CONVERGENCE OF SOLITARY-WAVE SOLUTIONS IN A PERTURBED BI-HAMILTONIAN DYNAMICAL SYSTEM. II. COMPLEX ANALYTIC BEHAVIOR AND CONVERGENCE TO NON-ANALYTIC SOLUTIONS.

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Abstract. In this part, we prove that the solitary wave solutions investigated in part I are extended as analytic functions in the complex plane, except at most countably many branch points and branch lines. We describe in detail how the limiting behavior of the complex singularities allows the creation of non-analytic solutions with corners and/or compact support.

This is the second in a series of two papers investigating the solitary wave solutions of the integrable model wave equation

$$u_t + \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + \frac{3}{\nu} u u_x + u u_{xxx} + 2 u_x u_{xx}. \tag{3.6}$$

(We adopt the notation and numbering of statements from part I.) The ordinary differential equation for travelling wave solutions $u(x,t) = \phi(x - ct)$ is

$$(\alpha + c)\phi' + (\beta + c\nu + \phi)\phi''' + \frac{3}{\nu}\phi\phi' + 2\phi'\phi'' = 0. \tag{3.7}$$

Substituting $\phi = \phi_a + a$, where a is the undisturbed fluid depth for our solitary wave solutions, and integrating the resulting equation twice, leads to the first order equation

$$\nu(\phi_a + \beta + c\nu + a)(\phi_a')^2 = -\phi_a^2(\phi_a + 3a + \nu(\alpha + c)) \tag{3.17}$$

To understand why analytic solitary wave solutions converge to non-analytic functions, such as compactons and peakons, having singularities on the real axis, we shall extend the solitary wave solutions described in Theorems 3.1 and 3.2 in Part I to functions defined in the complex plane to study singularity distribution of these functions. This method

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not only provides another way to prove the last two theorems, but also makes it clear that singularities of solitary wave solutions are approaching the real axis in the process of convergence. Thus, roughly speaking, the singularities of compactons or peakons come from those complex singularities of analytic solitary wave solutions, which are close to the real axis.

The explicit form (3.3) of solitary wave solutions of the KdV equation shows that they are restriction to the real axis of meromorphic functions with countably many poles in the complex plane so that their analytic extension is unique. In contrast to these functions, extensions of solitary wave solutions under our consideration do not have poles but branch points. These branch points play an important role in the formation of singularities of compactons and peakons. The analytic extension of these solitary wave solutions enables us to understand the loss in analyticity of their limiting compacton or peakon solutions.

For any complex number $z \in \mathbb{C}$, the real part and the imaginary part of z are denoted by $\Re z$ and $\Im z$, respectively. The real part $u(x,y)$ and imaginary part $v(x,y)$ of an analytic function $w = F(z) = F(x+iy) = u(x,y) + iv(x,y)$ will be called the velocity potential and stream function respectively. The level sets of the velocity potential, $u(x,y) = u_0$, and the stream function, $v(x,y) = v_0$, are called the equipotentials and streamlines of F, respectively. Finally, $\log w$ is the single-valued branch of the natural logarithmic function $\operatorname{Log} w$, defined as $\log w = \log|w| + i\arg w$ with $-\pi < \arg w \le \pi$.

4. Analytic extensions of solitary wave solutions for $\nu > 0$.

We shall consider the solitary wave solutions in *Case I* , when $\nu > 0$, and *Case II* , when $\nu < 0$, separately because of their different structures as functions defined on the complex plane. We begin with the compacton case where $\nu > 0$. Under the assumption of Theorem 3.1, Equation (3.17) has an orbitally unique and analytic solitary wave solution ϕ_a. Rescaling (3.17) by $\phi_a(x) = -\nu(\alpha + c + \frac{3a}{\nu})\varphi(x)$, we reduce it to the equation

$$(\delta\varphi + \epsilon)(\varphi')^2 = \varphi^2(1-\varphi), \tag{4.1}$$

where $\delta = \nu$ and $\epsilon = -(\beta + c\nu + a)/(\alpha + c + \frac{3a}{\nu})$. The phase plane portrait of (4.1) indicates that its solitary wave solution φ is a positive, even function with unit amplitude and decaying to zero at infinity. Therefore, as $x > 0$, the solution φ satisfies the integral equation

$$x = \int_\varphi^1 \frac{1}{\zeta}\sqrt{\frac{\delta\zeta+\epsilon}{1-\zeta}}\,d\zeta = \int_\varphi^1 \frac{\delta\zeta+\epsilon}{\zeta}\frac{d\zeta}{\sqrt{(\delta\zeta+\epsilon)(1-\zeta)}}.$$

Making the substitution

$$\zeta = \frac{\delta-\epsilon}{2\delta} + \frac{\delta+\epsilon}{2\delta}\sin\theta, \tag{4.2}$$

into the preceding integral yields the equation

$$-x = \sqrt{\epsilon}\,\log \frac{\tan\frac{\theta}{2} + \tan\frac{\theta_0}{2}}{1 + \tan\frac{\theta_0}{2}\tan\frac{\theta}{2}} + \sqrt{\delta}\,\big(\theta - \frac{\pi}{2}\big), \tag{4.3}$$

where θ_0 is a constant satisfying $\sin\theta_0 = \frac{\delta-\epsilon}{\delta+\epsilon}$ and $|\theta_0| < \frac{\pi}{2}$. Equation (4.3) expresses θ implicitly as a function of x, with range $-\theta_0 \le \theta \le \pi + \theta_0$. Another expression satisfied by the function θ is

$$\exp\Big[-\frac{1}{\sqrt{\epsilon}}\big(x + \sqrt{\delta}(\theta - \frac{\pi}{2})\big)\Big] = \frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}}. \tag{4.4}$$

We shall use (4.3) and (4.4) to discuss properties of the function θ in the following lemma. Then the transformation (4.2) will help find extension of the solitary wave solution φ to the complex plane.

Lemma 4.1. *The function θ has an extension $\Theta(z)$ which is a holomorphic function on the strip $\{z \in \mathbb{C};\ |\Im z| < \sqrt{\epsilon}\,\pi\}$ and continuous up to its boundary, such that $\Theta(z)$ maps the line segment $\{z = iy;\ y \in (-\sqrt{\epsilon}\,\pi, \sqrt{\epsilon}\,\pi)\}$ onto the line segment $\{\theta = \frac{\pi}{2} + i\eta;\ \eta \in (-\eta_\epsilon, \eta_\epsilon)\}$ with $\Theta(i\sqrt{\epsilon}\,\pi) = \frac{\pi}{2} - i\eta_\epsilon$, $\Theta(0) = \frac{\pi}{2}$ and $\Theta(-i\sqrt{\epsilon}\,\pi) = \frac{\pi}{2} + i\eta_\epsilon$ for some $\eta_\epsilon > 0$.*

Proof. If we consider the right-hand side of (4.3) as a function of θ, denoted by $-\Sigma(\theta)$, then $\Sigma(\theta)$ maps the interval $(-\theta_0, \pi + \theta_0)$ homeomorphically onto the real axis $\mathbb{R}$ with $\Sigma(-\theta_0) = \infty$, $\Sigma(\frac{\pi}{2}) = 0$ and $\Sigma(\pi + \theta_0) = -\infty$. Substituting $\theta = \frac{\pi}{2} + i\eta$ into the right-hand side of Equation (4.3) to extend the function Σ to the line $\{\theta = \frac{\pi}{2} + i\eta;\ \eta \in \mathbb{R}\}$, it follows that

$$\Sigma\big(\frac{\pi}{2} + i\eta\big) = -i\left(\sqrt{\delta}\,\eta + 2\sqrt{\epsilon}\tan^{-1}\big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta}{2}\big)\right), \tag{4.5}$$

which indicates that $i\Sigma(\frac{\pi}{2} + i\eta)$ is an odd and increasing function of η, mapping the real axis to itself homeomorphically. In consequence, Θ maps $\{z = iy;\ y \in [0, \sqrt{\epsilon}\pi]\}$ to the line segment $\{\theta = \frac{\pi}{2} + i\eta;\ \eta \in [-\eta_\epsilon, 0]\}$ and it maps $\{z = iy;\ y \in [-\sqrt{\epsilon}\pi, 0]\}$ to the line segment $\{\theta = \frac{\pi}{2} + i\eta;\ \eta \in [0, \eta_\epsilon]\}$, where $\Sigma(\frac{\pi}{2} + i\eta_\epsilon) = -i\sqrt{\epsilon}\pi$.

