The affine structure of jet bundles

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Abstract

The affine structure of jets of sections of a fibred manifold is reviewed. Intrinsic and coordinate characterisations are provided.

1. Equipment list

We work with infinitely differentiable objects to which we refer as smooth. We suppose the reader knows what a locally trivial fibre bundle is, and what a vector bundle is. The material here is extracted primarily from [**?**] and [Saunders 1989].

1.1. Fibred manifolds and their jet spaces. A ***fibred manifold*** is a surjective submersion $\pi\colon \mathsf{Y} \to \mathsf{X}$. If $\pi\colon \mathsf{Y} \to \mathsf{X}$ and $\pi'\colon \mathsf{Y}' \to \mathsf{X}'$ are fibred manifolds, a ***fibred morphism*** from Y to Y' is a pair (F, f) where $F\colon \mathsf{Y} \to \mathsf{Y}'$ and $f\colon \mathsf{X} \to \mathsf{X}'$ are smooth maps such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{Y} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathsf{Y}' \arrow[d, "\pi'"] \\ \mathsf{X} \arrow[r, "f"'] & \mathsf{X}' \end{tikzcd}$$

We will also say that F is a fibred morphism ***over f*** in this case.

The ***vertical bundle*** of a fibred manifold $\pi\colon \mathsf{Y} \to \mathsf{X}$ is the subbundle $\mathsf{V}\pi = \ker(T\pi)$ of TY.

A ***local section*** of a fibred manifold $\pi\colon \mathsf{Y} \to \mathsf{X}$ is a pair $(\xi, \mathcal{U})$ where $\mathcal{U} \subset \mathsf{X}$ is an open subset and where $\xi\colon \mathcal{U} \to \mathsf{Y}$ is a smooth map such that $\pi \circ \xi(x) = x$ for each $x \in \mathcal{U}$.[1] For $x \in \mathsf{X}$, a local section $(\xi, \mathcal{U})$ is a ***local section at x*** if $x \in \mathcal{U}$. Two local sections $(\xi_1, \mathcal{U}_1)$ and $(\xi_2, \mathcal{U}_2)$ at x are ***equivalent to order k*** if, in local coordinates, the first k derivatives of ξ_1 and ξ_2 agree at x. One can tediously check, using the higher-order Chain Rule, that this notion of equivalence is independent of choice of local coordinates. One can also check that this notion of equivalence defines an equivalence relation on the set of local sections at x, and we denote the equivalence class of a local section $(\xi, \mathcal{U})$ by $j_k\xi(x)$. We call $j_k\xi(x)$ the ***k-jet*** of ξ at x. We denote

$$\mathsf{J}_k\pi_x = \{ j_k\xi(x) \mid (\xi, \mathcal{U}) \text{ is a local section at } x \}, \quad \mathsf{J}_k\pi = \cup_{x \in \mathsf{X}} \mathsf{J}_k\pi_x.$$

The set $\mathsf{J}_k\pi_x$ is the set of ***k-jets of local sections at x***, and the set $\mathsf{J}_k\pi$ is the set of ***k-jets of local sections***, or the set of ***k-jets*** for short. We adopt the convention that $\mathsf{J}_0\pi = \mathsf{Y}$.

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[1] Note that the set of local sections of the form (ξ, M) may be empty, i.e., there may be no global sections.

