Connections for general group actions

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1 Introduction

Principal bundles serve as a powerful and elegant geometric framework for analyzing group actions and symmetry. Beyond their geometric origins, principal bundles play significant roles in the analysis of mechanical systems with symmetry, as well as the design of appropriate computational algorithms. A connection on a principal bundle is defined as an equivariant decomposition of vectors into infinitesimal group motions and complementary infinitesimal 'rigid' motions, which often facilitates the analysis of a system (see, e.g., [1, 2]). In some settings, the group motions are the crucial information, and often the only information that is sufficiently simple to lend itself to a rigorous analysis. For example, in the classic optimal control problem of a falling animal (e.g. a cat) righting itself in flight, the analysis focuses on the influence of the relative positions of the front and back halves of the animal on its inertia tensor; the complex details of the motion are not crucial to a clear understanding of the problem (see, e.g.,[3, 4], and the references therein). Similar situations arise in computational dynamics where the orientation of a body may be accurately computed even if the body deformations are not. On the other hand, there are many situations in which the information of interest is invariant under the group action. Thus, while in some circumstances it is appropriate to drop to the base manifold of the principal bundle, in others it may be more natural or more convenient to remain 'upstairs'.

A crucial limitation of the classical theory of connections is the requirement that the manifold in question be a principal bundle and so the group must act freely. Many important group actions fail to be free, e.g., transitive actions on homogeneous spaces, the rotation groups acting on $n \geq 3$-dimensional Euclidean space, affine and projective actions, etc. Systems with continuous isotropy arise in a wide variety of applications. In geometric mechanics, the action of a product group on a manifold of diffeomorphisms or embeddings, with one factor of the product acting by 'body' transformations, the other by 'spatial' transformations, [5, 6, 7], is not free. A sleeping top (one for which the axis of symmetry of the top is aligned with the axis of gravity) is a familiar

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instance of this. In this system, the infinitesimal versions of 'spatial' and 'body' rotations (resp. spin and precession) cannot be distinguished. A more substantial example, which reflects the same underlying geometric structure, is fluid flow in the absence of external forces. If the reference region for the fluid mass is axially symmetric, then any axisymmetric deformation has nontrivial isotropy corresponding to counter–balancing spatial and body rotations. Such states are analogous to the sleeping states of the Lagrange top; this common symmetry feature can be used to analyze the stability and bifurcation of both systems.

While one can no longer apply the classical theory of connections on principal bundles to non-free actions, one would still like to develop a comparable theory that will carry the many benefits of the usual theory over to this context. To this end, we introduce the concept of a *partial connection* that can be applied to general group actions. The key step is to focus our attention on the connection one-form that can be used to define the principal bundle connection. The classical Lie algebra-valued connection form defines an equivariant map from the tangent bundle of the manifold to the Lie algebra of the transformation group. In our approach, the connection form is more appropriately viewed as a projection of each tangent space onto the infinitesimal group orbit. Such projections can be defined even when the action fails to be free and lead to promising generalizations of several key constructs. The shift in focus from generators to projections allows us to broaden our search for appropriate forms, and we discover that the projections can be defined using smooth $\mathfrak{g}^*$–valued forms even at points in which the isotropy changes. This projection–based approach is inspired by the reduced energy–momentum method for stability and bifurcation analysis of relative equilibria of simple mechanical systems [8, 9] and, in particular, its generalization to non-free actions and regular Lagrangian systems [5, 6, 7]. In that setting, however, the decomposition of tangent vectors into their 'rigid' and 'internal' components is carried out only at relative equilibria. Our approach is modeled on the implementation of the simple mechanical connection by means of the momentum map, and is elucidated in Section 2.

The curvature of a connection can be interpreted as a measure of its non-integrability, and plays a crucial role in geometric mechanics, mathematical physics, and differential geometry. In optimal control theory, conditions for controllability can be described in terms of the curvature of a connection via the Ambrose-Singer Theorem, [10, 11]. Connection forms facilitate the study of holonomy, which is important in both classical and quantum mechanics, [11, 12, 13, 14]. In Section 3, we define a corresponding concept of curvature for a partial connections associated with sufficiently regular class of non-free actions. In Section 4, we generalize the classical involutivity result that the curvature is locally zero if and only if the connection is locally tangent to a cross section of the principal bundle, and show that a partial connection whose curvature lies entirely in the tangent spaces of the orbits of a specified isotropy subgroup is tangent to a slice of the group action.

Our results have, in part, been directly inspired by the equivariant approach to moving frames developed by Fels and Olver [15, 16]. They define a moving frame as an equivariant map from a manifold M to a group G acting (locally) freely on M. These maps, and their linearizations, can be used in the design of numerical integration schemes. If M is a principal bundle, then the trivialized linearization of a moving frame on M is a connection form. In prior work, [17, 18, 19], we developed a generalization, called *partial moving frames*, which play a similar role for non-free actions. Partial moving frames are mappings from a manifold M to a group G acting on M that are equivariant modulo isotropy. Like a genuine connection form, the trivialized linearization of a partial moving frame can be used to map vector fields on M into trivialized vector fields on G. These linearizations behave very much like our algebra–valued generalizations of connection forms, but they typically fail to be equivariant. By relaxing that condition to relative equivariance, i.e. equivariance with respect to a specified choice of representatives of the equivalence classes of the group modulo isotropy, we obtain a further generalization of connections that includes our motivating examples, and, in fact, served as our original definition of a partial connection. We describe some of the key features of relatively equivariant partial connections and the associated forms in Section 5.

This work was originally motivated by the need to develop symmetry-preserving numerical methods (geometric integrators) for solving ODEs and PDEs with nontrivial symmetry groups. Such methods rely on numerical integration schemes for initial value problems on Lie groups, cf. [20, 21, 22, 23, 24] and references therein. However, if the group action has continuous isotropy, the induced vector field on the group is not uniquely determined. While the true flows of different choices of vector field will yield the same solution curves back on the manifold, numerical approximations of these flows will typically yield different approximate discrete trajectories. Preliminary results in both toy problems and more substantial micromagnetic calculations are quite encouraging. Adapting the powerful machinery of connection forms and moving frames to the non-free context will be of fundamental importance in the further development of such numerical algorithms.

2 Partial connections

We shall assume throughout that a Lie group G acts continuously on a manifold M. For convenience, we introduce the following notation. Let $\Phi_g : M \to M$ and $\widehat{\Phi}_m : G \to M$ denote the maps

$$\widehat{\Phi}_m(g) := \Phi_g(m) := g \cdot m$$

and, given $\xi \in \mathfrak{g} = T_eG$, let ξ_M denote the vector field $\xi_M(m) := d\widehat{\Phi}_m \xi$, called the *infinitesimal generator* associated to ξ.

A connection on a principal bundle P is a differential system, i.e. a distribution, Γ satisfying $TP = \widetilde{\mathfrak{g}} \oplus \Gamma$ and $d\Phi_g \cdot \Gamma_p = \Gamma_{g \cdot p}$ for all $p \in P$ and $g \in G$. Here $\widetilde{\mathfrak{g}}$ denotes the differential system of the tangent spaces to the group orbits, i.e. $\widetilde{\mathfrak{g}}|_p = T_p(G \cdot p) = \operatorname{range} d_e\widehat{\Phi}_p$. Note that we will not explicitly indicate the basepoint of a differential structure when it is clear from the context. Specification of a connection Γ is equivalent to specification of an equivariant $\mathfrak{g}$–valued one–form α, called the *connection form*, satisfying

$$\alpha \circ d_e\widehat{\Phi}_p = \mathrm{id}, \qquad \text{i.e.} \quad \alpha_p(\xi_P(p)) = \xi \qquad \text{for all } \xi \in \mathfrak{g},$$

for all $p \in P$, where $\alpha_p = \alpha | T_pP$, the restriction of α to the fiber over p. By equivariance we mean that $\alpha \circ d\Phi_g = \mathrm{Ad}_g \circ \alpha$ for all $g \in G$. The connection Γ and connection form α are related by the condition $\ker \alpha = \Gamma$, i.e. $\ker \alpha_p = \Gamma|_p$ for all $p \in P$.

We shall retain most the properties of connections and connection forms given above in our proposed extension of connections to general actions; however, our connections need not be differential systems, since the dimensions of the group orbits need not be constant throughout the manifold. The connection form is not uniquely determined by the connection, since the isotropy algebra $\mathfrak{g}_m = \ker d_e\widehat{\Phi}_m$ can be nontrivial. However, equivariant assignments of complements Γ to $\widetilde{\mathfrak{g}}$ are in one to one correspondence with equivariant projections onto $\widetilde{\mathfrak{g}}$; specifically, the kernel of the projection is a complementary differential system. As we now show, projections onto $\widetilde{\mathfrak{g}}$ are naturally related to $\mathfrak{g}$–valued one forms.

Proposition 1 *Given a $\mathfrak{g}$–valued one–form α, define the map $\mathbb{P}_\alpha := d\widehat{\Phi} \circ \alpha : TM \to \widetilde{\mathfrak{g}}$, i.e. $\mathbb{P}_\alpha | T_m M = d\widehat{\Phi}_m \circ \alpha_m$. $\mathbb{P}_\alpha$ is an equivariant projection onto $\widetilde{\mathfrak{g}}$ if and only if $\alpha(\eta_M(m)) = \eta \quad mod \quad \mathfrak{g}_m$, i.e.*

$$\mathrm{range}\,(\mathbb{1} - \alpha \circ d_e\widehat{\Phi}_m) \subseteq \mathfrak{g}_m \tag{1}$$

for all $m \in M$, and α is equivariant modulo isotropy, i.e.

$$\Phi_g^* \alpha_m = \mathrm{Ad}_g \alpha_m \quad mod \quad \mathfrak{g}_{g\cdot m} \tag{2}$$

for all $g \in G$ and $m \in M$.

Any $\mathfrak{g}$–valued one–form satisfying (1) is discontinuous at singular points of M.

Proof: If α satisfies (1), it follows that $TM = \widetilde{\mathfrak{g}} \oplus \ker \mathbb{P}_\alpha$ and $\mathbb{P}_\alpha | \widetilde{\mathfrak{g}} = \mathbb{1}$; hence $\mathbb{P}_\alpha$ is a projection onto $\widetilde{\mathfrak{g}}$. On the other hand, if $\mathbb{P}_\alpha$ is an equivariant projection onto $\widetilde{\mathfrak{g}}$, then $\widetilde{\mathfrak{g}}|_m = \mathrm{range}\, d_e\widehat{\Phi}_m$ for all $m \in M$ and $\mathbb{1} = \mathbb{P}_\alpha | \widetilde{\mathfrak{g}}$ imply (1).

Fix $m \in M$ and $g \in G$. The identity $(\mathrm{Ad}_g \xi)_M (g \cdot m) = d_e(\Phi_g \circ \widehat{\Phi}_m)\xi$ for all $\xi \in \mathfrak{g}$ implies that any $\mathfrak{g}$–valued one–form α satisfies

$$\begin{aligned} d\widehat{\Phi}_{g\cdot m} \circ \left(\Phi_g^* \alpha - \mathrm{Ad}_g \alpha\right)_m &= d\widehat{\Phi}_{g\cdot m} \circ \alpha \circ d_m \Phi_g - d(\Phi_g \circ \widehat{\Phi}_m) \circ \alpha_m \\ &= \mathbb{P}_\alpha \circ d_m \Phi_g - d_m \Phi_g \circ \mathbb{P}_\alpha. \end{aligned}$$

Thus α is equivariant modulo isotropy if and only if $\mathbb{P}_\alpha$ is equivariant.

Given $m \in M$, define the map $\pi_m := \mathbb{1} - \alpha \circ d_e\widehat{\Phi}_m : \mathfrak{g} \to \mathfrak{g}$. Since $\mathfrak{g}_m = \ker d_e\widehat{\Phi}_m$, we have $\pi_m | \mathfrak{g}_m = \mathbb{1}$; hence if α satisfies (1), then $\mathrm{range}\, \pi_m = \mathfrak{g}_m$. Continuity of the action implies that $\dim \mathfrak{g}_m \leq \dim \mathfrak{g}_{m_0}$ on a sufficiently small neighborhood of a point m_0, with equality if and only if m_0 is regular, and the map $m \mapsto \pi_m$ is continuous if α is continuous. Hence if α is continuous at a point m_0, then

$$\dim \mathfrak{g}_m = \mathrm{rank}\,[\pi_m] \geq \mathrm{rank}\,[\pi_{m_0}] = \dim \mathfrak{g}_{m_0}$$

on a sufficiently small neighborhood of m_0. It follows that α can be continuous at m_0 only if m_0 is regular. ■

Proposition 1 implies that a $\mathfrak{g}$–valued form α determining a family of equivariant projections will be singular at points at which there is a jump in isotropy. Thus, unless all points of M are regular, and hence all group orbits in connected components of M are of equal dimension, complements to the tangent spaces to the group orbits do not appear to have 'natural' characterizations in terms of smooth $\mathfrak{g}$–valued one–forms. Rather than attempt to specify directly just what kinds of lapses in smoothness are permissible, we shall depart from the traditional approach and define our extension of connections utilizing smooth $\mathfrak{g}^*$–valued forms modeled on momentum maps. The following example typifies the situations that we will address.

