanding coordinate charts that are compatible.

**2.1.3 Functions and curves** One can talk of all manner of objects defined on a configuration space once one has the idea of coordinate charts. We shall start by talking about some of the simpler notions, those of functions and curves.

Let  $Q$  be a configuration manifold. A function  $f: Q \rightarrow \mathbb{R}$  is **differentiable** if for each  $q \in Q$  and for each coordinate chart  $(U, \phi)$  with  $q \in U$ , the function  $\mathbf{f}_\phi: \mathbf{U}_\phi \rightarrow \mathbb{R}$  defined by

$$\mathbf{f}_\phi(\mathbf{q}) = f(\phi^{-1}(\mathbf{q}))$$

is differentiable. The idea is very simple, of course. To test differentiability of a function on  $Q$ , one simply tests it in a coordinate chart. One can easily check that one does not have to check the condition for *every* chart, but that it suffices to find one for each point. Also, although our definition is one for differentiability, one can check for any order of differentiability, including simple continuity by our definition.

Now let us talk about curves. A **curve** is a map  $c: I \rightarrow Q$  from an interval  $I \subset \mathbb{R}$  with the property that for each  $t_0 \in I$  and for every coordinate chart  $(U, \phi)$  with  $c(t_0) \in U$ , the map  $\mathbf{c}_\phi: \mathbf{I}_\phi \rightarrow \mathbf{U}_\phi$  defined by  $\mathbf{c}_\phi(t) = c(\phi(t))$  is differentiable. Here  $\mathbf{I}_\phi \subset I$  is an interval with the property that  $t_0 \in \mathbf{I}_\phi$  and  $c(t) \in U$  for every  $t \in \mathbf{I}_\phi$ .

When representing both functions and curves in a single given coordinate chart, we will often engage in an abuse of notation and write a function as  $f(q^1, \dots, q^n)$  and a curve as  $t \mapsto (q^1(t), \dots, q^n(t))$ . The abuse of notation is that we are pretending that the single coordinate chart suffices to define the function and the curve, and this may not be the case.

## 2.2 Vector fields, one-forms, and Riemannian metrics

Lagrangian mechanics involves certain objects that transform in certain ways when one changes from a coordinate chart to another compatible one. There is a theory behind how one talks about such objects, and in this section we give a very cursory introduction to such things. For a more thorough account, we refer to [Abraham, Marsden, and Ratiu 1988]. Although there is some significant formalism to what we do here, it is entirely probable that the material we cover in this section is familiar, and we shall try as much as possible to make contact with more commonplace notions.

**2.2.1 Tangent vectors, tangent spaces, and the tangent bundle** After talking about configuration spaces, the next thing we must do is talk sensibly about “velocities” on configuration spaces. Again, things are complicated here by the fact that a configuration

space is not a nice Euclidean space, and so our presentation of velocities relies on the use of coordinate charts.

Let  $Q$  be a configuration space with  $(U_1, \phi_1)$  a coordinate chart for  $Q$  and  $q \in U_1$ . A  $\phi_1$ -**tangent vector at  $q$**  is an element  $X_{\phi_1} \in \mathbb{R}^n$ . This is too boring. What happens if we have another coordinate chart  $(U_2, \phi_2)$  where  $q \in U_2$ ? Well, we can certainly talk about a  $\phi_2$ -tangent vector  $X_{\phi_2}$  at  $q$ . What we want is a way to say that a  $\phi_1$ -tangent vector and a  $\phi_2$ -tangent vector are represent “the same thing.” To motivate how to do this, let us consider a curve  $c: [-\epsilon, \epsilon] \rightarrow Q$  with the property that  $c(0) = q$ . Suppose that the curve in the chart  $(U_1, \phi_1)$  is represented by  $t \mapsto (q^1(t), \dots, q^n(t))$  and that the curve in the chart  $(U_2, \phi_2)$  is represented by  $t \mapsto (\tilde{q}^1(t), \dots, \tilde{q}^n(t))$ . We wish to talk about the “velocity” of the curve  $c$  as it passes through  $q = c(0)$ . In the chart  $(U_1, \phi_1)$  this velocity is represented by

$$(\dot{q}^1(0), \dots, \dot{q}^n(0)) \in \mathbb{R}^n,$$

and in the chart  $(U_2, \phi_2)$  it is represented by

$$(\dot{\tilde{q}}^1(0), \dots, \dot{\tilde{q}}^n(0)) \in \mathbb{R}^n.$$

However, the chain rule dictates that

$$\dot{\tilde{q}}^i(0) = \sum_{j=1}^n \frac{\partial \tilde{q}^i}{\partial q^j}(\phi_1(q)) \dot{q}^j(0), \quad i = 1, \dots, n. \quad (2.4)$$

Before proceeding with our discussion proper, let us introduce an important notational convention. We shall write

$$\sum_{j=1}^n \frac{\partial \tilde{q}^i}{\partial q^j}(\phi_1(q)) \dot{q}^j(0) = \frac{\partial \tilde{q}^i}{\partial q^j}(\phi_1(q)) \dot{q}^j(0),$$

thus omitting the summation sign. The idea is that whenever you see a repeated index, one of which is a superscript and the other of which is a subscript, then summation will be implied, unless otherwise stated. Note that the  $j$  in  $\frac{\partial \tilde{q}^i}{\partial q^j}$  is a superscript in the denominator; these are regarded as subscripts! The details regarding the rules of the summation convention are included in Appendix A. Let us now proceed. From (2.4) we see that if we want a  $\phi_1$ -tangent vector  $X_{\phi_1}$  and a  $\phi_2$ -tangent vector  $X_{\phi_2}$  to be “the same,” we should require that

$$X_{\phi_2}^i = \frac{\partial \tilde{q}^i}{\partial q^j}(\phi_1(q)) X_{\phi_1}^j, \quad i = 1, \dots, n.$$

We note that the matrix with components  $\frac{\partial \tilde{q}^i}{\partial q^j}$ ,  $i, j = 1, \dots, n$ , is none other than the Jacobian matrix  $\mathbf{D}\phi_{12}(\phi_1(q))$ . Thus we say that  $X_{\phi_1}$  and  $X_{\phi_2}$  are **equivalent** if  $X_{\phi_2} = \mathbf{D}\phi_{12}(\phi_1(q)) \cdot X_{\phi_1}$ .

We may now formally define what we mean by a tangent vector.

**2.2.1 Definition** Let  $Q$  be a configuration space with  $q \in U$ . A **tangent vector at  $q$**  is a pair  $(X, S)$  where  $X \in \mathbb{R}^n$  is a  $\phi$ -tangent vector at  $q$  for some chart  $(U, \phi)$  and

$$S = \{(\tilde{X}, (\tilde{U}, \tilde{\phi})) \mid \tilde{X} \text{ is a } \tilde{\phi}\text{-tangent vector at } q \text{ that is equivalent to } X \text{ for some chart } (\tilde{U}, \tilde{\phi})\}.$$

The collection of all tangent vectors at  $q$  is denoted  $T_qQ$  and is called the **tangent space at  $q$** . The collection

$$\bigcup_{q \in Q} T_qQ$$

of all tangent spaces is called the **tangent bundle** and denoted  $TQ$ .  $\square$

Having stated this formal definition of a tangent vector, we will almost never use it. We will often refer to a tangent vector by writing it in a specific set of coordinates. Indeed, even its formal definition relies on an initial choice of coordinate chart for its definition. Some notation for writing tangent vectors is convenient. Let  $(U, \phi)$  be a coordinate chart with coordinates  $(q^1, \dots, q^n)$ . For each  $q \in U$  there are  $n$  distinguished tangent vectors given by

$$X_i = \mathbf{e}_i, \quad i = 1, \dots, n,$$

where  $\mathbf{e}_i$  is the  $i$ th standard basis vector for  $\mathbb{R}^n$ . We shall write  $X_i = \frac{\partial}{\partial q^i} \Big|_q$ . This may seem like strange notation, but as we shall see, it is *extremely* convenient. It is then possible to write any tangent vector  $X$  at  $q$  as

$$X = X^i \frac{\partial}{\partial q^i}$$

for some  $X^1, \dots, X^n \in \mathbb{R}$  called the **components** of  $X$ . Thus the tangent vectors  $\{\frac{\partial}{\partial q^1} \Big|_q, \dots, \frac{\partial}{\partial q^n} \Big|_q\}$  form a basis for the tangent space  $T_qQ$ .

Let us see how our basis vector are changed by changing coordinate charts. In the following result we let  $(q^1, \dots, q^n)$  be coordinates in a chart  $(U_1, \phi_1)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  be coordinates in a chart  $(U_2, \phi_2)$ .

2.2.2 Lemma  $\frac{\partial}{\partial q^i} \Big|_q = \frac{\partial \tilde{q}^j}{\partial q^i}(\phi_1(q)) \frac{\partial}{\partial \tilde{q}^j} \Big|_q, i = 1, \dots, n.$

*Proof* Let us work only in the intersection of the chart domains, and so take  $U = U_1 \cap U_2$ . At  $q \in U$  we have the tangent vector  $\frac{\partial}{\partial q^i} \Big|_q$  for fixed  $i \in \{1, \dots, n\}$ . We also have the basis of tangent vectors  $\{\frac{\partial}{\partial \tilde{q}^1} \Big|_q, \dots, \frac{\partial}{\partial \tilde{q}^n} \Big|_q\}$ . We can then write

$$\frac{\partial}{\partial q^i} \Big|_q = \xi^j \frac{\partial}{\partial \tilde{q}^j} \Big|_q$$

for a collection of numbers  $\xi^1, \dots, \xi^n$  on  $U$ . The tangent vectors on the left and right side of the equation are equivalent. This means that

$$\mathbf{D}\phi_{12}(\phi_1(q)) \left( \frac{\partial}{\partial q^i} \Big|_q \right) = \xi^j(q) \frac{\partial}{\partial \tilde{q}^j} \Big|_q.$$

Now note that the matrix representing the Jacobian  $\mathbf{D}\phi_{12}(\phi_1(q))$  is exactly  $\frac{\partial \tilde{q}^i}{\partial q^k}$ ,  $i, k = 1, \dots, n$ . To apply the Jacobian to  $\frac{\partial}{\partial q^i}$ , we apply the matrix  $\frac{\partial \tilde{q}^i}{\partial q^k}$  to the components of the vector field  $\frac{\partial}{\partial q^i}$ . But the components of this vector field are

$$\delta_i^k = \begin{cases} 1, & i = k \\ 0, & i \neq k. \end{cases}$$

Therefore

$$\xi^j(q) = \frac{\partial \tilde{q}^j}{\partial q^k} \Big|_q \delta_i^k = \frac{\partial \tilde{q}^j}{\partial q^i} \Big|_q,$$

which is as we claimed.  $\blacksquare$

Note that this result agrees with what the notation looks like! That is to say, if you were to treat the partial derivative notation in terms of differential operators rather than vector fields, it would look just like as stated in the lemma.

Let's see how this plays out in a simple example.

**2.2.3 Example** (Example 2.1.5 cont'd) Recall that we had defined two coordinate charts on  $Q = \mathbb{R}^2$ ;  $(U_1, \phi_1)$  were the standard Cartesian coordinates, and  $(U_2, \phi_2)$  were polar coordinates. We shall define tangent vectors in each coordinate chart, and show that they are equivalent.

Let  $q = (x, y)$  so that  $\phi_1(q) = (x, y)$  and  $\phi_2(q) = (\sqrt{x^2 + y^2}, \text{atan}(x, y))$ . We define the  $\phi_1$ -tangent vector at  $q$  by

$$X_{\phi_1} = -y \frac{\partial}{\partial x} \Big|_q + x \frac{\partial}{\partial y} \Big|_q,$$

and the  $\phi_2$ -tangent vector by

$$X_{\phi_2} = \frac{\partial}{\partial \theta} \Big|_q.$$

We shall show that these tangent vectors are equivalent. By Lemma 2.2.2 we have

$$\begin{aligned} \frac{\partial}{\partial r} \Big|_q &= \frac{\partial x}{\partial r} \frac{\partial}{\partial x} \Big|_q + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} \Big|_q \\ &= \cos \theta \frac{\partial}{\partial x} \Big|_q + \sin \theta \frac{\partial}{\partial y} \Big|_q \\ &= \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} \Big|_q + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y} \Big|_q, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \theta} \Big|_q &= \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} \Big|_q + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} \Big|_q \\ &= -r \sin \theta \frac{\partial}{\partial x} \Big|_q + r \cos \theta \frac{\partial}{\partial y} \Big|_q \\ &= -y \frac{\partial}{\partial x} \Big|_q + x \frac{\partial}{\partial y} \Big|_q. \end{aligned}$$

Therefore, we directly see that

$$X_{\phi_2} \frac{\partial}{\partial \theta} \Big|_q = -y \frac{\partial}{\partial x} \Big|_q + x \frac{\partial}{\partial y} \Big|_q = X_{\phi_1},$$

which means that the two tangent vectors are indeed equivalent. Understand, however, that  $X_{\phi_1}$  is defined on  $U_1$  which is all of  $Q$  in this example, while  $X_{\phi_2}$  is only defined on the strict subset  $U_2$  of  $Q$ .

Note that in Cartesian coordinates the vectors are tangent to circles, so it makes sense that in polar coordinates, the tangent vectors would have only a component in the  $\theta$  direction.  $\square$

From Lemma 2.2.2 it also follows that we may determine from one another the components of the same vector field represented in different coordinates.



**2.2.4 Lemma** *Let  $X$  be a tangent vector at  $q \in Q$  and let  $(q^1, \dots, q^n)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  be coordinates for  $Q$ . If  $X^1, \dots, X^n$  are the components for  $X$  in the coordinates  $(q^1, \dots, q^n)$ , then*

$$\frac{\partial \tilde{q}^1}{\partial q^{j_1}}(\phi_1(q))X^{j_1}, \dots, \frac{\partial \tilde{q}^n}{\partial q^{j_n}}(\phi_1(q))X^{j_n}$$

*are the components of  $X$  in the coordinates  $(\tilde{q}^1, \dots, \tilde{q}^n)$ .*

*Proof* Let  $\tilde{X}^1, \dots, \tilde{X}^n$  be the components of  $X$  in the coordinates  $(\tilde{q}^1, \dots, \tilde{q}^n)$ . We then have

$$X^i \frac{\partial}{\partial q^i} \Big|_q = \tilde{X}^j \frac{\partial}{\partial \tilde{q}^j} \Big|_q$$

on the intersection of the two chart domains. However, from Lemma 2.2.2 this means that

$$X^i \frac{\partial \tilde{q}^j}{\partial q^i}(\phi_1(q)) \frac{\partial}{\partial \tilde{q}^j} \Big|_q = \tilde{X}^j \frac{\partial}{\partial \tilde{q}^j} \Big|_q.$$

Since the vector fields  $\left\{ \frac{\partial}{\partial \tilde{q}^1} \Big|_q, \dots, \frac{\partial}{\partial \tilde{q}^n} \Big|_q \right\}$  are linearly independent, this means that  $\tilde{X}^j = \frac{\partial \tilde{q}^j}{\partial q^i} X^i$ , as claimed.  $\blacksquare$

Let us see how this works out in our polar coordinate example.

**2.2.5 Example** (Example 2.1.5 cont'd) Again we take  $(U_1, \phi_1)$  to be Cartesian coordinates for  $Q = \mathbb{R}^2$  and we take  $(U_2, \phi_2)$  to be polar coordinates for  $Q$ . Also again, we take the vector field  $X$  with the property that

$$X_{\phi_1} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad X_{\phi_2} = \frac{\partial}{\partial \theta}.$$

We take  $(x, y)$  as the coordinates  $(q^1, q^2)$  and  $(r, \theta)$  as the coordinates  $(\tilde{q}^1, \tilde{q}^2)$ . Thus the components of  $X$  in the coordinates  $(x, y)$  are  $(-y, x)$  and the components in the coordinates  $(r, \theta)$  are  $(0, 1)$ . Following Lemma 2.2.4 we have

$$\frac{\partial r}{\partial x}(-y) + \frac{\partial r}{\partial y}(x) = \frac{x}{\sqrt{x^2 + y^2}}(-y) + \frac{y}{\sqrt{x^2 + y^2}}(x) = 0,$$

and

$$\frac{\partial \theta}{\partial x}(-y) + \frac{\partial \theta}{\partial y}(x) = -\frac{y}{x^2 + y^2}(-y) + \frac{x}{x^2 + y^2}(x) = 1.$$

Things are as they should be.  $\square$

We will very often wish to think of  $TQ$ , the collection of all tangent vectors, as an object onto itself, with its own set of coordinates. To set about doing this, we let  $(U, \phi)$  be a coordinate chart for  $Q$  with coordinates  $(q^1, \dots, q^n)$ . Let us agree to write a typical tangent vector in  $T_q Q$  as  $v_q$ , thereby emphasising the fact that it is “attached” to the point  $q$ . For any  $q \in Q$  and any  $v_q \in T_q Q$ , we may write

$$v_q = v^i \frac{\partial}{\partial q^i} \Big|_q$$

for some  $v^1, \dots, v^n \in \mathbb{R}$ . In this way, for any  $q \in Q$  and any  $v_q \in T_q Q$ , we can assign a unique set of coordinates  $(q^1, \dots, q^n, v^1, \dots, v^n)$ . We call these the **natural coordinates** for  $TQ$

associated with the coordinate chart  $(U, \phi)$ . Thus  $TQ$ , the set of all configurations and all velocities, has dimension  $2n$ . If we have an overlapping set of coordinates  $(\tilde{q}^1, \dots, \tilde{q}^n)$  then there is a similarly induced set of coordinates  $(\tilde{q}^1, \dots, \tilde{q}^n, \tilde{v}^1, \dots, \tilde{v}^n)$  for  $TQ$ . Quite clearly, by Lemma 2.2.4, the velocity coordinates are related by

$$\tilde{v}^i = \frac{\partial \tilde{q}^i}{\partial q^j} v^j, \quad i = 1, \dots, n. \quad (2.5)$$

Note that we wish to think of velocity as being an *independent* coordinate, and so shall steer away from using the symbols  $(\dot{q}^1, \dots, \dot{q}^n)$  to denote velocity, unless we actually are dealing with a curve  $t \mapsto (q^1(t), \dots, q^n(t))$ , in which case  $(\dot{q}^1(t), \dots, \dot{q}^n(t))$  have their usual meaning as time derivatives. To denote the velocity coordinates we shall use the symbols  $(v^1, \dots, v^n)$ . Note, however, that this is not entirely standard notation, although we feel it to be superior to using  $\dot{q}$  when  $\dot{q}$  is not really what is intended.

Let us see how this notation plays out for our running example.

**2.2.6 Example** (Example 2.1.5 cont'd) We have  $Q = \mathbb{R}^2$  with  $(U_1, \phi_1)$  the Cartesian coordinate chart and  $(U_2, \phi_2)$  the polar coordinate chart. Thus we have coordinates  $(x, y, v_x, v_y)$  and  $(r, \theta, v_r, v_\theta)$  for  $TQ$ . To see how these are related, we use (2.5). Thus we have

$$\begin{aligned} v_r &= \frac{\partial r}{\partial x} v_x + \frac{\partial r}{\partial y} v_y = \frac{x}{\sqrt{x^2 + y^2}} v_x + \frac{y}{\sqrt{x^2 + y^2}} v_y \\ v_\theta &= \frac{\partial \theta}{\partial x} v_x + \frac{\partial \theta}{\partial y} v_y = -\frac{y}{x^2 + y^2} v_x + \frac{x}{x^2 + y^2} v_y. \end{aligned}$$

Notice that this is just as velocities should behave in that we would also compute

$$\begin{aligned} \dot{r} &= \frac{x}{\sqrt{x^2 + y^2}} \dot{x} + \frac{y}{\sqrt{x^2 + y^2}} \dot{y} \\ \dot{\theta} &= -\frac{y}{x^2 + y^2} \dot{x} + \frac{x}{x^2 + y^2} \dot{y}. \end{aligned}$$

It generally does not hurt to think of  $v^i$  as being like  $\dot{q}^i$ , at least heuristically. However, when one wishes to be precise, this notation often betrays us.  $\square$

**2.2.2 Vector fields** Vector fields are extremely useful objects in many fields of applied mathematics, and they possess many surprising properties that may be used to answer all manner of interesting questions. We will get some exposure to the utility of the vector field in our discussion of mechanics. There are at least two ways in which one can think of a vector field, and we shall start with that suggested by the words “vector field.”

**Vector fields as...well...fields of vectors** Let  $Q$  be a configuration space. A *vector field* on  $Q$  is a map  $X: Q \rightarrow TQ$  with the property that  $X(q) \in T_qQ$ . The idea of a vector field is one with which you are undoubtedly familiar. The idea is that  $X$  assigns to each point  $q \in Q$  a point in the tangent space  $T_qQ$ , and this is to be thought of as a vector anchored at the point  $q$  (see Figure 2.5).

Let us see how we represent vector fields in coordinates. Choose a chart  $(U, \phi)$  for  $Q$  with coordinates  $(q^1, \dots, q^n)$ . The vectors  $\left\{ \frac{\partial}{\partial q^1} \Big|_q, \dots, \frac{\partial}{\partial q^n} \Big|_q \right\}$  form a basis for  $T_qQ$ , and let

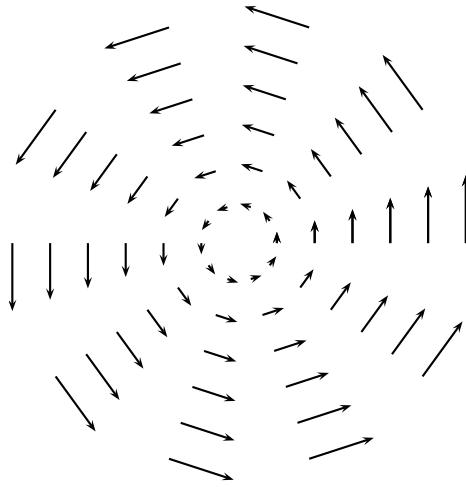


Figure 2.5 A planar vector field

us use these to define  $n$  vector fields  $\{\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\}$  on the subset  $U$  of  $Q$ . Then any vector field can be written as

$$X(q) = X^i(q) \frac{\partial}{\partial q^i}$$

for functions  $X^1, \dots, X^n$  on  $Q$ . Following what we did with tangent vectors, these are the **components** of  $X$  in the given coordinate chart. Of course, the components of a vector field obey the same transformation properties that tangent vectors obey when we change coordinate charts. Note that we will always ask that the components of a vector field be at least continuously differentiable. If the components are not, then we will generally disallow the resulting object from being a vector field.

**2.2.7 Example** (Example 2.1.5 cont'd) In Example 2.2.3 we looked at tangent vectors on  $Q = \mathbb{R}^2$  in Cartesian coordinates and polar coordinates. These may be regarded as vector fields merely by a change of notation. Thus we have a vector field  $X$  on  $Q$  that is represented as

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

in Cartesian coordinates, and by

$$X = \frac{\partial}{\partial \theta}$$

in polar coordinates. The Cartesian coordinate version of this vector field is the one depicted in Figure 2.5.  $\square$

**Vector fields and ordinary differential equations** In the above discussion of vectors, we were merely thinking of vectors as geometric objects that, by definition, obeyed certain properties with respect to coordinate changes. It turns out that vector fields also are in some sense equivalent to something you already know about: ordinary differential equations.

To make this connection, let  $X$  be a vector field on  $Q$ . A differentiable curve  $c: [0, T] \rightarrow Q$  is an **integral curve** for  $X$  if for every set of coordinates  $(q^1, \dots, q^n)$  for  $Q$  in which the

curve  $c$  has the representation  $t \mapsto (q^1(t), \dots, q^n(t))$ , the curve  $c$  satisfies

$$\begin{aligned}\dot{q}^1(t) &= X^1(q^1(t), \dots, q^n(t)) \\ &\vdots \\ \dot{q}^n(t) &= X^n(q^1(t), \dots, q^n(t)).\end{aligned}$$

Note therefore, that an integral curve for  $X$  is determined by its being a solution of an ordinary differential equation in every coordinate chart, with the right-hand side vector of the ordinary differential equation being the components of the vector field in that set of coordinates.

You may be used to trying to solve differential equations when you see them. This is an almost always futile exercise. It works for differential equations that are linear (but see Exercise E2.16), and for a small class of other examples, but when faced with a randomly selected differential equation, you are simply not going to be able to solve it. And even when you can get a closed form solution, it is often not entirely helpful. For example, one can solve the nonlinear pendulum equation in terms of elliptic functions. But so what? Unless you know a lot about elliptic functions, then you are not going to know very much about the behaviour of a pendulum by looking at its closed form solution. So how should we talk about differential equations in a sensible manner? The answer lies in trading off a *quantitative* for a *qualitative* description of the dynamics, and to do this, one needs some concepts that may be new. We will not be entirely thorough in our introduction to the qualitative aspects of ordinary equations as they relate to vector fields.

Let  $X$  be a vector field on  $Q$  and let  $q_0 \in Q$ . If we are working in a coordinate chart  $(U, \phi)$  for which  $q \in U$  then, if  $\mathbf{q}_0 = \phi(q_0) \in \mathbb{R}^n$ , we have a unique solution to the initial value problem

$$\begin{aligned}\dot{q}^1(t) &= X^1(q^1(t), \dots, q^n(t)), & q^1(0) &= q_0^1 \\ &\vdots \\ \dot{q}^n(t) &= X^n(q^1(t), \dots, q^n(t)), & q^n(0) &= q_0^n.\end{aligned}$$

This implies that there is a unique integral curve  $c$  with the property that  $c(0) = q_0$ . Now, this integral curve may not be extensible for all time, so we let

$$T(X, q_0) = \sup_{T \in \mathbb{R}_+} \{ \text{there exists an integral curve of } X \text{ through } q_0 \text{ defined on } [0, T] \}.$$

For many well-behaved systems,  $T(X, q_0) = \infty$ , but there are some innocuous systems for which  $T(X, q_0)$  is finite.<sup>1</sup> Now we define

$$D(X) = \{ (t, q) \in \mathbb{R}_+ \times Q \mid t < T(X, q) \}.$$

Thus for  $(t, q) \in D(X)$  there is an integral curve through  $q$  that can be defined for at least time  $t$ . The **flow** of  $X$  is the map  $F_X: D(X) \rightarrow Q$  defined by  $F_X(t, q_0) = q$  if there exists an integral curve  $c$  with  $c(0) = q_0$  and  $c(t) = q$ . The idea is that  $F_X(t, q_0)$  is where you end up when you start at  $q_0$  at time 0 and go along with the solution of the differential equation for time  $t$ .

Let us work this out for our example vector field.

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<sup>1</sup>If you have had a decent differential equations course, you will know, for example, that the solutions to the differential equation  $\dot{x} = x^2$  exist only for a finite time.

2.2.8 Example (Example 2.1.5 cont'd) We again have the two coordinate charts  $(U_1, \phi_1)$  and  $(U_2, \phi_2)$  that have their previous meanings. We also have the vector field  $X$  that in the coordinate chart  $(U_1, \phi_1)$  is given by

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y},$$

and in the coordinate chart  $(U_2, \phi_2)$  is given by

$$X = \frac{\partial}{\partial \theta}.$$

The differential equation in the coordinate chart  $(U_1, \phi_1)$  is then

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x, \end{aligned}$$

which is linear, and for the initial conditions  $(x(0), y(0))$  has the solution

$$(x(t), y(t)) = (x(0) \cos t - y(0) \sin t, x(0) \sin t + y(0) \cos t).$$

Note that solutions exist for all time, so for each  $(x, y) \in Q$  we have  $T(X, (x, y)) = \infty$ . The flow is then defined by

$$F_X(t, (x, y)) = (x \cos t - y \sin t, x \sin t + y \cos t).$$

Let us look at the same thing in the other coordinate chart. The differential equations in this coordinate chart are

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\theta} &= 1. \end{aligned}$$

Note that these equations are no longer linear, but are trivially solved for the initial condition  $(r(0), \theta(0))$  as

$$(r(t), \theta(t)) = (r(0), \theta(0) + t).$$

One might be tempted to say that from this one may deduce that  $T(X, (r(0), \theta(0))) = \infty$ , but be careful, because this coordinate chart does not cover all of  $Q$ . To in fact verify that all integral curves may be extended for all time, one actually needs to find another chart to extend the integral curves. But we do not need to do this here since we already know from our use of the coordinate chart  $(U_1, \phi_1)$  that we may extend all integral curves for infinite time.  $\square$

**2.2.3 One-forms** As we have seen, a vector field on  $Q$  is an object defined in a coordinate chart that has  $n$  components if  $n$  is the dimension of  $Q$ . We shall shortly see that a “one-form” also has this exact property. This will confuse a newcomer, and many a long-time practitioner has only a shady understanding of the difference between a one-form and a vector field. The fact of the matter is that they live in entirely different spaces. Thus, to get things started, let us look at something that you may be tempted to call a vector field, and show that it is actually *not* a vector field. Let  $f: Q \rightarrow \mathbb{R}$  be a differentiable function and let  $(U, \phi)$  be a coordinate chart for  $Q$ . We then have the “gradient” of  $f$  that we denote

by  $df$ . Let us agree not to call this the gradient, but the **differential** of  $f$ . Whatever we call it, it is certainly an object with  $n$  components, and these components are none other than

$$\frac{\partial f}{\partial q^1}, \dots, \frac{\partial f}{\partial q^n},$$

if  $(q^1, \dots, q^n)$  are coordinates in the chart  $(U, \phi)$ . Is this a vector field? To determine the answer to this question, we suppose that we have another set of overlapping coordinates  $(\tilde{q}^1, \dots, \tilde{q}^n)$  so that the components of the abstract object  $df$  in these new coordinates are

$$\frac{\partial f}{\partial \tilde{q}^1}, \dots, \frac{\partial f}{\partial \tilde{q}^n}$$

To relate the two different coordinate expressions, we use the chain rule:

$$\frac{\partial f}{\partial \tilde{q}^i} = \frac{\partial q^j}{\partial \tilde{q}^i} \frac{\partial f}{\partial q^j}.$$

But this is *not* how the components of a vector field should transform with respect to coordinate changes, cf. Lemma 2.2.4. Hold on a minute here! The components of a vector field in our indexing convention are supposed to be superscripts. But the index in  $\frac{\partial f}{\partial q^i}$  is a subscript (a superscript in the denominator). So things are really not looking good for  $df$  being a vector field. So what is it? It is a one-form!

Now let's get serious.

Recall from Appendix A that the dual space  $V^*$  to a vector space  $V$  is the collection of linear maps from  $V$  to  $\mathbb{R}$ . The dual space is a vector space with the same dimension as  $V$ . It turns out that the tangent spaces  $T_q Q$  to a configuration space  $Q$  are themselves vector spaces. This is clear in coordinates. Let  $(U, \phi)$  be a coordinate chart with coordinates  $(q^1, \dots, q^n)$  for  $Q$  and coordinates  $(q^1, \dots, q^n, v^1, \dots, v^n)$  for  $TQ$ . Then for fixed  $q_0 \in Q$  with  $(q_0^1, \dots, q_0^n) = \phi(q_0)$ , the tangent space  $T_{q_0} Q$  is represented by

$$\{(q_0^1, \dots, q_0^n, v^1, \dots, v^n) \mid (v^1, \dots, v^n)\}.$$

The vector space structure on  $T_q Q$  is inherited by the usual vector addition and scalar multiplication on  $\mathbb{R}^n$ .

With this background, let us look at the dual space to  $T_q Q$ .

**2.2.9 Definition** The dual space to  $T_q Q$  we denote by  $T_q^* Q$  and call the **cotangent space**. Following what we did with tangent vectors, let us agree to write a typical point in the cotangent space  $T_q^* Q$  as  $\alpha_q$ . An element  $\alpha_q \in T_q^* Q$  is a **covector at  $q$** . The collection

$$\bigcup_{q \in Q} T_q^* Q$$

of all cotangent spaces is called the **cotangent bundle** of  $Q$  and denoted  $T^* Q$ . □

Now let us give some coordinate notation for covectors to match that for tangent vectors. Let  $(U, \phi)$  be a coordinate chart for  $Q$  with coordinates  $(q^1, \dots, q^n)$ . There are then  $n$  distinguished covectors at  $q \in Q$  defined by

$$q \mapsto e^i, \quad i = 1, \dots, n,$$

where  $\{e^1, \dots, e^n\}$  is the basis for  $(\mathbb{R}^n)^*$  that is dual to the standard basis. Let us denote by  $dq^i|_q$ ,  $i = 1, \dots, n$ , these  $n$  covectors. As with tangent vectors, this notation is strange but convenient. Given an arbitrary covector  $\alpha_q \in T_q^*Q$ , we may write

$$\alpha_q = \alpha_i dq^i|_q$$

for some  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  called the **components** of  $\alpha_q$ . Note that we are still using the summation convention, but things for covectors look different from what we have for tangent vectors. The indices for the basis covectors are superscripts (they were subscripts for the basis tangent vectors) and the indices for the components of a covector are subscripts (they were superscripts for the components of a tangent vectors).

Now let us observe how the basis covectors and their components react to changes of coordinate. In the following lemma, we let  $(q^1, \dots, q^n)$  be coordinates in a chart  $(U_1, \phi_1)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  be coordinates in a chart  $(U_2, \phi_2)$ .

**2.2.10 Lemma**  $dq^i|_q = \frac{\partial q^i}{\partial \tilde{q}^j}(\phi_2(q)) d\tilde{q}^j|_q$ ,  $i = 1, \dots, n$ .

*Proof* One the intersection  $U = U_1 \cap U_2$  of the chart domains we have

$$\eta_j d\tilde{q}^j|_q = dq^i|_q$$

for some numbers  $\eta_1, \dots, \eta_n$ . Let us apply both sides of this equation to the tangent vector  $\frac{\partial}{\partial \tilde{q}^k}|_q$ :

$$\eta_k = dq^i|_q \left( \frac{\partial}{\partial \tilde{q}^k} \Big|_q \right) = dq^i|_q \left( \frac{\partial q^i}{\partial \tilde{q}^k}(\phi_2(q)) \frac{\partial}{\partial \tilde{q}^k} \Big|_q \right) = \frac{\partial q^i}{\partial \tilde{q}^k}. \quad \blacksquare$$

Now, it should be clear that the components of a covector transform as follows.

**2.2.11 Lemma** Let  $\alpha$  be a one-form on  $Q$  and let  $(q^1, \dots, q^n)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  be coordinates for  $Q$ . If  $\alpha_1, \dots, \alpha_n$  are the components for  $\alpha$  in the coordinates  $(q^1, \dots, q^n)$ , then

$$\frac{\partial q^{j_1}}{\partial \tilde{q}^1} \alpha_{j_1}, \dots, \frac{\partial q^{j_n}}{\partial \tilde{q}^n} \alpha_{j_n}$$

are the components of  $\alpha$  in the coordinates  $(\tilde{q}^1, \dots, \tilde{q}^n)$ .

Now that we understand what we mean by a covector, we can easily define a **one-form** on  $Q$  to be a map  $\alpha: Q \rightarrow T_q^*Q$  with the property that  $\alpha(q) \in T_q^*Q$ . Just as we did with vector fields, we can write a one-form in coordinates  $(q^1, \dots, q^n)$  as

$$\alpha = \alpha_i dq^i$$

where the functions  $\alpha^1, \dots, \alpha^n$  are the **components** of  $\alpha$ . As with vector fields, we require that the components of a one-form be at least continuously differentiable when represented in coordinates.

Let us look at an example.

**2.2.12 Example** (Example 2.1.5 cont'd) As usual in our running example, we have  $Q = \mathbb{R}^2$  with two coordinate charts  $(U_1, \phi_1)$  and  $(U_2, \phi_2)$ . First let's see how the basis one-forms in each coordinate chart are related. We have, by Lemma 2.2.10,

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos \theta dr - r \sin \theta d\theta,$$

and

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin \theta dr + r \cos \theta d\theta.$$

Now let us consider a one-form that is the differential of a function. We take as our function that defined in the coordinate chart  $(U_1, \phi_1)$  to be  $f(x, y) = \frac{1}{2}(x^2 + y^2)$ . In the chart  $(U_1, \phi_1)$  we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = x dx + y dy.$$

Note that  $dx$  and  $dy$  are not silly “infinitesimals,” but are linearly independent one-forms!

In polar coordinates we have  $f(r, \theta) = \frac{1}{2}r^2$ , so

$$df = r dr.$$

These components ought to be related as by Lemma 2.2.11, so let us check this directly. We compute

$$\frac{\partial x}{\partial r}(x) + \frac{\partial y}{\partial r}(y) = \cos \theta (r \cos \theta) + \sin \theta (r \sin \theta) = r$$

and

$$\frac{\partial x}{\partial \theta}(x) + \frac{\partial y}{\partial \theta}(y) = -r \sin \theta (\cos \theta) + r \cos \theta (\sin \theta) = 0,$$

and this is as it should be. □

Note that one-forms “eat” vector fields. That is, if we have a vector field  $X$  and a one-form  $\alpha$ , then we have a function on  $Q$  defined by

$$Q \ni q \mapsto \alpha(q)(X(q)) \in \mathbb{R}.$$

We denote this function by  $\langle \alpha; X \rangle$ , or  $\alpha(X)$ , or  $\alpha \cdot X$ , whichever pleases us. This should not be confusing since whenever you see a one-form and a vector field sitting beside one another, about the only thing they can do is combine to give a function!

**2.2.13 Example** (Example 2.1.5 cont'd) We have the vector field  $X$  defined in Cartesian coordinates by

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y},$$

and the one-form defined in Cartesian coordinates by

$$df = x dx + y dy.$$

In polar coordinates we have

$$X = \frac{\partial}{\partial \theta}, \quad df = r dr.$$

Thus, in Cartesian coordinate we compute

$$\langle df; X \rangle = x(-y) + y(x) = 0,$$

and similarly, in polar coordinates we have

$$\langle df; X \rangle = r(0) + 0(1) = 0.$$

Thus when we feed  $X$  to the one-form  $df$  we get zero in this case. In this case we can interpret this as follows: *The function  $f$  does not change in the direction of the vector field  $X$ .* □



Just as we can think of the tangent bundle  $TQ$  as an object having its own sets of coordinates, we can think of the cotangent bundle  $T^*Q$  as such an object as well. The way we do so is entirely analogous with how this is done in the tangent bundle case. Given a coordinate chart  $(U, \phi)$  for  $Q$  with coordinates  $(q^1, \dots, q^n)$ , and given  $q \in U$ , we can write a typical element  $\alpha_q \in T_q^*Q$  as

$$\alpha_q = p_i dq^i|_q$$

for some  $p_1, \dots, p_n \in \mathbb{R}$ . We think of  $(q^1, \dots, q^n, p_1, \dots, p_n)$  as coordinates for  $T^*Q$ , which we call the **natural coordinates** associated with the coordinate chart  $(U, \phi)$ . The coordinates for the “one-form part” transform like

$$\tilde{p}_i = \frac{\partial q^j}{\partial \tilde{q}^i} p_j, \quad i = 1, \dots, n. \quad (2.6)$$

This works out as we might expect in our polar coordinate example.

**2.2.14 Example** (Example 2.1.5 cont'd) We again have  $Q = \mathbb{R}^2$  with  $(U_1, \phi_1)$  the Cartesian coordinate chart and  $(U_2, \phi_2)$  the polar coordinate chart. Thus natural Cartesian coordinates for  $T^*Q$  are denoted  $(x, y, p_x, p_y)$  and natural polar coordinates are denoted by  $(r, \theta, p_r, p_\theta)$ . These coordinates are related by

$$\begin{aligned} p_r &= \frac{\partial x}{\partial r} p_x + \frac{\partial y}{\partial r} p_y = \frac{x}{\sqrt{x^2 + y^2}} p_x + \frac{y}{\sqrt{x^2 + y^2}} p_y \\ p_\theta &= \frac{\partial x}{\partial \theta} p_x + \frac{\partial y}{\partial \theta} p_y = -y p_x + x p_y. \end{aligned}$$

As we shall see in Section 2.9, these are what are called “conjugate momenta” in the Hamiltonian formalism.  $\square$

**2.2.15 Remark** The Examples 2.2.6 and 2.2.14 exhibit some interesting phenomenon which are general. If we write

$$\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{bmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix},$$

then we note, by direct calculation, that

$$\begin{pmatrix} p_r \\ p_\theta \end{pmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{bmatrix}^{-t} \begin{pmatrix} p_x \\ p_y \end{pmatrix}.$$

This will generally be the case, and is simply a reflection of the fact that the matrix with components  $\frac{\partial q^i}{\partial \tilde{q}^j}$ ,  $i, j = 1, \dots, n$ , is the inverse of the transpose of the matrix with components  $\frac{\partial \tilde{q}^i}{\partial q^j}$ ,  $i, j = 1, \dots, n$ , provided we agree that for  $\frac{\partial q^i}{\partial \tilde{q}^j}$  the “up” index is the row index and the “down” index is the column index, and that for  $\frac{\partial \tilde{q}^i}{\partial q^j}$  the “up” index is the column index and the “down” index is the row index. This is consistent with the usage if the equations (2.5) and (2.6) if they are to be thought of as matrix-vector multiplication.  $\square$

**2.2.4 Riemannian metrics** Let us now move briefly on to talking about another important object, particularly in mechanics. A **Riemannian metric** on a configuration space  $Q$  is an assignment of an inner product to each tangent space  $T_qQ$ . This seems straightforward enough. How do we represent a Riemannian metric in coordinates? Well, let  $(q^1, \dots, q^n)$  be coordinates for a chart  $(U, \phi)$  and let  $g$  be a Riemannian metric. Since  $g(q)$  is an inner product on the vector space  $T_qQ$ , it will, in particular, take any two vectors in  $T_qQ$  and return a number. If we feed  $g(q)$  two of the basis vectors from the set  $\{\frac{\partial}{\partial q^1}|_q, \dots, \frac{\partial}{\partial q^n}|_q\}$ , then we may define the  $n^2$  numbers  $g_{ij}(q)$ ,  $i, j = 1, \dots, n$ , by

$$g_{ij}(q) = g(q)\left(\frac{\partial}{\partial q^i}|_q, \frac{\partial}{\partial q^j}|_q\right).$$

These are the **components** of  $g(q)$  in the coordinate chart  $(U, \phi)$ . Are these components with respect to a basis? Yes, and let us describe this basis. For  $i, j = 1, \dots, n$  we define a map

$$dq^i|_q \otimes dq^j|_q: T_qQ \times T_qQ \rightarrow \mathbb{R}$$

by defining it on basis elements by

$$dq^i|_q \otimes dq^j|_q\left(\frac{\partial}{\partial q^k}|_q, \frac{\partial}{\partial q^\ell}|_q\right) = \begin{cases} 1, & i = k, j = \ell \\ 0, & \text{otherwise.} \end{cases}$$

If we feed  $dq^i|_q \otimes dq^j|_q$  two general vectors, say  $u = u^k \frac{\partial}{\partial q^k}|_q$  and  $v = v^\ell \frac{\partial}{\partial q^\ell}|_q$ , we declare by linearity that

$$dq^i|_q \otimes dq^j|_q(u, v) = u^i v^j.$$

Now with this notation we claim that  $g$  is represented in our set of coordinates by

$$g(q) = g_{ij}(q) dq^i|_q \otimes dq^j|_q.$$

Indeed, for  $u = u^k \frac{\partial}{\partial q^k}|_q$  and  $v = v^\ell \frac{\partial}{\partial q^\ell}|_q$  we have

$$g(q)(u, v) = g_{ij} u^i v^j,$$

and this indeed how the correct formula for an inner product.

Let us see how the bases and the components change when we change coordinates. We will not prove what we say here, as it is straightforward along the lines of Lemma 2.2.10 and Lemma 2.2.11. As usual, in the following result we are considering  $(q^1, \dots, q^n)$  as coordinates in a chart  $(U_1, \phi_1)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  as coordinates in a chart  $(U_2, \phi_2)$ .

**2.2.16 Lemma**  $dq^i \otimes dq^j = \frac{\partial q^i}{\partial \tilde{q}^k} \frac{\partial q^j}{\partial \tilde{q}^\ell} d\tilde{q}^k \otimes d\tilde{q}^\ell$ ,  $i, j = 1, \dots, n$ .

The components now change in the predictable manner.

**2.2.17 Lemma** Let  $g$  be a Riemannian metric on  $Q$  and let  $(q^1, \dots, q^n)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  be coordinates for  $Q$ . If  $g_{ij}$ ,  $i, j = 1, \dots, n$ , are the components for  $g$  in the coordinates  $(q^1, \dots, q^n)$ , then

$$\frac{\partial q^k}{\partial \tilde{q}^i} \frac{\partial q^\ell}{\partial \tilde{q}^j} g_{k\ell}, \quad i, j = 1, \dots, n,$$

are the components of  $g$  in the coordinates  $(\tilde{q}^1, \dots, \tilde{q}^n)$ .

The following example shows that this is all not so bad. In the example, we illustrate how one deals with Riemannian metrics in practice.

**2.2.18 Example** (Example 2.1.5 cont'd) We again take  $Q = \mathbb{R}^2$  with its two coordinate charts  $(U_1, \phi_1)$  and  $(U_2, \phi_2)$ . Let us define a Riemannian metric on  $Q$  by defining it in the chart  $(U_1, \phi_1)$ . We define

$$g = dx \otimes dx + dy \otimes dy.$$

This is something you already know about. Indeed, let  $X$  and  $Y$  be general vector fields given by

$$X = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}, \quad Y = Y_1 \frac{\partial}{\partial x} + Y_2 \frac{\partial}{\partial y}.$$

We then have

$$\begin{aligned} g(X, Y) &= (dx \otimes dx + dy \otimes dy) \left( X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}, Y_1 \frac{\partial}{\partial x} + Y_2 \frac{\partial}{\partial y} \right) \\ &= dx \otimes dx \left( X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}, Y_1 \frac{\partial}{\partial x} + Y_2 \frac{\partial}{\partial y} \right) + \\ &\quad dy \otimes dy \left( X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}, Y_1 \frac{\partial}{\partial x} + Y_2 \frac{\partial}{\partial y} \right) \\ &= X_1 Y_1 + X_2 Y_2. \end{aligned}$$

Here we have used the rules

$$dx \left( \frac{\partial}{\partial x} \right) = 1, \quad dx \left( \frac{\partial}{\partial y} \right) = 0, \quad dy \left( \frac{\partial}{\partial x} \right) = 0, \quad dy \left( \frac{\partial}{\partial y} \right) = 1.$$

Note that  $g$  is just the usual inner product on  $\mathbb{R}^2$ , but that we allow it to take arguments that depend on the point on  $Q$ —that is, its arguments are vector fields.

Now let us see how this Riemannian metric looks in polar coordinates. We shall do this in two ways. First let us use the change of basis formula Lemma 2.2.16. In practice, one does this as follows:

$$\begin{aligned} dx \otimes dx + dy \otimes dy &= d(r \cos \theta) \otimes d(r \cos \theta) + d(r \sin \theta) \otimes d(r \sin \theta) \\ &= \cos^2 \theta dr \otimes dr - r \sin \theta \cos \theta d\theta \otimes dr - r \sin \theta \cos \theta dr \otimes d\theta + r^2 \sin^2 \theta d\theta \otimes d\theta + \\ &\quad \sin^2 \theta dr \otimes dr + r \sin \theta \cos \theta dr \otimes d\theta + r \sin \theta \cos \theta d\theta \otimes dr + r^2 \cos \theta d\theta \otimes d\theta \\ &= dr \otimes dr + r^2 d\theta \otimes d\theta. \end{aligned}$$

Notice that this calculation goes just like it would if  $dx$  and  $dy$  were silly infinitesimals, but now you are doing something real!

The other way to reach the same answer is by using the formula in Lemma 2.2.17 for the change of components for a Riemannian metric. We compute

$$\frac{\partial x}{\partial r} \frac{\partial x}{\partial r}(1) + \frac{\partial x}{\partial r} \frac{\partial y}{\partial r}(0) + \frac{\partial y}{\partial r} \frac{\partial x}{\partial r}(0) + \frac{\partial y}{\partial r} \frac{\partial y}{\partial r}(1) = \cos^2 \theta + \sin^2 \theta = 1,$$

and

$$\frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta}(1) + \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta}(0) + \frac{\partial y}{\partial r} \frac{\partial x}{\partial \theta}(0) + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta}(1) = -r \sin \theta \cos \theta + r \sin \theta \cos \theta = 0,$$

and

$$\frac{\partial x}{\partial \theta} \frac{\partial x}{\partial r}(1) + \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r}(0) + \frac{\partial y}{\partial \theta} \frac{\partial x}{\partial r}(0) + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial r}(1) = r \sin \theta \cos \theta - r \sin \theta \cos \theta = 0,$$

and

$$\frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \theta}(1) + \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \theta}(0) + \frac{\partial y}{\partial \theta} \frac{\partial x}{\partial \theta}(0) + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \theta}(1) = r^2 \sin^2 \theta + r^2 \cos^2 \theta = r^2.$$

Thus

$$g = dr \otimes dr + r^2 d\theta \otimes d\theta,$$

just as before.

As you can see, it is often more straightforward in practice to use the change of basis to determine the components for a Riemannian metric in a new set of coordinates.  $\square$

Associated with a Riemannian metric are two useful pieces of notation. Since  $g$  is definite, the matrix with components  $g_{ij}$ ,  $i, j = 1, \dots, n$ , is invertible at each point in any coordinate system. This allows us to define an invertible map  $g^\flat: TQ \rightarrow T^*Q$  by

$$\langle g^\flat(v_q); u_q \rangle = g(v_q, u_q).$$

One readily checks (see Exercise E2.17) that in a set of coordinates  $(q^1, \dots, q^n)$  that

$$g^\flat\left(\frac{\partial}{\partial q^i}\right) = g_{ij} dq^j, \quad i = 1, \dots, n,$$

if  $g_{ij}$ ,  $i, j = 1, \dots, n$ , are the components of  $g$  in the given set of coordinates. Therefore, the representation of  $g^\flat$  in coordinates is

$$g^\flat(q^1, \dots, q^n, v^1, \dots, v^n) = (q^1, \dots, q^n, g_{1j}v^j, \dots, g_{nj}v^j). \quad (2.7)$$

Note in particular that  $g^\flat$  maps the tangent space  $T_q Q$  into the cotangent space  $T_q^* Q$ . As  $g$  is definite,  $g^\flat$  is invertible, and we write its inverse as  $g^\sharp: T^*Q \rightarrow TQ$ . If we let  $g^{ij}$ ,  $i, j = 1, \dots, n$ , denote the components of the matrix which is the inverse of the matrix with components  $g_{ij}$ ,  $i, j = 1, \dots, n$ , then

$$g^\sharp(q^1, \dots, q^n, p_1, \dots, p_n) = (q^1, \dots, q^n, g^{1j}p_j, \dots, g^{nj}p_j).$$

We call the maps  $g^\flat$  and  $g^\sharp$  the *musical isomorphisms*.

**2.2.19 Example** (Example 2.1.5 cont'd) This all works out quite simply in our running example where  $Q = \mathbb{R}^2$ . We, as previously,

$$g = dx \otimes dx + dy \otimes dy.$$

In the Cartesian coordinates, one readily verifies that

$$g^\flat(x, y, v_x, v_y) = (x, y, v_x, v_y).$$

This looks like we are not saying anything, but remember that on the left,  $(x, y, v_x, v_y)$  are coordinates for a point in  $TQ$ , whereas on the right they are coordinates for a point in  $T^*Q$ . We also clearly have

$$g^\sharp(x, y, p_x, p_y) = (x, y, p_x, p_y).$$

Here again, be careful in saying that nothing is happening. On the right,  $(x, y, p_x, p_y)$  are coordinates for a point in  $T^*Q$ , and on the left they are coordinates for a point in  $TQ$ .

This becomes less opaque when we represent  $g^b$  in polar coordinates:

$$g^b(r, \theta, v_r, v_\theta) = (r, \theta, v_r, r^2 v_\theta)$$

and

$$g^\sharp(r, \theta, p_r, p_\theta) = (r, \theta, p_r, \frac{1}{r^2} p_\theta).$$

Although in Cartesian coordinates the maps  $g^b$  and  $g^\sharp$  are “trivial,” they are less so in polar coordinates. However, both are simply coordinate representations for the same thing.  $\square$

## 2.3 A variational principle

Many important ideas in mechanics have a variational basis. The calculus of variations is a wide-ranging and complex subject, and we shall only be able to deal with it superficially. Nevertheless, the notion that the laws of nature act in such a way as to extremise some function is an important one.

**2.3.1 Lagrangians** Let  $Q$  be a configuration space with tangent bundle  $TQ$ . A **Lagrangian** is a function on  $\mathbb{R} \times TQ$ . Thus  $L$  is a function of time, position, and velocity. If  $(U, \phi)$  is a coordinate chart with coordinates  $(q^1, \dots, q^n)$  we might write  $L(t, q^1, \dots, q^n, v^1, \dots, v^n)$ , or  $L(t, \mathbf{q}, \mathbf{v})$  for short, if we are working in coordinates.

We will wish to evaluate Lagrangians along curves. To do so, we need to say how to regard the velocity vector along a curve as a coordinate independent object. Let us start with a coordinate chart  $(U, \phi)$  with coordinates  $(q^1, \dots, q^n)$  and suppose that a curve  $c$  on  $Q$  is defined in these coordinates by  $t \mapsto (q^1(t), \dots, q^n(t))$ . For fixed  $t_0$  we define the **tangent vector field** to  $c$  at  $t_0$  to be the tangent vector in  $T_{c(t_0)}Q$  that in our given set of coordinates is given by  $(\dot{q}^1, \dots, \dot{q}^n(t_0))$ . To see that this is indeed a tangent vector, we need only verify that it obeys Lemma 2.2.4 as concerns coordinate changes. But, if  $(\tilde{q}^1, \dots, \tilde{q}^n)$  is another set of coordinates with  $c$  represented by  $t \mapsto (\tilde{q}^1(t), \dots, \tilde{q}^n(t))$ , then by the chain rule we have, of course,

$$\tilde{q}^i(t_0) = \frac{\partial \tilde{q}^i}{\partial q^j}(\phi(c(t_0))) \dot{q}^j(t_0), \quad i = 1, \dots, n.$$

But this is, as it surely must be, exactly how a tangent vector should behave. Let us then denote this tangent vector by  $c'(t_0) \in T_{c(t_0)}Q$ . Therefore,  $t \mapsto c'(t)$  defines a tangent vector at each point along the curve  $c$ , and the tangent vector  $c'(t)$  is to be regarded as a measure of the velocity of the curve at time  $t$  (see Figure 2.6).

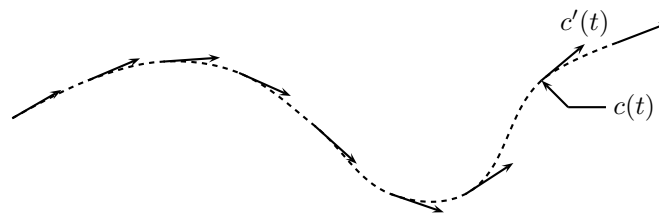


Figure 2.6 The tangent vector field to a curve

We will postpone physical examples of Lagrangians to Section 2.4 where we discuss by far the predominant class of Lagrangians.

**2.3.2 Variations** We will be wishing to minimise a certain function over a class of curves. To do this we need to say how to “vary” a curve in just the same way as one needs to be able to vary a function near a point to determine its derivative.

**2.3.1 Definition** Let  $c: [a, b] \rightarrow Q$  be a curve that is twice continuously differentiable. A *variation* of  $c$  is a map  $\sigma: [-\epsilon, \epsilon] \times [a, b] \rightarrow Q$  with the properties

- (i)  $\sigma(0, t) = c(t)$ ,
- (ii)  $\sigma(s, a) = c(a)$ , and
- (iii)  $\sigma(s, b) = c(b)$ .

The *infinitesimal variation* associated with a variation  $\sigma$  is the vector field defined at points along  $c$  by

$$\delta\sigma(t) = \left. \frac{d}{ds} \right|_{s=0} \sigma(s, t) \in T_{c(t)}Q. \quad \square$$

The idea is that a variation of  $c$  is a “wiggling” of  $c$ , and an infinitesimal variation corresponding to a certain manner of wiggling is a measure of the wiggle for small values of the “wiggle parameter.” Some intuition is as seen in Figure 2.7. Note that the endpoints of

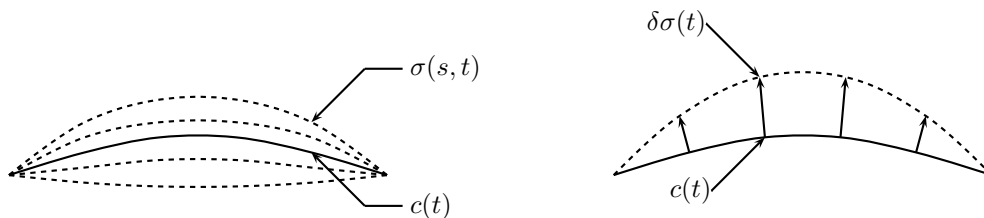


Figure 2.7 A variation (on the left) and an infinitesimal variation (on the right)

a variation, as we have defined it, remain stationary, so the infinitesimal variations vanish at the endpoints.

**2.3.3 Statement of the variational problem and Euler’s necessary condition** We let  $Q$  be a configuration space, and let  $a < b \in \mathbb{R}$  with  $q_a, q_b \in Q$ . We let

$$C^2([a, b], q_a, q_b) = \{c: [a, b] \rightarrow Q \mid c(a) = q_a, c(b) = q_b, \text{ and} \\ c \text{ is twice continuously differentiable}\}$$

be the collection of twice continuously differentiable curves that are defined on  $[a, b]$  and that start at  $q_a$  and end at  $q_b$ . Now we suppose that we have a Lagrangian on  $Q$  and define a function  $J_L: C^2([a, b], q_a, q_b) \rightarrow \mathbb{R}$  by

$$J_L(c) = \int_a^b L(t, c'(t)) dt,$$

where  $t \mapsto c'(t)$  is the tangent vector field of  $c$ . A curve  $c_0 \in C^2([a, b], q_a, q_b)$  *minimises*  $J_L$  if  $J_L(c_0) \leq J_L(c)$  for every  $c \in C^2([a, b], q_a, q_b)$ .

The following condition gives what might be regarded as a first-order necessary condition for a curve  $c \in C^2([a, b], q_a, q_b)$  to be a minimiser for  $J_L$ . Thus it should be thought of as being analogous to the first derivative condition for determining the minimum of a function.

**2.3.2 Theorem** *If a curve  $c \in C^2([a, b], q_a, q_b)$  minimises  $J_L$  then for any chart  $(U, \phi)$  with the property that the image of  $c$  intersects  $U$ , the representation  $t \mapsto \mathbf{q}(t)$  of  $c$  in that chart must satisfy*

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n,$$

at  $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$  for each  $t$  with the property that  $c(t) \in U$ .

*Proof* Let us first show that if  $c$  is a minimiser, then any subarc of  $c$  is also a minimiser. Thus let  $t_1, t_2 \in [a, b]$  satisfy  $a \leq t_1 < t_2 \leq b$ , and define a curve  $\tilde{c}: [t_1, t_2] \rightarrow Q$  by  $\tilde{c}(t) = c(t)$  (i.e.,  $\tilde{c}$  is the restriction of  $c$  to  $[t_1, t_2]$ ). With  $t_1$  and  $t_2$  as defined we can also define the set of curves  $C^2([t_1, t_2], c(t_1), c(t_2))$  analogously to our definition of  $C^2([a, b], q_a, q_b)$ . We also define a function  $\tilde{J}_L$  on the set of curves  $C^2([t_1, t_2], c(t_1), c(t_2))$  by

$$\tilde{J}_L(\tilde{c}) = \int_{t_1}^{t_2} L(t, \tilde{c}'(t)) dt.$$

We claim that if  $c$  is a minimiser for  $J_L$ , then  $\tilde{c}$  is a minimiser for  $\tilde{J}_L$ . Indeed, if  $\tilde{c}$  were *not* a minimiser for  $\tilde{J}_L$ , this could imply the existence of a curve  $\tilde{c}_1 \in C^2([t_1, t_2], c(t_1), c(t_2))$  with the property that  $\tilde{J}_L(\tilde{c}_1) < \tilde{J}_L(\tilde{c})$ . Now define a curve  $c_1: [a, b] \rightarrow Q$  by

$$c_1(t) = \begin{cases} c(t), & t \in [a, t_1] \\ \tilde{c}_1(t), & t \in [t_1, t_2] \\ c(t), & t \in [t_2, b]. \end{cases}$$

We then have

$$\begin{aligned} J_L(c_1) &= \int_a^{t_1} L(t, c'(t)) dt + \int_{t_1}^{t_2} L(t, \tilde{c}'_1(t)) dt + \int_{t_2}^b L(t, c'(t)) dt \\ &< \int_a^{t_1} L(t, c'(t)) dt + \int_{t_1}^{t_2} L(t, \tilde{c}'(t)) dt + \int_{t_2}^b L(t, c'(t)) dt \\ &= \int_a^b L(t, c'(t)) dt = J_L(c). \end{aligned}$$

This would complete our assertion that the restriction of  $c$  to any subinterval of  $[a, b]$  is itself a minimiser, but for the fact that the curve  $c_1$  may not be twice continuously differentiable, as it may fail to be differentiable at  $t_1$  and  $t_2$ . However, one may show (see Exercise E2.18) that it is possible to modify  $c_1$  slightly to a curve  $\tilde{c}_1$  so that  $\tilde{c}_1$  is twice continuously differentiable on  $[a, b]$ , and so that  $|J_L(\tilde{c}_1) - J_L(c_1)| \leq \epsilon$  for any  $\epsilon > 0$ . In this way, one can ensure that  $J_L(\tilde{c}_1) < J_L(c)$ , contradicting the fact that  $c$  is a minimiser.

The above argument says that  $c$  is a minimiser along every subinterval of  $[a, b]$  if it is a minimiser along the entire interval. Therefore, if  $(U, \phi)$  is a chart so that  $c(t) \in U$  for some  $t \in [a, b]$ , if  $c$  is a minimiser on  $[a, b]$ , it is also a minimiser for the curve restricted to an interval  $[t_1, t_2]$  with the property that  $c(t) \in U$  for  $t \in [t_1, t_2]$ . The upshot is that without loss of generality, we may suppose that the curve  $c$  lies in the domain of a chart  $(U, \phi)$ , and we do this for the remainder of the proof.

Let  $\sigma$  be a variation of  $c$  and let  $c_{\sigma, s} \in C^2([a, b], q_a, q_b)$  be defined by  $c_{\sigma, s}(t) = \sigma(s, t)$ . Consider the function  $f_\sigma$  defined on  $[-\epsilon, \epsilon]$  by  $f_\sigma(s) = J_L(c_{\sigma, s})$ . If  $c$  is a minimiser for  $J_L$  then

$s = 0$  should be a minimum of  $f_\sigma$  for every variation  $\sigma$  of  $c$ . We let  $\mathbf{q}_{\sigma,s}$  be the coordinate representative of  $c_{\sigma,s}$ . We have

$$\begin{aligned} 0 &= \left. \frac{d}{ds} \right|_{s=0} f_\sigma(s) = \left. \frac{d}{ds} \right|_{s=0} \int_a^b L(t, \mathbf{q}_{\sigma,s}(t), \dot{\mathbf{q}}_{\sigma,s}(t)) dt \\ &= \int_a^b \left( \left. \frac{\partial L}{\partial q^i} \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0} + \left. \frac{\partial L}{\partial v^i} \frac{d\dot{q}_{\sigma,s}^i(t)}{ds} \right|_{s=0} \right) dt. \end{aligned}$$

Now note that

$$\left. \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0} = \left. \frac{d}{ds} \right|_{s=0} \frac{dq_{\sigma,s}^i}{dt} = \left. \frac{d}{dt} \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0}.$$

Therefore

$$\begin{aligned} 0 &= \left. \frac{d}{ds} \right|_{s=0} f_\sigma(s) = \int_a^b \left( \left. \frac{\partial L}{\partial q^i} \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0} + \left. \frac{\partial L}{\partial v^i} \frac{d}{dt} \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0} \right) dt \\ &= \int_a^b \left( \left. \frac{\partial L}{\partial q^i} \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0} - \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) \right) \frac{dq_{\sigma,s}^i(t)}{ds} \Big|_{s=0} dt + \left. \frac{\partial L}{\partial v^i} \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0} \Big|_{t=b}^{t=a} \\ &= \int_a^b \left( \left. \frac{\partial L}{\partial q^i} \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0} - \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) \right) \frac{dq_{\sigma,s}^i(t)}{ds} \Big|_{s=0} dt \end{aligned}$$

where in the next to last step we have used integration by parts, and in the last step we have used the fact that an infinitesimal variation vanishes at the endpoints. Since

$$\int_a^b \left( \left. \frac{\partial L}{\partial q^i} \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0} - \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) \right) \frac{dq_{\sigma,s}^i(t)}{ds} \Big|_{s=0} dt = 0$$

for every variation, this means that  $\left. \frac{dq_{\sigma,s}^i(t)}{ds} \right|_{s=0}$  can be arbitrary, and so the only way for the integral to vanish is for

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right)$$

to be zero when evaluated at  $(q, \mathbf{q}(t), \dot{\mathbf{q}}(t))$ . ■

The equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n,$$

are called the **Euler-Lagrange equations**, and they are what we will be primarily concerned with in the remainder of these notes. Though innocent enough in appearance, they hide an enormous amount of beautiful structure. Note that the Euler-Lagrange equations are only necessary conditions. A solution to the Euler-Lagrange equations may or may not be an actual minimiser. The common language is to call a curve in  $C^2(q_a, q_b, [a, b])$  an **extremal** when it satisfies the Euler-Lagrange equations. This reflects the fact that the Euler-Lagrange equations are essentially analogous to the first derivative conditions in calculus.

Let's look at a very simple example.

**2.3.3 Example** (Example 2.1.5 cont'd) We work again with  $Q = \mathbb{R}^2$  and its two coordinate charts  $(U_1, \phi_1)$  and  $(U_2, \phi_2)$ . We define a Lagrangian on  $Q$  by defining it in the coordinate chart  $(U_1, \phi_1)$  to be

$$L(t, x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2).$$



Suppose that we have a curve  $t \mapsto (x(t), y(t))$  that is a minimiser for  $J_L$  for some interval and with some endpoints. Then the Euler-Lagrange equations say that this curve must satisfy

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial L}{\partial x} &= \frac{d}{dt}v_x = \dot{v}_x(t) = \ddot{x}(t) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial v_y}\right) - \frac{\partial L}{\partial y} &= \frac{d}{dt}v_y = \dot{v}_y(t) = \ddot{y}(t) = 0.\end{aligned}$$

Thus the Euler-Lagrange equations are the simple second-order differential equations

$$\ddot{x}(t) = 0, \quad \ddot{y}(t) = 0,$$

and these have solution  $x(t) = x(0) + \dot{x}(0)t$  and  $y(t) = y(0) + \dot{y}(0)t$ . These describe straight lines in the plane  $Q = \mathbb{R}^2$ .

Now let us look at the same example in the polar coordinate chart  $(U_2, \phi_2)$ . First we need to determine the Lagrangian in these coordinates. Using the transformation rule (2.5) we have

$$\begin{aligned}v_x &= \frac{\partial x}{\partial r}v_r + \frac{\partial x}{\partial \theta}v_\theta = \cos \theta v_r - r \sin \theta v_\theta \\ v_y &= \frac{\partial y}{\partial r}v_r + \frac{\partial y}{\partial \theta}v_\theta = \sin \theta v_r + r \cos \theta v_\theta.\end{aligned}$$

With these relations we readily compute

$$v_x^2 + v_y^2 = v_r^2 + r^2 v_\theta^2$$

so that

$$L = \frac{1}{2}(v_r^2 + r^2 v_\theta^2).$$

The Euler-Lagrange equations are then

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial v_r}\right) - \frac{\partial L}{\partial r} &= \frac{d}{dt}v_r - r(t)v_\theta^2(t) = \dot{v}_r(t) - r(t)v_\theta^2(t) = \ddot{r}(t) - r(t)\dot{\theta}^2(t) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial v_\theta}\right) - \frac{\partial L}{\partial \theta} &= \frac{d}{dt}(r^2 v_\theta) = r^2 \dot{v}_\theta(t) + 2r(t)\dot{r}(t)v_\theta(t) = r^2(t)\ddot{\theta}(t) + 2r(t)\dot{r}(t)\dot{\theta}(t) = 0.\end{aligned}$$

That is, the Euler-Lagrange equations are the two coupled second-order differential equations

$$\ddot{r}(t) - r(t)\dot{\theta}^2(t) = 0, \quad r^2(t)\ddot{\theta}(t) + 2r(t)\dot{r}(t)\dot{\theta}(t) = 0.$$

These equations are less amenable to solution by inspection than their Cartesian coordinate counterparts. However, since we know the Cartesian solutions and the change of coordinate formulae, we may in principal write the solutions to the polar coordinate equations. However, to do so is straightforward, and not a very interesting display of symbology.  $\square$

**2.3.4 The Euler-Lagrange equations and changes of coordinate** In the previous example, we wrote the Euler-Lagrange equations in two different sets of coordinates, and somehow we think that the two sets of equations should represent the same thing. Let us explore exactly what is meant by this. First of all, we remark that the *problem* of minimising  $J_L$  over curves in  $C^2([a, b], q_a, q_b)$  obviously does not depend upon any choice of coordinates. Now, if the *problem statement* is not dependent on coordinate chart, then so should be any statements relating to its solution. Thus we should hope that the Euler-Lagrange equations are in some sense coordinate invariant. Indeed they are, and in just the way of the following result.

**2.3.4 Proposition** *Let  $Q$  be a configuration space with  $(q^1, \dots, q^n)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  coordinates for  $Q$ . If  $L$  is a Lagrangian on  $Q$  then*

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \tilde{v}^i} \right) - \frac{\partial L}{\partial \tilde{q}^i} = \frac{\partial q^j}{\partial \tilde{q}^i} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial q^j} \right) - \frac{\partial L}{\partial q^j} \right].$$

*In particular, the Euler-Lagrange equations are satisfied in the coordinates  $(q^1, \dots, q^n)$  if and only if they are satisfied in the coordinates  $(\tilde{q}^1, \dots, \tilde{q}^n)$ .*

*Proof* Note that by (2.5) we have

$$\frac{\partial v^j}{\partial \tilde{v}^i} = \frac{\partial q^j}{\partial \tilde{q}^i}, \quad \frac{\partial v^j}{\partial \tilde{q}^i} = \frac{\partial^2 q^j}{\partial \tilde{q}^i \partial \tilde{q}^k} \tilde{v}^k$$

Therefore, using the chain rule, we have

$$\begin{aligned} \frac{\partial L}{\partial \tilde{v}^i} &= \frac{\partial L}{\partial v^j} \frac{\partial v^j}{\partial \tilde{v}^i} + \frac{\partial L}{\partial q^j} \frac{\partial q^j}{\partial \tilde{v}^i} \\ &= \frac{\partial L}{\partial v^j} \frac{\partial q^j}{\partial \tilde{q}^i} \\ \frac{\partial L}{\partial \tilde{q}^i} &= \frac{\partial L}{\partial v^j} \frac{\partial v^j}{\partial \tilde{q}^i} + \frac{\partial L}{\partial q^j} \frac{\partial q^j}{\partial \tilde{q}^i} \\ &= \frac{\partial L}{\partial v^j} \frac{\partial^2 q^j}{\partial \tilde{q}^i \partial \tilde{q}^k} \tilde{v}^k + \frac{\partial L}{\partial q^j} \frac{\partial q^j}{\partial \tilde{q}^i}. \end{aligned}$$

Now note that

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \tilde{v}^i} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial v^j} \right) \frac{\partial q^j}{\partial \tilde{q}^i} + \frac{\partial L}{\partial v^j} \frac{\partial^2 q^j}{\partial \tilde{q}^i \partial \tilde{q}^k} \dot{\tilde{q}}^k.$$

Therefore, simple addition verifies the first part of the proposition. As for the second assertion, it follows from the first since the matrix with components  $\frac{\partial q^j}{\partial \tilde{q}^i}$ ,  $i, j = 1, \dots, n$ , is invertible. ■

**2.3.5 Important remark** Note that the components of the Euler-Lagrange equations transform just like the components of a one-form on  $Q$ . However, the Euler-Lagrange equations are *not* a one-form, because their coefficients depend time, velocity, and acceleration. A utilisation of this fact is to be found in [Lewis 2000c]. □

Let's see how this plays out in our running example.

**2.3.6 Example** (Example 2.1.5 cont'd) We have, as usual,  $Q = \mathbb{R}^2$  with its Cartesian and polar coordinate charts. In Example 2.3.3 we derived the Euler-Lagrange equations for a particular Lagrangian in both sets of coordinates. Let us verify that these equations are indeed related as in Proposition 2.3.4.

In Cartesian coordinates the Euler Lagrange equations were

$$\ddot{x} = 0, \quad \ddot{y} = 0,$$

and in polar coordinates we had

$$\ddot{r} - r\dot{\theta}^2 = 0, \quad r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0.$$

We compute

$$\begin{aligned} \frac{\partial x}{\partial r} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} \right] + \frac{\partial y}{\partial r} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial v_y} \right) - \frac{\partial L}{\partial y} \right] &= \cos \theta (\ddot{x}) + \sin \theta (\ddot{y}) \\ &= \cos \theta (\cos \theta \ddot{r} - 2 \sin \theta \dot{r} \dot{\theta} - r \cos \theta \dot{\theta}^2 - r \sin \theta \ddot{\theta}) + \\ &\quad \sin \theta (\sin \theta \ddot{r} + 2 \cos \theta \dot{r} \dot{\theta} - r \sin \theta \dot{\theta}^2 + r \cos \theta \ddot{\theta}) \\ &= \ddot{r} - r \dot{\theta}^2. \end{aligned}$$

Similarly we compute

$$\begin{aligned} \frac{\partial x}{\partial \theta} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} \right] + \frac{\partial y}{\partial \theta} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial v_y} \right) - \frac{\partial L}{\partial y} \right] &= -r \sin \theta (\ddot{x}) + r \cos \theta (\ddot{y}) \\ &= -r \sin \theta (\cos \theta \ddot{r} - 2 \sin \theta \dot{r} \dot{\theta} - r \cos \theta \dot{\theta}^2 - r \sin \theta \ddot{\theta}) + \\ &\quad r \cos \theta (\sin \theta \ddot{r} + 2 \cos \theta \dot{r} \dot{\theta} - r \sin \theta \dot{\theta}^2 + r \cos \theta \ddot{\theta}) \\ &= r^2 \ddot{\theta} + 2r \dot{r} \dot{\theta}. \end{aligned}$$

Thus the components do indeed transform as they ought to, although a direct verification of this, even in a simple example, is a not entirely pleasant task.  $\square$

## 2.4 Simple mechanical systems

“Simple” mechanical systems are not so named because they behave in a particularly simple manner. The usage of the word simple here has a precise meaning, and the fact of the matter is that the vast majority of mechanical systems one encounters are “simple” by our usage of the word. Simple mechanical systems are characterised by having a special sort of Lagrangian.

**2.4.1 Kinetic energy** The most important feature of a simple mechanical system is that it has kinetic energy. This means something very particular in the terminology we introduced in Section 2.2.4. Let  $g$  be a Riemannian metric on  $Q$ . The **kinetic energy** associated with  $g$  is the function  $K$  on  $TQ$  defined by  $K(v_q) = \frac{1}{2}g(v_q, v_q)$ . *Note that we need a Riemannian metric to define kinetic energy!* This is why we introduced the concept of a Riemannian metric. However, when working with an example, one can often readily determine by elementary methods its kinetic energy in a set of coordinates  $(q^1, \dots, q^n)$ , without knowing what is the Riemannian metric in these coordinates. But the fact of the matter is that the Riemannian metric is there. To recover it proceed as follows. The kinetic energy you derive will be a function  $K(q^1, \dots, q^n, v^1, \dots, v^n)$ . The components  $g_{ij}$ ,  $i, j = 1, \dots, n$ , of the corresponding Riemannian metric are then determined by

$$g_{ij} = \frac{\partial^2 K}{\partial v^i \partial v^j}, \quad i, j = 1, \dots, n.$$

Note that if you apply this formula and the components for  $g$  turn out to involve the velocities, then you have either made a mistake, or the system you are dealing with is not a simple mechanical system.

Let us see how this works in our simple example.

2.4.1 Example (Example 2.1.5 cont'd) We now add some physics to this example. The configuration space  $Q = \mathbb{R}^2$  we take to be the configuration space for a particle of mass  $m$  moving in a plane. Thus we have chosen an appropriate coordinate system for a Galilean spacetime to reduce ourselves to this situation. In Cartesian coordinates, the kinetic energy of such a mass is simply

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(v_x^2 + v_y^2).$$

One then computes

$$g_{ij} = \frac{\partial^2 K}{\partial v^i \partial v^j} = m\delta_{ij}, \quad i, j = 1, \dots, n.$$

Therefore, the kinetic energy of a particle in the plane is in exact correspondence with the standard Riemannian metric on  $\mathbb{R}^2$ , scaled by  $m > 0$ .

Although we have illustrated this correspondence via the Cartesian coordinate system, it does not rely on this system of coordinates. Let us see how this pans out in polar coordinates. Clearly the particle example is essentially like the Example 2.2.18 as concerns the Riemannian metric, and like Example 2.3.3 as concerns the Lagrangian. Referring to those examples, the Riemannian metric  $g$  in polar coordinates is

$$g = m(dr \otimes dr + r^2 d\theta \otimes d\theta),$$

and we also have  $K = \frac{1}{2}m(v_r^2 + r^2 v_\theta^2)$  so that

$$\frac{\partial^2 K}{\partial v^i \partial v^j} = \begin{cases} m, & i = 1, \\ mr^2, & i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

This demonstrates that  $\frac{\partial^2 K}{\partial v^i \partial v^j} = g_{ij}$ , even in polar coordinates.  $\square$

2.4.2 Remarks 1. Since we have a 1–1 correspondence between the concept of kinetic energy and a Riemannian metric, let us agree to use the latter as basic, since, as we shall see, a Riemannian metric possesses many interesting properties.

2. In very simple examples like the previous one, one can see that it is a bit overkill to formally compute  $\frac{\partial^2 K}{\partial v^i \partial v^j}$  in order to determine the components of the Riemannian metric; one can simply look at the expression for kinetic energy and write down the corresponding metric components. However, for more complicated system formed on multiple components, one often will write the kinetic energy of each component, then taking the total kinetic energy to be the sum of the component energies. In this case, the formula  $g_{ij} = \frac{\partial^2 K}{\partial v^i \partial v^j}$  can be used, perhaps in a symbolic manipulation language, to produce the metric components with relative ease.  $\square$

**2.4.2 Potential energy** The notion of potential energy is quite simple as concerns its basic definition: a **potential function** is a function  $V$  on the configuration space  $Q$ . In particular, it is a function independent of velocity.