Let us outline how one puts local coordinates on $\mathsf{J}_k\pi$. Let $(\mathcal{V},\psi)$ be an adapted chart for Y and let $(\mathcal{U},\phi)$ be the corresponding chart for X. Let us denote coordinates for $(\mathcal{V},\psi)$ by $(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^n\times\mathbb{R}^m$. Let $(\xi,\mathcal{U})$ be a local section whose local representative is denoted by $\boldsymbol{x}\mapsto(\boldsymbol{x},\boldsymbol{\xi}(\boldsymbol{x}))$, so defining $\boldsymbol{\xi}\colon\mathcal{U}\to\mathbb{R}^m$. Let $x\in\mathcal{U}$ and denote $\boldsymbol{x}=\phi(x)$. The equivalence class in $\mathsf{J}_k\pi_x$ containing ξ is then uniquely determined by the first k derivatives of the local representative of ξ, i.e., by $\boldsymbol{\xi}(\boldsymbol{x}),\boldsymbol{D}\boldsymbol{\xi}(\boldsymbol{x}),\boldsymbol{D}^2\boldsymbol{\xi}(\boldsymbol{x}),\dots,\boldsymbol{D}^k\boldsymbol{\xi}(\boldsymbol{x})$. Note that $\boldsymbol{\xi}(\boldsymbol{x})\in\mathbb{R}^m$ and that $\boldsymbol{D}^j\boldsymbol{\xi}(\boldsymbol{x})\in L^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ for $j\in\{1,\dots,k\}$, where $L^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ denotes the set of symmetric multilinear maps from $\prod_{j=1}^j\mathbb{R}^n$ into $\mathbb{R}^m$. Thus we define a chart $(j_k\mathcal{V},j_k\psi)$ for $\mathsf{J}_k\pi$ by $j_k\mathcal{V}=\cup_{x\in\mathcal{U}}\mathsf{J}_k\pi_x$ and

$$j_k\psi(j_k\xi(x))=(\boldsymbol{x},\boldsymbol{\xi}(x),\boldsymbol{D}\boldsymbol{\xi}(\boldsymbol{x}),\dots,\boldsymbol{D}^k\boldsymbol{\xi}(\boldsymbol{x}))\in\mathbb{R}^n\times\mathbb{R}^m\times L^1_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)\times\cdots\times L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m).$$

One can then verify that, if $\{(\mathcal{V}_a,\psi_a)\}_{a\in A}$ is an atlas of adapted charts for Y, then $\{(j_k\mathcal{V}_a,j_k\psi_a)\}_{a\in A}$ is an atlas for $\mathsf{J}_k\pi$. The tedious thing to check is that the overlap condition is satisfied, but this follows from the higher-order Chain Rule. This gives the differentiable structure for $\mathsf{J}_k\pi$.

Now define a projection $\pi_k\colon\mathsf{J}_k\pi\to\mathsf{X}$ by $\pi_k(j_k\xi(x))=x$. For $k,l\in\mathbb{N}$ satisfying $k>l$ we also have projections $\pi^k_l\colon\mathsf{J}_k\pi\to\mathsf{J}_l\pi$ defined by the fact that if two local sections are equivalent to order k, then they are also equivalent to order l. By writing these projections in local coordinates, one can see that they are surjective submersions. Thus we speak of $\mathsf{J}_k\pi$ as the bundle of k-jets, since it has the structure of a fibred manifold.

1.2. Fibred products. A fibred product is most often used to fabricate a fibre bundle with a desired base space, and with fibres being those of an existing fibre bundle.

1.1 DEFINITION: (Fibred product) Let $f\colon\mathsf{M}\to\mathsf{X}$ and $g\colon\mathsf{N}\to\mathsf{X}$ be smooth maps. The ***fibred product*** of M and N with respect to f and g is the subset

$$\mathsf{M}\times_{\mathsf{X}}\mathsf{N}=\{(u,v)\in\mathsf{M}\times\mathsf{N}\mid\ f(u)=g(v)\}$$

of $\mathsf{M}\times\mathsf{N}$. We also define the projections $f^*g\colon\mathsf{M}\times_{\mathsf{X}}\mathsf{N}\to\mathsf{M}$, $g^*f\colon\mathsf{M}\times_{\mathsf{X}}\mathsf{N}\to\mathsf{N}$, and $f\times_{\mathsf{X}}g\colon\mathsf{M}\times_{\mathsf{X}}\mathsf{N}\to\mathsf{X}$ by $f^*g(u,v)=u$, $g^*f(u,v)=v$, and $f\times_{\mathsf{X}}g(u,v)=f(u)$ (or, equivalently, $f\times_{\mathsf{X}}g(u,v)=g(v)$), respectively. •

In general, the fibred product will have no useful structure. However, the following important case will come up for us.