Let M be a Riemannian manifold and let G be a subgroup of the group of (local) isometries of M. The orthogonal complement $\widetilde{\mathfrak{g}}^{\perp}$ to $\widetilde{\mathfrak{g}}$ in TM will be the prototypical example of a partial connection. If the action of G is free and proper, and hence M is a principal bundle, the associated connection form is known as the *simple mechanical connection form* and is given as follows: Define the equivariant $\mathfrak{g}^*$–valued one–form μ by

$$\mu(v) \cdot \xi := \langle v, \xi_M(m) \rangle_m$$

for $v \in T_m M$. Note that $\widetilde{\mathfrak{g}}^{\perp} = \ker \mu$. The form μ is the momentum map associated to the G action and the Lagrangian $L(v) = \frac{1}{2} \|v\|^2$. The *locked inertia tensor* $\chi : M \to \mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$ is given by

$$(\chi(m)\xi) \cdot \eta = \langle \xi_M(m), \eta_M(m) \rangle_m .$$

The locked inertia tensor is singular precisely at points with continuous isotropy. Thus if M is a principal bundle, then $\chi(m)$ is invertible for all m and $\alpha_m = \chi(m)^{-1}\mu_m$ is the simple mechanical connection form. Note that the $\mathfrak{g}$–valued one–form α is defined via the $\mathfrak{g}^*$–valued one–form μ and associated map χ; thus, in this setting, the $\mathfrak{g}^*$–valued form can be regarded as more 'directly' linked to the group action and the geometry of the problem than the $\mathfrak{g}$–valued form. At a point m with continuous isotropy, the relation $\chi(m)\alpha_m = \mu_m$ does not uniquely determine α_m. However, any $\mathfrak{g}$–valued one–form satisfying $\chi\, \alpha = \mu$ will satisfy $\widetilde{\mathfrak{g}}^{\perp}|_m = \alpha_m^{-1}(\mathfrak{g}_m)$ and hence will qualify as a generalized connection form. Note, however, that in many situations it appears to be more convenient to work directly with the maps μ and χ, since these maps are both smooth, while α will have singularities at points where the isotropy jumps.

Example: $SO(3)$ acting on $\mathbb{R}^3$

For the sake of concreteness, we now specialize the situation described above to the case $M = \mathbb{R}^3$ and $G = SO(3)$, the group of rotations of $\mathbb{R}^3$. This action is not free: any nonzero vector m is fixed by the circle of rotations about m, while the origin is fixed by the entire group $SO(3)$. The group orbit $G \cdot m$ through m is the sphere of radius $||m||$ centered at the origin; $\widetilde{\mathfrak{g}}|_m = \left\{ \xi \times m : \xi \in \mathbb{R}^3 \right\}$ is the space of infinitesimal rotations of m. The orthogonal complements to $\widetilde{\mathfrak{g}}$ satisfy

$$\widetilde{\mathfrak{g}}^{\perp}|_m = \begin{cases} \text{span}[m] & m \neq 0 \\ \mathbb{R}^3 & m = 0 \end{cases} .$$

The angular momentum can be regarded as a $so(3)^*$–one–form $\mu(v) = m \times v$ for $v \in T_m\mathbb{R}^3$, with $\ker \mu = \widetilde{\mathfrak{g}}^{\perp}$. More generally, given any smooth function $q : \mathbb{R} \to \mathbb{R}$ that is strictly positive on $\mathbb{R}^+$,

$$\mu^q(v) := q(||m||^2) m \times v, \tag{3}$$

for all $v \in T_m M$, also satisfies $\ker \mu^q = \widetilde{\mathfrak{g}}^{\perp}$. The associated inertia tensors $\chi^q(m) := \mu^q \circ d_e \widehat{\Phi}_m \in \mathbb{R}^{3\times 3}$ satisfy

$$\chi^q(m)\xi = \begin{cases} q(||m||^2)||m||^2 \mathbb{P}_{\perp}\xi & m \neq 0 \\ 0 & m = 0 \end{cases}$$

where $\mathbb{P}_\perp|T_mM$ denotes orthogonal projection onto $\mathrm{span}[m]^\perp$. Any smooth function $f:\mathbb{R}^3\backslash\{0\}\to\mathbb{R}$ determines a $\mathbb{R}^3$–valued one–form

$$\alpha(v)=\begin{cases} ||m||^{-2}m\times v+f(m)\,\langle m,v\rangle\,m & v\in T_m\mathbb{R}^3,\ m\neq 0\\ 0 & v\in T_0\mathbb{R}^3\end{cases}\tag{4}$$

that in turn determines an equivariant projection $\mathbb{P}_\alpha$, with $\ker\mathbb{P}_\alpha=\widetilde{\mathfrak{g}}^\perp$. Note that α is discontinuous at the origin for any f, while the generalized angular momentum $\mu^q=\chi^q\alpha$ is everywhere smooth.

Motivated by the role of the momentum map in our prototypical example, we introduce an alternative to $\mathfrak{g}$–valued connection forms; we specify a differential system complementary to the tangent spaces of the group orbits by means of a $\mathfrak{g}^*$–valued form that is required to remain smooth even at points at which a jump in isotropy occurs. We first show that an equivariant and appropriately nondegenerate $\mathfrak{g}^*$–valued form determines a projection onto the tangent spaces of the group orbits, just as the connection form does.

Proposition 2 *An equivariant $\mathfrak{g}^*$–valued one–form μ satisfies $TM=\widetilde{\mathfrak{g}}\oplus\ker\mu$ if and only if the associated equivariant map $\chi:M\to\mathfrak{L}(\mathfrak{g},\mathfrak{g}^*)$ given by*

$$\chi(m):=\mu\circ d_e\widehat{\Phi}_m\tag{5}$$

satisfies

$$\ker\chi(m)=\mathfrak{g}_m\qquad\text{and}\qquad\mathrm{range}\,\chi(m)=\mathrm{range}\,\mu_m\tag{6}$$

for all $m\in M$.

For each $m\in M$, we can define the isomorphism $\gamma(m):\mathrm{range}\,\chi(m)\to\widetilde{\mathfrak{g}}$ by

$$\gamma(m)(\nu)=\xi_M(m)\qquad\text{for any }\xi\in\mathfrak{g}\text{ satisfying }\chi(m)\xi=\nu\tag{7}$$

and equivariant projection $\mathbb{P}_\mu:=\gamma\circ\mu:TM\to\widetilde{\mathfrak{g}}$.

Proof: Equivariance of μ implies that $\ker\mu_{g\cdot m}=d\Phi_g(\ker\mu_m)$ and

$$\chi(g\cdot m)=\mathrm{Ad}^*_{g^{-1}}\circ\chi(m)\circ\mathrm{Ad}_{g^{-1}}$$

for all $m\in M$ and $g\in G$. If μ satisfies $TM=\widetilde{\mathfrak{g}}\oplus\ker\mu$, then

$$\ker\chi(m)=\ker\mu\circ d_e\widehat{\Phi}_m=\ker d_e\widehat{\Phi}_m=\mathfrak{g}_m$$

and $\mathrm{range}\,\mu_m=\mathrm{range}\,(\mu|\widetilde{\mathfrak{g}})|_m=\mathrm{range}\,\chi(m)$ for all $m\in M$.

On the other hand, if χ given by (5) satisfies (6), then the nondegeneracy condition $\ker\chi(m)=\mathfrak{g}_m$ implies that $\mu|\widetilde{\mathfrak{g}}$ is injective, while $\mathrm{range}\,\chi(m)=\mathrm{range}\,\mu_m$ implies that for any $m\in M$ and $v\in T_mM$, there exists $\xi\in\mathfrak{g}$ such that

$$0=\mu(v)-\chi(m)\xi=\mu(v-\xi_M(m)).$$

Hence $TM=\widetilde{\mathfrak{g}}\oplus\ker\mu$.

The map γ is well–defined, since $\ker \chi(m) = \ker d_e\widehat{\Phi}_m$, and equivariant. Equivariance of $\mathbb{P}_\mu$ thus follows from the equivariance of μ. For any $m \in M$,

$$\mathbb{P}_\mu \circ d_e\widehat{\Phi}_m = \gamma(m) \circ \mu \circ d_e\widehat{\Phi}_m = \gamma(m) \circ \chi(m) = d_e\widehat{\Phi}_m,$$

so $\mathbb{P}_\mu|\widetilde{\mathfrak{g}} = \mathbb{1}$. Since $\mathbb{P}_\mu|\ker \mu = 0$ and $TM = \widetilde{\mathfrak{g}} \oplus \ker \mu$, $\mathbb{P}_\mu$ is a projection onto $\widetilde{\mathfrak{g}}$. ■

We now have at hand the appropriate notion of smoothness to formulate our extension of connections to general actions.

Definition 1 *An* equivariant partial connection *is a (singular) equivariant differential system* Γ *satisfying* $TM = \widetilde{\mathfrak{g}} \oplus \Gamma$ *and either* Γ *is smooth or* $\Gamma = \ker \mu$ *for some smooth* $\mathfrak{g}^*$*–valued one–form.*

An equivariant dual connection form *is a smooth equivariant* $\mathfrak{g}^*$*–valued one–form* μ *on* M *satisfying* $T_mM = \widetilde{\mathfrak{g}} \oplus \ker \mu_m$ *for all* $m \in M$*.*

An equivariant partial connection form *is a* $\mathfrak{g}$*–valued one–form* α *on* M *such that the map* $\mathbb{P}_\alpha = d\widehat{\Phi} \circ \alpha$ *is an equivariant projection onto* $\widetilde{\mathfrak{g}}$ *and* $\Gamma := \ker \mathbb{P}_\alpha$ *determines an equivariant partial connection.*

An equivariant inertia factor *is an equivariant map* $\chi : M \to \mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$ *satisfying* $\ker \chi(m) = \widetilde{\mathfrak{g}}|_m$ *for all* $m \in M$*.*

From now on, with the exception of §5, we shall drop the adjective 'equivariant' and simply refer to partial connections, dual connection forms, etc. Note that if M is finite dimensional, m is a regular point of M, and Γ is a partial connection, then there is a neighborhood $\mathcal{U}$ of m such that $\Gamma|\mathcal{U}$ is a smooth differential system. (Either Γ is a priori smooth or there is a smooth form μ such that $\Gamma = \ker \mu$; the rank of μ, and hence the dimension of Γ, is constant on a sufficiently small neighborhood of a regular point.)

We shall rely primarily on dual connection forms in our calculations. The following proposition, which is proved in the appendix, presents some of the fundamental links between dual connection forms, partial connections, and partial connection forms.

Theorem 1 *(i) A dual connection form* μ *determines a partial connection* $\Gamma = \ker \mu$ *and inertia factor* $\chi = \mu \circ d_e\widehat{\Phi}$*.*

(ii) An equivariant (singular) differential system Γ *satisfying* $TM = \widetilde{\mathfrak{g}} \oplus \Gamma$ *is a partial connection if there is inertia factor* χ *such that the equivariant* $\mathfrak{g}^*$*–valued one–form* μ *given by*

$$\mu|\Gamma := 0 \qquad \text{and} \qquad \mu \circ d_e\widehat{\Phi}_m := \chi(m) \quad \text{for all } m \in M \tag{8}$$

is smooth, and hence a dual connection form.

(iii) A $\mathfrak{g}$*–valued one–form* α *is a partial connection form if there is an inertia factor* χ *such that* $\mu = \chi\,\alpha$ *is a dual connection form with inertia factor* χ*.*

We shall call a pair (α, χ) consisting of a partial connection form α and a inertia factor χ such that $\mu := \chi\,\alpha$ is a dual connection form a *partial connection pair.* Note that, given a partial connection pair, the inertia factor is not needed to specify the associated partial connection;

however, it converts a typically singular form to a smooth one, and thus will play a crucial role in our development of the curvature form.

Remark: Note that in contrast to the situation for classical connection forms, a partial connection form need not satisfy $\ker \mathbb{P}_\alpha | T_m M = \ker \alpha_m$. For example, the one–form (4) on $\mathbb{R}^3$ satisfies

$$\operatorname{range} \alpha_m \cap \mathfrak{g}_m = \{0\} \tag{9}$$

if and only if $f \equiv 0$. However, given an arbitrary partial connection form α, we can construct a partial connection form satisfying (9) and determining the same partial connection as α. Specifically, set $\tilde{\alpha} := \alpha \circ \mathbb{P}_\alpha$. By construction, $\mathbb{P}_{\tilde{\alpha}} = \mathbb{P}_\alpha$ for all $m \in M$. Equivariance modulo isotropy of $\tilde{\alpha}$ follows from that of α and the associated projection; specifically, $d\Phi_g \circ \mathbb{P}_\alpha = \mathbb{P}_\alpha \circ d\Phi_g$ implies that

$$\operatorname{range} (\Phi_g^* \tilde{\alpha} - \operatorname{Ad}_g \tilde{\alpha})_m = \operatorname{range} (\Phi_g^* \alpha - \operatorname{Ad}_g \alpha) \circ \mathbb{P}_\alpha | T_m M \subseteq \mathfrak{g}_{g \cdot m}$$

for any $m \in M$ and $g \in G$.