Replacing x with $x + iy$ and substituting $\theta = \xi + i\eta$ into Equation (4.4), one obtains the equation

$$\frac{\sin\theta_0\cosh\eta + \sin\xi + i\cos\theta_0\sinh\eta}{\cosh\eta + \cos(\xi - \theta_0)} = \exp\Big[\frac{-1}{\sqrt{\epsilon}}\big(x + \sqrt{\delta}(\xi - \frac{\pi}{2}) + i(y + \sqrt{\delta}\eta)\big)\Big]. \tag{4.6}$$

Comparing norms and angles on both sides of (4.6) leads to the equations,

$$\frac{\cosh\eta - \cos(\xi + \theta_0)}{\cosh\eta + \cos(\xi - \theta_0)} = \exp\Big[-\frac{2}{\sqrt{\epsilon}}\big(x + \sqrt{\delta}(\xi - \frac{\pi}{2})\big)\Big], \tag{4.7}$$

and

$$\sin\xi = -\sin\theta_0\cosh\eta - \cos\theta_0\sinh\eta\cot\frac{y+\sqrt{\delta}\eta}{\sqrt{\epsilon}}. \tag{4.8}$$

For each $y \in (0, \sqrt{\epsilon}\,\pi)$, the graph of the streamline (4.8) is symmetric with respect to the line $\{\frac{\pi}{2} + i\eta; \eta \in \mathbb{R}\}$ and is the reflection of the streamline for $y = -y_0$ with respect to the real axis. Moreover, Σ is a one-to-one mapping of the streamline connecting the points $(-\theta_0, 0)$ and $(\pi + \theta_0, 0)$ in the lower-half plane onto the line $\{x + iy_0; x \in \mathbb{R}\}$ for each $y_0 \in (0, \sqrt{\epsilon}\pi)$ with $\Sigma(\pi + \theta_0) = -\infty + iy_0$, $\Sigma(\frac{\pi}{2} + i\eta_0) = iy_0$ and $\Sigma(-\theta_0) = \infty + iy_0$. The streamline $\sqrt{\epsilon}\pi = y(\xi, \eta)$ of the function Σ, consisting of the line segments $\{\xi;\ \xi \in [-\frac{\pi}{2}, -\theta_0)\}$ and $\{\xi;\ \xi \in (\pi + \theta_0, \frac{3\pi}{2}]\}$ and the curve in the lower-half plane connecting the points $(-\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$ as shown in Figure 7, is homeomorphic to the line $\{x+i\sqrt{\epsilon}\pi; x \in \mathbb{R}\}$. Whereas for each $x_0 \in \mathbb{R}$, the equipotential $x(\xi, \eta) = x_0$ is symmetric with respect to the ξ-axis, and it is the reflection of the equipotential $x(\xi, \eta) = -x_0$ with respect to the line $\xi = \pi/2$. Furthermore, Σ maps the equipotential $x = x(\xi, \eta)$ homeomorphically to the line $\{x + iy;\ y \in (-\infty, \infty)\}$ for each fixed $x \in (0, \sqrt{\delta}\,\pi)$ with $\Sigma(\frac{\pi}{2} - \frac{x}{\sqrt{\delta}} - i\infty) = x + i\infty$ and $\Sigma(\frac{\pi}{2} - \frac{x}{\sqrt{\delta}} + i\infty) = x - i\infty$. Also, Σ is a one-to-one mapping of the equipotential $x = x(\xi, \eta)$ onto the line segment $\{x + iy;\ y \in (-\sqrt{\epsilon}\,\pi, \sqrt{\epsilon}\,\pi)\}$ for each $x \in [\sqrt{\delta}\,\pi, \infty)$.

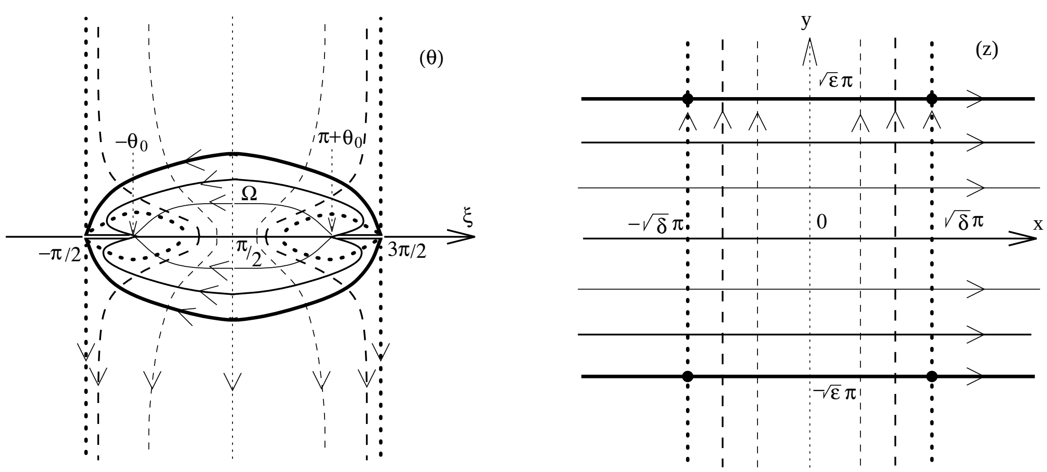

Fig. 7. Streamlines and equipotentials of the function $z = \Sigma(\theta)$

As a result of the above discussion, the function $\Sigma(\theta)$ is seen as a conformal mapping of the domain Ω onto the strip $\{x+iy;\ -\infty < x < \infty, |y| < \sqrt{\epsilon}\,\pi\}$, where Ω is bounded above by the streamline $-\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ consisting of the line segments $\{\xi;\ \xi \in [-\frac{\pi}{2}, -\theta_0)\}$ and $\{\xi;\ \xi \in (\pi+\theta_0, \frac{3\pi}{2})\}$, and the thickest solid curve in the upper-half plane shown in Figure 7, and Ω is bounded below by the streamline $\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ also consisting of the line segments

$\{\xi;\ \xi \in [-\frac{\pi}{2}, -\theta_0)\}$ and $\{\xi;\ \xi \in (\pi+\theta_0, \frac{3\pi}{2})\}$, as well as the thickest solid curve in the lower-half plane illustrated in Figure 7. Therefore, the inverse $\theta(x)$ of the the function $\Sigma(\theta)$ has an analytic extension $\Theta(z)$ to the strip $\{x+iy;\ -\infty < x < \infty, |y| < \sqrt{\epsilon}\,\pi\}$ and continuous up to its boundary ([3], Thm. 14.18). Then the transformation $\varphi = \frac{\delta-\epsilon}{2\delta} + \frac{\delta+\epsilon}{2\delta}\sin\theta$ leads to the conclusion that the solitary wave solution φ of (4.1) has an analytic extension to the same strip. $\square$