2.4.3 Example (Example 2.1.5 cont'd) Let us resume talking about a mass  $m$  moving about in the plane. Now let us suppose that the plane is oriented in such a way that the force of gravity acts in the  $y$ -direction in Cartesian coordinates. In this case, our elementary

physics tells us that potential energy should be a “storage function” for the energy that the system has when it is at a given height. This, of course, requires us to declare some height as being the zero height, where the body has no gravitational potential energy. This choice is arbitrary, so let us choose it to be  $y_0$ . The gravitational potential energy is then  $V = ma_g(y - y_0)$  where  $a_g$  is the acceleration due to gravity (e.g.,  $a_g \approx 13.24\text{m/sec}^2$  on the surface of Neptune). One of the drawbacks of the classical (due to Einstein, I expect) use of  $g$  for a Riemannian metric is that one cannot now use that symbol for the acceleration due to gravity.

Let us see what this function looks like in polar coordinates. Define  $r_0 \in \mathbb{R}_+$  and  $\theta_0 \in (-\pi, \pi)$  by  $y_0 = r_0 \sin \theta_0$ . This supposes that  $y_0 \neq 0$ . We then have

$$y - y_0 = r \sin \theta - r_0 \sin \theta_0,$$

so the gravitational potential in polar coordinates is

$$V = ma_g(r \sin \theta - r_0 \sin \theta_0).$$

Note that, even in the Newtonian world, this is really only an approximation of the gravitational force (see Exercise E2.24).  $\square$

Although the concept of potential energy is benign enough, there is actually a reason why it is a wee bit subtle, and we will touch on that soon.

**2.4.3 The Euler-Lagrange equations for simple mechanical systems** We now have the data to say what we mean by a simple mechanical system.

**2.4.4 Definition** A *simple mechanical system* is a triple  $(Q, g, V)$  where

- (i)  $Q$  is a configuration space,
- (ii)  $g$  is a Riemannian metric on  $Q$ , and
- (iii)  $V$  is a potential energy function on  $Q$ .

The *Lagrangian* associated to a simple mechanical system  $(Q, g, V)$  is the function on  $TQ$  defined by

$$L(v_q) = \frac{1}{2}g(v_q, v_q) - V(q). \quad \square$$

One might well ask, “Why use a minus sign on the potential?” Indeed, the “natural” quantity seems to be the total energy  $\frac{1}{2}g(v_q, v_q) + V(q)$ . The reasons for using the minus sign cannot be explicated in a coherent manner until we look at the Lagrange-d’Alembert principle in Section 2.5. For now, let us just say that we use it because it works.

Let us look at the Euler-Lagrange equations for simple mechanical systems. To do so, given a potential function  $V$  on  $Q$ , define a vector field  $\text{grad } V$  on  $Q$  by

$$\text{grad } V(q) = g^\sharp(dV(q)).$$

Note that since  $dV$  is a one-form,  $\text{grad } V$  is indeed a vector field as claimed. In coordinates we have

$$\text{grad } V = g^{ij} \frac{\partial V}{\partial q^j} \frac{\partial}{\partial q^i}.$$

Now we may state the following result.

**2.4.5 Proposition** *Let  $(Q, g, V)$  be a simple mechanical system with associated Lagrangian  $L$ . Let  $c: I \rightarrow Q$  be a curve which is represented by  $t \mapsto (q^1(t), \dots, q^n(t))$  in a coordinate chart  $(U, \phi)$ . The following statements are equivalent:*

- (i)  $t \mapsto (q^1(t), \dots, q^n(t))$  satisfies the Euler-Lagrange equations for the Lagrangian  $L$ ;
- (ii)  $t \mapsto (q^1(t), \dots, q^n(t))$  satisfies the second-order differential equation

$$\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = -(\text{grad } V)^i, \quad i = 1, \dots, n,$$

where  $\Gamma_{jk}^i$ ,  $i, j, k = 1, \dots, n$ , are functions of  $q$  defined by

$$\Gamma_{jk}^i = \frac{1}{2} g^{i\ell} \left( \frac{\partial g_{j\ell}}{\partial q^k} + \frac{\partial g_{k\ell}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell} \right).$$

*Proof* In coordinates we have

$$L = \frac{1}{2} g_{jk} v^j v^k - V(q).$$

We therefore have

$$\frac{\partial L}{\partial v^\ell} = g_{\ell j} v^j, \quad \frac{\partial L}{\partial q^\ell} = \frac{1}{2} \frac{\partial g_{jk}}{\partial q^\ell} v^j v^k - \frac{\partial V}{\partial q^\ell}.$$

Therefore the Euler-Lagrange equations are

$$g_{\ell j} \ddot{q}^j + \left( \frac{\partial g_{\ell j}}{\partial q^k} - \frac{1}{2} \frac{\partial g_{jk}}{\partial q^\ell} \right) \dot{q}^j \dot{q}^k + \frac{\partial V}{\partial q^\ell} = 0. \quad (2.8)$$

Since the expression  $\dot{q}^j \dot{q}^k$  is symmetric with respect to transposing the indices  $j$  and  $k$ , only the symmetric part of the expression

$$A_{\ell jk} \triangleq \frac{\partial g_{\ell j}}{\partial q^k} - \frac{1}{2} \frac{\partial g_{jk}}{\partial q^\ell}$$

with respect to the indices  $j$  and  $k$  will contribute to the expression

$$\left( \frac{\partial g_{\ell j}}{\partial q^k} - \frac{1}{2} \frac{\partial g_{jk}}{\partial q^\ell} \right) \dot{q}^j \dot{q}^k.$$

But the symmetric part of  $A_{\ell jk}$  with respect to  $j$  and  $k$  is

$$\frac{1}{2} (A_{\ell jk} + A_{\ell kj}) = \frac{1}{2} \left( \frac{\partial g_{j\ell}}{\partial q^k} + \frac{\partial g_{k\ell}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell} \right).$$

The result now follows by multiplying (2.8) by  $g^{i\ell}$ . ■

We shall see in the next section the meaning of the functions  $\Gamma_{jk}^i$ ,  $i, j, k = 1, \dots, n$ . They represent a very important object in the study of simple mechanical systems.

Let us see how the description of the Euler-Lagrange equations plays out for our running example.

**2.4.6 Example** (Example 2.1.5 cont'd) We take as our simple mechanical system  $(Q = \mathbb{R}^2, g, V)$  where  $g$  is the Riemannian metric of Example 2.4.1 and  $V$  is the potential function of Example 2.4.3.

In Cartesian coordinates, the resulting Lagrangian on  $TQ$  is given by

$$L(x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2) - ma_g(y - y_0).$$

The Euler-Lagrange equations are then readily computed to be

$$m\ddot{x} = 0, \quad m\ddot{y} + ma_g = 0. \quad (2.9)$$

Given that

$$g = m(dx \otimes dx + dy \otimes dy)$$

in these coordinates, one readily determines that the functions  $\Gamma_{jk}^i$  are zero for all  $i, j, k = 1, 2$ . One also sees that

$$\text{grad } V = a_g \frac{\partial}{\partial y}.$$

Therefore the equations of part (ii) of Proposition 2.4.5 read

$$\ddot{x} = 0, \quad \ddot{y} = -a_g,$$

and these equations are obviously equivalent to the equations (2.9).

Now let us also check that things work out in polar coordinates. In these coordinates we have

$$L(r, \theta, v_r, v_\theta) = \frac{1}{2}m(v_r^2 + r^2v_\theta^2) - ma_g(r \sin \theta - r_0 \sin \theta_0).$$

The Euler-Lagrange equations are then

$$m\ddot{r} - mr\dot{\theta}^2 + ma_g \sin \theta = 0, \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + ma_g r \cos \theta = 0. \quad (2.10)$$

To check the equations of part (ii) of Proposition 2.4.5, we note that

$$g = m(dr \otimes dr + r^2 d\theta \otimes d\theta),$$

and a simple, slightly tedious, computation gives

$$\Gamma_{\theta\theta}^r = -r, \quad \Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \frac{1}{r},$$

with the remaining  $\Gamma$ 's being zero. One also computes

$$\text{grad } V = a_g \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} a_g \cos \theta \frac{\partial}{\partial \theta}.$$

Therefore the equations of part (ii) of Proposition 2.4.5 are

$$\ddot{r} - r\dot{\theta}^2 = -a_g \sin \theta, \quad \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = -\frac{1}{r}a_g \cos \theta.$$

These equations are clearly equivalent to the equations (2.10).

Note that the equations are independent of the datum  $y_0$  from which potential energy was measured.  $\square$

**2.4.4 Affine connections** In the previous section, we saw arise the functions  $\Gamma_{jk}^i$ ,  $i, j, k = 1, \dots, n$ , defined in a set of coordinates for a configuration space  $Q$ . We shall now address the question, “What are these functions?” The answer may seem weird, but it is nonetheless the answer, dictated by the physics of the problem.

In order to make the definition, we need a few simple concepts. If  $X$  is a vector field on  $Q$  and  $f$  is a function on  $Q$ , then  $fX$  is also a vector field on  $Q$  with  $(fX)(q) = f(q)X(q)$  for each  $q \in Q$ . Thus  $fX$  is simply the vector field with the vectors scaled by the values of  $f$ . Also, given a vector field  $X$  and a function  $f$  we define the **Lie derivative** of  $f$  with respect to  $X$  to be the function on  $Q$  defined by

$$\mathcal{L}_X f(q) = \langle \mathbf{d}f(q); X(q) \rangle.$$

In coordinates  $(q^1, \dots, q^n)$  we have

$$\mathcal{L}_X f = X^i \frac{\partial f}{\partial q^i},$$

thus  $\mathcal{L}_X f$  may be known to you as the “directional derivative.”

With this language, we make the following definition.

**2.4.7 Definition** An **affine connection** on a configuration space  $Q$  is an assignment to each pair of vector fields  $X$  and  $Y$  on  $Q$  a vector field  $\nabla_X Y$  and the assignment has the properties:

- (i) the map  $(X, Y) \mapsto \nabla_X Y$  is bilinear with respect to multiplication by real numbers;
- (ii) for each pair of vector fields  $X$  and  $Y$  and for each function  $f$  on  $Q$ , we have  $\nabla_{fX} Y = f(\nabla_X Y)$ ;
- (iii) for each pair of vector fields  $X$  and  $Y$  and for each function  $f$  on  $Q$ , we have  $\nabla_X (fY) = f(\nabla_X Y) + (\mathcal{L}_X f)Y$ .

We also call  $\nabla_X Y$  the **covariant derivative** of  $Y$  with respect to  $X$ . □

What can this possibly have to do with mechanics?! Well, let us see what an affine connection looks like in coordinates. Let  $(U, \phi)$  be a coordinate chart with coordinates  $(q^1, \dots, q^n)$ . Given a pair of coordinate vector fields  $\frac{\partial}{\partial q^i}$  and  $\frac{\partial}{\partial q^j}$  their covariant derivative is also a vector field. Therefore,  $\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j}$  must be a linear combination of the basis vector fields  $\{\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\}$ . Thus there exists  $n$  functions  $\Gamma_{ij}^k$ ,  $k = 1, \dots, n$ , so that

$$\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \Gamma_{ij}^k \frac{\partial}{\partial q^k}. \quad (2.11)$$

As this must be true for *each*  $i, j = 1, \dots, n$ , this defines  $n^3$  functions  $\Gamma_{ij}^k$ ,  $i, j, k = 1, \dots, n$ , of the coordinates  $(q^1, \dots, q^n)$  which are called the **Christoffel symbols** for the affine connection  $\nabla$  in the coordinate chart  $(U, \phi)$ . Note that the affine connection is determined uniquely in a coordinate chart by its Christoffel symbols. This is made clear by the following result.

**2.4.8 Proposition** Let  $\nabla$  be an affine connection on a configuration space  $Q$  and let  $X$  and  $Y$  be vector fields on  $Q$ . If  $(U, \phi)$  is a coordinate chart with coordinates  $(q^1, \dots, q^n)$  then we have

$$\nabla_X Y = \left( \frac{\partial Y^k}{\partial q^i} X^i + \Gamma_{ij}^k X^i Y^j \right) \frac{\partial}{\partial q^k}$$

on  $U$ , where  $\Gamma_{ij}^k$ ,  $i, j, k = 1, \dots, n$ , are the Christoffel symbols for  $\nabla$  in the coordinate chart  $(U, \phi)$ .



*Proof* This follows by the properties of an affine connection. Indeed we have

$$\begin{aligned}\nabla_X Y &= \nabla_{X^i \frac{\partial}{\partial q^i}} Y^j \frac{\partial}{\partial q^j} \\ &= X^i Y^j \nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} + X^i \mathcal{L}_{\frac{\partial}{\partial q^i}} Y^j \frac{\partial}{\partial q^j} \\ &= \left( X^i Y^j \Gamma_{ij}^k + X^i \frac{\partial Y^k}{\partial q^i} \right) \frac{\partial}{\partial q^k},\end{aligned}$$

just as claimed. ■

Now we see that  $\Gamma$ 's may come up as claimed, but it is still not clear how the  $\Gamma$ 's of Proposition 2.4.5 are related to the Christoffel symbols. To see how this relationship is established, we will associate with a Riemannian metric, i.e., with kinetic energy, a unique affine connection. In order to carry this out, we need to have at hand the concept of a **Lie bracket** between two vector fields  $X$  and  $Y$  on  $Q$ . This is defined to be the vector field  $[X, Y]$  on  $Q$  which, in any set of coordinates  $(q^1, \dots, q^n)$  is given by

$$[X, Y] = \left( \frac{\partial Y^i}{\partial q^j} X^j - \frac{\partial X^i}{\partial q^j} Y^j \right) \frac{\partial}{\partial q^i}. \quad (2.12)$$

Of course, for this definition to make sense, one must check that this definition does not depend on coordinates, and we leave this straightforward calculation to the reader (Exercise E2.15). The Lie bracket is an extremely important object in differential geometry, and we will put it to rather pedestrian usage here in our discussion of affine connections. However, in Sections 2.6 and 4.5, we shall see a more interesting application of the Lie bracket.

Let us proceed with our discussion. The following result is not difficult to prove, but its proof is perhaps a bit sophisticated for our purposes here. We refer the reader to, for example, [Kobayashi and Nomizu 1963] for a proof.

**2.4.9 Theorem** *Let  $g$  be a Riemannian metric on a configuration space  $Q$ . There exists a unique affine connection  $\overset{g}{\nabla}$  on  $Q$  with the following properties:*

- (i)  $\overset{g}{\nabla}_X Y - \overset{g}{\nabla}_Y X = [X, Y]$  for all vector fields  $X$  and  $Y$  on  $Q$ ;
- (ii)  $\mathcal{L}_Z(g(X, Y)) = g(\overset{g}{\nabla}_Z X, Y) + g(X, \overset{g}{\nabla}_Z Y)$  for all vector fields  $X, Y$ , and  $Z$  on  $Q$ .

Furthermore, in a set of coordinates  $(q^1, \dots, q^n)$  for  $Q$ , the Christoffel symbols for  $\overset{g}{\nabla}$  are given by

$$\Gamma_{jk}^i = \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial q^k} + \frac{\partial g_{kl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right).$$

The affine connection  $\overset{g}{\nabla}$  is the **Levi-Civita connection** corresponding to the Riemannian metric  $g$ .

Now we at last see the meaning of the functions  $\Gamma_{jk}^i$  in the statement of Proposition 2.4.5. They are the Christoffel symbols of an affine connection, called the Levi-Civita connection, which one may associate with the Riemannian metric defining the kinetic energy. Let us look a little more closely at the differential equation of part (ii) of Proposition 2.4.5, concentrating on the case when the potential energy function is zero. The differential equation in question is then,

$$\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0, \quad i = 1, \dots, n,$$

where we now acknowledge explicitly the dependence of the  $\Gamma$ 's on  $g$ . Clearly, there is nothing preventing us from talking about this equation when we do not use the Christoffel symbols for the Levi-Civita connection, but rather Christoffel symbols for a general affine connection. To this end, let  $\nabla$  be a general affine connection on  $Q$ . A curve  $c: I \rightarrow Q$  is called a **geodesic** for  $\nabla$  when for any  $t_0 \in I$  and any coordinate chart  $(U, \phi)$  around  $c(t_0)$  with coordinates  $(q^1, \dots, q^n)$ , the coordinate representation  $t \mapsto (q^1(t), \dots, q^n(t))$  satisfies the differential equation

$$\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0, \quad i = 1, \dots, n,$$

where  $\Gamma_{jk}^i$ ,  $i, j, k = 1, \dots, n$ , are the Christoffel symbols of  $\nabla$  in the coordinate chart  $(U, \phi)$ . Of course, for this to make sense, the definition needs to be shown to be independent of coordinates. We leave this only slightly tedious computation to the reader (Exercise E2.26). Note, however, that each of the terms in the geodesic equations are *not* coordinate independent! Thus one should not write  $\ddot{q}^i$  as a tangent vector, since it does not obey a transformation law like a tangent vector should! To talk about acceleration as a vector one genuinely needs an affine connection. In this case, the notation one uses for the quantity which in coordinates is  $\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k$  is  $\nabla_{c'(t)} c'(t)$ , and a geodesic is a curve  $c$  for which  $\nabla_{c'(t)} c'(t) = 0$ .

We may now offer the following coordinate independent restatement of Proposition 2.4.5.

**2.4.10 Theorem** *Let  $(Q, g, V)$  be a simple mechanical system with associated Lagrangian  $L$ . Let  $c \in C^2([a, b], q_a, q_b)$ . The following statements are equivalent:*

- (i)  $c$  satisfies the Euler-Lagrange equations for the Lagrangian  $L$ ;
- (ii)  $c$  satisfies

$$\overset{g}{\nabla}_{c'(t)} c'(t) = -\text{grad } V(c(t)).$$

Let us see how this looks in our running example.

**2.4.11 Example** (Example 2.1.5 cont'd) The  $Q = \mathbb{R}^2$  example we have been using, although simple, is ample to illustrate why one cannot regard acceleration as a tangent vector. On  $Q$  we consider a curve  $c: \mathbb{R} \rightarrow Q$  defined in Cartesian coordinates by

$$c(t) = (\cos t, \sin t).$$

Thus  $c$  describes a circle. In Cartesian coordinates we then have  $(x(t), y(t)) = (\cos t, \sin t)$  so that

$$(\dot{x}(t), \dot{y}(t)) = (-\sin t, \cos t), \quad (\ddot{x}(t), \ddot{y}(t)) = (-\cos t, -\sin t).$$

In polar coordinates the curve is represented by  $(r(t), \theta(t)) = (1, t)$ , but note that  $r(t)$  and  $\theta(t)$  are only defined for  $t \in (-\pi, \pi)$ . We readily compute

$$(\dot{r}(t), \dot{\theta}(t)) = (0, 1), \quad (\ddot{r}(t), \ddot{\theta}(t)) = (0, 0).$$

Therefore the acceleration along the curve is zero in polar coordinates, but nonzero in polar coordinates. This precludes acceleration from being a tangent vector since it cannot be both zero and nonzero.

Now let us look at the acceleration represented by the Levi-Civita connection corresponding to the Riemannian metric  $g$  from Example 2.4.1. In Cartesian coordinates, the Christoffel symbols are zero so we have

$$\overset{g}{\nabla}_{c'(t)} c'(t) = \ddot{x} \frac{\partial}{\partial x} + \ddot{y} \frac{\partial}{\partial y} = -\cos t \frac{\partial}{\partial x} - \sin t \frac{\partial}{\partial y}.$$

In polar coordinates, the Christoffel symbols were computed in Example 2.4.1, and using these we have

$$\overset{g}{\nabla}_{c'(t)}c'(t) = (\ddot{r} - r\dot{\theta}^2)\frac{\partial}{\partial r} + (\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta})\frac{\partial}{\partial \theta} = -\frac{\partial}{\partial r}.$$

Thus in both Cartesian and polar coordinates, the quantity  $\overset{g}{\nabla}_{c'(t)}c'(t)$  is a vector of “unit length” (if  $m = 1$ , remove the quotes) pointing towards the origin (see Figure 2.8). The

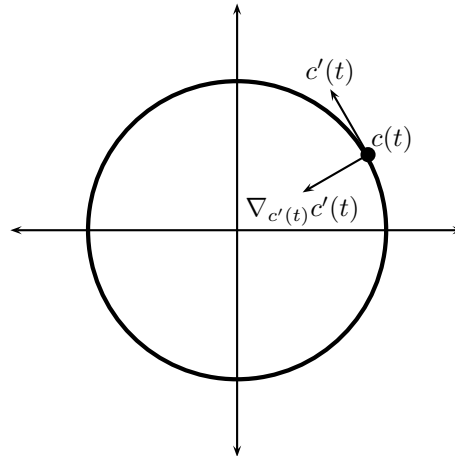


Figure 2.8 Velocity and “geometric acceleration” along a curve

quantity  $\overset{g}{\nabla}_{c'(t)}c'(t)$  is sometimes called the **geometric acceleration**. Note that since the geometric acceleration is nonzero, the curve  $c$  is not a geodesic. Well, of course it is not: geodesics, as we have seen, are straight lines for this affine connection.  $\square$

**2.4.12 Remark** One often sees the Euler-Lagrange equations for a simple mechanical Lagrangian  $L(v_q) = \frac{1}{2}g(v_q, v_q) - V(q)$  written as

$$M(q)\ddot{q} + C(q, \dot{q}) + K(q) = 0,$$

or something similar to this, where  $M(q)$  is a matrix whose entries are  $g_{ij}(q)$  in some set of coordinates. In our manner of writing the equations of motion, the term  $C(q, \dot{q})$  is given by  $g_{il}\overset{g}{\Gamma}_{jk}^l\dot{q}^j\dot{q}^k$ , and is called by some authors “Coriolis forces.” This, as we can see, is nonsensical in a certain sense. The term  $C(q, \dot{q})$  makes no sense by itself. Indeed, as Example 2.4.11 shows, this term in the equations can be zero in one set of coordinates, and nonzero in another set of coordinates. The term  $C(q, \dot{q})$  should not be taken by itself as meaningful.  $\square$

## 2.5 Forces in Lagrangian mechanics

If one wishes to talk about control theory for mechanical systems, as we do, then the natural “inputs” for these systems are forces. In this section we discuss forces generally, concentrating on the various nature of forces which can arise in mechanical systems. We begin with a discussion of how forces appear in the Euler-Lagrange equations.

**2.5.1 The Lagrange-d’Alembert principle** The appearance of forces in Newtonian mechanics is done using the laws Newton laid out as motivation. How one should go about

adding forces to the Euler-Lagrange equations is not altogether obvious. First let us say that a **force** on a configuration space  $Q$  is a map  $F: \mathbb{R} \times TQ \rightarrow T^*Q$  which has the property that  $F(t, v_q) \in T_q^*Q$ . Thus a force is a function of time, position, and velocity, and for each fixed value  $(t, v_q)$  of these arguments, it is a covector in the cotangent space  $T_q^*Q$ . Just why a force should naturally be regarded as a covector rather than a tangent vector is a question best answered by saying, “Because that’s the way it works out.” Some motivation may be drawn from Proposition 2.3.4 which indicates how that the Euler-Lagrange equations themselves change like the components of a one-form under coordinate changes. In any event, we can write a force in coordinates as

$$F(t, \mathbf{q}, \mathbf{v}) = F_i(t, \mathbf{q}, \mathbf{v})dq^i,$$

for some function  $F_1, \dots, F_n$  of time, configuration, and velocity, called the **components** of the force  $F$ .

The Lagrange-d’Alembert principle, often simply called “d’Alembert’s principle,” tells us how a force  $F$  should appear in the Euler-Lagrange equations.

**2.5.1 Definition** Let  $L$  be a Lagrangian on a configuration space  $Q$  and let  $F$  be a force. A curve  $c: [a, b] \rightarrow Q$  satisfies the **Lagrange-d’Alembert principle** for the force  $F$  if for every variation  $\sigma: [-\epsilon, \epsilon] \times [a, b] \rightarrow Q$  we have

$$\left. \frac{d}{ds} \right|_{s=0} \int_a^b L(t, \frac{d}{dt}\sigma(s, t)) dt + \int_a^b \langle F(t, c'(t)); \delta\sigma(t) \rangle dt = 0. \quad \square$$

The idea behind this definition is that when the external forces are not present, they make no contribution in the second term in Definition 2.5.1. However, when they are present, they do work, and this is evidently properly accounted for in the variational formulation which we call the Lagrange-d’Alembert principle.

Let us see what are the implications of the Lagrange-d’Alembert principle for the equations describing the motion of a Lagrangian system in the presence of forces.

**2.5.2 Proposition** Let  $L$  be a Lagrangian on  $Q$  with  $F$  a force on  $Q$ . A curve  $c: [a, b] \rightarrow Q$  satisfies the Lagrange-d’Alembert principle if and only if for any coordinate chart  $(U, \phi)$  which intersects the image of  $c$ , the coordinate representation  $t \mapsto (q^1(t), \dots, q^n(t))$  satisfies

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = F_i, \quad i = 1, \dots, n, \quad (2.13)$$

where  $F_1, \dots, F_n$  are the components of  $F$ . The equations (2.13) are the **forced Euler-Lagrange equations**.

*Proof* Fix a coordinate chart  $(U, \phi)$  which intersects the image of  $c$ . By considering variations of  $c$  whose infinitesimal variation vanishes outside a coordinate chart, we may suppose that  $c(t) \in U$  for each  $t$ . Let  $\sigma$  be a variation of  $c$  with  $c_{\sigma, s}$  the curve defined by  $c_{\sigma, s}(t) = \sigma(s, t)$ . Denote the coordinate representation of  $c_{\sigma, s}$  by  $\mathbf{q}_{\sigma, s}$ . Then, by following the calculations of Theorem 2.3.2, we arrive at

$$\int_a^b \left( \frac{\partial L}{\partial q^i} \frac{dq_{\sigma, s}^i(t)}{ds} \Big|_{s=0} - \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) + F_i \right) \frac{dq_{\sigma, s}^i(t)}{ds} \Big|_{s=0} dt = 0.$$

Since this must hold for every variation, the result follows. ■

Thus the manner in which one adds forces to the Euler-Lagrange equations, governed by the Lagrange-d'Alembert principle, is very simple: one merely sticks the components of the force on the right-hand side of the Euler-Lagrange equations. The question of why this is the right thing to do often results in a pointless metaphysical discussion. The real reason is, “Because it works,” meaning that the Lagrange-d'Alembert principle agrees with measurement to a tolerable degree. Mathematical descriptions of our physical world must be a slave to empirical data.

**2.5.2 Potential forces** A potential force is one resulting from potential energy. More usefully, given a potential function  $V$  on  $Q$ , its **potential force** is the force given by  $F(t, v_q) = -dV(q)$ . In particular, potential forces are independent of time and velocity.<sup>2</sup> However, potential forces are more than simply independent of time and velocity. They have a useful property with respect to the “work” done by a mechanical system. We need to define work. One normally takes as the definition of work “force times distance.” This is fine for defining work done on straight line paths in Euclidean space. However, our setting is more general than this in several ways. First, we are working with general configuration spaces. Second, on a general configuration space, there is no such thing as a straight line path. Thus we are led to the following definition. On a configuration space  $Q$ , we define the **work** done by a force  $F$  along a curve  $c: I \rightarrow Q$  to be the quantity

$$W(F, c) = \int_I \langle F(t, c'(t)); c'(t) \rangle dt. \quad (2.14)$$

One should be able to readily see that this properly generalises the classical notion of work (see Exercise E2.27).

Now with this notion of work, we have the following characterisation of a potential force. A curve  $c: [a, b] \rightarrow Q$  is **closed** if  $c(a) = c(b)$ .

**2.5.3 Proposition** *Let  $F$  be a force on a configuration space  $Q$ . The following statements are equivalent:*

- (i) *the work done by  $F$  on any closed curve is zero;*
- (ii)  *$F$  is a potential force.*

*Proof* The implication (ii)  $\implies$  (i) is readily made. Indeed, if  $F$  is a potential force, then  $F(t, v_q) = -dV(q)$  for some function  $V$ . We then have

$$\begin{aligned} W(F, c) &= \int_a^b \langle F(t, c'(t)); c'(t) \rangle dt \\ &= - \int_a^b \langle dV(c(t)); c'(t) \rangle dt \\ &= - \int_a^b \frac{dV(c(t))}{dt} dt \\ &= - (V(c(b)) - V(c(a))) = 0, \end{aligned}$$

for every closed curve  $c: [a, b] \rightarrow Q$ .

---

<sup>2</sup>One can actually define potential forces to be time-dependent by allowing  $V$  to depend on time. Sometimes it is interesting to do this, but we shall not do so.

The implication (i)  $\implies$  (ii) goes as follows. Suppose that  $F$  is a potential force and let  $q_0 \in Q$ . For  $q \in Q$  define  $V(q) \in \mathbb{R}$  by

$$V(q) = - \int_0^1 \langle F(c(t)); c'(t) \rangle dt,$$

where  $c: [0, 1] \rightarrow Q$  is any curve satisfying  $c(0) = q_0$  and  $c(1) = q$ . We will first show that the function  $V$  defined in this manner does not depend on the choice made for the curve  $c$ . Let  $c_1, c_2: [0, 1] \rightarrow Q$  be two curves satisfying  $c_1(0) = c_2(0) = q_0$  and  $c_1(1) = c_2(1) = q$ . Define a curve  $\sigma: [0, 1] \rightarrow Q$  by

$$\sigma(t) = \begin{cases} c_1(2t), & t \in [0, \frac{1}{2}] \\ c_2(2 - 2t), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Note that  $\sigma(0) = \sigma(1)$  so that

$$\int_0^1 \langle F(\sigma(t)); \sigma'(t) \rangle dt = 0.$$

However, we also have

$$\begin{aligned} \int_0^1 \langle F(\sigma(t)); \sigma'(t) \rangle dt &= \int_0^{\frac{1}{2}} \langle F(\sigma(t)); \sigma'(t) \rangle dt + \int_{\frac{1}{2}}^1 \langle F(\sigma(t)); \sigma'(t) \rangle dt \\ &= \int_0^1 \langle F(c_1(t)); c_1'(t) \rangle dt + \int_1^0 \langle F(c_2(t)); c_2'(t) \rangle dt, \end{aligned}$$

from which we conclude that

$$\int_0^1 \langle F(c_1(t)); c_1'(t) \rangle dt = \int_0^1 \langle F(c_2(t)); c_2'(t) \rangle dt.$$

Thus our definition of  $V$  is independent of curve, as desired.

Finally, let us show that  $F = -dV$ , thus making  $F$  a potential force. Let  $(U, \phi)$  be a coordinate chart around  $q \in Q$  and suppose that  $\phi(q) = \mathbf{0}$ . Let  $B_\epsilon(\mathbf{0})$  be the ball of radius  $\epsilon$  centred at the origin in  $\mathbb{R}^n$ , and suppose that  $B_\epsilon(\mathbf{0}) \subset \phi(U)$ . Let  $\mathbf{v} \in \mathbb{R}^n$  and let  $\mathbf{x}_1 = \frac{\epsilon}{\|\mathbf{v}\|} \mathbf{v}$  be the point on  $B_\epsilon(\mathbf{0})$  which lies in the direction of  $\mathbf{v}$ . Denote  $q_1 = \phi^{-1}(\mathbf{x}_1)$ . Let  $c: [0, 1] \rightarrow Q$  be a curve with the properties

1.  $c(0) = q_0$ ;
2.  $c(\frac{1-\epsilon}{\|\mathbf{v}\|}) = q_1$ ;
3.  $\phi \circ c(t) = \frac{\|\mathbf{v}\|(1-t)}{\|\mathbf{v}\| + \epsilon - 1} \mathbf{x}_1$ ,  $t \in [\frac{1-\epsilon}{\|\mathbf{v}\|}, 1]$ .

Now we define  $\psi: [\frac{1-\epsilon}{\|\mathbf{v}\|}, 1] \rightarrow \mathbb{R}$  by

$$\psi(t) = \int_{\frac{1-\epsilon}{\|\mathbf{v}\|}}^t \langle \mathbf{F}(\phi \circ c(s)); \mathbf{v} \rangle ds,$$

where  $\mathbf{F}$  is the representation of  $F$  in the coordinate chart  $(U, \phi)$ . Let  $V_\phi: B_\epsilon(\mathbf{0}) \rightarrow \mathbb{R}$  be defined by  $V_\phi(\mathbf{x}) = V \circ \phi^{-1}(\mathbf{x})$ . Note that with  $\psi$  defined as we have done, we have

$$V_\phi(t\mathbf{v}) = \psi(1 - t), \quad t \in [0, \frac{\epsilon}{\|\mathbf{v}\|}].$$

Therefore we have

$$\langle dV_\phi(\mathbf{0}); \mathbf{v} \rangle = -\left. \frac{d}{dt} \right|_{t=1} \psi(t).$$

By the definition of  $\psi$  this gives

$$\langle dV_\phi(\mathbf{0}); \mathbf{v} \rangle = -\langle \mathbf{F}(\mathbf{0}); \mathbf{v} \rangle.$$

If  $v_q \in T_q Q$  is the tangent vector whose coordinate representation in the chart  $(U, \phi)$  is  $\mathbf{v}$ , then this means that

$$\langle dV(q); v_q \rangle = -\langle F(q); v_q \rangle.$$

Since our above construction can be done for any  $\mathbf{v} \in \mathbb{R}^n$  this shows that  $F(q) = -dV(q)$ , as desired. ■

Of course, we have already seen a potential function, but let us explicitly identify the force associated with it.

**2.5.4 Example** (Example 2.1.5 cont'd) We consider the gravitational potential introduced in Example 2.4.3. In Cartesian coordinates we had  $V(x, y) = ma_g(y - y_0)$ , so the potential force is simply

$$F = -dV = -ma_g dy.$$

In polar coordinates we had  $V = ma_g(r \sin \theta - r_0 \sin \theta_0)$ , so that the potential force in these coordinates is

$$F = -dV = -ma_g(\sin \theta dr + r \cos \theta d\theta).$$

Were one to derive from “first principles” the formula for the potential force, one would have some work to do, but using the blind calculations, it is quite simple. □

**2.5.3 Dissipative forces** A force  $F$  on a configuration space  $Q$  is a **dissipative force** if  $\langle F(t, v_q); v_q \rangle \leq 0$  for every  $(t, v_q) \in \mathbb{R} \times TQ$ . A dissipative force is **strictly dissipative** if strict inequality holds for each  $v_q \in TQ$ . Dissipative forces “dissipate energy.” Although we have talked about kinetic energy and potential energy, we have not really talked about the total energy of a Lagrangian system. Certainly, for a simple mechanical system  $(Q, g, V)$  one is inclined to define the total energy to be the sum of the kinetic and potential energy:  $E(v_q) = \frac{1}{2}g(v_q, v_q) + V(q)$ . This can be generalised for arbitrary Lagrangians.

Rayleigh  
dissipation  
function

Let  $L: \mathbb{R} \times TQ \rightarrow \mathbb{R}$  be a Lagrangian on a configuration space  $Q$ . For  $(t, q) \in \mathbb{R} \times Q$  let  $L_{(t,q)}: T_q Q \rightarrow \mathbb{R}$  be the function  $L_{(t,q)}(v_q) = L(t, v_q)$ . Thus  $L_{(t,q)}$  is simply obtained by restricting  $L$  to a given time and configuration, leaving velocity free. This function on  $T_q Q$  can then be differentiated with respect to velocity, and let us denote the derivative at  $v_q$  by  $FL(t, v_q)$ . In natural coordinates  $(q^1, \dots, q^n, v^1, \dots, v^n)$  for  $TQ$ ,  $FL$  is given by

$$FL(t, \mathbf{q}, \mathbf{v}) = \frac{\partial L}{\partial v^i}(t, \mathbf{q}, \mathbf{v}),$$

the derivative with respect to velocity.

**2.5.5 Lemma**  $FL(t, v_q) \in T_q^* Q$ .

*Proof* We need to show that the coordinate expression transforms in the correct way when we change coordinates. If  $(q^1, \dots, q^n)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  are coordinates in overlapping coordinate charts, then we have

$$\frac{\partial L}{\partial \tilde{v}^i} = \frac{\partial L}{\partial q^j} \frac{\partial q^j}{\partial \tilde{v}^i} + \frac{\partial L}{\partial v^j} \frac{\partial v^j}{\partial \tilde{v}^i}.$$

But  $q^j$ ,  $j = 1, \dots, n$ , is independent of  $\tilde{v}^i$ ,  $i = 1, \dots, n$  (the  $q$ 's are only functions of the  $\tilde{q}$ 's, not the  $\tilde{v}$ 's). Also, we have

$$v^j = \frac{\partial q^j}{\partial \tilde{q}^k} v^k,$$

so that

$$\frac{\partial v^j}{\partial \tilde{v}^i} = \frac{\partial q^j}{\partial \tilde{q}^i}.$$

The result now follows from Lemma 2.2.11. ■

Thus we have defined a map  $\mathbf{FL}: \mathbb{R} \times TQ \rightarrow \mathbb{R} \times T^*Q$  which has the property that  $\mathbf{FL}(t, v_q) \in T^*_q Q$ . This map is called the **Legendre transformation** associated with the Lagrangian  $L$ . The **energy** we now define to be the function  $\mathbf{EL}: \mathbb{R} \times TQ \rightarrow \mathbb{R}$  given by

$$\mathbf{EL}(t, v_q) = \langle \mathbf{FL}(t, v_q); v_q \rangle - L(t, v_q).$$

In natural coordinates  $(q^1, \dots, q^n, v^1, \dots, v^n)$  for  $TQ$  we have

$$\mathbf{EL} = \frac{\partial L}{\partial v^i} v^i - L. \quad (2.15)$$

Let us verify that this gives the usual expression for energy when  $L$  is the Lagrangian for a simple mechanical system.

**2.5.6 Proposition** *If  $(Q, g, V)$  is a simple mechanical system with  $L$  the associated Lagrangian, then*

- (i)  $\mathbf{FL}(t, v_q) = g^b(v_q)$  and
- (ii)  $\mathbf{EL}(t, v_q) = \frac{1}{2}g(v_q, v_q) + V(q)$ .

*Proof* In coordinates we have  $L(\mathbf{q}, \mathbf{v}) = \frac{1}{2}g_{ij}v^i v^j - V$  so that

$$\mathbf{FL} = \frac{\partial L}{\partial v^i} dq^i = g_{ij}v^j dq^i.$$

That is to say,  $\mathbf{FL}(t, v_q) = g^b(v_q)$ . The definition of energy now gives

$$\begin{aligned} \mathbf{EL}(t, v_q) &= \langle \mathbf{FL}(t, v_q); v_q \rangle - L(t, v_q) \\ &= \langle g^b(v_q); v_q \rangle - \frac{1}{2}g(v_q, v_q) + V(q) \\ &= \frac{1}{2}g(v_q, v_q) + V(q), \end{aligned}$$

where we have used the defining properties of  $g^b$ . ■

Let us now show that a dissipative force dissipates energy, which is the whole point of defining energy at this point.



**2.5.7 Proposition** *Let  $L$  be a time-independent Lagrangian on a configuration space  $Q$  and let  $F$  be a dissipative force. If a curve  $c: [a, b] \rightarrow Q$  satisfies the Lagrange-d'Alembert principle for the Lagrangian  $L$  and the force  $F$ , then*

$$\frac{d}{dt} \mathbf{E}L(t, c'(t)) \leq 0$$

for every  $t \in [a, b]$ .

*Proof* The proof is easily carried out in coordinates, using the fact that  $L$  is time-independent. Using the coordinate expression (2.15) for  $\mathbf{E}L$  we have

$$\begin{aligned} \frac{d}{dt} \mathbf{E}L(t, c'(t)) &= \frac{\partial \mathbf{E}L}{\partial q^i} \dot{q}^i + \frac{\partial \mathbf{E}L}{\partial v^i} \dot{v}^i \\ &= \frac{\partial^2 L}{\partial v^i \partial v^j} \dot{v}^i \dot{q}^j + \frac{\partial^2 L}{\partial q^i \partial v^j} \dot{q}^i \dot{q}^j + \frac{\partial L}{\partial v^i} \dot{v}^i - \frac{\partial L}{\partial q^i} \dot{q}^i - \frac{\partial L}{\partial v^i} \dot{v}^i. \end{aligned} \quad (2.16)$$

Since the Euler-Lagrange equations hold, we may expand them fully to give

$$\frac{\partial^2 L}{\partial v^i \partial v^j} \dot{v}^j + \frac{\partial^2 L}{\partial v^i \partial q^j} \dot{q}^j - \frac{\partial L}{\partial q^i} = F_i, \quad i = 1, \dots, n. \quad (2.17)$$

Comparing the expressions (2.16) and (2.17) gives the coordinate independent expression, true for general forces,

$$\frac{d}{dt} \mathbf{E}L(t, c'(t)) = \langle F(t, c'(t)); c'(t) \rangle.$$

If  $F$  is a dissipative force, the result clearly follows. ■

This yields the following corollary of independent interest. It states that for an unforced Lagrangian system, energy is conserved.

**2.5.8 Corollary** *If  $L$  is a time-independent Lagrangian on a configuration space  $Q$  then the function  $\mathbf{E}L$  is constant along solutions of the Euler-Lagrange equations.*

Let's look at our simple example.

**2.5.9 Example** (Example 2.1.5 cont'd) We return to our example of a mass  $m$  moving in the plane  $Q = \mathbb{R}^2$ . We consider three examples of dissipative forces.

1. We take as an example of a dissipative force the force defined in Cartesian coordinates by

$$F_\delta(t, x, y, v_x, v_y) = -\delta(v_x dx + v_y dy),$$

where  $\delta > 0$ . That this is indeed a dissipative force follows since

$$\langle F_\delta(t, x, y, v_x, v_y); (v_x, v_y) \rangle = -\delta(v_x^2 + v_y^2),$$

and this quantity is negative except when  $v_x = v_y = 0$ . This kind of dissipative force is often called **viscous friction**, and is characterised by its being a linear function of velocity. It is often a good model when the contact is "lubricated."

We can also represent this same force in polar coordinates. Trying to do this from “first principles” is a bit annoying. But just writing the above force in polar coordinates is quite simple: we compute

$$\begin{aligned} v_x dx + v_y dy &= (v_r \cos \theta - r v_\theta \sin \theta)(\cos \theta dr - r \sin \theta d\theta) + \\ &\quad (v_r \sin \theta + r v_\theta \cos \theta)(\sin \theta dr + r \cos \theta d\theta) \\ &= v_r dr + r^2 v_\theta d\theta. \end{aligned}$$

Therefore the viscous dissipation force in polar coordinates is

$$F(t, r, \theta, v_r, v_\theta) = -\delta(v_r dr + r^2 v_\theta d\theta).$$

2. Another example of a dissipative force is given by

$$F_{K,\delta}(t, x, y, v_x, v_y) = \begin{cases} K(dx + dy), & v_x = v_y = 0 \\ -\delta(v_x dx + v_y dy), & \text{otherwise,} \end{cases}$$

where both  $K$  and  $\delta$  are positive numbers. This kind of dissipative force is discontinuous, and is called **sticking friction**. It is commonly observed when there is no lubrication, and the contact between surfaces needs to be “broken” before motion can commence.

We may again compute the polar coordinate version of this force. Doing the calculations gives

$$F_{K,\delta}(t, r, \theta, v_r, v_\theta) = \begin{cases} K((\cos \theta + \sin \theta)dr + r(\cos \theta - \sin \theta)d\theta), & v_x = v_y = 0 \\ -\delta(v_r dr + r^2 v_\theta d\theta), & \text{otherwise.} \end{cases}$$

Again, if one were to try to derive such a force in polar coordinates using first principles, it would not be a trivial task. But by understanding the force as a one-form, and so as a coordinate independent object, the calculations become systematic.

3. A final example of a dissipative force is what is termed **rolling friction**. This is the friction encountered when two surfaces are in rolling contact, and is intended to model the force required as one surface deflects as the other rolls across it. No matter, rolling friction is modelled by a force

$$F_\alpha(t, x, y, v_x, v_y) = \begin{cases} 0, & v_x = v_y = 0 \\ -\alpha \frac{v_x}{\sqrt{v_x^2 + v_y^2}} dx - \alpha \frac{v_y}{\sqrt{v_x^2 + v_y^2}} dy, & \text{otherwise.} \end{cases}$$

Note that this force is independent of the magnitude of velocity, and points in the opposite “direction” as velocity.

As always, we may compute the rolling friction force in polar coordinate, and it is readily determined to be

$$F_\alpha(t, x, y, v_x, v_y) = \begin{cases} 0, & v_x = v_y = 0 \\ -\alpha \frac{v_r}{\sqrt{v_r^2 + r^2 v_\theta^2}} dr - \alpha \frac{r^2 v_\theta}{\sqrt{v_r^2 + r^2 v_\theta^2}} d\theta, & \text{otherwise.} \end{cases}$$

We shall leave to the reader the simple task of verifying that both sticking and rolling friction meet our criteria of being dissipative forces.  $\square$

**2.5.4 Forces for simple mechanical systems** When dealing with simple mechanical systems, because of their special structure, one can write forces in a slightly different manner, making them compatible with the writing of the equations using the affine connection formalism. The idea is quite simple, and is encapsulated in the following result, which gives a nice coordinate free way of expressing the forced equations for a simple mechanical system.

**2.5.10 Proposition** *Let  $(Q, g, V)$  be a simple mechanical system with Lagrangian  $L$ , and let  $F$  be a force on  $Q$ . A curve  $c: [a, b] \rightarrow Q$  satisfies the Lagrange-d'Alembert principle for  $F$  if and only if*

$$\nabla_{c'(t)}^g c'(t) = -\text{grad } V(c(t)) + g^\sharp(F(t, c'(t))). \quad (2.18)$$

*Proof* This follows directly from Proposition 2.5.2 and the computations of Proposition 2.4.5. ■

As always, this is readily illustrated via our running example.

**2.5.11 Example** (Example 2.1.5 cont'd) Let us consider the particle of mass  $m$  of Example 2.4.1 with the potential function of Example 2.4.3 and the viscous friction force of Example 2.5.9. Therefore, in Cartesian coordinates, the force on the mass is

$$F = -\delta(v_x dx + v_y dy).$$

If we wished, we could consider the potential force as part of the total force, or we could leave it in the Lagrangian as the potential energy function; the equations are the same, of course. Just make sure you do not account for the potential force twice! The forced Euler-Lagrange equations, using the Lagrangian  $L = \frac{1}{2}m(v_x^2 + v_y^2) - ma_g(y - y_0)$  are therefore

$$\begin{aligned} m\ddot{x} &= -\delta\dot{x} \\ m\ddot{y} + ma_g &= -\delta\dot{y}. \end{aligned} \quad (2.19)$$

On the other hand, we compute

$$g^\sharp(F) = -\frac{\delta}{m} \left( v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} \right).$$

Therefore the equation (2.18) is given by

$$\begin{aligned} \ddot{x} &= -\frac{\delta}{m}\dot{x} \\ \ddot{y} &= -a_g - \frac{\delta}{m}\dot{y}. \end{aligned} \quad (2.20)$$

Clearly the two sets of differential equations (2.19) and (2.20) are equivalent.

Now let us do the same thing in polar coordinates. In polar coordinates the dissipative force is

$$F = -\delta(v_r dr + r^2 v_\theta d\theta).$$

Therefore, the forced Euler-Lagrange equations with Lagrangian

$$L = \frac{1}{2}m(v_r^2 + r^2 v_\theta^2) - ma_g(r \sin \theta - r_0 \sin \theta_0)$$

are

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + ma_g \sin \theta &= -\delta\dot{r} \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + ma_g r \cos \theta &= -\delta r^2 \dot{\theta}. \end{aligned} \quad (2.21)$$

To compare these equations with equations (2.18), we compute

$$g^\sharp(F) = -\frac{\delta}{m} \left( v_r \frac{\partial}{\partial r} + v_\theta \frac{\partial}{\partial \theta} \right).$$

Therefore we may compute the equations (2.18), with the aid of our computations in Example 2.4.6, to be

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -a_g \sin \theta - \frac{\delta}{m} \dot{r} \\ \ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} &= -\frac{1}{r} a_g \cos \theta - \frac{\delta}{m} \dot{\theta}. \end{aligned} \tag{2.22}$$

One readily sees that equations (2.21) and (2.22) are indeed equivalent.  $\square$

## 2.6 Constraints in mechanics

The subject of constraints in mechanics seems to be one which lends itself readily to notational confusion. However, this need not be the case. With our notion of an abstract configuration space and its tangent bundle, it is quite easy to give a coherent definition of a constraint, and then write the equations of motion in the presence of a constraint. Thus you will find the treatment in this section brief compared to the often seen treatment, and devoid of the bizarre posturing one typically encounters.

**2.6.1 Definitions** For us, a *constraint* will be a specification of an affine subspace  $C_q$  of the tangent space  $T_q Q$  for each  $q \in Q$ . Recall from Example 1.1.4 that as  $C_q$  is an affine subspace, there exists a subspace  $\ell(C_q)$  of  $T_q Q$  so that

$$C_q = \{ \eta_q + u_q \mid u_q \in \ell(C_q) \}$$

for some vector  $\eta_q \in T_q Q$ . We call  $\ell(C_q)$  the *linear part* of the constraint. A constraint is *linear* if  $C_q = \ell(C_q)$  for each  $q \in Q$ . For simplicity, we will ask that  $\dim(\ell(C_q))$  be independent of  $q$  and that for each  $q_0 \in Q$  there exists a neighbourhood  $U$  of  $q_0$  and linearly independent vector fields  $X_1, \dots, X_r$  on  $U$  so that

$$\ell(C_q) = \text{span}_{\mathbb{R}} \{ X_1(q), \dots, X_r(q) \}, \quad q \in U,$$

and a vector field  $\eta$  on  $Q$  so that  $C_q = \eta(q) + \ell(C_q)$ . The integer  $r$  is the *rank* of the constraint, denoted  $\text{rank}(C)$ . These assumptions are almost always (but not always) satisfied for physical systems. Note that it may not be possible to find vector fields  $X_1, \dots, X_r$  defined on *all* of  $Q$  which have this property. We shall typically denote a constraint by  $C$  where, as expected,

$$C = \bigcup_{q \in Q} C_q.$$

A curve  $c: I \rightarrow Q$  *satisfies* the constraint  $C$  if  $c'(t) \in C_{c(t)}$  for each  $t \in I$ . Thus we see that what a constraint does is places restrictions on the allowable velocities.

An example here might do some good.

**2.6.1 Example** Let us introduce a new example into the mix. As this is the first time we have seen the example, let us take this opportunity to describe its configuration space and its Lagrangian.

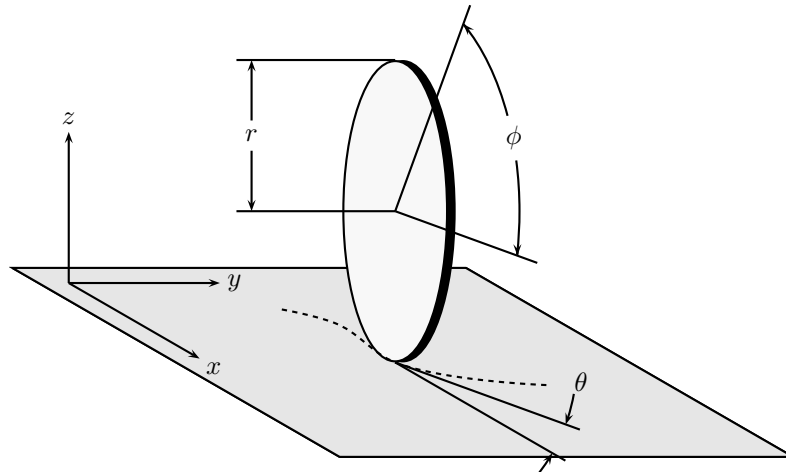


Figure 2.9 A rolling disk

The example is a disk rolling upright on a flat surface as shown in Figure 2.9. The configuration space for the disk is  $Q = \mathbb{R}^2 \times \mathbb{T}^2$ , where  $\mathbb{R}^2$  gives the point where the disk touches the plane and where the angles in  $\mathbb{T}^2$  describe the two angles of rotation, “roll” and “spin.” For coordinates on  $Q$ , we shall take  $(x, y, \theta, \phi)$  as shown in Figure 2.9.

We suppose the disk to have a uniform mass distribution with support in a vertical plane. Thus its inertia tensor has two equal eigenvalues as seen in Proposition 1.5.9. We denote the inertia for the “spin” motion by  $J$  and the inertia for the “roll” motion by  $I$ . The mass of the body we denote by  $m$ . The kinetic energy for the disk is then

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2,$$

giving the associated Riemannian metric as

$$g = m(dx \otimes dx + dy \otimes dy) + Id\theta \otimes d\theta + Jd\phi \otimes d\phi.$$

We suppose the system to be without potential, which will be the case if gravity acts in the  $z$ -direction in Figure 2.9.

Now let us describe the constraints. We ask that the disk roll without slipping on the plane. This will place constraints on the velocities allowed to us, as we shall now see. In Figure 2.10, we view the disk from directly above. A little thought with this picture, and

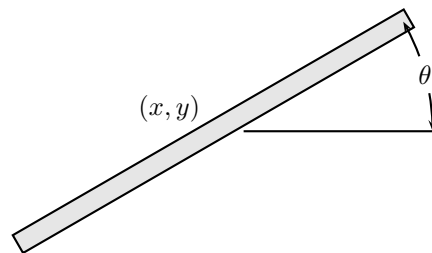


Figure 2.10 The rolling disk from above

one realises that the condition that the disk roll without slipping may be expressed as the

condition

$$\dot{x} = r \cos \theta \dot{\phi}, \quad \dot{y} = r \sin \theta \dot{\phi},$$

where  $r$  is the radius of the disk. This means that the points  $(x, y, \theta, \phi, v_x, v_y, v_\theta, v_\phi)$  in  $TQ$  which are allowed by the constraints must satisfy

$$v_x - r \cos \theta v_\phi = 0, \quad v_y - r \sin \theta v_\phi = 0.$$

We now must put this into the form we have given for a constraint. That is, we must write down the set of allowable velocities at each configuration  $(x, y, \theta, \phi)$ . We shall do this by writing down vector fields on  $Q$  which span the linear part of the constraint distribution at each point. To do this, we note that a vector field  $X$  given by

$$X = X_x \frac{\partial}{\partial x} + X_y \frac{\partial}{\partial y} + X_\theta \frac{\partial}{\partial \theta} + X_\phi \frac{\partial}{\partial \phi}$$

will satisfy the constraints if and only if the vector  $(X_x, X_y, X_\theta, X_\phi)$  lies in the kernel of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -r \cos \theta \\ 0 & 1 & 0 & -r \sin \theta \end{bmatrix}.$$

One readily ascertains that two such vector fields are

$$X_1 = r \cos \theta \frac{\partial}{\partial x} + r \sin \theta \frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}, \quad X_2 = \frac{\partial}{\partial \theta}.$$

We therefore have

$$\mathcal{C}_{(x,y,\theta,\phi)} = \text{span}_{\mathbb{R}} \{X_1(x, y, \theta, \phi), X_2(x, y, \theta, \phi)\}.$$

We note that this is therefore a linear constraint. □

**2.6.2 Holonomic and nonholonomic constraints** We have already seen that a constraint is linear when  $\mathcal{C}_q = \ell(\mathcal{C}_q)$  for each  $q \in Q$ . Linear constraints are by far the predominant type, although others certainly occur. Within the set of linear constraints, however, there is an important distinction which can be made. Let  $q_0 \in Q$ . Denote by  $M_{q_0}$  the set of points  $q \in Q$  for which there exists a piecewise differentiable curve  $c: [0, 1] \rightarrow Q$ , satisfying the constraints, with the property that  $c(0) = q_0$  and  $c(1) = q$ . Thus  $M_{q_0}$  is the set of points reachable from  $q_0$  with curves which satisfy the constraint. The set of points  $M_{q_0}$  will typically be some smooth surface in  $Q$  running through the point  $q_0$ , and so  $M_{q_0}$  will have a well-defined dimension. We shall suppose that this dimension is independent of the point  $q_0$ , something which is frequently true. A linear constraint  $\mathcal{C}$  is **holonomic** if the  $\dim(M_{q_0}) = \text{rank}(\mathcal{C})$ . A constraint which is not holonomic is called **nonholonomic**. Thus, the idea with an holonomic constraint is that one can only access as many dimensions in configuration space as directions are allowed by the constraints. Perhaps a newcomer would expect that this would always be the case. But the fact of the matter is that many linear constraints are in fact nonholonomic. If this were not so, you would not be able to park your car. Indeed, your car can be thought of as having a configuration space of dimension 5 (at least for present purposes): (1) 3 dimensions for the position and orientation of the car (say  $(x, y, \theta)$ ); (2) 1 dimension for the steering wheel angle; (3) 1 dimension for the drive wheel angle i.e., what makes the car move forward. However, you have direct access to only 2 of the 5 directions, one via the steering angle, and the other via the drive wheel angle.

Nonetheless, by properly using these two directions, you can move your car anywhere you want!

Let us give a simple example of an holonomic constraint.

**2.6.2 Example** We work with  $Q = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ . We exclude the origin, as we shall see, because we want the constraint we define to satisfy the property that  $\text{rank}(\mathbf{C})$  be independent of  $q$ . For  $(x, y) \in Q$  we define

$$\mathbf{C}_{(x,y)} = \text{span}_{\mathbb{R}} \left\{ -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right\}.$$

Thus  $\mathbf{C}$  is a linear constraint of rank 1. We shall show that it is holonomic by explicitly constructing the set of points reachable from any point  $(x_0, y_0)$ . Let  $t \mapsto (x(t), y(t))$  be a curve starting at  $(x_0, y_0)$  which satisfies the constraint. This means that  $(\dot{x}(t), \dot{y}(t)) \in \mathbf{C}_{(x(t), y(t))}$ . Since  $\mathbf{C}_{(x,y)}$  is 1-dimensional, this means that there exists  $a(t)$  so that

$$(\dot{x}(t), \dot{y}(t)) = a(t) (-y(t), x(t)).$$

However, this simply means that  $(x(t), y(t))$  satisfy the differential equation

$$\begin{aligned} \dot{x} &= -a(t)y \\ \dot{y} &= a(t)x. \end{aligned}$$

Supposing that  $a(t) \neq 0$  we may rescale time by  $\tau(t) = \int_0^t a(s) ds$ . We then have

$$\begin{aligned} \frac{dx}{d\tau} &= \dot{x} \frac{dt}{d\tau} = -y \\ \frac{dy}{d\tau} &= \dot{y} \frac{dt}{d\tau} = x. \end{aligned}$$

Thus we have rendered the equation linear with constant coefficients:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The curve  $\tau \mapsto (x(\tau), y(\tau))$  then satisfies

$$\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = e^{\mathbf{A}\tau} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{bmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Thus the set of points reachable from  $(x_0, y_0)$  by curves which satisfy the constraint must lie on the circle of radius  $\sqrt{x_0^2 + y_0^2}$  with centre at the origin. This is illustrated in Figure 2.11. Notice that the subspace  $\mathbf{C}_q \subset T_q Q$  is tangent to the set  $M_{q_0}$  of reachable points. This is generally the case.  $\square$

Note that with an holonomic constraint, we may as well specify  $M_{q_0}$  since the tangent vectors in  $\mathbf{C}_{q_0}$  will always simply be tangent to  $M_{q_0}$ . Indeed, one often *does* simply specify the subset of accessible points  $M_{q_0}$  rather than specifying the vectors tangent to this set. In such cases, one directly restricts the configuration space to  $M_{q_0}$ , and so in some sense, one may as well just take  $M_{q_0}$  to be the configuration space. The following example, following from Example 2.6.2, illustrates this.

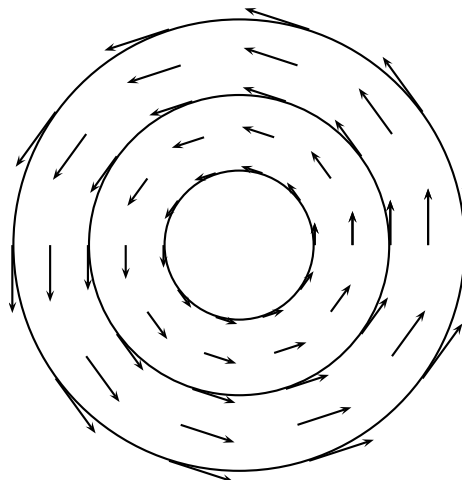


Figure 2.11 An holonomic constraint

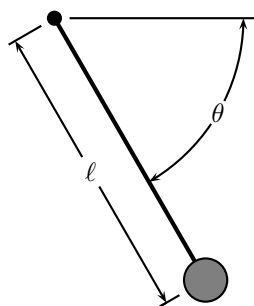


Figure 2.12 The classical pendulum

**2.6.3 Example** (Example 2.6.2 cont'd) We suppose that we wish to model a classical pendulum. Thus the model we use is that of a point mass at the end of a massless rod of length  $\ell$  (see Figure 2.12). We shall declare that the configuration space for the system is  $Q = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ , and then add the constraint that all configurations  $(x, y)$  of the system satisfy  $\sqrt{x^2 + y^2} = \ell$ . Note, however, that the set of all points which satisfy the constraint is exactly the set  $M_{q_0}$  for the constraint  $C$  of Example 2.6.2, provided that one chooses  $q_0 = (x_0, y_0)$  to satisfy  $\sqrt{x_0^2 + y_0^2} = \ell$ . Thus, in principle, there is not much difference between writing the constraint as  $C$  or by asking that all points satisfy the relation  $\sqrt{x^2 + y^2} = \ell$ .

It is also now obvious that by taking  $Q = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ , we have a configuration space that is simply too large for the problem. To fix this, we simply take our actual configuration space to be

$$\tilde{Q} = \{(x, y) \in Q \mid \sqrt{x^2 + y^2} = \ell\}.$$

The smaller configuration space is therefore  $\mathbb{S}^1$  in this example. As we shall see, it is of no consequence whether one chooses to define a holonomic constraint by specifying  $C$  or by specifying the smaller configuration space.  $\square$

Let us now address the question of how one may determine whether a given linear constraint is holonomic or nonholonomic. By using the definition, or by following Example 2.6.2, we would have to construct all curves which start at a given point, and try to determine



the dimension of the set of points reached in this manner. Clearly this is not an attractive proposition. The following result, which we shall not prove, is a famous one, and relies on the Lie bracket which we defined in coordinates in (2.12). We refer to [van der Schaft and Maschke 1994] for a proof of this theorem.

**2.6.4 Frobenius's Theorem** *A linear constraint  $C$  is holonomic if and only if the vector field  $[X, Y]$  takes values in  $C$  for every pair of vector fields  $X$  and  $Y$  taking their values in  $C$ .*

The idea, then, is simple. Although the statement of the result suggests that we need to take every pair of vector fields taking values in  $C$  and check their Lie bracket, in fact it suffices to choose a set of vector fields which form a basis, possibly only locally, for  $C$ .

This is best illustrated with an example.

**2.6.5 Example** (Example 2.6.1 cont'd) We continue on with our example of the rolling disk. As we have seen, the constraint in this example is linear, and has as basis the following two vector fields:

$$X_1 = r \cos \theta \frac{\partial}{\partial x} + r \sin \theta \frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}, \quad X_2 = \frac{\partial}{\partial \theta}.$$

Theorem 2.6.4 tells us that to check whether the constraint is holonomic, we need only take the Lie bracket of the vector fields  $X_1$  and  $X_2$ ; if the resulting vector field satisfies the constraint, then the constraint is holonomic, if it does not then the constraint is nonholonomic. The expression (2.12) for the Lie bracket tells us that the vector of components of the Lie bracket  $[X_1, X_2]$  is the Jacobian of  $X_2$  times  $X_1$  minus the Jacobian of  $X_1$  times  $X_2$ . Doing this gives

$$\begin{aligned} DX_2 \cdot X_1 - DX_1 \cdot X_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \cos \theta \\ r \sin \theta \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -r \sin \theta & 0 \\ 0 & 0 & r \cos \theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} r \sin \theta \\ -r \cos \theta \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Since the Lie bracket does not satisfy the constraint, the constraint is nonholonomic.

That the constraint should be nonholonomic in this example is not surprising, and let us explain why. Start the disk with the initial configuration  $(0, 0, 0, 0)$  and with zero velocity. Were the constraint to be holonomic, then, from this configuration, we should only be able to reach a 2-dimensional subset of other configurations. However, it is possible to *explicitly* give a 3-dimensional subset of reachable configurations. There are two obvious directions which are reachable. The disk can “spin” and “roll.” Therefore, we can reach any configuration of the form  $(r\theta, 0, \theta, \phi)$  where  $\theta$  and  $\phi$  are arbitrary. However, by combining a spin and a roll, one can see that it is possible to make the disk move in a circle of any radius. Therefore, it is possible to reach any configuration of the type  $(0, y, \pi, 2\pi\frac{y}{r})$  where  $y$  is arbitrary (the angle  $2\pi\frac{y}{r}$  is the angle through which the disk would have to roll in a semi-circular arc so that the point of contact would reach the point  $(0, y)$ ). This then explicitly demonstrates a 3-dimensional subset of configurations which are reachable from  $(0, 0, 0, 0)$ , thereby precluding the constraint from being holonomic.  $\square$

A question which now arises is, “What does  $M_{q_0}$  look like when the constraint is *not* holonomic?” We know that when it is holonomic, the set of configurations available from a given configuration is restricted. Is the set of available configurations *unrestricted* when the constraint is nonholonomic? The answer is, “Maybe.” The fact is that the set of available configurations *can* in principle be described using constructions involving the Lie bracket,<sup>3</sup> but we postpone this description until Section 4.5.1 where it will be given a control theoretic context. We will say here, however, that the dimension of the set of accessible configurations can be pretty arbitrary, ranging from a minimum possible value of  $\text{rank}(\mathbf{C})$  to a maximum possible value of  $\dim(Q)$ . For the rolling disk example, it turns out that all points in  $Q$  are accessible from any point with curves which satisfy the constraint.

Let us finish off this section by giving a useful example of an holonomic constraint. This may seem like something of a contrived and silly example, but we shall see in Section 2.7 that it is actually quite useful.

**2.6.6 Example** We let  $Q = (\mathbb{R}^3)^6$  be six copies of  $\mathbb{R}^3$ . Let  $r_1, r_2,$  and  $r_3$  be positive real numbers. We shall construct an holonomic constraint by directly specifying the subset  $\tilde{Q}$  of points which we will consider. Given  $(\mathbf{x}_1, \dots, \mathbf{x}_6) \in Q$  define  $\mathbf{y}_i = \mathbf{x}_i - \mathbf{x}_{i+3}, i = 1, 2, 3$ . We shall say that a point  $q = (\mathbf{x}_1, \dots, \mathbf{x}_6) \in Q$  is in the subset  $\tilde{Q}$  if it satisfies the conditions

1.  $\|\mathbf{y}_i\| = 2r_i, i = 1, 2, 3,$
2. the vectors  $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$  are orthogonal,
3. the matrix

$$\left[ \mathbf{y}_1 \mid \mathbf{y}_2 \mid \mathbf{y}_3 \right]$$

has positive determinant, and

4. the three lines

$$\ell_i = \{ \mathbf{x}_i + t\mathbf{y}_i \mid t \in \mathbb{R} \}, \quad i = 1, 2, 3,$$

have a point of intersection.

The picture is that there is a point  $O \in \mathbb{R}^3$  (the point of intersection  $\ell_1 \cap \ell_2 \cap \ell_3$ ) with the property that the points  $\mathbf{x}_i$  and  $\mathbf{x}_{i+3}$  lie a distance  $r_i$  from  $O$ , and that the lines  $\ell_i, i = 1, 2, 3$  are orthogonal. An attempt to illustrate this is Figure 2.13.

What we are trying to account for here is the configurations of a rigid body using the fact, demonstrated in Example 1.5.10–3, that a general rigid body with nondegenerate inertia tensor is dynamically equivalent to six appropriately placed points in  $\mathbb{R}^3$ . Therefore, the set  $\tilde{Q}$  we are claiming to be the configuration space of six point masses which are rigidly constrained so that they form a rigid body. We should therefore be able to provide an identification of  $\tilde{Q}$  with  $SE(3)$ , since the latter is the set of configurations of a rigid body. Indeed, this is easily done as follows. Let  $(\mathbf{x}_1, \dots, \mathbf{x}_6) \in \tilde{Q}$  and define  $\mathbf{y}_i = \mathbf{x}_i - \mathbf{x}_{i+3}, i = 1, 2, 3$ . Since the vectors  $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$  are orthogonal and since the matrix

$$\left[ \mathbf{y}_1 \mid \mathbf{y}_2 \mid \mathbf{y}_3 \right]$$

has positive determinant, the vectors  $\left\{ \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|}, \frac{\mathbf{y}_2}{\|\mathbf{y}_2\|}, \frac{\mathbf{y}_3}{\|\mathbf{y}_3\|} \right\}$  form a positively oriented orthonormal basis for  $\mathbb{R}^3$ . Therefore, there exists a unique matrix  $\mathbf{R} \in SO(3)$  with the property that  $\mathbf{R}\mathbf{e}_i = \frac{\mathbf{y}_i}{\|\mathbf{y}_i\|}, i = 1, 2, 3$ , where  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  is the standard basis for  $\mathbb{R}^3$ . Also, the point  $O = \ell_1 \cap \ell_2 \cap \ell_3 \in \mathbb{R}^3$  is uniquely defined by our point in  $\tilde{Q}$ . This therefore gives a unique

<sup>3</sup>The result is often call “Chow’s theorem” [Chow 1939].

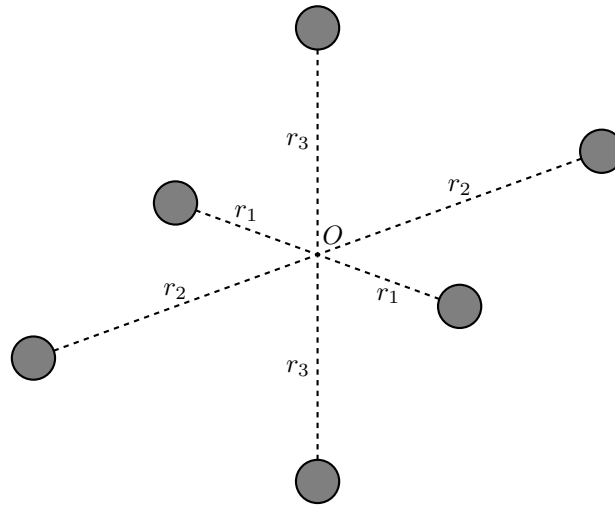


Figure 2.13 Six points in  $\mathbb{R}^3$  satisfying the constraint of Example 2.6.6

element  $(\mathbf{R}, O)$  of  $SE(3)$  which we can associate to each point in  $\tilde{Q}$ . One can also easily provide a unique point in  $\tilde{Q}$  for each element of  $SE(3)$ , which establishes that indeed  $\tilde{Q}$  is the configuration space for a rigid body.  $\square$

**2.6.3 The Euler-Lagrange equations in the presence of constraints** Up til now, our discussion of constraints has focussed on how to write constraints and how to classify them. We have not told you how to actually give dynamical equations which describe the motion of a constrained system. It turns out that to do this, the correct thing to do is to add forces to the Euler-Lagrange equations which are “constraint forces.” These constraint forces have the property that they maintain the constraint. However, they should also have some other properties which ensure that the resulting equations do in fact agree with physical observations.

To provide the magical characterisation of constraint forces, for a constraint  $C$ , and a curve  $c: [a, b] \rightarrow Q$  which satisfies the constraint, a **virtual displacement** along  $c$  is an assignment of a tangent vector  $v(t) \in T_{c(t)}Q$  for each  $t \in [a, b]$  with the property that  $v(t) \in \ell(C_{c(t)})$ . We then define a **constraint force** along  $c$  to be an assignment of a covector  $\alpha(t) \in T_{c(t)}^*Q$  for each  $t \in [a, b]$  which satisfies  $\langle F(t); v(t) \rangle = 0$  for every virtual displacement  $v$ .

**2.6.7 Remarks 1.** A natural question to ask is why virtual displacements are not required to satisfy  $v(t) \in C_{c(t)}$  rather than  $v(t) \in \ell(C_{c(t)})$ . Of course, the two competing definitions would agree when the constraints are linear, but when they are not, there is a genuine difference. Some authors choose to call  $v$  a “possible displacement” when  $v(t) \in C_{c(t)}$ . The introduction of the additional terminology seems to have nothing to recommend it, however. I have not seen a reasonable description of *why* virtual displacements need to be defined the way we have defined them. Thus we must resort to the old mantra of, “Because it agrees with the physics.”

2. For linear constraints, where  $C = \ell(C)$ , constraint forces have the property that they do no work (in the sense of our definition (2.14)) on curves which satisfy the constraints.

However, for constraints which are *not* linear, the constraint forces *will* generally do work on curves which satisfy the constraint.  $\square$

We may now say what it means for a curve to be a solution for a Lagrangian system with a constraint.

**2.6.8 Definition** Let  $Q$  be a configuration space. A **constrained Lagrangian system** on  $Q$  is a triple  $(L, F, C)$  where  $L$  is a Lagrangian on  $Q$ ,  $F$  is a force on  $Q$ , and  $C$  is a constraint on  $Q$ . A curve  $c: [a, b] \rightarrow Q$  is a **solution to the constrained Lagrangian system**  $(L, F, C)$  if there exists a constraint force  $\alpha$  so that together  $c$  and  $\alpha$  satisfy

- (i)  $c$  satisfies the constraint  $C$  and
- (ii)  $c$  satisfies the Lagrange-d'Alembert principle for the force  $F + \alpha$ .  $\square$

Thus, in coordinates  $(q^1, \dots, q^n)$  for  $Q$ , the constrained equations are simply

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = F_i + \alpha_i, \quad i = 1, \dots, n,$$

plus the requirement that  $t \mapsto (q^1(t), \dots, q^n(t))$  satisfies the constraints. Note that the constraint force  $\alpha$  is an unknown in this equation. To make this more concrete, let us represent the constraint equations in a slightly different manner. For  $q \in Q$ , define a subspace of  $T_q^*Q$  by

$$\text{ann}(\ell(C_q)) = \{ \alpha_q \in T_q^*Q \mid \langle \alpha_q; v_q \rangle = 0 \text{ for every } v_q \in \ell(C_q) \}.$$

We also choose a vector field  $\eta$  on  $Q$  so that

$$C_q = \{ \eta(q) + u_q \mid u_q \in \ell(C_q) \}. \quad (2.23)$$

Thus  $\eta(q)$  provides a vector to “shift” the subspace  $\ell(C_q)$  to get the affine subspace  $C_q$ . The condition that a curve  $c: I \rightarrow Q$  satisfy the constraint, i.e., that  $c'(t) \in C_{c(t)}$ , may be written as  $c'(t) - \eta(c(t)) \in \ell(C_{c(t)})$  which is in turn the same as requiring that

$$\langle \alpha; c'(t) - \eta(c(t)) \rangle = 0, \quad \forall \alpha \in \text{ann}(\ell(C_q)).$$

To represent this in a set of coordinates we choose  $n - r$  one-forms  $\alpha^1, \dots, \alpha^{n-r}$  on a coordinate neighbourhood so as to be a basis for  $\text{ann}(\ell(C))$ . A coordinate representation,  $t \mapsto (q^1(t), \dots, q^n(t))$ , of a curve satisfies the constraint if and only if

$$\alpha_i^a \dot{q}^i = \alpha_i^a \eta^i, \quad a = 1, \dots, n - r.$$

Thus we arrive at the following coordinate characterisation of the solutions for a constrained Lagrangian system.

**2.6.9 Proposition** Let  $(L, F, C)$  be a constrained Lagrangian system on  $Q$ . A curve  $c: [a, b] \rightarrow Q$  is a solution to the constrained Lagrangian system  $(L, F, C)$  if and only if for every chart  $(U, \phi)$  for which  $U$  intersects the image of  $c$ , the coordinate representation,  $t \mapsto \mathbf{q}(t)$ , of  $c$  satisfies the equations

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} &= F_i + \lambda_a \alpha_i^a, & i = 1, \dots, n, \\ \alpha_i^a \dot{q}^i &= \alpha_i^a \eta^i, & a = 1, \dots, n - r, \end{aligned}$$

where

- (i)  $\alpha^1, \dots, \alpha^{n-r}$  are one-forms on  $U$  forming a basis for  $\text{ann}(\ell(\mathbf{C}))$ ,
- (ii)  $\eta$  is a vector field satisfying (2.23), and
- (iii) where  $\lambda_1, \dots, \lambda_{n-r}$  are functions of  $t$  to be determined, and are called **Lagrange multipliers**.

*Proof* The only thing we have not already shown is that the unknown constraint force should be expressible as  $\alpha = \lambda_a \alpha^a$ . However, this is clear since by our definition of constraint forces, and since  $\alpha^1, \dots, \alpha^{n-r}$  form a basis for  $\text{ann}(\ell(\mathbf{C}))$ . ■

**2.6.10 Remark** The  $2n - r$  equations of Proposition 2.6.9 have both a differential and an algebraic component. Indeed, equations of this type are often called DAE's, for "differential algebraic equations." As such, methods for solution, including numerical methods, exist. □

Let us see how this works out for our rolling disk example. It is really quite simple, as we shall see.

**2.6.11 Example** (Example 2.6.1 cont'd) We work with the coordinates  $(x, y, \theta, \psi)$  as given in Example 2.6.1. Let us first determine one forms  $\alpha^1, \alpha^2$  which form a basis for  $\text{ann}(\ell(\mathbf{C}))$ . We have practically done this already. Recall that admissible velocities were specified to satisfy the relations

$$v_x - r \cos \theta v_\phi = 0, \quad v_y - r \sin \theta v_\phi = 0.$$

From this, we can immediately "read off" the one-forms as

$$\alpha^1 = dx - r \cos \theta d\phi, \quad \alpha^2 = dy - r \sin \theta d\phi.$$

Using the system Lagrangian  $L = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}Iv_\theta^2 + \frac{1}{2}Jv_\phi^2$ , we then determine the equations of Proposition 2.6.9 to be

$$\begin{aligned} m\ddot{x} &= \lambda_1 \\ m\ddot{y} &= \lambda_2 \\ I\ddot{\theta} &= 0 \\ J\ddot{\phi} &= -\lambda_1 r \cos \theta - \lambda_2 r \sin \theta \\ \dot{x} - r \cos \theta \dot{\phi} &= 0 \\ \dot{y} - r \sin \theta \dot{\phi} &= 0. \end{aligned} \tag{2.24}$$

Let us not deal right now with the matter of solving these equations, but leave this until Chapter 3.