3 Curvature

The curvature Ω of a classical connection on a principal bundle is typically defined as the covariant derivative of the unique associated connection form α, i.e. $\Omega = \nabla \alpha = \mathbb{P}^* \mathrm{d}\alpha$, where $\mathbb{P}$ denotes projection onto the connection. When extending this notion to partial connections, we encounter two key difficulties: first, the nonuniqueness of the partial connection form; second, the singularity of partial connection forms at points where there is a jump in isotropy. Ambiguities regarding elements in the isotropy subalgebras motivate us to adopt the convention that the curvature takes values in the vertical structure $\widetilde{\mathfrak{g}}$, rather than in the algebra $\mathfrak{g}$. Complications arising from singularities are less easily resolved, but we identify a large family of dual connection forms for which a natural version of curvature can be defined at all points.

Following the standard definition, we define the *covariant derivative* ∇ of a (vector–valued) k–form ν with respect to a partial connection Γ as

$$\nabla \nu(v_0, \ldots, v_k) = \mathrm{d}\nu(\mathbb{P}_\Gamma v_0, \ldots, \mathbb{P}_\Gamma v_k)$$

for any $v_j \in T_m M$, $j = 0, \ldots, k$, and $m \in M$. We first consider the situation closest to the standard one, namely a partial connection form at a regular point. A partial connection form is smooth at regular points; hence we define the *curvature* of a partial connection form α at a regular point m as

$$\Omega(u, v) = d\widehat{\Phi}_m(\nabla \alpha(u, v)) \tag{10}$$

for any vectors u, $v \in T_m M$.

To extend our definition of curvature to singular points, we will replace partial connection forms, which are discontinuous at singular points. with dual connection forms, which are everywhere smooth. If (α, χ) is a partial connection pair associated to a partial connection Γ, we have

$$\chi \nabla \alpha = \nabla(\chi \alpha) - \nabla \chi \wedge \alpha \circ \mathbb{P}_\Gamma \tag{11}$$

at regular points. The right hand side is well-defined even at points at which α fails to be differentiable. This suggests that we use (11) to define the curvature, replacing the dual connection form $\chi\,\alpha$ with a general dual connection form μ as appropriate. However, it need not be the case that the right hand side of (11) lie in the range of $\chi(m)$. Before addressing this issue, we show that the wedge product appearing in (11) is identically zero, yielding $\chi\,\nabla\alpha = \nabla(\chi\,\alpha)$ whenever the left hand side is defined.

Lemma 1 *Let μ be a dual connection form with inertia factor χ. Then* range $\nabla\chi(m) \subseteq \mathrm{Ann}\,\mathfrak{g}_m$*, the annihilator of $\mathfrak{g}_m$.*

Proof: Equivariance of μ implies that $\eta_M(\mu) = -\mathrm{ad}^*_\eta\mu$ for any $m \in M$ and $\eta \in \mathfrak{g}$. Hence $\mu(\eta_M) = \chi\,\eta$ implies that

$$\eta_M \,\lrcorner\, \mathrm{d}\mu = \eta_M(\mu) - \mathrm{d}(\mu(\eta_M)) = -\mathrm{ad}^*_\eta\mu - \mathrm{d}\chi\,\eta. \tag{12}$$

If $\zeta \in \mathfrak{g}_m$, and hence $\zeta_M(m) = 0$, and $u \in \Gamma|_m = \ker\mu_m$, then (12) yields $0 = \mathrm{d}\chi(u)\zeta$. ■

Combining (10), (11), and Lemma 1, given a dual connection pair (α, χ), we see that at a regular point m

$$\Omega_m = d\widehat{\Phi}_m(\nabla_m\alpha) = \gamma(m)(\chi(m)\nabla_m\alpha) = \gamma(m)(\nabla_m(\chi\,\alpha)).$$

Note that the right hand side is well–defined even at singular points, provided that

$$\mathrm{range}\,\nabla_m(\chi\,\alpha) \subseteq \mathrm{range}\,\chi(m).$$

Motivated by this observation, we introduce the following definitions:

Definition 2 *A dual connection form μ is* docile *at m if* range $\nabla_m\mu \subseteq$ range μ_m*. If μ is docile at m, then the* curvature *of μ at m is $\Omega_m := \gamma(m)\circ\nabla\mu_m$, where γ is given by (7).*

If a partial connection form α is differentiable at m, then the curvature *of α at m is given by (10). A partial connection pair (α, χ) is* docile *at m if the dual connection form $\chi\,\alpha$ is docile at m. If (α, χ) is docile at m, then the* curvature *of (α, χ) at m equals that of $\chi\,\alpha$.*

In the classical setting, there is a unique connection form associated to a given connection. Thus the curvature can naturally be regarded as data of the connection itself, not just of the connection form. In general, there is not a unique partial connection form or dual connection form associated to a given partial connection; hence curvature is not a priori determined by the partial connection, rather than a specific form. We shall show in §5 that at regular points of M, i.e. points at which the dimensions of the group orbits are locally constant, the curvature of a partial connection is well–defined. However, at singular points it can easily occur that two dual connection forms determining the same partial connection fail to share docility. For example, the dual connection forms (3) for the action of $SO(3)$ on $\mathbb{R}^3$ satisfy

$$\mathrm{d}\mu^q(0)(u_0, v_0) = 2\,q(0)\,u_0 \times v_0,$$

while range $\mu^q(0) = \{0\}$. If $q(0) \neq 0$, then $\mathrm{d}\mu^q(0)(u_0, v_0) \notin$ range $\mu^q(0)$ for any u_0 and v_0 that are not parallel. Thus the dual connection forms μ^q discussed above are docile if and only if

$q(0) = 0$, in which case the curvature at the origin is zero. This example suggests that in some circumstances we can 'tame' a given dual connection form that fails to be docile, obtaining a docile dual connection form whose curvature agrees with that of the original form wherever the original form is docile.

Remark 1 Consider two inertia factors χ and $\tilde{\chi}$ compatible with a partial connection Γ, in the sense that (8) determines smooth forms μ and $\tilde{\mu}$, and satisfying $\tilde{\chi} = \sigma \circ \chi$ for some smooth map $\sigma : M \to \mathcal{L}(\mathfrak{g}^*, \mathfrak{g}^*)$. If Ω (respectively $\tilde{\Omega}$) denotes the curvature of μ (respectively $\tilde{\mu}$), then

$$\nabla \tilde{\mu} = \sigma \, \nabla \mu + \nabla \sigma \wedge (\mu \circ \mathbb{P}_\Gamma) = \sigma \, \nabla \mu$$

and $\tilde{\gamma} \circ \sigma = \gamma$ imply that

$$\tilde{\Omega} = \tilde{\gamma} \circ \nabla \tilde{\mu} = \gamma \circ \nabla \mu = \Omega$$

wherever both μ and $\tilde{\mu}$ are docile.

If $\tilde{\mu}$'s domain of docility is larger than that of μ, it may be preferable to replace μ by $\tilde{\mu}$. For example, assume that $\mathfrak{g}$ is a inner product space and μ is a dual connection form such that $\chi(m)$ is symmetric for every $m \in M$. Let $\mu^\sharp$ denote the $\mathfrak{g}$–valued one–form satisfying $\langle \mu^\sharp(v), \xi \rangle = \mu(v) \cdot \xi$ for all $\xi \in \mathfrak{g}$ and $v \in TM$; the dual connection form $\tilde{\mu} = \chi \circ \mu^\sharp$ is docile on all of M. Since μ and $\tilde{\mu}$ determine the same partial connection and have the same curvature wherever the curvature of μ is defined, the form $\tilde{\mu}$ is, by some standards, the preferable one to use when analysing the partial connection. This situation frequently arises when μ is the momentum map determined by the kinetic energy on a Riemannian manifold χ is the 'locked inertia tensor', as in our prototypical example. We investigate such an example in the next section. ■

Using the equivariance properties of dual connection forms, we can derive the analogs of the classical structure equations for dual connection forms and partial connection pairs.

Proposition 3 *Given a dual connection form μ that is docile at m and tangent vectors u, $v \in T_m M$, let ξ, $\eta \in \mathfrak{g}$ satisfy $\mu(u) = \chi(m)\xi$ and $\mu(v) = \chi(m)\eta$. Then*

$$\Omega(u, v) + [\xi, \eta]_M(m) = \gamma(m)(\mathrm{d}\mu(u, v) - \mathrm{d}\chi(u)\eta + \mathrm{d}\chi(v)\xi).$$

Given a partial connection pair (α, χ), define the $\mathfrak{g}$–valued curvature form $\Omega^\alpha := \alpha \circ \Omega$ at points where (α, χ) is docile. Then

$$\chi(\Omega^\alpha + \alpha \wedge \alpha) = \mathrm{d}(\chi \, \alpha) - \mathrm{d}\chi \wedge \alpha,$$

where $(\alpha \wedge \alpha)(u, v) = [\alpha(u), \alpha(v)]$ for all u, $v \in TM$. In particular, if $\mathrm{d}\chi(m) = 0$, we recapture the classical structure equations modulo $\mathfrak{g}_m$ at m.

Proof: Equivariance of χ implies that $\xi_M(\chi) = -\mathrm{ad}^*_\xi \circ \chi - \chi \circ \mathrm{ad}_\xi$ for all $\xi \in \mathfrak{g}$. Thus (12) implies that

$$\mathrm{d}\mu(\xi_M, \eta_M) = -\mathrm{ad}^*_\xi(\chi\eta) + \mathrm{ad}^*_\eta(\chi\xi) - \chi[\xi, \eta].$$

and hence

$$\begin{aligned} \nabla \mu(u, v) &= \mathrm{d}\mu(u - \xi_M(m), v - \eta_M(m)) \\ &= \mathrm{d}\mu(u, v) - \mathrm{d}\chi(u) \, \eta + \mathrm{d}\chi(v) \, \xi - \chi(m)[\xi, \eta]. \end{aligned}$$

■

3.1 An example: the combined left–right action of a subgroup on a Lie group

Let G be a Lie group, with Lie algebra $\mathfrak{g}$ equipped with an Ad–invariant inner product $\langle\ ,\ \rangle$, and let H be a subgroup of G. Let $H \times H$ act on G by

$$(h, k) \cdot g = hgk^{-1}.$$

Given a subspace $V \subseteq \mathfrak{g}$, let $V^\perp$ denote its orthogonal complement and $\mathbb{P}_V : \mathfrak{g} \to V$ the orthogonal projection onto V. We take as our partial connection $\Gamma = (\mathfrak{h} \times \mathfrak{h})^\perp$, the orthogonal complement to the tangent to the group orbit with respect to the right–invariant metric induced on G by $\langle\ ,\ \rangle$. If we identify $T_g G$ with $\mathfrak{g}$ by right trivialization, as we shall throughout this example, then the infinitesimal generator of $(\eta, \zeta) \in \mathfrak{h} \times \mathfrak{h}$ is

$$(\eta, \zeta)_G(g) = \eta - \mathrm{Ad}_g \zeta.$$

If we let $\mathrm{Ad}_g(\mathfrak{h})$ denote the image of $\mathfrak{h}$ under the adjoint action of g, it follows that the isotropy subalgebra of g is

$$(\mathfrak{h} \times \mathfrak{h})_g = (\mathbb{1} \times \mathrm{Ad}_{g^{-1}})(\mathfrak{h} \cap \mathrm{Ad}_g(\mathfrak{h})).$$

The momentum map μ associated to the action of $H \times H$ on G and the Lagrangian $L(\xi) = \frac{1}{2}|\xi|^2$ is a dual connection form compatible with $\Gamma = \widetilde{(\mathfrak{h} \times \mathfrak{h})}^\perp$. However, as we shall show, μ fails to be docile at points in the normalizer $N(H)$ of H. Hence we work with the 'tamed' dual connection form ν associated to μ, as described in Remark 1. The form ν has curvature

$$\Omega_g(\xi, \omega) = (\mathbb{P}_{\mathfrak{h}} - \mathbb{P}_{\mathrm{Ad}_g \mathfrak{h}})[\mathbb{P}_{\Gamma_g} \xi, \mathbb{P}_{\Gamma_g} \eta] \tag{13}$$

for all $g \in G$ and ξ, $\omega \in \mathfrak{g}$. It follows that

$$\operatorname{rank} \Omega_g \leq 2(\dim \mathfrak{h} - \dim(\mathfrak{h} \cap \mathrm{Ad}_g(\mathfrak{h}))).$$

In particular, $\Omega_g = 0$ if $g \in N(H)$.