An immediate consequence of Lemma 4.1 is the existence of a cuspon as a weak solution of Equation (4.1). When $\eta = 0$, $y = \sqrt{\epsilon}\,\pi$ and $-\frac{\pi}{2} < \xi < -\theta_0$, it follows from (4.6) that the equation

$$\frac{\sin\xi + \sin\theta_0}{1+\cos(\xi-\theta_0)} = -e^{-\frac{1}{\sqrt{\epsilon}}\left(x+\sqrt{\delta}(\xi-\frac{\pi}{2})\right)}. \tag{4.9}$$

implicitly determines the value of the function Σ on the line segment $\{\xi;\ \xi \in (-\frac{\pi}{2}, -\theta_0)\}$ which is mapped to the line $\{x + i\sqrt{\epsilon}\,\pi;\ x \in (\sqrt{\delta}\,\pi, \infty)\}$. Extending the function Σ on the real axis from $\xi = -\frac{\pi}{2}$ to the left up to the point $\xi = -\pi + \theta_0$, one may also realize that Σ maps $\{\xi;\ \xi \in (-\pi+\theta_0, -\theta_0)\}$ homeomorphically to the line $\{x + i\sqrt{\epsilon}\,\pi;\ x \in (-\infty, \infty)\}$. This leads us to the discovery of the cuspon solution of Equation (4.1), as stated in the following corollary.

Corollary 4.2. *Equation (4.1) has a weak solution φ_p in the sense of Definition 3.1. Moreover, φ_p is an even and negative function, continuous on the real axis, monotonically increasing on the positive x-axis and approaching zero at infinity, thereby representing a wave of depression. The derivative φ_p' has a discontinuity at the minimum of φ_p.*

Proof. It follows from (4.9) that the equation

$$x = -\sqrt{\delta}(\xi - \frac{\pi}{2}) - \sqrt{\epsilon}\log\Big(\frac{-(\sin\xi + \sin\theta_0)}{1+\cos(\xi-\theta_0)}\Big)$$

determines a function of ξ, denoted by $x = \Sigma_p(\xi)$ for $-\pi + \theta_0 < \xi < -\theta_0$. Since

$$\frac{dx}{d\xi} = -\frac{\sqrt{\delta}(\sin\xi + 1)}{\sin\xi + \sin\theta_0}, \tag{4.10}$$

$\frac{dx}{d\xi} > 0$ on the interval $(-\pi+\theta_0, -\theta_0)$, and thus Σ_p is an increasing function whose graph is symmetric with respect to the point $(-\frac{\pi}{2}, \sqrt{\delta}\,\pi)$, having an inflection point at $\xi = -\frac{\pi}{2}$ and asymptotes $\xi = -\pi + \theta_0$ and $\xi = -\theta_0$. Therefore, the inverse of Σ_p, denoted by $\xi = \xi(x)$, is also an increasing function, symmetric with respect to the point $(\sqrt{\delta}\,\pi, -\frac{\pi}{2})$ with $\xi'(\sqrt{\delta}\,\pi) = \infty$. Let

$$\varphi_p(x) = \frac{\delta-\epsilon}{2\delta} + \frac{\delta+\epsilon}{2\delta}\sin\big(\xi(x+\sqrt{\delta}\pi)\big). \tag{4.11}$$

Because of the symmetry $-\pi - \xi(x) = \xi(2\sqrt{\delta}\,\pi - x)$, we see that φ_p is an even function having a cusp at $x = 0$ such that $\lim\limits_{x \to 0^-} \varphi_p'(x) = -\infty$ and $\lim\limits_{x \to 0^+} \varphi_p'(x) = \infty$. Moreover, φ_p is continuous. Then the translation invariance of Equation (4.1) shows that φ_p is the cuspon solution satisfying properties stated in this lemma. It is also worth noticing that so called "cuspon solution" in this case is represented by two unbounded orbits illustrated in Figure 2. □

In order to find further extension of $\Theta(z)$, we need to study other properties of the function Σ defined on the complex plane, such as its singularities, streamlines $y = y(\xi, \eta)$ for $|y| > \sqrt{\epsilon}\,\pi$ and zeros of its derivative given by (4.10). We summarize these properties as follows.

(i) *Singularities of* Σ *and zeros of* $\frac{d\Sigma}{d\theta}$. There are two countable sets of distinguished points on the ξ axis: the critical points where $\frac{d\Sigma}{d\theta} = 0$, and so Σ is not angle-preserving occur at $\mathcal{O} = \{-\frac{\pi}{2} + 2n\pi;\ n = 0, \pm 1, \pm 2, \cdots\}$. The singularities of Σ are at $\mathcal{S} = \{-\theta_0 + 2n\pi, \theta_0 + (2n+1)\pi;\ n = 0, \pm 1, \pm 2, \cdots\}$. Let Γ_+ (respectively Γ_-) be a closed Jordan curve containing only the singular point $\xi = \theta_0 + (2n+1)\pi$ (respectively $\xi = -\theta_0 + 2n\pi$), but no other points in $\mathcal{S}$. Integrating $\frac{d\Sigma}{d\theta}$ once around $\Gamma_\pm$ in the counterclockwise direction yields

$$\int_{\Gamma_\pm} -\frac{\sqrt{\delta}(\sin\xi + 1)}{\sin\xi + \sin\theta_0} d\xi = \pm i 2\pi\sqrt{\epsilon}.$$

Therefore, singularities of Σ are branch points of infinite order.

(ii) *A single-valued branch* Σ_0 *of* Σ. Notice that if we integrate $\frac{d\Sigma}{d\xi}$ on a closed path Γ_2 whose interior contains only two adjacent singularities $(2n-1)\pi + \theta_0$ and $2n\pi - \theta_0$ of Σ for some integer n, then

$$\int_{\Gamma_2} -\frac{\sqrt{\delta}(\sin\xi + 1)}{\sin\xi + \sin\theta_0} d\xi = -i 2\pi\sqrt{\epsilon} + i 2\pi\sqrt{\epsilon} = 0,$$

i.e. the value of the function Σ does not change after a complete circuit around the two singular points $(2n-1)\pi + \theta_0$ and $2n\pi - \theta_0$. Therefore, to find a single-valued branch Σ_0 of the multi-valued function Σ, one may define countably many branch lines on the ξ-axis by $\{\xi;\ (2n-1)\pi + \theta_0 \le \xi \le 2n\pi - \theta_0\}$ for $n = 0, \pm 1, \pm 2, \cdots$. Let Σ_0 be defined to take the same value as the function Σ on the domain Ω bounded by the streamlines $-\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ and $\sqrt{\epsilon}\,\pi = y(\xi, \eta)$, and contained in the strip $\{\xi + i\eta;\ -\frac{\pi}{2} < \xi < \frac{3\pi}{2}, \eta \in \mathbb{R}\}$ as illustrated in Figure 7. Then using the property that $x + iy = \Sigma(\xi + i\eta)$ if and only if $x - 2n\pi\sqrt{\delta} + iy = \Sigma(\xi + i\eta + 2n\pi)$, one can extend the function Σ_0 to the domain