1.2 PROPOSITION: (Fibred products of fibred manifolds) *If $\pi_1\colon\mathsf{Y}_1\to\mathsf{X}$ and $\pi_2\colon\mathsf{Y}_2\to\mathsf{X}$ are fibred manifolds, then the following statements hold:*

(i) *$\mathsf{Y}_1\times_{\mathsf{X}}\mathsf{Y}_2$ is a closed submanifold of $\mathsf{Y}_1\times\mathsf{Y}_2$;*

(ii) *$\pi_1^*\pi_2\colon\mathsf{Y}_1\times_{\mathsf{X}}\mathsf{Y}_2\to\mathsf{Y}_1$, $\pi_2^*\pi_1\colon\mathsf{Y}_1\times_{\mathsf{X}}\mathsf{Y}_2\to\mathsf{Y}_2$, and $\pi_1\times_{\mathsf{X}}\pi_2\colon\mathsf{Y}_1\times_{\mathsf{X}}\mathsf{Y}_2\to\mathsf{X}$ are fibred manifolds.*

Note that we can regard $\mathsf{Y}_1\times_{\mathsf{X}}\mathsf{Y}_2$ as a fibred manifold in three different ways, so it is important to understand which way is intended in any situation.

In the case of fibred manifolds, we have the following useful feature of fibred products.

1.3 PROPOSITION: (Pull-back fibred manifold) *Let $\pi\colon\mathsf{Y}\to\mathsf{X}$ be a fibred manifold and let $f\colon\mathsf{M}\to\mathsf{X}$ be a smooth map. The fibred product $\mathsf{Y}\times_{\mathsf{X}}\mathsf{M}$ is a fibred manifold over M with projection $f^*\pi$, and we denote $f^*\mathsf{Y}=\mathsf{Y}\times_{\mathsf{X}}\mathsf{M}$. Moreover, if $\pi\colon\mathsf{E}\to\mathsf{X}$ is a vector bundle, then so too is $f^*\pi\colon f^*\mathsf{E}\to\mathsf{M}$.*

The resulting fibred manifold is denoted by $f^*\pi\colon f^*\mathsf{Y} \to \mathsf{M}$ and is called the ***pull-back*** of Y to M.

1.3. Affine spaces and affine bundles. An affine space is a generalisation of a vector space where, roughly speaking, some of the structure of the vector space is stripped away.

1.4 DEFINITION: (Affine space) Let V be a vector space. An ***affine space*** modelled on V is a set A with a map $\Phi\colon \mathsf{V} \times \mathsf{A} \to \mathsf{A}$ satisfying the following properties:

(i) $\Phi(0_{\mathsf{V}}, x) = x$ for every $x \in \mathsf{A}$;

(ii) $\Phi(v_1 + v_2, x) = \Phi(v_1, \Phi(v_2, x))$ for every $v_1, v_2 \in \mathsf{V}$ and $x \in \mathsf{A}$;

(iii) for $x_1, x_2 \in \mathsf{A}$ there exists $v \in \mathsf{V}$ such that $x_2 = \Phi(v, x_1)$;

(iv) if, for any $x \in \mathsf{A}$, $\Phi(v, x) = x$, then $v = 0_{\mathsf{V}}$. •

We immediately dispense with the notation Φ and denote $\Phi(v, x) = x + v$. Also, if $x_1, x_2 \in \mathsf{A}$, then we denote by $x_2 - x_1 \in \mathsf{V}$ the (necessarily unique) element of V for which $x_2 = x_1 + (x_2 - x_1)$. Note that the expression $x_1 + x_2$ is undefined for $x_1, x_2 \in \mathsf{A}$: one cannot add points in an affine space, but can only add to them vectors from V.

The notion of an affine space can be adapted to the notion of an affine bundle.