If we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ using $\langle\ ,\ \rangle$, then the momentum map μ satisfies

$$\mu_g(\xi) = (\mathbb{P}_{\mathfrak{h}} \xi, -\mathbb{P}_{\mathfrak{h}} \mathrm{Ad}_{g^{-1}} \xi) = (\mathbb{P}_{\mathfrak{h}} \xi, -\mathrm{Ad}_{g^{-1}} \mathbb{P}_{\mathrm{Ad}_g \mathfrak{h}} \xi),$$

with associated correction factor

$$\chi(g) = \begin{pmatrix} \mathbb{1} & -\mathbb{P}_{\mathfrak{h}} \mathrm{Ad}_g \\ -\mathbb{P}_{\mathfrak{h}} \mathrm{Ad}_{g^{-1}} & \mathbb{1} \end{pmatrix}.$$

We first compute the exterior derivative of μ. If we let X_ξ denote the right invariant vector field associated to $\xi \in \mathfrak{g}$, then

$$X_\xi(\mu(X_\omega))(g) = (0, \mathbb{P}_{\mathfrak{h}} \mathrm{Ad}_{g^{-1}}[\xi, \omega])$$

and $[X_\xi, X_\omega] = -X_{[\xi,\omega]}$ imply that

$$\mathrm{d}\mu_g(\xi, \omega) = (X_\xi(\mu(X_\omega)) - X_\omega(\mu(X_\xi)) - \mu([X_\xi, X_\omega]))(g) = (\mathbb{P}_{\mathfrak{h}}[\xi, \omega], \mathbb{P}_{\mathfrak{h}} \mathrm{Ad}_{g^{-1}}[\xi, \omega]).$$

Note that $\operatorname{range}[\nabla\mu_g] \subseteq \operatorname{range}[\chi(g)]$ need not hold for all $g \in G$. In particular, if $g \in N(H)$, then $\mathbb{P}_{\mathfrak{h}}$ commutes with $\mathrm{Ad}_{g^{-1}}$; hence, in this case,

$$\mathrm{d}\mu_g(\xi,\omega) = (\mathbb{1} \times \mathrm{Ad}_{g^{-1}})\mathbb{P}_{\mathfrak{h}}[\xi,\omega]$$

for all ξ, $\omega \in \mathfrak{g}$, while $\operatorname{range}\chi(g) = (\mathbb{1} \times (-\mathrm{Ad}_{g^{-1}}))\mathfrak{h}$. Thus μ fails to be docile at points in $N(H)$ unless $\mathfrak{h}^\perp$ is a subalgebra of $\mathfrak{g}$. Hence we replace μ with the 'tamed' dual connection form $\nu = \chi \circ \mu$, as in Remark 1. Given our identification of $\mathfrak{h}$ and $\mathfrak{h}^*$, the map $\sharp$ is trivial.

We now show that the curvature Ω of ν satisfies (13). Given ξ, $\omega \in \mathfrak{g}$, let

$$[\xi', \omega'] = \alpha + \beta + \gamma + \delta,$$

where $\alpha \in \mathfrak{h} \cap (g \cdot \mathfrak{h})^\perp$, $\beta \in \mathfrak{h}^\perp \cap g \cdot \mathfrak{h}$, $\gamma \in \mathfrak{h} \cap g \cdot \mathfrak{h}$, and $\delta \in \Gamma_g$, and let ξ' and ω' denote the projections of ξ and ω into $\Gamma|_g$. Then

$$(\mathbb{P}_{\mathfrak{h}} - \mathbb{P}_{\mathrm{Ad}_g \mathfrak{h}})[\xi', \omega'] = (\alpha + \gamma) - (\beta + \gamma) = \alpha - \beta.$$

Observe that

$$\mu_g(\alpha - \beta) = (\mathbb{P}_{\mathfrak{h}}(\alpha - \beta), -\mathrm{Ad}_{g^{-1}}\mathbb{P}_{\mathrm{Ad}_g \mathfrak{h}}(\alpha - \beta)) = (\alpha, \mathrm{Ad}_{g^{-1}}\beta),$$

while

$$\nabla\mu_g(\xi,\omega) = (\mathbb{P}_{\mathfrak{h}} \times \mathrm{Ad}_{g^{-1}}\mathbb{P}_{\mathrm{Ad}_g \mathfrak{h}})[\xi', \omega'] = (\alpha + \gamma, \mathrm{Ad}_{g^{-1}}(\beta + \gamma)).$$

Thus

$$\nabla\mu_g(\xi,\omega) - \mu_g(\alpha - \beta) = (\gamma, \mathrm{Ad}_{g^{-1}}\gamma) \in \ker\chi(g),$$

and hence

$$\nabla\nu_g(\xi,\omega) = \chi(g)\nabla\mu_g(\xi,\omega) = \chi(g)\mu_g(\alpha - \beta) = \nu_g(\alpha - \beta).$$

As a simple application of the formulas derived above, we now provide an example of a point with nontrivial isotropy and nonzero curvature. We take as our manifold the group $SU(3)$ and select as our subgroup H the two–torus of diagonal matrices in $SU(3)$. We work with the orthogonal basis $\{\delta_1, \delta_2, \sigma_1, \sigma_2, \sigma_3, \xi_1, \xi_2, \xi_3\}$, where $\delta_1 = \mathrm{diag}(i, -i, 0)$, $\delta_2 = \mathrm{diag}(i, i, -2i)$, σ_j has i in the $k\ell$ and ℓk positions and zeroes elsewhere, and ξ_j has 1 in the $k\ell$ position, -1 in the ℓk position, and zeroes elsewhere; here $jk\ell$ is a cyclic permutation of 123. Note that $\{\delta_1, \delta_2\}$ is a basis for the two–torus H, while $\{\xi_1, \xi_2, \xi_3\}$ is a basis for the rotation group $SO(3)$.

We compute the curvature at an element g of $SO(3)$ corresponding to a rotation through an angle θ about the vertical axis, $\theta \neq \frac{n\pi}{2}$ for any $n \in \mathbb{Z}$. The adjoint action of g fixes δ_2 and maps δ_1 to $\cos 2\theta\, \delta_1 + \sin 2\theta\, \sigma_3$. Hence

$$(\mathfrak{h} \times \mathfrak{h})_g = \mathrm{span}\,\{\delta_2\}, \quad (\widetilde{\mathfrak{h} \times \mathfrak{h}})|_g = \mathrm{span}\,\{\delta_1, \delta_2, \sigma_3\}, \quad \text{and} \quad \Gamma|_g = \mathrm{span}\,\{\sigma_1, \sigma_2, \xi_1, \xi_2, \xi_3\}.$$

The commutators of the basis elements of $\Gamma|_g$ are $[\sigma_1, \sigma_2] = \xi_3$, $[\xi_j, \xi_k] = \xi_\ell$, where $jk\ell$ is a cyclic permutation of 123, and

$$[\sigma_j, \xi_k] = \begin{cases} (-1)^j \delta_1 + \delta_2 & j = k \neq 3 \\ -\sigma_3 & 3 \neq j \neq k \neq 3 \\ (-1)^j \sigma_{j'} & j \neq k = 3 \end{cases},$$

where $jj' = 12$ or 21. We have $(\mathbb{P}_{\mathfrak{h}} - \mathbb{P}_{\mathrm{Ad}_g \mathfrak{h}})\eta = \frac{1}{2}(\langle \eta, \sigma_3 \rangle \sigma_3 - \langle \eta, \delta_1 \rangle \delta_1)$ for all $\eta \in su(3)$. Hence the nontrivial elements of the curvature at g are

$$\Omega_g(\sigma_j, \xi_k) = \begin{cases} (-1)^j \delta_1 & j = k \neq 3 \\ -\sigma_3 & 3 \neq j \neq k \neq 3 \end{cases}$$

and the curvature at g has rank two, with $\operatorname{range} \Omega_g = \operatorname{span}\{\delta_1, \sigma_3\}$.

4 Curvature and involutivity

The classical formula $\Omega(X, Y) = \alpha([Y, X])$ for the curvature Ω of a connection form α, where X and Y are horizontal vector fields, can be rephrased as the assertion that the curvature of a connection measures the extent to which the connection is involutive. We can easily show that at regular points of M with respect to proper actions the curvature of a partial connection does not depend on the choice of partial or dual connection form used to characterize the differential system and this classical involutivity relation is satisfied. The development of an analogous expression for the curvature at singular points involves the construction of a smooth differential system, the system of 'almost horizontal' vectors, containing the singular partial connection on a neighborhood of a singular point.

The curvature at a regular point can be expressed in terms of the Lie bracket of horizontal vector fields, as in the classical case. This expression leads directly to a correspondence between (locally) zero curvature and involutivity of the partial connection Γ, and hence the existence of maximal integral manifolds tangent to Γ.

Theorem 2 *Let Ω denote the curvature of a dual connection form or partial connection form compatible with the partial connection Γ. Given any horizontal vector fields X and Y, the vector field $\Omega(X, Y) + [X, Y]$ is horizontal.*

If Ω is identically zero on some open set containing only regular points, then Γ is involutive on that set.

Proof: Let $Z = [X, Y]$. We first consider a dual connection form μ. We have

$$\nabla\mu(X, Y) = \mathrm{d}\mu(X, Y) = X(\mu(Y)) - Y(\mu(X)) - \mu([X, Y]) = -\mu(Z)$$

and hence

$$Z + \Omega(X, Y) = Z - \gamma(\mu(Z)) = (\mathbb{1} - \mathbb{P}_\mu)Z = \mathbb{P}_\Gamma Z.$$

Given a partial connection form α, analogous arguments show that

$$Z + \Omega(X, Y) = Z - d\widehat{\Phi}_m(\alpha(Z)) = (\mathbb{1} - \mathbb{P}_\alpha)Z = \mathbb{P}_\Gamma Z.$$

As was previously discussed, if m is a regular point, then Γ is a smooth differential system near m and hence is spanned by horizontal vector fields on some neighborhood of m. Thus the curvature on this neighborhood is determined by the relation $\Omega(X, Y) = (\mathbb{P}_\Gamma - \mathbb{1})[X, Y]$ for horizontal vector fields X and Y. In particular, if Ω is identically zero on such a neighborhood, then Γ is involutive there. ■

The key feature of the proof of Theorem 2 is the existence of vector fields generating Γ near m. If m is a singular point, then the dimension of $\Gamma|_m$ is greater than that of $\Gamma|_n$ at nearby points n. Thus we cannot find a set of horizontal vector fields on a neighborhood of m that span $\Gamma|_m$, and hence cannot directly invoke the identity relating exterior derivatives and commutators of vector fields. At singular points we must relax the notion of horizontality, obtaining a condition that yields a smooth differential system even in the neighborhood of a point at which a jump in isotropy occurs. We enlarge the partial connection by including a 'rotated' copy of the tangent space to the G_{m_0} orbit at each point. These spaces are rotated so as to yield a smooth differential system near the given singular point. By analogously rotating the dual connection form, we obtain a form whose kernel is the desired differential system. Using this adapted form, we can mimic the argument of Theorem 2 at singular points under an appropriate hypothesis on the adapting map.

Definition 3 *An* adaptor ϕ *for* m_0 *is a smooth* G_{m_0}*–equivariant map* $\phi : \mathfrak{U} \to G$ *on a* G_{m_0}*–invariant neighborhood* $\mathfrak{U}$ *of* m_0 *in* M *satisfying* $\mathrm{Ad}_{\phi(m)}\mathfrak{g}_{m_0} \supseteq \mathfrak{g}_m$ *for all* $m \in \mathfrak{U}$*, and* $\phi(m_0) \in G_{m_0}$*. Here* G_{m_0}*–equivariance is with respect to conjugation in* G*, so that* $\phi(g \cdot m) = g\,\phi(m)g^{-1}$ *for all* $m \in S$ *and* $g \in G_{m_0}$*.*

Given a singular point m_0*, a partial connection* Γ*, and an adaptor* ϕ *for* m_0*,* $\Xi := \Gamma \oplus \mathrm{Ad}_\phi \mathfrak{g}_{m_0}$ *is the* almost horizontal *differential system for* m_0*.*

Fix a dual connection form μ *for the partial connection* Γ *and a* G_{m_0}*–equivariant projection* π *on* $\mathfrak{g}$ *with* $\ker \pi = \mathfrak{g}_{m_0}$*. Define the* G_{m_0}*–equivariant map* $\chi_\phi : \mathfrak{U} \to \mathfrak{L}(\mathfrak{g}, \mathfrak{g}^*)$*, where* $\mathfrak{U}$ *is the domain of the adaptor* ϕ*, by*

$$\chi_\phi(m) := \chi(m) \circ \mathrm{Ad}_{\phi(m)}.$$

Using χ_ϕ*, define the projections* $\pi_\phi(m) := \chi_\phi(m) \circ \pi \circ \chi_\phi(m)^{-1}$ *and the* adapted dual connection form $\widetilde{\mu}_m := \pi_\phi(m) \circ \mu_m$.

Note that $\chi_\phi(m)^{-1}(\mu(v))$ is defined modulo $\mathrm{Ad}_{\phi(m)^{-1}}(\mathfrak{g}_m) \subseteq \mathfrak{g}_{m_0} = \ker \pi$, and hence π_ϕ and $\widetilde{\mu}$ are well–defined. The adapted dual connection form $\widetilde{\mu}$ is a constant rank, G_{m_0}–equivariant $\mathfrak{g}^*$–valued one form with $\ker \widetilde{\mu} = \Xi$. Hence Ξ is a smooth differential system. Finally, note that $\pi_\phi \circ \chi_\phi = \chi_\phi \circ \pi$.