$\Omega_n = \{\xi + i\eta + 2n\pi;\ \xi + i\eta \in \Omega\}$ for $n = \pm 1, \pm 2, \cdots$. To get a complete portrait of the function Σ_0, let us consider its streamlines $y_0 = y(\xi, \eta)$ for $|y_0| > \sqrt{\epsilon}\,\pi$. Using an argument similar to that used to consider the streamline $\sqrt{\epsilon}\,\pi = y(\xi, \eta)$, one may show that the streamline is decreasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and increasing on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$ with $\eta'(-\frac{\pi}{2}) = \eta'(\frac{\pi}{2}) = \eta'(\frac{3\pi}{2}) = 0$, where η is a function of ξ determined by the streamline (4.8) and $\eta(\frac{\pi}{2})$ is determined by the equation $y_0 = -i\Sigma_0\big(\frac{\pi}{2} + i\eta\big)$ as shown in (4.5). Then the property of the stream function $y(\xi, \eta) = y(\xi + 2\pi, \eta)$ shows that for each $y_0 \in (-\infty, -\sqrt{\epsilon}\,\pi) \cup (\sqrt{\epsilon}\,\pi, \infty)$, the streamline $y_0 = y(\xi, \eta)$ determines a function $\eta = \eta(\xi)$ defined on the real axis such that $\eta(\xi)$ is a continuous and periodic function with the period 2π. Using the Cauchy-Riemann equations for x and y leads to the conclusion that Σ_0 is a one-to-one mapping of the streamline $y_0 = y(\xi, \eta)$ onto the line $y = y_0$.

Remark. For any integer n, we find that Σ maps $\overline{\Omega_n}$ onto the strip $\{|\Im z| < \sqrt{\epsilon}\,\pi\}$,. Therefore Σ can be defined as a conformal mapping from a Riemann surface obtained by excluding Ω_n, $n = \pm 1, \pm 2, \cdots$, onto a domain containing the strip $\{|\Im z| < \sqrt{\epsilon}\,\pi\}$, whose inverse is the extension of the function $\Theta(z)$ defined in Lemma 4.1. It is also important to realize that $\Theta(z)$ has four singularities on the boundary of the strip $\{|\Im z| < \sqrt{\epsilon}\,\pi\}$, even though Θ can be extended up to the boundary continuously. We demonstrate this fact in the following lemma.

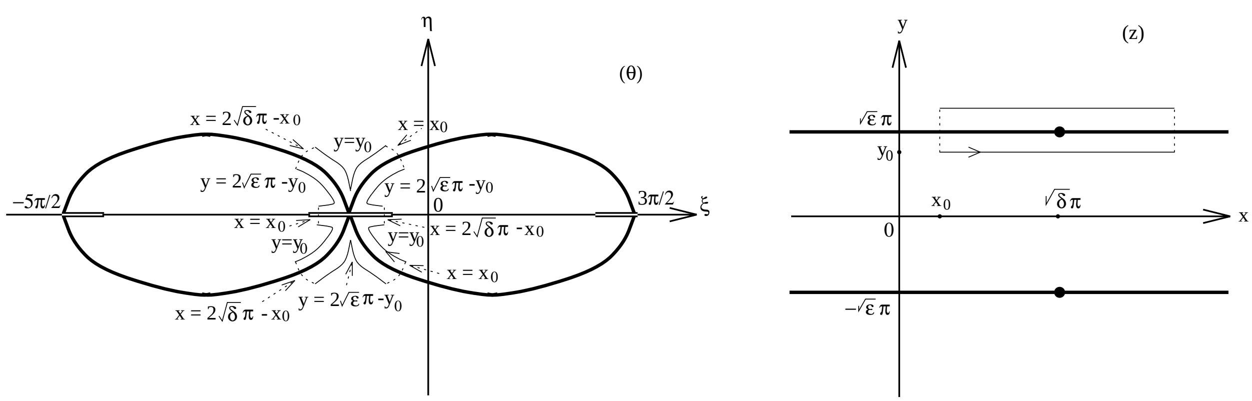

Fig. 8. A sketch of the path γ in the θ-plane and the corresponding path $\Sigma(\gamma)$ in the z-plane

Lemma 4.3. *$\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ and $-\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ are singularities of the function $\Theta(z)$ and each of them is a branch point of order three.*

Proof. We choose a closed path γ consisting of segments of streamlines and equipotentials of Σ in the θ-plane, as illustrated in Figure 8, which starts from the point $\theta = \xi_0 + i\eta_0$ on the streamline $y_0 = y(\xi, \eta)$ such that $-\frac{\pi}{2} < \xi_0 < 0$, $\eta_0 < 0$, $x_0 + iy_0 = \Sigma_0(\xi_0 + i\eta_0)$ with $0 < x_0 < \sqrt{\delta}\,\pi$ and $0 < y_0 < \sqrt{\epsilon}\,\pi$, and both $\sqrt{\delta}\,\pi - x_0$ and $\sqrt{\epsilon}\,\pi - y_0$ are sufficiently small. If one goes along the path γ counterclockwise for a complete circuit, then the corresponding trace described by the function Σ in the z-plane finishes three complete circuits of the rectangle $\Sigma(\gamma)$. Since the path γ is contained in the strip $\{\xi + i\eta;\ -\pi < \xi < 0, \eta \in \mathbb{R}\}$, and sine function $\sin\theta$ has the period 2π, $\sqrt{\delta}\,\pi + i\sqrt{\epsilon}\,\pi$ is also a branch point of order three of the function $\varphi = \frac{\delta+\epsilon}{2\delta}(\sin\Theta + \sin\theta_0)$. In the similar way, one may show that the other three points $\sqrt{\delta}\,\pi - i\sqrt{\epsilon}\,\pi$ and $-\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ are also branch points of order three of both functions Θ and φ. $\square$

One may also question whether Θ has other singularities on the lines $\{x \pm i\sqrt{\epsilon}\,\pi;\ x \in \mathbb{R}\}$. As a matter of fact, it follows from the derivative of Σ shown in (4.10) and the streamlines $-\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ and $\sqrt{\epsilon}\,\pi = y(\xi, \eta)$ portrayed in Figure 7 that $\Theta(z)$ has a local analytic extension from the interior of the strip $\{z \in \mathbb{C};\ |\Im z| < \sqrt{\epsilon}\,\pi\}$ to each point on the lines $\{x + i\sqrt{\epsilon}\,\pi;\ x \in \mathbb{R}\}$ and $\{x - i\sqrt{\epsilon}\,\pi;\ x \in \mathbb{R}\}$ except the four branch points $\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ and $-\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$. Therefore, to extend $\Theta(z)$ beyond these two lines, one needs to define branch lines connecting these branch points. Different definitions of branch lines lead to different extensions of Θ to the complex plane. In the following theorems, we discuss two distinct extensions of $\Theta(z)$ beyond the strip $\{z \in \mathbb{C};\ |\Im z| \le \sqrt{\epsilon}\,\pi\}$.