1.5 DEFINITION: (Affine bundle) Let $\sigma\colon \mathsf{V} \to \mathsf{X}$ be a vector bundle. An ***affine bundle*** modelled on V is a locally trivial fibre bundle $\pi\colon \mathsf{A} \to \mathsf{X}$ with a smooth map $\Phi\colon \mathsf{V} \times_{\mathsf{X}} \mathsf{A} \to \mathsf{A}$ such that

(i) Φ is a fibred morphism over id_{X} and

(ii) for each $x \in \mathsf{X}$, the map $\Phi|\sigma^{-1}(x) \times \pi^{-1}(x)$ makes $\pi^{-1}(x)$ an affine space modelled on the vector space $\sigma^{-1}(x)$. •

1.4. Germs of functions. Let $x \in \mathsf{X}$. A ***local function*** at x is a pair $(f, \mathcal{U})$, where $\mathcal{U}$ is a neighbourhood of x and where $f\colon \mathcal{U} \to \mathbb{R}$ is a smooth function. Two local functions $(f_1, \mathcal{U}_1)$ and $(f_2, \mathcal{U}_2)$ at x are ***equivalent*** if there exists a neighbourhood $\mathcal{U}$ of x such that $\mathcal{U} \subset \mathcal{U}_1$, $\mathcal{U} \subset \mathcal{U}_2$, and $f_1|\mathcal{U} = f_2|\mathcal{U}$. This notion of equivalence is an equivalence relation, and the set of equivalence classes is called the set of ***germs*** of functions at x. We denote by $\mathscr{C}^\infty_x(\mathsf{X})$ the set of germs of functions at x, and we denote an element of $\mathscr{C}^\infty_x(\mathsf{X})$ by $[f]_x$. Let us denote

$$\mathscr{M}_x = \{\, [f]_x \in \mathscr{C}^\infty_x(\mathsf{X}) \mid \ f(x) = 0 \}\,.$$

For those who like algebra, $\mathscr{C}^\infty_x(\mathsf{X})$ is a commutative unit ring with addition and multiplication defined by

$$[f]_x + [g]_x = [f + g]_x, \quad [f]_x[g]_x = [fg]_x.$$

Moreover, if we define scalar multiplication over $\mathbb{R}$ by $a[f_x] = [af]_x$, then $\mathscr{C}^\infty_x(\mathsf{X})$ is a $\mathbb{R}$-algebra. One can easily check that these definitions are independent of representative. One can also show that $\mathscr{M}_x$ is an ideal of $\mathscr{C}^\infty_x(\mathsf{X})$, and is moreover the unique maximal ideal in $\mathscr{C}^\infty_x(\mathsf{X})$. Let us denote by $\mathscr{M}^k_x$ the k-fold product of the ideal $\mathscr{M}_x$ with itself. Thus

$$\mathscr{M}^k_x = \{[f^1_1 \cdots f^1_k]_x + \cdots + [f^l_1 \cdots f^l_k]_x \mid [f^i_j]_x \in \mathscr{M}_x(\mathsf{X}),\ i \in \{1, \dots, l\},\ j \in \{1, \dots, k\},\ l \in \mathbb{N}\}.$$

Note that if $k < l$ then $\mathscr{M}^l_x \subset \mathscr{M}^k_x$ (essentially because the product of two functions vanishing at x is also a function vanishing at x). One can show that the functions in $\mathscr{M}^k_x$

are exactly those which, when their Taylor expansion about x is computed in a coordinate chart about x, have a Taylor series that begins at degree k (see [Golubitsky and Guillemin 1973, Lemma 3.9]).

The following result gives the most important feature of the ideals $\mathscr{M}_x^k$ for us. We denote by $[f]_{x,k}$ the equivalence class of $[f]_x \in \mathscr{M}_x^k$ in $\mathscr{M}_x^k/\mathscr{M}_x^{k+1}$.

1.6 Theorem: (Characterisation of $\mathscr{M}_x^k/\mathscr{M}_x^{k+1}$) *There exists a unique isomorphism (of* $\mathbb{R}$*-vector spaces) from* $\mathscr{M}_x^k/\mathscr{M}_x^{k+1}$ *to* $S_k(\mathsf{T}_x^*\mathsf{X})$ *such that* $[f]_{x,k}$ *is mapped to the* k*th derivative of* f *at* x *(this defining an element of* $S_k(\mathsf{T}_x^*\mathsf{X})$ *since* f *vanishes to order* k*).*

Once one believe our assertion, stated before the theorem, about the character of $\mathscr{M}_x^k$, the theorem is more or less obvious.