**2.6.12 Remark** In our above discussion we have presented a method for computing equations of motion for Lagrangian systems with constraints. There are other methods available for determining equations of motion. Some of these are now listed.

1. **The Poincaré equations:** This method is one which eliminates the need for Lagrange multipliers, by incorporating the equations of constraint into the equations of motion by introducing "quasi-velocities." This method receives a quite coherent treatment in the recent book of Talman [2000].
2. **The Gibbs-Appell equations:** These equations use not only "quasi-velocities," but "quasi-accelerations." In the original formulation of Appell [1900a, 1900b] and Gibbs [1879] (yes, the free energy guy), the treatment is for particles and rigid bodies. A generalisation to arbitrary Lagrangian systems is provided by Lewis [1996].

3. **Gauss's principle of least constraint:** This method is of a variational nature, although Gauss [1829] was able to cast the problem so that the variational methods are finite dimensional. That this method bears a strong resemblance to the Gibbs-Appell equations is flushed out in a general setting by Lewis [1996].
4. **The “vakonomic method”:** Notice that the Lagrange-d'Alembert principle, unlike the characterisation Theorem 2.3.2 of the Euler-Lagrange equations, is not given in the form of a variational problem. That is, the Lagrange-d'Alembert principle does not produce equations which are necessary conditions for the minimisation of a functional on a class of curves. Since Definition 2.6.8 gives the definition of the solution of a constrained Lagrangian system in terms of the Lagrange-d'Alembert principle in which the external force is a constraint force, it follows that one may expect that the solutions to a constrained system, even without external forces which are not constrained forces, are not curves which minimise a functional on a class of curves. This is indeed the case, and is seen by some as a weakness of Definition 2.6.8. In an attempt to circumvent this difficulty, Kozlov [1983] gives a variational formulation of constrained motion, calling the resulting equations of “variational axiomatic kind,” leading to the name “vakonomic.” The problem is that these equations are genuinely different from the equations of Definition 2.6.8, as is shown in a simple example in [Lewis and Murray 1995]. There has been a heated debate over which of the methods, the vakonomic method or any of the methods equivalent to Definition 2.6.8, are correct [e.g., Kharlomov 1992, Kozlov 1992]. In Section 2.7 we shall show that the equations derived from Definition 2.6.8 can be derived from Newtonian mechanics. Therefore, if the vakonomic method is correct, it is so in violation of Newtonian mechanics. This constitutes, one should think, the death knell for the vakonomic method, at least as concerns its validity for modelling “normal” mechanical systems.

This is an incomplete list, and some people who read it will be offended in the extreme at the omission of their favourite method. However, with the exception of the “vakonomic method,” all methods are equivalent. Depending on the problem one is looking at, and what one wishes to do, one or the other of the available methods may be the most suitable.  $\square$

**2.6.4 Simple mechanical systems with constraints** Let us briefly specialise our discussion of constraints to simple mechanical systems with Lagrangians of the form  $L(v_q) = \frac{1}{2}g(v_q, v_q) - V(q)$ . To present our main result, we need the notion of orthogonal projection. We let  $(Q, g, V)$  be a simple mechanical system and we let  $C$  be a constraint on  $Q$ . We let  $\ell(C)^\perp$  be the  $g$ -orthogonal complement to  $\ell(C)$ . Thus for each  $q \in Q$  we have a subspace of  $T_qQ$  defined by

$$\ell(C_q)^\perp = \{v_q \in T_qQ \mid g(v_q, u_q) = 0 \text{ for all } u_q \in \ell(C_q)\}.$$

Now any vector  $v \in TQ$  can be written as  $v = v_1 + v_2$  with  $v_1 \in \ell(C)$  and  $v_2 \in \ell(C)^\perp$ . We then define a map  $P_C: TQ \rightarrow TQ$  which sends a vector  $v$  to its component  $v_2 \in \ell(C)^\perp$ .

Let us see how to compute all of these objects in our rolling disk example.

**2.6.13 Example** (Example 2.6.1 cont'd) For the rolling disk we have  $Q = \mathbb{R}^2 \times \mathbb{T}^2$  and we were using coordinates  $(x, y, \theta, \phi)$ . The Riemannian metric in these coordinates is

$$g = m(dx \otimes dx + dy \otimes dy) + Id\theta \otimes d\theta + Jd\phi \otimes d\phi,$$

and the one-forms which form a basis for  $\text{ann}(\ell(\mathbf{C}))$  are

$$\alpha^1 = dx - r \cos \theta d\phi, \quad \alpha^2 = dy - r \sin \theta d\phi.$$

Referring to Exercise E2.35, we see that the vector fields  $g^\sharp(\alpha^1)$  and  $g^\sharp(\alpha^2)$  form a basis for  $\ell(\mathbf{C})^\perp$ . To compute the components of these vector fields, we may use matrix notation. The components of  $g^\sharp(\alpha^1)$  are then

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & J \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -r \cos \theta \end{bmatrix} = \begin{bmatrix} \frac{1}{m} \\ 0 \\ 0 \\ -\frac{r}{J} \cos \theta \end{bmatrix},$$

so that

$$g^\sharp(\alpha^1) = \frac{1}{m} \frac{\partial}{\partial x} - \frac{r}{J} \cos \theta \frac{\partial}{\partial \phi}.$$

We similarly compute

$$g^\sharp(\alpha^2) = \frac{1}{m} \frac{\partial}{\partial y} - \frac{r}{J} \sin \theta \frac{\partial}{\partial \phi}.$$

This then gives a basis for  $\ell(\mathbf{C})^\perp$ , so effectively describing it.

Let us now turn to the computation of the orthogonal projection  $P_{\mathbf{C}}$ . This is actually a not entirely trivial thing to compute, but let us go through the steps. From Example 2.6.1 we have the vector fields

$$X_1 = r \cos \theta \frac{\partial}{\partial x} + r \sin \theta \frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}, \quad X_2 = \frac{\partial}{\partial \theta}$$

as a basis for  $\ell(\mathbf{C})$ . We also have the vector fields

$$X_3 = \frac{1}{m} \frac{\partial}{\partial x} - \frac{r}{J} \cos \theta \frac{\partial}{\partial \phi}, \quad X_4 = \frac{1}{m} \frac{\partial}{\partial y} - \frac{r}{J} \sin \theta \frac{\partial}{\partial \phi}$$

as a basis for  $\ell(\mathbf{C})^\perp$ . Therefore, any tangent vector  $v$  can be written as

$$v = v^1 X_1 + v^2 X_2 + v^3 X_3 + v^4 X_4$$

for some suitable coefficients  $(v^1, v^2, v^3, v^4)$ . The orthogonal projection of  $v$  onto  $\ell(\mathbf{C})^\perp$  will then be simply

$$P_{\mathbf{C}}(v) = v^3 X_3 + v^4 X_4. \quad (2.25)$$

But it remains to do the computations. To determine  $(v^1, v^2, v^3, v^4)$  we must use the change of basis formula. Thus we write

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \underbrace{\begin{bmatrix} r \cos \theta & r \sin \theta & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{m} & 0 & 0 & -\frac{r}{J} \cos \theta \\ 0 & \frac{1}{m} & 0 & -\frac{r}{J} \sin \theta \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{bmatrix}.$$

With the  $4 \times 4$  matrix  $\mathbf{P}$  as defined above, if we write

$$v = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_\theta \frac{\partial}{\partial \theta} + v_\phi \frac{\partial}{\partial \phi} = v^1 X_1 + v^2 X_2 + v^3 X_3 + v^4 X_4,$$

then we may determine  $(v^1, v^2, v^3, v^4)$  in terms of  $(v_x, v_y, v_\theta, v_\phi)$  using the change of basis formula:

$$\begin{aligned} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{bmatrix} &= \mathbf{P}^{-t} \begin{bmatrix} v_x \\ v_y \\ v_\theta \\ v_\phi \end{bmatrix} \\ &= \frac{1}{mr^2 + J} \begin{bmatrix} mr(\cos \theta v_x + \sin \theta v_y) + Jv_\phi \\ (mr^2 + J)v_\theta \\ (mJ + \frac{1}{2}m^2r^2(1 - \cos 2\theta))v_x - \frac{1}{2}m^2r^2 \sin 2\theta - mrJ \cos \theta v_\phi \\ -\frac{1}{2}m^2r^2 \sin 2\theta v_x + (mJ + \frac{1}{2}m^2r^2(1 + \cos 2\theta))v_y - mrJ \sin \theta v_\phi \end{bmatrix}. \end{aligned}$$

Using (2.25) we may now compute the components of  $P_C(v)$  relative to the basis  $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\}$  as

$$\frac{1}{mr^2 + J} \underbrace{\begin{bmatrix} J + mr^2 \sin^2 \theta & -mr^2 \sin \theta \cos \theta & 0 & -rJ \cos \theta \\ -mr^2 \sin \theta \cos \theta & J + mr^2 \cos^2 \theta & 0 & -rJ \sin \theta \\ 0 & 0 & 0 & 0 \\ -mr \cos \theta & -mr \sin \theta & 0 & mr^2 \end{bmatrix}} \begin{bmatrix} v_x \\ v_y \\ v_\theta \\ v_\phi \end{bmatrix}.$$

Thus the object with the brace under it is the matrix representation of  $P_C$  at the point  $q = (x, y, \theta, \phi)$  relative to the basis  $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\}$  for  $T_q Q$ .  $\square$

The example illustrates that unless we have to, we really do not want to participate in computing the projection  $P_C$ , and certainly we would wish to have at hand a computer to do the algebra. But, it is in principle computable.

The tedium of the previous example notwithstanding, let us see what our notation using  $\ell(C)^\perp$  and  $P_C$  gives us. The following result is useful because it gives us a way of writing the equations for constrained systems in a coordinate independent manner *without* resorting to a variational statement.

**2.6.14 Proposition** *Let  $(Q, g, V)$  be a simple mechanical control system with Lagrangian  $L$ , let  $F$  be a force on  $Q$ , and let  $C$  be a constraint on  $Q$ . Let  $\eta$  be the unique vector field with the properties that  $\eta(q) \in \ell(C_q)^\perp$  and that*

$$C_q = \{\eta(q) + v_q \mid v_q \in \ell(C_q)\}.$$

*Then a curve  $c: [a, b] \rightarrow Q$  is a solution for the constrained system  $(L, F, C)$  if and only for each  $t \in [a, b]$  there exists  $\lambda(t) \in \ell(C_{c(t)})^\perp$  so that*

$$\begin{aligned} \overset{g}{\nabla}_{c'(t)} c'(t) &= -\text{grad } V(c(t)) + g^\sharp(F(t, c'(t))) + \lambda(t) \\ P_C(c'(t)) &= \eta(c(t)). \end{aligned}$$

*Proof* The result will follow from Propositions 2.5.10 and 2.6.9 provided we can show that the constraint force  $\alpha$  in Definition 2.6.8 has the property that  $g^\sharp(\alpha(c(t))) \in \ell(C)^\perp$ . However, this follows since  $g^\sharp: T^*Q \rightarrow TQ$  maps  $\text{ann}(\ell(C))$  exactly onto  $\ell(C)^\perp$ . This fact follows from Exercise E2.35.  $\blacksquare$



The essential content of the result is that for simple mechanical systems, when we apply  $g^\sharp$  to a constraint force, the resulting vector field is  $g$ -orthogonal to the linear part of the constraint distribution. This will be helpful for us when we look at establishing a correspondence between Newtonian and Lagrangian mechanics in Section 2.7.

Let us verify Proposition 2.6.14 for our rolling disk example.

**2.6.15 Example** (Example 2.6.1 cont'd) We have yet to compute the Christoffel symbols for the rolling disk, but this is easy since the coefficients of the Riemannian metric in the coordinates  $(x, y, \theta, \phi)$  are constant, implying that all Christoffel symbols are zero. In Example 2.6.13 we determined that the vector fields

$$X_3 = \frac{1}{m} \frac{\partial}{\partial x} - \frac{r}{J} \cos \theta \frac{\partial}{\partial \phi}, \quad X_4 = \frac{1}{m} \frac{\partial}{\partial y} - \frac{r}{J} \sin \theta \frac{\partial}{\partial \phi}$$

were a basis for  $\ell(\mathbf{C})^\perp$ . Therefore, the vector field  $\lambda(t)$  of Proposition 2.6.14 must be of the form  $\lambda(t) = \lambda^1(t)X_3(c(t)) + \lambda^4(t)X_4(c(t))$  for some functions  $\lambda^3, \lambda^4: [a, b] \rightarrow \mathbb{R}$ . Therefore, we may write the differential equation of Proposition 2.6.14 as

$$\begin{aligned} \ddot{x} &= \frac{1}{m} \lambda^3 \\ \ddot{y} &= \frac{1}{m} \lambda^4 \\ \ddot{\theta} &= 0 \\ \ddot{\phi} &= -\frac{r}{J} \cos \theta \lambda^3 - \frac{r}{J} \sin \theta \lambda^4. \end{aligned}$$

These equations are obviously equivalent to the first four of the equations (2.24). To these differential equations, we must append the constraint equations, which, in Proposition 2.6.14 are the equations  $P_C(c'(t))$ . Since we have computed  $P_C$  in Example 2.6.13, we can in principle write down these equations, but let us not do this, as it is ugly, and at this stage pointless.  $\square$

**2.6.16 Remark** The computations in this section with our disk example make one wonder about the actual utility of Proposition 2.6.14. While the equations in abstract form look nice, in practice to compute  $P_C$  is burdensome. It is true that if one's only objective is to write equations of motion, then computing  $P_C$  is not recommended. However, it turns out that the formulation of the equations of motion in Proposition 2.6.14 has some interesting benefits, particularly in the study of control theory for these systems. It turns out that using the equations of Proposition 2.6.14 as background, one may put the equations of motion for a constrained simple mechanical control system into affine connection form, but using an affine connection which is *not* the Levi-Civita connection associated with the kinetic energy Riemannian metric. The control theory is explained in the paper [Lewis 2000b], while the affine connection background is explained in [Lewis 1998], motivated by earlier papers [e.g., Bloch and Crouch 1998, Synge 1928]. We shall deal with this briefly in Section 3.8.3.  $\square$

**2.6.5 The Euler-Lagrange equations for holonomic constraints** As a final element of our discussion of constrained systems, let us demonstrate that for holonomic constraints defined by their constraint surface  $M_{q_0}$ , one may genuinely restrict attention to  $M_{q_0}$ , forgetting that  $M_{q_0}$  sits inside some larger configuration space.

**2.6.17 Theorem** *Let  $(L, F, C)$  be a constrained Lagrangian system and let  $C$  be holonomic. Let  $q_0 \in Q$  and define  $\tilde{Q} = M_{q_0}$  to be the set of points accessible from  $q_0$  by curves satisfying the constraint and let  $T\tilde{Q}$  by the subset of  $TQ$  consisting of vectors tangent to  $\tilde{Q}$ . Denote by  $\tilde{L}$  the restriction of  $L$  to  $T\tilde{Q}$ .*

*For a curve  $c: [a, b] \rightarrow Q$  with the property that  $\text{image}(c) \subset \tilde{Q}$ , the following statements are equivalent:*

- (i) *a curve  $c: [a, b] \rightarrow Q$  is a solution to the constrained Lagrangian system  $(L, C, F)$  with  $c(a) \in \tilde{Q}$ ;*
- (ii) *there exists a force  $\tilde{F}$  on  $\tilde{Q}$  so that  $c: [a, b] \rightarrow \tilde{Q}$  satisfies the Lagrange-d'Alembert principle for  $\tilde{F}$  on  $\tilde{Q}$ .*

*Proof* Let  $c(t)$  be a point on  $\text{image}(c)$  and let  $(q^1, \dots, q^n)$  be coordinates around  $c(t)$ . Let us choose the coordinates so that  $(q^1, \dots, q^r)$  are coordinates for  $\tilde{Q}$  and so that  $q^{n-r+1} = \dots = q^n = 0$  on  $\tilde{Q}$ . (That this can be done is true, but is not obvious from anything we have said here.) The curve  $c$  in these coordinates then looks like  $t \mapsto (q^1(t), \dots, q^r(t), 0, \dots, 0)$ . Since the vector fields  $\frac{\partial}{\partial q^i}$ ,  $i = 1, \dots, r$  are tangent to  $\tilde{Q}$ , we can use  $(q^1, \dots, q^r, v^1, \dots, v^r)$  as coordinates for  $T\tilde{Q}$ . We then have

$$\tilde{L}(t, q^1, \dots, q^r, v^1, \dots, v^r) = L(t, q^1, \dots, q^r, 0, \dots, 0, v^1, \dots, v^r, 0, \dots, 0).$$

We also note that  $\alpha^1 = dq^{n-r+1}, \dots, \alpha^{n-r} = dq^n$  form a basis for  $\text{ann}(\ell(C))$ . We may now use the equations of Proposition 2.6.9 to represent in coordinates the solution of the constrained system:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial v^a} \right) - \frac{\partial L}{\partial q^a} &= F_a, & a = 1, \dots, r \\ \frac{d}{dt} \left( \frac{\partial L}{\partial v^b} \right) - \frac{\partial L}{\partial q^b} &= F_b + \lambda_b, & b = n - r + 1, \dots, n \\ \dot{q}^b &= 0, & b = n - r + 1, \dots, n. \end{aligned}$$

Since  $q^{n-r+1}(t) = \dots = q^n(t) = 0$  we have

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial v^b} \right) &= \\ \frac{\partial L}{\partial t}(t, q^1(t), \dots, q^r(t), 0, \dots, 0, \dot{q}^1(t), \dots, \dot{q}^r(t), 0, \dots, 0), & \quad b = 1, \dots, n - r + 1, \dots, n. \end{aligned}$$

For similar reasons we have

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial v^a} \right) &= \frac{\partial L}{\partial t} + \frac{\partial^2 L}{\partial v^a \partial v^i} \dot{q}^i + \frac{\partial^2 L}{\partial v^a \partial q^i} \dot{q}^i \\ &= \frac{\partial L}{\partial t} + \frac{\partial^2 L}{\partial v^a \partial v^c} \dot{q}^c + \frac{\partial^2 L}{\partial v^a \partial q^c} \dot{q}^c, & a = 1, \dots, r. \end{aligned}$$

where the index  $i$  runs from 1 to  $n$ , and the index  $c$  runs from 1 to  $r$ , and where all partial derivatives are evaluated at  $(t, q^1(t), \dots, q^r(t), 0, \dots, 0, \dot{q}^1(t), \dots, \dot{q}^r(t), 0, \dots, 0)$ . One therefore sees that, in fact

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^a} \right) = \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial v^a} \right).$$

Similarly,

$$\frac{\partial L}{\partial q^a} = \frac{\partial \tilde{L}}{\partial q^a}$$

when the left-hand side has the argument  $(t, q^1(t), \dots, q^r(t), 0, \dots, 0, \dot{q}^1(t), \dots, \dot{q}^r(t), 0, \dots, 0)$ . Therefore, if we define a force on  $\tilde{Q}$  by

$$\tilde{F}(t, q^1, \dots, q^r, v^1, \dots, v^r) = F(t, q^1, \dots, q^r, 0, \dots, 0, v^1, \dots, v^r, 0, \dots, 0),$$

the result follows. ■

This seems perhaps a bit intimidating, but it is very simple. The idea is that when the constraint is holonomic, then we may well restrict to the subset  $\tilde{Q}$  specified by the constraint. Once we do this, we can also restrict the Lagrangian, and in doing so, we can then write the unconstrained Euler-Lagrange equations on  $\tilde{Q}$ . The question then is, are the equations we get the same as the equations we get if we apply the definition of a constrained solution? The above theorem answers this question in the affirmative.

This is easily illustrated with an example.

**2.6.18 Example** (Examples 2.1.5 and 2.6.2 cont'd) We look again at the pendulum example of Example 2.6.3. Thus we work with  $Q = \mathbb{R}^2 \setminus \{\mathbf{0}\}$  and the holonomic constraint  $C$  which gives rise to the subset

$$\tilde{Q} = \{(x, y) \in Q \mid \sqrt{x^2 + y^2} = \ell\}$$

as in Example 2.6.3. By the nature of  $\tilde{Q}$ , obviously polar coordinates are the better coordinates for the job. Indeed, in polar coordinates we simply have

$$\tilde{Q} = \{(r, \theta) \mid r = \ell\}.$$

Therefore, since  $r$  is fixed by restricting to  $\tilde{Q}$ , we can simply use  $\theta$  as a coordinate. We then have

$$T\tilde{Q} = \{(r, \theta, v_r, v_\theta) \mid r = \ell, v_r = 0\}$$

as the description of the tangent bundle  $T\tilde{Q}$  in polar coordinates. Thus we may use  $(\theta, v_\theta)$  as coordinates for  $T\tilde{Q}$ . The restricted kinetic energy will then be  $\frac{1}{2}m\ell^2 v_\theta^2$ , simply the restriction of the kinetic energy to  $T\tilde{Q}$ . Since the pendulum also needs potential energy, we also need to restrict the gravitational potential function. But from Example 2.4.3 we can easily see that the restriction of the potential function to  $\tilde{Q}$  is  $ma_g(\ell \sin \theta - r_0 \sin \theta_0)$ . Thus the restricted Lagrangian, as a function of  $(\theta, v_\theta)$ , is

$$L(\theta, v_\theta) = \frac{1}{2}m\ell^2 v_\theta^2 - ma_g(\ell \sin \theta - r_0 \sin \theta_0).$$

The Euler-Lagrange equations for this Lagrangian are

$$m\ell^2 \ddot{\theta} + ma_g \ell \cos \theta = 0. \tag{2.26}$$

Let us now compare this with what happens when we write the equations using Lagrange multipliers. First we need to find a basis for  $\text{ann}(\ell(C))$ . But this is simple since in polar coordinates, the vector field  $\frac{\partial}{\partial \theta}$  is a basis for  $C$ . Therefore  $dr$  will be a basis for  $\text{ann}(C)$ . Thus we take  $\alpha^1 = dr$ . The equations of Proposition 2.6.9 then read

$$\begin{aligned} m\ddot{r} - r\dot{\theta}^2 &= \lambda_1 \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + ma_g \ell \cos \theta &= 0 \\ \dot{r} &= 0. \end{aligned}$$

From the first equation, using the fact that  $r(t) = \ell$ , we read off  $\lambda_1 = -\ell\dot{\theta}^2$  which determines the constraint force. From the second equation, we directly see that it is equivalent to (2.26), using again the fact that  $r(t) = \ell$ .  $\square$

## 2.7 Newton's equations and the Euler-Lagrange equations

By now, we have presented a rather general setting for Lagrangian mechanics, and although we have seen that things appear to be reasonable in a couple of examples, the discerning reader will question the very idea of Lagrangian mechanics as modelling the physical world, and will be crying out once more for Newton's mechanics. This is hardly surprising since the contents of the chapter to this point have been a compression of about 250 years of intense work by many of the best mathematicians and mechanicians during that time. To attempt to quell some of these anxieties, in this section we shall prove that Lagrangian mechanics and Newtonian mechanics agree, at least when both can be applied. Clearly, Lagrangian mechanics is more general, deriving as it does from an abstract variational principle on a general configuration space. Thus we will provide a fairly general Newtonian setting for mechanics, and then show that the laws of motion of Newton are the same as the Euler-Lagrange equations.

**2.7.1 Lagrangian mechanics for a single particle** We shall start with something very simple. We consider a single particle of mass  $m$  moving in  $\mathbb{R}^3$  and subject to a force and a constraint. Thus we have fixed a coordinate system  $\phi$  for some Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ , and so the  $\mathbb{R}^3$  we are considering is the first component of the canonical Galilean spacetime  $\mathbb{R}^3 \times \mathbb{R}$ . Let us first deal with the Newtonian mechanics for our particle. We shall consider the particle to have a force  $\mathbf{F}$  acting on it. Thus, in the Newtonian setting,  $\mathbf{F}$  is a map from  $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$  to  $\mathbb{R}^3$  which sends a triple  $(t, \mathbf{x}, \mathbf{v})$  of time, position, and velocity to a force. We also consider a constraint on the particle. Thus we constrain the velocities of the particle to satisfy equations of the form

$$\mathbf{c}^a(\mathbf{x}) \cdot \mathbf{v} = f^a(\mathbf{x}), \quad a = 1, \dots, 3 - r.$$

for maps  $\mathbf{c}^a: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and  $f^a: \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $a = 1, \dots, 3 - r$ . Here  $r \in \{0, 1, 2, 3\}$ , depending on how many constraints we place on the particle. Note that this form of writing the constraints follows along lines like the constraint equations of Proposition 2.6.9. This type of constraint will take into account any of the various ways in which the motion of a particle may be constrained; for example

1. the particle may be constrained to move on a surface in  $\mathbb{R}^3$ ,
2. the particle may be constrained to follow a prescribed path,
3. the particle may be constrained to slide on the surface of a rigid body, or
4. the particle may have its velocity constrained by some external source.

One may wish to consider generalisations where the maps  $\mathbf{c}^1, \dots, \mathbf{c}^{3-r}$  are time-dependent. This, however, is easily done—one simply sticks a  $t$  in the argument of  $\mathbf{c}^a$ ,  $a = 1, \dots, 3 - r$ , at each stage. We shall not worry about whether constraints are holonomic or nonholonomic—of the examples of constraints listed above, many are holonomic. As we have seen, we may think of an holonomic constraint as being nonholonomic. Therefore we will keep all constraints in the general nonholonomic form which allows holonomic constraints as a

special case. In Newtonian mechanics, experimental observation leads one to the conclusion that forces of constraint are always orthogonal to the direction of the linear part of the constraint, with orthogonality being with respect to the standard inner product on  $\mathbb{R}^3$ . That is to say, if you, for example, looked at any of the four constraint situations enumerated above, you would measure that they behaved as if they were subjected to a force orthogonal to the linear part of the constraint. Thus a constraint force will have the form

$$\mathbf{R} = \lambda_1 \mathbf{c}^1(\mathbf{x}) + \cdots + \lambda_{3-r} \mathbf{c}^{3-r}(\mathbf{x}),$$

for some  $\lambda_1, \dots, \lambda_{3-r}$ . Newton's equations are now written as

$$\begin{aligned} m\ddot{\mathbf{x}} &= \mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}}) + \lambda_1 \mathbf{c}^1(\mathbf{x}) + \cdots + \lambda_{3-r} \mathbf{c}^{3-r}(\mathbf{x}) \\ \mathbf{c}^a(\mathbf{x}) \cdot \dot{\mathbf{q}} &= f^a(\mathbf{x}), \quad a = 1, \dots, 3-r, \end{aligned} \quad (2.27)$$

where the functions of  $t, \lambda_1, \dots, \lambda_{3-r}$ , are to be determined.

This completes the Newtonian description of the dynamics of a single forced and constrained particle. For the Lagrangian description, let us fix upon using the canonical Cartesian coordinates  $(x, y, z)$  for  $Q = \mathbb{R}^3$  which give  $(x, y, z, v_x, v_y, v_z)$  as the natural coordinates for  $TQ$ . The kinetic energy of the particle is then

$$K = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2),$$

giving the Riemannian metric

$$g = m(dx \otimes dx + dy \otimes dy + dz \otimes dz)$$

on  $Q$ . We can subtract from this a potential function to get the classical Lagrangian, but since the result is simply an external force, but one of a specific type, we shall omit the potential function from the Lagrangian, and consider its appearance as part of the external force. Indeed, now is a good time to describe a force in our Lagrangian setup. By our definition, a force is to be a  $T^*Q$ -valued function with coefficients which may depend on time, position, and velocity. Thus we write a force as

$$F = F_x dx + F_y dy + F_z dz,$$

where  $F$ , as well as the coefficients  $F_x, F_y$ , and  $F_z$ , have the argument  $(t, x, y, z, v_x, v_y, v_z)$ . A constraint is then specifiable, as in Proposition 2.6.9, by a set of equations of the form

$$\alpha_i^a(x, y, z)v^i = \beta^a(x, y, z), \quad a = 1, \dots, 3-r,$$

for functions  $\alpha_i^a, a = 1, \dots, 3-r, i = 1, 2, 3$ , and  $\beta^a, a = 1, \dots, 3-r$ , on  $Q = \mathbb{R}^3$ . By Proposition 2.6.9, the solution to the constrained system then satisfies the equations

$$\begin{aligned} m\ddot{x} &= F_x + \lambda_a \alpha_x^a \\ m\ddot{y} &= F_y + \lambda_a \alpha_y^a \\ m\ddot{z} &= F_z + \lambda_a \alpha_z^a \\ \alpha_i^a \dot{q}^i &= \beta^a, \quad a = 1, \dots, 3-r. \end{aligned} \quad (2.28)$$

However, if we define

$$\begin{aligned} \mathbf{x} &= (x, y, z), \quad \mathbf{v} = (v_x, v_y, v_z) \\ \mathbf{F}(t, \mathbf{x}, \mathbf{v}) &= (F_x(t, x, y, z, v_x, v_y, v_z), F_y(t, x, y, z, v_x, v_y, v_z), F_z(t, x, y, z, v_x, v_y, v_z)) \\ \mathbf{c}^a(\mathbf{x}) &= (\alpha_x^a(x, y, z), \alpha_y^a(x, y, z), \alpha_z^a(x, y, z)), \end{aligned}$$

then clearly the equations (2.27) and (2.28) are equivalent; indeed they are identical!

Thus we have shown the following:

*For a single particle moving in  $\mathbb{R}^3$  subjected to a general constraint<sup>4</sup> and a general force, the equations of Newtonian mechanics and Lagrangian mechanics agree.*

### 2.7.2 Lagrangian mechanics for multi-particle and multi-rigid body systems

Of course, the simplification of the previous section is too simple to be really all that useful. Most mechanical systems one encounters are formed of various interconnections of particles and rigid bodies. For example, a look through the exercises at the end of the chapter will yield several mechanical systems which are built up of particles and rigid bodies. For each of these, one could apply Newtonian methods, with appropriate constraints to model the interconnectedness, and arrive a set of governing equations. One could then choose coordinates for the systems, possibly taking into account any holonomic constraints, and then derive the Euler-Lagrange equations. The resulting set of equations would be equivalent to those derived using Newtonian mechanics. In this section we shall prove that this is generally the case. The first thing we need to do is simplify things by considering only particles and not rigid bodies. We do this by resorting to Example 2.6.6 which shows that a rigid body with nondegenerate inertia tensor can be regarded as six particles with an holonomic constraint keeping them in a certain configuration. Using this fact, we can model *any* collection of interconnected particles and rigid bodies by a collection of particles which are appropriately constrained.

Let us proceed then by modelling a set of particles subject to arbitrary forces and constraints in a Newtonian setting. We shall suppose that we have  $N$  particles with masses  $m_1, \dots, m_N$ . To describe the system we use  $(\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^3)^N$  to provide the positions of the  $N$  masses. Each mass is subject to a force which may depend on time, and on the position and velocity of the other  $N$  masses. Thus the force on mass  $i$  is written  $\mathbf{F}_i(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{v}_1, \dots, \mathbf{v}_N)$ . The collection of masses is also subject to constraints. Note that we do not constrain each mass independently since one can expect a constraint to relate two or more of the masses. Thus we have constraint equations of the form

$$\mathbf{c}_1^a(\mathbf{x}_1, \dots, \mathbf{x}_N)\mathbf{v}_1 + \dots + \mathbf{c}_N^a(\mathbf{x}_1, \dots, \mathbf{x}_N)\mathbf{v}_N = f^a(\mathbf{x}_1, \dots, \mathbf{x}_N), \quad a = 1, \dots, 3N - r, \quad (2.29)$$

for maps  $\mathbf{c}_i^a: (\mathbb{R}^3)^N \rightarrow \mathbb{R}^3$ ,  $i = 1, \dots, N$ ,  $a = 1, \dots, 3N - r$ , and  $f^a: (\mathbb{R}^3)^N \rightarrow \mathbb{R}$ ,  $a = 1, \dots, 3N - r$ . Here  $r \in \{0, \dots, 3N - r\}$  depends on the number of constraints which we apply to the system of particles. As with the single particle, we can readily make the constraints time-dependent, but let us agree not to do this. The constraints place a force on each mass. Again, we simply assert that the force of constraint on the  $i$ th mass is given by a force of the form

$$\mathbf{R}_i = \lambda_1 \mathbf{c}_i^1(\mathbf{x}_1, \dots, \mathbf{x}_N) + \dots + \lambda_{3N-r} \mathbf{c}_i^{3N-r}(\mathbf{x}_1, \dots, \mathbf{x}_N)$$

for some  $\lambda_1, \dots, \lambda_{3N-r}$ . This is one of those facts that is empirically verified.

With the nature of the forces of constraint at hand, we can now immediately write down

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<sup>4</sup>Some, including the author, may disagree that the constraint we consider is the most general. However, it is quite general, particularly if one does the business of formally sticking in a  $t$  in various place to make the constraints possibly time-dependent.

Newton's equations. These are

$$\begin{aligned}
\ddot{\mathbf{x}}_1 &= \mathbf{F}_1(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \dot{\mathbf{x}}_1, \dots, \dot{\mathbf{x}}_N) + \sum_{a=1}^{3N-r} \lambda_a \mathbf{c}_1^a(\mathbf{x}_1, \dots, \mathbf{x}_N) \\
&\vdots \\
\ddot{\mathbf{x}}_N &= \mathbf{F}_N(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \dot{\mathbf{x}}_1, \dots, \dot{\mathbf{x}}_N) + \sum_{a=1}^{3N-r} \lambda_a \mathbf{c}_N^a(\mathbf{x}_1, \dots, \mathbf{x}_N) \\
\mathbf{c}_1^1(\mathbf{x}_1, \dots, \mathbf{x}_N) \dot{\mathbf{x}}_1 + \dots + \mathbf{c}_N^1(\mathbf{x}_1, \dots, \mathbf{x}_N) \dot{\mathbf{x}}_N &= f^1(\mathbf{x}_1, \dots, \mathbf{x}_N), \quad a = 1, \dots, 3N - r.
\end{aligned} \tag{2.30}$$

As usual, the functions of time,  $\lambda_1, \dots, \lambda_{3N-r}$  are to be determined.

For the Lagrangian description, we take  $Q = (\mathbb{R}^3)^N$  and use coordinates  $(x_1, y_1, z_1, \dots, x_N, y_N, z_N)$ . We shall write the corresponding natural coordinates for  $TQ$  as  $(x_1, y_1, z_1, \dots, x_N, y_N, z_N, v_{x_1}, v_{y_1}, v_{z_1}, \dots, v_{x_N}, v_{y_N}, v_{z_N})$ . The kinetic energy of the system is

$$K = \frac{1}{2} m_1 (v_{x_1}^2 + v_{y_1}^2 + v_{z_1}^2) + \dots + \frac{1}{2} m_N (v_{x_N}^2 + v_{y_N}^2 + v_{z_N}^2).$$

This gives the Riemannian metric

$$g = m_1(dx_1 \otimes dx_1 + dy_1 \otimes dy_1 + dz_1 \otimes dz_1) + \dots + m_N(dx_N \otimes dx_N + dy_N \otimes dy_N + dz_N \otimes dz_N).$$

A general force is written as

$$F = F_{x_1} dx_1 + F_{y_1} dy_1 + F_{z_1} dz_1 + \dots + F_{x_N} dx_N + F_{y_N} dy_N + F_{z_N} dz_N,$$

where the coefficients are functions of time, as well as functions of all the coordinates and their velocities. A general constraint, in the Lagrangian formalism, can be written using Proposition 2.6.9 as

$$\alpha_i^a(q) v^i = \beta^a(q),$$

where  $q$  denotes a point in  $Q = (\mathbb{R}^3)^N$ . One now directly applies Proposition 2.6.9 to get the equations of motion in the Lagrangian setting as

$$\begin{aligned}
m_1 \ddot{x}_1 &= F_{x_1} + \sum_{a=1}^{3N-r} \lambda_1 \alpha_1^a, & m_1 \ddot{y}_1 &= F_{y_1} + \sum_{a=1}^{3N-r} \lambda_1 \alpha_2^a, & m_1 \ddot{z}_1 &= F_{z_1} + \sum_{a=1}^{3N-r} \lambda_1 \alpha_3^a \\
&\vdots \\
m_N \ddot{x}_N &= F_{x_N} + \sum_{a=1}^{3N-r} \lambda_1 \alpha_{3N-2}^a, & m_N \ddot{y}_N &= F_{y_N} + \sum_{a=1}^{3N-r} \lambda_1 \alpha_{3N-1}^a, & m_N \ddot{z}_N &= F_{z_N} + \sum_{a=1}^{3N-r} \lambda_1 \alpha_{3N}^a \\
\alpha_i^a(q) \dot{q}^i &= \beta^a(q), \quad a = 1, \dots, 3N - r.
\end{aligned} \tag{2.31}$$

Now we define

$$\begin{aligned}
\mathbf{x}_i &= (x_i, y_i, z_i), \quad i = 1, \dots, N, & \mathbf{v}_i &= (v_{x_i}, v_{y_i}, v_{z_i}), \quad i = 1, \dots, N \\
\mathbf{F}_i &= (F_{x_i}, F_{y_i}, F_{z_i}), \quad i = 1, \dots, N \\
\mathbf{c}_i^a &= (\alpha_{3i-2}^a, \alpha_{3i-1}^a, \alpha_{3i}^a), \quad i = 1, \dots, N, \quad a = 1, \dots, 3N - r,
\end{aligned}$$



where we have omitted arguments for convenience. As with the single particle, one readily sees that the equations (2.30) and (2.31) are identical. Thus we have demonstrated the following:

*For an arbitrary number of particles and rigid bodies moving in  $\mathbb{R}^3$  subjected to a general constraint and a general force, the equations of Newtonian mechanics and Lagrangian mechanics agree.*

**2.7.1 Remark** The above calculations are harder to write down than they are to understand. This is in contrast to the usual situation where much work is made of demonstrating the equivalence of the Newtonian and Lagrangian formalism. Part of the reason for this is that most authors deal explicitly with holonomic constraints and introduce a set of coordinates to parameterise the set  $M_{q_0}$ . However, as we have demonstrated in Theorem 2.6.17, it matters not whether one restricts to the subset of  $Q$  defined by the holonomic constraints as far as the Lagrangian formalism is concerned. It therefore suffices to consider holonomic constraints as nonholonomic constraints, and if all one is after is a demonstration of the equivalence of the Newtonian and Lagrangian formalisms, then this is sufficient.  $\square$

## 2.8 Euler's equations and the Euler-Lagrange equations

In our above discussion of the equivalence of the Newtonian and the Lagrangian formulations for mechanics, we cheated somewhat when considering rigid bodies in that we modelled a rigid body as a collection of particles subject to constraints. While this is correct, it deprives us of any insight in understanding how Euler's equations for rigid body motion are related to the Euler-Lagrange equations. Let us address this point by demonstrating explicitly the equivalence between the two sets of equations for a single rigid body in the absence of constraints and forces.

Thus in this section we consider a rigid body  $(\mathcal{B}, \mu)$  moving in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$  via the rigid motion  $\Sigma = (\Psi, \nu)$ . We consider a centre of mass observer  $\mathcal{O} \in \text{Ob}(\Sigma)$ . By choosing a coordinate system  $\phi \in \text{Coor}(\mathcal{O})$ , the equations which govern the rigid body are, by Proposition 1.6.12,

$$\begin{aligned}\dot{\mathbf{R}}(t) &= \mathbf{R}(t)\hat{\mathbf{\Omega}}(t) \\ \dot{\mathbf{r}}(t) &= \mathbf{R}(t)\mathbf{V}(t) \\ \mathbb{I}_c(\dot{\mathbf{\Omega}}(t)) &= (\mathbb{I}_c(\mathbf{\Omega}(t))) \times \mathbf{\Omega}(t) \\ \mu(\mathcal{B})\dot{\mathbf{V}}(t) &= \mu(\mathcal{B})\mathbf{V}(t) \times \mathbf{\Omega}(t),\end{aligned}$$

where  $t \mapsto (\mathbf{R}(t), \mathbf{r}(t))$  are coordinate representations of the rigid motion, and where  $\mathbf{\Omega}$ ,  $\mathbf{V}$ , and  $\mathbb{I}_c$  are coordinate representations of the body angular velocity, the body linear velocity, and the inertia tensor. While these equations do indeed describe the motion of the rigid body, the problem is that it is not so easy to compare them to the Euler-Lagrange equations of the rigid body since the body velocities  $\mathbf{\Omega}$  and  $\mathbf{V}$  are *not* expressible as derivatives of coordinates. Actually, you will recall that with our choice of a centre of mass observer and a coordinate system adapted to this observer, the linear body velocity  $\mathbf{V}(t)$  is in fact  $\dot{\mathbf{r}}(t)$ , and so is a legitimate Lagrangian velocity. However, the body angular velocity, being defined by  $\mathbf{\Omega}(t) = \mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t)$  is not actually a legitimate Lagrangian velocity. This is not entirely obvious, but is true (Exercise E2.37). This then raises the question about what is the best way to do Lagrangian mechanics for a rigid body. Let us now turn to this very issue.



**2.8.1 Lagrangian mechanics for a rigid body** We retain the setup in the introduction to the section, and we try to fit this setup into our Lagrangian scheme. The configuration space for the rigid body is clearly  $Q = SE(3)$ . We shall represent a point in  $SE(3)$  by  $(\mathbf{R}, \mathbf{r}) \in SO(3) \times \mathbb{R}^3$ . We need to assign a Lagrangian to the system. The rigid body has kinetic energy. To define the kinetic energy, we integrate the kinetic energy of each point in the rigid body via its mass distribution. The following result records the answer.

**2.8.1 Lemma** *Let  $(\mathcal{B}, \mu)$  be a rigid body in a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, \mathbf{V}, \mathbf{g}, \tau)$  which undergoes a rigid motion  $\Sigma = (\Psi, \nu)$ . Let  $\mathcal{O} \in \text{Ob}(\Sigma)$  and suppose that  $\phi$  is a coordinate system adapted to  $\mathcal{O}$ . If  $t \mapsto (\mathbf{R}(t), \mathbf{r}(t))$  are the coordinate representations of the rigid motion,  $t \mapsto \boldsymbol{\Omega}(t)$  and  $t \mapsto \mathbf{V}(t)$  are the coordinate representations of the body angular and linear velocity, and  $\mathbb{I}_c$  is the coordinate representation of the inertia tensor about the centre of mass, then the kinetic energy of the rigid body along the rigid motion is*

$$t \mapsto \frac{1}{2}\mu(\mathcal{B}) \|\mathbf{V}(t)\|^2 + \frac{1}{2}g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}(t)), \boldsymbol{\Omega}(t)).$$

*Proof* Suppose that  $\mathcal{B} \subset \mathcal{E}(s_0)$ . If  $T_\Sigma$  is the motion associated with the rigid motion  $\Sigma$ , then the kinetic energy is by definition

$$\frac{1}{2} \int \|T'_\Sigma(s, x)\| \, d\mu.$$

However, with the notation we have introduced for our coordinate representation of the rigid motion, the coordinate form of  $T_\Sigma$  is

$$\mathbf{T}_\Sigma(t, \mathbf{x}) = \mathbf{R}(t)\mathbf{x} + \mathbf{r}(t),$$

since the observer at each instant is at the origin in the coordinate system  $\phi$ . Here  $\mathbf{x}$  is a point in the rigid body  $\phi_{s_0}(\mathcal{B})$ . Therefore the kinetic energy at time  $t$  is

$$K(t) = \frac{1}{2} \int \|\dot{\mathbf{R}}(t)\mathbf{x} + \dot{\mathbf{r}}(t)\|^2 \, d\mu.$$

We have  $\dot{\mathbf{R}}(t) = \mathbf{R}(t)\hat{\boldsymbol{\Omega}}(t)$  and  $\dot{\mathbf{r}}(t) = \mathbf{R}(t)\mathbf{V}(t)$  using the definitions of body velocity. Thus we compute

$$\begin{aligned} K(t) &= \frac{1}{2} \int \|\mathbf{R}(t)\mathbf{V}(t)\|^2 \, d\mu + \frac{1}{2} \int \|\mathbf{R}(t)(\boldsymbol{\Omega}(t) \times \mathbf{x})\|^2 \, d\mu + \\ &\quad \int g_{\text{can}}(\mathbf{R}(t)\mathbf{V}(t), \mathbf{R}(t)(\boldsymbol{\Omega}(t) \times \mathbf{x})) \, d\mu \\ &= \frac{1}{2}\mu(\mathcal{B}) \|\mathbf{V}(t)\|^2 + \frac{1}{2} \int g_{\text{can}}(\mathbf{x} \times (\boldsymbol{\Omega}(t) \times \mathbf{x}), \boldsymbol{\Omega}(t)) \, d\mu \\ &= \frac{1}{2}\mu(\mathcal{B}) \|\mathbf{V}(t)\|^2 + \frac{1}{2}g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}(t)), \boldsymbol{\Omega}(t)). \end{aligned}$$

Here we have used Lemma 1.5.2(ii), the vector identity (1.19), and the definition of the inertia tensor. ■

As an immediate corollary to the lemma, we have the following description of the Riemannian metric describing the rigid body kinetic energy. Note that in the statement of the corollary, we are writing a point in  $TSO(3)$  as  $(\mathbf{R}, \mathbf{A})$ . Thus  $\mathbf{A}$  is a matrix representing a tangent vector in the tangent space  $T_{\mathbf{R}}SO(3)$ ; one may wish to think of  $\mathbf{A}$  as being  $\dot{\mathbf{R}}$ . We also write  $\mathbf{A}^\vee$  for the image of a  $3 \times 3$  skew-symmetric matrix  $\mathbf{A}$  under the inverse of the map  $\hat{\cdot}$ .

**2.8.2 Corollary** *The Riemannian metric on  $Q = SE(3)$  describing the kinetic energy of Lemma 2.8.1 is given by*

$$g_{\mathcal{B}}(\mathbf{r}, \mathbf{R})((\mathbf{v}_1, \mathbf{A}_1), (\mathbf{v}_2, \mathbf{A}_2)) = \mu(\mathcal{B})g_{\text{can}}(\mathbf{v}_1, \mathbf{v}_2) + g_{\text{can}}(\mathbb{I}_c((\mathbf{R}^{-1}\mathbf{A}_1)^\vee), (\mathbf{R}^{-1}\mathbf{A}_2)^\vee).$$

*Proof* From the expression of Lemma 2.8.1 for the rigid body kinetic energy, the only thing we need to show is that any tangent vector  $\mathbf{A} \in T_{\mathbf{R}}SO(3)$  can be written as  $\mathbf{R}\hat{\boldsymbol{\Omega}}$  for some  $\hat{\boldsymbol{\Omega}} \in \mathbb{R}^3$ . But this follows from the proof of Proposition 1.4.5 where we showed that if  $t \mapsto \mathbf{R}(t)$  is a curve in  $SO(3)$ , then the tangent vector  $\dot{\mathbf{R}}(t)$  in the tangent space  $T_{\mathbf{R}(t)}SO(3)$  always has the form  $\mathbf{R}(t)\hat{\boldsymbol{\Omega}}(t)$  for some  $\hat{\boldsymbol{\Omega}}(t) \in \mathbb{R}^3$ . Since this is true for *any* curve  $t \mapsto \mathbf{R}(t)$ , it must be true for *any* tangent vector at any point. ■

Lemma 2.8.1 provides us, then, with a description of the kinetic energy of a rigid body in terms of the body velocities. What one would like to do is use this expression as the Lagrangian for the rigid body since we are assuming there are no external forces or constraints. However, as we have seen, the body angular velocity is not a normal Lagrangian velocity, so we cannot use it as we use “ $v$ ” in the Euler-Lagrange equations. This leaves us with having to use coordinates for  $SE(3)$  to describe the motion of a rigid body in the Lagrangian setting. Since  $SE(3) = SO(3) \times \mathbb{R}^3$  this amounts really to choose coordinates for  $SO(3)$ . For this, one could use the Euler angles  $(\alpha, \beta, \gamma)$  of Example 2.1.3–3. One would then have to express the velocities  $(v_\alpha, v_\beta, v_\gamma)$  in terms of the body angular velocity to get the kinetic energy expressed in a form where one could apply the Euler-Lagrange equations. To do this, one could differentiate the expression (2.3) with respect to  $t$  (suppose that  $(\alpha, \beta, \gamma)$  were functions of  $t$ ) and then use the relation  $\dot{\mathbf{R}}(t) = \mathbf{R}(t)\hat{\boldsymbol{\Omega}}(t)$ . One would then have a Lagrangian given in terms of coordinates for  $TSO(3)$ , and it would then be possible to compute the Euler-Lagrange equations. While this is possible, it is not very pleasant, and is not a very agreeable way to prove that the Euler equations are equivalent to the Euler-Lagrange equations.

In the next section we illustrate a rather more slick way to show the equivalence we are after.

**2.8.2 A modified variational principle** The idea we give here is due to Marsden and Scheurle [1993] and is also explained in the book [Marsden and Ratiu 1999]. What we shall do is, rather than compute the Euler-Lagrange equations for a rigid body, we will employ the variational principle of Section 2.3 to deduce the equivalence of the Euler equations for a rigid body with the Euler-Lagrange equations. The result is the following important theorem.

**2.8.3 Theorem** *On  $Q = SE(3)$  consider the Riemannian metric  $g_{\mathcal{B}}$  given by Corollary 2.8.2 and let  $L$  be the corresponding Lagrangian function on  $TSO(3)$ . Define a function  $\ell$  on  $\mathbb{R}^3 \times \mathbb{R}^3$  by*

$$\ell(\mathbf{v}, \boldsymbol{\omega}) = \frac{1}{2}\mu(\mathcal{B})\|\mathbf{v}\|^2 + \frac{1}{2}g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\omega}), \boldsymbol{\omega}).$$

*The following statements are equivalent:*

- (i)  $t \mapsto (\mathbf{R}(t), \mathbf{r}(t))$  is a solution of the Euler-Lagrange equations with Lagrangian  $L$ ;
- (ii)  $t \mapsto (\mathbf{R}^{-1}(t)\dot{\mathbf{r}}(t), \mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t))$  satisfies the Euler equations.

*Furthermore, the above two statements are implied by the following:*

(iii)  $t \mapsto (\mathbf{R}^{-1}(t)\dot{\mathbf{r}}(t), \mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t))$  minimises the functional

$$(\mathbf{v}(t), \boldsymbol{\omega}(t)) \mapsto \int_a^b \ell(\mathbf{v}(t), \boldsymbol{\omega}(t)) dt$$

over curves  $t \mapsto (\boldsymbol{\omega}(t), \mathbf{v}(t)) \in \mathbb{R}^3 \times \mathbb{R}^3$  where the variations have the property that their infinitesimal variations satisfy

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \boldsymbol{\omega}(s, t) &= \boldsymbol{\eta}(t) + \boldsymbol{\omega}(t) \times \boldsymbol{\eta}(t) \\ \frac{d}{ds} \Big|_{s=0} \mathbf{v}(s, t) &= \dot{\mathbf{u}}(t) + \boldsymbol{\omega}(t) \times \mathbf{u}(t) - \boldsymbol{\eta}(t) \times \mathbf{v}(t), \end{aligned}$$

where  $\mathbf{u}$  and  $\boldsymbol{\eta}$  vanish at the endpoints.

*Proof* Let us first show that (iii) is equivalent to the variational principle of Theorem 2.3.2. Thus, we let  $\mathbf{g}: [a, b] \ni t \mapsto (\mathbf{R}(t), \mathbf{r}(t))$  be a curve on  $SE(3)$  and define a curve on  $\mathbb{R}^3 \times \mathbb{R}^3$  by

$$\boldsymbol{\xi}(t) = (\boldsymbol{\Omega}(t), \mathbf{V}(t)) = (\mathbf{R}^{-1}(t)\dot{\mathbf{r}}(t), (\mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t))^\vee).$$

Let  $\boldsymbol{\sigma}: (s, t) \mapsto (\mathbf{r}(s, t), \mathbf{R}(s, t))$  be a variation of a curve  $c$ . We shall first show that the infinitesimal variation corresponding to  $\boldsymbol{\sigma}$  gives rise to an infinitesimal variation of  $\boldsymbol{\xi}$  which satisfies the conditions of (iii). Define  $\boldsymbol{\eta}(t) = \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{R}(s, t)$ . Using the fact that  $\frac{d}{dt} \mathbf{R}^{-1}(t) = -\mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t)\mathbf{R}^{-1}(t)$ , we then have

$$\dot{\boldsymbol{\eta}}(t) = -\mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t)\mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{R}(s, t) + \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \dot{\mathbf{R}}(s, t)$$

We then compute

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \boldsymbol{\Omega}(s, t) &= -\mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{R}(s, t) \mathbf{R}^{-1}(t) \dot{\mathbf{R}}(t) + \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \dot{\mathbf{R}}(s, t) \\ &= \dot{\boldsymbol{\eta}} + (\mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t)) \left( \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{R}(s, t) \right) - \\ &\quad \left( \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{R}(s, t) \right) (\mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t)) \\ &= \dot{\boldsymbol{\eta}}(t) + \hat{\boldsymbol{\Omega}}(t)\dot{\boldsymbol{\eta}}(t) - \dot{\boldsymbol{\eta}}(t)\hat{\boldsymbol{\Omega}}(t) \\ &= \dot{\boldsymbol{\eta}}(t) + \boldsymbol{\Omega}(t) \times \boldsymbol{\eta}(t). \end{aligned}$$

In the last step we have used Exercise E1.11. We also need to show that variations of  $\mathbf{V}(t)$  have the specified form. If we let  $\mathbf{u}(t) = \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{r}(s, t)$  we compute

$$\dot{\mathbf{u}}(t) = -\mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t)\mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{r}(s, t) + \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \dot{\mathbf{r}}(s, t).$$

Therefore

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \mathbf{V}(s, t) &= -\mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{R}(s, t) \mathbf{R}^{-1}(t) \dot{\mathbf{r}}(t) + \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \dot{\mathbf{r}}(s, t) \\ &= \dot{\mathbf{u}}(t) + \mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t) \left( \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{r}(s, t) \right) - \\ &\quad \left( \mathbf{R}^{-1}(t) \frac{d}{ds} \Big|_{s=0} \mathbf{R}(s, t) \right) \mathbf{R}^{-1}(t) \dot{\mathbf{r}}(t) \\ &= \hat{\boldsymbol{\Omega}}(t)\mathbf{u}(t) - \dot{\boldsymbol{\eta}}(t)\mathbf{V}(t) \\ &= \dot{\mathbf{u}}(t) + \boldsymbol{\Omega}(t) \times \mathbf{u}(t) - \boldsymbol{\eta}(t) \times \mathbf{V}(t). \end{aligned}$$

Now, since we have

$$\int_a^b \ell(\mathbf{V}(s, t), \boldsymbol{\Omega}(s, t)) dt = \int_a^b L(\mathbf{R}(s, t), \dot{\mathbf{R}}(s, t)) dt,$$

it follows that

$$\frac{d}{ds} \Big|_{s=0} \int_a^b L(\mathbf{R}(s, t), \dot{\mathbf{R}}(s, t)) dt$$

will vanish for arbitrary variations if and only if

$$\frac{d}{ds} \Big|_{s=0} \int_a^b \ell(\mathbf{V}(s, t), \boldsymbol{\Omega}(s, t)) dt$$

vanishes for all variations of the form specified in part (iii) of the theorem.

By Theorem 2.3.2, the above calculations also show that (i) is implied by (iii).

Let us show that (iii) implies (ii). Suppose that we have a variation of  $t \mapsto (\mathbf{V}(t), \boldsymbol{\Omega}(t))$  satisfying the conditions of part (iii) of the theorem. We then compute

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \int_a^b \ell(\mathbf{V}(s, t), \boldsymbol{\Omega}(s, t)) dt &= \int_a^b \left( \frac{\partial \ell}{\partial \boldsymbol{\omega}} \frac{d}{ds} \Big|_{s=0} \boldsymbol{\Omega}(s, t) + \frac{\partial \ell}{\partial \mathbf{v}} \frac{d}{ds} \Big|_{s=0} \mathbf{V}(s, t) \right) dt \\ &= \int_a^b \left( \frac{\partial \ell}{\partial \boldsymbol{\omega}} (\dot{\boldsymbol{\eta}}(t) + \boldsymbol{\Omega}(t) \times \boldsymbol{\eta}(t)) + \right. \\ &\quad \left. \frac{\partial \ell}{\partial \mathbf{v}} (\dot{\mathbf{u}}(t) + \boldsymbol{\Omega}(t) \times \mathbf{u}(t) - \boldsymbol{\eta}(t) \times \mathbf{V}(t)) \right) dt \\ &= \int_a^b \left( -\frac{d}{dt} \frac{\partial \ell}{\partial \boldsymbol{\omega}} \boldsymbol{\eta}(t) + \frac{\partial \ell}{\partial \boldsymbol{\omega}} (\boldsymbol{\Omega}(t) \times \boldsymbol{\eta}(t)) - \right. \\ &\quad \left. \frac{d}{dt} \frac{\partial \ell}{\partial \mathbf{v}} \mathbf{u}(t) + \frac{\partial \ell}{\partial \mathbf{v}} (\boldsymbol{\Omega}(t) \times \mathbf{u}(t) - \boldsymbol{\eta}(t) \times \mathbf{V}(t)) \right) dt \\ &= \int_a^b \left( -\frac{d}{dt} \frac{\partial \ell}{\partial \boldsymbol{\omega}} + \frac{\partial \ell}{\partial \boldsymbol{\omega}} \times \boldsymbol{\Omega}(t) + \frac{\partial \ell}{\partial \mathbf{v}} \times \mathbf{V}(t) \right) \boldsymbol{\eta}(t) dt + \\ &\quad \int_a^b \left( -\frac{d}{dt} \frac{\partial \ell}{\partial \mathbf{v}} + \frac{\partial \ell}{\partial \mathbf{v}} \times \boldsymbol{\Omega}(t) \right) \mathbf{u}(t) dt, \end{aligned}$$

where all partial derivatives are evaluated at  $(\boldsymbol{\Omega}(t), \mathbf{V}(t))$ . Using the given expression for  $\ell$  we have

$$\frac{\partial \ell}{\partial \mathbf{v}}(\boldsymbol{\Omega}(t), \mathbf{V}(t)) = \mu(\mathcal{B})\mathbf{V}(t), \quad \frac{\partial \ell}{\partial \boldsymbol{\omega}}(\boldsymbol{\Omega}(t), \mathbf{V}(t)) = \mathbb{I}_c(\boldsymbol{\Omega}(t)).$$

Therefore, using the fact that

$$\frac{d}{ds} \Big|_{s=0} \int_a^b \ell(\mathbf{V}(s, t), \boldsymbol{\Omega}(s, t)) dt = 0$$

for arbitrary variations gives

$$\begin{aligned} \mathbb{I}_c(\dot{\boldsymbol{\Omega}}(t)) &= (\mathbb{I}_c(\boldsymbol{\Omega}(t))) \times \boldsymbol{\Omega}(t) \\ \mu(\mathcal{B}) \dot{\mathbf{V}}(t) &= \mu(\mathcal{B}) \mathbf{V}(t) \times \boldsymbol{\Omega}(t). \end{aligned}$$

However, these are exactly the Euler equations, and thus this completes the proof.  $\blacksquare$

This result provides a useful application of variational methods, although Theorem 2.8.3 strictly follows from the results of Section 2.7.

## 2.9 Hamilton's equations

The Hamiltonian formalism (named for Sir William Rowan Hamilton (1805–1865)) provides another setting in which to study the problems of mechanics. Some like to quibble over which is the “best” setting. We shall adopt the position that each provides advantages which can be exploited, and circumstances often dictate which, if any, approach is best. However, since the topic under discussion is Lagrangian mechanics, we will marginalise Hamiltonian mechanics only for this reason. There are many excellent texts which give a modern overview of Hamiltonian mechanics and dynamics [Abraham and Marsden 1978, Arnol'd 1989, Libermann and Marle 1987, Talman 2000].

Let  $L: \mathbb{R} \times TQ \rightarrow \mathbb{R}$  be a Lagrangian on  $Q$ . Recall that in Section 2.5.3 we had defined the Legendre transformation as a map  $\mathbf{FL}: \mathbb{R} \times TQ \rightarrow \mathbb{R} \times T^*Q$  which in a set of coordinates is given by

$$(t, q^1, \dots, q^n, v^1, \dots, v^n) \mapsto \left( t, q^1, \dots, q^n, \frac{\partial L}{\partial v^1}, \dots, \frac{\partial L}{\partial v^n} \right).$$

We wish to think of the Legendre transformation as a change of coordinates, and compute the form of the Euler-Lagrange equations in these coordinates. This can only be done for certain Lagrangians. To wit, a Lagrangian  $L$  is **hyperregular**<sup>5</sup> when  $\mathbf{FL}$  is a diffeomorphism; that is, when  $\mathbf{FL}$  is invertible and it and its inverse are smooth. The following result ensures that a large number of important Lagrangians are hyperregular.

**2.9.1 Proposition** *If  $(Q, g, V)$  is a simple mechanical system with associated Lagrangian  $L$ , the Legendre transformation  $\mathbf{FL}$  is hyperregular.*

*Proof* This is easily seen in coordinates where

$$\mathbf{FL}(t, q^1, \dots, q^n, v^1, \dots, v^n) = (t, q^1, \dots, q^n, g_{1j}v^j, \dots, g_{nj}v^j).$$

From the coordinate expression (2.7) for the map  $g^\flat: TQ \rightarrow T^*Q$ , we see that  $\mathbf{FL} = g^\flat$ . Since  $g^\flat$  is invertible with inverse  $g^\sharp: T^*Q \rightarrow TQ$ , the result follows. ■

Corresponding to a hyperregular Lagrangian  $L: \mathbb{R} \times TQ \rightarrow \mathbb{R}$  on  $Q$  define the corresponding **Hamiltonian**  $H_L: \mathbb{R} \times T^*Q \rightarrow \mathbb{R}$  by

$$H_L(t, \alpha_q) = \mathbf{EL}(t, \mathbf{FL}^{-1}(\alpha_q)) = \langle \alpha_q, \mathbf{FL}^{-1}(\alpha_q) \rangle - L(t, \mathbf{FL}^{-1}(\alpha_q)).$$

The following result gives the form of “Hamilton's equations,” and how they are related to the Euler-Lagrange equations.

**2.9.2 Theorem** *Let  $L$  be a hyperregular Lagrangian on a configuration space  $Q$  with  $H_L$  the corresponding Hamiltonian. Let  $c$  be a curve on  $Q$  and let  $(U, \phi)$  be a coordinate chart for  $Q$  which intersects  $\text{image}(c)$ . Denote natural coordinates for  $TQ$  in this coordinate chart by  $(q^1, \dots, q^n, v^1, \dots, v^n)$  and denote natural coordinates for  $T^*Q$  by  $(q^1, \dots, q^n, p_1, \dots, p_n)$ .*

*The following statements concerning  $c$  are equivalent:*

(i)  *$c$  satisfies the Euler-Lagrange equations*

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n;$$

---

<sup>5</sup>We have bypassed the notion of merely being **regular**. Regularity occurs when for each  $v_q \in TQ$  there exists a neighbourhood  $U$  of  $v_q$  so that  $\mathbf{FL}|_U$  is a diffeomorphism onto its image. Regularity may be easily shown to be equivalent to the  $n \times n$  matrix with components  $\frac{\partial^2 L}{\partial v^i \partial v^j}$ ,  $i, j = 1, \dots, n$ , being invertible.

(ii) the curve  $t \mapsto \mathbf{FL}(c'(t))$  satisfies **Hamilton's equations**

$$\begin{aligned}\dot{q}^i &= \frac{\partial H_L}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H_L}{\partial q^i},\end{aligned}\quad i = 1, \dots, n.$$

*Proof* In coordinates we have

$$H_L(t, \mathbf{q}, \mathbf{p}) = p_i v^i - L(t, \mathbf{q}, \mathbf{v}),$$

where we think of  $\mathbf{v}$  as being expressed in terms of  $\mathbf{p}$  using the inverse of the Legendre transformation. We thus compute

$$\begin{aligned}dH_L &= \frac{\partial H_L}{\partial q^i} dq^i + \frac{\partial H_L}{\partial p_i} dp_i + \frac{\partial H_L}{\partial t} dt \\ &= v^i dp_i + p_j \frac{\partial v^j}{\partial q^i} dq^i + p_j \frac{\partial v^j}{\partial p_i} dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial v^j} \frac{\partial v^j}{\partial p_i} dp_i - \frac{\partial L}{\partial t} dt.\end{aligned}$$

Now we use the fact that  $v^j$  is independent of  $q^i$  and the fact that  $p_j = \frac{\partial L}{\partial v^j}$ ,  $j = 1, \dots, n$ , to simplify the above expression to

$$\frac{\partial H_L}{\partial q^i} dq^i + \frac{\partial H_L}{\partial p_i} dp_i + \frac{\partial H_L}{\partial t} dt = v^i dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial t} dt,$$

where again  $\mathbf{v}$  is to be thought of as a function of  $\mathbf{p}$  via the inverse of the Legendre transformation. This gives us the equalities

$$\frac{\partial H_L}{\partial q^i} = -\frac{\partial L}{\partial q^i}, \quad \frac{\partial H_L}{\partial p_i} = v^i.$$

Now suppose that the curve  $c$  satisfies the Euler-Lagrange equations. In this case,  $v^i = \dot{q}^i$ ,  $i = 1, \dots, n$ , so the first  $n$  of Hamilton's equations hold. Also, since  $p_i = \frac{\partial L}{\partial v^i}$ ,  $i = 1, \dots, n$ , and since the Euler-Lagrange equations hold, we have

$$\frac{\partial H_L}{\partial q^i} = -\frac{\partial L}{\partial q^i} = -\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) = -\dot{p}_i, \quad i = 1, \dots, n.$$

Thus the second  $n$  of Hamilton's equations also hold.

Now suppose that Hamilton's equations hold. One then easily sees that

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n.$$

This completes the proof. ■

The coordinates  $p_i$ ,  $i = 1, \dots, n$ , are called **conjugate momenta** in the Hamiltonian formalism.

Let us see what Hamilton's equations look like for our planar particle example.

**2.9.3 Example** (Example 2.1.5 cont'd) We first deal with Cartesian coordinates where the Lagrangian is, from Examples 2.4.1 and 2.4.3,

$$L(t, x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2) - ma_g(y - y_0).$$

The Legendre transformation is

$$\begin{aligned} \mathbf{FL}(t, x, y, v_x, v_y) &= (t, x, y, mv_x, mv_y) \\ \implies \mathbf{FL}^{-1}(t, x, y, p_x, p_y) &= (t, x, y, \frac{1}{m}p_x, \frac{1}{m}p_y). \end{aligned}$$

From Proposition 2.5.6 we have

$$\mathbf{EL}(t, x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2) + ma_g(y - y_0)$$

which then, by the definition of the Hamiltonian, gives

$$H_L(t, x, y, p_x, p_y) = \frac{1}{2m}(p_x^2 + p_y^2) + ma_g(y - y_0).$$

We thus readily compute Hamilton's equations in Cartesian coordinates to be

$$\begin{aligned} \dot{x} &= \frac{\partial H_L}{\partial p_x} = \frac{p_x}{m} \\ \dot{y} &= \frac{\partial H_L}{\partial p_y} = \frac{p_y}{m} \\ \dot{p}_x &= -\frac{\partial H_L}{\partial x} = 0 \\ \dot{p}_y &= -\frac{\partial H_L}{\partial y} = -ma_g. \end{aligned}$$

One may readily verify that these equations have the same solution as their Lagrangian counterpart of Example 2.4.6.

Now let us do the same thing in polar coordinates where the Lagrangian is

$$L(t, r, \theta, v_r, v_\theta) = \frac{1}{2}m(v_r^2 + r^2v_\theta^2) - ma_g(r \sin \theta - r_0 \sin \theta_0).$$

The Legendre transformation is then readily derived to be

$$\begin{aligned} \mathbf{FL}(t, r, \theta, v_r, v_\theta) &= (t, r, \theta, mv_r, mr^2v_\theta) \\ \implies \mathbf{FL}^{-1}(t, r, \theta, p_r, p_\theta) &= (t, r, \theta, \frac{1}{m}p_r, \frac{1}{mr^2}p_\theta). \end{aligned}$$

The energy is easily seen to be given by

$$\mathbf{EL}(t, r, \theta, v_r, v_\theta) = \frac{1}{2}m(v_r^2 + r^2v_\theta^2) + ma_g(r \sin \theta - r_0 \sin \theta_0).$$

One then determines that in polar coordinates we have

$$H_L(t, r, \theta, v_r, v_\theta) = \frac{1}{2m}(p_r^2 + \frac{1}{r^2}p_\theta^2) + ma_g(r \sin \theta - r_0 \sin \theta_0).$$

Hamilton's equations in polar coordinates are then

$$\begin{aligned} \dot{r} &= \frac{\partial H_L}{\partial p_r} = \frac{p_r}{m} \\ \dot{\theta} &= \frac{\partial H_L}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_r &= -\frac{\partial H_L}{\partial r} = ma_g \sin \theta - \frac{p_\theta^2}{mr^3}. \end{aligned}$$

Theorem 2.9.2 tells us that these equations must be the same as the Euler-Lagrange equations we derived in polar coordinates in Example 2.4.6. While the correspondence is easily established above in Cartesian coordinates, it is less transparent in polar coordinates. Indeed, although the moving between the Euler-Lagrange and the Hamiltonian equations are affected merely by a change of coordinates, there is something not completely transparent happening.  $\square$

**2.9.4 Remarks 1.** For simple mechanical systems  $(Q, g, V)$ , as is the example we just worked out, it is possible to describe the Hamiltonian quite explicitly. For each  $q \in Q$  we may define an inner product  $g^{-1}(q)$  on the cotangent space  $T_q^*Q$  by

$$g^{-1}(q)(\alpha_q, \beta_q) = g(g^\sharp(\alpha_q), g^\sharp(\beta_q)).$$

One readily verifies that the matrix for this inner product with respect to a coordinate basis  $\{dq^1|_q, \dots, dq^n|_q\}$  for  $T_q^*Q$  is simply the matrix with components  $g^{ij}$ ,  $i, j = 1, \dots, n$  (recall that the matrix with components  $g^{ij}$ ,  $i, j = 1, \dots, n$ , is defined to be the inverse of the matrix with components  $g_{ij}$ ,  $i, j = 1, \dots, n$ ). With this inner product on each cotangent space, the Hamiltonian corresponding to the Lagrangian  $L(v_q) = \frac{1}{2}g(v_q, v_q) - V(q)$  is given by  $H_L(\alpha_q) = g^{-1}(\alpha_q, \alpha_q) + V(q)$ .

An important distinction to make between the Lagrangian and the Hamiltonian formalisms comes up when describing the equations of motion associated with each. In the Lagrangian setting, Proposition 2.4.5 provides a convenient equivalent expression for the Euler-Lagrange equations involving the Levi-Civita affine connection associated with the Riemannian metric  $g$ . For Hamilton's equations, there is no such convenient expression.

2. It is possible to include forces and constraints in an Hamiltonian setting, although we do not pursue this here [see Weber 1986]
3. As a final comment on the Hamiltonian approach, let us say that there is an enormous amount of literature available which expounds upon its many virtues. A good review is in the book [Liebermann and Marle 1987]. An account of control theory for Hamiltonian control systems can be found in Chapter 12 of the book [Nijmeijer and van der Schaft 1990].  $\square$

## 2.10 Conservation laws

In our discussion of rigid body dynamics in Section 1.6, we used the fact that spatial and angular momentum are conserved for a free rigid body to derive the Galilean Euler equations for rigid bodies. The idea of quantities being conserved by the motion of a system is an important one, and in the Lagrangian setting has its most basic formulation in Noether's theorem, (Emmy Amalie Noether (1882–1935)) which we address in this section.

The formulation of Noether's theorem requires the notion that a Lagrangian be “invariant” with respect to a vector field. Let us try to be clear about what that means. Let  $Q$  be a configuration space and let  $L: TQ \rightarrow \mathbb{R}$  be a time-independent Lagrangian on  $Q$ . For a vector field  $X$  on  $Q$ , recall that the integral curve of  $X$  through  $q_0 \in Q$  is the unique curve  $t \mapsto c(t)$  with the property that  $c(0) = q_0$  and that  $c'(t) = X(c(t))$ . (Also recall that  $c'(t)$  denotes the tangent vector field of  $c$ . Thus  $c'(t) \in T_{c(t)}Q$  gives the “velocity” of the curve at time  $t$ .) The Lagrangian  $L$  is *invariant* under the vector field  $X$  if for each integral curve  $t \mapsto c(t)$ , the function  $t \mapsto L(c'(t))$  is independent of  $t$ . The following result gives an easy way to determine when a Lagrangian is invariant under a vector field. The proof, although



readable with the tools we have developed thus far, perhaps relies on concepts just out of range of what we are doing.

**2.10.1 Lemma** *Let  $L: TQ \rightarrow \mathbb{R}$  be time-independent Lagrangian on a configuration space  $Q$ , and let  $X$  be a vector field on  $Q$ . Let  $(U, \phi)$  be a coordinate chart for  $Q$  with coordinates  $(q^1, \dots, q^n)$ .  $L$  is invariant under  $X$  if and only if*

$$\frac{\partial L}{\partial q^i} X^i + \frac{\partial L}{\partial v^i} \frac{\partial X^i}{\partial q^j} v^j = 0. \quad (2.32)$$

*Proof* Note that the curve  $c'(t)$  is the solution to the differential equation given in coordinates by

$$\begin{aligned} \dot{q}^i &= X^i \\ \dot{v}^i &= \frac{\partial X^i}{\partial q^j} v^j, \quad i = 1, \dots, n. \end{aligned} \quad (2.33)$$

To verify this, one need only differentiate the first  $n$  of these equations using the chain rule to get the second  $n$  equations. Therefore, the equation (2.32) merely states that the directional derivative of  $L$  in the direction  $(X^1, \dots, X^n, \frac{\partial X^1}{\partial q^j} v^j, \dots, \frac{\partial X^n}{\partial q^j} v^j)$  is zero. But this means that  $L$  is constant along solutions of the differential equations (2.33), which is exactly what is meant by  $L$  being invariant under  $X$ , by definition. ■

Let us see how this works out in an example.

**2.10.2 Example** (Example 2.1.5 cont'd) We take  $Q = \mathbb{R}^2$  with the Lagrangian given in Cartesian coordinates by

$$L(x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2) - ma_g(y - y_0).$$

Note here the Lagrangian is time-independent. Let us show that this Lagrangian is invariant under the vector field  $X = \frac{\partial}{\partial x}$ . One readily sees that the integral curve of  $X$  through the point  $(\tilde{x}, \tilde{y}) \in Q$  is given by  $c: t \mapsto (\tilde{x} + t, \tilde{y}) \in Q$ . To arrive at this, one simply solves the initial value problem

$$\dot{x} = 1, \quad \dot{y} = 0, \quad x(0) = \tilde{x}, \quad y(0) = \tilde{y}.$$

Therefore, the corresponding curve  $t \mapsto c'(t)$  is given by  $t \mapsto (\tilde{x} + t, \tilde{y}, 1, 0) \in TQ$ . We then see that the function

$$t \mapsto L(c'(t)) = \frac{1}{2}m - ma_g(\tilde{y} - y_0)$$

is indeed independent of  $t$ , thus verifying that  $L$  is indeed invariant under  $X$ , by directly employing the definition.

We may arrive at the same conclusion using Lemma 2.10.1. Indeed, this is merely a simple calculation. We note that the components of  $X$  are  $X^1 = 1$  and  $X^2 = 0$ . Therefore the terms in the expression (2.33) which involve the derivatives of the components of  $X$  vanish. In this case, the expression (2.33) reduces to  $\frac{\partial L}{\partial x} X^1$  which is zero since  $L$  is independent of  $x$ . In this way, and somewhat more straightforwardly, we again show that  $L$  is invariant under  $X$ . □

The computations in the above example illustrate a commonly encountered situation. If  $(U, \phi)$  is a coordinate chart, and a Lagrangian  $L$  in this set of coordinates is independent of one of the coordinates, say  $q^1$ , then we say  $q^1$  is a **cyclic coordinate** for  $L$ . As in the

previous example, one may show that if  $q^1$  is a cyclic coordinate, then  $L$  is invariant under the vector field  $X = \frac{\partial}{\partial q^1}$ .

Now let us turn to investigating the implications of  $L$  being invariant under some vector field. The following result shows that if  $L$  admits a vector field  $X$  under which it is invariant, then this implies the existence of a “constant of motion.”

**2.10.3 Theorem (Noether’s Theorem)** *Let  $L$  be a time-independent Lagrangian on  $Q$  and let  $X$  be a vector field on  $Q$ . Define a function  $P_{L,X}: TQ \rightarrow \mathbb{R}$  by  $P_{L,X}(v_q) = \langle \mathbf{FL}(v_q); X(q) \rangle$ . If  $L$  is invariant under  $X$ , then the function  $t \mapsto P_{L,X}(c'(t))$  is independent of  $t$  for each solution  $t \mapsto c(t)$  of the Euler-Lagrange equations.*

*Proof* Fix a solution  $t \mapsto c(t)$  of the Euler-Lagrange equations. For simplicity, suppose that  $c(t)$  is defined for all  $t \in \mathbb{R}$  and that all integral curves of  $X$  can be infinitely extended. Define a map  $\Phi: \mathbb{R} \times \mathbb{R} \rightarrow Q$  by asking that  $\Phi(s, t) = F_X(s, c(t))$  where  $F_X$  is the flow of  $X$ . Thus  $\Phi(s, t)$  is the point on  $Q$  where  $c(t)$  gets mapped to after time  $s$  under the flow of  $X$ .