Proposition 4 *If* G *acts properly, then there is an adaptor for any point in* M*. Given a partial connection* Γ *and a point* $m_0 \in M$*, there is an adaptor* ϕ *for* m_0 *satisfying* $\phi(m_0) = e$ *and* $d\phi(\Gamma|_{m_0}) \subseteq \mathfrak{g}_{m_0}$*.*

The proof of Proposition 4, which makes use of some technical results related to slices, is given in the appendix.

Remark: We could work with the *adapted connection form* $\widetilde{\alpha}(v) := \mathrm{Ad}_{\phi(m)}\pi\,\chi_\phi(m)^{-1}(\mu(v))$, rather than the adapted dual connection form $\widetilde{\mu}$. Either $\widetilde{\mu}$ or $\widetilde{\alpha}$ can be used to define a G_{m_0}–equivariant projection $\mathbb{P}_\phi$ on $T\mathfrak{U}$, with $\ker \mathbb{P}_\phi = \Xi$, by $\mathbb{P}_\phi := \gamma \circ \widetilde{\mu}$ or $\mathbb{P}_\phi := d_e\widehat{\Phi}_\circ\widetilde{\alpha}$.

We now generalize the 'lack of involutivity' characterization of curvature given in Theorem 2 to singular points, replacing the partial connection Γ with the almost horizontal differential structure Ξ.

Theorem 3 *If μ is docile and ϕ is an adaptor for m_0 satisfying*

$$[d^{\natural_L}\phi(\Xi|_m), \mathfrak{g}_{m_0}] \subseteq \mathfrak{g}_{m_0} \tag{14}$$

where $d^{\natural_L}_m\phi = d_m(L_{\phi(m)^{-1}} \circ \phi)$ is the left trivialization of the linearization of ϕ at m, for some m near m_0, then for any almost horizontal vector fields X and Y, $\Omega(X,Y) + [X,Y]$ is almost horizontal at m.

If (14) holds for all m on a neighborhood $\mathcal{V}$ of m_0 and $\operatorname{range}\Omega \subseteq \Xi$ *on $\mathcal{V}$, then Ξ is involutive on $\mathcal{V}$.*

The key ingredient in the proof of Theorem 3 is an expression for the differential of the adapted form $\widetilde{\mu}$ in terms of the covariant derivative of the dual connection form μ. We assume here, and throughout the remainder of this section, that the adapted inertia factor χ_ϕ has a differentiable restricted pseudo–inverse ι; specifically, that there is a differentiable map $\iota : \mathcal{U} \to \mathcal{L}(\mathfrak{g}^*, \mathfrak{g})$ such that

$$\pi = \pi \circ \iota(m) \circ \chi_\phi(m)$$

for all $m \in \mathcal{U}$. (Note that $\ker \chi_\phi(m) = \mathrm{Ad}_{\phi(m)^{-1}}\mathfrak{g}_m \subseteq \mathfrak{g}_{m_0} = \ker \pi$.) If we set

$$\widetilde{\pi}_\phi(m) := \chi_\phi(m) \circ \pi \circ \iota(m)$$

for all $m \in \mathcal{U}$, then $\pi_\phi(m) = \widetilde{\pi}_\phi(m)|\operatorname{range}\chi_\phi(m)$, and hence $\widetilde{\mu} = \widetilde{\pi}_\phi \circ \mu$.

Lemma 2 *Let $u, v \in T_m\mathcal{U}$, and let ξ and $\eta \in \mathfrak{g}$ satisfy $\chi_\phi(m)\xi = \mu(u)$ and $\chi_\phi(m)\eta = \mu(v)$. If μ is docile at m and ϕ satisfies (14), then*

$$\begin{aligned} d\widetilde{\mu}(u,v) - \pi_\phi(m)\nabla\mu(u,v) &= d\chi_\phi(u)\pi\eta - d\chi_\phi(v)\pi\xi \\ &\quad + \chi_\phi(m)\pi([d^{\natural_L}\phi(v), \xi] - [d^{\natural_L}\phi(u), \eta] + [\xi, \eta]). \end{aligned}$$

Proof: Leibniz's Rule implies that

$$d\widetilde{\mu}(u,v) = d\widetilde{\pi}_\phi(u)\mu(v) - d\widetilde{\pi}_\phi(v)\mu(u) + \widetilde{\pi}_\phi(m)d\mu(u,v). \tag{15}$$

Linearizing $\pi = \pi \circ \iota \circ \chi_\phi$ yields

$$0 = \pi \circ (d\iota(v) \circ \chi_\phi(m) + \iota(m) \circ d\chi_\phi(v)).$$

Thus

$$\begin{aligned} d\widetilde{\pi}_\phi(v)\mu(u) &= d\widetilde{\pi}_\phi(v)\chi_\phi(m)\xi \\ &= (d\chi_\phi(v)\pi\iota(m) + \chi_\phi(m)\pi d\iota(v))\chi_\phi(m)\xi \\ &= d\chi_\phi(v)\pi\xi - \widetilde{\pi}_\phi(m)d\chi_\phi(v)\xi. \end{aligned}$$

Next, we have

$$d\chi_\phi(v)\xi = d\chi(v)\mathrm{Ad}_{\phi(m)}\xi + \chi_\phi(m)[d^{\natural_L}\phi(v), \xi].$$

Entirely analogous expressions hold when u and v are exchanged. Proposition 3 implies that

$$d\mu(u,v) = \nabla\mu(u,v) + d\chi(u)\mathrm{Ad}_{\phi(m)}\eta - d\chi(v)\mathrm{Ad}_{\phi(m)}\xi + \chi_\phi(m)[\xi, \eta].$$

Substituting these expressions into (15) and regrouping terms yields the desired expression. ■

Proof of Theorem 3: Let X and Y be almost horizontal vector fields. We apply Lemma 2, taking $u = X(m)$ and $v = Y(m)$. The condition $u, v \in \Xi|_m$ implies that $\xi, \eta \in \mathfrak{g}_{m_0} = \ker \pi$, and hence $\pi_\phi \nabla \mu(u, v) = d\widetilde{\mu}(u, v)$. Thus $\widetilde{\mu}(X) = \widetilde{\mu}(Y) = 0$ implies

$$0 = d\widetilde{\mu}(X, Y) + \widetilde{\mu}([X, Y]) = \pi_\phi(\nabla \mu(X, Y) + \mu([X, Y])).$$

Since $\ker \pi_\phi = \chi_\phi(\mathfrak{g}_{m_0})$, calculations analogous to those used in the proof of Theorem 2 yield

$$\Omega(X, Y) + [X, Y] = \mathbb{P}_\Gamma[X, Y] \qquad \text{mod} \quad \widetilde{\mathrm{Ad}_\phi \mathfrak{g}_{m_0}} = \widetilde{\mathfrak{g}} \cap \Xi;$$

hence $\Gamma \subseteq \Xi$ implies that $\Omega(X, Y) + [X, Y]$ takes values in Ξ.

Involutivity of Ξ when $\operatorname{range} \Omega \subseteq \Xi$ follows immediately from the first part of the theorem. ■

Corollary 1 *If G acts properly, the curvature at a singular point m_0 is determined by the equation $\Omega(X, Y)(m_0) = (\mathbb{P}_\Gamma - \mathbb{1})[X, Y](m_0)$ for almost horizontal vector fields X and Y.*

Proof: Proposition 4 guarantees the existence of an adaptor ϕ for m_0 satisfying (14) at m_0. Since Ξ is a smooth differential structure spanned by almost horizontal vector fields, Theorem 3 determines the curvature modulo $\widetilde{\mathrm{Ad}_\phi \mathfrak{g}_{m_0}}$ at points near m_0 satisfying (14). Hence, since $\widetilde{\mathrm{Ad}_\phi \mathfrak{g}_{m_0}}|_{m_0}$ is trivial, the curvature at m_0 is entirely determined by Theorem 3. ■

We now show that the classical result that the curvature is identically zero on some neighborhood of a given point if and only if the horizontal differential system is tangent to a local cross section through that point can be generalized to partial connections under appropriate hypotheses on the isotropy subgroups. By a local cross section, we mean a submanifold S_0 such that $G \cdot S_0$ contains a neighborhood of m_0 in M and $g \cdot m \in S_0$ for $m \in S_0$ only if $g \cdot m = m$.

Corollary 2 *If G acts properly on M, Ω equals zero near m_0, and there is an adaptor ϕ satisfying*

$$G_m = \phi(m) G_{m_0} \phi(m)^{-1} \qquad \text{for all } m \text{ near } m_0, \tag{16}$$

then Γ is tangent to a local cross section through m_0.

The proof of Corollary 2 is rather technical and is in part modeled on a proof for a similar result for slices given in [25]. Hence it is relegated to the appendix.

Remark: If $G_m = \exp(\mathfrak{g}_m)$ for all m in a neighborhood of a regular point m_0 of a proper action, then equivariance of the exponential map implies that any adaptor ϕ for m_0 satisfies (16).

A slice generalizes the notion of a local cross section, allowing some overlap of the slice and the group orbits near singular points. Specifically, a slice at m_0 is a submanifold S through m_0 satisfying

(i) $T_{m_0} M = T_{m_0} S \oplus \widetilde{\mathfrak{g}}|_{m_0}$ and

(ii) $T_m M = T_m S + \widetilde{\mathfrak{g}}|_m$ for all $m \in S$

(iii) if $m \in S$ and $g \in G$, then $g \cdot m \in S$ if and only if $g \in G_{m_0}$.

Corollary 3 *If G acts properly, G_{m_0} is a normal subgroup of G, $G_m \subseteq G_{m_0}$ for all m in a neighborhood of m_0, and* range $\Omega \subseteq \widetilde{\mathfrak{g}_{m_0}}$ *on that neighborhood, then some neighborhood of m_0 in the integral submanifold of Ξ containing m_0 is a slice.*

The following lemma establishes the essential information regarding slices used in the proof of Corollary 3. These results are minor variations of standard results and for the most part we utilize straightforward modifications of the proofs given in [25]; hence the proof of the Lemma is given in the appendix.

Lemma 3 *(i) Given an involutive G_{m_0}–equivariant differential system $\Delta \supseteq \widetilde{\mathfrak{g}_{m_0}}$ on a G_{m_0}–invariant neighborhood of m_0, there is an G_{m_0}–invariant integral submanifold of Δ containing m_0.*

(ii) If G acts properly and S is a G_{m_0}–invariant submanifold of M containing m_0 and satisfying $T_{m_0} M = T_{m_0} S \oplus \widetilde{\mathfrak{g}}|_{m_0}$, there is a neighborhood S_0 of m_0 in S such that S_0 is a slice through m_0.

Proof of Corollary 3: Since G is normal and $G_m \subseteq G_{m_0}$ for all m near m_0, we can use the trivial adaptor $\phi \equiv e$. Theorem 3 implies that $\Xi = \Gamma \oplus \widetilde{\mathfrak{g}_{m_0}}$ is involutive near m_0.

Let S' denote the maximal integral manifold of the almost horizontal system containing m_0. Lemma 3.i and the G_{m_0}–equivariance of Ξ imply that S' is G_{m_0}–invariant. Lemma 3.ii and

$$T_{m_0} M = (\Gamma \oplus \widetilde{\mathfrak{g}})|_{m_0} = T_{m_0} S' + \widetilde{\mathfrak{g}}|_{m_0}$$

imply that some G_{m_0}–invariant neighborhood S of m_0 in S' is a slice through m_0. ■

Example: $S^1 \times S^1$ acting on $SO(3)$

As an application of Corollary 3, we consider a special case of §3.1, with $G = SO(3)$ and $H \approx S^1$ consisting of rotations about a given axis σ. We identify $so(3)$ with $\mathbb{R}^3$ using the cross product and take the standard Euclidean inner product as our inner product on $\mathbb{R}^3$. The horizontal subspaces are one dimensional at points without continuous isotropy, while all points with continuous isotropy are elements of the normalizer of H. Hence the curvature is identically zero.