2. The affine structure of jet bundles

In this section we indicate how to give the fibred manifold $\pi_{k-1}^k\colon \mathsf{J}_k\pi \to \mathsf{J}_{k-1}\pi$ the structure of an affine bundle. As we shall see, the vector bundle on which the affine bundle is modelled is $\pi_{k-1}^* S_k(\mathsf{T}^*\mathsf{X}) \otimes (\pi_0^{k-1})^*\mathsf{V}\pi$.

2.1. The intrinsic version. We let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold, $x_0 \in \mathsf{X}$, $(\xi, \mathcal{U})$ a local section at x_0, $[f]_{x_0} \in \mathscr{M}_{x_0}^k$, and $v \in \mathsf{V}_{\xi(x_0)}\pi$. Let us denote the representative of $[f]_{x_0}$ in $\mathscr{M}_{x_0}^k/\mathscr{M}_{x_0}^{k+1} \simeq S_k(\mathsf{T}_{x_0}^*\mathsf{X})$ by $[f]_{x_0,k}$. We consider a deformation of ξ, by which we mean a smooth map $\sigma\colon \mathcal{U} \times I \to \mathsf{Y}$ such that

1. $I \subset \mathbb{R}$ is an interval for which $0 \in \operatorname{int}(I)$,
2. $\sigma(x, 0) = \xi(x)$ for all $x \in \mathcal{U}$, and
3. $\pi \circ \sigma(x, t) = x$ for all $(x, t) \in \mathcal{U} \times I$.

Thus, for fixed $t \in I$, the map $x \mapsto \sigma(x, t)$ defines a local section. Let us additionally suppose that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sigma(x_0, t) = v,$$

this making sense since the curve $t \mapsto \sigma(x_0, t)$ has its image in the fibre over x_0. Now, with this data, define a local section $(\xi_{[f]_{x_0,k},v}, \mathcal{U})$ by $\xi_{[f]_{x_0,k},v}(x) = \sigma(x, f(x))$.

The following lemma records the useful fact about this local section.

2.1 Lemma: (Property of deformation) *The* k*-jet of* $\xi_{[f]_{x_0,k},v}$ *at* x_0 *is uniquely determined by*

(i) the k*-jet of* ξ *at* x*,*

(ii) $[f]_{x_0,k} \in \mathscr{M}_{x_0}^k/\mathscr{M}_{x_0}^{k+1} \simeq S_k(\mathsf{T}_{x_0}^*\mathsf{X})$*, and*

(iii) $v \in \mathsf{V}_{\xi(x_0)}\pi$*.*

Moreover, $j_{k-1}\xi_{[f]_{x_0,k},v}(x_0) = j_{k-1}\xi(x_0)$*.*

We shall adopt the suggestive notation

$$j_k\xi_{[f]_{x_0,k},v}(x_0) = j_k\xi(x_0) + [f]_{x_0,k} \otimes v.$$

The lemma tells us that at least the object on the left only depends on the objects used on the right. The next result tells us that the notation is justified, and that in fact the previous equation exactly expresses the affine structure of the set of k-jets having equal $k-1$-jets.

2.2 THEOREM: (The affine structure of jet bundles) *For each* $p \in \mathsf{J}_{k-1}\pi_{x_0}$*, the set* $(\pi^k_{k-1})^{-1}(p)$ *is an affine space modelled on* $S_k(\mathsf{T}^*_{x_0}\mathsf{X}) \otimes \mathsf{V}_{\pi_0^{k-1}(p)}\pi$*, and the affine structure satisfies*

$$j_k\xi(x_0) + [f]_{x_0,k} \otimes v = j_k\xi_{[f]_{x_0,k},v}(x_0).$$

In particular, $\pi^k_{k-1}\colon \mathsf{J}_k\pi \to \mathsf{J}_{k-1}\pi$ *is an affine bundle modelled on* $\pi^*_{k-1}S_k(\mathsf{T}^*\mathsf{X}) \otimes (\pi_0^{k-1})^*\mathsf{V}\pi$*.*

We do not intend to offer coordinate-independent proofs of these results. The most direct proofs are in coordinates, and we next state coordinate versions of the preceding constructions.