**1 Lemma** *For each  $s \in \mathbb{R}$ , the curve  $t \mapsto \Phi(s, t)$  is a solution of the Euler-Lagrange equations.*

*Proof* We use the variational characterisation of the Euler-Lagrange equations of Theorem 2.3.2. Since  $c$  is a solution of the Euler-Lagrange equations, for  $a < b$  we know that  $c$  is an extremal of the functional

$$\tilde{c} \mapsto \int_a^b L(\tilde{c}'(t)) dt$$

over curves  $\tilde{c} \in C^2(c(a), c(b), [a, b])$ . Let us denote by  $c_s$  the curve  $t \mapsto \Phi(s, t)$ . Since the Lagrangian is invariant under  $X$ , the function  $L(c'_s(t))$  is independent of  $s$ . Therefore,  $c_s$  is an extremal of the functional

$$\tilde{c} \mapsto \int_a^b L(\tilde{c}'(t)) dt$$

over curves  $\tilde{c} \in C^2(c_s(a), c_s(b), [a, b])$ , which means that  $c_s$  is a solution of the Euler-Lagrange equations. ▼

For the remainder of the proof, we work in a coordinate chart for  $Q$  with coordinates  $(q^1, \dots, q^n)$ . In this case,  $\Phi$  becomes a map from  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}^n$ . By the lemma, for each fixed  $s$  we have

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i}(\Phi(s, t), \dot{\Phi}(s, t)) \right) - \frac{\partial L}{\partial q^i}(\Phi(s, t), \dot{\Phi}(s, t)) = 0, \quad i = 1, \dots, n, \quad (2.34)$$

where  $\dot{\Phi}$  denotes the derivative of  $\Phi$  with respect to  $t$ . Since  $L(\Phi(s, t), \dot{\Phi}(s, t))$  is independent of  $s$  we have

$$\frac{d\Phi^i(s, t)}{ds} \frac{\partial L}{\partial q^i}(\Phi(s, t), \dot{\Phi}(s, t)) + \frac{d\dot{\Phi}^i(s, t)}{ds} \frac{\partial L}{\partial v^i}(\Phi(s, t), \dot{\Phi}(s, t)) = 0. \quad (2.35)$$

Substituting (2.34) into (2.35) gives

$$\frac{d\Phi^i(s, t)}{ds} \frac{d}{dt} \left( \frac{\partial L}{\partial v^i}(\Phi(s, t), \dot{\Phi}(s, t)) \right) + \frac{d\dot{\Phi}^i(s, t)}{ds} \frac{\partial L}{\partial v^i}(\Phi(s, t), \dot{\Phi}(s, t)) = 0.$$

However, using the chain rule we write

$$\frac{d\dot{\Phi}^i(s, t)}{ds} \frac{d}{dt} \left( \frac{\partial L}{\partial v^i}(\Phi(s, t), \dot{\Phi}(s, t)) \right) + \frac{d\dot{\Phi}^i(s, t)}{ds} \frac{\partial L}{\partial v^i}(\Phi(s, t), \dot{\Phi}(s, t)) = \frac{d}{dt} \left( \frac{d\dot{\Phi}^i(s, t)}{ds} \frac{\partial L}{\partial v^i}(\Phi(s, t), \dot{\Phi}(s, t)) \right),$$

which gives the result after evaluation at  $s = 0$  since  $\left. \frac{d}{ds} \right|_{s=0} \Phi(s, t) = X(c(t))$ . ■

Noether's theorem is often quite easy to apply. Let us do so for our simple example.

**2.10.4 Example** (Example 2.10.2 cont'd) We resume with the situation when the Lagrangian on  $Q = \mathbb{R}^2$  given in Cartesian coordinates by

$$L(x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2) - ma_g(y - y_0)$$

is invariant under the vector field  $X = \frac{\partial}{\partial x}$ . Theorem 2.10.3 tells us that the function on  $TQ$  given by

$$\langle \mathbf{F}L(v_q); X(q) \rangle = \frac{\partial L}{\partial v_x} = mv_x$$

is conserved. We recognise this simply as the linear momentum of the particle in the  $x$ -direction. Note that if we choose  $X = \frac{\partial}{\partial y}$ , then the Lagrangian is no longer left invariant under  $X$  (check this!)

We could proceed to work through the above case in polar coordinates, but this is actually not very pleasant. Let us instead simplify the Lagrangian and look at another type of conservation law. We remove the potential energy from the Lagrangian so that in Cartesian coordinates we have

$$L(x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2).$$

This Lagrangian will still be invariant under the vector field used above, but it is now invariant under other vector fields. Let us take the vector field  $X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$  and show that  $L$  is invariant under  $X$ . Since  $L$  is independent of  $x$  and  $y$ , and since  $X^1$  is independent of  $x$  and  $X^2$  is independent of  $y$ , one readily computes that the expression (2.33) is given by

$$\frac{\partial L}{\partial v_x} \frac{\partial X^1}{\partial y} v_y + \frac{\partial L}{\partial v_y} \frac{\partial X^2}{\partial x} v_x = mv_x(-1)v_y + mv_y(1)v_x = 0.$$

This verifies that  $L$  is indeed invariant under  $X$ . Noether's theorem tells us that the corresponding conserved quantity is

$$\frac{\partial L}{\partial v_x}(-y) + \frac{\partial L}{\partial v_y}(x) = m(xv_y - yv_x).$$

If you are in the know, you recognise this as the angular momentum of the particle about the origin.

Let us now look at the situation in polar coordinates where

$$L(r, \theta, v_r, v_\theta) = \frac{1}{2}m(v_r^2 + r^2v_\theta^2).$$

As we saw in Example 2.2.3, the vector field  $X$  is given by  $\frac{\partial}{\partial \theta}$ . Also note that  $\theta$  is a cyclic coordinate! Therefore, had we been working in polar coordinates in the first place, the task of verifying that  $L$  is invariant under  $X$  would have followed immediately. In any case, the conserved quantity in polar coordinates is, by Theorem 2.10.3,

$$\frac{\partial L}{\partial v_\theta}(1) = mr^2 v_\theta.$$

Perhaps this is more readily identified as the angular momentum of the particle about the origin. Indeed, the angular inertia of the particle is  $mr^2$ , and its angular velocity is  $v_\theta$ , so the angular momentum is angular inertia times angular velocity.

Note that only when we used polar coordinates did we see that the situation was one where a cyclic coordinate was involved. Typically, one cannot expect to be in the cyclic situation, although it does come up often.  $\square$

**2.10.5 Remark** If in some set of coordinates  $q^1$  is cyclic for the Lagrangian  $L$ , then we see that the conserved quantity is  $\frac{\partial L}{\partial v^1}$ . But this is trivial, actually. If  $L$  is independent of  $q^1$ , then the Euler-Lagrange equations directly give

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^1} \right) = 0,$$

which is exactly the conservation law of Theorem 2.10.3. Thus one can view Noether's theorem as a generalisation of cyclic coordinates.  $\square$

## Exercises

- E2.1 Suppose a mass is constrained to move in a three-dimensional Galilean sub-spacetime  $\mathcal{F}$  of a Galilean spacetime  $\mathcal{G} = (\mathcal{E}, V, g, \tau)$ .
- (a) What is the configuration space for the system?
  - (b) How would one choose “Cartesian coordinates” for the system? Over which portions of the configuration space are these coordinates valid? What is the range of validity of these coordinates?
  - (c) How would one choose “polar coordinates” for the system? Over which portions of the configuration space are these coordinates valid? What is the range of validity of these coordinates?
  - (d) If the particle has mass  $m$ , what is its kinetic energy in Cartesian coordinates?
  - (e) What is the kinetic energy of the particle in polar coordinates?
- E2.2 Consider a particle constrained to move on the surface of a cylinder.
- (a) What is the configuration space for the system?
  - (b) Find a set of coordinates for the configuration space, making sure you state *exactly* how they coordinatise the space, and their range of validity.
  - (c) Is it possible to find a single set of coordinates that are valid on the entire configuration space?
  - (d) Is it possible to find vector fields that form a basis each point of the configuration space?
  - (e) If the particle has mass  $m$ , what is the kinetic energy for the system in the coordinates you specified?
- E2.3 Let  $\mathcal{B}$  be a rigid body whose mass distribution has its support contained in a line (see Example 1.5.1–2).
- (a) What is the configuration space for the system?
  - (b) Find a set of coordinates for the configuration space, making sure you state *exactly* how they coordinatise the space, and their range of validity.
  - (c) Is it possible to find a single set of coordinates that are valid on the entire configuration space?
  - (d) Is it possible to find two vector fields that are linearly independent at each point of the configuration space?
  - (e) Show that a single number  $I$  describes the inertia tensor of the system.
  - (f) With the coordinates you have chosen, provide an expression for the kinetic energy of the system.
- E2.4 Consider a pendulum swinging atop a cart constrained to move in a line (Figure E2.1).
- (a) What is the configuration space of the system?
  - (b) Define a coordinate system for the configuration space.
- E2.5 Consider a pendulum attached to a radial arm by a universal joint. The base of the radial arm is constrained to move in a line. See Figure E2.2.
- (a) What is the configuration space of the system?
  - (b) Define a coordinate system for the configuration space.
- E2.6 Consider the two-axis gyroscope of Figure E2.3.
- (a) Determine the configuration space for the system.

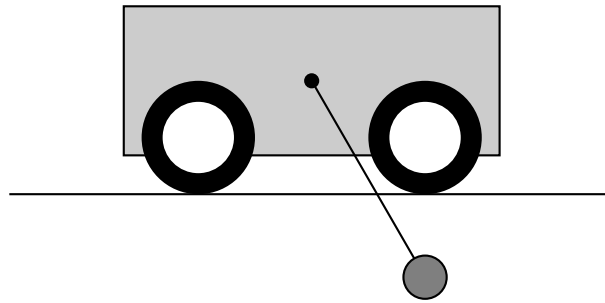


Figure E2.1 Pendulum on cart

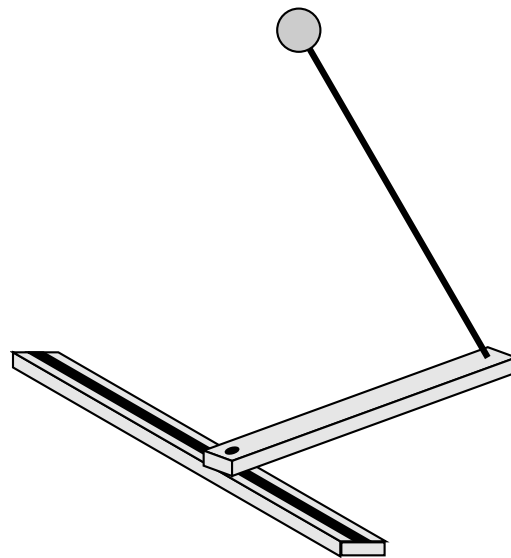


Figure E2.2 Pendulum on arm on linear track

- (b) Define a coordinate system for the configuration space.
- E2.7 Consider a point mass on the end of a massless leg, rotating on a rigid body fixed at a point in space (see Figure E2.4).
- Determine the configuration space for the system.
  - Define a coordinate system for the configuration space.
- E2.8 Consider the planar rigid body depicted in Figure E2.5.
- Determine the configuration space for the system.
  - Define a coordinate system for the configuration space.
- E2.9 Denote points on  $\mathbb{S}^1 \subset \mathbb{R}^2$  by  $(\cos \theta, \sin \theta)$  where  $\theta$  takes values in  $[0, 2\pi)$ . Let  $f: \mathbb{S}^1 \rightarrow \mathbb{R}$  be the function defined by  $f(\cos \theta, \sin \theta) = \theta$ . Is  $f$  a continuous function on  $\mathbb{S}^1$ ?
- E2.10 Consider the map  $c: \mathbb{R} \rightarrow \mathbb{S}^1$  defined by  $c(t) = (\cos t, \sin t)$ . Is  $c$  continuous? differentiable?
- E2.11 Define a vector field, any vector field, on  $\mathbb{S}^1$ .
- E2.12 Let  $Q = \mathbb{R}$  with  $(x)$  the coordinate in the standard coordinate chart. Define a vector field on  $Q$  by  $X = x^2 \frac{\partial}{\partial x}$ .
- Determine the flow of  $X$ .

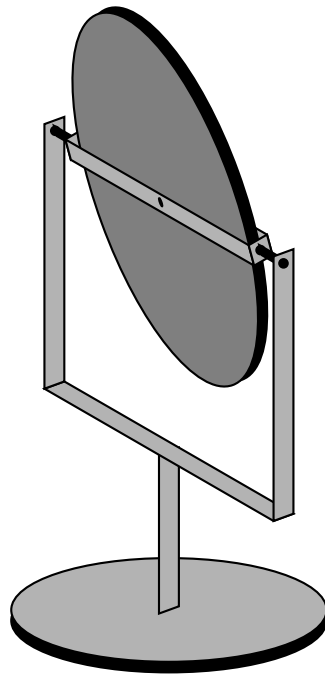


Figure E2.3 A two-axis gyroscope

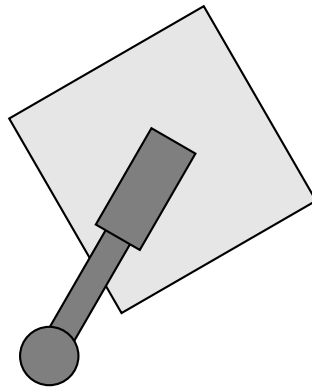


Figure E2.4 A “robotic leg”

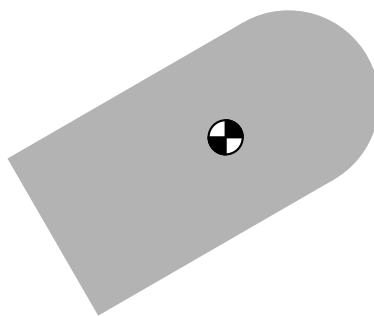


Figure E2.5 A planar rigid body

Now consider another coordinate chart  $(V, \psi)$  for  $Q$  defined  $V = Q$  and  $\psi(x) = e^x$ .

- (b) Express  $X$  in this new coordinate chart.
- (c) Determine the flow of  $X$  in this new coordinate chart.

E2.13 Is it possible to define two vector fields  $X_1$  and  $X_2$  on  $\mathbb{S}^2$  with the property that  $\{X_1(q), X_2(q)\}$  are linearly independent for every  $q \in Q$ ?

E2.14 Let  $Q = \mathbb{R}^2$  with  $(x, y)$  the coordinates in the standard coordinate chart. Define a vector field on  $Q$  by  $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ .

- (a) Determine the flow of  $X$ .
- (b) Determine the components of  $X$  in the polar coordinate chart.
- (c) Determine the flow of  $X$  in the polar coordinate chart.

E2.15 Let  $Q$  be a configuration space with  $(U, \phi)$  a coordinate chart. For vector fields  $X$  and  $Y$  with components  $(X^1, \dots, X^n)$  and  $(Y^1, \dots, Y^n)$  in the given chart, we propose the following two expressions,

$$\begin{aligned} \frac{\partial Y^i}{\partial q^j} X^j + \frac{\partial X^i}{\partial q^j} Y^j, \quad i = 1, \dots, n, \\ \frac{\partial Y^i}{\partial q^j} X^j - \frac{\partial X^i}{\partial q^j} Y^j, \quad i = 1, \dots, n, \end{aligned}$$

as being the components of a vector field on  $Q$ . Which, if either, are actually the components of a vector field?

E2.16 When we allow arbitrary coordinate charts, the notion of a linear ordinary differential equation loses its distinctness. Let  $Q = \mathbb{R}^2$  and consider on  $Q$  with the standard coordinate system  $(U_1, \phi_1)$  defined in the usual manner by  $U_1 = \mathbb{R}^2$  and  $\phi_1(x, y) = (x, y)$ . In these coordinates define a vector field

$$X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial x}.$$

- (a) Determine the flow of  $X$ .

Now consider the coordinate chart  $(U_2, \phi_2)$  where  $U_2 = \mathbb{R}^2$  and  $\phi_2(x, y) = (y, x)$ .

- (b) Write the vector field  $X$  in this coordinate chart.
- (c) Obtain the flow in these coordinates. Do you think you could have solved the differential equations were you given them without knowing how they came about?

E2.17 Let  $Q$  be a configuration space with  $g$  a Riemannian metric on  $Q$ . Show that for a coordinate chart  $(U, \phi)$  with coordinates  $(q^1, \dots, q^n)$  and for a fixed  $q \in U$ , the linear map  $g^b(q): T_q Q \rightarrow T_q^* Q$  has components  $g_{ij}$ ,  $i, j = 1, \dots, n$ , with respect to the bases  $\{\frac{\partial}{\partial q^i}|_q, \dots, \frac{\partial}{\partial q^n}|_q\}$  and  $\{dq^1|_q, \dots, dq^n|_q\}$  for  $T_q Q$  and  $T_q^* Q$ , respectively. Also show that the components of  $g^\sharp(q): T_q^* Q \rightarrow T_q Q$  with respect to these same bases are  $g^{ij}$ ,  $i, j = 1, \dots, n$ .

E2.18 Let  $c \in C^2([a, b], q_a, q_b)$  and let  $[t_1, t_2] \subset [a, b]$ . Let  $\tilde{c} \in C^2([t_1, t_2], c(t_1), c(t_2))$  and define a curve  $c_1: [a, b] \rightarrow Q$  by

$$c_1(t) = \begin{cases} c(t), & t \in [a, t_1] \\ \tilde{c}_1(t), & t \in [t_1, t_2] \\ c(t), & t \in [t_2, b]. \end{cases}$$



Show that for any  $\epsilon > 0$  there exists a curve  $\tilde{c}_1 \in C^2([a, b], q_a, q_b)$  so that  $|J_L(\tilde{c}_1) - J_L(c_1)| \leq \epsilon$ .

**Hint:** You may find it helpful to use a function like

$$f(x) = \begin{cases} \exp(-1/(1-x^2)), & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

E2.19 For the system of Exercise E2.4 do the following.

- Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
- Determine the Christoffel symbols for the Riemannian metric.
- Determine the potential function.
- Using this data, write the Euler-Lagrange equations for the system.

E2.20 For the system of Exercise E2.5 do the following.

- Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
- Determine the Christoffel symbols for the Riemannian metric.
- Determine the potential function.
- Using this data, write the Euler-Lagrange equations for the system.

E2.21 For the system of Exercise E2.6 do the following.

- Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
- Determine the Christoffel symbols for the Riemannian metric.
- Determine the potential function.
- Using this data, write the Euler-Lagrange equations for the system.

E2.22 For the system of Exercise E2.7 do the following.

- Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
- Determine the Christoffel symbols for the Riemannian metric.
- Determine the potential function.
- Using this data, write the Euler-Lagrange equations for the system.

E2.23 For the system of Exercise E2.8 do the following.

- Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
- Determine the Christoffel symbols for the Riemannian metric.
- Determine the potential function.
- Using this data, write the Euler-Lagrange equations for the system.

E2.24 Exercise on how the Newtonian potential gives the linear approximation.

E2.25 Let  $Q = \mathbb{R}$  and let  $F: \mathbb{R} \times TQ \rightarrow T^*Q$  be a force which is independent of time and velocity. Show that  $F$  is a potential force. Show that this is not true when  $Q = \mathbb{R}^2$ .

E2.26 Let  $\nabla$  be an affine connection on a configuration space  $Q$  and let  $(U_1, \phi_1)$  and  $(U_2, \phi_2)$  be overlapping coordinate charts with coordinates  $(q^1, \dots, q^n)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$ . If  $\Gamma_{jk}^i$ ,  $i, j, k = 1, \dots, n$  are the Christoffel symbols for  $\nabla$  in the coordinate chart  $(U_1, \phi_1)$

and  $\tilde{\Gamma}_{jk}^i$ ,  $i, j, k = 1, \dots, n$ , are the Christoffel symbols for  $\nabla$  in the coordinate chart  $(U_2, \phi_2)$ , show that

$$\tilde{\Gamma}_{jk}^i = \frac{\partial \tilde{q}^i}{\partial q^\ell} \frac{\partial q^r}{\partial \tilde{q}^j} \frac{\partial q^s}{\partial \tilde{q}^k} \Gamma_{rs}^\ell + \frac{\partial \tilde{q}^i}{\partial q^\ell} \frac{\partial^2 q^\ell}{\partial \tilde{q}^j \partial \tilde{q}^k}.$$

Show that as a result of this, the geodesic equations

$$\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0, \quad i = 1, \dots, n,$$

are independent of coordinates. What can you say about the coordinate independence of each of the separate terms “ $\ddot{q}^i$ ” and “ $\Gamma_{jk}^i \dot{q}^j \dot{q}^k$ ” in the geodesic equation?

E2.27 Argue that the quantity defined in equation (2.14) generalises the usual freshman notion of “work.”

E2.28 Let  $Q = \mathbb{R}^2 \setminus \{(0, 0)\}$  and define a force on  $Q$  by

$$F(t, x, y, v_x, v_y) = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

(a) Is  $F$  a potential force? Why or why not?

(b) If  $\tilde{Q} = \{(x, y) \in Q \mid x > 0\}$ , is  $F$  a potential force on  $\tilde{Q}$ ? Why or why not?

E2.29 This is a continuation of Exercise E2.4. Suppose that a torque is applied to the base of the pendulum. What is this force, written as a one-form on  $Q$ ?

E2.30 This is a continuation of Exercise E2.5. Suppose that two forces are applied to the system, one a torque at the base of the arm to rotate the arm, and the other a linear force moving the base down the track. What are these forces, written as one-forms on  $Q$ ?

E2.31 This is a continuation of Exercise E2.6. Suppose that a torque is applied which rotates the gyro in its frame. What is this force, written as a one-form on  $Q$ ?

E2.32 This is a continuation of Exercise E2.7. Suppose that we actuate the system with two inputs as follows. We may apply a linear force to the mass, so extending its distance from the pivot, and we have a motor atop the rigid body which we can use to actuate the angle of the arm relative to the body. What are these forces, written as one-forms on  $Q$ ?

E2.33 This is a continuation of Exercise E2.8. Suppose a force  $\mathbf{F}$  is applied to the body at a point a distance  $h$  from the centre of mass. Write this force as a one-form on  $Q$ .

E2.34 Show using Frobenius’s theorem that if  $C$  is a linear constraint with the property that  $\text{rank}(C) = 1$ , then  $C$  is holonomic.

E2.35 Let  $C$  be a constraint on  $Q$  and let the one-forms  $\alpha^1, \dots, \alpha^{n-r}$  be a basis for  $\text{ann}(\ell(C))$ . Show that the vector fields  $g^\sharp(\alpha^1), \dots, g^\sharp(\alpha^{n-r})$  form a basis for  $\ell(C)^\perp$ .

E2.36 Consider a ball rolling on a flat horizontal table rotating with uniform angular velocity  $\Omega$  with respect to some inertial frame (see Figure E2.6).

(a) What is the configuration space for the system?

Suppose that the ball has radius  $r$ , mass  $m$ , and isotropic inertia tensor so that all principal inertias are equal to  $I$ .

(b) Use the spatial angular velocity  $\boldsymbol{\omega}$  of the ball to express the constraints of the system.

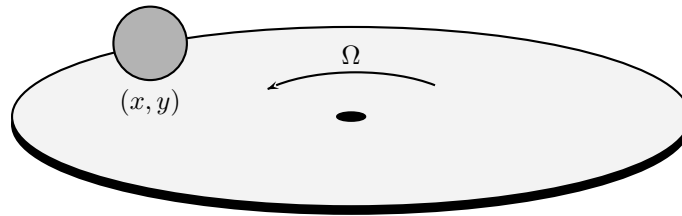


Figure E2.6 Ball rolling on rotating table

- (c) Using the definition of the spatial angular velocity, turn the constraints from the previous part of the problem into a constraint of the type we discuss. Is the constraint linear? Why or why not?
- (d) Write the Lagrangian for the system in terms of the spatial angular velocities.
- (e) Use the previous part of the problem, give an expression for the Lagrangian as a function on the tangent bundle of the configuration space. Use Exercise E1.11(e) to make for a nicer expression of the rotational kinetic energy.

You will observe that this Lagrangian is not altogether a pleasant one, and that writing the Euler-Lagrange equations is not a pleasing prospect.

- (f) However, use the fact that the Euler-Lagrange equations are equivalent to Newton's equations to obtain the equations of motion in terms of the spatial angular velocities and the position of contact of the ball with the table.

E2.37 Show that the body angular velocity of a rigid body is not a vector in the tangent space to the configuration space  $Q = SO(3)$ .

E2.38 Let  $Q$  be a configuration space and consider a function  $H: \mathbb{R} \times T^*Q \rightarrow \mathbb{R}$ . In a coordinate chart, define a map  $\mathbf{F}H: \mathbb{R} \times T^*Q \rightarrow \mathbb{R} \times TQ$  by

$$\mathbf{F}H(t, \mathbf{q}, \mathbf{p}) = \left( \mathbf{q}, \frac{\partial H}{\partial \mathbf{p}} \right).$$

- (a) Show that  $\mathbf{F}H$  is well-defined, i.e., that the above definition is independent of coordinates, and that  $\mathbf{F}H(t, \alpha_q) \in T_qQ$ .

Call the function  $H$  **hyperregular** if  $\mathbf{F}H$  is a diffeomorphism. For a hyperregular  $H$ , define a function  $L_H: \mathbb{R} \times TQ \rightarrow \mathbb{R}$  by

$$L_H(t, v_q) = \langle \mathbf{F}H^{-1}(v_q); v_q \rangle - H(t, \mathbf{F}H^{-1}(v_q)).$$

- (b) State and prove the natural result corresponding to Theorem 2.9.2.
- (c) Show that for a hyperregular Lagrangian,  $H_L$  is a hyperregular Hamiltonian and that  $\mathbf{F}L^{-1} = \mathbf{F}H_L$ .
- (d) Show that for a hyperregular Hamiltonian,  $L_H$  is a hyperregular Lagrangian and that  $\mathbf{F}H^{-1} = \mathbf{F}L_H$ .



# Chapter 3

## Lagrangian dynamics

Essentially, the previous two chapters dealt with writing the equations of motion, and had very little to say about the behaviour of the solutions of the resulting equations. Generally, one can say very little about the nature of solutions to the Euler-Lagrange equations. Therefore, in this section we look at special cases, and introduce *qualitative* methods. The study of the dynamics of mechanical systems fills an enormous number of pages, but despite this, very little is known about the behaviour of such systems. Thus we start our discussion in this section with some rather general ideas. Only when we get to Section 3.4 are we able to apply the general ideas to specific, and very simple, systems. It is only in these very special cases that one is able approach anything like a complete understanding to the dynamics of a class of systems. For these systems, the general ideas of the earlier sections of the chapter become more tangible, so it is hoped that the reader will have some patience in waiting for the application of these ideas.

### 3.1 The Euler-Lagrange equations and differential equations

We begin our investigation of Lagrangian dynamics with a discussion of how one obtains differential equations from the Euler-Lagrange equations. For those who think that the Euler-Lagrange equations *are* differential equations, we bring you the following example.

**3.1.1 Example** Consider the Lagrangian on  $Q = \mathbb{R}$  defined by  $L(t, x, v) = v - x$ . The Euler-Lagrange equation (there is only one) is then

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial x} = 1 = 0.$$

Clearly this is nonsense, and in this case there can be no solution for the Euler-Lagrange equations. □

You will say that the preceding example is contrived, as surely it is. However, it nonetheless points out the need for there to be some consideration given to the matter of when the Euler-Lagrange equations *have* solutions. The following result is perhaps not surprising given Theorem 2.9.2.

**3.1.2 Proposition** *If  $L$  is a hyperregular Lagrangian on  $Q$ , then for each  $q_0 \in Q$  and for each  $v_0 \in T_{q_0}Q$  there exists  $T > 0$  and a solution  $c: [t_0, t_0 + T] \rightarrow Q$  to the Euler-Lagrange equations for  $L$  with the property that  $c'(t_0) = v_0$ . Furthermore, if we have  $\tilde{T} > 0$  and a curve  $\tilde{c}: [t_0, t_0 + \tilde{T}] \rightarrow Q$  with the properties that  $\tilde{c}$  is a solution to the Euler-Lagrange equations and that  $\tilde{c}'(t_0) = v_0$ , then  $c(t) = \tilde{c}(t)$  for all  $t \in [t_0, t_0 + T] \cap [t_0, t_0 + \tilde{T}]$ .*

*Proof* We fix a coordinate chart  $(U, \phi)$  around  $q$ . Since  $L$  is hyperregular, the map  $\mathbf{FL}$ , given in coordinates by,

$$(q^1, \dots, q^n, v^1, \dots, v^n) \mapsto \left( q^1, \dots, q^n, \frac{\partial L}{\partial v^1}, \dots, \frac{\partial L}{\partial v^n} \right)$$

is invertible. By the inverse function theorem, this means that the Jacobian of the map must also be invertible. The Jacobian will be a matrix of the form

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n \times n} \\ \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix},$$

where  $\mathbf{A}_1$  is the matrix with components  $\frac{\partial^2 L}{\partial v^i \partial q^j}$ ,  $i, j = 1, \dots, n$ , and where  $\mathbf{A}_2$  is the matrix with components  $\frac{\partial^2 L}{\partial v^i \partial v^j}$ ,  $i, j = 1, \dots, n$ . In order that this Jacobian be invertible, it is necessary and sufficient that  $\mathbf{A}_2$  be invertible when evaluated at any  $(\mathbf{q}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$ .

Now let us look at the Euler-Lagrange equations. We expand them to arrive at

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = \frac{\partial^2 L}{\partial v^i \partial v^j} \dot{v}^j + \frac{\partial^2 L}{\partial v^i \partial q^j} \dot{q}^j + \frac{\partial^2 L}{\partial v^i \partial t} - \frac{\partial L}{\partial q^i}, \quad i = 1, \dots, n.$$

Therefore, the Euler-Lagrange equations, when  $L$  is hyperregular, are equivalent to

$$\ddot{q}^k = (\mathbf{A}_2^{-1})^{ki} \left( \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial v^i \partial q^j} \dot{q}^j - \frac{\partial^2 L}{\partial v^i \partial t} \right), \quad k = 1, \dots, n.$$

Here  $(\mathbf{A}_2^{-1})^{ki}$ ,  $k, i = 1, \dots, n$ , are the components of the inverse of  $\mathbf{A}_2$ . Thus we have shown that the Euler-Lagrange equations are equivalent to a second-order differential equation when the Lagrangian is hyperregular, and thus the result follows by the standard existence and uniqueness theory for ordinary differential equations. ■

The above result has the following corollary which gets us out of the woods as concerns worrying about existence and uniqueness of solutions for simple mechanical systems.

**3.1.3 Corollary** *If  $(Q, g, V)$  is a simple mechanical system with Lagrangian  $L$ , then for each  $q_0 \in Q$  and for each  $v_0 \in T_{q_0}Q$  there exists  $T > 0$  and a solution  $c: [t_0, t_0 + T] \rightarrow Q$  to the Euler-Lagrange equations for  $L$  with the property that  $c'(t_0) = v_0$ . Furthermore, if we have  $\tilde{T} > 0$  and a curve  $\tilde{c}: [t_0, t_0 + \tilde{T}] \rightarrow Q$  with the properties that  $\tilde{c}$  is a solution to the Euler-Lagrange equations and that  $\tilde{c}'(t_0) = v_0$ , then  $c(t) = \tilde{c}(t)$  for all  $t \in [t_0, t_0 + T] \cap [t_0, t_0 + \tilde{T}]$ .*

*Proof* This follows directly from Proposition 2.9.1. ■

Also note that the differential equations derived in Proposition 2.4.5 for simple mechanical systems also serve as proof of the previous result. In any case, the bottom line is that the equations describing the behaviour of simple mechanical systems, at least in the absence of forces and constraints, satisfy the usual existence and uniqueness properties of ordinary differential equations.

Let us explore this just a little bit further in order that we ensure ourselves that we have left some stones unturned. Indeed, the immediate question raised by Proposition 3.1.2 is whether hyperregularity of  $L$  is equivalent to there being a unique solution to the Euler-Lagrange equations through each point  $v_q \in TQ$ . The answer is that while hyperregularity is, as we have seen, sufficient for existence and uniqueness, it is not necessary. Some contrived examples illustrate this.

3.1.4 Examples 1. On  $Q = \mathbb{R}$  consider the Lagrangian  $L(t, x, v) = e^v$ . This Lagrangian is not hyperregular since the Legendre transformation

$$(x, v) \mapsto (x, e^v)$$

is not surjective; points of the form  $(x, a)$  are not in the image of the Legendre transformation when  $a < 0$ . However, the Euler-Lagrange equation is determined to be  $e^x \ddot{x} = 0$ , which is clearly equivalent to  $\ddot{x} = 0$ . Thus the Euler-Lagrange equations possess unique solutions for any initial condition, despite the Lagrangian not being hyperregular.

2. On  $Q = \mathbb{R}$  consider the Lagrangian  $L(t, x, v) = xv$ . One determines that the Euler-Lagrange equations are then simply  $0 = 0$ . In this case, *any* function of  $t$  will be a solution! Thus, the Euler-Lagrange equations possess solutions, but they are not unique.

3.

□ Example where injective

The above examples suggest that if we are unwilling to place restrictions on our Lagrangians, we can expect the Euler-Lagrange equations to be generally weird. However, Corollary 3.1.3 assures us that if we are dealing with typical Lagrangians from physics, then things are as they are in the standard theory of differential equations.

## 3.2 Linearisations of Lagrangian systems

The first thing one does with pretty much *any* general ordinary differential equation is seek fixed points, and linearise about them. This is a good idea for Lagrangian systems as well, and as we shall see, linearisation of Lagrangian systems yields a structure which is quite special.

**3.2.1 Linear Lagrangian systems** When we linearise a general differential equation, the linearisation of the system is, by construction, a linear differential equation. Similarly, when we linearise Euler-Lagrange equations, we end up with “linear Euler-Lagrange equations.” Therefore, it is helpful to know what these are.

In order to define the equations governing the behaviour of a linear Lagrangian system, it is helpful to have at hand some notation. Let  $V$  be a  $\mathbb{R}$ -vector space (let us agree not to confuse  $V$  in this discussion with the potential function  $V$ ). A map  $A: V \times V \rightarrow \mathbb{R}$  is **bilinear** if

$$A(u_1 + u_2, v_1 + v_2) = A(u_1, v_1) + A(u_1, v_2) + A(u_2, v_1) + A(u_2, v_2)$$

for all  $u_1, u_2, v_1, v_2 \in V$  and if

$$A(au, bv) = ab A(u, v)$$

for all  $a, b \in \mathbb{R}$  and  $u, v \in V$ . A bilinear map  $A$  is **symmetric** (resp. **skew-symmetric**) if  $A(u, v) = A(v, u)$  (resp.  $A(u, v) = -A(v, u)$ ) for all  $u, v \in V$ . We all know of an example of a symmetric bilinear map, and that is an inner product which further has the property that  $A(u, u) > 0$  for  $u \neq 0$ . Using our notation for inner products as motivation, for a symmetric bilinear map  $A: V \times V \rightarrow \mathbb{R}$  define a map  $A^\flat: V \rightarrow V^*$  by asking that  $A(u, v)$  satisfy  $\langle A^\flat(u); v \rangle = A(u, v)$ . If  $\{e_1, \dots, e_n\}$  is a basis for  $V$  with dual basis  $\{e^1, \dots, e^n\}$  for  $V^*$ , then the **matrix** of  $A$  in the given basis is defined to be the  $n \times n$  matrix with components

$$A_{ij} = A(e_i, e_j), \quad i, j = 1, \dots, n.$$

Note, of course, that  $A_{ij} = A_{ji}$ ,  $i, j = 1, \dots, n$ . One then readily verifies (see, for example, Exercise E2.17) that

$$A(v^i e_i) = A_{ij} v^i e^j,$$

thus giving a component expression for  $A^\flat$ . If  $A$  is skew-symmetric, then one may define  $A^\flat: V \rightarrow V^*$  in exactly the same way, except that the matrix with components  $A_{ij}$ ,  $i, j = 1, \dots, n$ , will now be skew-symmetric.

Now we may make our definition concerning linear Lagrangian systems.

**3.2.1 Definition** Let  $V$  be an  $n$ -dimensional  $\mathbb{R}$ -vector space. A **linear Lagrangian system** on  $V$  is a triple  $(M, C, K)$  where  $M, C, K: V \times V \rightarrow \mathbb{R}$  are bilinear maps with  $M$  and  $K$  being symmetric and  $C$  being skew-symmetric. The **Lagrangian** associated with a linear Lagrangian system  $(M, C, K)$  is the function  $L: V \times V \rightarrow \mathbb{R}$  defined by

$$L(u, v) = \frac{1}{2}M(v, v) + \frac{1}{2}C(u, v) - \frac{1}{2}K(u, u).$$

A curve  $t \mapsto c(t) \in V$  is a **solution** of the linear Lagrangian system  $(M, C, K)$  if  $c$  satisfies the differential equation

$$M^\flat(\ddot{c}(t)) + C^\flat(\dot{c}(t)) + K^\flat(c(t)) = 0. \quad \square$$

Note that our notion of a solution for a linear Lagrangian system seems disjointed from the usual notion of solutions for Euler-Lagrange equations. Let us state a result to make the connection. In stating the result, it is helpful to note that for a  $\mathbb{R}$ -vector space  $V$  we have  $TV \simeq V \times V$ . We shall simply state this as a fact, and not prove it, although this is easily done in an appropriate setting. Using this fact, a point  $v_q$  in  $TV$  is written as  $(q, v) \in V \times V$ .

**3.2.2 Proposition** Let  $V$  be a  $\mathbb{R}$ -vector space and let  $(M, C, K)$  be a linear Lagrangian system on  $V$  with Lagrangian  $L$  defined on  $V \times V \simeq TQ$ . A curve  $t \mapsto c(t) \in V$  is a solution to the linear Lagrangian system  $(M, C, K)$  if and only if it is a solution to the Euler-Lagrange equations for  $L$ .

*Proof* We have

$$L(q, v) = \frac{1}{2}M(v, v) + \frac{1}{2}C(q, v) - \frac{1}{2}K(q, q).$$

Let  $\{e_1, \dots, e_n\}$  be a basis for  $V$  with  $M_{ij}$ ,  $C_{ij}$ , and  $K_{ij}$ ,  $i, j = 1, \dots, n$ , the matrices for  $M$ ,  $C$ , and  $K$  in this basis. We compute

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = M_{ji} \ddot{q}^j + \frac{1}{2} C_{ji} \dot{q}^j - \frac{1}{2} C_{ij} \dot{q}^j + K_{ji} q^j, \quad i = 1, \dots, n.$$

The result now follows since  $C_{ji} = -C_{ij}$ ,  $i, j = 1, \dots, n$ . ■

In the case when  $M$  is invertible then its inverse is denoted  $M^\sharp: V^* \rightarrow V$ . Also, in this case we may write the equations which are satisfied by a linear Lagrangian system as the following set of  $2n$  first-order differential equations:

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & \text{id}_V \\ -M^\sharp \circ K^\flat & -M^\sharp \circ C^\flat \end{bmatrix} \begin{pmatrix} q \\ v \end{pmatrix},$$



where  $\text{id}_V$  is the identity map on  $V$ . Note that  $M^\sharp \circ K^\flat$  and  $M^\sharp \circ C^\flat$  are simply linear transformations of  $V$ . Recall that the behaviour of the linear system is determined in large part by the eigenvalues of the linear transformation  $A_{(M,C,K)}: V \times V \rightarrow V \times V$  which is represented by

$$A_{(M,C,K)} = \begin{bmatrix} 0 & \text{id}_V \\ -M^\sharp \circ K^\flat & -M^\sharp \circ C^\flat \end{bmatrix}.$$

The following result characterises these eigenvalues.

**3.2.3 Proposition** *Let  $(M, C, K)$  be a linear Lagrangian system on a vector space  $V$ , and suppose that  $M^\flat$  is invertible. If  $\lambda \in \mathbb{C}$  is an eigenvalue of  $A_{(M,C,K)}$  then*

- (i)  $-\lambda$  is an eigenvalue of  $A_{(M,C,K)}$ ;
- (ii)  $\bar{\lambda}$  is an eigenvalue of  $A_{(M,C,K)}$ ;
- (iii)  $-\bar{\lambda}$  is an eigenvalue of  $A_{(M,C,K)}$ ;

Furthermore, the multiplicity of the eigenvalues  $\lambda$ ,  $-\lambda$ ,  $\bar{\lambda}$ , and  $-\bar{\lambda}$  are equal.

*Proof* On the vector space  $V^2 = V \times V$  define a bilinear map  $J_{M,C}: V^2 \times V^2 \rightarrow \mathbb{R}$  by

$$J_{M,C}((u_1, u_2), (v_1, v_2)) = M(u_1, v_2) - M(u_2, v_1) + C(v_1, u_1).$$

Clearly  $J$  is skew-symmetric. If  $M^\flat$  is invertible then  $J_{M,C}$  is nondegenerate in the sense that if  $J_{M,C}((u_1, u_2), (v_1, v_2)) = 0$  for all  $(v_1, v_2) \in V^2$ , then  $(u_1, u_2) = (0, 0)$ . Indeed, if  $J_{M,C}((u_1, u_2), (v_1, v_2)) = 0$  for all  $(v_1, v_2) \in V^2$  then

$$M(u_1, v_2) - M(u_2, v_1) = 0$$

for all  $(v_1, v_2) \in V^2$ . Taking  $v_1 = 0$ , this means that  $M(u_1, v_2) = \langle M^\flat(u_1); v_2 \rangle = 0$  for all  $v_2 \in V$ . This implies that  $M^\flat(u_1) = 0$  and since  $M^\flat$  is invertible, this means that  $u_1 = 0$ . Taking  $v_2 = 0$  we see that we must have  $M(u_2, v_1) - C(v_1, u_1) = 0$  for every  $v_1 \in V$ . Since  $u_1$  has already been demonstrated to be zero, it follows that  $M(u_2, v_1) = \langle M^\flat(u_2); v_1 \rangle = 0$  for every  $v_1 \in V$ , from which we deduce that  $u_2 = 0$ .

Next we compute

$$\begin{aligned} & J_{M,C}(A_{(M,C,K)}(u_1, u_2), (v_1, v_2)) \\ &= J_{M,C}((u_2, -M^\sharp \circ K^\flat(u_1) - M^\sharp \circ C^\flat(u_2)), (v_1, v_2)) + C(v_1, u_2) \\ &= M(u_2, v_2) + M(M^\sharp \circ K^\flat(u_1) + M^\sharp \circ C^\flat(u_2), v_1) + C(v_1, u_2) \\ &= M(u_2, v_2) + K(u_1, v_1) + C(u_2, v_1) + C(v_1, u_2) \\ &= M(v_2, u_2) + K(v_1, u_1) + C(v_2, u_1) - C(v_2, u_1) \\ &= M(M^\sharp \circ K^\flat(v_1) + M^\sharp \circ C^\flat(v_2), u_1) + M(v_2, u_2) - C(v_2, u_1) \\ &= -M(u_1, -M^\sharp \circ K^\flat(v_1) - M^\sharp \circ C^\flat(v_2)) + M(v_2, u_2) - C(v_2, u_1) \\ &= -J_{M,C}((u_1, u_2), (v_2, -M^\sharp \circ K^\flat(v_1) - M^\sharp \circ C^\flat(v_2))) \\ &= -J_{M,C}((u_1, u_2), A_{(M,C,K)}(v_1, v_2)). \end{aligned}$$

Let us now use this fact to prove the result.

Suppose that  $\lambda \in \mathbb{C}$  is an eigenvalue as stated. Then there exists a vector  $v = (v_1, v_2) \in V^2$  so that  $A_{(M,C,K)}(v) = \lambda v$ . This means that for every  $\tilde{v} \in V^2$  we have

$$\begin{aligned} & J_{M,C}(\tilde{v}, A_{(M,C,K)}(v) - \lambda v) = 0 \\ \implies & -J_{M,C}(A_{(M,C,K)}(\tilde{v}), v) - \lambda J_{M,C}(\tilde{v}, v) = 0 \\ \implies & J_{M,C}((A_{(M,C,K)} + \lambda \text{id}_{V^2})\tilde{v}, v) = 0. \end{aligned} \tag{3.1}$$

This proves that  $v$  is an eigenvector for  $A_{(M,C,K)}$  with eigenvalue  $\lambda$  then  $v$  is orthogonal to  $\text{image}(A_{(M,C,K)} + \lambda \text{id}_{V^2})$  with orthogonality taken with respect to  $J_{M,C}$ . This renders the following lemma helpful.

**1 Lemma** *Let  $U$  be a finite-dimensional  $\mathbb{R}$ -vector space with  $J: U \times U \rightarrow \mathbb{R}$  a nondegenerate skew-symmetric bilinear map. For a linear map  $B: U \rightarrow U$  define  $\hat{B}: U \rightarrow U$  by*

$$J(\hat{B}(u_1), u_2) = J(u_1, B(u_2)).$$

*Then the following facts hold:*

- (i) *a vector  $u \in U$  satisfies  $J(B(\tilde{u}), u) = 0$  for every  $\tilde{u} \in U$  if and only if  $u \in \ker(\hat{B})$ —that is, the  $J$ -orthogonal complement to  $\text{image}(B)$  is equal to  $\ker(\hat{B})$ ;*
- (ii)  $\dim(\ker(B)) = \dim(\ker(\hat{B}))$ .

*Proof* (i) We have

$$\begin{aligned} \{u \in U \mid J(B(\tilde{u}), u) = 0 \text{ for every } \tilde{u} \in U\} &= \{u \in U \mid J(\tilde{u}, \hat{B}(u)) = 0 \text{ for every } \tilde{u} \in U\} \\ &= \{u \in U \mid \hat{B}(u) = 0\} = \ker(\hat{B}). \end{aligned}$$

(ii) Choose a basis  $\{u_1, \dots, u_m\}$  for  $V$ . Let  $J_{ij}$ ,  $i, j = 1, \dots, m$ , be the matrix for  $J$  in this basis, let  $B_j^i$ ,  $i, j = 1, \dots, m$ , be the matrix for  $B$  in this basis, and let  $\hat{B}_j^i$ ,  $i, j = 1, \dots, m$ , be the matrix for  $\hat{B}$  in this basis. We then have, by the definition of  $\hat{B}$ ,

$$J_{ki} \hat{B}_j^k = J_{jk} B_i^j, \quad i, j = 1, \dots, m.$$

Letting  $J^{ij}$ ,  $i, j = 1, \dots, m$ , be the components of the inverse of the matrix with components  $J_{ij}$ ,  $i, j = 1, \dots, m$ , we then have

$$\hat{B}_j^i = J^{ki} B_k^\ell J_{j\ell}, \quad i = 1, \dots, m.$$

Writing this in matrix form, it reads  $\hat{B} = J^{-t} B J^t$ . Now since  $J$  is nondegenerate,  $J$  is invertible, which implies that  $B$  and  $\hat{B}$  have the same rank.  $\blacktriangledown$

Using the lemma with  $U = V^2$ ,  $J = J_{M,C}$ , and  $B = A_{(M,C,K)} + \lambda \text{id}_{V^2}$ , we see that if  $v$  is an eigenvector for  $A_{(M,C,K)}$  with eigenvalue  $\lambda$  then  $v \in \ker((A_{(M,C,K)} + \lambda \text{id}_{V^2})^\wedge)$ . Also by the lemma, this implies that there exists a nonzero vector  $v' \in \ker(A_{(M,C,K)} + \lambda \text{id}_{V^2})$ , and thus  $v'$  is an eigenvector with eigenvalue  $-\lambda$ . That  $\bar{\lambda}$  and  $-\bar{\lambda}$  are eigenvalues follows from the fact that  $V$  is a real vector space.

For the final assertion of the proposition, note that the reasoning of (3.1), along with part (i) of the above lemma, demonstrates that

$$\dim(\ker(A_{(M,C,K)} - \lambda \text{id}_{V^2})) = \dim(\ker((A_{(M,C,K)} - \lambda \text{id}_{V^2})^\wedge)),$$

and so the assertion about the multiplicities then follows from part (ii) of the above lemma.  $\blacksquare$

**3.2.4 Remark** The above proof, although straightforward in nature, relies on the introduction of the mysterious bilinear map  $J_{M,C}$ . This may seem like something of a rabbit pulled from a hat, and indeed it is without some extra context. On the  $\mathbb{R}$ -vector space  $V \times V^*$  consider the skew-symmetric bilinear map  $J: (V \times V^*)^2 \rightarrow \mathbb{R}$  defined by

$$J((v_1, \alpha^1), (v_2, \alpha^2)) = \alpha^2(v_1) - \alpha^1(v_2).$$

Note that this map is defined without the assistance of additional structure. Now consider a linear Lagrangian system  $(M, C, K)$  on  $V$ . The Legendre transformation for the corresponding Lagrangian  $L$  is the map

$$\mathbf{FL}(q, v) = (q, M^b(v) + \frac{1}{2}C^b(q)).$$

The map  $J_{M,C}$  defined in the proof of Proposition 3.2.3 is readily determined to be exactly

$$J_{M,C}((u_1, u_2), (v_1, v_2)) = J(\mathbf{FL}(u_1, u_2), \mathbf{FL}(v_1, v_2)).$$

Thus  $J_{M,C}$  is “the same” as  $J$ . The object  $J$  is well studied, and is called a **symplectic structure** on the vector space  $V \times V^*$ . The mapping  $A_{(M,C,K)}$  is then an example of an **infinitesimally symplectic mapping**. These are well studied, and here we only give a glimpse of their properties. Williamson [1937] gives a classification of the Jordan normal forms for such linear mappings.  $\square$

The position of the eigenvalues for various situations is shown in Figure 3.1. Let us

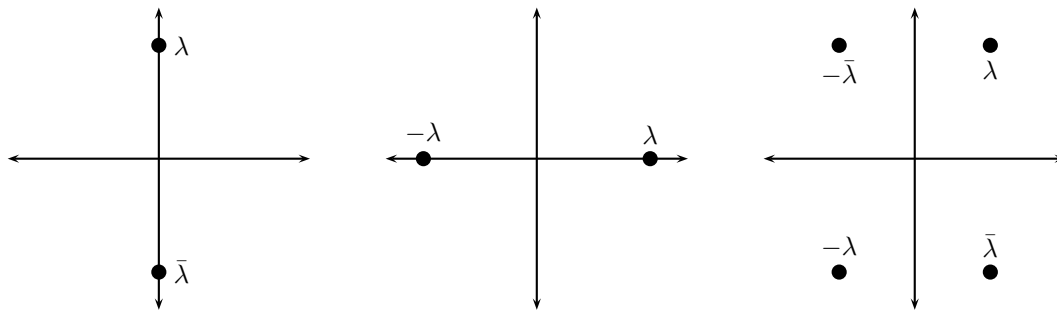


Figure 3.1 Imaginary eigenvalues (left), real eigenvalues (middle), and general complex eigenvalues (right) for a linear Lagrangian system

illustrate the conclusions of Proposition 3.2.3 in the simplest possible case.

**3.2.5 Example** We take  $V = \mathbb{R}$  and define a linear mechanical system on  $V$  by  $M = m$ ,  $C = 0$ ,  $K = k$ , for  $m, k \in \mathbb{R}$ . Note that skew-symmetric linear mappings on a 1-dimensional vector space are necessarily zero. We then have, provided that  $m \neq 0$ ,

$$A_{(M,C,K)} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}.$$

We have three cases.

1. If  $\frac{k}{m}$  is positive, then we write  $\frac{k}{m} = \omega_0^2$ . The eigenvalues of  $A_{(M,C,K)}$  are readily computed to be  $\lambda_1 = i\omega_0$  and  $\lambda_2 = -i\omega_0$ . Thus, in this case, both eigenvalues are purely imaginary, and they obviously have the property of being complex conjugates of one another.
2. If  $\frac{k}{m}$  is negative, that we write  $\frac{k}{m} = -\omega_0^2$  and we determine that the eigenvalues of  $A_{(M,C,K)}$  are  $\lambda_1 = \omega_0$  and  $\lambda_2 = -\omega_0$ . Thus we see that when we have a real eigenvalue, we also have its negative, as predicted by Proposition 3.2.3.
3. The final case is when  $\frac{k}{m} = 0$  when the eigenvalues are both zero. These also obviously satisfy the conditions of Proposition 3.2.3.  $\square$

Let us finally address the question of what are the implications of Proposition 3.2.3 for the dynamics of linear Lagrangian systems, at least when  $M^b$  is invertible.

The reader will at this point wish to recall some of the basic ideas assembled in Section C.2 concerning fixed points of general ordinary differential equations. In particular, we will at this point make reference to the notions of stability and asymptotic stability of a fixed point.

Recall that the **algebraic multiplicity** of an eigenvalue is the multiplicity of the eigenvalue as a root of the characteristic polynomial, and that the **geometric multiplicity** of an eigenvalue is the dimension of the eigenspace for that eigenvalue. The algebraic multiplicity always bounds the geometric multiplicity from above. The following result gives the stability properties of a linear Lagrangian system. The proof relies on some elementary facts from the theory of linear differential equations with constant coefficients. A rapid review of this material may be found in Section C.1. The notions of stability we use are to be found in Section C.2.

**3.2.6 Proposition** *Let  $(M, C, K)$  be a linear Lagrangian system on a vector space  $V$  and suppose that  $M^b$  is invertible. Consider the differential equation*

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & \text{id}_V \\ -M^\# \circ K^b & -M^\# \circ C^b \end{bmatrix} \begin{pmatrix} q \\ v \end{pmatrix}. \quad (3.2)$$

A point  $(q_0, v_0) \in V^2$  is a fixed point of (3.2) if and only if  $v_0 = 0$  and  $q_0 \in \ker(K^\#)$ .

Let  $(q_0, v_0)$  be a fixed point of (3.2). The following statements hold:

- (i)  $(q_0, v_0)$  is not asymptotically stable;
- (ii)  $(q_0, v_0)$  is stable if and only if all eigenvalues of  $A_{(M,C,K)}$  lie on the imaginary axis and have equal algebraic and geometric multiplicities.

*Proof* That  $(q_0, v_0) \in V^2$  is a fixed point of (3.2) if and only if  $v_0 = 0$  and  $q_0 \in \ker(K^\#)$  follows directly from the form of  $A_{(M,C,K)}$ .

(i) This follows since a linear system is asymptotically stable if and only if all eigenvalues have negative real part. For a linear Lagrangian system, if an eigenvalue has negative real part, then Proposition 3.2.3 guarantees that there will also be an eigenvalue with positive real part.

(ii) From part (i) we see that for a fixed point to be stable, all eigenvalues of  $A_{(M,C,K)}$  should lie on the imaginary axis. Recall from the Jordan canonical form theorem that if the geometric multiplicity is less than the algebraic multiplicity, then for some initial conditions the solution will have the form of a linear combination of terms like

$$t^k \cos \omega t, \quad t^k \sin \omega t,$$

for some  $\omega \geq 0$  and  $k \geq 1$ . This precludes stability as such solutions are not bounded as  $t \rightarrow \infty$ . ■

Our previous simple example is enough to illustrate the consequences of the previous result.

**3.2.7 Example** (Example 3.2.5 cont'd) We again take  $V = \mathbb{R}$  and take the linear Lagrangian system with  $M = m$ ,  $C = 0$ , and  $K = k$ . We had

$$A_{(M,C,K)} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}.$$

Recall that the solution to (3.2) is

$$\begin{pmatrix} q(t) \\ v(t) \end{pmatrix} = e^{A_{(M,C,K)}t} \begin{pmatrix} q(0) \\ v(0) \end{pmatrix}.$$

Again, we consider three cases.

1. We first take  $\frac{k}{m} > 0$  and write  $\frac{k}{m} = \omega_0^2$ . In this case, the only fixed point is  $(0, 0)$  since  $K^\sharp$  is invertible. A straightforward computation gives

$$e^{A_{(M,C,K)}t} = \begin{bmatrix} \cos(\omega_0 t) & \frac{\sin(\omega_0 t)}{\omega_0} \\ -\omega_0 \sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}.$$

Clearly, by choosing the initial condition small enough, one can ensure that solutions remain inside a ball of any specified radius. Thus  $(0, 0)$  is a stable fixed point. However, it is not asymptotically stable.

2. Next take  $\frac{k}{m} < 0$  and write  $\frac{k}{m} = -\omega_0^2$ . Again the only fixed point is  $(0, 0)$ . One also computes

$$e^{A_{(M,C,K)}t} = \begin{bmatrix} \cosh(\omega_0 t) & \frac{\sinh(\omega_0 t)}{\omega_0} \\ -\omega_0 \sinh(\omega_0 t) & \cosh(\omega_0 t) \end{bmatrix}.$$

Note that now there will be initial conditions which result in unbounded solutions. Therefore, the system is not stable.

3. Finally we take  $k = 0$ . In this case any point  $(q_0, 0)$  will be a fixed point since  $K = 0$ . One computes

$$e^{A_{(M,C,K)}t} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Therefore any initial condition with  $v(0) \neq 0$  will result in unbounded solutions. This precludes any of the infinite number of fixed points from being stable. Note here that the eigenvalues lie on the imaginary axis—they are both zero—but since the geometric multiplicity is only one, Proposition 3.2.6 implies that fixed points will be unstable.  $\square$

**3.2.8 Remark** We have studiously avoided talking about the case when  $M^\sharp$  is not invertible. In such cases, the differential equations

$$M\ddot{q} + C\dot{q} + Kq = 0$$

are “implicit” in that one cannot solve explicitly for the derivative and stick it on the left-hand side as one likes to do with differential equations. In this situation, the concerns of Section 3.1 come into play. That is to say, matters of existence and uniqueness become of concern. Rather than getting involved in such matters, we will concentrate on describing the dynamics in those cases when the Euler-Lagrange equations give well-behaved differential equations.  $\square$

**3.2.2 Equilibria for Lagrangian systems** In order to linearise, we need something to linearise about, and such things are titled “equilibria.” To be specific, let  $L$  be a Lagrangian on a configuration space  $Q$ . A point  $q_0 \in Q$  is an *equilibrium point* for  $L$  if the trivial curve  $t \mapsto q_0$  is a solution of the Euler-Lagrange equations for  $L$ . The following result gives a useful characterisation of equilibrium points for time-independent Lagrangians. For the statement of this result, we use the notation  $0_q$  to denote the zero vector in the tangent space  $T_q Q$ .

**3.2.9 Proposition** *A point  $q_0$  is an equilibrium point for a time-independent Lagrangian  $L$  on a configuration space  $Q$  if and only if  $\mathbf{dEL}(0_{q_0}) = 0$ .*

*Proof* We work in a set of coordinates  $(q^1, \dots, q^n)$  and denote by  $\mathbf{q}_0$  the image of  $q_0$  in this set of coordinates. Recall that in coordinates we have

$$\mathbf{EL} = \frac{\partial L}{\partial v^i} v^i - L$$

so that

$$\begin{aligned} \mathbf{dEL} &= \frac{\partial^2 L}{\partial q^j \partial v^i} v^i dq^j + \frac{\partial^2 L}{\partial v^j \partial v^i} v^i dv^j + \frac{\partial L}{\partial v^i} dv^i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial v^i} dv^i \\ &= \frac{\partial^2 L}{\partial q^j \partial v^i} v^i dq^j + \frac{\partial^2 L}{\partial v^j \partial v^i} v^i dv^j - \frac{\partial L}{\partial q^i} dq^i. \end{aligned}$$

Thus  $\mathbf{dEL}(0_{q_0}) = 0$  if and only if

$$\frac{\partial L}{\partial q^i}(\mathbf{q}_0, \mathbf{0}) = 0, \quad i = 1, \dots, n. \quad (3.3)$$

However, this Euler-Lagrange equations are

$$\frac{\partial^2 L}{\partial v^j \partial v^i} \ddot{q}^j + \frac{\partial^2 L}{\partial q^j \partial v^i} \dot{q}^j - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n.$$

Thus  $\mathbf{q}_0$  is an equilibrium point if and only if (3.3) holds, and thus follows the proposition. ■

This fact relating the energy to equilibrium points will be important when we talk about dynamics near equilibrium points later in this section.

We shall now in show that when the Lagrangian is time-independent, one may define a linear Lagrangian system on the vector space  $T_{q_0}Q$ . Suppose that  $L$  is a time-independent Lagrangian on a configuration space  $Q$  and let  $q_0 \in Q$  be an equilibrium point for  $L$ . Let us choose a coordinate chart  $(U, \phi)$  around  $q_0$  with coordinates  $(q^1, \dots, q^n)$ , and suppose that  $\phi(q_0) = \mathbf{q}_0$ . Let us try to obtain an approximate expression for solutions of the Euler-Lagrange equations which start near  $q_0$ . Let  $t \mapsto \mathbf{q}(t)$  be a coordinate representation of such a solution. We therefore have the Euler-Lagrange equations:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n.$$

Let us Taylor expand everything about  $(\mathbf{q}, \mathbf{v}) = (\mathbf{q}_0, \mathbf{0})$ . We have

$$\begin{aligned} &\frac{d}{dt} \left( \frac{\partial^2 L}{\partial q^j \partial v^i} (q^j(t) - q_0^j) + \frac{\partial^2 L}{\partial v^j \partial v^i} (\dot{q}^j(t) - \dot{q}_0^j) + \dots \right) - \\ &\frac{\partial^2 L}{\partial q^j \partial q^i} (q^j(t) - q_0^j) - \frac{\partial^2 L}{\partial v^j \partial q^i} (\dot{q}^j(t) - \dot{q}_0^j) + \dots = 0 \\ \implies &\frac{\partial^2 L}{\partial v^j \partial v^i} \xi^j(t) + \left( \frac{\partial^2 L}{\partial q^j \partial v^i} - \frac{\partial^2 L}{\partial v^j \partial q^i} \right) \dot{\xi}^j(t) - \frac{\partial^2 L}{\partial q^j \partial q^i} \xi^j(t) + \dots = 0, \end{aligned}$$

where  $\xi^i(t) = q^i(t) - q_0^i$ ,  $i = 1, \dots, n$ . Here all partial derivatives have been evaluated at each step at  $(\mathbf{q}_0, \mathbf{0})$ , and we have used the fact that  $t \mapsto \mathbf{q}_0$  is a solution of the Euler-Lagrange

equations. From this expression we may define a linear Lagrangian system  $(\mathbf{M}_L, \mathbf{C}_L, \mathbf{K}_L)$  on  $\mathbb{R}^n$  by asking that the matrix representations of these bilinear maps satisfy

$$\begin{aligned} (\mathbf{M}_L)_{ij} &= \frac{\partial^2 L}{\partial v^i \partial v^j}(\mathbf{q}_0, \mathbf{0}), \quad i, j = 1, \dots, n \\ (\mathbf{C}_L)_{ij} &= \frac{\partial^2 L}{\partial q^i \partial v^j}(\mathbf{q}_0, \mathbf{0}) - \frac{\partial^2 L}{\partial v^i \partial q^j}(\mathbf{q}_0, \mathbf{0}), \quad i, j = 1, \dots, n \\ (\mathbf{K}_L)_{ij} &= -\frac{\partial^2 L}{\partial q^i \partial q^j}(\mathbf{q}_0, \mathbf{0}), \quad i, j = 1, \dots, n. \end{aligned} \quad (3.4)$$

This is, of course, not very satisfying because it only works in a fixed coordinate chart. The following result indicates how that can be made to be coordinate independent.

**3.2.10 Proposition** *Let  $L$  be a time-independent Lagrangian on  $Q$  and let  $q_0$  be an equilibrium point for  $L$ . There exists bilinear maps  $M_L(q_0), C_L(q_0), K_L(q_0): T_{q_0}Q \times T_{q_0}Q \rightarrow \mathbb{R}$  with  $M_L(q_0)$  and  $K_L(q_0)$  symmetric, and with  $C_L(q_0)$  skew-symmetric, and having the property that with respect to any set of coordinates  $(q^1, \dots, q^n)$  they are represented as in (3.4).*

*Proof* Let  $(q^1, \dots, q^n)$  and  $(\tilde{q}^1, \dots, \tilde{q}^n)$  be two sets of coordinates around  $q_0$ . Suppose that  $q_0$  is mapped to  $\mathbf{q}_0$  and  $\tilde{\mathbf{q}}_0$  in the two sets of coordinates. Using the chain rule and evaluating at  $(\mathbf{q}_0, \mathbf{0})$  we compute

$$\begin{aligned} \frac{\partial^2 L}{\partial \tilde{v}^i \partial \tilde{v}^j}(\tilde{\mathbf{q}}_0, \mathbf{0}) &= \left( \frac{\partial^2 L}{\partial v^k \partial v^\ell} \frac{\partial q^k}{\partial \tilde{q}^i} \frac{\partial q^\ell}{\partial \tilde{q}^j} \right)(\mathbf{q}_0, \mathbf{0}) \\ \frac{\partial^2 L}{\partial \tilde{q}^i \partial \tilde{v}^j}(\tilde{\mathbf{q}}_0, \mathbf{0}) &= \left( \frac{\partial^2 L}{\partial q^\ell \partial v^k} \frac{\partial q^\ell}{\partial \tilde{q}^i} \frac{\partial q^k}{\partial \tilde{q}^j} \right)(\mathbf{q}_0, \mathbf{0}) + \left( \frac{\partial L}{\partial v^k} \frac{\partial^2 q^k}{\partial \tilde{q}^i \partial \tilde{q}^j} \right)(\mathbf{q}_0, \mathbf{0}) \\ \frac{\partial^2 L}{\partial \tilde{q}^i \partial \tilde{q}^j}(\tilde{\mathbf{q}}_0, \mathbf{0}) &= \left( \frac{\partial^2 L}{\partial q^k \partial q^\ell} \frac{\partial q^k}{\partial \tilde{q}^i} \frac{\partial q^\ell}{\partial \tilde{q}^j} \right)(\mathbf{q}_0, \mathbf{0}) + \left( \frac{\partial L}{\partial q^k} \frac{\partial^2 q^k}{\partial \tilde{q}^i \partial \tilde{q}^j} \right)(\mathbf{q}_0, \mathbf{0}). \end{aligned}$$

We therefore have

$$\frac{\partial^2 L}{\partial \tilde{q}^i \partial \tilde{v}^j}(\tilde{\mathbf{q}}_0, \mathbf{0}) - \frac{\partial^2 L}{\partial \tilde{v}^i \partial \tilde{q}^j}(\tilde{\mathbf{q}}_0, \mathbf{0}) = \left[ \left( \frac{\partial^2 L}{\partial q^\ell \partial v^k} - \frac{\partial^2 L}{\partial v^\ell \partial q^k} \right) \frac{\partial q^\ell}{\partial \tilde{q}^i} \frac{\partial q^k}{\partial \tilde{q}^j} \right](\mathbf{q}_0, \mathbf{0}), \quad i, j = 1, \dots, n.$$

Furthermore, one see directly from the Euler-Lagrange equations (cf. equation (3.3)) that is  $q_0$  is an equilibrium point then

$$\frac{\partial L}{\partial q^k}(\mathbf{q}_0, \mathbf{0}) = 0, \quad k = 1, \dots, n.$$

This implies that

$$\frac{\partial^2 L}{\partial \tilde{q}^i \partial \tilde{q}^j}(\tilde{\mathbf{q}}_0, \mathbf{0}) = \left( \frac{\partial^2 L}{\partial q^k \partial q^\ell} \frac{\partial q^k}{\partial \tilde{q}^i} \frac{\partial q^\ell}{\partial \tilde{q}^j} \right)(\mathbf{q}_0, \mathbf{0}), \quad i = 1, \dots, n.$$

Now suppose that  $v_1, v_2 \in T_{q_0}Q$  are two tangent vectors which we write as

$$u_a = u_a^i \frac{\partial}{\partial q^i} = \tilde{u}_a^i \frac{\partial}{\partial \tilde{q}^i}, \quad a, b = 1, 2.$$

Using Lemma 2.2.2 we directly compute

$$\begin{aligned}\frac{\partial^2 L}{\partial v^i \partial v^j} u_1^i u_2^j &= \frac{\partial^2 L}{\partial \tilde{v}^i \partial \tilde{v}^j} \tilde{u}_1^i \tilde{u}_2^j \\ \left( \frac{\partial^2 L}{\partial q^i \partial v^j} - \frac{\partial^2 L}{\partial v^i \partial q^j} \right) u_1^i u_2^j &= \left( \frac{\partial^2 L}{\partial \tilde{q}^i \partial \tilde{v}^j} - \frac{\partial^2 L}{\partial \tilde{v}^i \partial \tilde{q}^j} \right) \tilde{u}_1^i \tilde{u}_2^j \\ \frac{\partial^2 L}{\partial q^i \partial q^j} u_1^i u_2^j &= \frac{\partial^2 L}{\partial \tilde{q}^i \partial \tilde{q}^j} \tilde{u}_1^i \tilde{u}_2^j.\end{aligned}$$

This shows that the representation of  $M_L(q_0)$ ,  $C_L(q_0)$ , and  $K_L(q_0)$  in coordinates by the expressions (3.4) is indeed independent of coordinates. ■

With the above result at hand, we define the **linearisation** of a Lagrangian  $L$  at an equilibrium point  $q_0$  to be the linear Lagrangian system  $(M_L(q_0), C_L(q_0), K_L(q_0))$  on the vector space  $T_{q_0}Q$ . For brevity, if  $M_L^b(q_0)$  is invertible, let us denote  $A_L(q_0) = A_{(M_L(q_0), C_L(q_0), K_L(q_0))}$ . The question now becomes, “What is the relationship between the dynamics of the linear Lagrangian system  $(M_L(q_0), C_L(q_0), K_L(q_0))$  and the dynamics of the Euler-Lagrange equations for  $L$  near  $q_0$ ?” In Section C.2 we provide some results which address this question in the case of a general ordinary differential equation (the Hartman-Grobman Theorem and the Stable and Unstable Manifold Theorem). Let us summarise these results in the context of Lagrangian systems. For precise statements, we refer to Section C.2.

**3.2.11 Summary** *Let  $L$  be a hyperregular, time-independent Lagrangian on a configuration space  $Q$  and let  $q_0 \in Q$  be an equilibrium point for  $L$ . Let  $E^s(q_0)$  be the subspace of  $T_{q_0}Q \oplus T_{q_0}Q$  consisting of the generalised eigenvectors of  $A_L(q_0)$  for eigenvalues with negative real part, and let  $E^u(q_0)$  be the subspace of  $T_{q_0}Q \oplus T_{q_0}Q$  consisting of the generalised eigenvectors of  $A_L(q_0)$  for eigenvalues with positive real part.*

- (i) *If  $A_L(q_0)$  has no eigenvalues on the imaginary axis, then the dynamics of the Euler-Lagrange equations near  $q_0$  “look like” the dynamics of the linear Lagrangian system  $(M_L(q_0), C_L(q_0), K_L(q_0))$ .*
- (ii) *There exists subsets  $W^s(q_0)$  and  $W^u(q_0)$  of  $TQ$  which are invariant under the dynamics of the Euler-Lagrange equations, and solutions starting on  $W^s(q_0)$  tend to  $q_0$  as  $t \rightarrow \infty$  and solutions on  $W^u(q_0)$  tend to  $q_0$  as  $t \rightarrow -\infty$ . Furthermore, the tangent space to  $W^s(q_0)$  at  $0_{q_0}$  is  $E^s(q_0)$  and the tangent space of  $W^u(q_0)$  at  $0_{q_0}$  is  $E^u(q_0)$ .*

We shall encounter various applications of these ideas as we proceed. In particular, in Section 3.3 we see how these results give some, although not much, insight into the stability of equilibrium points. We shall also see in Section 3.4 that the ideas of Summary 3.2.11 allow a great deal of insight into the dynamics of very simple systems.

### 3.3 Stability of Lagrangian equilibria

First the reader should recall the definitions of stability and asymptotic stability in Section C.2. Of course, these notions are applicable to Lagrangian systems only when the Euler-Lagrange equations can be put into ordinary differential equation form, e.g., when the Lagrangian is hyperregular. In this case, if  $q_0 \in Q$  is an equilibrium point, then  $0_{q_0}$  is a fixed point for the corresponding differential equations on  $TQ$ . In this case, we thus say that  $q_0$  is a **stable** (resp. **asymptotically stable**, resp. **spectrally stable**, resp. **linearly stable**,



resp. **linearly asymptotically stable**) equilibrium point if  $0_{q_0}$  is a stable (resp. asymptotically stable, resp. spectrally stable, resp. linearly stable, resp. linearly asymptotically stable) fixed point of the corresponding differential equations.<sup>1</sup>

The following result is the only easy general result one can get for Lagrangian systems, and even it relies on some nontrivial ideas. For example, we refer to Appendix B for the notion of the Hessian of a critical point of a function.

**3.3.1 Theorem** *Let  $L$  be an hyperregular, time-independent Lagrangian on a configuration space  $Q$ . If  $q_0$  is an equilibrium point for  $L$  then*

- (i)  $q_0$  is not asymptotically stable,
- (ii) if  $A_L(q_0)$  has an eigenvalue with nonzero real part, then  $q_0$  is unstable, and
- (iii)  $q_0$  is stable if  $0_{q_0}$  is either a strict local maximum or a strict local minimum of  $\mathbf{EL}$ ,
- (iv)  $q_0$  is stable if  $\text{Hess}(\mathbf{EL})(0_{q_0})$  is either positive or negative-definite.

*Idea of Proof* (i) The proof here is not quite within our reach. It relies on the fact that the flow of the Euler-Lagrange equations preserves a natural volume on  $TQ$ . The idea is that if an equilibrium point is to be stable, then a ball around  $0_{q_0}$  would have to be shrunk under the flow, and this is not possible if the flow is volume preserving.

(ii) If  $A_L(q_0)$  has an eigenvalue with nonzero real part, then by Proposition 3.2.3 it must have an eigenvalue with positive real part. By the Unstable Manifold Theorem, there exists a subset  $W^u(q_0)$  on  $TQ$  on which all solutions of the Euler-Lagrange equations tend to  $0_{q_0}$  as  $t \rightarrow -\infty$ . But this implies that the solutions must move away from  $0_{q_0}$  as  $t$  increases, and so will leave any sufficiently small neighbourhood of  $0_{q_0}$  in finite time. This means that  $q_0$  is unstable.

(iii) We fix a coordinate chart  $(U, \phi)$  for  $Q$  with coordinates  $(q^1, \dots, q^n)$ . If we suppose that  $\phi(q_0) = \mathbf{0}$ , then we may as well suppose that  $Q$  is an open subset  $U$  in  $\mathbb{R}^n$  containing a neighbourhood of  $\mathbf{0}$ . In this case we have  $TQ = U \times \mathbb{R}^n$ . Define

$$B_r = \{(\mathbf{q}, \mathbf{v}) \in TQ \mid \|\mathbf{q}\| + \|\mathbf{v}\| \leq r\}$$

$$\partial B_r = \{(\mathbf{q}, \mathbf{v}) \in TQ \mid \|\mathbf{q}\| + \|\mathbf{v}\| = r\}.$$

Now let  $\epsilon > 0$  and  $r \in (0, \epsilon]$  and let

$$\alpha = \min \{ \mathbf{EL}(\mathbf{q}, \mathbf{v}) \mid (\mathbf{q}, \mathbf{v}) \in \partial B_r \}.$$

For  $\beta \in (0, \alpha)$  let

$$\Omega_\beta = \{(\mathbf{q}, \mathbf{v}) \in TQ \mid \mathbf{EL}(\mathbf{q}, \mathbf{v}) \leq \beta\}.$$

We claim that  $\Omega_\beta$  lies in the interior of  $B_r$ . Indeed, suppose that  $(\mathbf{q}, \mathbf{v}) \in \partial B_r$ . Then, by definition of  $\alpha$ ,  $\mathbf{EL}(\mathbf{q}, \mathbf{v}) \geq \alpha$ , and since  $\beta < \alpha$ , we cannot have  $(\mathbf{q}, \mathbf{v}) \in \Omega_\beta$ . Since  $B_r$  is bounded, since  $\Omega_\beta \subset B_r$ , and since  $\Omega_\beta$  is closed, we have  $\mathbf{EL}^{-1}(\beta) \subset \Omega_\beta$  as a compact subset of  $U$ . Thus we have shown that in a neighbourhood of a strict local minimum of  $\mathbf{EL}$ , the level sets of  $\mathbf{EL}$  are compact. Since  $\mathbf{EL}$  is constant along solutions of the Euler-Lagrange equations (see Corollary 2.5.8), this implies that solutions which start near  $0_{q_0}$  remain near  $0_{q_0}$ , implying stability. The same argument, with a change of inequalities from “less than”

<sup>1</sup>To make sense of these definitions, at least as they are related to the definitions of Section C.2, one must choose a particular coordinate chart on  $Q$ . However, it is easy to show that if a stability property holds in one coordinate chart, it will hold in any coordinate chart.

to “greater than” shows that the same argument works if  $0_{q_0}$  is a strict local maximum of  $\mathbf{E}L$ .

(iv) This follows from part (iii) and the fact that definiteness of  $\text{Hess}(\mathbf{E}L)(0_{q_0})$  implies that  $0_{q_0}$  is a strict local minimum (positive-definiteness) or a strict local maximum (negative-definiteness). ■

Part (iii) is essentially due to Aleksandr Mikhailovich Liapunov (1857–1918). Part (iv) of the theorem is due in part to Dirichlet [1846] (Johann Peter Gustav Lejeune Dirichlet (1805–1859)).

Since asymptotic stability is not possible, we are left with deciding whether an equilibrium point is stable. Although part (iv) gives a sufficient condition, it is not necessary. The following example provides a system for which  $\text{Hess}(\mathbf{E}L)(0_{q_0})$  is not definite, but for which the system is nonetheless stable.

**3.3.2 Example** We take  $Q = \mathbb{R}^2$  and define a Lagrangian by

$$L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 - v_y^2) - \frac{1}{2}(x^2 - y^2).$$

Note that this is a Lagrangian for a linear Lagrangian system, so we can explicitly determine the stability of the system from the eigenvalues, and if need be, the eigenvectors using Proposition 3.2.6. However, for the moment let us ignore the fact that the system is linear, and proceed as if it were not.

The energy is readily determined to be

$$\mathbf{E}L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 - v_y^2) + \frac{1}{2}(x^2 - y^2).$$

We compute

$$d\mathbf{E}L = v_x dv_x - v_y dv_y + x dx - y dy.$$

Therefore there is only one equilibrium point, and it is  $q_0 = (x_0, y_0) = (0, 0)$ . The linearisation of  $L$  about  $q_0$  is given by

$$M_L(q_0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C_L(q_0) = \mathbf{0}, \quad K_L(q_0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We thus compute

$$A_L(q_0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of  $A_L(q_0)$  are readily computed to be  $\{i, i, -i, -i\}$ . Thus the eigenvalues are on the imaginary axis, and so it is *possible* that the equilibrium point be stable. However, we cannot conclude this from part (iv) of Theorem 3.3.1 since we compute that the matrix for  $\text{Hess}(\mathbf{E}L)(0_{q_0})$  is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Thus  $\text{Hess}(\mathbf{E}L)(0_{q_0})$  is neither positive nor negative-definite.

However, one *can* verify that the equilibrium point is stable. Here we take advantage of the fact that the system is linear. We have computed its eigenvalues, and from these we cannot yet conclude stability since they have algebraic multiplicity 2. By Proposition 3.2.6 we need to show that the geometric multiplicity is also 2, but this is something which we can readily check. Thus this example is one which violates the sufficient condition of part (iv) of Theorem 3.3.1, but which is still stable.  $\square$

From the previous example, one may be compelled to conjecture that a sufficient condition for stability would be that the eigenvalues for the linearised system should lie on the imaginary axis. After some thought, and a reference to Proposition 3.2.6, one may wish to alter this conjecture to say that the eigenvalues should also have equal geometric and algebraic multiplicities. That is, one would be led to the following.

**3.3.3 False Conjecture** *An equilibrium point  $q_0$  for a hyperregular Lagrangian is stable if and only if its linearised system  $(M_L(q_0), C_L(q_0), K_L(q_0))$  is stable.*

This conjecture is false in both directions. Let us first give a nontrivial example which indicates that the “if” assertion in the conjecture is false. The following example, apparently due to Cherry [1928], indicates that an equilibrium point with a stable linearisation can be unstable.