Theorem 4.4. *Let Σ_0 be the single-valued branch of Σ defined in (ii), and let*

$$D_0 = \bigcup_{n \neq 0} \overline{\Omega}_n \cup \{\xi;\ -\infty < \xi \le -\theta_0\} \cup \{\xi;\ \pi + \theta_0 \le \xi < \infty\},$$

where $\overline{\Omega}_n$ is the same as defined in (ii). Then the restriction of Σ_0 to the domain $X_0 = \mathbb{C} \setminus D_0$ is a conformal mapping of X_0 onto the domain

$$Y_0 = \mathbb{C} \setminus \left(\{x + i\sqrt{\epsilon}\,\pi;\ |x| \ge \sqrt{\delta}\,\pi\} \cup \{x - i\sqrt{\epsilon}\,\pi;\ |x| \ge \sqrt{\delta}\,\pi\}\right),$$

and the inverse Θ_0 of the function Σ_0 is an analytic extension of $\Theta(z)$ to the manifold Y_0 such that $\Theta_0(z)$ has countably many branch points $(2k+1)\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ for integers $k = 0, \pm 1, \pm 2 \cdots$ and when $k \neq -1, 0$, the singularities $(2k+1)\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi$ are located at the upper side of the branch line $\{x + i\sqrt{\epsilon}\,\pi;\ |x| \ge \sqrt{\delta}\,\pi\}$ and the lower side of the branch line $\{x - i\sqrt{\epsilon}\,\pi;\ |x| \ge \sqrt{\delta}\,\pi\}$, satisfying $\Theta_0\big((2k+1)\sqrt{\delta}\,\pi \pm i\sqrt{\epsilon}\,\pi\big) = -(2k+1)\pi + \frac{\pi}{2}$.

Proof. It follows from Lemma 4.1 an the discussion in (*ii*) that $\Sigma_0|_{X_0}$ is a one-to-one mapping of X_0 onto Y_0. Therefore, the inverse of $\Sigma_0|_{X_0}$, denoted by $\Theta_0(z)$, is an analytic function on Y_0, and its restriction to the strip $\{x + iy;\ x \in \mathbb{R}, |y| < \sqrt{\epsilon}\,\pi\}$ is $\Theta(z)$. $\square$

Theorem 4.5. *Let* $Y_1 = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{x + i\sqrt{\epsilon}\,(2n+1)\pi;\ |x| \le \sqrt{\delta}\,\pi\}$*, and let* X_1 *be the Riemann surface formed by infinitely many layers of the domain* Ω *defined in Lemma 4.1, having branch lines* $\{\xi;\ -\frac{\pi}{2} \le \xi \le -\theta_0\}$ *and* $\{\xi;\ \pi+\theta_0 \le \xi \le \frac{3\pi}{2}\}$*, and being pasted in such a way that on any layer of the Riemann surface, if one goes across any of the two branch lines from the lower half plane, one gets to the next lower layer of the Riemann surface; whereas if one goes across any of the branch lines from the upper-half plane, one arrives at the adjacent upper layer of the Riemann surface. Then there exists a conformal mapping of* X_1 *onto the Riemann surface* Y_1 *such that its inverse* Θ_1 *is an analytic extension of* Θ *from the strip* $\{x + iy;\ x \in \mathbb{R}, |y| < \sqrt{\epsilon}\,\pi\}$ *to* Y_1*, and* Θ_1 *is continuous up to the boundary of* Y_1*. Moreover,* Θ_1 *has infinitely many branch points* $\pm\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi$ *for* $n = 0, \pm 1, \pm 2, \cdots$*, such that each pair of branch points* $\pm\sqrt{\delta}\,\pi + i(2n+1)\sqrt{\epsilon}\,\pi$ *are connected by the branch line* $b_n = \{x + i(2n+1)\sqrt{\epsilon}\,\pi;\ |x| \le \sqrt{\delta}\,\pi\}$ *which is regarded as a line segment having an upper side and a lower side.*

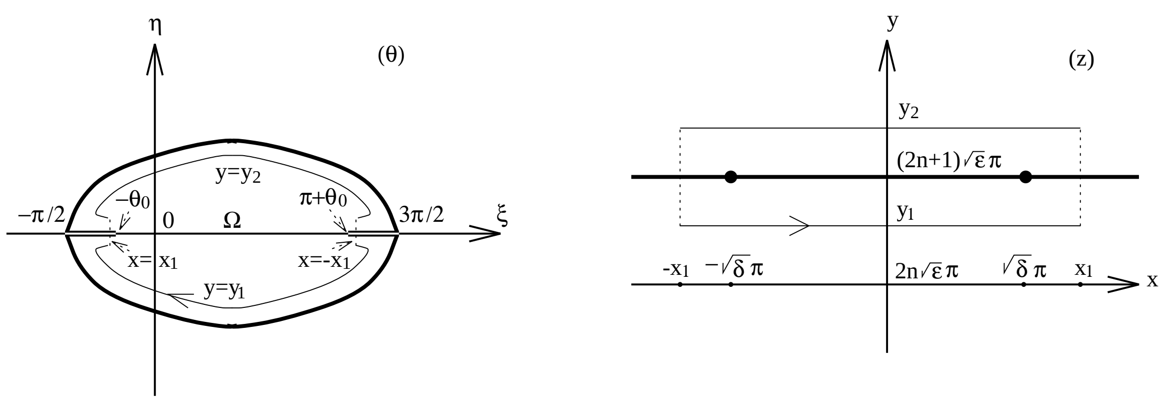

Fig. 9. A sketch of the path γ_1 in the θ-plane and the corresponding path $\Sigma(\gamma_1)$ in the z-plane

Proof. It follows from Lemma 4.1 and (ii) that Θ is a homeomorphism of the strip $\{x + iy;\ x \in \mathbb{R}, |y| < \sqrt{\epsilon}\,\pi\}$ onto Ω, whose inverse is a branch Σ_0 of the function Σ. There are also countably many other branches of Σ, denoted by Σ_n for $n = \pm 1, \pm 2, \cdots$, such that $\Sigma_n(\theta) = \Sigma_0(\theta) + i2\sqrt{\epsilon}\,n\pi$ for any $\theta \in \Omega$ and Σ_n is a homeomorphism of $\Omega^n = \Omega$ onto the strip $S_n = \{z \in \mathbb{C};\ (2n-1)\sqrt{\epsilon}\,\pi < \Im z < (2n+1)\sqrt{\epsilon}\,\pi\}$. If one chooses a path, denoted by γ_1 as shown in Figure 9, and extends the value of Σ_n by starting from a point on γ_1 in the lower half plane, going clockwise along γ_1 and coming back to the same point after a complete circuit, then the corresponding values of Σ_n form a complete circuit of a rectangle in the z-plane. Therefore, if one defines the branch line $\{x + i(2n+1)\sqrt{\epsilon}\,\pi, |x| \le \sqrt{\delta}\,\pi\}$, then one may extend the inverse Σ_n^{-1} of Σ_n from the strip S_n to to the strip S_{n+1} uniquely and analytically. Hence, we may use the domain Ω^n of the branch Σ_n to construct the Riemann surface X_1 for Σ and obtain a conformal mapping of X_1 onto Y_1 (ref. [1]), whose inverse Θ_1 is an analytic extension of Θ to the domain Y_1. $\square$

Remark. It follows from the definition of the function Θ_1 in the last theorem that Θ_1 can also be regarded as a periodic function with the period $T = i2\sqrt{\epsilon}\pi$, mapping each strip $\overline{S_n}$ homeomorphically onto the domain $\overline{\Omega}$. In Section 6, we will use this property to discuss how solitary wave solutions extended as analytic functions defined on a Riemann surface converge to either a compacton or a solitary wave solution of the KdV equation as ϵ or δ approaches zero.