2.2. The coordinate version. In this section we give the coordinate construction of the section $\xi_{[f]_{x_0,k},v}$ from the preceding section, and show that it does indeed have the properties stated.

Since we wish to work locally, let us simply suppose that $\mathsf{Y} = \mathcal{U} \times \mathcal{W}$ with $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^m$ open sets, that $\mathsf{X} = \mathcal{U}$, and that $\pi\colon \mathcal{U} \times \mathcal{W} \to \mathcal{U}$ is projection onto the first factor. Let $\boldsymbol{x}_0 \in \mathcal{U}$, let $[f]_{\boldsymbol{x}_0} \in \mathscr{M}^k_{\boldsymbol{x}_0}$, and let $\boldsymbol{v} \in \mathbb{R}^m$ (note that vertical vectors are, essentially, vectors in $\mathbb{R}^m$). Now consider a section ξ given by $\xi(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}))$, defining $\boldsymbol{\xi}\colon \mathcal{U} \to \mathcal{W}$. A deformation of ξ is then a map $\sigma\colon \mathcal{U} \times I \to \mathcal{V}$ of the form $\sigma(\boldsymbol{x}, t) = (\boldsymbol{x}, \boldsymbol{\sigma}(\boldsymbol{x}, t))$ where $\boldsymbol{\sigma}(\boldsymbol{x}, 0) = \boldsymbol{\xi}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{U}$. We suppose that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\boldsymbol{\sigma}(\boldsymbol{x}_0, t) = \boldsymbol{v}.$$

To see that such a deformation exists, we could take $\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{x} + t\boldsymbol{v}$. But we wish to allow σ to be *any* deformation with this property. Now we define a section $\xi_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}$ by $\xi_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}(\boldsymbol{x}) = \xi(\boldsymbol{x}, f(\boldsymbol{x}))$, and write

$$\xi_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}(\boldsymbol{x})),$$

so defining $\boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}\colon \mathcal{U} \to \mathcal{V}$. Note that

$$\boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}, f(\boldsymbol{x})).$$

Let us compute the first k derivatives of $\boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}$. First the first derivative. We have

$$\frac{\partial \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^i}(\boldsymbol{x}) = \frac{\partial \boldsymbol{\sigma}}{\partial x^i}(\boldsymbol{x}, f(\boldsymbol{x})) + \frac{\partial \boldsymbol{\sigma}}{\partial t}(\boldsymbol{x}, f(\boldsymbol{x}))\frac{\partial f}{\partial x^i}(\boldsymbol{x}).$$

Since σ is a deformation of ξ and since $f(\boldsymbol{x}_0) = 0$,

$$\frac{\partial \boldsymbol{\sigma}}{\partial x^i}(\boldsymbol{x}_0, f(\boldsymbol{x}_0)) = \frac{\partial \boldsymbol{\xi}}{\partial x^i}(\boldsymbol{x}_0).$$

If $k = 1$ then we have

$$\frac{\partial \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^i}(\boldsymbol{x}_0) = \frac{\partial \boldsymbol{\xi}}{\partial x^i}(\boldsymbol{x}_0) + \boldsymbol{v}\frac{\partial f}{\partial x^i}(\boldsymbol{x}_0).$$

By Theorem 1.6 it follows that $[f]_{\boldsymbol{x}_0,1}$ is uniquely determined by $\frac{\partial f}{\partial x^i}(\boldsymbol{x}_0)$, $i \in \{1,\dots,n\}$. Thus we see that Lemma 2.1 follows in this case.