**3.3.4 Example** We take  $Q \subset \mathbb{R}^2$  (we will say exactly what  $Q$  is shortly) and use the Lagrangian

$$L(x, y, v_x, v_y) = \frac{1}{4(4 + 2\sqrt{2}x + y^2)^2} \left( -32x^4 - 8\sqrt{2}x^3(8 + 3y^2) - 4x^2(16 + 12y^2 + 3y^4 - 2v_x^2) + 2(4 + y^2)(4y^2 + y^4 + 2v_x^2 - 2\sqrt{2}yv_xv_y - 4v_y^2) + x(-\sqrt{2}y^6 + 2\sqrt{2}y^2(8 + v_x^2) - 16yv_xv_y + 16\sqrt{2}(v_x^2 - v_y^2)) \right).$$

Clearly this is no ordinary Lagrangian! One can readily ascertain that  $q_0 = (0, 0)$  is an equilibrium point, and that the above Lagrangian is well-defined on a neighbourhood of  $q_0$ . Thus we take  $Q = B_r(\mathbf{0})$  for some sufficiently small  $r$ . A computation with a computer gives

$$M_L(q_0) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}, \quad C_L(q_0) = \mathbf{0}_2, \quad K_L(q_0) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of  $A_L(q_0)$  are readily computed to be  $\{i, -i, 2i, -2i\}$ . Therefore the linearised system is stable by Proposition 3.2.6.

However, Cabral and Meyer [1999] provide a general result which demonstrates that the equilibrium point  $q_0$  is *unstable*. This result is a long way beyond our grasp. However, in Figure 3.2 we show some plots of solutions projected from  $TQ$  to  $Q$ . Note that the linear solution starts and remains close to the equilibrium point. Indeed, we can compute this solution in closed form to verify this. However, the nonlinear solution, starting at the same initial condition, zips around the equilibrium point, and then shoots away. If you believe numerics, this means something, but one really needs to resort to Cabral and Meyer for the “truth.”  $\square$

As a final example to illustrate the complex relationship between the dynamics of a linearised system and the full dynamics of the system, we give an example where the linearisation is unstable, but the full system is stable. Thus the “only if” assertion in the above false conjecture is also wrong.

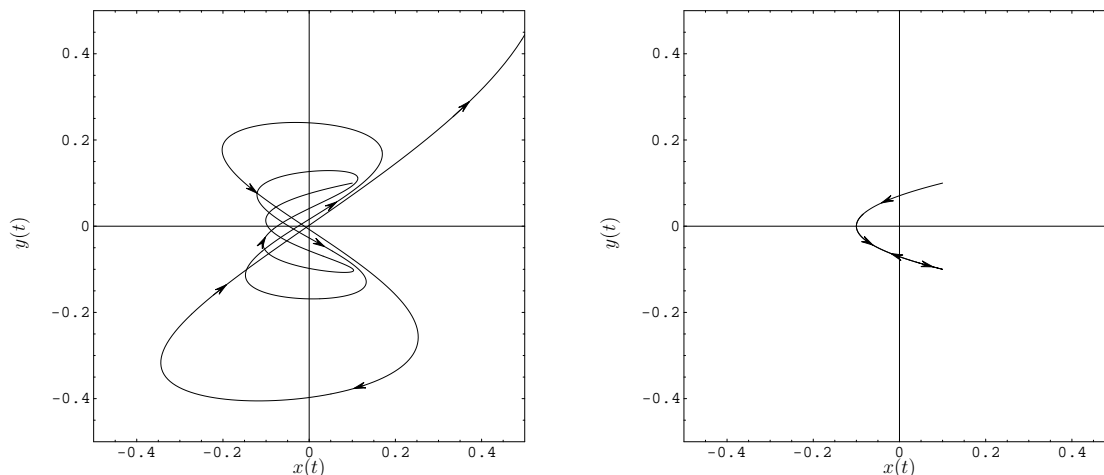


Figure 3.2 Solutions near an unstable equilibrium point with stable linearisation. On the left is the nonlinear solution for the initial condition  $x(0) = \frac{1}{10}$ ,  $y(0) = \frac{1}{10}$ ,  $\dot{x}(0) = 0$ , and  $\dot{y}(0) = 0$ . On the right is the linear solution for the same initial condition.

**3.3.5 Example** We consider  $Q = \mathbb{R}$  and on  $Q$  take the Lagrangian  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{4}q^4$ . This system has energy  $\mathbf{E}L = \frac{1}{2}v^2 + \frac{1}{4}q^4$ . We compute

$$d\mathbf{E}L = vdv + q^3dq$$

Therefore, by Proposition 3.2.9, the system has one equilibrium point at  $q_0 = 0$ . We compute

$$M_L(q_0) = \frac{\partial^2 L}{\partial v^2}(q, 0) = 1, \quad C_L(q_0) = \frac{\partial^2 L}{\partial q \partial v}(q, 0) - \frac{\partial^2 L}{\partial v \partial q}(q, 0) = 0,$$

$$K_L(q_0) = \frac{\partial^2 L}{\partial q^2}(q, 0) = 0.$$

Thus we have

$$A_L(q_0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This matrix has two zero eigenvalues, so does not satisfy the hypotheses of the Hartman-Grobman Theorem. Let us compare the linear and the nonlinear dynamics. It is possible, of course, to explicitly solve the linear differential equation

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} q \\ v \end{pmatrix}$$

using the matrix exponential:

$$\begin{pmatrix} q(t) \\ v(t) \end{pmatrix} = e^{A_L(q_0)t} \begin{pmatrix} q(0) \\ v(0) \end{pmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{pmatrix} q(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} v(0)t + q(0) \\ v(0) \end{pmatrix}.$$

To allow a comparison with the solutions of the nonlinear system, let us represent this solution by plotting the curves  $t \mapsto (q(t), v(t))$  for various initial conditions. The result is shown in Figure 3.3.

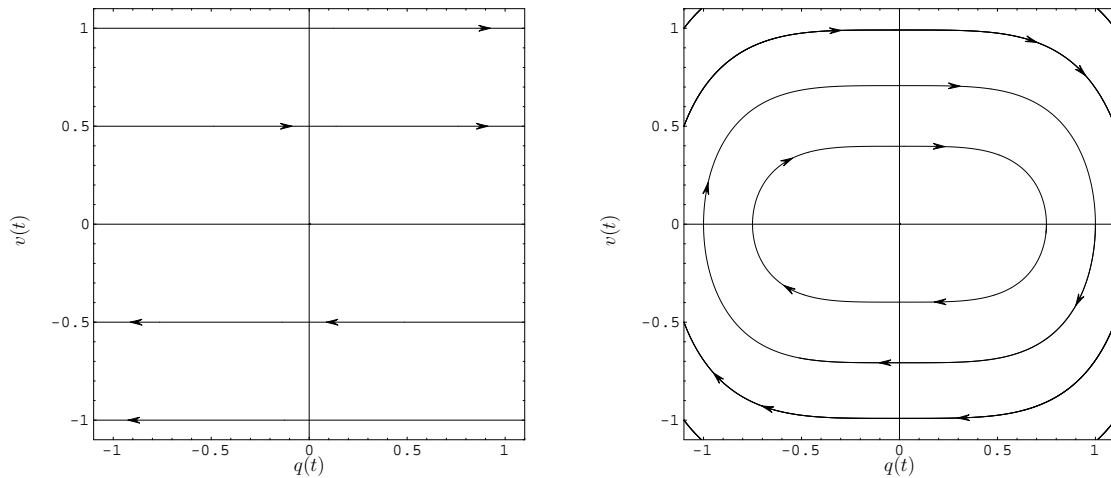


Figure 3.3 For the Lagrangian  $L = \frac{1}{2}v^2 - \frac{1}{4}q^4$ , on the left is the linearised dynamics for the equilibrium  $q_0 = 0$  and on the right is the actual dynamics

Now let us look at the actual dynamics. In this case the differential equations are

$$\ddot{q} + q^3 = 0.$$

While it actually *is* possible to obtain a closed form expression for the solution of this differential equation, to do so is rather pointless, at least as concerns trying to understand what the system is doing. In Section 3.4 we shall present a way of looking at systems of this type. For now, let us merely say that the flow is represented in Figure 3.3. The key thing to observe here is that the linearised system and the actual system behave nothing like one another. In particular, the origin is an unstable fixed point for the linearised system, but is stable, although not asymptotically stable, for the nonlinear system.  $\square$

These examples illustrate the difficulty one has in even doing something apparently simple like determining the stability of equilibrium points for Euler-Lagrange equations. Matters such as these are currently active areas of research. Indeed, the reason why Example 3.3.4 is so contrived is that for systems with model physical systems, the matter of stability of equilibria is unknown.

**3.3.1 Equilibria for simple mechanical systems** Let us see how the above development specialises to the case when the Lagrangian is derived from a simple mechanical system  $(Q, g, V)$ . In this case, the general discussion above concerning the relation of energy to equilibrium points and their stability boils down to looking only at the potential function. As we shall see, things are not as depressingly complicated for simple mechanical systems as they are for general systems.

First of all, let us look at how to find equilibrium points for a simple mechanical system.

**3.3.6 Proposition** *Let  $(Q, g, V)$  be a simple mechanical system with Lagrangian  $L$ . A point  $q_0 \in Q$  is an equilibrium point for  $L$  if and only if  $dV(q_0) = 0$ .*

*Proof* This is easily verified in coordinates where

$$\mathbf{E}L(\mathbf{q}, \mathbf{v}) = \frac{1}{2}g_{ij}v^i v^j + V(\mathbf{q}).$$

We then have

$$d\mathbf{E}L(\mathbf{q}, \mathbf{v}) = g_{ij}v^j dv^i + \frac{1}{2} \frac{\partial g_{ij}}{\partial q^k} v^i v^j dq^k + \frac{\partial V}{\partial q^i} dq^i,$$

where all partial derivatives are evaluated at  $(\mathbf{q}, \mathbf{v})$ . Thus we obtain

$$d\mathbf{E}L(\mathbf{q}, \mathbf{0}) = \frac{\partial V}{\partial q^i}(\mathbf{q})dq^i,$$

and from this the proposition easily follows. ■

Thus when one defines the linear Lagrangian system for a simple mechanical system at an equilibrium point  $q_0$ , one knows that  $q_0$  is a critical point of the potential function. Therefore, at  $q_0$  the Hessian of  $V$ ,  $\text{Hess}(V)(q_0)$ , is well-defined. With this in mind, we state the following result, telling us the nature of the linearisation for simple mechanical systems.

**3.3.7 Proposition** *Let  $(Q, g, V)$  be a simple mechanical system with Lagrangian  $L$ , and let  $q_0 \in Q$  be an equilibrium point for  $L$ . The linearised Lagrangian system at  $q_0$  is then*

$$M_L(q_0) = g(q_0), \quad C_L(q_0) = 0, \quad K_L(q_0) = \text{Hess}(V)(q_0).$$

*Proof* This is straightforward in coordinates. We have

$$L = \frac{1}{2}g_{k\ell}v^k v^\ell - V,$$

and so we compute

$$\begin{aligned} \frac{\partial^2 L}{\partial v^i \partial v^j} &= g_{ij} \\ \frac{\partial^2 L}{\partial q^i \partial v^j} &= \frac{\partial g_{j\ell}}{\partial q^i} v^\ell \\ \frac{\partial^2 L}{\partial v^i \partial q^j} &= \frac{\partial g_{i\ell}}{\partial q^j} v^\ell \\ \frac{\partial^2 L}{\partial q^i \partial q^j} &= \frac{1}{2} \frac{\partial^2 g_{k\ell}}{\partial q^i \partial q^j} v^k v^\ell - \frac{\partial^2 V}{\partial q^i \partial q^j}. \end{aligned}$$

The result follows after taking  $\mathbf{v} = \mathbf{0}$ . ■

Now let us see how this rather specific form of the linearised system leads to corresponding simplifications in the computation of eigenvalues. You will wish to refer to Appendix A for the notion of the index and the rank of a symmetric bilinear mapping.

**3.3.8 Proposition** *Let  $(Q, g, V)$  be a simple mechanical system with associated Lagrangian  $L$  and let  $q_0 \in Q$  be an equilibrium point for  $L$ . Let  $(M_L(q_0) = g(q_0), C_L(q_0) = 0_V, K_L(q_0) = \text{Hess}(V)(q_0))$  be the linearisation of  $L$  at  $q_0$ . A complex number  $\lambda$  is an eigenvalue of  $A_L(q_0)$  if and only if  $\lambda = \sqrt{-\ell}$  where  $\ell$  is an eigenvalue of  $M_L^\sharp \circ K_L^\flat$ . Furthermore, the eigenvalues of  $M_L^\sharp \circ K_L^\flat$  are real, and*

- (i) the number of zero eigenvalues is  $n - \text{rank}(K_L(q_0))$ ,
- (ii) the number of negative eigenvalues is  $\text{ind}(K_L(q_0))$ , and
- (iii) the number of positive eigenvalues is  $\text{rank}(K_L(q_0)) - \text{ind}(K_L(q_0))$ ,

where  $n = \dim(Q)$ .

*Proof* For brevity, denote  $V = T_{q_0}Q$ ,  $M = M_L(q_0)$ , and  $K = K_L(q_0)$ . Then  $\lambda \in \mathbb{C}$  is an eigenvalue of  $A_L(q_0)$  if and only if

$$\det \begin{bmatrix} -\lambda \text{id}_V & \text{id}_V \\ -M^\# \circ K^\flat & -\lambda \text{id}_V \end{bmatrix} = \det(\lambda^2 \text{id}_V + M^\# \circ K^\flat) = 0.$$

This shows that  $\lambda$  is an eigenvalue for  $A_L(q_0)$  if and only if  $-\lambda^2$  is an eigenvalue for  $M^\# \circ K^\flat$ . Next we claim that the eigenvalues of  $M^\# \circ K^\flat$  are real. This will follow if we can show that  $M^\# \circ K^\flat$  is symmetric with respect to an inner product on  $V$ . However, we compute

$$\begin{aligned} M(M^\# \circ K^\flat(v_1), v_2) &= \langle K^\flat(v_1); v_2 \rangle \\ &= K(v_1, v_2) = K(v_2, v_1) = \langle K^\flat(v_2); v_1 \rangle = M(M^\# \circ K^\flat(v_2), v_1). \end{aligned}$$

This shows that  $M^\# \circ K^\flat$  is symmetric with respect to the inner product  $M$ . This means that  $-\lambda^2$  must be real.

Finally, it remains to prove that the stated relationship between the eigenvalues of  $M^\# \circ K^\flat$  and the index of  $K$ . First of all, note that if we choose a basis  $\{e_1, \dots, e_n\}$  for  $V$  which is  $M$ -orthonormal, then the matrix of  $M^\# \circ K^\flat$  in this basis is  $\delta^{ik} K_{kj}$ ,  $i, j = 1, \dots, n$ . Let us denote this matrix by  $\mathbf{K}$ . Therefore, the eigenvalues of  $M^\# \circ K^\flat$  are the eigenvalues of  $\mathbf{K}$ . Let  $\ell_1, \dots, \ell_n$  be the eigenvalues of  $\mathbf{K}$  ordered so that  $\ell_1, \dots, \ell_k$  are positive,  $\ell_{k+1}, \dots, \ell_{k+m}$  are negative, and  $\ell_{k+m+1}, \dots, \ell_n$  are zero. For  $i \in \{1, \dots, k+m\}$  let  $\mathbf{v}_i$  be the eigenvector corresponding to  $\ell_i$ , and having the property that  $\mathbf{v}_i^t \mathbf{K} \mathbf{v}_i = 1$ . For  $i \in \{k+m+1, \dots, n\}$  let  $\mathbf{v}_i$  be any collection of orthogonal eigenvectors for the eigenvalue 0. In the basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  the matrix for  $K$  is then

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

since the basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is orthogonal with respect to the standard inner product. Therefore the index of  $K$  is  $k$  and the rank of  $K$  is  $n - k - m$ , and thus our claim follows. ■

Note that *all* eigenvalues for the linearisation of a simple mechanical system lie on either the real or imaginary axis. There can be no eigenvalue configurations as illustrated on the right in Figure 3.1.

The following result turns the above statement about eigenvalues into a statement about stability of the linearised system.

**3.3.9 Proposition** *Let  $(Q, g, V)$  be a simple mechanical system with Lagrangian  $L$ , and let  $q_0$  be an equilibrium point for  $L$ . Let  $(M_L(q_0) = g(q_0), C_L(q_0) = 0_V, K_L(q_0) = \text{Hess}(V)(q_0))$  be the linearisation of  $L$  about  $q_0$ . The following statements hold:*

- (i)  $q_0$  is spectrally stable if and only if  $\text{ind}(K_L(q_0)) = 0$ ;
- (ii)  $q_0$  is linearly stable if and only if  $\text{rank}(K_L(q_0)) - \text{ind}(K_L(q_0)) = n$ ;
- (iii)  $q_0$  is stable if  $q_0$  is a strict local minimum of  $V$ ;
- (iv)  $q_0$  is stable if  $\text{rank}(K_L(q_0)) - \text{ind}(K_L(q_0)) = n$ ;

(v)  $q_0$  is unstable if  $\text{ind}(K_L(q_0)) > 0$ .

*Proof* Throughout the proof, for brevity denote  $M = M_L(q_0)$ ,  $K_L(q_0) = K$ , and  $V = T_{q_0}Q$ .

(i) This follows since there are no eigenvalues of  $A_L(q_0)$  in  $\mathbb{C}_+$  if and only if  $M^\# \circ K$  has no negative eigenvalues. From part (iii) of Proposition 3.3.8 the result then follows.

(ii) For linear stability we must first of all have all eigenvalues of  $A_L(q_0)$  on the imaginary axis. This means all eigenvalues of  $M^\# \circ K^b$  must be nonnegative. We shall show that they must further be strictly positive, and that this is in fact sufficient for stability. So first suppose that all eigenvalues of  $M^\# \circ K^b$  are strictly positive. The only problem which can arise is when we have eigenvalues whose algebraic and geometric multiplicities are not equal. To resolve this, let  $\{e_1, \dots, e_n\}$  be a basis in which the matrix of  $M^\# \circ K^b$  is diagonal, with the diagonal entries therefore being the eigenvalues. Let  $\ell \in \mathbb{R}$  be an eigenvalue of  $M^\# \circ K^b$  and suppose that the basis  $\{e_1, \dots, e_n\}$  is arranged so that the matrix for  $M^\# \circ K^b$  has the form

$$\begin{bmatrix} \ell \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$$

where  $\mathbf{D}$  is a diagonal matrix with diagonal entries  $d_i$ ,  $i = 1, \dots, n-k$ , none of whose entries are  $\ell$ . Corresponding to the eigenvalue  $\ell$  of  $M^\# \circ K^b$  are the eigenvalues  $\pm\sqrt{-\ell}$  for  $A_L(q_0)$ .

Let us look at the eigenvalue  $\lambda = \sqrt{-\ell}$ . In the basis we have for  $V$ , the matrix for  $A_L(q_0) - \lambda \text{id}_{V^2}$  has the form

$$\begin{bmatrix} -\lambda \mathbf{I}_k & \mathbf{0} & \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\lambda \mathbf{I}_{n-k} & \mathbf{0} & \mathbf{I}_{n-k} \\ -\ell \mathbf{I}_k & \mathbf{0} & -\lambda \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} & \mathbf{0} & -\lambda \mathbf{I}_{n-k} \end{bmatrix}.$$

First suppose that  $\ell \neq 0$ . By performing the row operations  $R_{n+i} + \sqrt{-\ell}R_i$ ,  $i = 1, \dots, k$ , we obtain the matrix

$$\begin{bmatrix} -\lambda \mathbf{I}_k & \mathbf{0} & \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\lambda \mathbf{I}_{n-k} & \mathbf{0} & \mathbf{I}_{n-k} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} & \mathbf{0} & -\lambda \mathbf{I}_{n-k} \end{bmatrix}.$$

Now, performing the row operations  $R_{n+k+i} - \frac{d_i}{\sqrt{-\ell}}R_{k+i}$ ,  $i = 1, \dots, n-k$ , we obtain the matrix

$$\begin{bmatrix} -\lambda \mathbf{I}_k & \mathbf{0} & \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\lambda \mathbf{I}_{n-k} & \mathbf{0} & \mathbf{I}_{n-k} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\lambda \mathbf{I}_{n-k} - \frac{1}{\sqrt{-\ell}}\mathbf{D} \end{bmatrix}.$$

Thus we see that  $\dim(\ker(A_L(q_0) - \lambda \text{id}_{V^2})) = k$ .

Now we take the case where  $\lambda = -\sqrt{\ell}$ . Here we have the matrix for  $A_L(q_0) - \lambda \text{id}_{V^2}$  again as

$$\begin{bmatrix} -\lambda \mathbf{I}_k & \mathbf{0} & \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\lambda \mathbf{I}_{n-k} & \mathbf{0} & \mathbf{I}_{n-k} \\ -\ell \mathbf{I}_k & \mathbf{0} & -\lambda \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} & \mathbf{0} & -\lambda \mathbf{I}_{n-k} \end{bmatrix}.$$

By performing the row operations  $R_{n+i} + \sqrt{-\ell}R_i$ ,  $i = 1, \dots, k$ , followed by the row operations



$R_{n+k+i} + \frac{d_i}{\sqrt{-\ell}} R_{k+i}$ ,  $i = 1, \dots, n - k$ , we get the matrix

$$\begin{bmatrix} -\lambda \mathbf{I}_k & \mathbf{0} & \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & -\lambda \mathbf{I}_{n-k} & \mathbf{0} & \mathbf{I}_{n-k} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\lambda \mathbf{I}_{n-k} + \frac{1}{\sqrt{-\ell}} \mathbf{D} \end{bmatrix},$$

which again has rank  $2n - k$ . Thus we have shown that nonzero eigenvalues for  $M^\sharp \circ K^b$  give rise to eigenvalues for  $A_L(q_0)$  whose geometric and algebraic multiplicities are equal. By Proposition 3.2.6, this proves that  $q_0$  is linearly stable if all eigenvalues of  $V$  are positive.

Finally, suppose that  $\ell$  is a zero eigenvalue for  $M^\sharp \circ K^b$  of algebraic multiplicity  $k$ . By Proposition 3.2.3, this gives rise to a zero eigenvalue of  $A_L(q_0)$  of multiplicity  $2k$ . We choose a basis  $\{e_1, \dots, e_n\}$  for  $V$  as above so that the matrix for  $A_L(q_0)$  has the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-k} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The rank of this matrix is  $k$ , which shows that zero eigenvalues of the linearisation of a simple mechanical system *never* have equal algebraic and geometric multiplicities. Therefore, by Proposition 3.2.6, if  $M^\sharp \circ K^b$  has a zero eigenvalue,  $q_0$  is linearly unstable. This part of the proposition now follows from Proposition 3.3.8(i).

(iii) From Theorem 3.3.1(iii), we need only show that if  $q_0$  is a strict local minimum of  $V$ , then  $0_{q_0}$  is a strict local minimum of  $\mathbf{E}L$ . Suppose that  $0_{q_0}$  is not a strict local minimum of  $\mathbf{E}L$ . This means that there exists a  $v_q$  nearby  $0_{q_0}$  with the property that

$$\mathbf{E}L(v_q) = \frac{1}{2}g(v_q, v_q) + V(q) \leq V(q_0).$$

Since  $g$  is positive-definite, this would imply that there exists a  $q$  nearby  $q_0$  with the property that  $V(q_0) - V(q) \geq 0$ . This means that  $q_0$  is not a strict local minimum for  $V$ .

(iv) If  $\text{rank}(K) - \text{ind}(K) = n$  then  $\text{Hess}(V)(q_0)$  is positive-definite. However, since  $\mathbf{E}L(v_q) = \frac{1}{2}g(v_q, v_q) + V(q)$ , we compute in coordinates

$$\text{Hess}(\mathbf{E}L)(0_{q_0}) = g_{ij} dv^i \otimes dv^j + (\text{Hess}(V)(q_0))_{ij} dq^i \otimes dq^j.$$

Therefore  $\text{Hess}(\mathbf{E}L)(0_{q_0})$  is definite if and only if  $\text{Hess}(V)(q_0)$  is positive-definite. From this we determine, from Theorem 3.3.1(iv) that  $q_0$  is stable if  $\text{Hess}(V)(q_0)$  is positive-definite, from which follows our result.

(v) Recall from Proposition 3.3.8(iii) that  $A_L(q_0)$  has a nonzero real eigenvalue if and only if  $\text{ind}(K_L(q_0)) > 0$ . The result now follows from Theorem 3.3.1(ii). ■

Thus our False Conjecture 3.3.3 is partially true for simple mechanical systems, as is borne out by the following important result.

**3.3.10 Corollary** *An equilibrium point for a simple mechanical system is stable if it is linearly stable.*

The converse assertion is still false, and Example 3.3.5 is a counterexample where an equilibrium point is stable, but not linearly stable. However, it *is* true that if one disallows equilibrium points where  $\text{Hess}(V)(q_0)$  is degenerate, then linear stability is necessary and sufficient for stability of simple mechanical systems. Let us record this as another corollary.

**3.3.11 Corollary** *Let  $\mathcal{S}$  be the class of simple mechanical systems, all of whose equilibrium points  $q_0$  have the property that  $\text{Hess}(V)(q_0)$  is nondegenerate. Then an equilibrium point  $q_0$  for a simple mechanical system  $(Q, g, V) \in \mathcal{S}$  is stable if and only if it is linearly stable.*

In nonlinear analysis, functions all of whose critical points are nondegenerate are called **Morse functions**.

Note that our above discussion of stability of equilibrium points for simple mechanical systems has a simple interpretation as follows.

**3.3.12 Corollary** *Let  $(Q, g, V)$  be a simple mechanical system with  $q_0$  an equilibrium point for the associated Lagrangian. The following statements hold:*

- (i)  $q_0$  is stable if  $q_0$  is a strict local minimum of the potential  $V$ ;
- (ii) if  $\text{Hess}(V)(q_0)$  is nondegenerate, then  $q_0$  is unstable if and only if  $q_0$  is a local maximum or a saddle point for the potential  $V$ .

*Proof* The first statement is simply a restatement of Proposition 3.3.9(iii). The second assertion follows from the classical results (see Appendix B) stating that if  $\text{Hess}(V)(q_0)$  is nondegenerate, then  $q_0$  is a local maximum of  $V$  if and only if  $\text{ind}(\text{Hess}(V)(q_0)) = 0$ , i.e., if and only if  $V$  is positive-definite. ■

The picture one might have in mind here is shown in Figure 3.4. If one wished, one

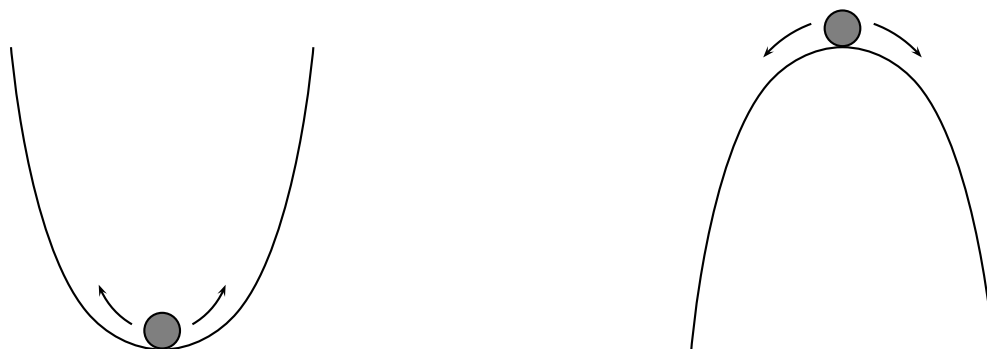


Figure 3.4 Stability and local minima/maxima of the potential

could turn this into some meta-principle where a physical system will seek to minimise its energy. But such a principle might not be that helpful in really understanding what is going on. . . The question of what happens when  $q_0$  is not a *strict* local minimum of the potential function is explored by [Laloy and Peiffer \[1982\]](#). I am not aware of any result stating that if  $q_0$  is degenerate then one may still conclude that  $q_0$  being a strict local maximum implies instability of  $q_0$ .

### 3.4 The dynamics of one degree of freedom systems

The previous section was very general in nature, and the examples we gave were for the most part “cooked” to show how some of our results are the best one can generally expect, without a lot of work. We have deliberately set aside any detailed discussion of specific physical systems, because there are really very few of these we can handle. The vast majority of systems we have any sort of complete understanding of are examples with

a one-dimensional configuration space, and in this section we will see that the behaviour of these are, in principal, simple.

**3.4.1 General one degree of freedom systems** It is a fact that essentially<sup>2</sup> the only one-dimensional configuration spaces are  $Q = \mathbb{R}$  and  $Q = \mathbb{S}^1$ . If  $Q = \mathbb{S}^1$  one may suppose without loss of generality that  $Q = \mathbb{R}$  by simply assuming that everything is  $2\pi$ -periodic in the configuration coordinate. Thus throughout this section we suppose that  $Q = \mathbb{R}$ . We shall also denote by  $q$  the standard coordinate on  $Q = \mathbb{R}$  and use  $(q, v)$  as the induced standard coordinates for  $TQ$ .

The key simplification comes from the fact that for time-independent Lagrangians, energy is conserved (Corollary 2.5.8). Therefore, for a given initial condition  $(q(0), \dot{q}(0)) \in TQ$ , each point  $(q(t), \dot{q}(t))$  along the solution to the Euler-Lagrange equations will satisfy  $(q(t), v(t)) \in \mathbf{EL}^{-1}(e)$  where  $e = \mathbf{EL}(q(0), \dot{q}(0))$  and where

$$\mathbf{EL}^{-1}(e) = \{(q, v) \in TQ \mid \mathbf{EL}(q, v) = e\}$$

(note that  $\mathbf{EL}^{-1}(e)$  is not the inverse of  $e$  under  $\mathbf{EL}$ !) An important question is then, “What do the level sets  $\mathbf{EL}^{-1}(e)$  of the energy look like?” For one degree of freedom systems, it is the answer to this question which leads to our being able to give these systems a rather thorough treatment. The following result contains some key features of the level sets  $\mathbf{EL}^{-1}(e)$ . The statement of this result requires that we know what a real analytic function is, and the proof requires some techniques which would take us slightly far afield. However, the reader will, I hope, be satisfied by our saying that the vast majority of Lagrangians which are smooth and associated with physical systems are actually real analytic.

**3.4.1 Theorem** *Let  $L: TQ \rightarrow \mathbb{R}$  be a real analytic Lagrangian with  $Q = \mathbb{R}$ . For  $e \in \text{image}(\mathbf{EL})$ , the following statements hold:*

- (i) *if  $d\mathbf{EL}(q, v) \neq 0$  for each  $(q, v) \in \mathbf{EL}^{-1}(e)$  then  $\mathbf{EL}^{-1}(e)$  is a smooth, nonintersecting one-dimensional surface in  $TQ$ ;*
- (ii) *if  $d\mathbf{EL}(q, v) = 0$  for some  $(q, v) \in \mathbf{EL}^{-1}(e)$  then  $\mathbf{EL}^{-1}(e)$  is a union of smooth one-dimensional surfaces and points in  $TQ$ ;*
- (iii) *if  $\mathbf{EL}^{-1}(e)$  contains an open subset of  $TQ$  then  $\mathbf{EL}$  is the constant function given by  $\mathbf{EL}(q, v) = e$ .*

The first assertion in the theorem is true even when  $L$  is not real analytic, but the other two statements require real analyticity.

**3.4.2 Examples** Let us look at some simple examples of Lagrangians to verify that we understand what the above statements mean.

1. First we take  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2$ . We then have  $\mathbf{EL}(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2$ . Clearly  $\mathbf{EL}^{-1}(0) = \{(0, 0)\}$ . If  $e < 0$  then  $\mathbf{EL}^{-1}(e) = \emptyset$ . If  $e > 0$  then  $\mathbf{EL}^{-1}(e)$  is the circle of radius  $\sqrt{2e}$ . The level sets are depicted in Figure 3.5.

Let us see how this lines up against Theorem 3.4.1. We compute

$$d\mathbf{EL}(q, v) = vdv + qdq.$$

Therefore  $d\mathbf{EL}(q, v) = 0$  if and only if  $(q, v) = (0, 0)$ . Therefore, by Theorem 3.4.1,  $\mathbf{EL}^{-1}$  should be a smooth one-dimensional surface for  $e > 0$ , and indeed it is. For  $e = 0$ ,

<sup>2</sup>By “essentially,” we mean “up to diffeomorphism.”

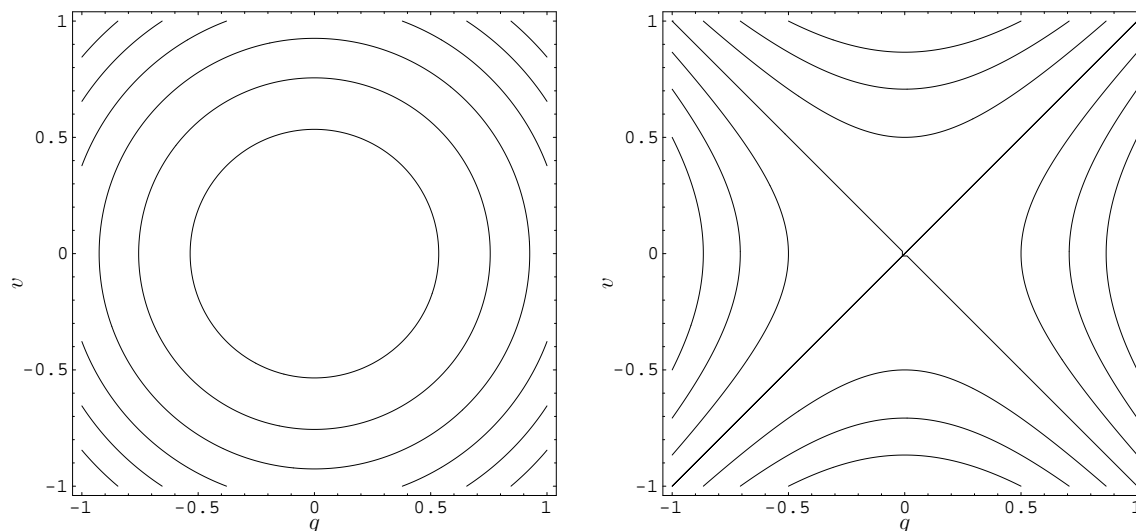


Figure 3.5 Level sets for  $\mathbf{EL}$  for  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2$  (left) and  $L = \frac{1}{2}v^2 + \frac{1}{2}q^2$  (right).

Theorem 3.4.1 says that  $\mathbf{EL}^{-1}(0)$  should be the union of one-dimensional surfaces and points. Since  $\mathbf{EL}^{-1}(0) = \{(0, 0)\}$ , it is a single point in this case.

- Next we take  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2$ . We have  $\mathbf{EL} = \frac{1}{2}v^2 - \frac{1}{2}q^2$ . Note that  $\text{image}(\mathbf{EL}) = \mathbb{R}$ , so  $\mathbf{EL}^{-1}(e)$  is nonempty for each  $e \in \mathbb{R}$ . The level sets are things you plotted as a child, and there you noticed that they were hyperbolae, except when  $e = 0$ . If  $e > 0$  then the hyperbolae are “up and down” mirror images of one another, and when  $e < 0$  they are “left and right” mirror images of one another. When  $e = 0$  the points  $(q, v)$  in  $\mathbf{EL}^{-1}(0)$  satisfy  $q^2 = v^2$ . Therefore the level set  $\mathbf{EL}^{-1}(0)$  consists of the two lines with slopes  $\pm 1$  passing through the origin. This is shown in Figure 3.5.

Let us make sure we understand that this is the same as what is predicted by Theorem 3.4.1. For  $e \neq 0$  the level set is comprised of two nice one-dimensional surfaces. For  $e = 0$  the level set is comprised of the four lines

$$\begin{aligned} \ell_1 &= \{(a, a) \mid a > 0\} \\ \ell_2 &= \{(a, a) \mid a < 0\} \\ \ell_3 &= \{(a, -a) \mid a > 0\} \\ \ell_4 &= \{(a, -a) \mid a < 0\}, \end{aligned}$$

along with the point  $\{(0, 0)\}$ . For various reasons which are perhaps not transparent, we do not wish to allow our one-dimensional surfaces to intersect, and for this reason we make four nonintersecting lines out of the two intersecting lines, and then throw in the point of intersection as something extra.  $\square$

In the next section we shall see many not quite so trivial illustrations of Theorem 3.4.1.

Let us now turn to the investigation of equilibrium points for single degree of freedom systems. Whereas in the general setting of Section 3.2 we were primarily concerned with stability, which is a qualitative notion, for single degree of freedom systems, one can be a bit more ambitious and try to get a somewhat more quantitative understanding of the system behaviour near an equilibrium point  $q_0$ . Thus we let  $q_0$  be an equilibrium point for a time-independent Lagrangian on  $Q = \mathbb{R}$ . By Proposition 3.2.9 it follows that  $d\mathbf{EL}(q_0, 0) = 0$ .

This puts us in the situation of part (ii) of Theorem 3.4.1. Thus the level set through  $(q_0, 0)$  may not be a smooth one-dimensional surface. We also note that the linearisation at  $q_0$  is simply defined

$$m = M_L(q_0) = \frac{\partial^2 L}{\partial v \partial v}(q_0, 0) \in \mathbb{R}, \quad k = K_L(q_0) = \frac{\partial^2 L}{\partial q \partial q}(q_0, 0) \in \mathbb{R},$$

since  $C_L(q_0) = 0$ . Thus the linearisation is defined by two numbers  $m$  and  $k$ . The following result gives some key properties of an equilibrium configuration in terms of the numbers  $m$  and  $k$ .

**3.4.3 Proposition** *Let  $L$  be a time-independent Lagrangian on  $Q = \mathbb{R}$  and let  $q_0$  be an equilibrium point for  $L$ . If  $m = M_L(q_0)$  and  $k = K_L(q_0)$ , then the following statements hold:*

- (i) *if  $L$  is hyperregular then  $m = M_L(q_0) \neq 0$ ;*
- (ii) *Hess( $\mathbf{E}L$ )( $q_0, 0$ ) is nondegenerate if and only if both  $m$  and  $k$  are nonzero.*

*Proof* (i) This follows from the fact that if  $L$  is hyperregular, then the derivative of  $\mathbf{F}L$  must be invertible. Indeed, the Jacobian for the mapping  $\mathbf{F}L$  has the matrix representation

$$\begin{bmatrix} 1 & 0 \\ \frac{\partial^2 L}{\partial q \partial v} & \frac{\partial^2 L}{\partial v^2} \end{bmatrix}.$$

Thus we see that hyperregularity implies that  $\frac{\partial^2 L}{\partial v^2} \neq 0$ , which in particular implies that  $m = M_L(q_0) \neq 0$ .

(ii) As

$$\mathbf{E}L(q, v) = \frac{\partial L}{\partial v}v - L,$$

we directly compute (skipping the calculation) that

$$\text{Hess}(\mathbf{E}L)(q_0, 0) = \frac{\partial^2 L}{\partial v^2}(q_0, 0)dv \otimes dv - \frac{\partial^2 L}{\partial q^2}(q_0, 0)dq \otimes dq.$$

Therefore the Hessian is nondegenerate if and only if both  $m$  and  $k$  are nonzero, as stated. ■

Now let us look at the nature of the dynamics near an equilibrium point  $q_0$  where  $\mathbf{E}L(q_0, 0)$  is nondegenerate. First let us look at the *exact* dynamics of the linearised system. We have obtained the closed form solution to the linearised system already (cf. Example 3.2.7). Here we are after a more descriptive characterisation of the solutions.

**3.4.4 Proposition** *Let  $L$  be a time-independent Lagrangian on  $Q = \mathbb{R}$  and suppose that  $q_0$  is an equilibrium point where Hess( $\mathbf{E}L$ )( $q_0, 0$ ) is nondegenerate. Denote  $m = M_L(q_0)$  and  $k = K_L(q_0)$ . We have the following two cases.*

- (i)  $\frac{k}{m} > 0$ : *Nonequilibrium solutions  $t \mapsto q(t)$  have the property that  $(q(t), \dot{q}(t))$  lies on the ellipses defined by  $v^2 + \frac{k}{m}q^2 = \frac{2e}{m}$ .*
- (ii)  $\frac{k}{m} < 0$ : *Nonequilibrium solutions come in three forms:*
  - (a) *solutions  $t \mapsto q(t)$  with the property that  $(q(t), \dot{q}(t))$  lies on the line of slope  $\sqrt{-k/m}$  through the origin and which satisfy  $\lim_{t \rightarrow -\infty} q(t) = 0$ ;*
  - (b) *solutions  $t \mapsto q(t)$  with the property that  $(q(t), \dot{q}(t))$  lies on the line of slope  $-\sqrt{-k/m}$  through the origin and which satisfy  $\lim_{t \rightarrow \infty} q(t) = 0$ ;*

(c) solutions  $t \mapsto q(t)$  for which  $(q(t), \dot{q}(t))$  lies on one of the hyperbolae defined by  $v^2 + \frac{k}{m}q^2 = \frac{2e}{m}$ .

*Proof* In each case, the result simply follows by conservation of energy; solutions must lie on the level sets  $\mathbf{E}L^{-1}(e)$ ,  $e \in \mathbb{R}$ . Since the energy for the linearised Lagrangian is given by  $\frac{m}{2}v^2 + \frac{k}{2}q^2$ , when  $\frac{k}{m} > 0$  these level sets are ellipses, and when  $\frac{k}{m} < 0$ , these level sets are hyperbolae. In part (ii), the statements about the limiting values of  $q(t)$  follow if we note that if  $\dot{q}(t) > 0$  then  $q(t)$  must be increasing at time  $t$ , and if  $\dot{q}(t) < 0$  then  $q(t)$  must be decreasing at time  $t$ . ■

The situation of Proposition 3.4.4 is illustrated in Figure 3.6. Note, of course, that the

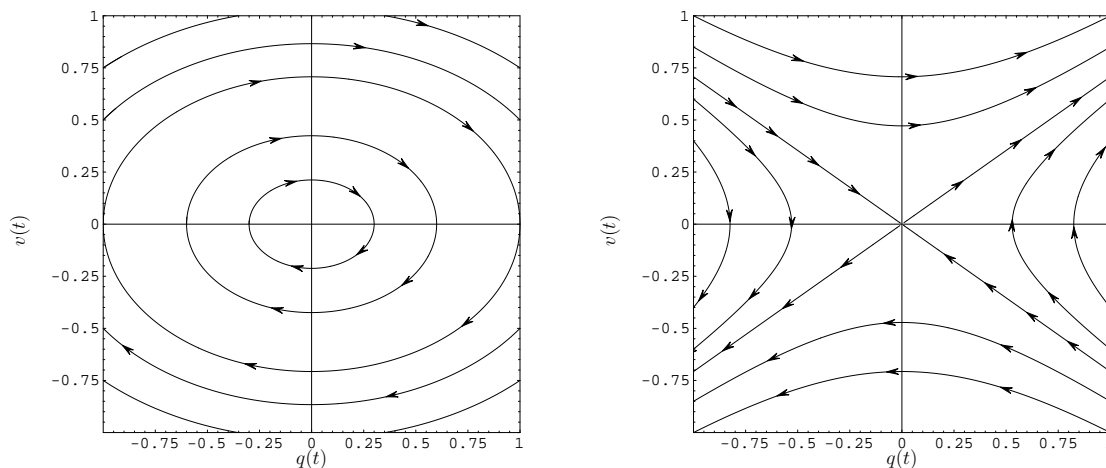


Figure 3.6 Solutions to one degree of freedom linear Lagrangian systems. On the left,  $\frac{k}{m} > 0$  and on the right,  $\frac{k}{m} < 0$ . In each case  $|\frac{k}{m}| = \frac{1}{2}$ .

solutions follow curves which look a lot like the level sets we plotted in Example 3.4.2. All we have done is added arrows to indicate that these are curves which are parameterised by time.

Now one can ask whether the dynamics of the nonlinear system “look like” the dynamics of the linearised system. We still restrict our discussion to the case where  $\text{Hess}(\mathbf{E}L)(q_0, 0)$  is nondegenerate. We consider the two cases.

1. When  $\frac{k}{m} < 0$ , the answer is given to us by the Hartman-Grobman Theorem since in this case the eigenvalues of  $A_L(q_0)$  all have nonzero real part; indeed they are real and nonzero. By the Hartman-Grobman Theorem we may assert that the solutions nearby the equilibrium point behave like those of the linearised system. Furthermore, we can appeal to the Stable and Unstable Manifold Theorem to assert the existence of stable and unstable manifolds which emit from the point  $(q_0, 0) \in TQ$  and which are tangent to the lines of slope  $\pm\sqrt{-k/m}$ .
2. When  $\frac{k}{m} > 0$  then we can appeal to neither the Hartman-Grobman Theorem nor the Stable and Unstable Manifold Theorem to help us in understanding the nature of the solutions. However, we *can* appeal to Morse’s Lemma which tells us, since  $\text{Hess}(\mathbf{E}L)(q_0, 0)$

is nondegenerate, that the solutions must lie on one-dimensional subsets of  $TQ$  which are like circles. So, even in the case when  $A_L(q_0)$  has imaginary eigenvalues, in the single degree of freedom case we can understand the nature of solutions near nondegenerate equilibrium points by looking at the linearisation. *But this only works for single degree of freedom systems!* This is evident from Example 3.3.4 which has two degrees of freedom.

Let us see how to apply this in an example. We choose an example which is not a simple mechanical system, since we shall specifically deal with such systems in the next section.

**3.4.5 Example** We take  $L = \frac{1}{2}v^2 + \frac{1}{4}v^4 - \frac{1}{2}q^2 + \frac{1}{3}q^3$ . We then compute

$$\begin{aligned} \mathbf{E}L &= \frac{\partial L}{\partial v}v - L = \frac{1}{2}v^2 + \frac{3}{4}v^4 + \frac{1}{2}q^2 - \frac{1}{3}q^3 \\ \implies \mathbf{E}L &= (q - q^2)dq + (v + 3v^3)dv. \end{aligned}$$

Therefore, equilibrium points occur when  $q(1 - q) = 0$ . Thus there are two equilibrium points  $q_1 = 0$  and  $q_2 = 1$ . We compute

$$\begin{aligned} m_1 &= \frac{\partial L}{\partial v^2}(q_1, 0) = 1, & k_1 &= \frac{\partial L}{\partial q^2}(q_1, 0) = 1 \\ m_2 &= \frac{\partial L}{\partial v^2}(q_2, 0) = 1, & k_2 &= \frac{\partial L}{\partial q^2}(q_2, 0) = -1. \end{aligned}$$

Referring to Proposition 3.4.4, we see that for the linearised system at  $q_1$  the solutions lie on ellipses (in fact circles in this case), and for the linearised system at  $q_2$  the solutions lie on hyperbolae. This then tells us the local nature of the solutions near these equilibria since  $\text{Hess}(\mathbf{E}L)$  is nondegenerate at each of the equilibrium points.

While this describes solutions near the equilibria, it does not provide a complete description of the solutions. To provide such a description, one would need to understand the structure of all level sets of  $\mathbf{E}L$ . For general Lagrangians, even single degree of freedom Lagrangians, this can be problematic. Thus we simply give the numerically plotted set of solution curves in Figure 3.7. Note that in this problem the stable and unstable manifolds of the equilibrium point  $q_2$ , while they look locally like their linear counterparts, behave quite differently when we move away from the equilibrium point. Indeed, the stable and unstable manifolds of this equilibrium point intersect! This is something which cannot happen in a linear system.  $\square$

**3.4.6 Remarks 1.** It is interesting to observe that our approach to describing the solutions in this section has been done without explicit reference to the differential equations! This perhaps seems very strange indeed. However, it serves to illustrate the important point that when dealing with Lagrangian mechanics, and indeed with many other complicated systems, often it is beneficial to concentrate on the other aspects of the problem rather than concentrating on the differential equations themselves. This is because it is not of much value to obtain actual solutions to these differential equations since such solutions, when obtainable, involve complicated functions. It is often more useful to have at hand the qualitative information our described methods yield. However, some diehards may prefer elliptic functions, or approximations of elliptic functions, in lieu of a nice picture like Figure 3.7.

2. A picture such as Figure 3.7 is called a **phase portrait** for the system. It is possible to draw these effectively only for single degree of freedom systems. However, for systems



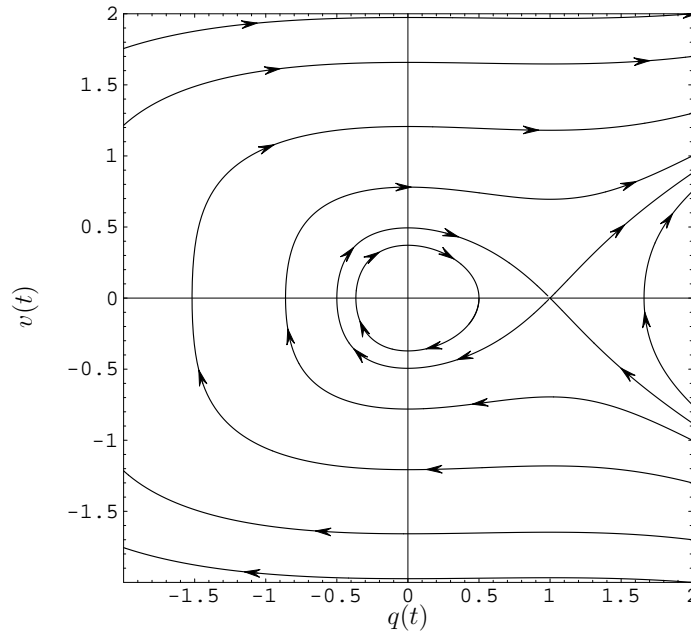


Figure 3.7 Solution curves for the Lagrangian  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{4}v^4 - \frac{1}{2}q^2 + \frac{1}{3}q^3$

with multiple degrees of freedom, it is still an effective way to think of solutions, rather than thinking of them as functions of time.  $\square$

**3.4.2 Simple mechanical systems with one degree of freedom** Now let us specialise the somewhat general discussion of the previous section to the case of a simple mechanical system with a single degree of freedom. Let us begin by demonstrating that in this case we may make some fairly extreme simplifications.

First note that the most general simple mechanical system on  $Q = \mathbb{R}$  has the Lagrangian

$$L(q, v) = \frac{1}{2}g(q)v^2 - V(q)$$

for functions  $g: Q \rightarrow \mathbb{R}_+$  and  $V: \mathbb{R} \rightarrow \mathbb{R}$ . The Euler-Lagrange equations for this Lagrangian are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial q} = g(q)\ddot{q} + \frac{1}{2}g'(q)\dot{q}^2 + V'(q) = 0.$$

If we introduce a new coordinate  $\tilde{q}$  for  $\mathbb{R}$  defined by

$$\tilde{q}(q) = \int_0^q \sqrt{g(s)} \, ds,$$

then we determine that the equality

$$\sqrt{g(q)}\ddot{q} + \frac{g'(q)}{2\sqrt{g(q)}}\dot{q}^2 + \frac{V'(q)}{\sqrt{g(q)}} = \ddot{\tilde{q}} + \tilde{V}'(\tilde{q})$$



holds, where  $\tilde{V}(\tilde{q}) = V(q(\tilde{q}))$ . The equations on the right are exactly the Euler-Lagrange equations for the Lagrangian  $\tilde{L}(\tilde{q}, \tilde{v}) = \frac{1}{2}\tilde{v}^2 - \tilde{V}(\tilde{q})$ . Thus we have proved the following result.

**3.4.7 Lemma** *If  $(Q, g, V)$  is a simple mechanical system with  $Q = \mathbb{R}$ , then we may choose a coordinate  $\tilde{q}$  for  $Q$  so that in this coordinate system the Lagrangian is  $\frac{1}{2}\tilde{v}^2 - V(\tilde{q})$ .*

Therefore, without loss of generality, in the remainder of this section we shall assume that we are working with a simple mechanical system on  $Q = \mathbb{R}$  with Lagrangian  $L(q, v) = \frac{1}{2}v^2 - V(q)$ .

Most of what is interesting about such systems can be gleaned from the potential function  $V$ . Indeed, from what we have done in general, we may make the following statements:

1. from Proposition 3.3.6, a point  $q_0 \in Q$  is an equilibrium point if and only if  $V'(q_0) = 0$ ;
2.  $\text{Hess}(\mathbf{E}L)(q_0, 0)$  is nondegenerate if and only if  $V''(q_0) \neq 0$ ;
3. from Proposition 3.4.4(i), if  $q_0$  is an equilibrium point and if  $V''(q_0) > 0$  then solutions near the equilibrium point lie on ellipses in  $TQ$ ;
4. from Proposition 3.4.4(ii), if  $q_0$  is an equilibrium point and if  $V''(q_0) < 0$  then solutions near the equilibrium point lie on either (1) hyperbolae in  $TQ$  or on (2) curves which approach the point  $(q_0, 0) \in TQ$  as  $t$  goes to either  $+\infty$  or  $-\infty$ .

There is, in fact, a very simple intuitive rule which takes into account all of the above, and makes it very easy to draw the phase portraits for single degree of freedom simple mechanical systems. The rule is this.

**3.4.8 Phase portraits for single degree of freedom simple mechanical systems** *Do the following:*

(i) *Draw the graph of the potential function  $V$ .*

*Imagine a ball rolling without friction in a well with the shape of the potential function you have drawn (see Figure 3.8). To get the essential features of the phase portrait, imagine*

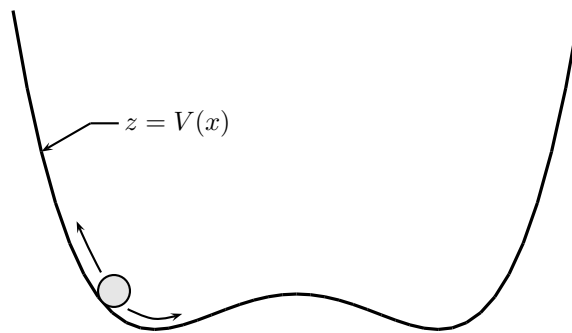


Figure 3.8 Ball rolling in a potential well

*placing the ball at various spots in the potential well and watching its motion as follows.*

- (ii) *Locate the points where the ball will remain if placed there with zero velocity. These are exactly at the critical points of  $V$ .*
- (iii) *If a critical point is a strict local minimum of  $V$ , then the ball will roll around in the potential well as long as it is not given too much energy (i.e., too much initial velocity).*

- (iv) If the critical point is a strict local maximum of  $V$ , then there will be trajectories which either
- (a) have just the right energy so that as  $t \rightarrow \infty$  the ball will end up sitting at the local maximum, or
  - (b) have just the right energy so that as  $t \rightarrow -\infty$  the ball will end up sitting at the local maximum.

In all of the preceding, remember that the ball will display conservation of energy. Therefore, if it is at rest at height  $h$ , if it is at rest at any other time, then it must still be at height  $h$ .

Note that the above rules are an analogy, and are useful for generating qualitative phase portraits. It is not quite true that the phase portraits are *exactly* the same for a simple mechanical system with potential  $V$  and a ball rolling well of shape  $V$ . However, they do share the same qualitative features, and this is all we are after.

Let us see how this works for some examples. Let us begin with the linear examples.

**3.4.9 Examples 1.** We first take  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2$ . The potential function is  $V = \frac{1}{2}q^2$ , and the graph is given in Figure 3.9, along with the corresponding phase portrait. One should verify that all the artifacts alluded to in the recipe above do indeed hold in this simple example.

2. Next we take  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2$ , so that  $V(q) = -\frac{1}{2}q^2$ . This potential function and the corresponding phase portrait are shown in Figure 3.9. If you can understand this simple problem using the rolling ball analogy, then you should be able to pretty much do any problem, provided you can plot the potential function.  $\square$

The above two examples are simple, and we already knew what their phase portraits should look like. All we have done is provide an interpretation for them using an analogy of a ball rolling in a potential well. Let us look at some examples for which we do not already know the answer.

**3.4.10 Examples 1.** First we take the Lagrangian  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2 - \frac{1}{4}q^4$ , with potential function  $V(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4$ . This potential function is plotted in Figure 3.10. Let us point out the most interesting features, and see that we understand their relationship with the rolling ball analogy.

- (a) There are three equilibrium points, one corresponding to the local maximum of the potential at  $q_1 = 0$ , and two at the two local minima  $q_2 = -1$  and  $q_3 = 1$ . At these points, the ball will sit at rest in the potential well.
- (b) Since the equilibrium point at  $q_1 = 0$  is at a maximum of the potential well, this will be an unstable point. Any small perturbation of the ball from this rest point will result in it zipping off away from the equilibrium point.
- (c) There are two curves coming into the equilibrium point at  $q_1 = 0$  which correspond to giving the ball just the right amount of energy that it will make it to the local maximum of the potential in infinite time. This is the stable manifold of the equilibrium point.
- (d) There are also two curves leaving the equilibrium point at  $q_1 = 0$  which correspond to solutions where the ball was at the equilibrium point at time  $-\infty$ . This is the unstable manifold of the equilibrium point.

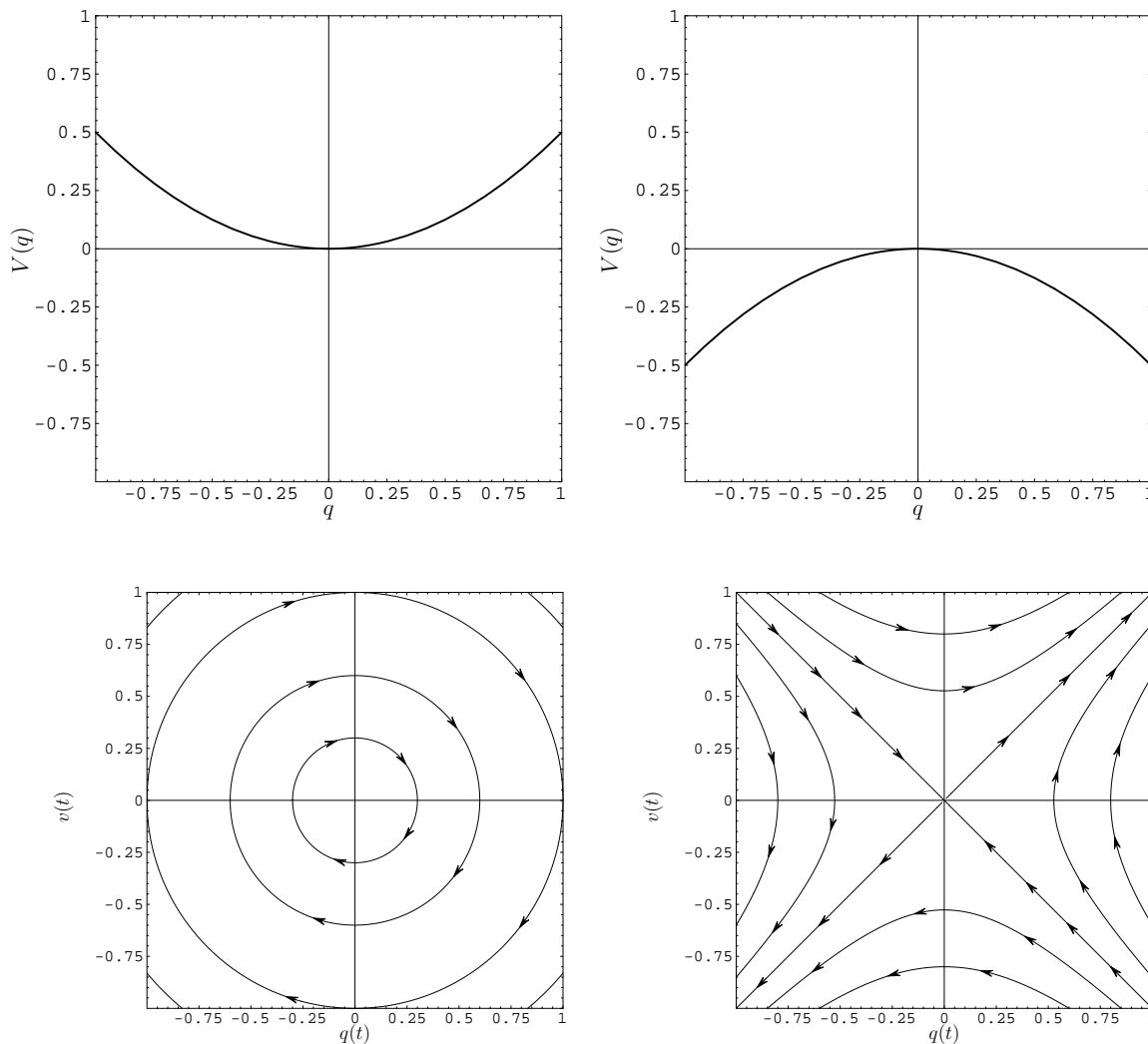


Figure 3.9 Potential wells and phase portraits for the Lagrangians

$$L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2 \text{ (left) and } L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2 \text{ (right)}$$

- (e) The branches of the stable and unstable manifolds join with one another on the left and right sides of the equilibrium point at  $q_1 = 0$ . This corresponds to the fact that if you nudge the ball infinitesimally (whatever that means) from its unstable equilibrium point, it will rush away, then get trapped by the potential again, and wind up coming back to where it started.
- (f) All other solutions are periodic, and there are essentially three types of these. There are the periodic solutions whose energy is insufficient to get them out of the two small wells in the potential. This accounts for two types: one type on the left and one on the right of the equilibrium point at  $q_1 = 0$ . The other type consists of those solutions whose energy is enough that they do not remain in the two small potential wells, but zip right through both of these.
2. *Example 2.6.3 cont'd:* We look now at the simple pendulum. In Example 2.6.18 we showed that this system has configuration space  $Q = \mathbb{S}^1$  and in the standard coordinate

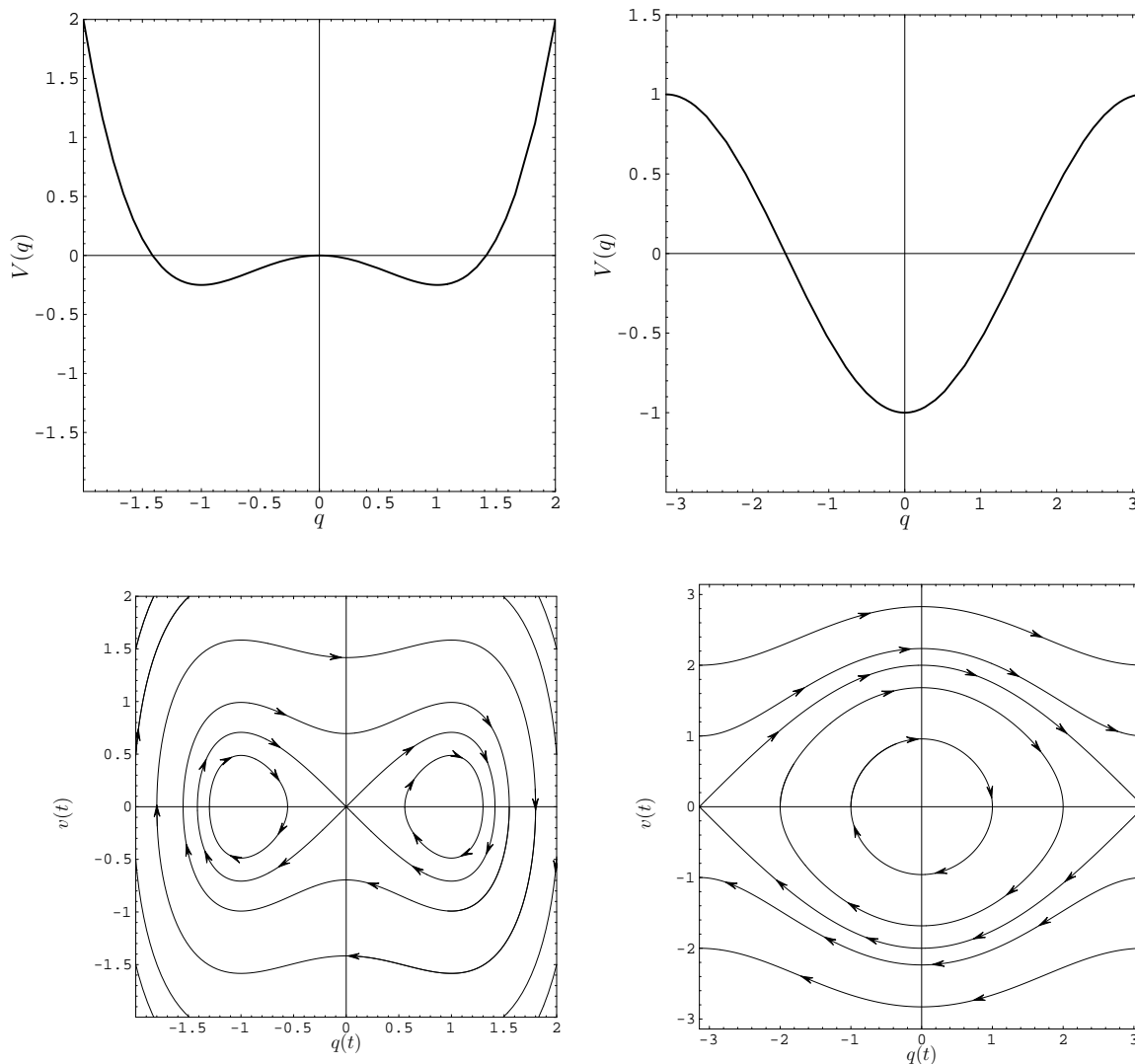


Figure 3.10 Potential wells and phase portraits for the Lagrangians  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2 - \frac{1}{4}q^4$  (left) and  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{3}q^3$  (right)

$\theta$  for  $\mathbb{S}^1$  the Lagrangian is

$$\frac{1}{2}m\ell^2 v_\theta^2 - ma_g(\ell \sin \theta - r_0 \sin \theta_0).$$

For simplicity, let us work in units where  $\frac{g}{\ell} = 1$ . Also, since the equations of motion only depend on the derivative of the potential, we can eliminate the constant term from the potential. Thus we end up with the simplified Lagrangian  $\frac{1}{2}v_\theta^2 - \sin \theta$ . As we mentioned at the beginning of this section, we can think of this not as a Lagrangian on  $\mathbb{S}^1$ , but as a Lagrangian on  $\mathbb{R}$  which is  $2\pi$  periodic. To make for a more attractive phase portrait (thank you, hindsight), we use the coordinate  $q = \theta + \frac{\pi}{2}$ . In this case we have our Lagrangian in the form we like:

$$L(q, v) = \frac{1}{2}v^2 + \cos q.$$

The potential function is therefore  $V(q) = -\cos q$ . The potential and the phase portrait are plotted in Figure 3.10. We plot for  $q \in (-\pi, \pi]$ . Let us point out the interesting features, here pointing out the correspondence to pendulum motions, although the rolling ball analogy still works.

- (a) There are two equilibrium points, one at  $q_1 = 0$  and another at  $q_2 = \pi$ . One may wish to say that there is also one at  $-\pi$ , but for the pendulum, this is the same as the one at  $\pi$ : both correspond to the pendulum being in the upright equilibrium configuration.
- (b) The equilibrium point at  $q_2 = \pi$  is unstable. There are two solutions which go into the unstable equilibrium point, corresponding to the pendulum having just the right amount of energy that it will end up at rest at the upright configuration. These form the stable manifold of the equilibrium point at  $q_2 = \pi$ .
- (c) There are also two curves which come out of the unstable equilibrium. These correspond to solutions which were at the upright configuration at  $t = -\infty$ . These form the unstable manifold of the equilibrium point at  $q_2 = \pi$ .
- (d) If we think in terms of the configuration space as being  $\mathbb{S}^1$ , then the stable and unstable manifolds intersect. This corresponds to the pendulum starting in its unstable configuration at  $t = -\infty$ , then swinging up to its unstable configuration from the other side at  $t = +\infty$ .
- (e) The other solutions fall into two categories. Solutions with not enough energy to swing past the upright configuration are the periodic solutions in the potential well. Other solutions are periodic if we think of  $\mathbb{S}^1$  being the configuration space, and correspond to motions of the pendulum where it just keeps swinging around, and around, and around.  $\square$

### 3.5 Lagrangian systems with dissipative forces

In the above analysis on Lagrangian dynamics, we have made no reference to either forces or constraints. That is, we have thus far only talked about “pure” Lagrangian dynamics. It is simply not possible to engage in a systematic study of Lagrangian dynamics in the presence of *arbitrary* forces. This is simply too general a situation to allow a significant discussion. However, we can say something useful when the forces are dissipative, so let us address this situation.

Before we begin, we need to make a few remarks about solutions of Euler-Lagrange equations in the presence of external forces. It is readily shown (see Exercise E3.1) that as long as the Lagrangian is hyperregular, the curves which satisfy the Lagrange-d’Alembert principle for a Lagrangian  $L$  and a force  $F$  satisfy an ordinary differential equation. Therefore, for hyperregular Lagrangians, one has existence and uniqueness of curves satisfying the Lagrange-d’Alembert principle. One also has the same notion of an *equilibrium point* as being a point  $q_0 \in Q$  for the curve  $t \mapsto q_0$  satisfies the Lagrange-d’Alembert principle. One may also immediately transfer the notions of *stability*, and *asymptotic stability* to the forced case. In the discussion below, we take all this for granted.

**3.5.1 The LaSalle Invariance Principle for dissipative systems** In the study of mechanical systems with dissipation, there is a natural notion of what is generally called a Liapunov function. This is a concept which exists in general, however, we will only see it in our mechanical context. Throughout this section we will make the assumption that for each

point  $v_q \in TQ$ , the solution to the Euler-Lagrange equations with initial condition  $c'(0) = v_q$  exists for all forward and backward time. This will very often be the case, but it needs to be an hypothesis.

From Proposition 2.5.7 we know that energy will decrease (or at least not increase) along the solutions of the Euler-Lagrange equations in the presence of dissipative forces. Intuitively, one would expect that this would mean that one should eventually end up at a minimum of the energy. To make this precise and have a respectable level of generality requires a few moments effort.

The first thing we need to do is precisely characterise the set of points to which a solution of the Euler-Lagrange equations tends.

**3.5.1 Definition** Let  $L$  be a hyperregular Lagrangian on a configuration space  $Q$  and let  $F$  be a force on  $Q$ . For  $v_q \in TQ$  let  $\omega(L, F, v_q)$  be the set of points  $u_{\tilde{q}} \in TQ$  with the property that there exists a sequence  $\{t_k\}_{k \in \mathbb{Z}_+} \subset \mathbb{R}$  with the properties

- (i)  $0 < t_1 < \dots < t_k < \dots$ ,
- (ii)  $\lim_{k \rightarrow \infty} t_k = \infty$ , and
- (iii)  $\lim_{k \rightarrow \infty} c'(t_k) = u_{\tilde{q}}$ ,

where  $c$  satisfies the Lagrange-d'Alembert principle and  $c'(0) = v_q$ . We shall call  $\omega(v_q, L, F)$  the **positive limit set of  $v_q$** .  $\square$

The idea of this definition is simply that the set  $\omega(L, F, v_q)$  is the set of points to which the solution tends as  $t \rightarrow \infty$ . Some examples using problems we have already worked out illustrate the idea.

**3.5.2 Examples** All the systems we consider for now are unforced, single degree of freedom systems.

1. *Example 3.4.9-1 cont'd:* We take  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2$ . There are two cases to consider.
  - (a) First we take  $v_q = (0, 0)$ . In this case,  $\omega(v_q, L, 0) = \{(0, 0)\}$  trivially, since this point is an equilibrium point.
  - (b) Next we take an arbitrary point  $v_q = (q, v) \in TQ$ . In this case we have

$$\omega(v_q, L, 0) = \{(\tilde{q}, \tilde{v}) \in TQ \mid \tilde{q}^2 + \tilde{v}^2 = q^2 + v^2\}.$$

That is, the positive limit set is the circle containing  $(q, v)$ . Since the solution to the Euler-Lagrange equations in this case lies on this circle,  $\omega(v_q, L, 0)$  must be contained in the circle, and it is easy to see from the definition that every point on the circle is also a point in  $\omega(v_q, L, 0)$ . In fact, it will always be the case that the positive limit set for points along periodic solutions will be the entire periodic orbit. If you do not see why this is so, then think on it.

2. *Example 3.4.9-2 cont'd:* Next we take  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2$ . Here we have a few cases to look at.
  - (a) For  $v_q = (0, 0)$  we have  $\omega(v_q, L, 0) = \{(0, 0)\}$  since  $(0, 0)$  is an equilibrium point.
  - (b) If  $v_q$  lies on the line which flows into the origin, i.e., if  $v_q = (a, -a)$  for some  $a \neq 0$ , then  $\omega(v_q, L, 0) = \{(0, 0)\}$ . This is because the solution curve passing through these points approaches  $(0, 0)$  as  $t \rightarrow \infty$ . It therefore stands to reason that  $\{(0, 0)\}$  should be the only point in the positive limit set.

- (c) For all other points  $v_q$  than covered in the previous two cases,  $\omega(v_q, L, 0) = \emptyset$ . This is because the solution curves through all other points become unbounded as  $t \rightarrow \infty$ .
3. *Example 3.4.10–1 cont'd:* The next case we look at is nonlinear, and we take  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2 - \frac{1}{4}q^4$ . Again, we have a few cases.
- (a) For the three equilibrium points,  $v_q \in \{(-1, 0), (0, 0), (1, 0)\}$ , we have as usual  $\omega(v_q, L, 0) = \{v_q\}$ .
- (b) For a point  $v_q$  on the stable (and therefore also the unstable) manifold for  $(0, 0)$ , we have  $\omega(v_q, L, 0) = \{(0, 0)\}$  since all such points tend to  $(0, 0)$  as  $t \rightarrow \infty$ .
- (c) The solution through all other points are periodic, and so for a point  $v_q$  on such a periodic orbit, the positive limit set is equal to the set of points along the periodic orbit.
4. *Example 3.4.10–2 cont'd:* Finally, we look at the pendulum whose Lagrangian we reworked to look like  $L(q, v) = \frac{1}{2}v^2 + \cos q$ . In the following discussion, we are thinking of  $Q$  as being  $\mathbb{S}^1$ .
- (a) As usual, if  $v_q$  is one of the two equilibrium points  $(0, 0)$  or  $(0, \pi)$ , then  $\omega(v_q, L, 0) = \{v_q\}$ .
- (b) If  $v_q$  is a point on the stable (and therefore the unstable) manifold through  $(\pi, 0)$ , then  $\omega(v_q, L, 0) = \{(\pi, 0)\}$ . This is for the reason, as we have seen, that the solution curve through such points tends to the point  $(\pi, 0)$ .
- (c) The remaining points lie on periodic orbits, so for such points  $v_q$  we have the positive limit set as the collection of points along the periodic orbit.  $\square$

**3.5.3 Remark** The above examples all exhibit simple limiting behaviour in that the sets  $\omega(v_q, L, F)$  are quite benign for these systems: when defined, they are either points or periodic orbits. This is an artifact of these systems being time-independent systems in the plane. There is a general result, known as the ***Poincaré-Bendixson Theorem***, which classifies all possibilities for  $\omega(v_q, L, F)$ , provided that  $L$  and  $F$  are time-independent and defined on a one-dimensional configuration space (actually, the theorem is stated in the more general setting of differential equations in the plane). Things are not so rosy in higher dimensions where the limiting sets can have rather exotic behaviour.  $\square$

We need some other notions, but these may be easier to come to grips with than the positive limit set. Let  $L$  be a Lagrangian on  $Q$  and let  $F$  be a force on  $Q$ . A subset  $\Omega \subset TQ$  is ***invariant*** for  $L$  and  $F$  if for every  $v_q \in \Omega$  and for every  $T > 0$  we have  $c(T) \in \Omega$  and  $c(-T) \in \Omega$  where  $t \mapsto c(t)$  satisfies the Lagrange-d'Alembert principle for  $L$  and  $F$  with  $c'(0) = v_q$ . The idea here is simply that all solutions with initial condition in  $\Omega$  remain in  $\Omega$  in both forward and backward time.

The following technical result is not of intrinsic importance, so can be bypassed by readers only interested in *using* the results of this section. The proof of the lemma in our general context is not within our reach. We refer to [Muñoz-Lecanda and Yaniz-Fernández 2000] for the general result, and to [Khalil 1996] for a proof when  $Q = \mathbb{R}^n$ .

**3.5.4 Lemma** *Let  $L$  be a time-independent hyperregular Lagrangian on a configuration space  $Q$  and let  $F$  be a time-independent force on  $Q$ . For  $v_q \in TQ$  let  $c$  be the curve satisfying the Lagrange-d'Alembert principle for  $L$  and  $F$  and for which  $c'(0) = v_q$ . If there exists a compact subset of  $TQ$  containing the image of the map  $t \mapsto c'(t)$  then  $\omega(v_q, L, F)$  is a compact set, invariant for  $L$  and  $F$ .*



The following result shows that for a class of dissipative systems, the sets  $\omega(v_q, L, F)$  have a structure which can be characterised in terms of the force and the energy of the Lagrangian.

**3.5.5 Theorem** *Let  $L$  be a hyperregular time-independent Lagrangian on a configuration space  $Q$  and let  $F$  be a time-independent dissipative force. For  $v_q \in TQ$  let  $c$  be the curve satisfying the Lagrange-d'Alembert principle for  $L$  and  $F$  and for which  $c'(0) = v_q$ . If there exists a compact subset of  $TQ$  containing the image of the map  $t \mapsto c'(t)$  then*

$$\omega(v_q, L, F) \subset \{u_{\tilde{q}} \in TQ \mid \langle F(u_{\tilde{q}}); u_{\tilde{q}} \rangle = 0\}.$$

*Proof* By Proposition 2.5.7,  $\frac{d}{dt} \mathbf{EL}(c'(t)) \leq 0$ . Since  $c'(t)$  is contained in a compact set, this means that the limit  $\lim_{t \rightarrow \infty} \mathbf{EL}(c'(t))$  exists. Denote this limit by  $e$ . Now let  $u_{\tilde{q}} \in \omega(v_q, L, F)$ . There exists a sequence  $\{t_k\}$  with the property that  $\lim_{k \rightarrow \infty} t_k = \infty$  and  $\lim_{k \rightarrow \infty} c'(t_k) = u_{\tilde{q}}$ . As  $\mathbf{EL}$  is continuous, this implies that  $\mathbf{EL}(u_{\tilde{q}}) = \lim_{k \rightarrow \infty} \mathbf{EL}(c'(t_k))$ . This shows that  $\mathbf{EL}$  takes the value  $e$  on every point of  $\omega(v_q, L, F)$ . Since  $\omega(v_q, L, F)$  is invariant for  $L$  and  $F$  by Lemma 3.5.4, if  $\tilde{c}(t)$  is the curve satisfying the Lagrange-d'Alembert principle for  $L$  and  $F$  and satisfying  $\tilde{c}'(0) = u_{\tilde{q}}$ , then

$$\left. \frac{d}{dt} \mathbf{EL}(\tilde{c}(t)) \right|_{t=0} = \lim_{t \rightarrow 0} \frac{\mathbf{EL}(\tilde{c}(t)) - \mathbf{EL}(u_{\tilde{q}})}{t} = \lim_{t \rightarrow 0} \frac{e - e}{t} = 0.$$

However, in the proof of Proposition 2.5.7 we showed that

$$\frac{d}{dt} \mathbf{EL}(\tilde{c}(t)) = \langle F(\tilde{c}'(t)); \tilde{c}'(t) \rangle.$$

Thus we have shown that for points in  $\omega(v_q, L, F)$  we have  $\langle F(\tilde{c}'(t)); \tilde{c}'(t) \rangle = 0$ , as claimed. ■

This theorem is a specialisation to Euler-Lagrange equations of a more general result called the **LaSalle Invariance Principle** [LaSalle 1968]. As we shall see in Section 4.3, it is the basis of an entire branch of control theory for mechanical systems. The above theorem also leads immediately to some useful corollaries which we now list.