5. Analytic extensions of solitary wave solutions for $\nu < 0$.

As we have seen in Section 3, the dynamical structure of Equation (1.1) changes noticeably when the sign of the parameter ν changes from positive to negative. In this section, we shall show that solitary wave solutions in case $\nu < 0$ have a different singularity distribution from the solitary wave solutions in case $\nu > 0$.

Suppose that the coefficients of Equation (3.17) satisfy the conditions

$$-(\beta + c\nu) < -\frac{\nu}{3}(\alpha + c) < a < \frac{1}{2}(\beta - \nu\alpha) \quad \text{and} \quad \nu < 0.$$

Then (3.17) has an orbitally unique and analytic solitary wave solution ϕ_a defined on the real axis as demonstrated in Section 3. Rescaling (3.17) by the transformation $\phi_a = [3a + \nu(\alpha + c)]\varphi$ reduces it to the equation

$$(\delta\varphi + \rho)(\varphi')^2 = \varphi^2(\varphi + 1), \tag{5.1}$$

where $\delta = -\nu > 0$, $\rho = \frac{-\nu(\beta + c\nu + a)}{3a + \nu(\alpha + c)} > \delta$. Then the corresponding solitary wave solution of (5.1) is symmetric with respect to its depression, having the amplitude $A = 1$ with $\varphi < 0$ and monotonically approaching zero at infinity. Since solutions to (5.1) are translation invariant, without loss of generality, we let the solitary wave solution φ be an even function, which satisfies the integral equation $-x = \int_{-1}^{\varphi} \frac{\sqrt{\delta\varphi + \rho}\, d\varphi}{\varphi\sqrt{\varphi + 1}}$ for $x \geq 0$. Substituting

$$\varphi = \frac{\rho - \delta}{2\delta}\cosh t - \frac{\rho + \delta}{2\delta} \tag{5.2}$$

into the above integral yields the following expressions

$$-x = \int_0^t \frac{\sqrt{\delta}(\cosh t + 1)}{\cosh t - \cosh t_0}\, dt = \sqrt{\delta}\, t + \sqrt{\rho} \log \frac{\tanh \frac{t_0}{2} - \tanh \frac{t}{2}}{\tanh \frac{t_0}{2} + \tanh \frac{t}{2}}, \tag{5.3}$$

where $\cosh t_0 = \frac{\rho+\delta}{\rho-\delta}$, $t_0 > 0$. Let

$$\Delta(t) = -\left(\sqrt{\delta}\, t + \sqrt{\rho} \log \frac{\tanh \frac{t_0}{2} - \tanh \frac{t}{2}}{\tanh \frac{t_0}{2} + \tanh \frac{t}{2}}\right). \tag{5.4}$$

We shall extend the function Δ to the complex plane, which in turn will provide information to find an analytic extension of the inverse $t = \Xi(x)$ of Δ, satisfying Equation (5.3).

Since

$$\frac{\tanh \frac{t_0}{2} - \tanh \frac{t}{2}}{\tanh \frac{t_0}{2} + \tanh \frac{t}{2}} = e^{-\frac{x+\sqrt{\delta} t}{\sqrt{\rho}}},$$

replacing t and x with $\xi + i\eta$ and $x + iy$ in the above equation, respectively, and rewriting its left-hand side as a sum of its real part and imaginary part, one obtains

$$\frac{\cosh t_0 \cos\eta - \cosh\xi - i \sinh t_0 \sin\eta}{\cosh(t_0+\xi) - \cos\eta} = e^{-\frac{x+\sqrt{\delta}\xi + i(y+\sqrt{\delta}\eta)}{\sqrt{\rho}}}.$$

Comparing angles and norms on both sides of this equation yields the two relations

$$\frac{\sinh t_0 \sin\eta}{\cosh t_0 \cos\eta - \cosh\xi} = \tan\frac{y+\sqrt{\delta}\,\eta}{\sqrt{\rho}}, \quad \text{and} \quad \frac{\cosh(t_0-\xi) - \cos\eta}{\cosh(t_0+\xi) - \cos\eta} = e^{-\frac{2(x+\sqrt{\delta}\,\xi)}{\sqrt{\rho}}},$$

which provide equations of streamlines and equipotentials of the function Δ, respectively. Similar to the technique used to study the function Σ, starting from the imaginary axis $t = i\eta$ to extend $\Delta(t)$, one has the expression

$$y = -i\Delta(i\eta) = -\sqrt{\delta}\,\eta + \sqrt{\rho}\, 2n\pi + 2\sqrt{\rho} \tan^{-1}\Big(\sqrt{\frac{\rho}{\delta}} \tan\frac{\eta}{2}\Big), \tag{5.5}$$

for any $\eta \in \big((2n-1)\pi, (2n+1)\pi\big]$ with $n = 0, \pm1, \pm2, \cdots$, which leads to the determination of streamlines $y_0 = y(\xi, \eta)$ of the function Δ. As illustrated in Figure 10, when $y_0 \in (-(\sqrt{\rho} - \sqrt{\delta}\,)\pi, (\sqrt{\rho} - \sqrt{\delta}\,)\pi)$, the streamline $y_0 = y(\xi, \eta)$ is a smooth curve connecting the points $-t_0$ and t_0 on the ξ-axis, and Δ maps the streamline diffeomorphically to the line $\{x + iy_0,\ x \in \mathbb{R}\}$, on which $\Delta(-t_0) = -\infty + iy_0$, $\Delta(i\eta_0) = iy_0$, and $\Delta(t_0) = \infty + iy_0$, where y_0 and η_0 satisfy (5.5) with $n = 0$. If $y_0 = (\sqrt{\rho} - \sqrt{\delta}\,)\pi$, the streamline is a curve in the upper-half plane connecting points $t = -t_0, \pi i$ and t_0, having a corner at the point $t = \pi i$, and Δ is a homeomorphism of this streamline to the line $\{x + i(\sqrt{\rho} - \sqrt{\delta}\,)\pi;\ x \in \mathbb{R}\}$. On the other hand, for $y_0 = -(\sqrt{\rho} - \sqrt{\delta}\,)\pi$, the streamline is a curve symmetric to the streamline for $y_0 = (\sqrt{\rho} - \sqrt{\delta}\,)\pi$ with respect to the ξ-axis. For each $x_0 > 0$, the equipotential $x_0 = x(\xi, \eta)$ is a closed loop in the right-half plane which starts from a point to the right of the point t_0 on the ξ-axis, going around t_0 clockwise once for a complete circuit. In this way, Δ is a one-to-one map of the equipotential onto the line segment $\{x_0+iy;\ |y|<\sqrt{\rho}\pi\}$; while the equipotential $-x_0=x(\xi,\eta)$ is symmetric to the equipotential $x_0=x(\xi,\eta)$ with respect to the η-axis. Since the proof of these facts and the following lemma concerning analytic extensions of Δ and its inverse function is similar to that of Lemma 4.1, we omit it to avoid tedious details here.

Lemma 5.1. *Let Δ be the function defined in (5.4). Then as a homeomorphism of the line segment $\{\xi;\ -t_0<\xi<t_0\}$ onto the real axis, Δ has an analytic extension to the region D bounded by the streamline $(\sqrt{\rho}-\sqrt{\delta}\,)\pi=y(\xi,\eta)$, connecting points $t=-t_0,\pi i$ and t_0, and the streamline $-(\sqrt{\rho}-\sqrt{\delta}\,)\pi=y(\xi,\eta)$, connecting points $t=-t_0,-\pi i$ and t_0. Moreover, the extension of Δ is a one-to-one mapping of D onto the strip*

$$\mathcal{S}_0=\{x+iy;\ x\in\mathbb{R},|y|<(\sqrt{\rho}-\sqrt{\delta}\,)\pi\},$$

whose inverse $t=\Xi(z)$ is an analytic function defined on the strip $\mathcal{S}_0$ and continuous up to its boundary such that $t=\Xi(z)$ satisfies Equation (5.3). In consequence, the transformation (5.2) leads to the analytic extension of the solitary wave solution φ of (5.1) to the strip $\mathcal{S}_0$ as well.