If $k > 1$ then $\frac{\partial f}{\partial x^i}(\boldsymbol{x}_0) = 0$ since f vanishes to order k, and so

$$\frac{\partial \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^i}(\boldsymbol{x}_0) = \frac{\partial \boldsymbol{\xi}}{\partial x^i}(\boldsymbol{x}_0).$$

Now let us compute the second derivative:

$$\begin{aligned}\frac{\partial^2 \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^j \partial x^i}(\boldsymbol{x}, f(\boldsymbol{x})) = \frac{\partial^2 \boldsymbol{\sigma}}{\partial x^j \partial x^i}(\boldsymbol{x}, f(\boldsymbol{x})) + \frac{\partial^2 \boldsymbol{\sigma}}{\partial t \partial x^i}(\boldsymbol{x}, f(\boldsymbol{x}))\frac{\partial f}{\partial x^j}(\boldsymbol{x}, f(\boldsymbol{x})) \\ + \frac{\partial^2 \boldsymbol{\sigma}}{\partial x^j \partial t}(\boldsymbol{x}, f(\boldsymbol{x}))\frac{\partial f}{\partial x^i}(\boldsymbol{x}) + \frac{\partial \boldsymbol{\sigma}}{\partial t}(\boldsymbol{x}, f(\boldsymbol{x}))\frac{\partial^2 f}{\partial x^j \partial x^i}(\boldsymbol{x}).\end{aligned}$$

If $k = 2$ then the terms involving the first derivatives of f vanish at $\boldsymbol{x}_0$, and we are left with

$$\frac{\partial^2 \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^j \partial x^i}(\boldsymbol{x}_0, f(\boldsymbol{x}_0)) = \frac{\partial^2 \boldsymbol{\xi}}{\partial x^j \partial x^i}(\boldsymbol{x}_0, f(\boldsymbol{x}_0)) + \boldsymbol{v}\frac{\partial^2 f}{\partial x^j \partial x^i}(\boldsymbol{x}_0).$$

Again, Theorem 1.6 gives Lemma 2.1 in this case.

It is hopefully now somewhat clear that one can fabricate an inductive proof to show that, for any $k \in \mathbb{N}$, we have

$$\begin{aligned}\frac{\partial \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^{i_1}}(\boldsymbol{x}_0) &= \frac{\partial \boldsymbol{\xi}}{\partial x^{i_1}}(\boldsymbol{x}_0), \\ \frac{\partial^2 \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^{i_1}\partial x^{i_2}}(\boldsymbol{x}_0) &= \frac{\partial^2 \boldsymbol{\xi}}{\partial x^{i_1}\partial x^{i_2}}(\boldsymbol{x}_0), \\ &\vdots \\ \frac{\partial^{k-1} \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^{i_1}\cdots\partial x^{i_{k-1}}}(\boldsymbol{x}_0) &= \frac{\partial^{k-1} \boldsymbol{\xi}}{\partial x^{i_1}\cdots\partial x^{i_{k-1}}}(\boldsymbol{x}_0) \\ \frac{\partial^{k} \boldsymbol{\xi}_{[f]_{\boldsymbol{x}_0,k},\boldsymbol{v}}}{\partial x^{i_1}\cdots\partial x^{i_{k}}}(\boldsymbol{x}_0) &= \frac{\partial^{k} \boldsymbol{\xi}}{\partial x^{i_1}\cdots\partial x^{i_{k}}}(\boldsymbol{x}_0) + \boldsymbol{v}\frac{\partial^k f}{\partial x^{i_1}\cdots\partial x^{i_k}}(\boldsymbol{x}_0).\end{aligned} \tag{2.1}$$

An appeal to Theorem 1.6 then gives Lemma 2.1.

To prove Theorem 2.2 is now more or less straightforward, given (2.1). Indeed, it really amounts to the observation that the vector spaces $L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ (where the objects $\frac{\partial^k \boldsymbol{\xi}}{\partial x^{i_1}\cdots\partial x^{i_k}}(\boldsymbol{x}_0)$ live) and $S_k((\mathbb{R}^n)^*)\otimes\mathbb{R}^m$ (where the objects $\boldsymbol{v}\frac{\partial^k f}{\partial x^{i_1}\cdots\partial x^{i_k}}(\boldsymbol{x}_0)$ live) are isomorphic.

References

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