First we adopt the same hypotheses as Theorem 3.5.5 and let  $\Omega$  be the largest subset of  $TQ$  which is invariant under  $L$  and  $F$  and for which

$$\Omega \subset \{u_{\tilde{q}} \in TQ \mid \langle F(u_{\tilde{q}}); u_{\tilde{q}} \rangle = 0\}.$$

A curve  $t \mapsto c(t)$  on  $Q$  **approaches**  $\Omega$  if for every open subset  $U$  of  $Q$  containing  $\Omega$  there exists a  $T > 0$  so that  $c'(t) \in U$  for every  $t > T$ . The following corollary is sometimes what is given as the LaSalle Invariance Principle.

**3.5.6 Corollary** *Under the same hypotheses as Theorem 3.5.5 and with  $\Omega$  as just defined,  $c$  approaches  $\Omega$ .*

*Proof* Suppose that  $c$  does not approach  $\Omega$ . Then there exists an open subset  $U$  of  $Q$  containing  $\Omega$  and  $T > 0$  so that  $c'(t) \notin U$  for every  $t > T$ . In particular for  $u_{\tilde{q}} \in \omega(v_q, L, F)$  there exists a neighbourhood  $\tilde{U}$  of  $u_{\tilde{q}}$  so that  $c'(t) \notin \tilde{U}$  for each  $t > T$ . But this contradicts the definition of  $\omega(v_q, L, F)$ . ■

The next corollary is the basic result which the LaSalle Invariance Principle may be thought of as generalising. For this result, call a force **strictly dissipative** if it is dissipative and if  $\langle F(t, v_q); v_q \rangle = 0$  only if  $v_q = 0_q$ .



**3.5.7 Corollary** *Let  $L$  be a time-independent hyperregular Lagrangian on  $Q$  and  $F$  a time-independent strictly dissipative force on  $Q$  with the property that  $F(0_q) = 0$  for each  $q \in Q$ . Suppose that  $D \subset TQ$  is a compact set, invariant for  $L$  and  $F$ . Let  $q_0 \in D$  have the property that  $0_{q_0}$  is a strict global minimum for  $\mathbf{EL}|_D$ . If  $q_0$  is the only equilibrium point for  $L$  in  $D$  then  $q_0$  is asymptotically stable.*

*Proof* First of all, since  $q_0$  is a strict global minimum for  $\mathbf{EL}$ , it must be the case that  $d\mathbf{EL}(0_{q_0}) = 0$ . Since  $F(0_{q_0}) = 0$  it follows that  $q_0$  is an equilibrium point for the forced system. Thus  $\{0_{q_0}\}$  is an invariant set. Furthermore, since  $q_0$  is the only equilibrium point in  $D$  and since  $F$  is strictly dissipative,  $\{0_{q_0}\}$  must be the largest invariant set contained in the set

$$\{v_q \in D \mid \langle F(v_q); v_q \rangle = 0\}.$$

It now follows from Corollary 3.5.6 that  $\omega(v_q, L, F) = \{0_{q_0}\}$  for every  $v_q \in D$ . ■

Before we look at some examples of the LaSalle Invariance Principle, let us quickly specialise the previous result to simple mechanical systems. As expected, the result becomes expressible in terms of the potential function.

**3.5.8 Corollary** *Let  $(Q, g, V)$  be a simple mechanical system with Lagrangian  $L$  and let  $F$  be a strictly dissipative force on  $Q$  with the property that  $F(0_q) = 0$  for each  $q \in Q$ . Suppose that  $D \subset TQ$  is a compact set, invariant for  $L$  and  $F$ . Let  $q_0 \in D$  have the property that  $0_{q_0}$  is a strict global minimum for  $V|_D$ . If  $q_0$  is the only equilibrium point for  $L$  in  $D$  then  $q_0$  is asymptotically stable.*

The following remark addresses the matter of some of the more questionably verifiable of the hypotheses of the above results.

**3.5.9 Important remark** *In our statement of Theorem 3.5.5 and its corollaries, hypotheses are required which ensure that the solution(s) to the forced Euler-Lagrange equations are bounded. These hypotheses become unnecessary in the not infrequently encountered situation when the level sets of  $\mathbf{EL}$  are compact. That is to say, if one replaces*

1. *in Theorem 3.5.5 the hypothesis that  $c'(t)$  remain in a compact set, and*
2. *in Corollary 3.5.7 the hypothesis that we restrict to a compact set  $D$ ,*

*with the hypothesis that  $\mathbf{EL}$  have compact level sets, then the results still hold.* □

**3.5.2 Single degree of freedom case studies** In this section we present some simple examples which illustrate the general development of the previous section.

**3.5.10 Examples** Each of the examples is a single degree of freedom simple mechanical system, and we take the dissipative force in each case to be  $F(q, v) = -\frac{1}{5}v$ , which is a strictly dissipative force. In each case, we shall produce a numerically generated phase plot, and make sure we understand how it agrees with the LaSalle Invariance Principle and its corollaries.

1. *Example 3.4.9–1 cont'd:* The first example we take is that with Lagrangian  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2$ . The phase portrait for the system is shown in Figure 3.11. Note that  $\mathbf{EL} = \frac{1}{2}v^2 + \frac{1}{2}q^2$  so the level sets of  $\mathbf{EL}$  are compact (they are circles). Thus, by Remark 3.5.9, we may dispense with the hypotheses of boundedness of trajectories and compactness of domains, and directly apply the results. Indeed, Corollary 3.5.8 applies in this case. Since  $q_0 = 0$  is the only equilibrium point for  $L$ , and since  $q_0$  is a strict global minimum

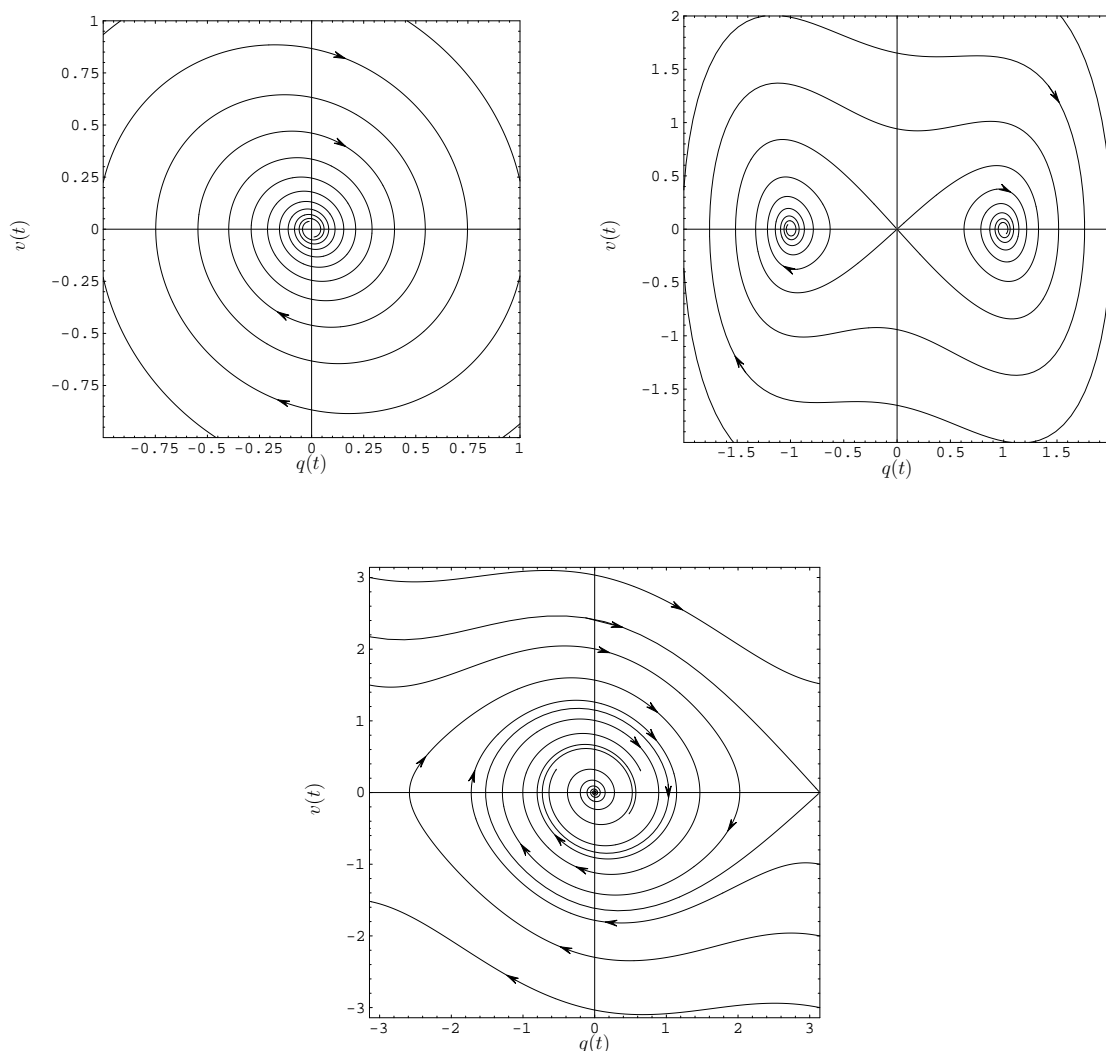


Figure 3.11 Phase portraits for dissipative systems with Lagrangians  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2$  (top left),  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2 - \frac{1}{4}q^4$  (top right), and  $L(q, v) = \frac{1}{2}v^2 + \cos q$  (bottom). In each case the dissipative force is  $F(q, v) = -\frac{1}{5}v$ .

for  $V$ , Corollary 3.5.8 implies that  $q_0$  is an asymptotically stable equilibrium point. This is clearly seen from the phase portrait as well.

2. *Example 3.4.10–1 cont'd:* Next we take the Lagrangian  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2 - \frac{1}{4}q^4$ . The phase portrait is shown in Figure 3.11. Note that the level sets of the energy are again compact, so we need no special hypotheses concerning bounded trajectories or compact domains. Note that there are three equilibrium points, so we cannot apply Corollary 3.5.8. Let us apply Corollary 3.5.6. We have

$$\{v_q \in TQ \mid \langle F(v_q); v_q \rangle = 0\} = \{(q, 0) \mid q \in \mathbb{R}\}.$$

The maximal invariant set  $\Omega$  contained in this set is then comprised of the three fixed points. Thus, in the notation of Corollary 3.5.6,  $\Omega = \{(-1, 0)\} \cup \{(0, 0)\} \cup \{(1, 0)\}$ . It then follows from Corollary 3.5.6 that for each  $(q, v) \in TQ$ , the positive limit set must be

one of these three points. And indeed from Figure 3.11 we see that all initial conditions end up in one of the fixed points. In fact, almost all trajectories end up in one of the two stable fixed points at  $(-1, 0)$  and  $(1, 0)$ . The only initial conditions which end up at the unstable fixed point at  $(0, 0)$  are those which lie on the stable manifold for this point.

3. *Example 3.4.10–2 cont'd:* As our final example, we take  $L(q, v) = \frac{1}{2}v^2 + \cos q$ . The phase portrait is shown in Figure 3.11, and the discussion goes much as in the previous example.  $\square$

The mantra of the examples in this section seems to be that dissipation will make a stable equilibrium point into an asymptotically stable equilibrium point. This is true for the kinds of systems we have been looking at, but care must be taken when extending this as a general principle. In [Lewis and Murray 1995] a mechanical system is investigated for which the introduction of dissipation causes instability of previously stable motions. This phenomenon is examined in a general setting by Bloch, Krishnaprasad, Marsden, and Ratiu [1993]. See also Exercise E3.13.

## 3.6 Rigid body dynamics

While we spent a significant portion of Chapter 1 talking about rigid bodies, and how they move in space, and in Section 2.8 we demonstrated that Euler's equations for a rigid body are equivalent to the Euler-Lagrange equations, we have yet to say anything about the actual solutions to the rigid body equations. Thus in this section we tackle this problem, describing these solutions in the absence of external forces.

The differential equations we concentrate on then are those for the body velocities about the centre of mass. We make the assumption that the rigid motion undergone by the rigid body is observed by a compatible observer. Thus, you will recall, the equations we look at are

$$\begin{aligned}\dot{\mathbf{R}}(t) &= \mathbf{R}(t)\hat{\boldsymbol{\Omega}}(t) \\ \dot{\mathbf{r}}(t) &= \mathbf{R}(t)\mathbf{V}(t) \\ \mathbb{I}_c(\dot{\boldsymbol{\Omega}}(t)) &= (\mathbb{I}_c(\boldsymbol{\Omega}(t))) \times \boldsymbol{\Omega}(t) \\ \mu(\mathcal{B})\dot{\mathbf{V}}(t) &= \mu(\mathcal{B})\mathbf{V}(t) \times \boldsymbol{\Omega}(t),\end{aligned}\tag{3.5}$$

where  $(\mathbf{R}, \mathbf{r}, \boldsymbol{\Omega}, \mathbf{V}) \in SO(3) \times (\mathbb{R}^3)^3$ . The meat of these equations is in the third equation

$$\mathbb{I}_c(\dot{\boldsymbol{\Omega}}(t)) = (\mathbb{I}_c(\boldsymbol{\Omega}(t))) \times \boldsymbol{\Omega}(t).\tag{3.6}$$

Indeed, once one has the solution  $t \mapsto \boldsymbol{\Omega}(t)$  to this equation, the equation for the body velocity  $\mathbf{V}$  is then a time-dependent linear differential equation, as is the differential equation for  $\mathbf{R}$ . Finally, once one has  $t \mapsto \mathbf{V}(t)$  and  $t \mapsto \mathbf{R}(t)$ , one may directly integrate to get  $\mathbf{r}(t)$ . However, the equation for  $\boldsymbol{\Omega}$  is nonlinear, and its solutions are quite interesting, so that is where we shall focus our attention.

**3.6.1 Conservation laws and their implications** It turns out that everything we need to know about equations (3.5) may be derived from the conservation laws these equations possess. This is true even of the more difficult equation (3.6). The following result classifies all the conserved quantities.

**3.6.1 Theorem** *Let  $(\mathcal{B}, \mu)$  be a rigid body. For the equations (3.5), the following quantities are constant along solutions:*

- (i) the spatial linear momentum  $\mathbf{m}(t) = \mu(\mathcal{B}) \mathbf{R}(t) \mathbf{V}(t)$ ;
- (ii) the translational kinetic energy  $E_T(t) = \frac{1}{2} \mu(\mathcal{B}) \|\dot{\mathbf{r}}(t)\|^2$ ;
- (iii) the rotational kinetic energy  $E_R(t) = \frac{1}{2} g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}(t)), \boldsymbol{\Omega}(t))$ ;
- (iv) the length of the body angular momentum vector  $L_R(t) = \|\mathbb{I}_c(\boldsymbol{\Omega}(t))\|$ .

*Proof* (i) We compute

$$\begin{aligned} \dot{\mathbf{m}}(t) &= \mu(\mathcal{B}) \dot{\mathbf{R}}(t) \mathbf{V}(t) + \mu(\mathcal{B}) \mathbf{R}(t) \dot{\mathbf{V}}(t) \\ &= \mu(\mathcal{B}) \mathbf{R}(t) \hat{\boldsymbol{\Omega}}(t) \mathbf{V}(t) + \mu(\mathcal{B}) \mathbf{R}(t) \mathbf{V}(t) \times \boldsymbol{\Omega}(t) \\ &= \mathbf{0}. \end{aligned}$$

(ii) Here we use the fact that  $\mathbf{m}(t) = \mu(\mathcal{B}) \dot{\mathbf{r}}(t)$ . From (i) we thus conclude that  $\dot{\mathbf{r}}(t)$  is constant. From this the conclusion that  $E_T(t)$  is constant immediately follows.

(iii) We compute

$$\begin{aligned} \dot{E}_R(t) &= \frac{1}{2} g_{\text{can}}(\mathbb{I}_c(\dot{\boldsymbol{\Omega}}(t)), \boldsymbol{\Omega}(t)) + \frac{1}{2} g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}(t)), \dot{\boldsymbol{\Omega}}(t)) \\ &= g_{\text{can}}(\mathbb{I}_c(\dot{\boldsymbol{\Omega}}(t)), \boldsymbol{\Omega}(t)), \end{aligned}$$

using the fact that  $\mathbb{I}_c$  is symmetric with respect to  $g_{\text{can}}$ . Now we use (3.6) to give

$$\begin{aligned} \dot{E}_R(t) &= g_{\text{can}}((\mathbb{I}_c(\boldsymbol{\Omega}(t))) \times \boldsymbol{\Omega}(t), \boldsymbol{\Omega}(t)) \\ &= g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}(t)), \boldsymbol{\Omega}(t) \times \boldsymbol{\Omega}(t)) \\ &= \mathbf{0}, \end{aligned}$$

where we have used the vector identity (1.19).

(iv) From (iii) we know that if  $L_R(0) = 0$  then  $L_R(t) = 0$  for all  $t$ . Thus we may suppose that  $L_R(0) \neq 0$ . We compute

$$\begin{aligned} \frac{dL_R^2}{dt} &= 2g_{\text{can}}(\mathbb{I}_c(\dot{\boldsymbol{\Omega}}(t)), \mathbb{I}_c(\boldsymbol{\Omega}(t))) \\ &= 2g_{\text{can}}((\mathbb{I}_c(\boldsymbol{\Omega}(t))) \times \boldsymbol{\Omega}(t), \mathbb{I}_c(\boldsymbol{\Omega}(t))) \\ &= -2g_{\text{can}}(\boldsymbol{\Omega}(t), (\mathbb{I}_c(\boldsymbol{\Omega}(t))) \times (\mathbb{I}_c(\boldsymbol{\Omega}(t)))) \\ &= \mathbf{0}. \end{aligned}$$

But we also have  $\frac{dL_R^2}{dt} = 2L_R(t)\dot{L}_R(t)$ . Since we are assuming that  $L_R(0) \neq 0$  it follows that  $\dot{L}_R(0) = 0$ , and from this that  $L_R(t) = L_R(0)$  for all  $t$ , as desired.  $\blacksquare$

Some of the statements of this theorem are unsurprising, but some are perhaps not obvious.

**3.6.2 Remarks 1.** That the spatial linear momentum is conserved should be in no way surprising. After all, in Section 1.6 we used this very fact to *derive* the equations (3.5)! A consequence of this conservation law is that the centre of mass of a rigid body in the absence of forces moves at a uniform velocity.

2. That the total kinetic energy should be preserved is, of course, a consequence of conservation of energy for general Euler-Lagrange equations (cf. Proposition 2.5.7). It is not quite so obvious, perhaps, that the translational and rotational components should be individually preserved.

3. The most interesting conservation law is that for  $L_R$ , the length of the body angular momentum vector. It is this conservation law which will allow us to describe the solutions of the equation (3.6). Its presence is an artifact of the conservation of *spatial* angular momentum, and is reflective of similar statements in a more general setting, as presented in Section 4.4 of [Abraham and Marsden 1978]. Interestingly, this phenomenon appears in a different manner for a wide variety of physical problems. For example, Arnol'd [1966] uses a setting like this to provide an interpretation of “helicity” as a conservation law in the flow of an inviscid incompressible fluid. □ Check

The statement 1 is of sufficient interest to merit its separate statement.

**3.6.3 Corollary** *The centre of mass of a rigid moves at a constant velocity with respect to an observer which is compatible with the rigid motion.*

Let us now use the fact that  $E_R$  and  $L_R$  are constants of motion to obtain a description of the solutions to (3.6).

**3.6.4 Proposition** *Let  $(\mathcal{B}, \mu)$  be a rigid body and let  $\{I_1, I_2, I_3\}$  be the principal inertias with  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  the corresponding principal axes. We have the following four case, which classify all possibilities up to permutation of the principal inertias.*

- (i) *The principal inertias are strictly positive and not all equal: The solutions to (3.6) lie on the intersection of an ellipsoid*

$$\{\boldsymbol{\Omega} \in \mathbb{R}^3 \mid \|\mathbb{I}(\boldsymbol{\Omega})\| = C_1\}$$

*and an ellipsoid*

$$\{\boldsymbol{\Omega} \in \mathbb{R}^3 \mid g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}), \boldsymbol{\Omega}) = C_2\}$$

*for some  $C_1, C_2 > 0$ .*

- (ii)  $0 = I_1 < I_2 = I_3$ : *The solutions to (3.6) are all of the form  $t \mapsto \boldsymbol{\Omega}_0 + \omega(t)\mathbf{v}_1$  where  $\boldsymbol{\Omega}_0 \in \text{span}_{\mathbb{R}}\{\mathbf{v}_2, \mathbf{v}_3\}$  and where  $t \mapsto \omega(t) \in \mathbb{R}$  is arbitrary.*
- (iii)  $0 = I_1 = I_2 = I_3$ : *Any curve  $t \mapsto \boldsymbol{\Omega}(t)$  is a solution to (3.6).*

*Proof* (i) This follows directly from parts (iii) and (iv) of Theorem 3.6.1. In this case since  $\mathbb{I}_c$  is invertible, the set

$$\{\boldsymbol{\Omega} \in \mathbb{R}^3 \mid \|\mathbb{I}_c(\boldsymbol{\Omega})\| = C_1\}$$

and the set

$$\{\boldsymbol{\Omega} \in \mathbb{R}^3 \mid g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}), \boldsymbol{\Omega}) = C_2\}$$

are indeed ellipsoids.

(ii) Recall from Proposition 1.5.6(ii) that a single zero principal inertia implies that the other two principal inertias must be equal. If  $(x^1, x^2, x^3)$  denotes coordinates for  $\mathbb{R}^3$  with respect to the principal axes, then the condition (iii) of Theorem 3.6.1 reads

$$I((x^2)^2 + (x^3)^2) = \text{constant},$$

and the condition (iv) reads

$$I^2((x^2)^2 + (x^3)^2) = \text{constant},$$

where  $I = I_2 = I_3$ . Thus one cannot make a conclusive statement regarding the solutions of (3.6) from the conservation laws alone. However, substituting the inertias directly

into (3.6) gives the equations  $\dot{x}^1 = 0$ ,  $\dot{x}^2 = 0$ , and  $0 = 0$ . Thus  $x^1$  and  $x^2$  are constant, and  $x^3$  is arbitrary, just as stated in the proposition.

(iii) In this case we note that (3.6) is simply the identity  $\mathbf{0} = \mathbf{0}$ . Thus any curve in  $\mathbb{R}^3$  will indeed be a solution. ■

The only really interesting case is part (i) of the proposition. We shall give this more attention shortly, but for now let us make some general remarks about the other cases.

**3.6.5 Remarks 1.** The case (i) covers the most interesting cases, and we shall investigate this in detail in the next section.

2. The case (ii) corresponds, as shown in the proof of Proposition 1.5.6(ii), to the case where the support of  $\mu$  is contained in a line. The motion described in part (ii) of the above proposition simply says that the angular velocity about this line is irrelevant since the body has no inertia to resist this angular velocity. The equations are degenerate because three angular velocities are one too many in this case. One should properly describe the system with a two-dimensional set of angular velocities.

3. The most degenerate case, case (iii), corresponds to a rigid body which is a point mass. In this case, the remarks from the previous case apply to *all* angular velocities since the body has no angular inertia. Therefore the three angular velocities have no meaning for the motion of the body in this case. □

**3.6.2 The evolution of body angular momentum** Let us now proceed to discuss in more detail the case (i) of Proposition 3.6.4. Since the length of the body angular momentum is constant along solutions of (3.6), it is convenient to change coordinates from body angular velocity to body angular momentum, since in momentum space, all solutions lie on spheres of constant radius. The differential equation describing body angular momentum when  $\mathbb{I}_c$  is invertible is

$$\dot{\mathbf{M}}(t) = \mathbf{M}(t) \times (\mathbb{I}_c^{-1}(\mathbf{M}(t))). \quad (3.7)$$

It is this equation we proceed to analyse. First we note that the conservation laws for  $E_R$  and  $L_R$  in Theorem 3.6.1 when expressed in terms of body angular momentum imply that  $E_R(t) = \frac{1}{2}g_{\text{can}}(\mathbb{I}_c(\mathbf{M}(t)), \mathbf{M}(t))$  and  $L_R(t) = \|\mathbf{M}(t)\|$  are constant along solutions of (3.7). The sets of constant value for  $L_R$  are, of course, spheres, and the sets of constant value for  $E_R$  are ellipsoids. The solutions must therefore lie on the intersection of some such sphere and some such ellipsoid.

Let us proceed to formally analyse (3.7). First let us find its fixed points.

**3.6.6 Proposition** *Let  $(\mathcal{B}, \mu)$  be a rigid body with principal inertias  $\{I_1, I_2, I_3\}$  arranged so that  $0 < I_1 \leq I_2 \leq I_3$ , and let  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  be the corresponding principal axes. If  $\mathcal{F}(\mathcal{B}, \mu) \subset \mathbb{R}^3$  denotes the fixed points of equation (3.7) which occur on the sphere of radius  $R$ , then*

(i) *when  $I_1 < I_2 < I_3$  we have*

$$\mathcal{F}(\mathcal{B}, \mu) = \{\pm R\mathbf{v}_1\} \cup \{\pm R\mathbf{v}_2\} \cup \{\pm R\mathbf{v}_3\};$$

(ii) *when  $I_1 = I_2 < I_3$  we have*

$$\mathcal{F}(\mathcal{B}, \mu) = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \mid a_1^2 + a_2^2 = R^2\} \cup \{\pm R\mathbf{v}_3\};$$

(iii) when  $I_1 < I_2 = I_3$  we have

$$\mathcal{F}(\mathcal{B}, \mu) = \{\pm R\mathbf{v}_1\} \cup \{a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \mid a_2^2 + a_3^2 = R^2\};$$

(iv) when  $I_1 = I_2 = I_3$  we have

$$\mathcal{F}(\mathcal{B}, \mu) = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \mid a_1^2 + a_2^2 + a_3^2 = R^2\}.$$

*Proof* From (3.7) we see that  $\mathbf{M} \in \mathbb{R}^3$  is a fixed point if and only if  $\mathbf{M}$  and  $\mathbb{I}_c^{-1}(\mathbf{M})$  are collinear. Thus  $\mathbf{M}$  is a fixed point if and only if it is an eigenvector of  $\mathbb{I}_c^{-1}$  if and only if it is an eigenvector of  $\mathbb{I}_c$ . The result now follows by our discussion in the course of the proof of Proposition 1.5.8 about the eigenvectors of  $\mathbb{I}_c$  when it has equal eigenvalues. ■

Now let us determine the stability of the various fixed points by linearising, and computing the eigenvalues of the linearised system. Suppose that  $\mathbf{M}_0$  is a fixed point for (3.7). Then, as we saw in the proof of Proposition 3.6.6,  $\mathbf{M}_0$  must be an eigenvector for  $\mathbb{I}_c^{-1}$ , so we must have  $\mathbb{I}_c(\mathbf{M}_0) = I_a^{-1}\mathbf{M}_0$  for some principal inertia  $I_a$ ,  $a \in \{1, 2, 3\}$  (here we use the fact that the eigenvalues of an invertible matrix are the inverse of the eigenvalues of the matrix). Linearising (3.7) about such a fixed point gives the linearised equations

$$\begin{aligned} \dot{\boldsymbol{\xi}}(t) &= \boldsymbol{\xi}(t) \times (\mathbb{I}_c^{-1}(\mathbf{M}_0)) + \mathbf{M}_0 \times (\mathbb{I}_c^{-1}(\boldsymbol{\xi}(t))) \\ &= \mathbf{M}_0 \times (\mathbb{I}_c^{-1}(\boldsymbol{\xi}(t)) - I_a^{-1}\boldsymbol{\xi}(t)). \end{aligned}$$

Now let us write  $\boldsymbol{\xi}(t)$  in a principal basis we

$$\boldsymbol{\xi}(t) = \xi^1(t)\mathbf{v}_1 + \xi^2(t)\mathbf{v}_2 + \xi^3(t)\mathbf{v}_3,$$

and let us suppose that we have permuted the principal inertias so that  $a = 3$ . Then we have suppose that  $\mathbf{M}_0 = M_0\mathbf{v}_3$  for some  $M_0 \in \mathbb{R}$ . Since  $\mathbb{I}_c^{-1}(\mathbf{v}_3) = I_3^{-1}\mathbf{v}_3$ , we have  $\dot{\xi}^3(t) = 0$ . We then compute

$$\begin{aligned} \dot{\xi}^1(t) &= M_0(I_2^{-1} - I_3^{-1})\xi^2(t) \\ \dot{\xi}^2(t) &= M_0(I_3^{-1} - I_1^{-1})\xi^1(t). \end{aligned}$$

Thus the eigenvalues for the linearised system are 0 (which is an eigenvalue with eigenvector  $\mathbf{v}_3$ ) and the eigenvalues of the matrix

$$M_0 \begin{bmatrix} 0 & I_2^{-1} - I_3^{-1} \\ I_3^{-1} - I_1^{-1} & 0 \end{bmatrix}. \quad (3.8)$$

These latter eigenvalues are the roots of the polynomial

$$\lambda^2 + M_0^2(I_2^{-1} - I_3^{-1})(I_1^{-1} - I_3^{-1}). \quad (3.9)$$

Note that the automatic zero eigenvalue is not a surprise since its eigenvector is orthogonal to the sphere of constant length for body angular momentum. It is the very preservation of this sphere which makes this eigenvalue zero. The other two eigenvalues are, therefore, the ones which interest us as they are in directions tangent to the body momentum sphere. From (3.9) we may immediately deduce the following result.



**3.6.7 Proposition** *Let  $(\mathcal{B}, \mu)$  be a rigid body with principal inertias  $\{I_1, I_2, I_3\}$ , and suppose without loss of generality that  $I_1 \leq I_2$ . Let  $\mathbf{M}_0$  be a fixed point of (3.7) corresponding to the principal inertia  $I_3$  (i.e.,  $\mathbf{M}_0$  is an eigenvalue of  $\mathbb{I}_c^{-1}$  with eigenvalue  $I_3^{-1}$ ). Then 0 is an eigenvalue for the linearisation of (3.7) about  $\mathbf{M}_0$ , and the other eigenvalues are given as follows:*

- (i)  $I_1 < I_3 < I_2$ : eigenvalues are  $\pm M_0 \sqrt{(I_3^{-1} - I_2^{-1})(I_1^{-1} - I_3^{-1})}$ ;
- (ii)  $I_3 < I_1 < I_2$ : eigenvalues are  $\pm i M_0 \sqrt{(I_3^{-1} - I_2^{-1})(I_3^{-1} - I_1^{-1})}$ ;
- (iii)  $I_1 < I_2 < I_3$ : eigenvalues are  $\pm i M_0 \sqrt{(I_2^{-1} - I_3^{-1})(I_1^{-1} - I_3^{-1})}$ ;
- (iv)  $I_1 = I_2 < I_3$ : eigenvalues are  $\pm i M_0 \sqrt{(I_2^{-1} - I_3^{-1})(I_1^{-1} - I_3^{-1})}$ ;
- (v)  $I_3 < I_1 = I_2$ : eigenvalues are  $\pm i M_0 \sqrt{(I_3^{-1} - I_2^{-1})(I_3^{-1} - I_1^{-1})}$ ;
- (vi)  $I_1 < I_2 = I_3$ : all eigenvalues are zero;
- (vii)  $I_1 = I_3 < I_2$ : all eigenvalues are zero;
- (viii)  $I_1 = I_2 = I_3$ : all eigenvalues are zero.

It is now easy to turn this into the following result concerning stability of the linearisation of (3.7) about fixed points.

**3.6.8 Corollary** *Let  $(\mathcal{B}, \mu)$  be a rigid body. We have the following cases.*

- (i) *If the principal inertias of  $\mathcal{B}$  are distinct, then*
  - (a) *the linearisation for rotations about the intermediate principal axis are unstable and*
  - (b) *the linearisation for rotations about the largest and the smallest principal axes are stable.*
- (ii) *If there are two equal principal inertias, then the linearisation about any of the fixed points is unstable.*
- (iii) *If all principal inertias are equal, then the linearisation is stable.*

*Proof* The only thing that is in doubt are the cases where all the eigenvalues are zero. For these cases, we need to ensure that there are 2 linearly independent eigenvectors for the matrix (3.8). However, in all such cases, one verifies that the matrix (3.8) has rank 1 when only two eigenvalues are equal; this means that the linearisation is unstable. When all eigenvalues are equal, then the matrix (3.8) is identically zero, so it indeed has two linearly independent eigenvectors. ■

This then completes the linear stability analysis of the fixed points for (3.7). When the linearisation is unstable, this it is so by virtue of there being a positive real eigenvalue, so the Unstable Manifold Theorem ensures instability of the full equations at such fixed points. However, in all of the cases when the linearisation is stable, it is stable with all eigenvalues on the imaginary axis, so one cannot conclude stability of the full equations from the linearised equations (as seems to be the way with mechanical systems). However, since the solutions restricted to the spheres on must also lie on the ellipsoids of constant energy, one may use this to help determine the character of the full nonlinear equations. Indeed, if you are good at imagining such things, imagine fixing a sphere of some radius, and starting with an ellipsoid which sits in the interior of the sphere. Now gradually grow the ellipsoid. Eventually it will touch the sphere, and as the ellipsoid grows further, the points or lines of intersection will determine the solutions of (3.7). By growing the ellipsoid in this manner one obtains



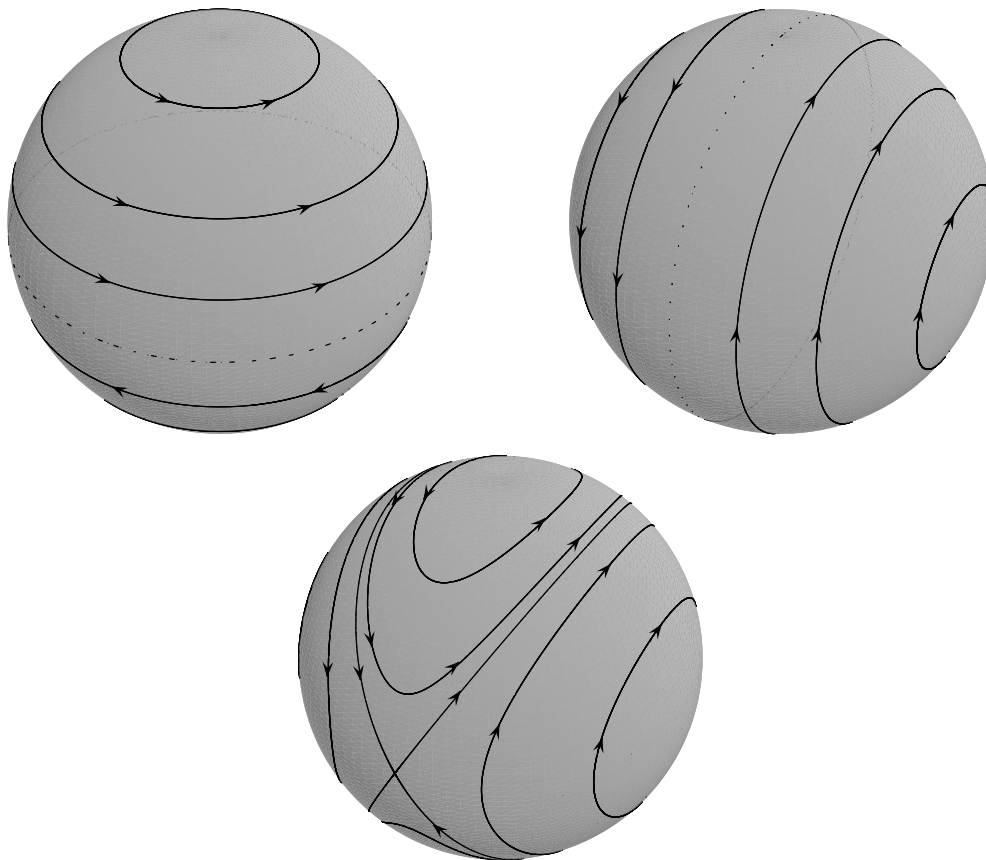


Figure 3.12 Solutions for equation (3.7) when  $I_1 = I_2 < I_3$  (top left), when  $I_1 < I_2 = I_3$  (top right), and when  $I_1 < I_2 < I_3$  (bottom)

all solutions as each point on the sphere will intersect *some* ellipsoid. In Figure 3.12 is depicted a numerically generated set of solution curves for (3.7) in the various cases. Note that in these pictures we always assume  $I_1 \leq I_2 \leq I_3$ , in order to reduce the number of cases we need to plot. This is in contrast with Proposition 3.6.7 and Corollary 3.6.8 where the principal inertias were considered arranged so that we are interested in the fixed points corresponding to the third principal inertia. Also, in Figure 3.12 we do not include the case where  $I_1 = I_2 = I_3$ . In this case, all points are fixed points, and so stable.

Let us summarise the conclusions one may derive from Figure 3.12.

**3.6.9 Proposition** *Let  $(\mathcal{B}, \mu)$  be a rigid body with principal inertias  $\{I_1, I_2, I_3\}$ , and suppose without loss of generality that  $I_1 \leq I_2$ . Let  $\mathbf{M}_0$  be a fixed point of (3.7) corresponding to the principal inertia  $I_3$  (i.e.,  $\mathbf{M}_0$  is an eigenvalue of  $\mathbb{I}_c^{-1}$  with eigenvalue  $I_3^{-1}$ ). The stability of  $\mathbf{M}_0$  is given as follows:*

- (i)  $I_1 < I_3 < I_2$ : *unstable*;
- (ii)  $I_3 < I_1 < I_2$ : *stable*;
- (iii)  $I_1 < I_2 < I_3$ : *stable*;
- (iv)  $I_1 = I_2 < I_3$ : *stable*;
- (v)  $I_3 < I_1 = I_2$ : *stable*;

- (vi)  $I_1 < I_2 = I_3$ : *unstable*;
- (vii)  $I_1 = I_3 < I_2$ : *unstable*;
- (viii)  $I_1 = I_2 = I_3$ : *stable*.

*Fixed points are never asymptotically stable.*

**3.6.3 Poinso't's description of a rigid body motion** Louis Poinso't (1777–1859) devised a clever way to visualise the solutions to (3.7). Poinso't's description tells us how the body itself moves *in space*, as opposed to Euler's equation which tells us how the angular momentum vector looks in a frame moving with the body. In this section, as in our previous discussion of rigid body dynamics, we fix a centre of mass observer adapted to the rigid motion, and a coordinate system adapted to this observer. The description of Poinso't relies on the inertial ellipsoid discussed in Section 1.5.3. With our conventions for the observer and the coordinate system, the inertial ellipsoid becomes the subset of  $\mathbb{R}^3$  given by

$$E(\mathcal{B}) = \{ \boldsymbol{\Omega} \in \mathbb{R}^3 \mid g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}), \boldsymbol{\Omega}) = 1 \}.$$

We think of the inertial ellipsoid as being attached to the body so that the axes of the ellipsoid align with the corresponding principal axes of the body. As the body moves in space, so too will the inertial ellipsoid move in space. It is this motion of the inertial ellipsoid which Poinso't describes. For simplicity, we assume that the centre of mass of the body is stationary, i.e., that the spatial linear momentum, which is conserved, is zero.

The following result gives Poinso't's description. You will recall that spatial angular momentum is constant along natural motions of a rigid body; indeed, we used this very property to *derive* the governing equations.

**3.6.10 Theorem** (Poinso't's description of rigid body motion) *Let  $(\mathcal{B}, \mu)$  be a rigid body and let the equations*

$$\begin{aligned} \dot{\mathbf{R}}(t) &= \mathbf{R}^{-1}(t)\boldsymbol{\Omega}(t) \\ \mathbb{I}_c(\dot{\boldsymbol{\Omega}}(t)) &= (\mathbb{I}_c(\boldsymbol{\Omega}(t))) \times \boldsymbol{\Omega}(t) \end{aligned}$$

*describe the angular motion of the body, and suppose that the centre of mass of the body is stationary. Also suppose that none of the principal inertias of the body are zero. The solution  $t \mapsto \mathbf{R}(t)$  has the property that the family of ellipsoids  $t \mapsto \mathbf{R}(t)(E(\mathcal{B}))$  rolls without slipping on a stationary plane  $P$  orthogonal to the (constant) spatial angular momentum vector  $\boldsymbol{\ell}$ . The plane  $P$  is called the **invariable plane**.*

*Proof* Suppose that the rigid motion has rotational kinetic energy  $e$  so that all motion takes place on the body angular momentum sphere of radius  $\sqrt{2e}$ . We work first in a body frame where the inertial ellipsoid is stationary. We define a plane  $P_t$  which is tangent to  $E(\mathcal{B})$  at the point  $\boldsymbol{\Omega}(t)$ . At the point  $\boldsymbol{\Omega}(t) \in E(\mathcal{B})$  we determine the that the normal to  $E(\mathcal{B})$  is given by

$$\text{grad}(g_{\text{can}}(\mathbb{I}_c(\boldsymbol{\Omega}), \boldsymbol{\Omega}))|_{\boldsymbol{\Omega}=\boldsymbol{\Omega}(t)} = 2\mathbb{I}_c(\boldsymbol{\Omega}(t)) = 2\mathbf{L}(t).$$

A translated plane  $P_t$  orthogonal to this normal vector should satisfy

$$P_t = \{ \mathbf{x} \in \mathbb{R}^3 \mid g_{\text{can}}(\mathbf{x}, 2\mathbb{I}_c(\boldsymbol{\Omega}(t))) = C \} \quad (3.10)$$

for some  $C \in \mathbb{R}$ . However, since the point  $\boldsymbol{\Omega}(t)$  must satisfy the constraint of conservation of rotational energy, we must have

$$g_{\text{can}}(\boldsymbol{\Omega}(t), 2\mathbb{I}_c(\boldsymbol{\Omega}(t))) = 2e.$$

Therefore, by the definition of the inertial ellipsoid, the point  $\boldsymbol{\Omega}(t)/\sqrt{2e}$  must lie in  $E(\mathcal{B})$ . Indeed we have

$$g_{\text{can}}(\mathbb{I}(\boldsymbol{\Omega}(t)/\sqrt{2e}), \boldsymbol{\Omega}(t)/\sqrt{2e}) = \frac{1}{2e} g_{\text{can}}(\mathbb{I}(\boldsymbol{\Omega}(t)), \boldsymbol{\Omega}(t)) = 1.$$

Therefore (3.10) defines a plane  $P_t$  tangent to the inertial ellipsoid at time  $t$  provided that we take  $C = \sqrt{2e}$ . Thus we have

$$P_t = \{ \boldsymbol{x} \in \mathbb{R}^3 \mid g_{\text{can}}(\boldsymbol{x}, \boldsymbol{L}(t)) = \sqrt{2e} \}.$$

We now move to a spatial frame. Fix the same time  $t$  and define the ellipsoid  $E_t(\mathcal{B}) = \boldsymbol{R}(t)(E(\mathcal{B}))$ . The body velocity  $\boldsymbol{\Omega}(t)$  gets mapped to the spatial velocity  $\boldsymbol{\omega}(t) = \boldsymbol{R}(t)(\boldsymbol{\Omega}(t))$ . The plane  $P_t$  is mapped to the plane  $P = \boldsymbol{R}(t)(P_t)$  which is therefore tangent to  $E_t(\mathcal{B})$  at the point  $\boldsymbol{\omega}(t)$ . Therefore  $P$  satisfies

$$\begin{aligned} P &= \{ \boldsymbol{y} \in \mathbb{R}^3 \mid g_{\text{can}}(\boldsymbol{L}(t), \boldsymbol{R}^{-1}(t)(\boldsymbol{y})) = \sqrt{2e} \} \\ &= \{ \boldsymbol{y} \in \mathbb{R}^3 \mid g_{\text{can}}(\boldsymbol{R}(t)\boldsymbol{L}(t), (\boldsymbol{y})) = \sqrt{2e} \} \\ &= \{ \boldsymbol{y} \in \mathbb{R}^3 \mid g_{\text{can}}(\boldsymbol{\ell}(t), (\boldsymbol{y})) = \sqrt{2e} \}. \end{aligned}$$

Now since  $\boldsymbol{\ell}(t)$  is constant, this shows that the plane  $P$  is independent of  $t$ . Thus we have shown that the ellipsoids  $E_t(\mathcal{B})$  do indeed roll along the plane  $P$ . It remains to be shown that they roll without slipping.

This latter fact will follow if we note that the point of contact of  $E_t(\mathcal{B})$  with the plane  $P$  lies at a point  $\boldsymbol{\omega}(t)/\sqrt{2e}$  which is on the line of instantaneous rotation of the body. Since all points along this line have zero instantaneous spatial velocity, our result follows. ■

A picture corresponding to Poincaré's description is given in Figure 3.13. We refer to [Arnol'd 1989] for a few applications of Poincaré's description of rigid body motion to determining the character of this motion. Also see Exercise E3.7.

## 3.7 Geodesic motion

In our analysis of the stability of equilibrium points Section 3.2—and particularly in our analysis of simple mechanical systems in Section 3.3.1—we paid little attention to the case where  $K_L(q_0)$  and  $C_L(q_0)$  are both zero. However, for the class of simple mechanical systems with no potential force, this will *always* be the case, cf. Proposition 3.3.7. Therefore, none of our preceding stability analysis has much relevance for this important class of systems. In this section we will study such systems and say what little we can about them; they are a very complicated class of system, interestingly made more complicated in some ways by the absence of a potential function.

**3.7.1 Basic facts about geodesic motion** In Section 2.4.4 we showed how the equations of motion for a simple mechanical system  $(Q, g, 0)$  with a zero potential function are geodesics of a particular affine connection—the Levi-Civita connection—which is associated with the Riemannian metric  $g$ . However, it is of no particular advantage to specialise to the case of the Levi-Civita connection, so in this section we consider a general affine connection  $\nabla$  on  $Q$ . Recall that in a coordinate chart one defines the Christoffel symbols  $\Gamma_{jk}^i$ ,

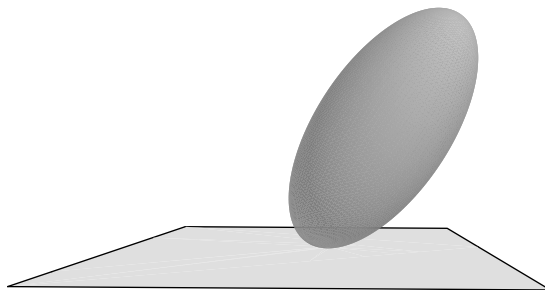


Figure 3.13 Poincaré's description of the motion of a rigid body

$i, j, k = 1, \dots, n$ , as in (2.11). Geodesics are then curves on  $Q$  which in coordinates satisfy the second-order differential equation

$$\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0, \quad i = 1, \dots, n.$$

From this equation, we immediately have the following result. The result uses facts about geodesic normal coordinates which we do not know about (see [Kobayashi and Nomizu 1963]).

**3.7.1 Proposition** *For a simple mechanical system  $(Q, g, 0)$  with zero potential function, all points in  $Q$  are equilibrium points, none are linearly stable, and none are stable for the nonlinear system.*

*Proof* That all points in  $Q$  are equilibrium points follows from Proposition 3.3.6. From Proposition 3.3.7 we see that at every equilibrium point, i.e., every point  $q_0 \in Q$ , we have

$$A_L(q_0) = \begin{bmatrix} 0 & \text{id}_{T_{q_0}Q} \\ 0 & 0 \end{bmatrix}.$$

Thus all eigenvalues of  $A_L(q_0)$  are zero, but  $\dim(\ker(A_L(q_0))) = n$  so the fixed points are linearly unstable since the geometric and algebraic multiplicities of the eigenvalues on the imaginary axis are not the same. Finally, let us show that the equilibria are not stable for the full system. This follows from the fact that about each point  $q_0 \in Q$  we may find a “normal” set of coordinates  $(q^1, \dots, q^n)$  with the property that any geodesic  $t \mapsto c(t)$  with initial condition  $c'(0) = v_{q_0} \in T_{q_0}Q$  has the form  $t \mapsto vt$  for some  $v \in \mathbb{R}^n$ . Such a curve

will leave any sufficiently small neighbourhood of  $q_0$  in finite time, implying the equilibrium point is unstable. ■

Clearly such systems must be of interest for things other than their equilibria, since equilibria are abundant and all unstable. The following result gives a distinctive property for solutions of the geodesic equations. Note that the result is not generally true for solutions of the Euler-Lagrange equations for arbitrary Lagrangians.

**3.7.2 Proposition** *Let  $\nabla$  be an affine connection on a configuration space  $Q$ . Let  $c: [0, T] \rightarrow Q$  be a curve and for  $\lambda > 0$  define a new curve  $c_\lambda: [0, \frac{T}{\lambda}] \rightarrow Q$  by  $c_\lambda(t) = c(\lambda t)$ . Then, if  $c$  is a geodesic, so is  $c_\lambda$ .*

*Proof* In coordinates represent  $c$  by  $t \mapsto (q^1(t), \dots, q^n(t))$  so that we have

$$\frac{d^2 q^i}{dt^2} + \Gamma_{jk}^i \frac{dq^j}{dt} \frac{dq^k}{dt} = 0, \quad i = 1, \dots, n.$$

Now define  $\tau = \lambda t$  and compute

$$\frac{d^2 q^i}{d\tau^2} + \Gamma_{jk}^i \frac{dq^j}{d\tau} \frac{dq^k}{d\tau} = \lambda^2 \left( \frac{d^2 q^i}{dt^2} + \Gamma_{jk}^i \frac{dq^j}{dt} \frac{dq^k}{dt} \right), \quad i = 1, \dots, n,$$

which gives the result. ■

Note that with the notation of the preceding result we have  $c'_\lambda(0) = \lambda c'(0)$ . The idea of the result is therefore that if one scales the initial velocity of a given geodesic, the resulting geodesic will follow the same path as the original, but will simply do so more quickly or more slowly, depending on whether  $\lambda > 1$  or  $\lambda < 1$ .

**3.7.2 The Jacobi metric** It turns out that even when the potential function is nonzero, one may still describe the motion of a simple mechanical system as geodesic motion. That this is possible is a discovery of Carl Gustav Jacob Jacobi (1804–1851). The principal observation is that since kinetic energy is always nonnegative, if a solution  $t \mapsto c(t)$  has total energy  $e$ , then the potential energy should satisfy  $V(c(t)) \leq e$ . For this reason, solutions of total energy  $e$  will evolve in the subset

$$\bar{Q}_e = \{q \in Q \mid V(q) \leq e\}.$$

Let us also denote

$$Q_e = \{q \in Q \mid V(q) < e\}.$$

Note that on  $Q_e$  the function  $q \mapsto e - V(q)$  is strictly positive. Therefore, given a Riemannian metric  $g$  on  $Q$ , at points  $q \in Q_e$ ,  $g_e(q) = (e - V(q))g(q)$  defines an inner product on  $T_q Q$ . Therefore, this defines a Riemannian metric  $g_e$  on  $Q_e$  called the **Jacobi metric**.

We now state the relationship between the Riemannian metric  $g_e$  on  $Q_e$  and solutions of the Euler-Lagrange equations.

**3.7.3 Theorem** *Let  $(Q, g, V)$  be a simple mechanical system with Lagrangian  $L$  and let  $e \in \mathbb{R}$  have the property that  $Q_e \neq \emptyset$ . If  $v_q \in \mathbf{EL}^{-1}(e)$ , then the solution to the Euler-Lagrange equations with initial condition  $v_q$  agrees up to reparameterisation with the geodesic of the Jacobi metric with initial velocity  $v_q$ .*

*Proof* Let  $\overset{e}{\nabla}$  be the Levi-Civita connection associated with the Riemannian metric  $g_e$ . We begin with a computational lemma.

**1 Lemma** *Let  $e^{2\rho} = (e - V)$ , so defining  $\rho$ . Then we have*

$$\overset{e}{\nabla}_X Y = \overset{g}{\nabla}_X Y + \langle \mathbf{d}\rho; X \rangle Y + \langle \mathbf{d}\rho; Y \rangle X - g(X, Y) \text{grad } \rho.$$

*Proof* In coordinates we have

$$(\overset{e}{\nabla}_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \overset{e}{\Gamma}_{jk}^i X^j Y^k, \quad i = 1, \dots, n,$$

where  $\overset{e}{\Gamma}$  are the Christoffel symbols for  $\overset{e}{\nabla}$ . The result will then follow if we can show that

$$\overset{e}{\Gamma}_{jk}^i = \overset{g}{\Gamma}_{jk}^i + \frac{\partial \rho}{\partial q^\ell} X^\ell Y^i + \frac{\partial \rho}{\partial q^\ell} Y^\ell X^i - g_{\ell m} X^\ell Y^m g^{ir} \frac{\partial \rho}{\partial q^r}, \quad i, j, k = 1, \dots, n. \quad (3.11)$$

By definition of the Levi-Civita Christoffel symbols we have

$$\overset{e}{\Gamma}_{jk}^i = \frac{1}{2} (g_e)^{i\ell} \left( \frac{\partial (g_e)_{\ell j}}{\partial q^k} + \frac{\partial (g_e)_{\ell k}}{\partial q^j} - \frac{\partial (g_e)_{jk}}{\partial q^\ell} \right), \quad i, j, k = 1, \dots, n. \quad (3.12)$$

However, since  $(g_e)_{ij} = e^{2\rho} g_{ij}$ ,  $i, j = 1, \dots, n$ , we directly compute

$$\frac{\partial (g_e)_{ij}}{\partial q^k} = 2e^{2\rho} \frac{\partial \rho}{\partial q^k} g_{\ell j} + e^{2\rho} \frac{\partial g_{ij}}{\partial q^k}, \quad i, j, k = 1, \dots, n. \quad (3.13)$$

Plugging (3.13) into (3.12) gives (3.11) after some manipulation.  $\blacktriangledown$

Now let us proceed with the remainder of the proof. Fix a solution  $c$  to the Euler-Lagrange equations for  $L$ . We compute

$$\mathbf{d}(e^{2\rho}) = -\mathbf{d}V \quad \implies \quad 2e^{2\rho} \mathbf{d}\rho = -\mathbf{d}V \quad \implies \quad 2(e - V) \mathbf{d}\rho = -\mathbf{d}V. \quad (3.14)$$

By conservation of energy we also have

$$g(c'(t), c'(t)) = 2(e - V), \quad (3.15)$$

which using (3.14) gives

$$g(c'(t), c'(t)) \text{grad } \rho = -\text{grad } V. \quad (3.16)$$

Since  $c$  is a solution to the Euler-Lagrange equations we have

$$\overset{g}{\nabla}_{c'(t)} c'(t) = -\text{grad } V(c(t)).$$

Using Lemma 1 and (3.16) we obtain

$$\begin{aligned} \overset{e}{\nabla}_{c'(t)} c'(t) &= \overset{g}{\nabla}_{c'(t)} c'(t) + 2 \langle \mathbf{d}\rho; c'(t) \rangle c'(t) + g(c'(t), c'(t)) \text{grad } \rho \\ &= -\text{grad } V(c(t)) + 2 \langle \mathbf{d}\rho; c'(t) \rangle c'(t) + g(c'(t), c'(t)) \text{grad } \rho \\ &= 2 \langle \mathbf{d}\rho; c'(t) \rangle c'(t). \end{aligned}$$

Now reparameterise  $c$  with the time parameter  $\tau$  defined so that  $\frac{d\tau}{dt} = (e - V)$ . We then have

$$\frac{dc}{d\tau} = \frac{dt}{d\tau} c'(t) = \frac{1}{e - V} c'(t) = e^{-2\rho} c'(t).$$

We then compute

$$\begin{aligned} \overset{e}{\nabla} \frac{dc}{d\tau} &= \overset{e}{\nabla}_{e^{-2\rho} c'(t)} (e^{-2\rho} c'(t)) \\ &= e^{-2\rho} \overset{e}{\nabla}_{c'(t)} (e^{-2\rho} c'(t)) \\ &= e^{-4\rho} \overset{e}{\nabla}_{c'(t)} c'(t) + e^{-2\rho} (\mathcal{L}_{c'(t)} e^{-2\rho}) c'(t) \\ &= e^{-4\rho} (\overset{e}{\nabla}_{c'(t)} c'(t) - \langle \mathbf{d}\rho; c'(t) \rangle c'(t)) = 0. \end{aligned}$$

In deriving this formula, we have used the defining properties of an affine connection given in Definition 2.4.7. Note that we have shown that  $c$  is a geodesic of  $\overset{e}{\nabla}$  provided we parameterise it with the time parameter  $\tau$ . ■

Ong [1975] provides a nice discussion of various aspects of dynamics of simple mechanical systems using the Jacobi metric. The substance of the above proof comes from Ong. One could, I suppose, give an application of Theorem 3.7.3, but it would not be an exercise which would shed any more light on the subject. We should really think of Theorem 3.7.3 as an interesting theoretical tidbit, although it may have some useful dynamical consequences in some examples.

## 3.8 The dynamics of constrained systems

Let us complete our incomplete discussion of Lagrangian dynamics with a few words about the dynamics of constrained systems. Owing to the fact that the body of work concerning the dynamics of constrained systems is minimal, our discussion only touches upon a few basic questions. Some issues are dealt with more thoroughly by Neimark and Fufaev [1972], but even there the authors remark on the lack of a consistent approach to the subject of dynamics for systems with constraints.

**3.8.1 Existence of solutions for constrained systems** In Section 3.1 we saw that hyperregularity of  $L$  was sufficient to guarantee that the solutions to the Euler-Lagrange equations are determined by an ordinary differential equation, and so subject to the existence and uniqueness theory of ordinary differential equations. For constrained systems, matters are not so simple, as is illustrated by a simple example.

**3.8.1 Example** We consider  $Q = \mathbb{R}^2$  with the time-independent hyperregular Lagrangian  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 - v_y^2)$ . As a constraint we take the linear constraint

$$\mathcal{C}_{(x,y)} = \{(v_x, v_y) \mid v_x + v_y = 0\}.$$

The constraints are annihilated by the single one-form  $\alpha = dx - dy$ . Thus we have a perfectly well-behaved Lagrangian and a perfectly well-behaved constraint. However, when we put them together, we get the constrained equations of motion

$$\ddot{x} = 1, \quad -\ddot{y} = -1.$$

Therefore we must have  $\ddot{x} = \ddot{y}$ . But the constraint equations give us  $\ddot{x} = -\ddot{y}$ , so we must have  $\ddot{x} = \ddot{y} = 0$ . This, however, is an immediate contradiction of the equations of motion.  $\square$

It is not immediately obvious what is the problem with the previous example. However, after some thought, one sees that the difficulty lies in the fact that the quadratic form with matrix  $\frac{\partial^2 L}{\partial v^i \partial v^j}$  is degenerate on the subspace of virtual displacements. More specifically, the quadratic form represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is identically zero when evaluated on vectors in  $\text{span}_{\mathbb{R}} \{(1, -1)\}$ . Let us convince ourselves that this is indeed a source of problems by looking at the same constraint, but with a Lagrangian for which the quadratic form with matrix  $\frac{\partial^2 L}{\partial v^i \partial v^j}$  is *nondegenerate* on the subspace of virtual displacements.

**3.8.2 Example** (Example 3.8.1 cont'd) We maintain the constraint from Example 3.8.1 but consider a different Lagrangian. We take  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2)$ . In this case the equations of motion read

$$\ddot{x} = 1, \quad \ddot{y} = -1,$$

from which we derive  $\ddot{x} = -\ddot{y}$ . These equations are now compatible with the constraints, and one can verify directly that any curve of the form

$$x(t) = x(0) + vt, \quad y(t) = y(0) - vt$$

will satisfy the constraints for any  $v \in \mathbb{R}$ . Note that this version of our example simply models a particle of unit mass constrained (by an holonomic constraint) to move along lines in  $\mathbb{R}^2$  with slope  $-1$ .  $\square$

Let us now reveal that the above example generalises in a natural manner.

**3.8.3 Theorem** *Let  $L$  be a hyperregular Lagrangian on  $Q$  with  $C$  a constraint and  $F$  a force. If for each  $v_q \in TQ$  the quadratic form with components  $\frac{\partial^2 L}{\partial v^i \partial v^j}$ ,  $i, j = 1, \dots, n$ , is nondegenerate on the subspace  $\ell(C_q) \subset T_q Q$ , then the solutions to the constrained Lagrangian system  $(L, F, C)$  satisfy a second-order differential equation.*

*Proof* We work in a set of coordinates  $(q^1, \dots, q^n)$ . Let  $\alpha^1, \dots, \alpha^{n-r}$  be a basis for  $\text{ann}(\ell(C))$  in these coordinates, and let  $\eta$  be a vector field on  $Q$  with the property that  $C_q = \ell(C_q) + \eta(q)$ . The constrained equations of motion are then

$$\begin{aligned} \frac{\partial^2 L}{\partial v^i \partial v^j} \ddot{q}^j + \frac{\partial^2 L}{\partial v^i \partial q^j} \dot{q}^j + \frac{\partial^2 L}{\partial v^i \partial t} - \frac{\partial L}{\partial q^i} &= F_i + \lambda_a \alpha_i^a, \quad i = 1, \dots, n \\ \alpha_i^a \dot{q}^i &= \alpha_i^a \eta^i, \quad a = 1, \dots, n-r. \end{aligned}$$

Differentiating the second equation with respect to time gives

$$\alpha_i^a \ddot{q}^i + \frac{\partial \alpha_i^a}{\partial q^j} \dot{q}^i \dot{q}^j = \frac{\partial \alpha_i^a}{\partial q^j} \eta^i \dot{q}^j + \alpha_i^a \frac{\partial \eta^i}{\partial q^j} \dot{q}^j, \quad a = 1, \dots, n-r. \quad (3.17)$$

Since  $L$  is hyperregular we may solve the first of the equations of motion for  $\ddot{q}^i$ :

$$\ddot{q}^i = A^{i\ell} \left( \frac{\partial L}{\partial q^\ell} - \frac{\partial^2 L}{\partial v^\ell \partial q^j} \dot{q}^j - \frac{\partial^2 L}{\partial v^\ell \partial t} + F_\ell + \lambda_a \alpha_\ell^a \right), \quad i = 1, \dots, n, \quad (3.18)$$



where  $A^{i\ell}$ ,  $i, \ell = 1, \dots, n$ , are defined by

$$A^{i\ell} \frac{\partial^2 L}{\partial v^\ell \partial v^j} = \delta_j^i, \quad i, j = 1, \dots, n.$$

Substituting (3.18) into (3.17) and simplifying gives

$$\begin{aligned} \alpha_i^a A^{i\ell} \alpha_\ell^b \lambda_b &= \frac{\partial \alpha_i^a}{\partial q^j} \eta^i \dot{q}^j + \alpha_i^a \frac{\partial \eta^i}{\partial q^j} \dot{q}^j - \frac{\partial \alpha_i^a}{\partial q^j} \dot{q}^i \dot{q}^j - \\ &\alpha_i^a A^{i\ell} \left( \frac{\partial L}{\partial q^\ell} - \frac{\partial^2 L}{\partial v^\ell \partial q^j} \dot{q}^j - \frac{\partial^2 L}{\partial v^\ell \partial t} + F_\ell \right), \quad a = 1, \dots, n-r. \end{aligned} \quad (3.19)$$

Now we use a lemma to complete the final step.

**1 Lemma** *The  $(n-r) \times (n-r)$  matrix with components  $\alpha_i^a A^{i\ell} \alpha_\ell^b$ ,  $a, b = 1, \dots, n-r$ , is invertible.*

*Proof* Suppose that we have  $\beta_1, \dots, \beta_{n-r}$  so that

$$\alpha_i^a A^{i\ell} \alpha_\ell^b \beta_b = 0, \quad a = 1, \dots, n-r.$$

This would imply that  $A^{i\ell} \alpha_\ell^b \beta_b$ ,  $i = 1, \dots, n$ , are the components of the zero vector field in  $\ell(\mathbf{C})$  since  $\alpha^1, \dots, \alpha^{n-r}$  form a basis for  $\text{ann}(\ell(\mathbf{C}))$ . Since the quadratic form with matrix  $\frac{\partial^2 L}{\partial v^i \partial v^j}$ ,  $i, j = 1, \dots, n$ , is nondegenerate on  $\ell(\mathbf{C})$ , we have

$$\begin{aligned} A^{i\ell} \alpha_\ell^b \beta_b &= 0, \quad i = 1, \dots, n \\ \iff \frac{\partial^2 L}{\partial v^i \partial v^j} A^{i\ell} \alpha_\ell^b \beta_b &= \alpha_j^b \beta_b = 0, \quad i = j, \dots, n. \end{aligned}$$

Since the one-forms  $\alpha^1, \dots, \alpha^{n-r}$  are linearly independent, the result now follows. ▼

Using the lemma we may use (3.19) to solve for the Lagrange multipliers. Now we plug these Lagrange multipliers back into (3.18), and the result is a set of second-order differential equations, as desired. ■

**3.8.4 Remarks 1.** Of course, the value of the theorem is that it gives an existence and uniqueness result for constrained systems. Constrained systems are capable of a great deal of exotic behaviour, so it is consoling to have some not unreasonable conditions which guarantee that solutions exist and are unique.

2. We make use of “the quadratic form whose matrix is  $\frac{\partial^2 L}{\partial v^i \partial v^j}$ ,  $i, j = 1, \dots, n$ .” Well, this *is* an actual quadratic form, and we actually demonstrated this in the proof of Proposition 3.2.10. If you look through the proof, you will see that when we demonstrated that  $M_L(q_0)$  was independent of choice of coordinates, we never used the fact that we evaluated the quantities  $\frac{\partial^2 L}{\partial v^i \partial v^j}$ ,  $i, j = 1, \dots, n$ , at zero velocity. This quadratic form is actually a quite useful object. Of course, for simple mechanical systems, it is simply the Riemannian metric.

3. Speaking of Riemannian metrics, note that the hypotheses of the theorem are satisfied automatically if  $L$  is the Lagrangian for a simple mechanical system (see Corollary 3.8.5 below). □

The upshot of Remark 3 is worth recording on its own.

**3.8.5 Corollary** *If  $(Q, g, V)$  is a simple mechanical system with Lagrangian  $L$ ,  $C$  is a constraint on  $Q$ , and  $F$  is a force on  $Q$ , then whenever  $v_q \in C$  is an initial condition satisfying the constraints, there exists a solution to the constrained system  $(L, F, C)$  with initial condition  $v_q$ . Furthermore, any two such solutions will agree on the intersection of their domains.*

**3.8.2 Some general observations** In this section let us give answers to a few questions of the form, “What aspects of dynamics for unconstrained Lagrangian systems carry over to constrained systems?” In particular, we concentrate on two questions, one concerning conservation of energy, and the other concerning the nature of the stability of equilibrium points.

Let us first consider the situation concerning conservation of energy.

**3.8.6 Proposition** *Let  $L$  be a time-dependent Lagrangian on  $Q$  and  $C$  a constraint on  $Q$ . Suppose that  $t \mapsto c(t)$  is a solution to the constrained Lagrangian system  $(L, 0, C)$  with  $t \mapsto \alpha(t) \in \text{ann}(\ell(C_{c(t)}))$  the corresponding constraint force. If  $\mathbf{E}L$  denotes the energy of the system then we have*

$$\frac{d}{dt} \mathbf{E}L(t, c'(t)) = \langle \alpha(t); c'(t) \rangle.$$

*In particular, if the constraint  $C$  is linear, then energy is conserved.*

*Proof* We work in coordinates  $(q^1, \dots, q^n)$  with  $\alpha^1, \dots, \alpha^{n-r}$  a local basis for  $\text{ann}(\ell(C))$ . We also define a vector field  $\eta$  on  $Q$  so that  $C_q = \ell(C_q) + \eta(q)$ . Recall that the equations of motion are then

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} &= \lambda_a \alpha_i^a, & i = 1, \dots, n \\ \alpha_i^a \dot{q}^i &= \alpha_i^a \eta^i, & a = 1, \dots, n-r. \end{aligned}$$

Following the computations of Proposition 2.5.7 we determine from the first of these equations that

$$\frac{d}{dt} \mathbf{E}L(t, c'(t)) = \langle \alpha(t); c'(t) \rangle,$$

which is the first assertion of the proposition. The second assertion follows from the second of the equations of motion since if  $C$  is linear we may take  $\eta = 0$ . ■

Now let us consider the matter of stability of equilibrium points. We have seen in Section 3.2 that it is not possible for an equilibrium point of an unconstrained system to be asymptotically stable. However, this is not the case for systems with constraints, and so let us give an example of a constrained system with an asymptotically stable equilibrium point.

**3.8.7 Example** □

**3.8.3 Constrained simple mechanical systems** In this section we address the nature of the equations which govern the dynamics of simple mechanical systems with constraints. The observation we make in this section seems to have originated with [Synge \[1928\]](#).

We consider a simple mechanical system  $(Q, g, 0)$  with no potential energy, and a linear constraint  $C$ . The constructions we make are readily modified to include the potential

function and to include constraints which are not necessarily linear. However, the clutter of this extra generality serves only to obscure the basic idea we illustrate.

Recall that in Section 2.6.4 we had introduced the  $g$ -orthogonal projection  $P_C: TQ \rightarrow TQ$  onto the orthogonal complement of the constraint distribution. Using this projection, in Proposition 2.6.14 we had shown that solutions to the constrained system satisfy

$$\begin{aligned} \overset{g}{\nabla}_{c'(t)} c'(t) &= \lambda(t) \\ P_C(c'(t)) &= 0 \end{aligned} \quad (3.20)$$

for some vector field  $t \mapsto \lambda(t) \in C_{c(t)}^\perp$ , along  $c$  taking its values in the subspaces orthogonal to  $C$ . We wish to examine these equations, and eliminate the Lagrange multipliers in a clever way.

Our construction requires that we be able to differentiate the projection  $P_C$  in the correct fashion. To this end, for a vector field  $X$  on  $Q$ , let us define a map  $\overset{g}{\nabla}_X P_C: TQ \rightarrow TQ$  as follows: for any vector field  $Y$  on  $Q$  we require  $\overset{g}{\nabla}_X P_C$  to satisfy

$$(\overset{g}{\nabla}_X P_C)(Y) = \overset{g}{\nabla}_X(P_C(Y)) - P_C(\overset{g}{\nabla}_X Y).$$

In this manner we are able to define  $\overset{g}{\nabla}_X P_C$  using our knowledge of how to define  $\overset{g}{\nabla}$  on vector fields. Using this definition of  $\overset{g}{\nabla}_X P_C$  we may show that in coordinates we have

$$(\overset{g}{\nabla}_X P_C)_j^i = \frac{\partial P_j^i}{\partial q^k} X^k + \overset{g}{\Gamma}_{kl}^i P_j^l X^k - \overset{g}{\Gamma}_{kj}^l P_l^i X^k, \quad i, j = 1, \dots, n, \quad (3.21)$$

where  $P_j^i$ ,  $i, j = 1, \dots, n$ , are the components of  $P_C$ , and so are defined by

$$P_C\left(\frac{\partial}{\partial q^j}\right) = P_j^i \frac{\partial}{\partial q^i}, \quad j = 1, \dots, n.$$

The expression (3.21) can be troublesome to compute in practice, much as  $P_C$  can be a bother to calculate. However, the utility of what we do in this section lies in its theoretical content.

Let us see how to compute  $\overset{g}{\nabla}_X P_C$  in an example for which we have already computed  $P_C$ .

**3.8.8 Example** (Example 2.6.1 cont'd) We carry on looking at the rolling disk example where  $Q = \mathbb{R}^2 \times \mathbb{T}^2$ , and we use coordinates  $(x, y, \theta, \phi)$ . The constraints are defined by

$$\dot{x} = r \cos \theta \dot{\phi}, \quad \dot{y} = r \sin \theta \dot{\phi}.$$

In Example 2.6.13 we determined that the components  $P_j^i$ ,  $i, j = 1, \dots, 4$ , for the projection  $P_C$  are given by

$$\frac{1}{mr^2 + J} \begin{bmatrix} J + mr^2 \sin^2 \theta & -mr^2 \sin \theta \cos \theta & 0 & -rJ \cos \theta \\ -mr^2 \sin \theta \cos \theta & J + mr^2 \cos^2 \theta & 0 & -rJ \sin \theta \\ 0 & 0 & 0 & 0 \\ -mr \cos \theta & -mr \sin \theta & 0 & mr^2 \end{bmatrix}.$$

In Example 2.6.11 we saw that the Christoffel symbols are all identically zero for the Riemannian metric  $g$  for this problem. Therefore we may compute the components of  $\overset{g}{\nabla}_X P_C$

using (3.21). This is a tedious, unpleasant task, obviously best handled by a computer. By delegating the labour in this fashion, we determine that if

$$X = X_x \frac{\partial}{\partial x} + X_y \frac{\partial}{\partial y} + X_\theta \frac{\partial}{\partial \theta} + X_\phi \frac{\partial}{\partial \phi},$$

then the components of  $\overset{g}{\nabla}_X P$  are given by

$$\frac{1}{mr^2 + J} \begin{bmatrix} mr^2 \sin 2\theta X_\theta & -mr^2 \cos 2\theta X_\theta & 0 & rJ \sin \theta X_\theta \\ -mr^2 \cos 2\theta X_\theta & -mr^2 \sin 2\theta X_\theta & 0 & -rJ \cos \theta X_\theta \\ 0 & 0 & 0 & 0 \\ mr \sin \theta X_\theta & -mr \cos \theta X_\theta & 0 & 0 \end{bmatrix}.$$

While this final expression is bearable, the intervening calculations are less so.  $\square$

Now we believe that at least in principle we may compute this quantity  $\overset{g}{\nabla}_X P_C$ . Let us see why it is useful for us.

**3.8.9 Theorem** *Let  $(Q, g, 0)$  be a simple mechanical system with zero potential, and let  $C$  be a linear constraint on  $Q$ . A curve  $t \mapsto c(t)$  on  $Q$  is a solution of the equations (3.20) if and only if  $c'(0) \in C_{c(0)}$  and if  $c$  is a geodesic of the affine connection  $\overset{c}{\nabla}$  on  $Q$  defined by*

$$\overset{c}{\nabla}_X Y = \overset{g}{\nabla}_X Y + (\overset{g}{\nabla}_X P_C)(Y).$$

*Proof* Take the covariant derivative of the second of equations (3.20) with respect to  $c'(t)$  to get

$$\overset{g}{\nabla}_{c'(t)}(P_C(c'(t))) = (\overset{g}{\nabla}_{c'(t)} P_C)(c'(t)) + P_C(\overset{g}{\nabla}_{c'(t)} c'(t)) = 0. \quad (3.22)$$

Now apply  $P_C$  to the first of equations (3.20) to get

$$P_C(\overset{g}{\nabla}_{c'(t)} c'(t)) = P_C(\lambda(t)) = \lambda(t),$$

since  $\lambda$  is orthogonal to  $C$ . By (3.22) we now have

$$\lambda(t) = -(\overset{g}{\nabla}_{c'(t)} P_C)(c'(t)).$$

Substituting this back into the first of equations (3.20) we see that if  $c$  satisfies both of equations (3.20) then  $c'(0) \in C_{c(0)}$  and  $c$  satisfies the equation

$$\overset{g}{\nabla}_{c'(t)} c'(t) + (\overset{g}{\nabla}_{c'(t)} P_C)(c'(t)) = \overset{c}{\nabla}_{c'(t)} c'(t) = 0.$$

To show the reverse implication is a matter of simply reversing the above steps.  $\blacksquare$

Note that the geodesic equations for  $\overset{c}{\nabla}$  will read in coordinates as

$$\ddot{q}^i + \overset{g}{\Gamma}_{jk}^i \dot{q}^j \dot{q}^k + \frac{\partial P_j^i}{\partial q^k} X^k + \overset{g}{\Gamma}_{kl}^i P_j^l X^k - \overset{g}{\Gamma}_{kj}^l P_l^i X^k, \quad i = 1, \dots, n. \quad (3.23)$$

Thus one can in principle compute the geodesic equations for a given system. Certainly, one can ask a computer to do it for you.

- 3.8.10 **Remarks 1.** The essential statement of the above result is this: *The solutions for a potentialless simple mechanical system subject to a linear constraint are geodesics of an appropriately defined affine connection.* This justifies any of our previous constructions using general affine connections, rather than restricting to Levi-Civita connections.
2. A somewhat thorough exploration of the matters surrounding Theorem 3.8.9 are provided by Lewis [1998]. Implications to control theory may be found in [Lewis 2000b].
  3. Bullo and Žefran [2002] give a way to simplify the computations involved in computing the geodesics for the affine connection  $\overset{c}{\nabla}$ .  $\square$

As a final demonstration of what we are doing in this section, let us use Theorem 3.8.9 to provide equations which govern the motion of the rolling disk.

3.8.11 **Example** (Example 2.6.1 cont'd) Let us write the equations of motion for the rolling disk as geodesic equations as commanded by Theorem 3.8.9. In the coordinates  $(x, y, \theta, \phi)$  the equations (3.23) are computed as

$$\begin{aligned}\ddot{x} + \frac{1}{mr^2 + J} \left( mr^2 \sin 2\theta \dot{x}\dot{\theta} - mr^2 \cos 2\theta \dot{y}\dot{\theta} + Jr \sin \theta \dot{\theta}\dot{\phi} \right) &= 0 \\ \ddot{y} + \frac{1}{mr^2 + J} \left( mr^2 \cos 2\theta \dot{x}\dot{\theta} + mr^2 \sin 2\theta \dot{y}\dot{\theta} + Jr \cos \theta \dot{\theta}\dot{\phi} \right) &= 0 \\ \ddot{\theta} &= 0 \\ \ddot{\phi} + \frac{1}{mr^2 + J} \left( mr \sin \theta \dot{x}\dot{\theta} - mr \cos \theta \dot{y}\dot{\theta} \right) &= 0.\end{aligned}$$

Of course, the solutions to these equations are only physical solutions when the initial velocities satisfy the constraints. Although one can certainly choose other initial conditions, they will have no relation to the problem of the rolling disk.