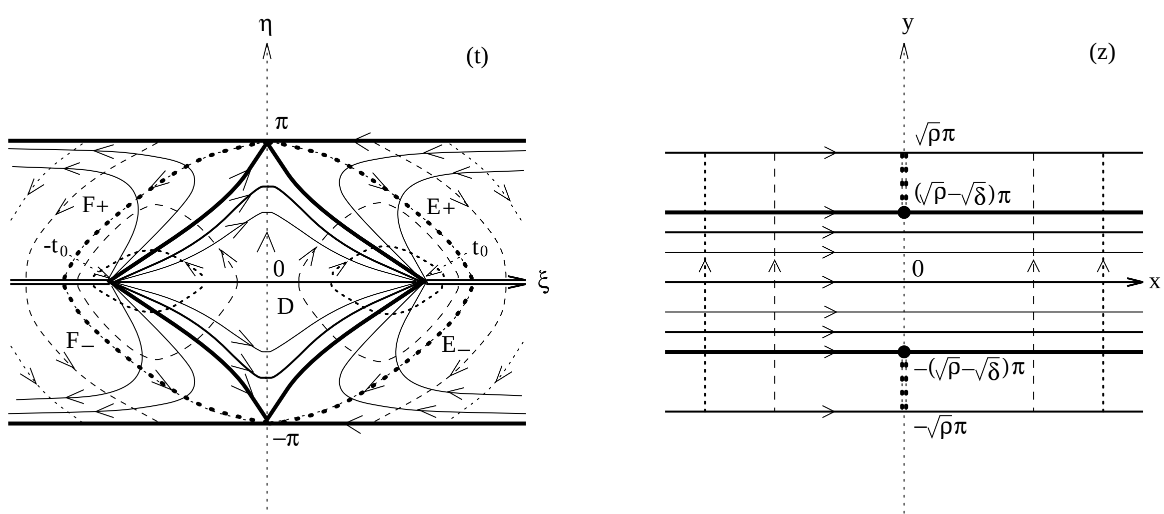

Fig. 10. Streamlines and equipotentials of the function Δ when $\delta<\rho\leq 3\delta$

In analogy with the discussion of solitary wave solutions in Section 4, we now summarize properties of the function Δ, including its singularities, critical points, and single-valued branches. This will lead us to find singularities of the function Ξ on the boundary of the trip $\mathcal{S}_0$ and its further extension to the complex plane.

(a) *Singularities of* Δ *and zeros of* $\frac{d\Delta}{dt}$. Clearly Δ has singularities $t = \pm t_0 + 2n\pi i$ for $n = 0, \pm 1, \pm 2, \cdots$. Let L_+ be a Jordan curve whose interior contains only one singularity $t = t_0 + 2n\pi i$ of Δ for some integer n, and let L_- be a closed path whose interior contains only the singularity $t = -t_0 + 2n\pi i$ of Δ. Integrating $\frac{d\Delta}{dt}$ along these curves yields

$$\int_{L_\pm} \frac{d\Delta}{dt} dt = \int_{L_\pm} \frac{\sqrt{\delta}(\cosh t + 1)}{\cosh t_0 - \cosh t} dt = \mp 2\pi \sqrt{\rho}\, i.$$

Therefore, $t = \pm t_0 + 2n\pi i$ are branch points of infinite order of Δ, and thus Δ is a multi-valued function. On the other hand, zeros of $\frac{d\Delta}{dt}$ are $t = (2k+1)\pi i$ for $k = 0, \pm 1, \pm 2, \cdots$, and Δ is not angle-preserving at these points. As illustrated in Figure 10, the streamlines $(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ and $-(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ joining the points $t = \pm t_0$ and forming the boundary of the region D have corners at $t = \pi i$ and $t = -\pi i$, respectively. As a matter of fact, Δ is not one-to-one in some neighbourhood of these points either, because $\eta = \pi i$ is also the streamline $(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ such that $\Delta(-\infty + \pi i) = \infty + (\sqrt{\rho} - \sqrt{\delta})\pi i$, $\Delta(\pi i) = (\sqrt{\rho} - \sqrt{\delta})\pi i$ and $\Delta(\infty + \pi i) = -\infty + (\sqrt{\rho} - \sqrt{\delta})\pi i$; while $\eta = -\pi i$ is another streamline $-(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ such that $\Delta(-\infty - \pi i) = \infty - (\sqrt{\rho} - \sqrt{\delta})\pi i$, $\Delta(-\pi i) = -(\sqrt{\rho} - \sqrt{\delta})\pi i$ and $\Delta(\infty - \pi i) = -\infty - (\sqrt{\rho} - \sqrt{\delta})\pi i$, which indicates that the open set we shall seek and use to construct the Riemann surface on which Δ is a conformal mapping has to be contained in the strip $\{\xi + \eta;\ \xi \in \mathbb{R}, |\eta| < \pi\}$.

(b) *A single-valued branch* Δ_0 *of* Δ. Since Δ has branch points $\pm t_0 + 2n\pi i$, first of all, we define branch lines by $l_n = \{\xi + 2n\pi i;\ |\xi| \geq t_0\}$ for $n = 0, \pm 1, \pm 2, \cdots$, so that the integration of $\frac{d\Delta}{dt}$ along any closed path contained in the open set $L = \mathbb{C} \backslash \bigcup_{-\infty}^{\infty} l_n$ is zero, *i.e.* values of the function Δ_0 can be uniquely determined on the domain L. Because $D \subset L$, the function Δ_0 is shown to be a homeomorphism of D onto the strip $\mathcal{S}_0$ in Lemma 5.1, whose streamlines and equipotentials are sketched in Figure 10. It is worth mentioning that in Figure 10 the curve E_- connecting the point $t = -\pi i$ and a point to the right of $t = t_0$ is part of the equipotential $0 = x(\xi, \eta)$ such that $\Delta_0(E_-) = \{iy;\ -\sqrt{\rho}\,\pi < y < -(\sqrt{\rho} - \sqrt{\delta}\,)\pi\}$, and the curve E_+ symmetric to E_- with respect to the ξ-axis is also a portion of the equipotential $0 = x(\xi, \eta)$ such that Δ_0 is a one-to-one mapping of E_+ onto the line segment $\{iy;\ (\sqrt{\rho} - \sqrt{\delta}\,)\pi < y < \sqrt{\rho}\,\pi\}$. On the other hand, the curves F_- and F_+ symmetric to E_- and E_+ with respect to the imaginary axis have the same image as E_- and E_+, respectively, *i.e.* both $E_- \cup \{i\eta;\ |\eta| \leq \pi\} \cup E_+$ and $F_- \cup \{i\eta;\ |\eta| \leq \pi\} \cup F_+$ are the equipotential $0 = x(\xi, \eta)$ such that Δ_0 maps each of them homeomorphically onto the line segment $\{iy;\ |y| \leq \sqrt{\rho}\,\pi\}$. In addition, solid lines passing through the curves E_+ and F_+ in Figure 10 are streamlines $y_0 = y(\xi, \eta)$ for some $y_0 \in ((\sqrt{\rho} - \sqrt{\delta}\,)\pi, \sqrt{\rho}\,\pi)$, and solid lines passing through the curves E_- and F_- are streamlines $y_0 = y(\xi, \eta)$ for some $y_0 \in (-\sqrt{\rho}\,\pi, -(\sqrt{\rho} - \sqrt{\delta}\,)\pi)$. For each of them, Δ_0 is a homeomorphism onto the line $\{x + iy_0;\, x \in \mathbb{R}\}$. Dotted and dashed lines to the right of curves E_- and E_+ are equipotentials $x_0 = x(\xi, \eta)$ for some $x_0 < 0$, which are mapped homeomorphically onto the lines $\{x_0 + iy;\, -\sqrt{\rho}\,\pi < y < -(\sqrt{\rho} - \sqrt{\delta}\,)\pi\}$ and $\{x_0 + iy;\, (\sqrt{\rho} - \sqrt{\delta}\,)\pi < y < \sqrt{\rho}\,\pi\}$, respectively. Symmetrically, dotted and dashed lines to the left of curves F_- and F_+ are equipotentials $x_0 = x(\xi, \eta)$ for some $x_0 > 0$, whose images are also line segments $\{x_0 + iy;\, -\sqrt{\rho}\,\pi < y < -(\sqrt{\rho} - \sqrt{\delta}\,)\pi\}$ and $\{x_0 + iy;\, (\sqrt{\rho} - \sqrt{\delta}\,)\pi < y < \sqrt{\rho}\,\pi\}$, respectively.