If $\hat{}: \mathbb{R}^3 \to so(3)$ is the standard identification of a three–vector with a skew–symmetric matrix, then $\mu(\hat{\xi} g) = (\langle \sigma, \xi \rangle, -\langle g\sigma, \xi \rangle)$, with inertia factor

$$\chi(g) = \begin{pmatrix} 1 & -r(g) \\ -r(g) & 1 \end{pmatrix},$$

where the invariant function $r : SO(3) \to \mathbb{R}$ is given by $r(g) := \langle \sigma, g\sigma \rangle$. The vectors $\nu_\pm := (1, \pm 1)$ are eigenvectors of $\chi(g)$, with eigenvalues $\lambda_\pm(g) = 1 \mp r(g)$. $g_\pm \in SO(3)$ has nontrivial isotropy if $g_\pm \sigma = \pm\sigma$, and hence $\mathfrak{g}_{g_\pm} = \mathrm{span}\{\nu_\pm\}$ and $\chi(g_\pm) = \nu_\mp \nu_\mp^T$. If we take $\pi = \frac{1}{2}\chi(g_\pm)$ as our projection, then $\pi\,\chi(g) = \lambda_{\mp}(g)\pi$. Thus we can take $\iota(g) = \frac{1}{\lambda_{\mp}(g)}\mathbb{1}$ as our restricted pseudo–inverse of χ and Corollary 3 implies there is a slice S through $g_{\pm}$ with

$$T_g S = \Xi_g = \left\{ \hat{\xi} g : \langle \xi, \sigma \pm g\sigma \rangle = 0 \right\}.$$

We can explicitly construct the slice S, using the Cayley transform

$$\mathrm{cay}(\eta) = (\mathbb{1} - \hat{\eta}/2)^{-1}(\mathbb{1} + \hat{\eta}/2)$$

from $\mathbb{R}^3 \approx so(3)$ to $SO(3)$ to define a coordinate map $\psi : \eta \mapsto \mathrm{cay}(\eta) g_0$ taking an open ball B_r about 0 in $\mathbb{R}^3$ onto an open neighborhood of g_0 in $SO(3)$. A curve $\eta(s) \in \mathbb{R}^3$ with $\eta(0) = 0$ and $\|\eta(s)\| < r$ for all s lies in S if $T_{\eta(s)}\psi\eta'(s) \in \Xi_{\eta(s)}$ for all s. This holds if and only if

$$0 = \tilde{\nu}(\psi(\eta(s))) T_{\eta(s)}\psi\eta'(s) = \left\langle \sigma \pm \psi(\eta(s))\sigma, d^{\natural}\psi(\eta(s))\eta'(s) \right\rangle. \tag{17}$$

To further simplify this expression, we make use of the identities $\mathbb{1} + \mathrm{cay}(\eta) = 2(\mathbb{1} - \frac{1}{2}\hat{\eta})^{-1}$, and hence

$$\sigma \pm \psi(\eta)\sigma = (\mathbb{1} + \mathrm{cay}(\eta))\sigma = (\mathbb{1} - \tfrac{1}{2}\hat{\eta})^{-1}\sigma,$$

and $d^{\natural}_{\eta}\mathrm{cay} = \frac{1}{1+\|\eta/2\|^2}(\mathbb{1} + \frac{1}{2}\hat{\eta})$. Inserting these expressions into (17), letting $\eta = \eta(s)$, yields the condition

$$0 = \left\langle (\mathbb{1} - \tfrac{1}{2}\hat{\eta})^{-1}\sigma, (\mathbb{1} + \tfrac{1}{2}\hat{\eta})\eta'(s) \right\rangle = \left\langle (\mathbb{1} + \tfrac{1}{2}\hat{\eta})^{T}(\mathbb{1} - \tfrac{1}{2}\hat{\eta})^{-1}\sigma, \eta'(s) \right\rangle = \left\langle \sigma, \eta'(s) \right\rangle.$$

Thus $S = \psi(B_r \cap \sigma^{\perp})$ is a slice through g_0. ■

5 β–relative equivariant partial connections

We now relax the equivariance condition on the horizontal projections to allow for isotropy, requiring equivariance only with respect to some elements of the group. This is, in fact, the setting in which we originally introduced partial connections; see [17, 18, 19]. Although the details of some calculations are more complicated in the β–equivariant setting, the central constructions and underlying strategies are essentially identical to those used in the fully equivariant setting.

We shall say that a map $\beta : G \times M \to G$ is a *slip map* if $\beta(g, m) \cdot m = g \cdot m$ for all $g \in G$ and $m \in M$. If G acts on manifolds M and N and β is a slip map for the action on M, we shall say that a map $F : M \to N$ is *β–relative equivariant* if

$$F(\beta(g, m) \cdot m) = \beta(g, m) \cdot F(m) \tag{18}$$

for all $g \in G$ and $m \in M$. Note that (18) can also be expressed in the form $F(g \cdot m) = \beta(g, m) \cdot F(m)$. (See [26] for a more general treatment of relative equivariance.)

The constructions of partial connections, dual and partial connection forms, etc. all carry over to the β–equivariant setting. Analogs of Propositions 1–3 hold for β–equivariant partial connections and forms. The inertia factor of a β–equivariant dual connection form is β–equivariant. The definition of the curvature of a β–equivariant connection form or dual connection form is entirely analogous to that of an equivariant form. Note, however, that Lemma 1 need not hold for β–equivariant dual connection forms with nontrivial slip maps. Expressions analogous to the structure equations of Proposition 3 can be obtained by means of straightforward, but rather tedious, calculations. Development of results analogous to those of §4 for relatively equivariant connections will be the subject of future work.

To motivate the introduction of the notion of β–equivariance, we first relate trivial principal bundles to moving frames, then argue that β–equivariance, rather than full equivariance, is the most that can be expected in the setting of a natural extension of moving frames to general actions. (See [17, 18, 19].)

Given a Lie group G acting on a manifold M, an equivariant map $\rho : M \to G$, i.e. a map satisfying

$$\rho(g \cdot m) = g\rho(m) \qquad \text{(left moving frame)} \qquad \text{or} \qquad \rho(m)g \qquad \text{(right moving frame)},$$

is called a *moving frame*. It is known that the existence of a moving frame implies that the group action is (locally) free. (See [15, 16].) Given a global cross–section $\mathcal{S}$ of a manifold P with a free G action, i.e. a transverse submanifold $\mathcal{S}$ such that for each $p \in P$ there is a unique $g \in G$ and $\tilde{p} \in \mathcal{S}$ such that $p = g \cdot \tilde{p}$ if and only if there is a moving frame ρ on P. If the action of G on M is globally free, then the existence of a moving frame implies that M is a trivial principal bundle. The base manifold can be naturally identified with the associated global cross–section $\sigma(M)$. It follows that in this situation, if $\rho : M \to G$ is a moving frame, then the right trivialization $d^\natural_m \rho = d_m(R_{\rho(m)} \circ \rho)$ of the linearization of ρ is a flat connection form on the principal bundle M. As we shall discuss below, the trivialized linearization of a generalization of moving frames to non–free actions yields a relatively equivariant partial connection form.

Example: A moving frame and connection on US^2

The rotation group $SO(3)$ acts transitively on S^2 and freely and transitively on the unit tangent bundle $US^2 = \{u \in TS^2 : \|u\| = 1\}$. We consider the map

$$\rho(u) := (m, u, m \times u) \tag{19}$$

taking $u \in U_m S^2$ to the orthogonal matrix determined by the positively oriented orthonormal basis $\{m, u, m \times u\}$ is a moving frame. (Note that by an abuse of notation, we will regard u both as a tangent vector to the sphere at m and as a unit vector in $\mathbb{R}^3$.)

Since $\rho(g\, u) = (g\, m, g\, u, g(m \times u)) = g\, \rho(u)$ for any $g \in SO(3)$, ρ is a left moving frame.

The trivialized linearization $d^\natural \rho$ of ρ satisfies

$$d^\natural \rho(\delta u) = m \times \delta m + \langle u \times \delta u, m \rangle\, m, \tag{20}$$

where $\delta u \in T_u US^2$, with $m = \pi(u)$ and $\delta m = d\pi\, \delta u$. (Here $\pi : US^2 \to S^2$ denotes the canonical projection.) This can be seen as follows: Let $u(\epsilon)$ be a curve in US^2 through u tangent to δu and set $m_\epsilon = \pi(u(\epsilon))$. If we identify each of these points with a vector in $\mathbb{R}^3$, then differentiating the relations $\|m(\epsilon)\| = \|u(\epsilon)\| = 1$ and $\langle m(\epsilon), u(\epsilon) \rangle = 0$ yields

$$\langle m, \delta m \rangle = \langle u, \delta u \rangle = \langle \delta m, u \rangle + \langle m, \delta u \rangle = 0.$$

Setting $z = m \times \delta m + \langle u \times \delta u, m \rangle\, m$, we obtain

$$d\rho(\delta u) = (\delta m, \delta u, \delta m \times u + m \times \delta u) = (z \times m, z \times u, z \times (m \times u)) = \hat{z}\rho(m),$$

where $\hat{z}$ denotes the skew–symmetric matrix satisfying $\hat{z}y = z \times y$ for any y, $z \in \mathbb{R}^3$.

In [17, 19, 18] we introduced the notion of a partial moving frame, which is equivariant modulo isotropy. To motivate this extension, we observe that full equivariance holds if and only if $\rho(g \cdot m)\rho(m)^{-1} = g$ for all $m \in M$ and $g \in G$ and relax this condition to allow for isotropy. Assume that G acts on M on the left. (If G acts on the right, then the roles of left and right should be reversed throughout.) We shall say that a (smooth) map $\phi : M \to G$ is a (left) *partial moving frame* if for any $g \in G$ and $m \in M$

$$\phi_g(m) := \phi(g \cdot m)\phi(m)^{-1} = g \mod G_m. \tag{21}$$

The quotient is with respect to right cosets; thus (21) is equivalent to

$$\phi_g(m) \cdot m = g \cdot m$$

for all $g \in G$ and $m \in M$. (A *partial moving frame* on a submanifold $\mathcal{S}$ of a manifold M with a G action is a map $\phi : \mathcal{S} \to G$ satisfying (21) for any $m \in \mathcal{S}$ and any $g \in G$ such that $g \cdot m \in \mathcal{S}$.)

To further motivate this construction, let us relate partial moving frames to cross–sections. Each partial moving frame $\phi : M \to G$ determines a global cross–section as follows. Define the map $\pi_\phi : M \to M$ by $\pi_\phi(m) := \phi(m)^{-1} \cdot m$. Condition (21) states that for any $m \in M$ and $g \in G$, there exists $h \in G_m$ satisfying $\phi(g \cdot m) = g\, h\, \phi(m)$, and thus

$$\pi_\phi(g \cdot m) = (g\, h\, \phi(m))^{-1} \cdot (g \cdot m) = \pi_\phi(m).$$

Thus each orbit $G \cdot m$ intersects the image of π_ϕ at precisely one point. However, a global section does not uniquely determine a partial moving frame, since any (smooth) map $\iota : M \to G$ satisfying $\iota(m) \in G_m$ for all $m \in M$ determines a new partial moving frame $\tilde{\phi}(m) := \phi(m)\iota(m)$ satisfying $\pi_{\tilde{\phi}} = \pi_\phi$.

Given a partial moving frame on a bundle $\pi : N \to M$ with an equivariant projection π, we can construct a partial moving frame on M using a section.

Proposition 5 *Let M and N be manifolds with G actions. If*

- $\pi : N \to M$ *is equivariant*
- $\sigma : M \to N$ *satisfies* $\pi \circ \sigma = \mathrm{id}$ *and* $\sigma(G \cdot m) \subseteq G \cdot \sigma(m)$ *for all* $m \in M$*,*

then a partial moving frame ϕ on N determines a partial moving frame $\tilde{\phi} := \phi \circ \sigma$ on M.

Proof: The inclusion $\sigma(G \cdot m) \subseteq G \cdot \sigma(m)$ implies that $\sigma(g \cdot m) = k\, \sigma(m)$ for some $k \in G$. Thus equivariance of π yields

$$g \cdot m = \pi(\sigma(g \cdot m)) = \pi(k \cdot \sigma(m)) = k \cdot \pi(\sigma(m)) = k \cdot m,$$

and hence $g^{-1}k \in G_m$. Combining this with

$$k^{-1}\tilde{\phi}_k(m) = k^{-1}\phi_k(\sigma(m)) \in G_{\sigma(m)} \subseteq G_m$$

gives $g^{-1}\tilde{\phi}_g(m) \in G_m$. ■

Example: Partial moving frames and connection forms on subsets of S^2

The projection π from US^2 to S^2 is clearly equivariant. Hence any unit vector field Y on a submanifold M of S^2 determines a partial moving frame $\phi = \rho \circ Y$ on M. It follows immediately from (20) that the partial moving connection form $d^\natural \phi$ determined by ϕ is

$$d^\natural \phi(\delta m) = m \times \delta m + \langle Y(m) \times (dY(\delta m)), m \rangle \, m. \tag{22}$$

The map ϕ_g associated to $g \in SO(3)$ is $\phi_g(m) = g \exp(\theta(g, m)\, m)$, where $\theta(g, m)$ denotes the angle between $g^{-1} Y(g\, m)$ and $Y(m)$. Thus $g^{-1} \phi_g(m)$ measures the failure of the vector field Y to commute with the action of g at m. To see this, let $u = g^{-1} Y(g\, m) \in U_m S^2$ and $\theta = \theta(g, m)$. Then $\phi(g\, m) = \rho(Y(g\, m)) = g\, \rho(u)$ and, using $\langle m, u \rangle = \langle g\, m, Y(g\, m) \rangle = 0$ and $\langle m \times u, m \times Y(m) \rangle = \langle u, Y(m) \rangle$, we obtain $\phi(m)^T \rho(u) = \exp(\theta\, e_1)$, where e_1 denotes the first standard Euclidean basis vector. Hence

$$\phi_g(m) = \phi(g\, m) \phi(m)^{-1} = g\, \rho(u) \phi(m)^{-1}.$$

Note that if $m(t)$ is a unit speed curve in S^2, then

$$d^\natural \rho(\ddot{m}) = m \times \dot{m} + \langle \dot{m} \times \ddot{m}, m \rangle \, m = m \times \dot{m} + k_g(m) m,$$

where $k_g(m(t))$ denotes the geodesic curvature of $m(t)$. Similarly, if the unit vector field Y is the normalization of a nonzero vector field X on some set $M \subset S^2$, i.e. $Y(m) = X(m) / \|X(m)\|$, then (22) yields

$$d^\natural \phi(X(m)) = m \times X(m) + k_g(m) \, \|X(m)\| \, m.$$

The relevance of partial moving frames to partial connections is established in the following result.