The equations can be simplified significantly by substituting the constraint equations, which we know must be satisfied. Thus we use the relations

$$\dot{x} = r \cos \theta \dot{\phi}, \quad \dot{y} = r \sin \theta \dot{\phi}$$

and after some straightforward manipulations we arrive at

$$\begin{aligned}\ddot{x} + r \sin \theta \dot{\theta}\dot{\phi} &= 0 \\ \ddot{y} - r \sin \theta \dot{\theta}\dot{\phi} &= 0 \\ \ddot{\theta} &= 0 \\ \ddot{\phi} &= 0.\end{aligned}$$

These equations are now straightforwardly solved (see Exercise E3.12).  $\square$

## Exercises

- E3.1 Show that if  $L$  is a hyperregular Lagrangian on a configuration space  $Q$  and  $F$  is a force on  $Q$ , then existence and uniqueness is guaranteed for curves which satisfy the Lagrange-d'Alembert principle for  $L$  and  $F$ . State a precise result which captures this phenomenon.
- E3.2 This exercise is a continuation of Exercise E2.4. In Figure E3.1 are shown the coordi-

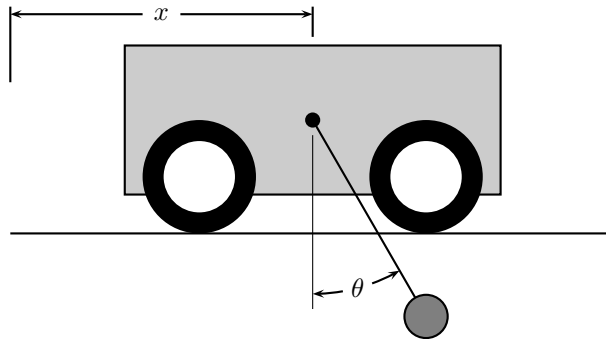


Figure E3.1 Coordinates for Exercise E2.4

ates for this example which will be used from now on.

Note that the system is a simple mechanical system. For this system, do the following.

- Determine all equilibrium points for the system.
  - For each equilibrium configuration  $q_0$ , compute  $M_L(q_0)$ ,  $C_L(q_0)$ , and  $K_L(q_0)$ .
  - For each equilibrium point  $q_0$ , determine whether it is linearly stable.
  - For each equilibrium point  $q_0$ , determine whether it is stable.
- E3.3 This exercise is a continuation of Exercise E2.5. In Figure E3.2 are shown the coordi-

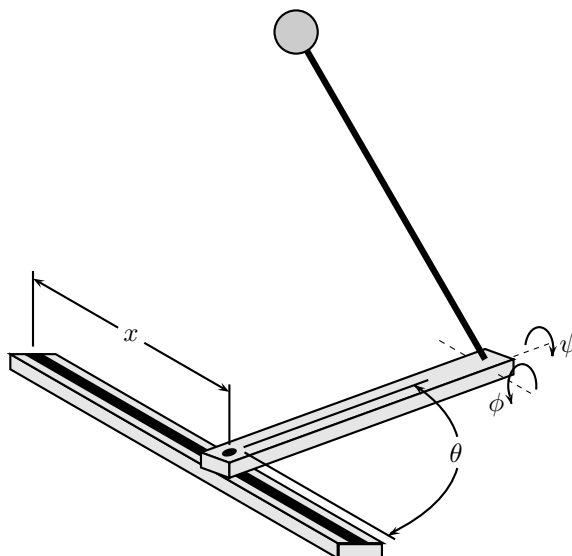


Figure E3.2 Coordinates for Exercise E2.5

dates for this example which will be used from now on. Choose  $\theta = 0$  so that the arm points “straight ahead,” and choose  $\phi = \psi = 0$  so that the pendulum is straight up.

Note that the system is a simple mechanical system. For this system, do the following.

- Determine all equilibrium points for the system.
- For each equilibrium configuration  $q_0$ , compute  $M_L(q_0)$ ,  $C_L(q_0)$ , and  $K_L(q_0)$ .
- For each equilibrium point  $q_0$ , determine whether it is linearly stable.
- For each equilibrium point  $q_0$ , determine whether it is stable.

E3.4 This exercise is a continuation of Exercise E2.6. In Figure E3.3 are the coordinates

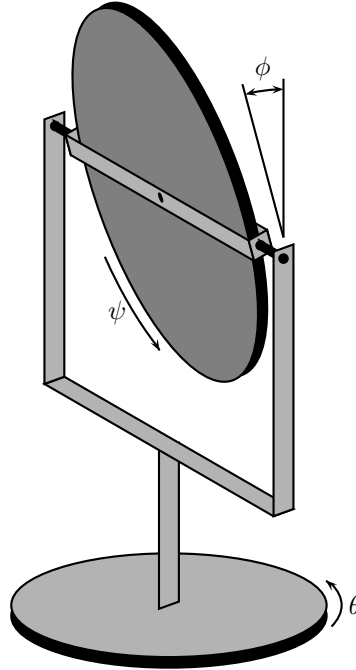


Figure E3.3 Coordinates for Exercise E3.4

for this example which will be used from now on.

Note that the system is a simple mechanical system. For this system, do the following.

- Determine all equilibrium points for the system.
- For each equilibrium configuration  $q_0$ , compute  $M_L(q_0)$ ,  $C_L(q_0)$ , and  $K_L(q_0)$ .
- For each equilibrium point  $q_0$ , determine whether it is linearly stable.
- For each equilibrium point  $q_0$ , determine whether it is stable.

E3.5 Let  $Q = \mathbb{R}^2$ . Since  $Q$  is a vector space, it makes sense to ask whether a Lagrangian on  $L$  is linear. For each of the following hyperregular Lagrangians, answer the following questions:

- Is the Lagrangian a Lagrangian for a linear system?
- Is the Lagrangian a Lagrangian for a simple mechanical system? If so, write the Riemannian metric and the potential function.
- Find all equilibrium points for the Lagrangian.
- Find the linearisation about each of the equilibrium points.

5. Is the linearisation stable? State which of the results from this chapter you applied to obtain your answer.
6. Is it possible, using the results given in this chapter, to determine whether the equilibrium is stable under the actual dynamics of the system? If it is possible, state the result which is applicable. If it is not possible,
  - (a) state why none of the results in this chapter apply, and
  - (b) see if you can figure out using your ingenuity whether the equilibrium point is stable (you can play with the equations, or try numerical simulation).

- (a)  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + (1 + x^2 + y^2)v_y^2) - \frac{1}{2}(x^2 - y^2)$ .
- (b)  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 - v_y^2) - \frac{1}{2}(x^2 + 4y^2)$ .
- (c)  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2) - \frac{1}{2}(xv_y - yv_x) + \frac{1}{2}(x^2 + y^2)$ .
- (d)  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2) + \frac{1}{2}(xv_x + yv_y) - \frac{1}{2}(x^2 + y^2)$ .
- (e)  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + 2v_y^2) - \frac{1}{2}(x^2 + \frac{1}{2}x^4 + y^2)$ .
- (f)  $L(x, y, v_x, v_y) = \frac{1}{2}(-v_x^2 + v_y^2) - \frac{1}{2}(y^2 - x^2 - \frac{1}{2}x^4)$ .
- (g)  $L(x, y, v_x, v_y) = \frac{1}{2}(2v_x^2 - v_y^2) - \frac{1}{2}(2x^2 + xy + 3y^2 - xy^2)$ .
- (h)  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + \frac{1}{2}v_x^4 + 2v_y^2) - \frac{1}{2}xy$ .
- (i)  $L(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2) - \frac{1}{2}(x + y^2)$ .

E3.6 For the following single degree of freedom Lagrangians, draw the corresponding phase portraits.

- (a)  $L(q, v) = \frac{1}{2}v^2$ .
- (b)  $L(q, v) = \frac{1}{2}v^2 - q$ .
- (c)  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{2}q^2 + \frac{1}{3}q^3$ .
- (d)  $L(q, v) = \frac{1}{2}v^2 + \frac{1}{2}q^2 - \frac{1}{3}q^3$ .
- (e)  $L(q, v) = \frac{1}{2}v^2 - \frac{1}{3}q^3$ .

E3.7 Take a typical hardcover book (by typical, it is meant that it has three quite distinct principal inertias, as most books do—avoid square books). You may wish to wrap an elastic band around it to keep it closed. Now flip the book once around the axis corresponding to its intermediate principal inertia.

- (a) Does the book return to its initial configuration?

We shall now endeavour to explain the phenomenon you observed by using the equations (3.7) which describe the evolution of angular momentum of the body.

- (b)

E3.8 A car sits at rest with its passenger door open at an angle of  $90^\circ$ . At time  $t = 0$  the driver accelerates with constant acceleration  $a$ .

- (a) Write down the Lagrangian associated to the door.
- (b) *Without writing down the Euler-Lagrange equations*, write an expression in terms of an integral for the time taken for the door to close.

**Hint:** *Is energy conserved for the Lagrangian you wrote down?*

E3.9 For each of the systems of Exercise E3.6, give a sketch of the phase portrait which would result from the addition of a dissipative force of the form  $F(q, v) = -\delta v$  for some smallish  $\delta$ . Interpret these phase portraits in light of the discussion of Section 3.5.1.

E3.10 This exercise is a continuation of Exercise E2.2. Suppose a particle of mass  $m$  is constrained to move on a cylinder of radius  $r$  in the absence of external forces.



- (a) In the coordinates used in Exercise E2.2, provide an expression for the kinetic energy Riemannian metric.
- (b) In these same coordinates, write the Euler-Lagrange equations.
- (c) In these same coordinates, determine the Christoffel symbols and write the geodesic equations for the kinetic energy Riemannian metric.
- (d) What do the geodesics look like from the previous part of the question.

E3.11 Let  $Q = \mathbb{R}_+ \times \mathbb{R}$  (here  $\mathbb{R}_+$  is the set of positive real numbers). Use standard Cartesian coordinates  $(x, y)$  for  $Q$ . On  $Q$  consider the Riemannian metric  $g$  specified in the given coordinates by

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

- (a) Determine the Christoffel symbols and write the geodesic equations for this Riemannian metric.
- (b) Show that geodesics fall into two classes:
  1. vertical lines;
  2. arcs of circles lying in  $Q$  with their centres on the  $x$ -axis.

Make sure that you demonstrate that these are the *only* possibilities.

**Hint:** For the second type of geodesic, try to substitute the parameterised equation for a circle into the geodesic equations, and see what comes out.

- (c) Using the discussion of Section 3.7.2, comment on the geodesics you have found as they relate to a simple mechanical system with a nonzero potential function. That is, relate the geodesics to a physical system with a nonzero potential.

E3.12 This problem refers to Example 3.8.11 for the rolling disk. Do the following.

- (a) Obtain explicit solutions to the rolling disk equations using the final form of the geodesic equations derived in Example 3.8.11.
- (b) Describe the resulting motions of the disk.
- (c) Verify that the energy of the system is constant along solutions. Should you have been able to predict this?
- (d) Are there any other functions of the configurations and their velocities which you can see are conserved?

E3.13 This exercise refers to Exercise E2.36. In part (f) of that exercise you obtained equations which govern the motion of the ball in terms of the ball's spatial angular velocity and the position of contact of the ball with the table.

- (a) Eliminate the spatial angular velocity from these equations to get equations only in the point of contact of the ball with the table.
- (b) Solve the equations from the previous part of the problem (this should be easy; if it is not, you have made an error), and plot a typical path for the point of contact as seen by an inertial observer.
- (c) Show explicitly that the energy of the ball is not constant along solutions. Should you have been able to guess that this was the case?

Now we add some dissipation to the problem. We consider the forces which might arise from air resistance as the surface of the ball rotates. A simple model for such an effect would be the addition of a force of the form  $\mathbf{F} = -\delta\boldsymbol{\omega}$  where  $\boldsymbol{\omega}$  is the spatial angular velocity.

- (d) Derive the equations of motion in terms of the spatial angular velocity and point of contact of the ball with the surface of the table, with the addition of the air resistance force.
- (e) As is part (a), eliminate the spatial angular velocity from your equations to get equations only in the position of the point of contact.
- (f) Observe that the equations you have are linear. Obtain the characteristic equation. Plug in some numbers, and numerically find the roots of this equation. What is the effect of the dissipative force on the motion of the ball?

## Chapter 4

# An introduction to control theory for Lagrangian systems

In this chapter we undertake rather a hodgepodge of control theoretic topics. The systematic investigation of control theory for mechanical systems is in its infancy, so one cannot expect much in the way of unity here. What's more, what *is* known is often beyond the scope of our presentation. What we try to do is indicate how the special structure of Lagrangian systems plays out when one employs some of the standard control theoretic techniques.

The most pleasure is apt to be derived from the material in this section by those who have some background in control. However, much of the material is self-contained. An excellent introduction to the mathematical foundations of control is the book by [Sontag \[1998\]](#). We refer to [\[Spong and Vidyasagar 1989\]](#) for more detail on the material in Section 4.2 on robot control. In Section 4.4 it would be helpful if the reader had some background in linear systems theory, and a good source for this is [\[Brockett 1970\]](#). The subject of nonlinear control is one with which an average reader may not be familiar. A good text is [\[Nijmeijer and van der Schaft 1990\]](#). The subject of passivity methods is dealt with in the book [\[Ortega, Loria, Nicklasson, and Sira-Ramirez 1998\]](#). We should blatantly state that the reader who forms an opinion about the subject matter in control theory based upon the material in this chapter will have a rather non-conventional view of the subject.

### 4.1 The notion of a Lagrangian control system

In this section we shall simply describe a general control problem in Lagrangian mechanics. Our setting here will be far too general for us to be able to say much about it, but it forms the basis for the specialising as we do in subsequent sections.

Let  $L$  be a Lagrangian on a configuration space  $Q$ . We consider a collection  $\mathcal{F} = \{F^1, \dots, F^m\}$  of forces on  $Q$ . The idea is that each one of these forces represents a direction in which we have control over our system. The total force which we command is then a linear combination of the forces  $F^1, \dots, F^m$ . Following the convention in control theory, we denote the coefficients in this linear combination by  $u_1, \dots, u_m$ . Thus the governing differential equations in a set of coordinates are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = u_a F_i^a, \quad i = 1, \dots, n. \quad (4.1)$$

We shall call a pair  $(L, \mathcal{F})$  as above a **Lagrangian control system**. Let us for simplicity make the assumption that the subspace of  $T_q Q$  spanned by  $F^1(t, v_q), \dots, F^m(t, v_q)$  is independent of  $q$ . If for each  $(t, v_q) \in \mathbb{R} \times TQ$  we have

$$\text{span}_{\mathbb{R}} \{F^1(t, v_q), \dots, F^m(t, v_q)\} = T_q Q,$$

we say that the system is **fully actuated**. Otherwise we say that it is **underactuated**. Generally, the issues surrounding the control of fully actuated systems are comparatively straightforward. Underactuated systems are quite challenging, on the other hand.

When one looks at equation (4.1) as a control system, the types of questions one is interested in differ from those one considers when one is interested in dynamics. One wishes, in control, to design the coefficients  $u_1, \dots, u_m$  so that the system behaves in a desired manner. If one designs the controls to be functions only of time,  $u_1(t), \dots, u_m(t)$ , one says the controls are **open loop**. The idea with open loop control is that you have analysed your problem and determined that the specified controls, when substituted into (4.1) will accomplish the desired objective. The problem is that this procedure is often not robust. It will suffer in the presence of model inaccuracies, uncertainty in initial conditions, etc. Thus one often wishes to have the controls somehow keep track of the current state using feedback, as well as perhaps the time, say,  $u_1(t, v_q), \dots, u_m(t, v_q)$ , so that if one is not quite doing what is expected, then the control can compensate. This is called a **closed loop** control.

Some examples of control problems are as follows.

1. *The reconfiguration problem:* Here one wishes to start the system at time  $t = 0$  at a state  $v_{q_1} \in TQ$  and design the controls  $u_1, \dots, u_m$  so that at time  $t = T > 0$  the system, upon solving the equation (4.1), will arrive at the state  $v_{q_2}$ .
2. *The trajectory tracking problem:* For this problem, one has a prescribed curve  $t \mapsto c_{\text{des}}(t) \in Q$  one wishes to follow. The objective is to design the controls so that one follows this curve, or follows it as closely as possible.
3. *The point stabilisation problem:* Here one has a point  $q_0 \in Q$  and one wishes to design the controls  $u_1, \dots, u_m$ , probably closed loop, so that  $q_0$  is an asymptotically stable equilibrium point for the equations (4.1).

Sometimes one will also have outputs for the system. These will be in the form of functions  $h^\alpha: \mathbb{R} \times TQ \rightarrow \mathbb{R}$ ,  $\alpha = 1, \dots, s$ . In such cases, one becomes not so much interested in the states in  $TQ$  as in the value of the outputs. Thus to the above problems, one may wish to add the following.

4. *The output tracking problem:* The problem here is something like the trajectory tracking problem, except that one is interested in making the *outputs* follow a desired path. Thus one has  $s$  desired functions of time,  $h_{\text{des}}^1, \dots, h_{\text{des}}^s$ , and one wishes to design the controls  $u_1, \dots, u_m$  so that the output follows these desired outputs, or at least does so as nearly as possible.
5. *The output stabilisation problem:* Here one has a fixed operating point for the output, say  $\mathbf{h}_0 \in \mathbb{R}^s$ , and one wishes to design controls in such a manner that outputs which start near  $\mathbf{h}_0$  end up at  $\mathbf{h}_0$  as  $t \rightarrow \infty$ . Thus  $\mathbf{h}_0$  becomes an asymptotically stable fixed point, in some sense.

## 4.2 “Robot control”

We place the title of this section in inverted commas because the subject of robot control is large, and there are many issues involved. We will only address the most basic ones. For a more thorough overview, see Chapter 4 of [Murray, Li, and Sastry 1994], or the book [Spong and Vidyasagar 1989].

Before we begin with the control theory proper, let us state what we mean by a system to which we may apply “robot control.” This is a rather limited class of problems. To be

precise, a Lagrangian control system  $(L, \mathcal{F})$  on  $Q$  is a **robotic control system** when

1.  $Q = \mathbb{R}^n$ , and the canonical coordinate chart has preference over all others,
2.  $L$  is time-independent and hyperregular,
3.  $(L, \mathcal{F})$  is fully actuated, and
4. unlimited force to the actuators is available.

Of course, the statement that the canonical coordinate chart is “preferred” is rather nebulous. Its presence as an assumption is to justify certain coordinate-dependent constructions made during the ensuing discussion.

It is perhaps useful to give some examples of robotic control systems, to ensure that our assumptions include what one might certainly wish to include in this class. First of all, consider an open chain planar robot with  $n$  links as depicted in Figure 4.1. By measuring

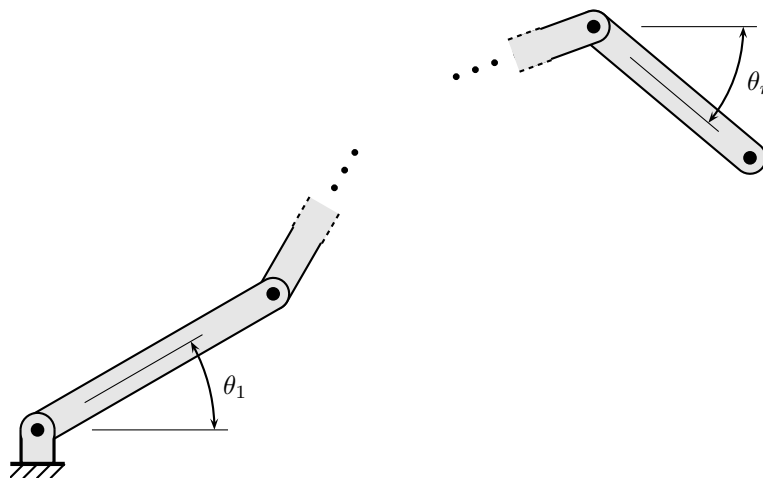


Figure 4.1 An open chain robot

the angle of each of link in some manner, the configuration space of such a robot is  $Q = \mathbb{T}^n$ . Note that  $\mathbb{T}^n \neq \mathbb{R}^n$ , so we seem to violate our first assumption of what a robotic control system should be. However, one typically closes one’s eyes to this, and pretends that  $\mathbb{S}^1$  is actually  $\mathbb{R}^1$ . That is, one allows angles to continually grow, rather than resetting them at  $2\pi$ . This may not be a problem, but one should be aware that, in fact, most robots do not quite satisfy the assumption 1 for robotic control systems. As for assumption 2, it is not a problem since the Lagrangians for the vast majority of, if not all, robotic control systems have Lagrangians which are derived from a simple mechanical system. The final assumption 3 is simply an assumption which may or may not hold in a given application. Most robots as would be found on a plant floor doing tasks such as welding or painting are typically fully actuated. More sophisticated “mobile” robots typically will not be fully actuated. Fully actuated systems, with the addition of the assumption 4, have the capacity to follow arbitrary paths in configuration space, making control somewhat simpler.

**4.2.1 The equations of motion for a robotic control system** Let us use the assumptions we give above to write down the form of the equations we will use for these systems. We let  $\mathbf{x} = (x^1, \dots, x^n)$  denote standard coordinates for  $Q = \mathbb{R}^n$ . Since the system is fully actuated, we may as well suppose that the input forces  $\{F^1, \dots, F^n\}$  are simply  $F^i = dx^i$ ,

$i = 1, \dots, n$ . Let us denote this set of forces by  $\mathcal{F}_{\text{can}}$ . The Euler-Lagrange equations are then

$$\frac{\partial^2 L}{\partial v^i \partial v^j} \ddot{x}^j + \frac{\partial^2 L}{\partial v^i \partial x^j} \dot{x}^j - \frac{\partial L}{\partial x^i} = u_i, \quad i = 1, \dots, n.$$

If we define a symmetric invertible matrix function  $\mathbf{M}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  to have components  $\frac{\partial^2 L}{\partial v^i \partial v^j}$ ,  $i, j = 1, \dots, n$ , and if we define a map  $\mathbf{N}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$\mathbf{N}(\mathbf{x}, \mathbf{v})_i = \frac{\partial^2 L}{\partial v^i \partial x^j} \dot{x}^j - \frac{\partial L}{\partial x^i}, \quad i = 1, \dots, n,$$

then the equations of motion can be written

$$\mathbf{M}(\mathbf{x}, \dot{\mathbf{x}}) \ddot{\mathbf{x}} + \mathbf{N}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{u}. \quad (4.2)$$

Let us specialise the notation further to the situation when the Lagrangian  $L$  is derived from a simple mechanical system  $(Q, g, V)$ , which will most often be the case. In this case one readily sees, first of all, that  $\mathbf{M}$  is independent of  $\mathbf{v}$ , and indeed is simply the matrix of components for the Riemannian metric  $g$  in the standard coordinates for  $\mathbb{R}^n$ . The term  $\mathbf{N}$  can also be effectively broken down. We define a map  $\mathbf{C}: \mathbb{R}^n \rightarrow L(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$  by

$$\mathbf{C}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)_i = \frac{1}{2} \left( \frac{\partial \mathbf{M}_{ij}}{\partial x^k} + \frac{\partial \mathbf{M}_{ik}}{\partial x^j} - \frac{\partial \mathbf{M}_{jk}}{\partial x^i} \right) v_1^j v_2^k.$$

Thus  $\mathbf{C}$  is related in a simple way to the Christoffel symbols for the Riemannian metric  $g$  in the standard coordinates. If  $dV$  is the vector of partial derivatives of the potential function  $V$ , then the equations (4.2) specialise to the equations

$$\mathbf{M}(\mathbf{x}) \ddot{\mathbf{x}} + \mathbf{C}(\mathbf{x})(\dot{\mathbf{x}}, \dot{\mathbf{x}}) + dV = \mathbf{u}. \quad (4.3)$$

The term  $\mathbf{C}(\mathbf{x})(\dot{\mathbf{x}}, \dot{\mathbf{x}})$  is often given the name ‘‘Coriolis forces.’’ Note that we violate our code of coordinate invariance in writing these equations. Indeed, in Remark 2.4.12 we explicitly advised *against* writing equations in the form (4.3)! However, we justify what we have done by saying that we have used our assumption 1 for a robotic control system where the standard coordinates for  $Q$  are preferred. Also, since we are supposed to be talking about robot control, we may as well do as most robot control folks do, and write the equations as (4.2) or (4.3).

Before we move on, let us give a useful property of the notation used in (4.3).

**4.2.1 Lemma** *With  $\mathbf{M}$  and  $\mathbf{C}$  as used in (4.3) we have  $\dot{\mathbf{x}}^t \dot{\mathbf{M}} \dot{\mathbf{x}} - 2 \dot{\mathbf{x}}^t (\mathbf{C}(\mathbf{x})(\dot{\mathbf{x}}, \dot{\mathbf{x}})) = 0$ .*

*Proof* We compute

$$\dot{\mathbf{M}}_{ij} = \frac{\partial \mathbf{M}_{ij}}{\partial x^k} \dot{x}^k.$$

It is then a straightforward computation using the definition of  $\mathbf{C}$  to verify that

$$\dot{\mathbf{x}}^t \dot{\mathbf{M}} \dot{\mathbf{x}} - 2 \dot{\mathbf{x}}^t (\mathbf{C}(\mathbf{x})(\dot{\mathbf{x}}, \dot{\mathbf{x}})) = \left( \frac{\partial \mathbf{M}_{jk}}{\partial x^i} \dot{x}^k - \frac{\partial \mathbf{M}_{ik}}{\partial x^j} \dot{x}^k \right) \dot{x}^i \dot{x}^j.$$

Since the expression in the brackets is skew-symmetric with respect to the indices  $i$  and  $j$ , the result follows.  $\blacksquare$

Lewis [1997] gives an interpretation of this result in terms of conservation of energy.

**4.2.2 Feedback linearisation for robotic systems** The title of this section is a bit pretentious. What we talk about in this section is sometimes referred to as the “computed torque” control law. It is, however, an example of a more general technique known as feedback linearisation, and this is why the title is as it is.

The setup for the computed torque control law, like that for the PD control law in the next section, is that one has a robotic control system  $(L, \mathcal{F}_{\text{can}})$  on  $Q$ , and a desired trajectory  $t \mapsto \mathbf{x}_{\text{des}}(t)$  for the configuration of the system. If the actual robot is evolving according to a curve  $t \mapsto \mathbf{x}(t)$  in configuration space, the **error**,  $t \mapsto \mathbf{e}(t)$ , is then the difference between where one is and where one wants to be:

$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}_{\text{des}}(t).$$

The idea is that one wishes to design a control law  $\mathbf{u}$  for (4.2) so that the error goes to zero as  $t \rightarrow \infty$ . The **computed torque control law** is defined by

$$\mathbf{u}_{\text{CT}} = \mathbf{M}(\mathbf{x}, \dot{\mathbf{x}})\ddot{\mathbf{x}}_{\text{des}} + \mathbf{N}(\mathbf{x}, \dot{\mathbf{x}}) - \mathbf{M}(\mathbf{x}, \dot{\mathbf{x}})(\mathbf{K}_v \dot{\mathbf{e}} + \mathbf{K}_p \mathbf{e}),$$

where  $\mathbf{K}_v, \mathbf{K}_p \in \mathbb{R}^{n \times n}$  are matrices, designed so as to accomplish a stated objective. Let us first get the form of the error dynamics using the computed torque control law.

**4.2.2 Proposition** *Let  $t \mapsto \mathbf{x}_{\text{des}}(t)$  be a curve in  $\mathbb{R}^n$ . With  $\mathbf{u}(t) = \mathbf{u}_{\text{CT}}(t)$  in (4.2), the error  $\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}_{\text{des}}(t)$  satisfies the differential equation*

$$\ddot{\mathbf{e}}(t) + \mathbf{K}_v \dot{\mathbf{e}}(t) + \mathbf{K}_p \mathbf{e}(t) = \mathbf{0}.$$

*Proof* Substituting  $\mathbf{u}_{\text{CT}}$  into (4.2), some straightforward manipulation yields

$$\mathbf{M}(\mathbf{x}, \dot{\mathbf{x}})(\ddot{\mathbf{e}}(t) + \mathbf{K}_v \dot{\mathbf{e}}(t) + \mathbf{K}_p \mathbf{e}(t)) = \mathbf{0}.$$

The result follows since  $\mathbf{M}(\mathbf{x}, \mathbf{v})$  is invertible for all  $\mathbf{v}$  and  $\mathbf{v}$ . ■

Now it is a simple matter to give a general form for  $\mathbf{K}_v$  and  $\mathbf{K}_p$  so that the error dynamics go to zero at  $t \rightarrow \infty$ .

**4.2.3 Proposition** *If  $\mathbf{K}_v$  and  $\mathbf{K}_p$  are symmetric and positive-definite, then the error dynamics in Proposition 4.2.2 satisfy  $\lim_{t \rightarrow \infty} \mathbf{e}(t) = \mathbf{0}$ .*

*Proof* Writing the differential equation for the error in first-order form gives

$$\begin{pmatrix} \dot{\mathbf{e}} \\ \mathbf{f} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{K}_p & \mathbf{K}_v \end{bmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix},$$

where  $\mathbf{f} = \dot{\mathbf{e}}$ . The result will follow if we can show that all eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{K}_p & \mathbf{K}_v \end{bmatrix}$$

have negative real part. Let  $\lambda \in \mathbb{C}$  be an eigenvalue with eigenvector  $(\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{C}^n \times \mathbb{C}^n$ . We must then have

$$\begin{pmatrix} \mathbf{v}_2 \\ -\mathbf{K}_p \mathbf{v}_1 - \mathbf{K}_v \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \lambda \mathbf{v}_1 \\ \lambda \mathbf{v}_2 \end{pmatrix}.$$

Let  $\langle \cdot, \cdot \rangle$  be the standard inner product on  $\mathbb{C}^n$ :  $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle = \bar{\mathbf{z}}_1^t \mathbf{z}_2$ , where  $\bar{\cdot}$  means complex conjugation. We then have

$$\begin{aligned} \langle \bar{\lambda} \mathbf{z}_1, \lambda \mathbf{v}_1 \rangle &= \langle \bar{\lambda} \mathbf{z}_1, \mathbf{v}_2 \rangle \\ &= \langle \mathbf{v}_1, \lambda \mathbf{v}_2 \rangle \\ &= \langle \mathbf{v}_1, -\mathbf{K}_p \mathbf{v}_1 - \mathbf{K}_v \mathbf{v}_2 \rangle \\ &= -\langle \mathbf{v}_1, \mathbf{K}_p \mathbf{v}_1 \rangle - \lambda \langle \mathbf{v}_1, \mathbf{K}_v \mathbf{v}_1 \rangle. \end{aligned}$$

We also have  $\langle \bar{\lambda} \mathbf{z}_1, \lambda \mathbf{v}_1 \rangle = \lambda^2 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle$ , thus giving us

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle \lambda^2 + \langle \mathbf{v}_1, \mathbf{K}_v \mathbf{v}_1 \rangle \lambda + \langle \mathbf{v}_1, \mathbf{K}_p \mathbf{v}_1 \rangle = 0.$$

Now note that if  $\mathbf{v}_1 = \mathbf{0}$  then we also have  $\mathbf{v}_2 = \mathbf{0}$ . Since eigenvectors are necessarily nonzero, this implies that  $\mathbf{v}_1 \neq \mathbf{0}$ . Thus  $\lambda$  satisfies the equation

$$a\lambda^2 + b\lambda + c = 0,$$

where  $a > 0$  since  $\mathbf{v}_1 \neq \mathbf{0}$ , and where  $b, c > 0$  since  $\mathbf{K}_p$  and  $\mathbf{K}_v$  are symmetric and positive-definite. We therefore conclude that  $\lambda$  has strictly negative real part, as desired. ■

The computed torque control law has the advantage, therefore, of being able to track accurately any desired reference trajectory. To do so, however, may require significant actuator forces. Nevertheless, it is an effective control strategy.

**4.2.3 PD control** It is possible to use some simple ideas from classical control for robotic systems, and perhaps the simplest controller which might be effective is PD control, or *Proportional-Derivative* control. Here one simply defines the control to be the sum of two terms, one proportional to the error, and one proportional to the derivative of the error. That is, for a given desired trajectory  $t \mapsto \mathbf{x}_{\text{des}}(t)$  in  $Q$ , we define

$$\mathbf{u}_{\text{PD}}(t) = -\mathbf{K}_p \mathbf{e}(t) - \mathbf{K}_v \dot{\mathbf{e}}(t),$$

for symmetric positive-definite matrices  $\mathbf{K}_p, \mathbf{K}_v \in \mathbb{R}^{n \times n}$ , and where  $\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}_{\text{des}}(t)$ , as before. The PD control law is not as effective as the computed torque controller for following arbitrary reference trajectories. This is a general failing of PD controllers, often alleviated by implementing integral control. However, the following result indicates that a PD controller will stabilise a robotic control system to a desired configuration when the Lagrangian is derived from a simple mechanical system. Since almost all robotic systems are in practice of this form, this is not at all a significant restriction.

**4.2.4 Proposition** *Let  $(L, \mathcal{F}_{\text{can}})$  be a robotic control system with  $L$  the Lagrangian for the simple mechanical system  $(Q, g, 0)$  with zero potential energy. If  $t \mapsto \mathbf{x}_{\text{des}}(t) = \mathbf{x}_0$  is a constant desired trajectory, then taking  $\mathbf{u}(t) = \mathbf{u}_{\text{PD}}(t)$  in (4.3) gives  $\lim_{t \rightarrow \infty} \mathbf{e}(t) = \mathbf{0}$ .*

*Proof* We will use the LaSalle Invariance Principle to show that  $\mathbf{x}_0$  is an asymptotically stable equilibrium point for (4.3) if we take  $\mathbf{u} = \mathbf{u}_{\text{PD}}$ . First note that the equations (4.3) with  $\mathbf{u} = \mathbf{u}_{\text{PD}}$  are

$$\mathbf{M}(\mathbf{x})\ddot{\mathbf{x}} + \mathbf{C}(\mathbf{x})(\dot{\mathbf{x}}, \dot{\mathbf{x}}) + \mathbf{K}_v \dot{\mathbf{x}} + \mathbf{K}_p(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}. \quad (4.4)$$



Now define a Liapunov function on  $Q$  by

$$G(\mathbf{x}, \mathbf{v}) = \frac{1}{2} \mathbf{v}^t \mathbf{M}(\mathbf{x}) \mathbf{v} + \frac{1}{2} (\mathbf{x}^t - \mathbf{x}_0^t) \mathbf{K}_p (\mathbf{x} - \mathbf{x}_0).$$

We compute

$$\begin{aligned} \dot{G}(\mathbf{x}, \dot{\mathbf{x}}) &= \dot{\mathbf{x}} \mathbf{M}(\mathbf{x}) \ddot{\mathbf{x}} + \frac{1}{2} \dot{\mathbf{x}} \dot{\mathbf{M}}(\mathbf{x}) \dot{\mathbf{x}} + \dot{\mathbf{x}} \mathbf{K}_p (\mathbf{x} - \mathbf{x}_0) \\ &= -\dot{\mathbf{x}} (\mathbf{C}(\mathbf{x}) (\dot{\mathbf{x}}, \dot{\mathbf{x}})) + \frac{1}{2} \dot{\mathbf{x}} \dot{\mathbf{M}}(\mathbf{x}) \dot{\mathbf{x}} - \dot{\mathbf{x}}^t \mathbf{K}_v \dot{\mathbf{x}} \\ &= -\dot{\mathbf{x}}^t \mathbf{K}_v \dot{\mathbf{x}}. \end{aligned}$$

Here we have used Lemma 4.2.1. Note that the closed-loop equations (4.4) are forced Euler-Lagrange equations with Lagrangian

$$\tilde{L}(\mathbf{x}, \mathbf{v}) = L(\mathbf{x}, \mathbf{v}) - \frac{1}{2} (\mathbf{x}^t - \mathbf{x}_0^t) \mathbf{K}_p (\mathbf{x} - \mathbf{x}_0).$$

and with external force  $\mathbf{F}(\mathbf{x}, \mathbf{v}) = \mathbf{K}_v \mathbf{v}$ . This force is strictly dissipative so we apply the LaSalle Invariance Principle. The LaSalle Invariance Principle states that the system will tend towards the set where  $\mathbf{F}(\mathbf{x}, \mathbf{v}) \cdot \mathbf{v} = 0$ . However, this set is comprised of those points in  $TQ$  where  $\mathbf{v} = \mathbf{0}$ . Now we note, again with the LaSalle Invariance Principle in mind, that the largest invariant set for  $L$  for which  $\mathbf{v} = \mathbf{0}$  is the set of critical points for  $\tilde{L}$  of the form  $(\bar{\mathbf{x}}, \mathbf{0})$ . However, for such critical points  $\bar{\mathbf{x}}$  must be a critical point for the potential function

$$\tilde{V}(\mathbf{x}) = \frac{1}{2} (\mathbf{x}^t - \mathbf{x}_0^t) \mathbf{K}_p (\mathbf{x} - \mathbf{x}_0).$$

Since  $\mathbf{x}_0$  is the only critical point for  $\tilde{V}$ , the result follows from Corollary 3.5.7. ■

Murray, Li, and Sastry [1994] provide an extension of the PD control we have defined here which allows one to asymptotically track general reference trajectories. Also, the ideas behind what we suggest here can be extended in a geometric framework which alleviates some of the inherent restrictions of the PD control we suggest in this section. We refer the reader to [Bullo and Murray 1999, Koditschek 1989] for ideas along these lines.

## 4.3 Passivity methods

Sorry, not this year. See [Ortega, Loria, Nicklasson, and Sira-Ramirez 1998].

## 4.4 Linearisation of Lagrangian control systems

The material in the previous section was geared towards fully actuated systems, and a special class of these, even. When confronted with a system with fewer controls than degrees of freedom, things become more difficult. For certain control problems, notably the point stabilisation problem, a good thing to try first is linearisation, just as when studying dynamics.

**4.4.1 The linearised system** When linearising (4.1), one wishes to linearise both with respect to control and state. In Section 3.2 we already linearised the left hand side of (4.1),

so it only remains to linearise the terms with the control. This, however, is trivial since this term is already linear in the control. Therefore, when linearising, one need only substitute the value of the equilibrium point into the forces. With this as motivation, we make the following definition.

**4.4.1 Definition** Let  $(L, \mathcal{F})$  be a Lagrangian control system on  $Q$  and let  $q_0$  be an equilibrium point for  $L$ . The **linearisation** of  $(L, \mathcal{F})$  is the quadruple  $(M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))$  where  $M_L(q_0)$ ,  $C_L(q_0)$ , and  $K_L(q_0)$  are as defined in Proposition 3.2.10, and where  $\mathcal{F}(q_0) = \{F^1(0_{q_0}), \dots, F^m(0_{q_0})\} \subset T_{q_0}^*Q$ .  $\square$

Corresponding to the linearisation is the following linear control system on  $T_{q_0}Q$ :

$$M_L(q_0)\ddot{q}(t) + C_L(q_0)\dot{q}(t) + K_L(q_0)q(t) = u_a F^a(0_{q_0}),$$

where  $t \mapsto q(t)$  is a curve on  $T_{q_0}Q$ . (We make a nasty abuse of notation here, so beware: we let  $q$  denote both points in  $Q$  and points in  $T_{q_0}Q$ .) In the same way as we did when linearising near equilibria in studying dynamics, let us suppose that  $L$  is hyperregular so that  $M_L^\sharp(q_0)$  is invertible. In this case, let us define a linear map  $B_L(q_0): \mathbb{R}^m \rightarrow T_{q_0}Q$  by

$$B_L(q_0)(u_1, \dots, u_m) = u_1 M_L^\sharp(q_0)(F^1(0_{q_0})) + \dots + u_m M_L^\sharp(q_0)(F^m(0_{q_0})).$$

We then arrive at the linear control system on  $T_{q_0}Q \oplus T_{q_0}Q$  given by

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & \text{id}_V \\ -M_L^\sharp(q_0) \circ K_L^\flat(q_0) & -M_L^\sharp(q_0) \circ C_L^\flat(q_0) \end{bmatrix} \begin{pmatrix} q \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ B_L(q_0) \end{pmatrix} u.$$

This is a linear control system in the best tradition. That is to say, it is a time-invariant system of the form

$$\dot{x} = Ax + Bu$$

where

$$\begin{aligned} x &= \begin{pmatrix} q \\ v \end{pmatrix} \\ A &= A_L(q_0) = \begin{bmatrix} 0 & \text{id}_V \\ -M_L^\sharp(q_0) \circ K_L^\flat(q_0) & -M_L^\sharp(q_0) \circ C_L^\flat(q_0) \end{bmatrix} \\ B &= B_{\mathcal{F}}(q_0) \triangleq \begin{pmatrix} 0 \\ B_L(q_0) \end{pmatrix}. \end{aligned}$$

There is an exorbitant amount of literature on such systems [e.g., Brockett 1970], and we only address a few of the more basic notions for such systems. Our emphasis, like it was when we were dealing with dynamics, is on deciding the relationship between the linearised system and the actual system near the equilibrium point in question.

**4.4.2 Controllability of the linearised system** A first basic question deals with controllability. The reader will recall that a linear control system

$$\dot{x} = Ax + Bu,$$

where  $t \mapsto x(t)$  is a curve in a vector space  $V$ , is **controllable** if for any  $x_1, x_2 \in V$  there exists a control  $u: [0, T] \rightarrow \mathbb{R}^m$  so that if the initial condition is  $x(0) = x_1$ , then  $x(T) = x_2$ .

In brief, the system is controllable if it can be steered from any state to any other state. The **Kalman rank condition** says that the system is controllable if and only if the matrix

$$[ \mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{N-1}\mathbf{B} ]$$

has rank equal to  $N = \dim(V)$ , where  $\mathbf{A} \in \mathbb{R}^{N \times N}$  and  $\mathbf{B} \in \mathbb{R}^{N \times m}$  are the matrix representations of  $A$  and  $B$  in some basis for  $V$ .

Let us see how this transfers to our Lagrangian setting. The first result we state is the most general, and to state it requires some notation. Let  $\mathcal{I}$  be the collection of multi-indices in  $\{1, 2\}$ . Thus an element  $I \in \mathcal{I}$  has the form  $I = (i_1, \dots, i_k)$  where  $i_a \in \{1, 2\}$  for  $a = 1, \dots, k$ . For a multi-index  $I$ , we write  $|I|$  for its length. Thus  $|(i_1, \dots, i_k)| = k$ . For a multi-index  $I \in \mathcal{I}$  we denote by  $|I|_1$  the number of 1's in  $I$ , and by  $|I|_2$  the number of 2's in  $I$ . Note that  $|I|_1 + |I|_2 = |I|$ . Now suppose that we choose a basis for  $T_{q_0}Q$  and denote by  $\mathbf{A}_1(q_0), \mathbf{A}_2(q_0) \in \mathbb{R}^{n \times n}$  the matrices of  $M_L^\sharp(q_0) \circ K_L^b(q_0)$  and  $M^\sharp(q_0) \circ C_L^b(q_0)$ , respectively, with respect to this basis. Now for  $k \in \mathbb{Z}_+$  define

$$\mathbb{R}^{n \times n} \ni \mathbf{M}^{(k)}(q_0) = \begin{cases} \sum_{j=k/2}^k \sum_{\substack{|I|=j \\ |I|_1=k-j}} \mathbf{A}_{i_1}(q_0) \cdots \mathbf{A}_{i_j}(q_0), & k \text{ even} \\ \sum_{j=(k+1)/2}^k \sum_{\substack{|I|=j \\ |I|_1=k-j}} \mathbf{A}_{i_1}(q_0) \cdots \mathbf{A}_{i_j}(q_0), & k \text{ odd.} \end{cases}$$

Let us give the first few of the matrices  $\mathbf{M}^{(k)}(q_0)$  so that the reader might begin to understand the cumbersome notation. We determine that

$$\begin{aligned} \mathbf{M}^{(1)}(q_0) &= \mathbf{A}_2(q_0) \\ \mathbf{M}^{(2)}(q_0) &= \mathbf{A}_1(q_0) + \mathbf{A}_2(q_0)^2 \\ \mathbf{M}^{(3)}(q_0) &= \mathbf{A}_1(q_0)\mathbf{A}_2(q_0) + \mathbf{A}_2(q_0)\mathbf{A}_1(q_0) + \mathbf{A}_2(q_0)^3 \\ \mathbf{M}^{(4)}(q_0) &= \mathbf{A}_1(q_0)^2 + \mathbf{A}_1(q_0)\mathbf{A}_2(q_0)^2 + \mathbf{A}_2(q_0)\mathbf{A}_1(q_0)\mathbf{A}_2(q_0) + \mathbf{A}_2(q_0)^2\mathbf{A}_1 + \mathbf{A}_2(q_0)^4. \end{aligned}$$

Hopefully this is enough to enable the reader to see how one proceeds, using the definitions for  $\mathbf{M}^{(k)}(q_0)$ .  $k > 0$ .

With this notation in hand, we may state our main result concerning controllability of linearisations.

**4.4.2 Theorem** *Let  $(L, \mathcal{F})$  be a Lagrangian control system on  $Q$  with  $L$  a hyperregular Lagrangian. Let  $q_0$  be an equilibrium point for  $L$  and let  $(M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))$  be the linearisation of  $(L, \mathcal{F})$  at  $q_0$ . The linearised system*

$$\dot{\mathbf{x}} = \mathbf{A}_L(q_0)\mathbf{x} + \mathbf{B}_{\mathcal{F}}(q_0)\mathbf{u}$$

*is controllable if and only if the matrix*

$$[ \mathbf{B}(q_0) \mid \mathbf{M}^{(1)}(q_0)\mathbf{B}(q_0) \mid \cdots \mid \mathbf{M}^{(2n-2)}(q_0)\mathbf{B}(q_0) ]$$

*has rank  $2n$ , where  $\mathbf{M}^{(k)}(q_0)$ ,  $k = 1, \dots, 2n - 2$  are as defined above with respect to some basis for  $T_{q_0}Q$ , and where  $\mathbf{B}(q_0)$  is the matrix of  $\mathbf{B}_L(q_0)$  with respect to the same basis.*

*Proof* The following lemma contains the bulk of the idea of the tedious proof.

**1 Lemma** Let  $\mathbf{A} \in \mathbb{R}^{2n \times 2n}$  and  $\mathbf{B} \in \mathbb{R}^{2n \times m}$  be given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix},$$

for  $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{n \times n}$  and  $\mathbf{B}_2 \in \mathbb{R}^{n \times m}$ . Let  $\mathbf{M}^{(0)} = \mathbf{I}_n$  and for  $k \in \mathbb{Z}_+$  define

$$\mathbf{M}^{(k)} = \begin{cases} \sum_{j=k/2}^k \sum_{\substack{|I|=j \\ |I_1|=k-j}} \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_j}, & k \text{ even} \\ \sum_{j=(k+1)/2}^k \sum_{\substack{|I|=j \\ |I_1|=k-j}} \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_j}, & k \text{ odd.} \end{cases}$$

Then, for  $k \in \mathbb{Z}_+$  we have

$$\mathbf{A}^k \mathbf{B} = \begin{bmatrix} \mathbf{M}^{(k-1)} \mathbf{B}_2 \\ \mathbf{M}^{(k)} \mathbf{B}_2 \end{bmatrix}.$$

*Proof* For  $k = 1$  we have  $\mathbf{M}^{(1)} = \mathbf{A}_2$  so that

$$\begin{pmatrix} \mathbf{M}^{(0)} \mathbf{B}_2 \\ \mathbf{M}^{(1)} \mathbf{B}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{B}_2 \\ \mathbf{A}_2 \mathbf{B}_2 \end{pmatrix}.$$

One readily verifies now that the lemma holds for  $k = 1$ . Suppose it true for  $k > 1$ . We then have

$$\mathbf{A}^{k+1} \mathbf{B} = \begin{pmatrix} \mathbf{M}^{(k)} \mathbf{B}_2 \\ \mathbf{A}_1 \mathbf{M}^{(k-1)} \mathbf{B}_2 + \mathbf{A}_2 \mathbf{M}^{(k)} \mathbf{B}_2 \end{pmatrix}.$$

First let us suppose that  $k$  is even. In this case we have

$$\begin{aligned} \mathbf{A}_1 \mathbf{M}^{(k-1)} + \mathbf{A}_2 \mathbf{M}^{(k)} &= \sum_{j=k/2}^{k-1} \sum_{\substack{|I|=j \\ |I_1|=k-j-1}} \mathbf{A}_1 \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_j} + \sum_{j=k/2}^k \sum_{\substack{|I|=j \\ |I_1|=k-j}} \mathbf{A}_2 \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_j} \\ &= \sum_{j=k/2}^{k-1} \sum_{\substack{|I|=j \\ |I_1|=k-j-1}} \mathbf{A}_1 \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_j} + \sum_{j=k/2}^{k-1} \sum_{\substack{|I|=j \\ |I_1|=k-j}} \mathbf{A}_2 \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_j} + \mathbf{A}_2^{k+1} \\ &= \sum_{j=(k+2)/2}^k \sum_{\substack{|I|=j \\ |I_1|=k-j+1}} \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_j} + \mathbf{A}_2^{k+1} \\ &= \sum_{j=(k+2)/2}^k \sum_{\substack{|I|=j \\ |I_1|=k-j+1}} \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_j} \\ &= \mathbf{M}^{(k+1)}. \end{aligned}$$

A similarly styled computation also shows that  $\mathbf{A}_1 \mathbf{M}^{(k-1)} + \mathbf{A}_2 \mathbf{M}^{(k)} = \mathbf{M}^{(k+1)}$  when  $k$  is odd. This now gives the result.  $\blacktriangledown$

To proceed with the proof of the theorem, note that the situation dealt with in the theorem is just as in the lemma with  $\mathbf{A}_1 = -\mathbf{A}_1(q_0)$ ,  $\mathbf{A}_2 = -\mathbf{A}_2(q_0)$ , and  $\mathbf{B}_2 = \mathbf{B}(q_0)$ . Furthermore, by the lemma we have

$$\left[ \mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \mathbf{A}^2\mathbf{B} \mid \cdots \mid \mathbf{A}^{2n-1}\mathbf{B} \right] = \left[ \begin{array}{c|c|c|c|c} \mathbf{0} & \mathbf{B}_2 & \mathbf{M}^{(1)}\mathbf{B}_2 & \cdots & \mathbf{M}^{(2n-2)}\mathbf{B}_2 \\ \mathbf{B}_2 & \mathbf{M}^{(1)}\mathbf{B}_2 & \mathbf{M}^{(2)}\mathbf{B}_2 & \cdots & \mathbf{M}^{(2n-1)}\mathbf{B}_2 \end{array} \right].$$

Let us define  $\mathbf{M}_1, \mathbf{M}_2 \in \mathbb{R}^{2n \times 2nm}$  by

$$\begin{aligned} \mathbf{M}_1 &= \left[ \mathbf{0} \mid \mathbf{B}_2 \mid \mathbf{M}^{(1)}\mathbf{B}_2 \mid \cdots \mid \mathbf{M}^{(2n-2)}\mathbf{B}_2 \right] \\ \mathbf{M}_2 &= \left[ \mathbf{B}_2 \mid \mathbf{M}^{(1)}\mathbf{B}_2 \mid \mathbf{M}^{(2)}\mathbf{B}_2 \mid \cdots \mid \mathbf{M}^{(2n-1)}\mathbf{B}_2 \right]. \end{aligned}$$

Note  $\text{image}(\mathbf{M}_1) \subset \text{image}(\mathbf{M}_2)$ . Therefore the rank of

$$\begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix}$$

will equal  $2n$  if and only if  $\text{rank}(\mathbf{M}_1) = n$ , as stated.  $\blacksquare$

Let us now specialise the above theorem to the important case when  $C_L(q_0) = 0$ . In this case, with our above notation with respect to a basis for  $T_{q_0}Q$ , we have  $\mathbf{A}_2(q_0) = \mathbf{0}$ . We now show, using Theorem 4.4.2, that the following condition for controllability of the linearised system holds.

**4.4.3 Corollary** *Let  $(L, \mathcal{F})$  be a Lagrangian control system on  $Q$  with  $L$  a hyperregular Lagrangian. Let  $q_0$  be an equilibrium point for  $L$  and let  $(M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))$  be the linearisation of  $(L, \mathcal{F})$  at  $q_0$ , supposing that  $C_L(q_0) = 0$ . The linearised system*

$$\dot{\mathbf{x}} = \mathbf{A}_L(q_0)\mathbf{x} + \mathbf{B}_{\mathcal{F}}(q_0)\mathbf{u}$$

*is controllable if and only if the matrix*

$$\left[ \mathbf{B}(q_0) \mid \mathbf{A}_1(q_0)\mathbf{B}(q_0) \mid \cdots \mid \mathbf{A}_1^{n-1}(q_0)\mathbf{B}(q_0) \right]$$

*has rank  $n$ , where  $\mathbf{A}_2(q_0)$  is the matrix of  $M_L^\sharp(q_0) \circ K_L^\flat(q_0)$  with respect to some basis for  $T_{q_0}Q$ , and where  $\mathbf{B}(q_0)$  is the matrix of  $\mathbf{B}_L(q_0)$  with respect to the same basis.*

*Proof* One verifies using the definition of  $\mathbf{M}^{(k)}(q_0)$  above, with  $\mathbf{A}_2(q_0) = \mathbf{0}$ , that

$$\mathbf{M}^{(k)}(q_0) = \begin{cases} \mathbf{0}, & k \text{ odd} \\ \mathbf{A}_1(q_0)^{k/2}, & k \text{ even.} \end{cases}$$

The result now follows directly from Theorem 4.4.2.  $\blacksquare$

**4.4.4 Remark** Corollary 4.4.3 holds in the important case where  $q_0$  is an equilibrium point for a simple mechanical system since, by Proposition 3.3.7,  $C_L(q_0) = 0$  at equilibrium points for simple mechanical systems. Also note that the condition for controllability of the linearised system simplifies significantly in this case since the matrix whose rank needs to be checked is essentially half the size.  $\square$

Let us look at a simple example on the basis of the previous remark.

**4.4.5 Example** (Example 2.6.3 cont'd) We consider the pendulum of Example 2.6.3, and now apply a torque at the base of the pendulum. One might intuitively expect the system to be controllable, but let us perform the linearisation analysis, just so we know how it goes.

The Lagrangian for the system we take to be  $L(q, v) = m\ell^2\frac{1}{2}v^2 + mla_g \cos q$ . Note that we reinsert units into the problem to make it a little more interesting. The Euler-Lagrange equations, with the torque at the base of the pendulum denoted by  $u$ , read

$$m\ell^2\ddot{q} + mla_g \sin \theta = u.$$

There are two equilibrium points,  $q_1 = 0$  and  $q_2 = \pi$ . We compute the linearisations of  $L$  at these points to be

$$\begin{aligned} M_L(q_1) &= m\ell^2, & K_L(q_1) &= mla_g \\ M_L(q_2) &= m\ell^2, & K_L(q_2) &= -mla_g, \end{aligned}$$

with  $C_L(q_1) = C_L(q_2) = 0$  by virtue of the system being a simple mechanical system. We then determine that

$$M_L^\sharp(q_1) \circ K_L^\flat(q_1) = \frac{a_g}{\ell}, \quad M_L^\sharp(q_2) \circ K_L^\flat(q_2) = -\frac{a_g}{\ell}.$$

We also have  $B_L(q_1) = B_L(q_2) = \frac{1}{m\ell^2}$ . Now, adopting the notation of Corollary 4.4.3, we find that

$$\begin{aligned} \left[ \mathbf{B}(q_1) \mid \mathbf{A}^1(q_1)\mathbf{B}(q_1) \right] &= \left[ \frac{1}{m\ell^2} \quad \frac{a_g}{m\ell} \right] \\ \left[ \mathbf{B}(q_2) \mid \mathbf{A}^1(q_2)\mathbf{B}(q_2) \right] &= \left[ \frac{1}{m\ell^2} \quad -\frac{a_g}{m\ell} \right]. \end{aligned}$$

In each case, of course the rank of the matrix has maximal rank 1. This example is perhaps too trivial to illustrate much, and the reader is encourage to try out the results on the more interesting examples in the exercises.  $\square$

An important corollary is the following which deals with simple mechanical systems with no potential. Recall that for such systems, every point  $q_0 \in Q$  is an equilibrium point.

**4.4.6 Corollary** *Let  $(L, \mathcal{F})$  be a Lagrangian control system with  $L$  the Lagrangian for a simple mechanical system  $(Q, g, 0)$  with zero potential. For any  $q_0 \in Q$ , the linearisation  $(M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))$  is controllable if and only if  $(L, \mathcal{F})$  is fully actuated.*

*Proof* In the notation of Theorem 4.4.2,  $\mathbf{A}_1(q_0) = \mathbf{A}_2(q_0) = \mathbf{0}$ . Therefore, using Corollary 4.4.3 the system is controllable if and only if the matrix  $\mathbf{B}(q_0)$  has rank  $n$ . But this will happen if and only if

$$\text{span}_{\mathbb{R}} \{F^1(0_{q_0}), \dots, F^m(0_{q_0})\} = T_{q_0}Q.$$

By our assumption of the independence of the dimension of the span of the forces  $F^1(v_q), \dots, F^m(v_q)$  on  $v_q \in TQ$ , the result follows.  $\blacksquare$

This result has important implications. It says that one cannot expect anything helpful to occur when linearising a simple mechanical system with a kinetic energy Lagrangian, except in the trivial case when we have full authority available to us with the controls. In these cases, we may resort to the methods of Section 4.2. In other cases, when the system is underactuated, things become rather complicated, and we mention a few ‘‘simple’’ ideas in Section 4.5.

Let us give another important application of Theorem 4.4.2. Let us suppose that we have a system which is subjected to a dissipative force which is linear in velocity, as is the case with viscous friction. The force we consider is thus of the form

$$F(v_q) = -R_q^b(v_q)$$

where  $R_q$  is a positive semi-definite quadratic form on  $T_qQ$ . Note that this is simply a generalisation of the viscous force given in a simple example in Example 2.5.9–1. With a dissipative force of this type, an equilibrium point for  $L$  will still be an equilibrium point with the addition of the dissipative force since this force vanishes when velocity is zero. Therefore, we may still linearise about equilibria for  $L$ , even in the presence of this dissipative force. To determine the controllability of the linearisation, we need a comparatively straightforward modification of the notation used in Theorem 4.4.2. Let  $\tilde{\mathbf{A}}_1(q_0) = \mathbf{A}_1(q_0)$  be as used in that theorem, and let  $\tilde{\mathbf{A}}_2(q_0)$  be the matrix for the linear map  $M_L^\sharp(q_0) \circ (C_L^b(q_0) + R_{q_0}^b)$ . Now define

$$\mathbb{R}^{n \times n} \ni \tilde{\mathbf{M}}^{(k)}(q_0) = \begin{cases} \sum_{j=k/2}^k \sum_{\substack{|I|=j \\ |I_1|=k-j}} \tilde{\mathbf{A}}_{i_1}(q_0) \cdots \tilde{\mathbf{A}}_{i_j}(q_0), & k \text{ even} \\ \sum_{j=(k+1)/2}^k \sum_{\substack{|I|=j \\ |I_1|=k-j}} \tilde{\mathbf{A}}_{i_1}(q_0) \cdots \tilde{\mathbf{A}}_{i_j}(q_0), & k \text{ odd.} \end{cases}$$

The following result tells us when the resulting linearisation of a system with viscous dissipation is controllable.

**4.4.7 Proposition** *Let  $(L, \mathcal{F})$  be a Lagrangian control system on  $Q$  with  $L$  a hyperregular Lagrangian, and let  $F$  be a dissipative force on  $Q$  of the form  $F(v_q) = R_q^b(v_q)$ , with  $R_q$  positive semi-definite, as above. If  $q_0$  is an equilibrium point for  $L$ , let  $(M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))$  be the linearisation of  $(L, \mathcal{F})$ . The linearisation of the system with the addition of the dissipative force  $F$  is controllable if and only if the matrix*

$$\left[ \mathbf{B}(q_0) \mid \tilde{\mathbf{M}}^{(1)}(q_0)\mathbf{B}(q_0) \mid \cdots \mid \tilde{\mathbf{M}}^{(2n-2)}(q_0)\mathbf{B}(q_0) \right]$$

has rank  $2n$ , where  $\tilde{\mathbf{M}}^{(k)}(q_0)$ ,  $k = 1, \dots, 2n - 2$  are as defined above with respect to some basis for  $T_{q_0}Q$ , and where  $\mathbf{B}(q_0)$  is the matrix of  $B_L(q_0)$  with respect to the same basis.

*Proof* We shall compute the linearisation of the system at the equilibrium point  $q_0$ . We do this by working in coordinates  $(q^1, \dots, q^n)$ . Following the computations preceding Proposition 3.2.10, we Taylor expand about  $(q_0, \mathbf{0})$ . The resulting expression is

$$\begin{aligned} \frac{\partial^2 L}{\partial v^j \partial v^i} \ddot{\xi}^j(t) + \left( \frac{\partial^2 L}{\partial q^j \partial v^i} - \frac{\partial^2 L}{\partial v^j \partial q^i} \right) \dot{\xi}^j(t) - \frac{\partial^2 L}{\partial q^j \partial q^i} \xi^j(t) + \cdots = \\ - R_{ij}(q_0) \dot{\xi}^j(t) + u_a F_i^a(q_0, \mathbf{0}), \quad i = 1, \dots, n, \end{aligned}$$

where  $\xi^i(t) = q^i(t) - q_0^i$ ,  $i = 1, \dots, n$ . Thus the linearised equations look like

$$M_L(q_0)\ddot{q} + C_L(q_0)\dot{q} + K_L(q_0)q = -R_{q_0}^b(q) + u_a F^a(0_{q_0}).$$

Here again we make the horrific abuse of notation of writing points in the tangent space  $T_{q_0}Q$  as  $q$ . Using the fact that  $L$  is hyperregular, we write these as first-order equations:

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & \text{id}_V \\ -M_L^\sharp(q_0) \circ K_L^b(q_0) & -M_L^\sharp(q_0) \circ (C_L^b(q_0) + R_{q_0}^b) \end{bmatrix} \begin{pmatrix} q \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ B_L(q_0) \end{pmatrix} u.$$

The result now follows directly from Lemma 1 of the proof of Theorem 4.4.2. ■

**4.4.3 On the validity of the linearised system** The previous section dealt with the linearisation of a Lagrangian system. However, given our discussion of Section 3.2 on linearisation as it relates to dynamics near equilibrium points, one would be well-justified in being suspicious about cavalierly making conclusions about Lagrangian control systems based upon properties of their linearised systems.

Things, however, are not as grim as they might be. The following result is Proposition 3.3 of [Nijmeijer and van der Schaft 1990], but translated into our context.

**4.4.8 Theorem** *Let  $(L, \mathcal{F})$  be a Lagrangian control system with hyperregular Lagrangian  $L$ , let  $q_0$  be an equilibrium point for  $L$ , and let  $(M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))$  be the linearisation of  $(L, \mathcal{F})$  at  $q_0$ . If the linearised system is controllable, then for every  $T, \epsilon > 0$  the set of points in  $TQ$  reachable from  $0_{q_0}$  in time  $T$  using controls satisfying  $\|u(t)\| < \epsilon$ ,  $t \in [0, T]$ , contains a neighbourhood of  $0_{q_0}$ .*

In other words, if the linearisation at  $q_0$  is controllable, then the system is controllable for small times, and for small controls, near  $q_0$ . This result is important, because it suggests that it may be valid to use the linear approximation, as long as we remain near the equilibrium point. Indeed, this is a very common technique for controlling systems near equilibrium points.

The converse of the question dealt with by Theorem 4.4.8, however, is not so straightforward. That is, it is certainly *not* true that if a system is controllable near  $q_0$  then its linearisation is also controllable. To determine the controllability properties of such systems, one must resort to nonlinear methods. It is towards this that we now turn our attention.

## 4.5 Control when linearisation does not work

In this section we give a crude flavour to the problem of how to attack the control problem for a mechanical system when one has found that the linearisation is not controllable. It is beyond the scope of what we are doing to deal in any depth with the problems here, and therefore we give only a brief sketch of some of the existing technology, and point the reader to the appropriate literature.

**4.5.1 Driftless nonlinear control systems** As a preparation to the techniques we will use to control mechanical systems, we look at a class of nonlinear system which is *not* mechanical. A **driftless control system** is a pair  $\Sigma = (Q, \mathcal{X})$  where  $\mathcal{X} = \{X_1, \dots, X_s\}$  is a collection of vector fields on  $Q$ . The control system we consider is given by

$$c'(t) = u^\alpha(t)X_\alpha(c(t)). \quad (4.5)$$

The controls we consider are, say, piecewise differentiable, although we may use more general controls. A **controlled trajectory** for a driftless system  $\Sigma = (Q, \mathcal{X})$  is a pair  $(c, u)$  where  $u: [0, T] \rightarrow \mathbb{R}^s$  is piecewise differentiable, and where  $c: [0, T] \rightarrow Q$  is such that the differential equation (4.5) is satisfied. We shall make the assumption that the dimension of the subspace  $X_q$  defined by

$$X_q = \text{span}_{\mathbb{R}} \{X_1(q), \dots, X_s(q)\}$$

is independent of  $q$ .

It is possible to give a complete description of controllability for such systems. First let us describe the type of controllability we consider. The driftless system  $\{X_1, \dots, X_s\}$  is



**controllable** if for  $q_1, q_2 \in Q$  there exists a controlled trajectory  $(c, u)$  defined on the interval  $[0, T]$  with the property that  $c(0) = q_1$  and  $c(T) = q_2$ . To determine the controllability conditions we let  $L(\mathcal{X})$  be the smallest subset of vector fields on  $Q$  with the property that  $\mathcal{X} \subset L(\mathcal{X})$  and with the property that  $[X, Y] \in L(\mathcal{X})$  for all vector fields  $X, Y \in L(\mathcal{X})$ . Now, for each  $q \in Q$ , define a subspace  $\overline{\text{Lie}}(\mathcal{X})_q$  by

$$\overline{\text{Lie}}(\mathcal{X})_q = \{X(q) \mid X \in L(\mathcal{X})\}.$$

To compute  $L(\mathcal{X})$  in practice, it often suffices to iteratively take Lie brackets of the vector fields  $X_1, \dots, X_s$  until one stops generating new directions. Thus one computes the vector fields  $[X_a, X_b]$ ,  $a, b = 1, \dots, s$ ,  $[X_a, [X_b, X_c]]$ ,  $a, b, c = 1, \dots, s$ , and so on. Typically this process will “terminate” in the sense that  $\overline{\text{Lie}}(\mathcal{X})_q$  will stop growing.

The following important result gives conditions on when a driftless system is controllable.

**4.5.1 Theorem (Chow [1939])** *A driftless control system  $\Sigma = (Q, \mathcal{X})$  is controllable if  $\overline{\text{Lie}}(\mathcal{X})_q = T_q Q$  for each  $q \in Q$ . If the vector fields  $\mathcal{X}$  are real analytic, then this condition is also necessary.*

Let us make some general comments concerning driftless systems.

**4.5.2 Remarks 1.** We name driftless systems as we do because they are a specific case of control systems of the type

$$c'(t) = X_0(c(t)) + u^\alpha(t)X_\alpha(c(t)),$$

where we now have no control over the vector field  $X_0$ , which is called the **drift vector field**. For driftless systems, of course, the drift vector field is zero. Systems with nonzero drift vector fields are *significantly* harder to deal with. In particular, there are no known necessary and sufficient conditions for controllability of the type given in Theorem 4.5.1 for systems with drift.

- Note that equation (4.5) essentially describes the set of curves  $t \mapsto c(t)$  whose tangent vectors lie in  $c'(t)$  lie in the subspace  $\text{span}_{\mathbb{R}}\{X_1(c(t)), \dots, X_s(c(t))\}$  for each  $t$ . In this way, one can think of (4.5) as describing a linear constraint on  $Q$ , exactly as we did in Section 2.6. With this interpretation, Theorem 4.5.1 gives conditions on when it is possible to connect any two points in  $Q$  with a curve satisfying the constraint.
- If one linearises (4.5) about a point  $q_0$ , the resulting linear control system on  $T_{q_0}Q$  is simply

$$\dot{q}(t) = B(q_0)u,$$

where  $B(q_0): \mathbb{R}^s \rightarrow T_{q_0}Q$  is defined by  $B(q_0)u = u^\alpha X_\alpha(q_0)$ . The Kalman rank condition (the “ $A$ ” matrix is zero) tells us that this linearisation is stable if and only if  $B(q_0)$  is surjective, i.e., if the tangent vectors  $X_1(q_0), \dots, X_s(q_0)$  generate  $T_{q_0}Q$ . Thus the system is controllable only in the trivial case where the inputs allow us to access all directions in  $Q$ .  $\square$

Now that we have dealt with the controllability question for driftless systems, let us look at how to handle some common control problems. First let us look at the problem of designing a control law which will stabilise the system to a desired point  $q_0$ . The following result states that the upshot of the observation Remark 4.5.2–3 is fatal as far as using feedback to stabilise a driftless system.

4.5.3 Theorem (Brockett [1983]) *Let  $q_0 \in Q$ . It is not possible to define a continuous function  $u: Q \rightarrow \mathbb{R}^s$  with the property that the closed-loop system for (4.5),*

$$c'(t) = u^\alpha(c(t))X_\alpha(c(t)),$$

*has  $q_0$  as an asymptotically stable fixed point.*

The theorem states that if we wish to use a closed-loop controller to stabilise the system (4.5), then the feedback must either be discontinuous, or time-dependent. Both approaches have been tried in the literature with some success. The papers [Bacciotti and Ceragioli 1999, Sontag 1999] provide a review of the use of discontinuous feedback to stabilise an equilibrium point. One of the problems surrounding the use of discontinuous feedback is that it becomes an issue to decide in what sense solutions are to be defined. The most common notion is “the sense of Fillipov” [Filippov 1984], but ideas of Clarke, [Clarke 1983] and [Clarke, Ledyaev, Stern, and Wolenski 1998], are also important.

When talking about stabilisation of driftless systems, the notion of “homogeneity” is important. A recent paper with an excellent survey of such issues in the introduction is [Morin, Pomet, and Samson 1999]. In that paper will also be found an algorithm for providing a stabilising control law based on the method of homogeneous approximation. Coron and Rosier [1994] show, roughly, that if a system can be stabilised by discontinuous feedback, then it can be stabilised by time-varying feedback.

Now let us turn to reviewing the literature concerning the question of constructing a control law which steers the system from one point to another. Of course, before attempting such a thing, one must ensure that the system is controllable. The state of knowledge here is not as advanced as it is for the stabilisation problem, and a review of what is known is presented in Chapter 8 of [Murray, Li, and Sastry 1994]. The paper [Murray and Sastry 1993] gives a methodology of using sinusoids to steer a class of systems. The class of systems are somewhat restricted, but it is sometimes possible to put a given system into the required form. Lafferriere and Sussmann [1991] provide another methodology based on “nilpotent approximations.” The methods here may be useful in the context of kinematically controllable systems which we discuss in Section 4.5.4.

**4.5.2 Affine connection control systems** The mechanical systems we consider are an important subclass of Lagrangian control systems whose Lagrangians are those for a simple mechanical system  $(Q, g, 0)$  with zero potential. We also make the assumption that the input forces are independent of time and velocity. Thus the input forces  $\mathcal{F} = \{F^1, \dots, F^m\}$  are simply one-forms on  $Q$ . As we saw in Proposition 2.5.10, the equations which govern the motion of this control system are then

$$\overset{g}{\nabla}_{c'(t)} c'(t) = u_a(t) g^\sharp(F^a(c(t))),$$

where  $\overset{g}{\nabla}$  is the Levi-Civita connection associated with  $g$ . It turns out that we loose nothing by generalising to the situation where the connection is arbitrary, and the inputs are arbitrary vector fields. To this end, we say that an **affine connection control system** is a triple  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$  where  $\nabla$  is an affine connection on  $Q$ , and  $\mathcal{Y} = \{Y_1, \dots, Y_m\}$  are vector fields on  $Q$ . The control equations are then, of course,

$$\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t)). \tag{4.6}$$

A **controlled trajectory** for an affine connection control system is a pair  $(c, u)$  where  $u: [0, T] \rightarrow \mathbb{R}^m$  is piecewise differentiable, and where  $c: [0, T] \rightarrow Q$  has the property that the differential equation (4.6) is satisfied. The act of even defining appropriate notions of controllability for (4.6) requires some ideas we do not wish to deal with in full generality [see Lewis and Murray 1997]. Let us agree to deal only with the simplified notion of controllability called **equilibrium controllability** where the system possesses this property if for each  $q_1, q_2 \in Q$ , there exists a controlled trajectory  $(c, u)$  defined on the interval  $[0, T]$  with the property that  $c'(0) = 0_{q_1}$  and  $c'(T) = 0_{q_2}$ . Roughly, the system is equilibrium controllable when it can be steered from any point at rest to any other point at rest. As we have been doing all along, we shall assume that the dimension of the subspace

$$Y_q = \text{span}_{\mathbb{R}} \{Y_1(q), \dots, Y_m(q)\}$$

is independent of  $q$ .

Note that (4.6) has a form somewhat similar to the equation (4.5) for driftless systems. However, the two types of systems are *in no way* the same! For example, the equations (4.6) are second-order, whereas the equations (4.5) are first-order. Despite the fact that the two types of equations are not equivalent, in some examples they are related in some way, and it is this fact which we will exploit in the remainder of the section.

**4.5.3 Mechanical systems which are “reducible” to driftless systems** It turns out that some, but certainly not all (in some sense very few indeed), mechanical systems can, in a limited sense, be thought of as driftless systems. Fortunately, these systems are ones for which linearisation is ineffective, so the connection with driftless systems provides an “in” to being able to do some control tasks for these systems.

The following definition establishes the type of correspondence we are after in this section.

**4.5.4 Definition** Let  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$  be an affine connection control system and let  $\Sigma = (Q, \mathcal{X})$  be a driftless system.  $\Sigma_{\text{aff}}$  is **reducible** to  $\Sigma$  if the following two conditions hold:

- (i) for each controlled trajectory  $(c, \tilde{u})$  for  $\Sigma$  defined on  $[0, T]$  with  $\tilde{u}$  differentiable and piecewise  $C^2$ , there exists a piecewise differentiable map  $u: [0, T] \rightarrow \mathbb{R}^m$  so that  $(c, u)$  is a controlled trajectory for  $\Sigma_{\text{aff}}$ ;
- (ii) for each controlled trajectory  $(c, u)$  for  $\Sigma_{\text{aff}}$  defined on  $[0, T]$  and with  $c'(0) \in X_q$ , there exists a differentiable and piecewise  $C^2$  map  $\tilde{u}: [0, T] \rightarrow \mathbb{R}^s$  so that  $(c, \tilde{u})$  is a controlled trajectory for  $\Sigma$ .  $\square$

The idea of the definition is quite simple. It establishes that there is a correspondence between controlled trajectories for the driftless system and the affine connection control system. In making the correspondence, one has to be careful of two things.

1. The same types of controls cannot be used for both  $\Sigma_{\text{aff}}$  and  $\Sigma$ . For the driftless system, if the input is discontinuous, this will imply that there will be instantaneous velocity jumps. Such phenomenon are not physically realisable for affine connection control systems since this would require infinite forces. This is because at points of an instantaneous velocity jump, acceleration will be infinite. This explains why in part (i) we need to add extra differentiability to the input for  $\Sigma$ .
2. It will not be possible to assign to every controlled trajectory of  $\Sigma_{\text{aff}}$  a controlled trajectory of  $\Sigma$ . This is clear since initial conditions for  $\Sigma_{\text{aff}}$  allow that  $c'(0)$  can be

arbitrary, whereas all controlled trajectories for  $\Sigma$  will have  $c'(0) \in X_{c(0)}$ . This explains the subsidiary condition on controlled trajectories for  $\Sigma_{\text{aff}}$  is part (ii) of Definition 4.5.4.

The following result gives the surprisingly simple answer to the question of when an affine connection control system is reducible to *some* driftless system.

**4.5.5 Theorem (Lewis [1999])** *An affine connection control system  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$  is reducible to a driftless system  $\Sigma = (Q, \mathcal{X})$  if and only if the following two conditions hold:*

- (i)  $X_q = Y_q$  for each  $q \in Q$ ;
- (ii)  $\nabla_X X(q) \in Y_q$  for every vector field  $X$  having the property that  $X(q) \in Y_q$  for every  $q \in Q$ .

The first condition is perhaps not surprising, but neither is it obvious. It states that when establishing whether  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$  is reducible to  $\Sigma = (Q, \mathcal{X})$  we may as well assume that  $\mathcal{X} = \mathcal{Y}$ . The meat of the theorem is the second condition, however. This condition is one which is readily checked. It is also true that there will be very few systems which satisfy this condition. Nevertheless, there appears to be an unnaturally large class of physical systems which meet the criterion of Theorem 4.5.5, so it is an interesting one as concerns applications.

Not only does Theorem 4.5.5 provide conditions for when a system is reducible to a driftless system, but it turns out that when the conditions of the theorem are met, it is comparatively easy to derive the controls for the affine connection control system from those for the driftless system. To state the result, we need to introduce the ***symmetric product*** which takes two vector fields  $X$  and  $Y$  on  $Q$  and returns another vector field  $\langle X : Y \rangle$  defined by

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

It is easy to show (see Exercise E4.4) that condition (ii) of Theorem 4.5.5 is equivalent to the statement that  $\langle X : Y \rangle(q) \in Y_q$  for every pair of vector fields  $X$  and  $Y$  with the property that  $X(q), Y(q) \in Y_q$ . With this, we state the following result.

**4.5.6 Proposition** *Let  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$  be an affine connection control system which is reducible to the driftless system  $(Q, \mathcal{Y})$ . Suppose that the vector fields  $\mathcal{Y} = \{Y_1, \dots, Y_m\}$  are linearly independent and define  $\gamma_{\text{ab}}^d : Q \rightarrow \mathbb{R}$ ,  $a, b, d = 1, \dots, m$ , by*

$$\langle Y_a : Y_b \rangle = \gamma_{\text{ab}}^d Y_d,$$

*which is possible by condition (ii) of Theorem 4.5.5. If  $(c, \tilde{u})$  is a controlled trajectory for the driftless system  $\Sigma$  then, if we define the control  $u$  by*

$$u^d(t) = \tilde{u}^a(t) \tilde{u}^b(t) (\dot{\tilde{u}}^d(t) + \frac{1}{2} \gamma_{\text{ab}}^d(c(t))), \quad d = 1, \dots, m,$$

*$(c, u)$  is a controlled trajectory for the affine connection control system  $\Sigma_{\text{aff}}$ .*

**Proof** By definition  $(c, \tilde{u})$  satisfy

$$c'(t) = \tilde{u}^a(t) Y_a(c(t)).$$

Therefore

$$\begin{aligned}
\nabla_{c'(t)}c'(t) &= \nabla_{c'(t)}(\tilde{u}^b(t)Y_b(c(t))) \\
&= \tilde{u}^b(t)\nabla_{c'(t)}Y_b(c(t)) + \dot{\tilde{u}}^b(t)Y_b(c(t)) \\
&= \tilde{u}^b(t)\nabla_{\tilde{u}^a(t)Y_a(c(t))}Y_b(c(t)) + \dot{\tilde{u}}^b(t)Y_b(c(t)) \\
&= \tilde{u}^a(t)\tilde{u}^b(t)\nabla_{Y_a(c(t))}Y_b(c(t)) + \dot{\tilde{u}}^b(t)Y_b(c(t)) \\
&= \tilde{u}^a(t)\tilde{u}^b(t)\frac{1}{2}(\nabla_{Y_a}Y_b + \nabla_{Y_b}Y_a) + \dot{\tilde{u}}^b(t)Y_b(c(t)) \\
&= \tilde{u}^a(t)\tilde{u}^b(t)(\dot{\tilde{u}}^d(t) + \frac{1}{2}\gamma_{ab}^d(c(t)))Y_d(t).
\end{aligned}$$

Now, if we define  $u$  as in the statement of the proposition, we have

$$\nabla_{c'(t)}c'(t) = u^d(t)Y_d(c(t)),$$

as desired. ■

Let us summarise the point of the development in this section.

**4.5.7 Affine connection control systems reducible to a driftless system** Suppose that an affine connection control system  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$  is reducible to a driftless system  $\Sigma = (Q, \mathcal{Y})$ . (Note that without loss of generality we may suppose the input vector fields are the same for  $\Sigma_{\text{aff}}$  and  $\Sigma$ .) First examine the control problem to see if it is one for which the design may be done for the driftless system  $\Sigma$ . Examples of control problems of this type are

1. steering from rest at a given configuration to another and
2. stabilising to a point, provided that all initial conditions have velocities in  $Y$ .

One then does the design for the problem, if possible, using the driftless system, making use of the literature given in Section 4.5.1. Once the controls are found for the driftless system, then one uses Proposition 4.5.6 to translate controls for the driftless system to controls for the affine connection control system. □

An example of an affine connection control system which can be reduced to a driftless system is given in Exercise E4.5.

**4.5.4 Kinematically controllable systems** Bullo and Lynch [2001] provide a notion which is weaker than the notion of equivalence to a driftless system we looked at in the last section. The notion they provide is applicable to a larger class of systems, and so is worth looking at as another means of approaching control for systems where linearisation methods are not applicable.

The idea of Bullo and Lynch is somewhat like the idea of reducibility presented in the previous section. There is a subtle difference, however, and this difference broadens the class of problems to which the methods can be applied, although the method of applicability is somewhat more restricted. The idea is that one starts with an affine connection control system  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$  (one not necessarily reducible to a driftless system) and asks if there are any vector fields on  $Q$  whose integral curves can be followed up to arbitrary parameterisation. Let us be precise about this. A vector field  $X$  on  $Q$  is a **decoupling vector field** for  $\Sigma_{\text{aff}}$  if for every integral curve  $t \mapsto c(t)$  for  $X$  and for every reparameterisation  $t \mapsto \tau(t)$  for  $c$ , there exists a control  $t \mapsto u(t)$  with the property that  $(c \circ \tau, u)$  is a controlled

trajectory.  $\Sigma_{\text{aff}}$  is **kinematically controllable** if there exists a collection of decoupling vector fields for  $\Sigma_{\text{aff}}$ ,  $\mathcal{X} = \{X_1, \dots, X_s\}$ , so that  $\overline{\text{Lie}}(\mathcal{X})_q = T_q Q$  for each  $q \in Q$ .

Before we provide the conditions for determining when a vector field is a decoupling vector field, let us say a few words about the implications of a system being kinematically controllable, and contrast it with the notion of reducibility introduced in the previous section. It is true that if a collection of vector fields  $\mathcal{X} = \{X_1, \dots, X_s\}$  has the property that  $\overline{\text{Lie}}(\mathcal{X})_q = T_q Q$  for each  $q \in Q$ , then one is able to connect any two points in  $Q$  with a curve which is a concatenation of integral curves of the vector fields from  $\mathcal{X}$ . Note that this is not quite what is implied by Chow's theorem. Chow's theorem gives conditions so that one can connect points with a curve whose tangent vector field lies in the subspace  $X_q$  at each point. The distinction here is an admittedly subtle one, but is one worth understanding. For example, it may be harder to construct a control law connecting two points if one is only allowed to follow integral curves of the given vector fields from  $\mathcal{X}$ . Also, the problem of stabilisation is one which is better adapted to the situation where one can use arbitrary curves with tangent vector fields in  $X$ . Therefore, in some ways the notion of possessing decoupling vector fields is not as useful as being reducible to a driftless system. In the latter case, one has greater freedom in determining the control laws available. However, the notion of decoupling vector fields is important because they can sometimes be applied in cases where the system is not reducible to a driftless system (see Exercise E4.6).

As a further note, we remark that if one wishes to avoid instantaneous velocity jumps, then one must switch between vector fields in  $\mathcal{X}$  at zero velocity. Therefore, if one wishes to use a control law designed for the system

$$c'(t) = \tilde{u}^\alpha(t) X_\alpha(c(t)),$$

then one must alter the parameterisation of the curves so that one always starts and ends with zero velocity when flowing along the segment of an integral curve for one of the vector fields from  $\mathcal{X}$ .

Now let us look at conditions which determine when a given vector field is a decoupling vector field.

**4.5.8 Proposition (Bullo and Lynch [2001])** *A vector field  $X$  is a decoupling vector field for an affine connection control system  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$  if and only if*

- (i)  $X(q) \in Y_q$  for each  $q \in Q$  and
- (ii)  $\nabla_X X(q) \in Y_q$  for each  $q \in Q$ .

*Proof* First suppose that (i) and (ii) hold. To show that a vector field  $X$  is a decoupling vector field, it suffices to show that for any function  $f: Q \rightarrow \mathbb{R}$  and any integral curve  $c$  for  $fX$ , there exists a control  $u$  so that  $(c, u)$  is a controlled trajectory for  $\Sigma_{\text{aff}}$ . Letting  $f$  and  $c$  be so chosen, we have

$$\begin{aligned} \nabla_{c'(t)} c'(t) &= \nabla_{fX} fX(c(t)) \\ &= f^2(c(t)) \nabla_X X(c(t)) + f(c(t)) (\mathcal{L}_X f(c(t))) X(c(t)). \end{aligned}$$

Now using (i) and (ii) we have

$$X(c(t)), \nabla_X X(c(t)) \in Y_{c(t)}.$$

Therefore, there exists  $t \mapsto u(t)$  so that

$$f^2(c(t)) \nabla_X X(c(t)) + f(c(t)) (\mathcal{L}_X f(c(t))) X(c(t)) = u^a(t) Y_a(c(t)),$$

and this shows that  $X$  is a decoupling vector field.

Now suppose that  $X$  is a decoupling vector field and let  $q \in Q$ . Let  $t \mapsto c(t)$  be the integral curve of  $X$  through  $q$  and suppose that  $c(0) = q$ . Define two reparameterisations,  $\tau_1$  and  $\tau_2$ , of  $c$  with the following properties:

1.  $\tau_1(0) = 0$ ,  $\tau_1'(0) = 1$ , and  $\tau_1''(0) = 0$ ;
2.  $\tau_2(0) = 0$ ,  $\tau_2'(0) = 1$ , and  $\tau_2''(0) = 1$ .

Let  $c_1 = c \circ \tau_1$  and  $c_2 = c \circ \tau_2$ . We then have

$$\begin{aligned} \nabla_{c_1'(t)} c_1'(t) &= \nabla_{c'(\tau_1(t))\tau_1'(t)} c'(\tau_1(t))\tau_1'(t) \\ &= (\tau_1'(t))^2 \nabla_{c'(\tau_1(t))} c'(\tau_1(t)) + \tau_1''(t) c'(\tau_1(t)). \end{aligned}$$

Evaluating this at  $t = 0$  gives

$$\nabla_{c_1'(t)} c_1'(t) \Big|_{t=0} = \nabla_X X(q).$$

Similarly for  $c_2$  we have

$$\nabla_{c_2'(t)} c_2'(t) \Big|_{t=0} = \nabla_X X(q) + X(q).$$

Since we are assuming that  $X$  is a decoupling vector field for  $\Sigma_{\text{aff}}$  we then have

$$\nabla_X X(q) = u_1^a Y_a(q), \quad \nabla_X X(q) + X(q) = u_2^a Y_a(q)$$

for some  $u_1, u_2 \in \mathbb{R}^m$ . This then clearly implies that  $\nabla_X X(q), X(q) \in Y_q$ , as in the statement of the proposition. ■

As with systems reducible to a driftless system, it is possible to relate the control laws used to move the system along a decoupling vector field to control laws for the affine connection control system.

**4.5.9 Proposition** *Let  $X$  be a decoupling vector field for the affine connection control system  $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$ , let  $t \mapsto c(t)$  be an integral curve for  $X$  and let  $t \mapsto \tau(t)$  be a reparameterisation for  $c$ . If  $t \mapsto u(t) \in \mathbb{R}^m$  is defined by*

$$u^a(t) Y_a(c \circ \tau(t)) = (\tau'(t))^2 \nabla_X X(c \circ \tau(t)) + \tau''(t) X(c \circ \tau(t)),$$

*then  $(c \circ \tau, u)$  is a controlled trajectory for  $\Sigma_{\text{aff}}$ .*

*Proof* Since  $c$  is an integral curve for  $X$ ,  $c'(t) = X(c(t))$ . We therefore have

$$\begin{aligned} \nabla_{(c \circ \tau)'(t)} (c \circ \tau)'(t) &= (\tau'(t))^2 \nabla_{c'(\tau(t))} c'(\tau(t)) + \tau''(t) c'(\tau(t)) \\ &= (\tau'(t))^2 \nabla_X X(c \circ \tau(t)) + \tau''(t) X(c \circ \tau(t)). \end{aligned}$$

The result now follows since  $X$  is a decoupling vector field so that

$$(\tau'(t))^2 \nabla_X X(c \circ \tau(t)) + \tau''(t) X(c \circ \tau(t)) \in Y_{c \circ \tau(t)}. \quad \blacksquare$$

Let us summarise how to deal with affine connection control systems which are kinematically controllable.



4.5.10 Strategy for affine connection control systems which are kinematically controllable First one must find a collection of decoupling vector fields. Tips on how to do this are given by [Bullo and Lynch \[2001\]](#). One then must decide whether the control problem is one which can be dealt with using the decoupling vector fields one has found. Here one must face the fact that for a system which is kinematically controllable, one can only move as one wishes *exactly* along the decoupling vector fields, and not along curves whose tangent vector field lies in their span. Typical problems which can be dealt with in this framework are path-planning problems. One then designs the control laws for the driftless system. In order to prevent instantaneous velocity changes, each segment of the control law for the driftless system must be reparameterised to stop and start at zero velocity. Then one uses [Proposition 4.5.9](#) to design the corresponding controls for the affine connection control system.  $\square$



## Exercises

- E4.1 This exercise is a continuation of Exercises E2.4, E2.29, and E3.2. For the equilibria you determined for this example, determine the linearised control system, and check whether it is controllable.
- E4.2 This exercise is a continuation of Exercises E2.5, E2.30, and E3.3. For the equilibria you determined for this example, determine the linearised control system, and check whether it is controllable.
- E4.3 This exercise is a continuation of Exercises E2.6, E2.31, and E3.4. For the equilibria you determined for this example, determine the linearised control system, and check whether it is controllable.
- E4.4 Show that condition (ii) of Theorem 4.5.5 is equivalent to the statement that  $\langle Y_a : Y_b \rangle(q) \in Y_q$  for every pair of vector fields  $Y_a, Y_b \in \mathcal{Y}$ .
- E4.5 This exercise is a continuation of Exercises E2.7, E2.22, and E2.32. Let us use the coordinates  $(r, \psi, \theta)$  as indicated in Figure E4.1. The system is a simple mechanical

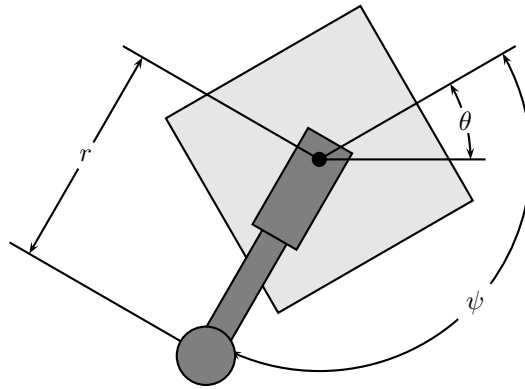


Figure E4.1 A “robotic leg”

system with zero potential energy, and so can be represented as an affine connection control system.

- (a) In the stated set of coordinates, write the vector fields  $\mathcal{Y}$  and from Exercise E2.22, write the Christoffel symbols.
- (b) Use the results of your previous answer to write the control equations in affine connection control system form.
- (c) Verify that the system is reducible to a kinematic system.
- (d) If  $(t \mapsto (r(t), \psi(t), \theta(t)), t \mapsto \tilde{u}(t))$  is a controlled trajectory for the driftless system  $(Q, \{Y_1, Y_2\})$ , find the control  $t \mapsto u(t)$  which makes  $(t \mapsto (r(t), \psi(t), \theta(t)), t \mapsto u(t))$  is a controlled trajectory for the affine connection control system.
- E4.6 This exercise is a continuation of Exercises E2.8 E2.23, and E2.33. Consider the planar rigid body depicted in Figure E4.2. Let  $\{e_1, e_2\}$  be an orthonormal frame fixed in an inertial coordinate system, and let  $\{f_1, f_2\}$  be an orthonormal frame fixed to the body at the centre of mass. Denote by  $(x, y)$  the location of the origin of the frame  $\{f_1, f_2\}$  (i.e., of the centre of mass) and let  $\theta$  be the orientation of  $\{f_1, f_2\}$  with respect to  $\{e_1, e_2\}$ . Thus  $(x, y, \theta)$  are coordinates for the rigid body. Suppose a force  $F$  is applied to the body at a point a distance  $h$  form the centre of mass along the body

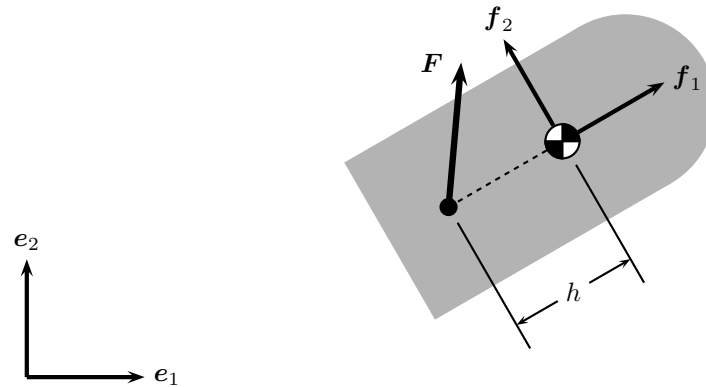


Figure E4.2 A controlled planar rigid body

$f_1$  direction as shown in the figure. Let  $F^1$  denote the component of  $F$  along the body  $f_1$ -direction and let  $F^2$  denote the component of  $F$  along the body  $f_2$ -direction.

(a) What are the two input forces for the system written as one-forms on  $Q$ ?

The system is a simple mechanical system with zero potential energy, and so can be represented as an affine connection control system.

(b) Write the vector fields  $\mathcal{V}$  in the same coordinates, and from Exercise E2.23, write the Christoffel symbols.

(c) Use the results of your previous answer to write the control equations in affine connection control system form.

(d) Verify that the system is not reducible to a kinematic system.

(e) Show that the two vector fields

$$X_1 = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}$$

$$X_2 = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} - \frac{mh}{J} \frac{\partial}{\partial \theta}$$

are decoupling vector fields for the system.

(f) Show that the system is kinematically controllable.

Define points  $(x_1, y_1, \theta_1), (x_2, y_2, \theta_2) \in Q$  by letting  $(x_i, y_i, \theta_i)$  be the point on the integral curve  $t \mapsto (x(t), y(t), \theta(t))$  for  $X_i$  which satisfies  $(x(0), y(0), \theta(0)) = (0, 0, 0)$  and  $(x(1), y(1), \theta(1)) = (x_i, y_i, \theta_i)$ ,  $i = 1, 2$ . That is to say, solve the differential equation for the vector field  $X_i$ ,  $i = 1, 2$ , with initial configuration  $(0, 0, 0)$ , and let  $(x_i, y_i, \theta_i)$ ,  $i = 1, 2$ , be the solution evaluated at  $t = 1$ .

(g) Construct control laws which start at  $(x(0), y(0), \theta(0))$  at rest, and steer the system to the point  $(x_i, y_i, \theta_i)$ ,  $i = 1, 2$ , also at rest. (That is, construct two control laws, each steering to one of the desired points.)

# Appendix A

## Linear algebra

The study of the geometry of Lagrangian mechanics requires that one be familiar with basic concepts in abstract linear algebra. The reader is expected to have encountered these concepts before, so this appendix serves as a refresher. We also use our discussion of linear algebra as our “in” to talking about the summation convention in a systematic manner. Since this gets used a lot, the reader may wish to take the opportunity to become familiar with it.

### A.1 Vector spaces

We shall suppose the reader to be familiar with the notion of a vector space  $V$  over a field  $K$ , particularly the field  $\mathbb{R}$  of real numbers, or the field  $\mathbb{C}$  of complex numbers. On such a vector space, one has defined the notions of **vector addition**,  $v_1 + v_2 \in V$ , between elements  $v_1, v_2 \in V$ , and the notion of **scalar multiplication**,  $av \in V$ , for a scalar  $a \in K$  and  $v \in V$ . There is a distinguished **zero vector**  $0 \in V$  with the property that  $0 + v = v + 0 = v$  for each  $v \in V$ . For the remainder of the section we take  $K \in \{\mathbb{R}, \mathbb{C}\}$ .

A subset  $U \subset V$  of a vector space is a **subspace** if  $U$  is closed under the operations of vector addition and scalar multiplication. If  $V_1$  and  $V_2$  are vector spaces, the **direct sum** of  $V_1$  and  $V_2$  is the vector space whose set is  $V_1 \times V_2$  (the Cartesian product), and with vector addition defined by  $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$  and scalar multiplication defined by  $a(v_1, v_2) = (av_1, av_2)$ . If  $U_1$  and  $U_2$  are subspaces of  $V$  we shall also write  $V = U_1 \oplus U_2$  if  $U_1 \cap U_2 = \{0\}$  and if every vector  $v \in V$  can be written as  $v = u_1 + u_2$  for some  $u_1 \in U_1$  and  $u_2 \in U_2$ .

A collection  $\{v_1, \dots, v_k\}$  of vectors is **linearly independent** if the equality

$$c_1v_1 + \dots + c_kv_k = 0$$

holds only when  $c_1 = \dots = c_k = 0$ . A set of vectors  $\{v_1, \dots, v_k\}$  **generates** a vector space  $V$  if every vector  $v \in V$  can be written as

$$v = c_1v_1 + \dots + c_kv_k$$

for *some* choice of constants  $c_1, \dots, c_k \in K$ . A **basis** for a vector space  $V$  is a collection of vectors which is linearly independent and which generates  $V$ . The number of vectors in a basis we call the **dimension** of  $V$ , and this is readily shown to be independent of choice of basis. A vector space is **finite-dimensional** if it possesses a basis with a finite number of elements. If  $\{e_1, \dots, e_n\}$  is a basis for  $V$ , we can write

$$v = v^1e_1 + \dots + v^ne_n$$

for some unique choice of  $v^1, \dots, v^n \in K$ , called the **components** of  $v$  relative to the basis. Here we begin to adopt the convention that components of vectors be written with index *up*. Let us use this chance to introduce the summation convention we shall employ.

**A.1.1 Basic premise of summation convention** *Whenever one sees a repeated index, one as a subscript and the other as a superscript, summation is implied.*

Thus, for example, we have

$$v^i e_i = \sum_{i=1}^n v^i e_i,$$

as summation over  $i$  is implied.

A map  $A: U \rightarrow V$  between is **k-linear** if  $A(au) = aA(u)$  and if  $A(u_1 + u_2) = A(u_1) + A(u_2)$  for each  $a \in k$  and  $u, u_1, u_2 \in U$ . The linear map  $\text{id}_V: V \rightarrow V$  defined by  $\text{id}_V(v) = v$ ,  $v \in V$ , is called the **identity map** for  $V$ . If  $\{f_1, \dots, f_m\}$  is a basis for  $U$  and  $\{e_1, \dots, e_n\}$  is a basis for  $V$ , for each  $i \in \{1, \dots, m\}$  we may write

$$A(f_a) = A_a^1 e_1 + \dots + A_a^n e_n$$

for some unique choice of constants  $A_a^1, \dots, A_a^n \in K$ . By letting  $a$  run from 1 to  $m$  we thus define  $nm$  constants  $A_a^i \in K$ ,  $i = 1, \dots, n$ ,  $a = 1, \dots, m$ , which we call the **matrix** of  $A$  relative to the two bases. If  $u \in U$  is written as

$$u = u^1 f_1 + \dots + u^m f_m,$$

one readily ascertains that

$$A(u) = \sum_{i=1}^n \sum_{a=1}^m A_a^i u^a e_i.$$

Thus the components of  $A(u)$  are written using the summation convention as  $A_a^1 u^a, \dots, A_a^n u^a$ .

Let us say a few more things about our summation convention.

**A.1.2 More properties of the summation convention** *Therefore, in our usual notion of matrix/vector multiplication, this renders the up index for A the row index, and the down index the column index. Note that we can also compactly write*

$$\sum_{i=1}^n \sum_{a=1}^m A_a^i u^a e_i = A_a^i u^a e_i.$$

The set of linear maps from a vector space  $U$  to a vector space  $V$  is itself a vector space which we denote  $L(U; V)$ . Vector addition in  $L(U; V)$  is given by

$$(A + B)(u) = A(u) + B(u),$$

and scalar multiplication is defined by

$$(aA)(u) = a(A(u)).$$

Note that what is being defined in these two equations is  $A + B \in L(U; V)$  in the first case, and  $aA \in L(U; V)$  in the second case. One verifies that  $\dim(L(U; V)) = \dim(U) \dim(V)$ .

Given a linear map  $A: U \rightarrow V$ , the **kernel** of  $A$  is the subspace

$$\ker(A) = \{u \in U \mid A(u) = 0\}$$

of  $U$ , and the **image** of  $A$  is the subspace

$$\text{image}(A) = \{A(u) \mid u \in U\}$$

of  $V$ . The **rank** of  $A$  is defined to be  $\text{rank}(A) = \dim(\text{image}(A))$ . The **rank-nullity formula** says that  $\dim(\ker(A)) + \text{rank}(A) = \dim(U)$ .

Of special interest are linear maps from a vector space  $V$  to itself:  $A: V \rightarrow V$ . In this case, an **eigenvalue** for  $A$  is an element  $\lambda \in K$  with the property that  $A(v) = \lambda v$  for some nonzero vector  $v$ , called an **eigenvector** for  $\lambda$ . To compute eigenvalues, one finds the roots of the **characteristic polynomial**  $\det(\lambda \text{id}_V - A)$  which has degree equal to the dimension of  $V$ . If  $K = \mathbb{C}$  this polynomial is guaranteed to have  $\dim(V)$  solutions, but it is possible that some of these will be repeated roots of characteristic polynomial. If  $\det(\lambda \text{id}_V - A) = (\lambda - \lambda_0)^k P(\lambda)$  for a polynomial  $P(\lambda)$  having the property that  $P(\lambda_0) \neq 0$ , then the eigenvalue  $\lambda_0$  has **algebraic multiplicity**  $k$ . The eigenvectors for an eigenvalue  $\lambda_0$  are nonzero vectors from the subspace

$$W_{\lambda_0} = \{v \in V \mid (A - \lambda_0 \text{id}_V)(v) = 0\}.$$

The **geometric multiplicity** of an eigenvalue  $\lambda_0$  is  $\dim(W_{\lambda_0})$ . We let  $m_a(\lambda_0)$  denote the algebraic multiplicity and  $m_g(\lambda_0)$  denote the geometric multiplicity of  $\lambda_0$ . It is always the case that  $m_a(\lambda_0) \geq m_g(\lambda_0)$ , and both equality and strict inequality can occur.

## A.2 Dual spaces

The notion of a dual space to a vector space  $V$  is extremely important for us. It is also a potential point of confusion, as it seems, for whatever reason, to be a slippery concept.

Given a finite-dimensional vector space  $V$  (let us agree to now restrict to vector spaces over  $\mathbb{R}$ ), the **dual space** to  $V$  is the set  $V^*$  of linear maps from  $V$  to  $\mathbb{R}$ . If  $\alpha \in V^*$ , we shall alternately write  $\alpha(v)$ ,  $\alpha \cdot v$ , or  $\langle \alpha; v \rangle$  to denote the image in  $\mathbb{R}$  of  $v \in V$  under  $\alpha$ . Note that since  $\dim(\mathbb{R}) = 1$ ,  $V^*$  is a vector space having dimension equal to that of  $V$ . We shall often call elements on  $V^*$  **one-forms**.

Let us see how to represent elements in  $V^*$  using a basis for  $V$ . Given a basis  $\{e_1, \dots, e_n\}$  for  $V$ , we define  $n$  elements of  $V^*$ , denoted  $e^1, \dots, e^n$ , by  $e^i(e_j) = \delta_j^i$ ,  $i, j = 1, \dots, n$ , where  $\delta_j^i$  denotes the **Kronecker delta**

$$\delta_j^i = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

The following result is important, albeit simple.

**A.2.1 Proposition** *If  $\{e_1, \dots, e_n\}$  is a basis for  $V$  then  $\{e^1, \dots, e^n\}$  is a basis for  $V^*$ , called the **dual basis**.*

**Proof** First let us show that the dual vectors  $\{e^1, \dots, e^n\}$  are linearly independent. Let  $c_1, \dots, c_n \in \mathbb{R}$  have the property that

$$c_i e^i = c_1 e^1 + \dots + c_n e^n = 0.$$

For each  $j = 1, \dots, n$  we must therefore have  $c_i e^i(e_j) = c_i \delta_j^i = c_j = 0$ . This implies linear independence. Now let us show that each dual vector  $\alpha \in V^*$  can be expressed as a linear combination of  $\{e^1, \dots, e^n\}$ . For  $\alpha \in V^*$  define  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  by  $\alpha_i = \alpha(e_i)$ ,  $i = 1, \dots, n$ . We claim that  $\alpha = \alpha_i e^i$ . To check this, it suffices to check that the two one-forms  $\alpha$  and  $\alpha_i e^i$  agree when applied to any of the basis vectors  $\{e_1, \dots, e_n\}$ . However, this is obvious since for  $j = 1, \dots, n$  we have  $\alpha(e_j) = \alpha_j$  and  $\alpha_i e^i(e_j) = \alpha_i \delta_j^i = \alpha_j$ . ■

If  $\{e_1, \dots, e_n\}$  is a basis for  $V$  with dual basis  $\{e^1, \dots, e^n\}$  then we may write  $\alpha \in V^*$  as  $\alpha = \alpha_i e^i$  for some uniquely determined  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ . If  $v \in V$  is expressed as  $v = v^i e_i$  then we have

$$\alpha(v) = \alpha_i e^i(v^j e_j) = \alpha_i v^j e^i(e_j) = \alpha_i v^j \delta_j^i = \alpha_i v^i.$$

Note that this makes the operation of feeding a vector to a one-form look an awful lot like taking the “dot product,” but it is in your best interests to refrain from thinking this way. One cannot take the dot product of objects in different spaces, and this is the case with  $\alpha(v)$  since  $\alpha \in V^*$  and  $v \in V$ . The proper generalisation of the dot product is given in Section A.3.

**A.2.2 More properties of the summation convention** *When we write a collection of elements of a vector space, we use subscripts to enumerate them, e.g.,  $v_1, \dots, v_k$ . For collections of elements of the dual space, we use superscripts to enumerate them, e.g.,  $\alpha^1, \dots, \alpha^k$ . The components of a vector with respect to a basis are written with indices as superscripts. The components of a dual vector with respect to a basis for the dual space are written with indices as subscripts.*

## A.3 Bilinear forms

We have multiple opportunities to define mechanical objects that are “quadratic.” Thus the notion of a bilinear form is a useful one in mechanics, although it is unfortunately not normally part of the background of those who study mechanics. However, the ideas are straightforward enough.

We let  $V$  be finite-dimensional  $\mathbb{R}$ -vector space. A **bilinear form** on  $V$  is a map  $B: V \times V \rightarrow \mathbb{R}$  with the property that for each  $v_0 \in V$  the maps  $v \mapsto B(v, v_0)$  and  $v \mapsto B(v_0, v)$  are linear. Thus  $B$  is “linear in each entry.” A bilinear form  $B$  is **symmetric** if  $B(v_1, v_2) = B(v_2, v_1)$  for all  $v_1, v_2 \in V$ , and **skew-symmetric** if  $B(v_1, v_2) = -B(v_2, v_1)$  for all  $v_1, v_2 \in V$ . If  $\{e_1, \dots, e_n\}$  is a basis for  $V$ , the **matrix** for a bilinear form  $B$  in this basis is the collection of  $n^2$  number  $B_{ij} = B(e_i, e_j)$ ,  $i, j = 1, \dots, n$ .  $B$  is symmetric if and only if  $B_{ij} = B_{ji}$ ,  $i, j = 1, \dots, n$ , and skew-symmetric if and only if  $B_{ij} = -B_{ji}$ ,  $i, j = 1, \dots, n$ .

**A.3.1 More properties of the summation convention** *Note that the indices for the matrix of a bilinear form are both subscripts. This should help distinguish bilinear forms from linear maps, since in the latter there is one index up and one index down. If  $B$  is a bilinear form with matrix  $B_{ij}$ , and if  $u$  and  $v$  are vectors with components  $u^i, v^i$ ,  $i = 1, \dots, n$ , then*

$$B(u, v) = B_{ij} u^i v^j.$$

An important notion attached to a symmetric or skew-symmetric bilinear form is a map from  $V$  to  $V^*$ . If  $B$  is a bilinear form which is either symmetric or skew-symmetric, we define

a map  $B^b: V \rightarrow V^*$  by indicating how  $B^b(v) \in V^*$  acts on vector in  $V$ . That is, for  $v \in V$  we define  $B^b(v) \in V^*$  to be defined by

$$\langle B^b v; u \rangle = B(u, v), \quad u \in V.$$

The **rank** of  $B$  is defined to be  $\text{rank}(B) = \dim(\text{image}(B^b))$ .  $B$  is **nondegenerate** if  $\text{rank}(B) = \dim(V)$ . In this case  $B^b$  is an isomorphism since  $\dim(V) = \dim(V^*)$ , and we denote the inverse by  $B^\sharp: V^* \rightarrow V$ .

**A.3.2 More properties of the summation convention** *If  $\{e_1, \dots, e_n\}$  is a basis for  $V$  with dual basis  $\{e^1, \dots, e^n\}$ , then  $B^b(v) = B_{ij} v^j e^i$ . If  $B$  is nondegenerate then  $B^\sharp(\alpha) = B^{ij} \alpha_j e_i$ , where  $B^{ij}$ ,  $i, j = 1, \dots, n$ , are defined by  $B^{ij} B_{jk} = \delta_k^i$ .*

This statement is the content of Exercise [E2.17](#).

For symmetric bilinear forms, there are additional concepts which will be useful for us. In particular, the following theorem serves as the definition for the index, and this is a useful notion.

**A.3.3 Theorem** *If  $B$  is a symmetric bilinear form on a vector space  $V$ , then there exists a basis  $\{e_1, \dots, e_n\}$  for  $V$  so that the matrix for  $B$  in this basis is given by*

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

*The number of nonzero elements on the diagonal of this matrix is the rank of  $B$ . The number of  $-1$ 's along the diagonal is called the **index** of  $B$ .*

This theorem is proved by the Gram-Schmidt argument. Note that the number of  $+1$ 's on the diagonal of the matrix in the theorem is given by  $\text{rank}(B) - \text{ind}(B)$ .

## A.4 Inner products

An important special case is when all elements on the diagonal in Theorem [A.3.3](#) have the same sign. If all diagonal elements are  $+1$  then we say  $B$  is **positive-definite** and if all diagonal elements are  $-1$  then we say  $B$  is **negative-definite**. Clearly,  $B$  is positive-definite if and only if  $B(v, v) > 0$  whenever  $v \neq 0$ , and  $B$  is negative-definite if and only if  $B(v, v) < 0$  whenever  $v \neq 0$ . A symmetric, positive-definite bilinear form is something with which you are doubtless familiar: it is an **inner product**.

The most familiar example of an inner product is the **standard inner product** on  $\mathbb{R}^n$ . We denote this inner product by  $g_{\text{can}}$  and recall that it is defined by

$$g_{\text{can}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x^i y^i,$$

if  $\mathbf{x} = (x^1, \dots, x^n)$  and  $\mathbf{y} = (y^1, \dots, y^n)$ . If  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is the *standard basis* for  $\mathbb{R}^n$ , i.e.,  $\mathbf{e}_i$  consists of all zeros, with the exception of a +1 in the  $i$ th entry, then one verifies that the matrix for  $g_{\text{can}}$  in this basis is the  $n \times n$  identity matrix, denoted  $\mathbf{I}_n$ .

On a vector space  $V$  with an inner product  $g$ , it is possible to define the notions of symmetry and skew-symmetry for linear transformations  $A: V \rightarrow V$ . To wit, a linear transformation  $A$  is ***g-symmetric*** if  $g(Av_1, v_2) = g(v_1, Av_2)$  for every  $v_1, v_2 \in V$ , and is ***g-skew-symmetric*** if  $g(Av_1, v_2) = -g(v_1, Av_2)$  for every  $v_1, v_2 \in V$ . Often, when the inner product is understood, we shall just say “symmetric” or “skew-symmetric.” However, one should be sure to understand that an inner product is necessary to make sense of these notions.

## A.5 Changes of basis

In our presentation of Lagrangian mechanics, often objects are characterised by how they alter under changes of coordinate. This is reflected on the linear level by changes of basis. Let us characterise how components of the objects discussed above behave when bases are changed.

We let  $V$  be a finite-dimensional  $\mathbb{R}$ -vector space with  $\mathcal{E} = \{e_1, \dots, e_n\}$  and  $\mathcal{F} = \{f_1, \dots, f_n\}$  bases for  $V$ . Since both  $\mathcal{E}$  and  $\mathcal{F}$  are bases, we may write

$$f_i = P_i^j e_j, \quad i = 1, \dots, n$$

and

$$e_i = Q_i^j f_j, \quad i = 1, \dots, n,$$

for  $2n^2$  constants  $P_j^i$  and  $Q_j^i$ ,  $i, j = 1, \dots, n$ . Furthermore we have

$$e_i = Q_i^j f_j = Q_i^j P_j^k e_k.$$

Since  $\mathcal{E}$  is linearly independent, this implies that

$$Q_i^j P_j^k = \delta_k^i, \quad i, k = 1, \dots, n.$$

Thus  $P_j^i$  and  $Q_j^i$ ,  $i, j = 1, \dots, n$ , are the components of matrices which are inverses of one another.

We may also find relations between the dual bases  $\mathcal{E}^* = \{e^1, \dots, e^n\}$  and  $\mathcal{F}^* = \{f^1, \dots, f^n\}$  for  $V^*$ . We may certainly write

$$f^i = A_j^i e^j, \quad i = 1, \dots, n,$$

for some constants  $A_j^i$ ,  $i, j = 1, \dots, n$ . We then have

$$\delta_j^i = f^i(f_j) = A_k^i e^k(P_j^\ell e_\ell) = A_k^i P_j^k, \quad i, j = 1, \dots, n.$$

Therefore, we conclude that  $A_j^i = Q_j^i$ ,  $i, j = 1, \dots, n$ , so that we have

$$f^i = Q_j^i e^j, \quad i = 1, \dots, n.$$

In like manner we have

$$e^i = P_j^i f^j, \quad i = 1, \dots, n.$$



Let  $v \in V$  and write

$$v = v^i e_i = \tilde{v}^i f_i.$$

Using the relation between the basis vectors we have

$$v^i e_i = v^i Q_i^j f_j = \tilde{v}^j f_j.$$

Since  $\mathcal{F}$  is linearly independent, this allows us to conclude that the components  $v^i$ ,  $i = 1, \dots, n$ , and  $\tilde{v}^i$ ,  $i = 1, \dots, n$ , are related by

$$\tilde{v}^j = v^i Q_i^j, \quad j = 1, \dots, n.$$

Similarly, if  $\alpha \in V^*$ , then we write

$$\alpha = \alpha_i e^i = \tilde{\alpha}_i f^i.$$

Proceeding as we did for vectors in  $V$ , we compute

$$\alpha_i e^i = \alpha_i P_j^i f^j = \tilde{\alpha}_j f^j.$$

Since  $\mathcal{F}^*$  is linearly independent we conclude that the components  $\alpha^i$ ,  $i = 1, \dots, n$ , and  $\tilde{\alpha}^i$ ,  $i = 1, \dots, n$ , are related by

$$\tilde{\alpha}_j = \alpha_i P_j^i, \quad j = 1, \dots, n.$$

Now let  $A: V \rightarrow V$  be a linear map. The matrix of  $A$  in the basis  $\mathcal{E}$ ,  $A_j^i$ ,  $i, j = 1, \dots, n$ , are defined by

$$Ae_i = A_j^i e_j, \quad i = 1, \dots, n.$$

Similarly, the matrix of  $A$  in the basis  $\mathcal{F}$ ,  $\tilde{A}_j^i$ ,  $i, j = 1, \dots, n$ , are defined by

$$Af_i = \tilde{A}_j^i f_j, \quad i = 1, \dots, n.$$

We write

$$\tilde{A}_i^\ell f_\ell = Af_i = P_i^j Ae_j = P_i^j A_j^k e_k = P_i^j A_j^k Q_k^\ell f_\ell, \quad i = 1, \dots, n.$$

Therefore, since  $\mathcal{F}$  is linearly independent, we have

$$\tilde{A}_i^\ell = P_i^j A_j^k Q_k^\ell, \quad i, \ell = 1, \dots, n.$$

Note that this is the usual **similarity** transformation.

Finally, let us look at a bilinear map  $B: V \times V \rightarrow \mathbb{R}$ . We let the matrix of  $B$  in the basis  $\mathcal{E}$  be defined by

$$B_{ij} = B(e_i, e_j), \quad i, j = 1, \dots, n,$$

and the matrix of  $B$  in the basis  $\mathcal{F}$  be defined by

$$\tilde{B}_{ij} = B(f_i, f_j), \quad i, j = 1, \dots, n.$$

Note that we have

$$\tilde{B}_{ij} = B(f_i, f_j) = B(P_i^k e_k, P_j^\ell e_\ell) = P_i^k P_j^\ell B_{k\ell}, \quad i, j = 1, \dots, n.$$

This relates for us the matrices of  $B$  in the two bases.



# Appendix B

## Differential calculus

Our approach to Lagrangian mechanics is to use a basic differential geometric formulation. In order to carry this out, the reader must be supposed familiar with some basic topic in real analysis. In this section we quickly review the prerequisite results.

### B.1 The topology of Euclidean space

We let  $\mathbb{R}^n$  be the collection of ordered  $n$ -tuples of real numbers. This is a vector space, of course, but in this section, and indeed throughout this appendix, we are not so interested in the algebraic structure as much as the analytic properties of  $\mathbb{R}^n$ .

A subset  $U \subset \mathbb{R}^n$  is **open** if for each  $\mathbf{x} \in U$  there exists  $\epsilon > 0$  so that the **open ball** of radius  $\epsilon$  centred at  $\mathbf{x}$ ,

$$B_\epsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\| < \epsilon\},$$

is contained in  $U$ . Here  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by  $\|\mathbf{x}\| = \sqrt{g_{\text{can}}(\mathbf{x}, \mathbf{x})}$ . A subset  $A \subset \mathbb{R}^n$  is **closed** if its complement  $\mathbb{R}^n \setminus A$  is open. Here

$$\mathbb{R}^n \setminus A = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \notin A\}.$$

A **limit point** of a subset  $S \subset \mathbb{R}^n$  is a point  $\mathbf{x} \in \mathbb{R}^n$  with the property that for each  $\epsilon > 0$  the sets

$$S \cap B_\epsilon(\mathbf{x}), \quad (\mathbb{R}^n \setminus S) \cap B_\epsilon(\mathbf{x})$$

are nonempty. One verifies that a set is closed if and only if it contains all of its limit points.

If  $S \subset \mathbb{R}^n$  is any subset and if  $A \subset S$ , we say  $A$  is **open relative to S** if for every  $\mathbf{x} \in A$  and for every  $\epsilon > 0$  the set  $B_\epsilon(\mathbf{x}) \cap S$  is contained in  $A$ . The set  $A \subset S$  is **closed relative to S** if  $S \setminus A$  is open.

A subset  $C \subset \mathbb{R}^n$  is **compact** if it is closed and bounded. One can show that this definition of compactness is equivalent to saying that for every collection of open sets  $\{U_\alpha\}_{\alpha \in A}$  indexed by an arbitrary index set  $A$  with the property that  $C \subset \cup_{\alpha \in A} U_\alpha$ , there exists a finite subset  $\{U_{\alpha_i}\}_{i \in \{1, \dots, N\}}$  of these open set with the property that  $C \subset \cup_{i \in \{1, \dots, N\}} U_{\alpha_i}$ . This is often abbreviated suggestively by the words, “every open cover of  $C$  possesses a finite subcover.” This equivalent notion of compactness has the advantage that it can be applied to more general situations where one has a notion of open sets, but no notion of boundedness.

The above terminology expresses the fact that  $\mathbb{R}^n$  is a “topological space.” Generally, a **topological space** is a pair  $(S, \mathcal{O})$  where  $S$  is a set and  $\mathcal{O}$  is a collection of subsets of  $S$  satisfying

1.  $\cup_{i \in I} U_i \in \mathcal{O}$  for an arbitrary collection  $\{U_i\}_{i \in I}$  of subsets from  $\mathcal{O}$ ,
2.  $\cap_{i=1}^k U_i \in \mathcal{O}$  for every finite collection  $\{U_1, \dots, U_k\}$  of subsets from  $\mathcal{O}$ , and
3.  $\emptyset \in \mathcal{O}$  and  $S \in \mathcal{O}$ .

## B.2 Mappings between Euclidean spaces

Let  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  be open sets and let  $f: U \rightarrow V$  be a map. The map  $f$  is **continuous at  $\mathbf{x}_0 \in U$**  if for every  $\epsilon > 0$  there exists  $\delta > 0$  so that  $\|\mathbf{x} - \mathbf{x}_0\| < \delta$  implies that  $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \epsilon$ . One can verify that this definition of continuity at  $\mathbf{x}_0$  is equivalent to asking that for every neighbourhood  $\tilde{V}$  of  $\mathbf{x}_0$ , the subset  $f^{-1}(\tilde{V})$  is an open subset of  $U$ . In this way one can define continuity only using the properties of open sets, without recourse to the notion of a norm. This allows one to talk about continuous functions in a more general setting. The map  $f$  is **continuous** if it is continuous at every  $\mathbf{x} \in U$ .

More stringent than mere continuity is differentiability. A map  $f: U \rightarrow V$  as above is **differentiable at  $\mathbf{x}_0 \in U$**  if there exists a linear map  $L \in L(\mathbb{R}^n; \mathbb{R}^m)$  so that the limit

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - L(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|}$$

exists. If the limit does exist, we shall denote the linear map  $L$  by  $Df(\mathbf{x}_0)$ . If  $f$  is differentiable at each  $\mathbf{x} \in U$  then  $f$  is **differentiable**, and we have defined a map  $Df: U \rightarrow L(\mathbb{R}^n; \mathbb{R}^m)$ . If this map is continuous, then  $f$  is **continuously differentiable** on  $U$ . Let us write

$$f(\mathbf{x}) = (f^1(\mathbf{x}), \dots, f^m(\mathbf{x}))$$

for  $m$  functions  $f^a: U \rightarrow \mathbb{R}$ . If  $f$  is differentiable, then the partial derivative of  $f^a$  with respect to  $x^i$  exists for each  $a = 1, \dots, m$  and  $i = 1, \dots, n$ . One then sees that the matrix of the linear map  $Df(\mathbf{x}_0)$  with respect to the standard bases for  $\mathbb{R}^n$  and  $\mathbb{R}^m$  is

$$\begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \cdots & \frac{\partial f^m}{\partial x^n} \end{bmatrix},$$

where all partial derivatives are evaluated at  $\mathbf{x}_0$ . This matrix and the linear map  $Df(\mathbf{x}_0)$  will interchangeably be referred to as the **Jacobian** of  $f$ .

Let  $f: U \rightarrow V$  be a continuously differentiable map between open sets  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$ . A point  $\mathbf{x}_0 \in U$  is a **critical point** for  $f$  if  $Df(\mathbf{x}_0)$  does not have maximal rank. A point  $\mathbf{y}_0 \in V$  is a **regular value** for  $f$  if there are no critical points in the set  $f^{-1}(\mathbf{y}_0)$ . If  $\mathbf{y}_0 \in V$  is not a regular value, it is a **critical value** for  $f$ .

## B.3 Critical points of $\mathbb{R}$ -valued functions

The general notions at the end of the previous section will be most interesting for us in the particular case where  $f$  takes its values in  $\mathbb{R}$ . In this case, we adopt the convention of writing  $Df(\mathbf{x}_0)$  as  $df(\mathbf{x}_0)$ . Therefore,  $\mathbf{x}_0$  is a critical point if and only if  $df(\mathbf{x}_0) = 0$ . Note that since  $df(\mathbf{x}_0) \in L(\mathbb{R}^n; \mathbb{R})$  we may say that  $df(\mathbf{x}_0) \in (\mathbb{R}^n)^*$ .

Since  $f$  is  $\mathbb{R}$ -valued, it makes sense to ask if a point  $\mathbf{x}_0 \in U$  is a local maximum or minimum for  $f$ . A necessary condition for a local maximum or minimum to occur at  $\mathbf{x}_0$  is, as you will recall, is that  $\mathbf{x}_0$  be a critical point. To assess further whether a critical point is a local maximum or minimum, one looks at the second derivative. The second derivative is to be thought of as the derivative of the map  $df: \mathbb{R}^n \rightarrow (\mathbb{R}^n)^*$ . If  $\{\mathbf{e}^1, \dots, \mathbf{e}^n\}$  is the basis for  $(\mathbb{R}^n)^*$  dual to the standard basis, then we may write the map  $df$  as

$$df(\mathbf{x}) = \frac{\partial f}{\partial x^1}(\mathbf{x})\mathbf{e}^1 + \cdots + \frac{\partial f}{\partial x^n}(\mathbf{x})\mathbf{e}^n.$$

To differentiate the map  $\mathbf{d}f$  we simply differentiate the components of  $\mathbf{d}f$  with respect to  $x^1, \dots, x^n$ . The resulting Jacobian is then represented by the square matrix of second partial derivatives

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} & \frac{\partial^2 f}{\partial x^1 \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^1 \partial x^n} \\ \frac{\partial^2 f}{\partial x^2 \partial x^1} & \frac{\partial^2 f}{\partial x^2 \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^2 \partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x^n \partial x^1} & \frac{\partial^2 f}{\partial x^n \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^n} \end{bmatrix}. \quad (\text{B.1})$$

When this is evaluated at a critical point  $\mathbf{x}_0$  for  $f$ , we call it the **Hessian** of  $f$  at  $\mathbf{x}_0$ , and denote it by  $\text{Hess}(f)(\mathbf{x}_0)$ . Note that we only define the Hessian at critical points for  $f$ . For reasons we do not get into here, we do not evaluate the matrix (B.1) at general points in  $U$ , although there is nothing in principle stopping us from doing this. Note that we adopt with our summation convention the convention that the superscripts in the denominator in the Hessian comprise subscripts. Thus the components of the Hessian have index down, and this leads us to believe that the Hessian should be a bilinear form, symmetric by virtue of equality of mixed partial derivatives. It is indeed the case that  $\text{Hess}(f)(\mathbf{x}_0)$  is a symmetric bilinear form,<sup>1</sup> and so we can speak of its rank and index. The critical point  $\mathbf{x}_0$  is **nondegenerate** if  $\text{rank}(\text{Hess}(f)(\mathbf{x}_0)) = n$ , and **degenerate** otherwise.

The following important result tells us what a function must “look like” near a nondegenerate critical point.

**B.3.1 Theorem (Morse’s Lemma)** *If  $\mathbf{x}_0 \in \mathbb{R}^n$  is a nondegenerate critical point of a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , then there exists a neighbourhood  $U$  of  $\mathbf{x}_0 \in \mathbb{R}^n$ , a neighbourhood  $V$  of  $\mathbf{0} \in \mathbb{R}^n$ , and a diffeomorphism  $\phi: U \rightarrow V$  with the property that*

$$f \circ \phi^{-1}(\mathbf{y}) = (y^1)^2 + \cdots + (y^k)^2 - (y^{k+1})^2 - \cdots - (y^n)^2.$$

From this result, we can conclude the following for a nondegenerate critical point  $\mathbf{x}_0$ :

1. if  $\text{ind}(\text{Hess}(f)(\mathbf{x}_0)) = n$  then  $\mathbf{x}_0$  is a strict local maximum for  $f$ ;
2. if  $\text{ind}(\text{Hess}(f)(\mathbf{x}_0)) = 0$  then  $\mathbf{x}_0$  is a strict local minimum for  $f$ ;
3. if  $0 < \text{ind}(\text{Hess}(f)(\mathbf{x}_0)) < n$  then  $\mathbf{x}_0$  is a saddle point for  $f$ ; in particular,  $\mathbf{x}_0$  is neither a local maximum nor a local minimum for  $f$ .

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<sup>1</sup>This is at the root of our refusal to evaluate the matrix (B.1) at non-critical points for  $f$ . When at general points, there is a certain sense in which the matrix (B.1) is *not* the matrix for a bilinear form.



# Appendix C

## Ordinary differential equations

Although the subject of these notes is not dynamical systems, in order to say some elementary but useful things about the behaviour of Lagrangian systems, it is essential to have on hand a collection of standard tools for handling differential equations. We assume that the reader knows what a differential equation *is*, and is aware of the existence and uniqueness theory for such.

### C.1 Linear ordinary differential equations

Although linear equations themselves are not necessarily interesting for us—few Lagrangian systems are actually linear—when linearising differential equations, one naturally obtains linear equations (of course). Thus we record some basic facts about linear differential equations.

Let  $V$  be an  $n$ -dimensional  $\mathbb{R}$ -vector space. A ***linear ordinary differential equation with constant coefficients*** on  $V$  is a differential equation of the form

$$\dot{x} = Ax, \quad x(0) = x_0 \tag{C.1}$$

for a curve  $t \mapsto x(t) \in V$  and where  $A: V \rightarrow V$  is a linear transformation. Given a linear transformation  $A$  we define a linear transformation  $e^A$  by

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This series may be shown to converge using the fact that the series for the usual exponential of a real variable converges. We next claim that the solution to (C.1) is  $x(t) = e^{At}x_0$ . To see this we simply substitute this proposed solution into the differential equation:

$$\begin{aligned} \dot{x}(t) &= \frac{d}{dt} \sum_{k=1}^{\infty} \frac{A^k t^k}{k!} \\ &= \sum_{k=1}^{\infty} \frac{k A^k t^{k-1}}{k!} \\ &= A \sum_{k=1}^{\infty} \frac{A^{k-1} t^{k-1}}{(k-1)!} \\ &= A \sum_{\ell=0}^{\infty} \frac{A^\ell t^\ell}{\ell!} = Ax(t). \end{aligned}$$

Thus  $x(t)$  satisfies the differential equation. It is also evident that  $x(t)$  satisfies the initial conditions, and so by uniqueness of solutions, we are justified in saying that the solution of the initial value problem (C.1) is indeed  $x(t) = e^{At}x_0$ .

Let us turn to the matter of computing  $e^{At}$ . First note that if  $P$  is an invertible linear transformation on  $V$  then one readily shows that  $e^{P(At)P^{-1}} = Pe^{At}P^{-1}$ . Thus  $e^{At}$  is independent of similarity transformations, and so we may simplify things by choosing a basis  $\{e_1, \dots, e_n\}$  for  $V$  so that the initial value problem (C.1) becomes the ordinary differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (\text{C.2})$$

in  $\mathbb{R}^n$ . To compute  $e^{At}$  one proceeds as follows. First one computes the eigenvalues for  $\mathbf{A}$ . There will be  $n$  of these in total, counting algebraic multiplicities and complex conjugate pairs. One treats each eigenvalue separately. For a real eigenvalue  $\lambda_0$  with algebraic multiplicity  $k = m_a(\lambda_0)$ , one must compute  $k$  linearly independent solutions. For a complex eigenvalue  $\lambda_0$  with algebraic multiplicity  $\ell = m_a(\lambda_0)$ , one must compute  $2\ell$  linearly independent solutions, since  $\bar{\lambda}_0$  is also necessarily an eigenvalue with algebraic multiplicity  $\ell$ .

We first look at how to deal with real eigenvalues. Let  $\lambda_0$  be one such object with algebraic multiplicity  $k$ . It is a fact that the matrix  $(\mathbf{A} - \lambda_0\mathbf{I}_n)^k$  will have rank  $n - k$ , and so will have a kernel of dimension  $k$  by the Rank-Nullity Theorem. Let  $\mathbf{u}_1, \dots, \mathbf{u}_k$  be a basis for  $\ker((\mathbf{A} - \lambda_0\mathbf{I}_n)^k)$ . We call each of these vectors a **generalised eigenvector**. If the geometric multiplicity of  $\lambda_0$  is also  $k$ , then the generalised eigenvectors will simply be the usual eigenvectors. If  $m_g(\lambda_0) < m_a(\lambda_0)$  then a generalised eigenvector may or may not be an eigenvector. Corresponding to each generalised eigenvector  $\mathbf{u}_i$ ,  $i = 1, \dots, k$ , we will define a solution to (C.2) by

$$\mathbf{x}_i(t) = e^{\lambda_0 t} \exp((\mathbf{A} - \lambda_0\mathbf{I}_n)t)\mathbf{u}_i.$$

Note that because  $\mathbf{u}_i$  is a generalised eigenvector, the infinite series  $\exp((\mathbf{A} - \lambda_0\mathbf{I}_n)t)\mathbf{u}_i$  will have only a finite number of terms—at most  $k$  in fact. Indeed we have

$$\exp((\mathbf{A} - \lambda_0\mathbf{I}_n)t)\mathbf{u}_i = \left( \mathbf{I}_n + t(\mathbf{A} - \lambda_0\mathbf{I}_n) + \frac{t^2}{2!}(\mathbf{A} - \lambda_0\mathbf{I}_n)^2 + \dots + \frac{t^{k-1}}{(k-1)!}(\mathbf{A} - \lambda_0\mathbf{I}_n)^{k-1} \right)\mathbf{u}_i,$$

since the remaining terms in the series will be zero. In any case, it turns out that the  $k$  vector functions  $\mathbf{x}_1(t), \dots, \mathbf{x}_k(t)$  so constructed will be linearly independent solutions of (C.2). This tells us how to manage the real case.

Now let us look at the complex case. Thus let  $\lambda_0$  be a complex eigenvalue (with nonzero imaginary part) of algebraic multiplicity  $\ell$ . This means that  $\bar{\lambda}_0$  will also be an eigenvalue of algebraic multiplicity  $\ell$  since  $\mathbf{A}$ , and hence  $P_{\mathbf{A}}(\lambda)$ , is real. Thus we need to find  $2\ell$  linearly independent solutions. We do this by following the exact same idea as in the real case, except that we think of  $\mathbf{A}$  as being a complex matrix for the moment. In this case it is still true that the matrix  $(\mathbf{A} - \lambda_0\mathbf{I}_n)^\ell$  will have an  $\ell$ -dimensional kernel, and we can take vectors  $\mathbf{u}_1, \dots, \mathbf{u}_\ell$  as a basis for this kernel. Note, however, that since  $(\mathbf{A} - \lambda_0\mathbf{I}_n)^\ell$  is complex, these vectors will also be complex. But the procedure is otherwise identical to the real case. One then constructs  $\ell$  complex vector functions

$$\mathbf{z}_j(t) = e^{\lambda_0 t} \exp((\mathbf{A} - \lambda_0\mathbf{I}_n)t)\mathbf{u}_j.$$

Each such complex vector function will be a sum of its real and imaginary parts:  $\mathbf{z}_j(t) = \mathbf{x}_j(t) + i\mathbf{y}_j(t)$ . It turns out that the  $2\ell$  real vector functions  $\mathbf{x}_1(t), \dots, \mathbf{x}_\ell(t), \mathbf{y}_1(t), \dots, \mathbf{y}_\ell(t)$  are linearly independent solutions to (C.2).



We still haven't gotten to the matrix exponential yet, but all the hard work is done. Using the above methodology we may in principle compute for any  $n \times n$  matrix  $\mathbf{A}$ ,  $n$  linearly independent solutions  $\mathbf{x}_1, \dots, \mathbf{x}_n(t)$ .<sup>1</sup> If we assemble the resulting solutions into the columns of a matrix  $\mathbf{X}(t)$ :

$$\mathbf{X}(t) = [ \mathbf{x}_1(t) \mid \cdots \mid \mathbf{x}_n(t) ],$$

the resulting matrix is an example of a fundamental matrix. Generally, a **fundamental matrix** is any  $n \times n$  matrix function of  $t$  whose columns form  $n$  linearly independent solutions to (C.2). What we have done above is give a recipe for computing a fundamental matrix (there are an infinite number of these). The following result connects the construction of a fundamental matrix with the matrix exponential.

**C.1.1 Theorem** *Given any fundamental matrix  $\mathbf{X}(t)$  we have  $e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$ .*

Thus, once we have a fundamental matrix, the computation of the matrix exponential is just algebra, although computing inverses of matrices of any size is a task best left to the computer.

One of the essential observations from the above discussion is that the behaviour of the solutions to the differential equation (C.1) are largely governed by the eigenvalues of  $A$ .

## C.2 Fixed points for ordinary differential equations

In this section we consider the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \tag{C.3}$$

for  $\mathbf{x}(t) \in \mathbb{R}^n$  and with  $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  a smooth map. A **fixed point** for (C.3) is a point  $\mathbf{x}_0$  for which  $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ . Thus if  $\mathbf{x}_0$  is a fixed point, the trivial curve  $t \mapsto \mathbf{x}_0$  is a solution to the differential equation. A fixed point  $\mathbf{x}_0$  is **stable** if for each  $\epsilon > 0$  there exists  $\delta > 0$  so that if  $\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta$ , then  $\|\mathbf{x}(t) - \mathbf{x}_0\| < \epsilon$  for all  $t > 0$ . The fixed point  $\mathbf{x}_0$  is **asymptotically stable** if there exists  $\delta > 0$  so that if  $\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta$  then  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_0$ . These notions of stability are often said to be “in the sense of Liapunov,” to distinguish them from other definitions of stability. In Figure C.1 we give some intuition concerning our definitions.

As a first pass at trying to determine when a fixed point is stable or asymptotically stable, one linearises the differential equation (C.3) about  $\mathbf{x}_0$ . Thus one has a solution  $\mathbf{x}(t)$  of the differential equation, and uses the Taylor expansion to obtain an approximate expression for the solution:

$$\begin{aligned} \frac{d}{dt}(\mathbf{x}(t) - \mathbf{x}_0) &= \mathbf{f}(\mathbf{x}_0) + \mathbf{D}\mathbf{f}(\mathbf{x}_0) \cdot (\mathbf{x}(t) - \mathbf{x}_0) + \cdots \\ \implies \dot{\boldsymbol{\xi}}(t) &= \mathbf{D}\mathbf{f}(\mathbf{x}_0) \cdot \boldsymbol{\xi}(t) + \cdots \end{aligned}$$

where  $\boldsymbol{\xi}(t) = \mathbf{x}(t) - \mathbf{x}_0$ . Thus linearisation leads us to think of the differential equation

$$\dot{\boldsymbol{\xi}}(t) = \mathbf{D}\mathbf{f}(\mathbf{x}_0) \cdot \boldsymbol{\xi}(t) \tag{C.4}$$

---

<sup>1</sup>Note that the solutions  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are those obtained from both real *and* complex eigenvalues. Therefore, the solutions denoted above as “ $\mathbf{y}_i(t)$ ” for complex eigenvalues will be included in the  $n$  linearly independent solutions, except now I am calling everything  $\mathbf{x}_j(t)$ .

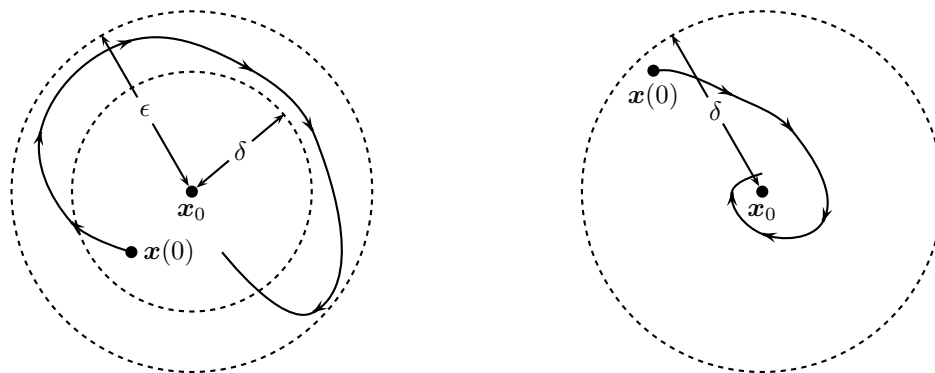


Figure C.1 A stable fixed point (left) and an asymptotically stable fixed point (right)

as somehow approximating the actual differential equation near the fixed point  $\mathbf{x}_0$ . Let us define notions of stability of  $\mathbf{x}_0$  which are related only to the linearisation. We say that  $\mathbf{x}_0$  is **spectrally stable** if  $\mathbf{Df}(\mathbf{x}_0)$  has no eigenvalues in the positive complex plane, and that  $\mathbf{x}_0$  is **linearly stable** (resp. **linearly asymptotically stable**) if the linear system (C.4) is stable (resp. asymptotically stable). Let us introduce the notation

$$\begin{aligned}\mathbb{C}_+ &= \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\} \\ \bar{\mathbb{C}}_+ &= \{z \in \mathbb{C} \mid \operatorname{Re}(z) \geq 0\} \\ \mathbb{C}_- &= \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\} \\ \bar{\mathbb{C}}_- &= \{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq 0\}.\end{aligned}$$

From our discussion of linear ordinary differential equations in Section C.1 we have the following result.

**C.2.1 Proposition** *A fixed point  $\mathbf{x}_0$  is linearly stable if and only if the following two conditions hold:*

- (i)  $\mathbf{Df}(\mathbf{x}_0)$  has no eigenvalues in  $\mathbb{C}_+$ , and
- (ii) all eigenvalues of  $\mathbf{Df}(\mathbf{x}_0)$  with zero real part have equal geometric and algebraic multiplicities.

*The point  $\mathbf{x}_0$  is linearly asymptotically stable if and only if all eigenvalues of  $\mathbf{Df}(\mathbf{x}_0)$  lie in  $\mathbb{C}_-$ .*

A question one can ask is how much stability of the linearisation has to do with stability of the actual fixed point. More generally, one can speculate on how the solutions of the linearisation are related to the actual solutions. The following important theorem due to Hartman and Grobman tells us when we can expect the solutions to (C.3) near  $\mathbf{x}_0$  to “look like” those of the linear system. The statement of the result uses the notion of a flow which we define in Section 2.2.

**C.2.2 Theorem (Hartman-Grobman Theorem)** *Let  $\mathbf{x}_0$  be a fixed point for the differential equation (C.3) and suppose that the  $n \times n$  matrix  $\mathbf{Df}(\mathbf{x}_0)$  has no eigenvalues on the imaginary axis. Let  $F_f$  denote the flow associated with the differential equation (C.3). Then there exists a neighbourhood  $V$  of  $\mathbf{0} \in \mathbb{R}^n$  and a neighbourhood  $U$  of  $\mathbf{x}_0 \in \mathbb{R}^n$ , and a homeomorphism (i.e., a continuous bijection)  $\phi: V \rightarrow U$  with the property that  $\phi(e^{\mathbf{Df}(\mathbf{x}_0)t}\mathbf{x}) = F_f(t, \phi(\mathbf{x}_0))$ .*

The idea is that when the eigenvalue *all* have nonzero real part, then one can say that the flow for the nonlinear system (C.3) looks like the flow of the linear system (C.4) in a neighbourhood of  $\mathbf{x}_0$ . When  $D\mathbf{f}(\mathbf{x}_0)$  has eigenvalues on the imaginary axis, one cannot make any statements about relating the flow of the nonlinear system with its linear counterpart. In such cases, one really has to look at the nonlinear dynamics, and this becomes difficult. A proof of the Hartman-Grobman theorem can be found in [Palis, Jr. and de Melo 1982].

Let us be more specific about the relationships which can be made about the behaviour of the nonlinear system (C.3) and its linear counterpart (C.4). Let  $E^s(\mathbf{x}_0)$  be the subspace of  $\mathbb{R}^n$  containing all generalised eigenvectors for eigenvalues of  $D\mathbf{f}(\mathbf{x}_0)$  in  $\mathbb{C}_-$ , and let  $E^u(\mathbf{x}_0)$  be the subspace of  $\mathbb{R}^n$  containing all generalised eigenvectors for eigenvalues of  $D\mathbf{f}(\mathbf{x}_0)$  in  $\mathbb{C}_+$ .  $E^s(\mathbf{x}_0)$  is called the **linear stable subspace** at  $\mathbf{x}_0$  and  $E^u(\mathbf{x}_0)$  is called the **linear unstable subspace** at  $\mathbf{x}_0$ . For the linearised system (C.4), the subspaces  $E^s(\mathbf{x}_0)$  and  $E^u(\mathbf{x}_0)$  will be invariant sets (i.e., if one starts with an initial condition in one of these subspaces, the solution of the differential equation will remain on that same subspace). Indeed, initial conditions in  $E^s(\mathbf{x}_0)$  will tend to  $\mathbf{0}$  as  $t \rightarrow \infty$ , and initial conditions in  $E^u(\mathbf{x}_0)$  will explode as  $t \rightarrow \infty$ . The following result says that analogues of  $E^s(\mathbf{x}_0)$  and  $E^u(\mathbf{x}_0)$  exist.

**C.2.3 Theorem (Stable and Unstable Manifold Theorem)** *Let  $\mathbf{x}_0$  be a fixed point for the differential equation (C.3).*

(i) *There exists a subset  $W^s(\mathbf{x}_0)$  of  $\mathbb{R}^n$  with the following properties:*

- (a)  $W^s(\mathbf{x}_0)$  is invariant under the flow  $F_{\mathbf{f}}$ ;
- (b)  $E^s(\mathbf{x}_0)$  forms the tangent space to  $W^s(\mathbf{x}_0)$  at  $\mathbf{x}_0$ ;
- (c)  $\lim_{t \rightarrow \infty} F_{\mathbf{f}}(t, \mathbf{x}) = \mathbf{x}_0$  for all  $\mathbf{x} \in W^s(\mathbf{x}_0)$ .

(ii) *There exists a subset  $W^u(\mathbf{x}_0)$  of  $\mathbb{R}^n$  with the following properties:*

- (a)  $W^u(\mathbf{x}_0)$  is invariant under the flow  $F_{\mathbf{f}}$ ;
- (b)  $E^u(\mathbf{x}_0)$  forms the tangent space to  $W^u(\mathbf{x}_0)$  at  $\mathbf{x}_0$ ;
- (c)  $\lim_{t \rightarrow -\infty} F_{\mathbf{f}}(t, \mathbf{x}) = \mathbf{x}_0$  for all  $\mathbf{x} \in W^u(\mathbf{x}_0)$ .

$W^s(\mathbf{x}_0)$  is called the **stable manifold** for the fixed point  $\mathbf{x}_0$ , and  $W^u(\mathbf{x}_0)$  is called the **unstable manifold** for the fixed point  $\mathbf{x}_0$ .

Again, we refer to [Palis, Jr. and de Melo 1982] for a proof. Some simple examples of stable and unstable manifolds can be found in Section 3.4. A picture of what is stated in this result is provided in Figure C.2. The idea is that the invariant sets  $E^s(\mathbf{x}_0)$  and  $E^u(\mathbf{x}_0)$  for the linear system do have counterparts in the nonlinear case. Near  $\mathbf{x}_0$  they follow the linear subspaces, but when we go away from  $\mathbf{x}_0$ , we cannot expect things to look at all like the linear case, and this is exhibited even in the simple examples of Section 3.4.

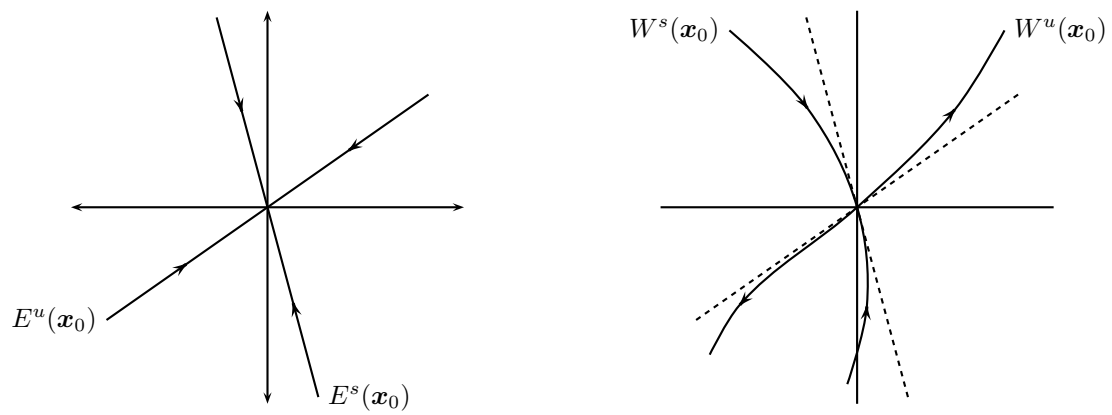


Figure C.2 Stable and unstable manifolds of a fixed point

# Appendix D

## Some measure theory

We refer the reader to [Cohn 1980] for more details on measure theory than we provide here.

In measure theory, it is helpful to have on hand the extended real numbers. Let us denote by  $[-\infty, \infty]$  the union of the real numbers  $\mathbb{R}$  with the two point sets with one element  $\{-\infty\}$  and  $\{\infty\}$ . This set is ordered in the following way. Points in  $(-\infty, \infty)$  adopt the usual order on  $\mathbb{R}$ , and we declare that  $-\infty < x$  and  $x < \infty$  for every  $x \in (-\infty, \infty)$ . We also declare that  $x + \infty = \infty$  and  $x + (-\infty) = -\infty$  for every  $x \in [-\infty, \infty)$ , and that  $\infty + \infty = \infty$ , thus defining addition in  $[-\infty, \infty]$  (we ask that addition be commutative and we decree that  $(-\infty) + \infty$  makes no sense). Multiplication on  $[-\infty, \infty]$  is defined by  $x \cdot \infty = \infty$  and  $x \cdot (-\infty) = -\infty$  for  $x > 0$ ,  $x \cdot \infty = -\infty$  and  $x \cdot (-\infty) = \infty$  for  $x < 0$ ,  $\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$ ,  $(-\infty) \cdot \infty = -\infty$ , and  $0 \cdot \infty = 0 \cdot (-\infty) = 0$ . We declare multiplication to be commutative. Note that  $[-\infty, \infty]$  is *not* a field!

A measure is applied to a certain class of subsets. Precisely, if  $S$  is a set, a  **$\sigma$ -algebra** is a collection  $\mathcal{A}$  of subsets of  $S$  with the properties

SA1.  $S \in \mathcal{A}$ ,

SA2. if  $A \in \mathcal{A}$  then  $S \setminus A \in \mathcal{A}$ ,

SA3. for each sequence  $\{A_i\}_{i \in \mathbb{Z}^+}$  in  $\mathcal{A}$ ,  $\cup_{i \in \mathbb{Z}^+} A_i \in \mathcal{A}$ , and

SA4. for each sequence  $\{A_i\}_{i \in \mathbb{Z}^+}$  in  $\mathcal{A}$ ,  $\cap_{i \in \mathbb{Z}^+} A_i \in \mathcal{A}$ .

D.0.4 Examples 1. If  $S$  is a set, the the collection  $2^S$  of all subsets of  $S$  is a  $\sigma$ -algebra.

2. Given an arbitrary collection  $\mathcal{F}$  of subsets of  $S$ , we may construct “the smallest  $\sigma$ -algebra containing  $\mathcal{F}$ .” We do this as follows. Let  $\mathcal{C}_{\mathcal{F}}$  be the collection of  $\sigma$ -algebras containing  $\mathcal{F}$ . Note that  $2^S$  is a  $\sigma$ -algebra containing  $\mathcal{F}$  so  $\mathcal{C}_{\mathcal{F}}$  is not empty. One then ascertains that

$$\bigcap_{\mathcal{A}} \{\mathcal{A} \in \mathcal{C}_{\mathcal{F}}\}$$

is itself a  $\sigma$ -algebra. Obviously it is the smallest  $\sigma$ -algebra containing  $\mathcal{F}$  by the very manner in which it was constructed. We call this  $\sigma$ -algebra that **generated** by  $\mathcal{F}$ .

3. We shall be interested in measures on  $\mathbb{R}^n$ . There is a natural  $\sigma$ -algebra on  $\mathbb{R}^n$  which we call the **Borel subsets**. This is the  $\sigma$ -algebra generated by the following collection of subsets:

(a) the closed subsets of  $\mathbb{R}^n$ ;

(b) the closed half-spaces

$$\{(x^1, \dots, x^i, \dots, x^n) \mid x^i \leq b \text{ for some } i = 1, \dots, n \text{ and } b \in \mathbb{R}\};$$

(c) the cubes  $\{(x^1, \dots, x^n) \mid a^i < x^i \leq b^i, i = 1, \dots, n\}$ .

As usual, we consider the standard topology on  $\mathbb{R}^n$  when making these definitions. In the sequel, unless we state otherwise, when we need a  $\sigma$ -algebra on  $\mathbb{R}^n$  we shall suppose it to be the collection of Borel subsets.  $\square$

If  $S$  is a set with  $\mathcal{A}$  a  $\sigma$ -algebra on  $S$ , then a **countably additive measure** (resp. a **finitely additive measure**) on  $S$  is a function  $\mu: \mathcal{A} \rightarrow [0, \infty]$  with the property that for every sequence  $\{A_i\}_{i \in \mathbb{Z}^+}$  (resp. for every finite collection  $\{A_1, \dots, A_n\} \subset \mathcal{A}$ ) we have

$$\mu\left(\bigcup_{i \in \mathbb{Z}^+} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad (\text{resp.} \quad \mu\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)),$$

and such that  $\mu(\emptyset) = 0$ .

Now suppose that  $S$  has a topology  $\mathcal{O}$ . Let  $Z$  denote the union of all open sets in  $S$  which have zero measure. This union exists, and is an open set. Thus its complement is closed, and we call this complement the **support** of  $\mu$ . Thus the support of a measure is the smallest closed set whose complement has zero measure. A **mass distribution** on  $S$  is a measure  $\mu$  which has compact support and which has the property that  $\mu(S) < \infty$ . Note that by the Heine-Borel theorem a mass distribution on  $\mathbb{R}^n$  (with its usual topology) has bounded support.

Now let us turn to integrating functions with respect to measures. A function  $f: S \rightarrow \mathbb{R}$  is **simple** if  $f(S) \subset \mathbb{R}$  is finite. If  $\mathcal{A}$  is a  $\sigma$ -algebra on  $S$  and  $\mu: \mathcal{A} \rightarrow [0, \infty]$  is a measure, then a simple function  $f$  on  $S$  taking values  $a_1, \dots, a_n$  is **measurable** if  $A_i \triangleq f^{-1}(a_i) \in \mathcal{A}$ ,  $i = 1, \dots, n$ . If  $a_i > 0$  for  $i = 1, \dots, n$ , then we define  $\int f \, d\mu = \sum_{i=1}^n a_i \mu(A_i)$ . If  $f$  is an arbitrary function taking values in  $[0, \infty]$  then we define

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu \mid g \text{ is a positive, measurable simple function} \right. \\ \left. \text{with } g(x) \leq f(x) \text{ for } x \in S \right\}.$$

For an arbitrary function  $f: S \rightarrow [-\infty, \infty]$  we may define functions  $f^+$  and  $f^-$  on  $S$  by

$$f^+(x) = \max\{0, f(x)\}, \quad f^-(x) = -\min\{0, f(x)\}.$$

We then define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

Of course,  $\int f \, d\mu$  may not exist. However, we shall say that  $f$  is **measurable** if  $f^{-1}((-\infty, x)) \in \mathcal{A}$  for every  $x \in \mathbb{R}$ , and one may show that if  $f$  is measurable then  $\int f^+ \, d\mu$  and  $\int f^- \, d\mu$  exist in  $[-\infty, \infty]$ . If these integrals are both finite then  $\int f \, d\mu$  is finite and  $f$  is said to be **integrable**. Given this definition of the integral of a function, one may then proceed to verify that it is linear:

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu, \quad \int (a f) \, d\mu = a \int f \, d\mu \quad (\text{D.1})$$

for  $f, g: S \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ . If  $V$  is a finite-dimensional vector space, then one may extend our definition of the integral to an  $V$ -valued function  $f: S \rightarrow V$  by choosing a basis for  $V$ , and integrating  $f$  component-wise. By (D.1) such an operation will be independent of basis.

Let us now consider how we might restrict a measure to a subset of  $S$ . Not just any subset will do; we must consider subsets contained in the corresponding  $\sigma$ -algebra. We let  $\mathcal{A}$  be a  $\sigma$ -algebra on  $S$  with  $T \in \mathcal{A}$ . We may define a  $\sigma$ -algebra  $\mathcal{A}_T$  on  $T$  by

$$\mathcal{A}_T = \{A \cap T \mid A \in \mathcal{A}\}.$$

If we further have a measure  $\mu: \mathcal{A} \rightarrow [-\infty, \infty]$  then this restricts to a measure  $\mu_T: \mathcal{A}_T \rightarrow [-\infty, \infty]$  defined by

$$\mu_T(A \cap T) = \mu(A \cap T).$$

This definition makes sense since  $T \in \mathcal{A}$ . One may then define the integral for functions restricted to  $T$  as above.





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