Theorem 4 *The trivialized linearization $d^\natural \phi$ of a partial moving frame $\phi : M \to G$ is a β–equivariant partial connection form, with associated slip map $\beta(g, m) = \phi_g(m)$.*

If we define $\pi_\phi(m) := \phi(m)^{-1} \cdot m$, then the partial connection Γ determined by $d^\natural \phi$ satisfies

$$\Gamma|_m = d(\Phi_{\phi(m)} \circ \pi_\phi)(T_m M)$$

for all $m \in M$.

We now develop some consequences of β–equivariance, some of which we will use in the proof of Proposition 4.

Lemma 4

(i) If $\beta_g : M \to G$ satisfies $\beta_g(m) \cdot m = g \cdot m$ for all $m \in M$, then

$$d_m \Phi_g - d_m \Phi_{\beta_g(m)} = d\hat{\Phi}_{g \cdot m} \circ d^\natural_m \beta_g \tag{23}$$

and

$$\text{range}\,(d^\natural \beta_g \circ d_e \hat{\Phi}_m - \text{Ad}_g + \text{Ad}_{\beta_g(m)}) \subseteq \mathfrak{g}_{g \cdot m} \tag{24}$$

for all $m \in M$.

(ii) A $\mathfrak{g}$–valued one–form α is β–equivariant modulo isotropy if and only if

$$\begin{aligned}\Phi_g^*\alpha(v) &= (\mathrm{Ad}_{\beta(g,m)}\alpha + d^\natural\beta_g)(v) \qquad \textit{mod} \qquad \mathfrak{g}_{g\cdot m}\\ &= (\mathrm{Ad}_g\alpha + d^\natural\beta_g\circ(\mathbb{1}-\mathbb{P}_\alpha)(v) \qquad \textit{mod} \qquad \mathfrak{g}_{g\cdot m},\end{aligned}$$

for all $g\in G$, $m\in M$, and $v\in T_MM$. In particular, α is fully equivariant modulo isotropy if and only if $d^\natural\beta_g\circ(\mathbb{1}-\mathbb{P}_\alpha)$ maps T_mM into $\mathfrak{g}_{g\cdot m}$ for all $g\in G$ and $m\in M$.

(iii) A $\mathfrak{g}^*$–valued one–form μ is β–equivariant if and only if

$$\Phi_g^*\mu(v) = (\mathrm{Ad}^*_{\beta(g,m)^{-1}}\mu + \chi(g\cdot m)d^\natural\beta_g)(v)$$

for all $g\in G$, $m\in M$, and $v\in T_mM$.

Proof: (i) Given $g\in G$, $m\in M$, and $v\in T_mM$, set $h=\beta_g(m)$ and $\zeta=d^\natural\beta_g(v)$. To prove (i), note that linearizing $\beta_g(m)\cdot m = g\cdot m$ with respect to m yields

$$d\Phi_g(v) = d_h\hat{\Phi}_m(d\beta_g(v)+d\Phi_h(v)).$$

Hence

$$d\hat{\Phi}_m(d\beta_g(v)) = d(\hat{\Phi}_m\circ R_h)\zeta = d\hat{\Phi}_{h\cdot m}\zeta = \zeta_M(g\cdot m).$$

Regrouping terms yields (23).

Combining (23) and the identity $d_e(\Phi_g\circ\widehat{\Phi}_m) = d\widehat{\Phi}_{g\cdot m}\circ\mathrm{Ad}_g$, for all $g\in G$ and $m\in M$, yields

$$d\widehat{\Phi}_{g\cdot m}\circ d^\natural\beta_g\circ d_e\widehat{\Phi}_m = (d\Phi_g - d\Phi_h)\circ d_e\widehat{\Phi}_m = d\widehat{\Phi}_{g\cdot m}\circ(\mathrm{Ad}_g-\mathrm{Ad}_h),$$

and hence $d^\natural\beta_g\circ d_e\widehat{\Phi}_m = (\mathrm{Ad}_g-\mathrm{Ad}_h)$ modulo $\mathfrak{g}_{h\cdot m}$.

(ii) and (iii) If α is β–equivariant modulo isotropy, then

$$\begin{aligned}\Phi_g^*\alpha(v) &= \alpha(d\Phi_g(v))\\ &= \alpha(d\Phi_{\beta(g,m)}(v) + d\widehat{\Phi}_{g\cdot m}d^\natural\beta_g(v))\\ &= \mathrm{Ad}_{\beta(g,m)}\alpha(v) + d^\natural\beta_g(v) \qquad \mathrm{mod} \quad \mathfrak{g}_{g\cdot m}\end{aligned} \tag{25}$$

yields the first equality of (ii). Analogously, if μ is a β–equivariant $\mathfrak{g}^*$–valued form, then

$$\Phi_g^*\mu(v) = \mu(d\Phi_{\beta(g,m)}(v) + d\widehat{\Phi}_{g\cdot m}d^\natural\beta_g(v)) = \mathrm{Ad}^*_{\beta(g,m)}\mu(v) + \chi(m)d^\natural\beta_g(v)).$$

To prove the second equality of (ii), we combine (24) and (25), obtaining

$$\Phi_g^*\alpha(v) = \mathrm{Ad}_g\alpha(v) + d^\natural\beta_g((\mathbb{1}-d\widehat{\Phi}_m\circ\alpha)(v)) \qquad \mathrm{mod} \quad \mathfrak{g}_{g\cdot m}.$$

Since $\mathbb{P}_\alpha|T_mM = d\widehat{\Phi}_m\circ\alpha_m$, this completes the proof. ■

Proof of Theorem 4: Condition (21) can be written as $R_{\phi(m)^{-1}}(\phi(\widehat{\Phi}_m(g))) = g$ modulo G_m. Linearizing with respect to the group element yields

$$d^\natural\phi(\xi_M(m)) = d(R_{\phi(m)^{-1}}\circ\phi\circ\widehat{\Phi}_m)\xi = \xi \qquad \mathrm{mod} \quad \mathfrak{g}_m$$

for all $\xi \in \mathfrak{g}$ and $m \in M$.

We now show that the trivialized linearization $d^\natural\phi$ of a partial moving frame $\phi : M \to G$ satisfies

$$d^\natural\phi(d\Phi_g(v)) = \mathrm{Ad}_{\phi_g(m)} d^\natural\phi(v) + d^\natural\phi_g(v)$$

for all $v \in T_mM$, and hence, applying Lemma 4, is β–equivariant. Given $v \in T_mM$, let $m(\epsilon)$ be a curve through m in M satisfying $\frac{d}{d\epsilon}m(\epsilon)|_{\epsilon=0} = v$. Then

$$\begin{aligned} d^\natural\phi(d\Phi_g(v)) &= d(R_{\phi(g\cdot m)^{-1}} \circ \phi \circ \Phi_g)(v) \\ &= \frac{d}{d\epsilon}\phi(g\cdot m_\epsilon)\phi(g\cdot m)^{-1}|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}\phi_g(m(\epsilon))\phi(m(\epsilon))\phi(m)^{-1}\phi_g(m)^{-1}|_{\epsilon=0} \\ &= \mathrm{Ad}_{\phi_g(m)} d^\natural\phi(v) + d^\natural\phi_g(v). \end{aligned}$$

The relatively equivariant analog of Proposition 1 thus implies that $d^\natural\phi$ determines a β–equivariant projection onto the tangent spaces of the group orbits.

Linearizing the relation $\phi(m)\cdot\pi_\phi(m) = m$ yields

$$d\Phi_{\phi(m)} \circ d_m\pi_\phi + d\widehat{\Phi}_{\pi_\phi(m)} d^\natural\phi = \mathbb{1},$$

and hence

$$d_m\pi_\phi = d\Phi_{\phi(m)^{-1}} \circ (\mathbb{1} - d\widehat{\Phi}_{\pi_\phi(m)} \circ d^\natural\phi) = d\Phi_{\phi(m)^{-1}} \circ \mathbb{P}_\Gamma,$$

where Γ is the partial connection determined by $d^\natural\phi$. Thus range $d_m\pi_\phi = d\Phi_{\phi(m)^{-1}}(\Gamma|_m)$. ■

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Appendix

Proposition 1

(i) A dual connection form μ determines a partial connection $\Gamma = \ker \mu$ and inertia factor $\chi = \mu \circ d_e\widehat{\Phi}$.

(ii) An equivariant (singular) differential system Γ satisfying $TM = \widetilde{\mathfrak{g}} \oplus \Gamma$ is a partial connection if there is inertia factor χ such that the equivariant $\mathfrak{g}^*$–valued one–form μ given by

$$\mu|\Gamma := 0 \qquad \text{and} \qquad \mu \circ d_e\widehat{\Phi}_m := \chi(m) \quad \text{for all } m \in M$$

is smooth, and hence a dual connection form.

(iii) A $\mathfrak{g}$–valued one–form α is a partial connection form if there is an inertia factor χ such that $\mu = \chi\,\alpha$ is a dual connection form with inertia factor χ.

Proof: (i) Equivariance of μ implies that Γ is an equivariant differential system; hence Γ is a partial connection.

(ii) $\ker(\mu \circ T_e\widehat{\Phi}_m) = \ker \chi(m) = \mathfrak{g}_m$ for all $m \in M$ implies that $\mu|\widetilde{\mathfrak{g}}$ is one-to-one, and hence $\ker \mu = \Gamma$. Hence Γ is a partial connection. Any tangent vector $v \in T_mM$ satisfies $v = \xi_M(m) + u$ for some $\xi \in \mathfrak{g}$ and $u \in \Gamma|_m$. Equivariance of Γ implies that $d\Phi_g u \in \Gamma|_{g\cdot m}$, while

$$d_e(\Phi_g \circ \widehat{\Phi}_m) = d\widehat{\Phi}_{g\cdot m} \circ \mathrm{Ad}_g = \chi(g \cdot m) \circ \mathrm{Ad}_g^* = \mathrm{Ad}_{g^{-1}}^* \chi(m)$$

mplies that

$$\Phi_g^*\mu(v) = \mu(d\Phi_g(\xi_M(m) + u)) = \mathrm{Ad}_{g^{-1}}^*\chi(m)\xi = \mathrm{Ad}_{g^{-1}}^*\mu(v)$$

for any $g \in G$. Thus μ is equivariant and, hence, if smooth, is a dual connection form.

(iii) If $\mu = \chi\,\alpha$ is a dual connection form with associated inertia factor χ, then

$$\chi(m) = \mu \circ d_e\widehat{\Phi}_m = \chi(m) \circ \alpha \circ d_e\widehat{\Phi}_m$$

and $\ker \chi(m) = \mathfrak{g}_m$ imply that α satisfies (1). Equivariance of μ and χ imply that

$$\chi\,\mathrm{Ad}_{g^{-1}}\Phi_g^*\alpha = \mathrm{Ad}_g^*\Phi_g^*\mu = \mu = \chi\,\alpha$$

and thus

$$\mathrm{range}\,(\mathrm{Ad}_{g^{-1}}\Phi_g^*\alpha - \alpha)_m \subseteq \ker \chi(m) = \mathfrak{g}_m,$$

i.e. that α is equivariant modulo isotropy. Hence Proposition 1 implies that α is a partial connection form. ■

Lemma 3

(i) Given an involutive G_{m_0}–equivariant differential system $\Delta \supseteq \widetilde{\mathfrak{g}_{m_0}}$ on a G_{m_0}–invariant neighborhood of m_0, there is an G_{m_0}–invariant integral submanifold of Δ containing m_0.

(ii) If G acts properly and S is a G_{m_0}–invariant submanifold of M containing m_0 and satisfying $T_{m_0}M = T_{m_0}S \oplus \widetilde{\mathfrak{g}}|_{m_0}$, there is a neighborhood S_0 of m_0 in S such that S_0 is a slice through m_0.

Proof: (i) Let N be a neighborhood of m_0 in the maximal integral manifold of Δ containing m_0 and let $v \in T_{g \cdot m}(G \cdot N)$ for some $g \in G$ and $m \in N$. There are curves $g(\epsilon) \in G$ and $m(\epsilon) \in N$, with $g(0) = g$, $m(0) = m$, such that

$$v = \frac{d}{d\epsilon} g(\epsilon) \cdot m(\epsilon)|_{\epsilon=0} = \xi_M(g \cdot m) + d_m\Phi_g(w),$$

where $\xi = \frac{d}{d\epsilon} g(\epsilon)|_{\epsilon=0}$ and $w = \frac{d}{d\epsilon} m(\epsilon)|_{\epsilon=0} \in \Delta|_m$. Hence equivariance of Δ implies that

$$v = d_m\Phi_g((\mathrm{Ad}_{g^{-1}}\xi)_M(m) + w) \in d_m\Phi_g(\Delta|_m) = \Delta|_{g \cdot m}.$$

Thus $T_{g \cdot m}(G \cdot N) \subseteq \Delta|_{g \cdot m}$. Maximality of N implies that $G \cdot N = N$.

(ii) To show that S is a slice, it suffices to show that there is some neighborhood S_0 of m_0 in S such that $(g \cdot S_0) \cap S_0 \neq \emptyset$ implies $g \in G_{m_0}$. Our proof largely follows that given in Duistermaat and Kolk. Consider a sequence (g_j, m_j) in $G \times S$ such that $g_j \cdot m_j \in S$ for all j and

$$\lim_{j \to \infty} g_j \cdot m_j = m_0 = \lim_{j \to \infty} m_j.$$

Passing to a convergent subsequence if necessary, let $g = \lim_{j \to \infty} g_j$; $g \cdot m_0 = \lim_{j \to \infty} g_j \cdot m_j = m_0$ implies that $g \in G_{m_0}$. If we set $h_j := g^{-1} g_j \notin G_{m_0}$, so that $\lim_{j \to \infty} h_j = e$, then the G_{m_0}–invariance of S implies that $h_j \cdot m_j \in S$ for all j.

The decompositions

$$\mathfrak{g} = \mathfrak{g}_{m_0} \oplus \mathfrak{h} \qquad \text{and} \qquad T_{m_0}M = \widetilde{\mathfrak{g}}|_{m_0} \oplus \Xi|_{m_0} = \tilde{\mathfrak{h}}|_{m_0} \oplus T_{m_0}S$$

and the Inverse Function Theorem imply that there are G_{m_0}–invariant neighborhoods $\mathcal{V}$ of 0 in $\mathfrak{h}$, $\mathcal{W}$ of e in G_{m_0}, and S_0 of m_0 in S such that the equivariant maps $\Psi_G : \mathcal{V} \times \mathcal{W} \to G$ and $\Psi_M : \mathcal{V} \times S_0 \to M$ given by

$$\Psi_G(\eta, g) := \exp(\eta) g \qquad \text{and} \qquad \Psi_M(\eta, m) := \exp(\eta) \cdot m$$

are diffeomorphisms onto their images. For any $\eta \in \mathcal{V}$, $g \in \mathcal{W}$, and $m \in S_0$, we have

$$\Psi_G(\eta, g) \cdot m = \Psi_M(\eta, g \cdot m).$$

For sufficiently large j, we have $m_j \in S_0$, $h_j \cdot m_j \in S_0$, and $h_j = \Psi_G(\eta_j, k_j)$ for some $k_j \in \mathcal{W}$ and $\eta_j \in \mathcal{V}$. Hence

$$\Psi_M(0, h_j \cdot m_j) = h_j \cdot m_j = \Psi_G(\eta_j, k_j) \cdot m_j = \Psi_M(\eta_j, k_j \cdot m_j).$$

Injectivity of Ψ_M implies that $\eta_j = 0$ and hence $h_j = k_j \in G_{m_0}$. ■

Before proving Proposition 4, we establish a close relationship between a certain class of nondegenerate adaptors and slices. Given an adaptor ϕ, if the maps $\rho_\phi(m) := d_e^\natural(\phi \circ \widehat{\Phi}_m)$ satisfy range $(\mathbb{1} - \rho_\phi(m)) \subseteq \mathfrak{g}_{m_0}$ for all $m \in \phi^{-1}(G_{m_0})$, then ϕ is *transversal.* Here $d^\natural f$ denotes the right trivialization of the linearization of a map $f : M \to G$, i.e. $d_m^\natural f = d_m(R_{f(m)^{-1}} \circ f)$. The following proposition shows that if the action is proper, then slices determine transversal adaptors and vice versa.

Proposition 6 *Assume that G acts properly on M.*

- *If S is a slice through m_0, there is a transversal adaptor ϕ for m_0 such that $\phi^{-1}(G_{m_0})$ is a neighborhood of m_0 in S. Given any $h \in G_{m_0}$ and any G_{m_0}–invariant complement $\mathfrak{h}$ of $\mathfrak{g}_{m_0}$, ϕ can be chosen so that $\phi(m_0) = h$ and* range $\rho_\phi(m_0) = \mathfrak{h}$.
- *If ϕ is a transversal adaptor for m_0, then some neighborhood of m_0 in $\phi^{-1}(G_{m_0})$ is a slice through m_0 and* range $\rho_\phi(m_0)$ *is a G_{m_0}–invariant complement of $\mathfrak{g}_{m_0}$.*

Proof: (i) Transversality of ϕ implies that

$$\text{range}\, d\phi_m + T_{\phi(m)} G_{m_0} \supseteq dR_{\phi(m)}(\text{range}\, \rho_\phi(m) + \mathfrak{g}_{m_0}) = dR_{\phi(m)}\mathfrak{g} = T_{\phi(m)} G$$

for all $m \in S := \phi^{-1}(G_{m_0})$. Hence S is a submanifold of M; G_{m_0}–equivariance of ϕ implies that S is G_{m_0}–invariant. Since ϕ is transversal, the restriction of $\rho_\phi(m_0)$ to $\mathfrak{h} := \text{range}\, \rho_\phi(m_0)$ is an isomorphism, and hence

$$T_{m_0} M = \ker d_{m_0}^\natural \phi \oplus \tilde{\mathfrak{h}}|_{m_0} = T_{m_0} S \oplus \widetilde{\mathfrak{g}}|_{m_0}.$$

Thus Lemma 3 implies that some neighborhood S_0 of m_0 in S is a slice through m_0. If $m \in S$, then $\phi(m) \in G_{m_0}$, and hence $G_{\phi(m)^{-1} \cdot m} \subseteq G_{m_0}$ implies that $G_m \subseteq G_{m_0}$. Finally, G_{m_0}–equivariance of ϕ and $\widehat{\Phi}_{m_0}$ imply that $\mathfrak{h}$ is G_{m_0}–invariant.
(ii) Restricting S_0 if necessary, there are G_{m_0}–invariant neighborhoods $\mathcal{V}$ of 0 in $\mathfrak{h}$, $\mathcal{W}$ of e in G_{m_0}, $\mathcal{G}$ of e in G, and $\mathcal{U}$ of m_0 in M such that the equivariant maps $\Psi_G : \mathcal{V} \times \mathcal{W} \to \mathcal{G}$ and $\Psi_M : \mathcal{V} \times S_0 \to \mathcal{U}$ given by $\Psi_G(\eta, h) := \exp(\eta) h$ and $\Psi_M(\eta, m) := \exp(\eta) \cdot m$ are diffeomorphisms satisfying

$$\Psi_G(\eta, h) \cdot m = \Psi_M(\eta, h \cdot m)$$

for all $\eta \in \mathcal{V}$, $h \in \mathcal{W}$, and $m \in \mathcal{U}$. If we set $\phi(\Psi_M(\eta, m)) := \exp(\eta)$, then $\phi | S_0 \equiv e$ and

$$\phi(h \cdot \Psi_M(\eta, m)) = \phi(\Psi_M(\text{Ad}_h \eta, h \cdot m)) = \exp(\text{Ad}_h \eta) = h\, \exp(\eta)\, h^{-1}$$

for $h \in G_{m_0}$. If $m \in \mathcal{U}$ and $g = \Psi_G(\eta, h) \in \mathcal{G}$, then

$$g^{-1} \cdot \phi(g \cdot m) = (\exp(\eta) h)^{-1} \phi(\Psi_M(\eta, h \cdot m)) = h^{-1} \in G_{m_0}.$$

Hence we can define $\psi_m : \mathcal{G} \to G_{m_0}$ by $\psi_m(g) := g^{-1}\phi(g \cdot m)$, with

$$d_e^{\natural}\psi_m = \mathrm{Ad}_{\phi(m)}(\mathbb{1} - \rho_\phi(m)).$$

Thus $\phi(m) \in G_{m_0}$ implies that range $(\mathbb{1} - \rho_\phi(m)) \subseteq \mathfrak{g}_{m_0}$, and hence ϕ is transversal. ■

Proposition 4 *If G acts properly, then there is an adaptor for any point in M. Given a partial connection Γ and a point $m_0 \in M$, there is an adaptor ϕ for m_0 satisfying $\phi(m_0) = e$ and $d\phi(\Gamma|_{m_0}) \subseteq \mathfrak{g}_{m_0}$.*

Proof: Equip M with a G_{m_0}–invariant metric and let S' denote the image of a G_{m_0}–invariant neighborhood of 0 in $\Gamma|_{m_0}$ under the associated exponential map. S' is G_{m_0}–invariant and satisfies

$$T_{m_0}S' \oplus \widetilde{\mathfrak{g}}|_{m_0} = \Gamma|_{m_0} \oplus \widetilde{\mathfrak{g}}|_{m_0} = T_{m_0}M.$$

Hence Lemma 3.ii implies that some neighborhood S of m_0 in S' is a slice. Proposition 6 now implies that there is a transversal adaptor ϕ for m_0 such that $\phi(m_0) = e$ and ϕ maps some neighborhood of m_0 in S into G_{m_0}; hence

$$d^{\natural_L}\phi(\Gamma|_{m_0}) = d\phi(T_{m_0}S) \subseteq T_{\phi(m_0)}G_{m_0} = \mathfrak{g}_{m_0}. \qquad ■$$

Corollary 2 *If G acts properly on M, Ω equals zero near m_0, and there is an adaptor ϕ satisfying*

$$G_m = \phi(m)G_{m_0}\phi(m)^{-1} \qquad \textit{for all } m \textit{ near } m_0, \tag{26}$$

then Γ is tangent to a local cross section through m_0.

Proof: Assume that Γ arises from a regular foliation. Let S' denote the leaf containing m_0. The decomposition

$$T_{m_0}M = \widetilde{\mathfrak{g}}|_{m_0} \oplus \Gamma|_{m_0} = \tilde{\mathfrak{h}}|_{m_0} \oplus T_{m_0}S'$$

and the Inverse Function Theorem imply that there are neighborhoods $\mathcal{V}$ of 0 in $\mathfrak{h}$ and S'' of m_0 in S' such that the map $\Psi_M : \mathcal{V} \times S'' \to M$ given by

$$\Psi_M(\eta, m) := \exp(\eta) \cdot m$$

is a diffeomorphism onto its image. Hence $G \cdot S'' \supseteq \Psi_M(\mathcal{V} \times S'')$ contains a neighborhood of m_0 in M.

To show that some neighborhood S of m_0 in S'' is a local cross section if there is an adaptor ϕ satisfying (26), it remains to be shown that $m \in S$ and $g \cdot m \in S$ implies $g \in G_m$. The map $F : \mathcal{V} \times G_{m_0} \times S' \to G$ given by

$$F(\eta, h, m) := \exp(\eta)\phi(m)h\phi(m)^{-1}$$

satisfies $F(\eta, h, m) \cdot m = \Psi_M(\eta, m)$ and

$$d_{(0,h,m_0)}F(\xi, dR_h\zeta, 0) = dR_{\phi(m_0)h\phi(m_0)^{-1}}(\xi + \mathrm{Ad}_{\phi(m_0)}\zeta)$$

for any $h \in G_{m_0}$. The decomposition $\mathfrak{g} = \mathfrak{g}_{m_0} \oplus \mathfrak{h}$ and the Implicit Function Theorem imply that there are neighborhoods $\mathcal{G}$ of h_0 in G and S of m_0 in S'' such that the maps $\eta : \mathcal{G} \times S \to \mathcal{V}$ and $h : \mathcal{G} \times S \to G_{m_0}$ satisfy $F(\eta(g, m), h(g, m), m) = g$ for all $g \in \mathcal{G}$ and $m \in S$.

Consider a sequence (g_j, m_j) in $G \times S$ such that $g_j \cdot m_j \in S$ for all j and

$$\lim_{j\to\infty} g_j \cdot m_j = m_0 = \lim_{j\to\infty} m_j.$$

Passing to a convergent subsequence if necessary, let $g = \lim_{j\to\infty} g_j$; $g \cdot m_0 = \lim_{j\to\infty} g_j \cdot m_j = m_0$ implies that $g \in G_{m_0}$. For sufficiently large j, we have $m_j \in S$, $g_j \cdot m_j \in S$, and $g_j \in \mathcal{G}$. Let $\eta_j = \eta(m_j, g_j)$ and $h_j = h(m_j, g_j)$. Then

$$\Psi_M(0, g_j \cdot m_j) = F(\eta_j, h_j, m_j) \cdot m_j = \Psi_M(\eta_j, m_j).$$

Injectivity of Ψ_M implies that $\eta_j = 0$ and $g_j \in G_{m_j}$. ■