Before finding a further extension of the function Ξ beyond the strip $\mathcal{S}_0$, we need to point out in the following lemma that Ξ has singularities on the boundary of $\mathcal{S}_0$, which has an effect on how to determine a further extension for Ξ.

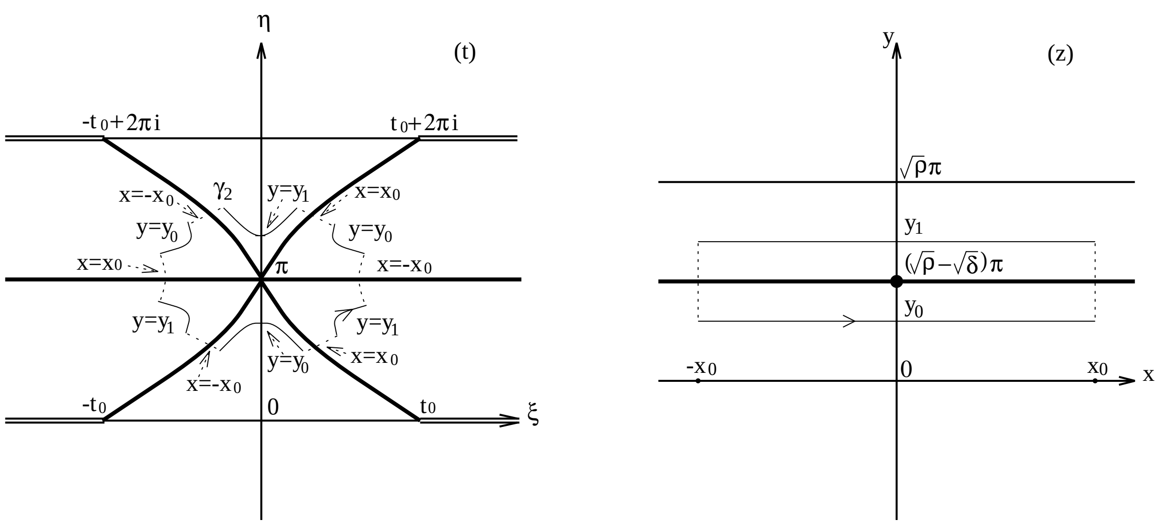

Fig. 11. The path γ_2 and the corresponding rectangle in the z-plane

Lemma 5.2. *The function Ξ has singularities at $z = \pm i(\sqrt{\rho} - \sqrt{\delta}\,)\pi$ which are branch points of order three.*

Proof. Since $\Xi\big((\sqrt{\rho} - \sqrt{\delta}\,)\pi i\big) = \pi i$ and $\Xi|_{\mathcal{S}_0} = (\Delta_0|_D)^{-1}$, to show that $z = (\sqrt{\rho} - \sqrt{\delta}\,)\pi i$ is a branch point of Ξ, we choose a closed path γ_2 consisting of segments of streamlines and equipotentials of the single-valued branch Δ_0 such that its interior contains $t = \pi i$ as sketched in Figure 11. As t moves around the circuit γ_2 once in a counterclockwise direction, its image $\Delta_0(t)$ traces three complete circuits of a rectangle in the z-plane with its interior containing the point $z = (\sqrt{\rho} - \sqrt{\delta}\,)\pi i$. It follows that the point $z = (\sqrt{\rho} - \sqrt{\delta}\,)\pi i$ is a branch point of order three of the function Ξ. Noticing that $\cosh t$ is a periodic function with the period $T = 2\pi i$, the solitary wave solution to (5.1) is given by the transformation $\varphi = \frac{\rho-\delta}{2\delta}(\cosh t - \cosh t_0)$ with $t = \Xi(z)$ and the path γ_2 is contained in the strip $\{\xi + i\eta;\ \xi \in \mathbb{R},\ \frac{\pi}{2} < \eta < \frac{3\pi}{2}\}$, one concludes that $z = (\sqrt{\rho} - \sqrt{\delta}\,)\pi i$ is also a branch point of order three of the solitary wave solution φ. The proof that $z = -(\sqrt{\rho} - \sqrt{\delta}\,)\pi i$ is a branch point of order three is similar. $\square$

It follows from Lemma 5.2 that any extension of solitary wave solutions of Equation (5.1) beyond the strip $\mathcal{S}_0$ will depend on how branch lines are defined in the complex plane. In the following theorems, we give two different definitions of branch lines and consequent extensions of $\Xi(z)$.

Theorem 5.3. *Let $\mathcal{D}_0$ be the open set bounded by the curves $E_\pm$, $F_\pm$ and branch lines $s_+ = \{t = \xi;\ t_0 \le \xi \le \xi_0\}$ and $s_- = \{t = \xi;\ -\xi_0 \le \xi \le -t_0\}$, where ξ_0 is the intersection of E_+ and the ξ-axis as shown in Figure 12. Then the function $t = \Xi(z)$ has a further extension $t = \Xi_1(z)$ to the domain*

$$\mathcal{Y}_1 = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{iy;\ \big((2n+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi \le y \le \big((2n+1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi\},$$

such that $t = \Xi_1(z)$ is a homeomorphism of the open set

$$\begin{aligned}\mathcal{E}_n =& \{x + iy;\ x \in \mathbb{R},\ (2n-1)\sqrt{\rho}\,\pi < y < (2n+1)\sqrt{\rho}\,\pi\} \\ & \quad \setminus \{iy;\ (2n-1)\sqrt{\rho}\,\pi \le y \le \big((2n-1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi, \\ & \qquad\quad \textit{or}\ \big((2n+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi \le y \le (2n+1)\sqrt{\rho}\,\pi\}\end{aligned}$$

onto $\mathcal{D}_n = \mathcal{D}_0$ for $n = 0, \pm1, \pm2, \cdots$. Moreover Ξ is a conformal mapping of $\mathcal{Y}_1$ onto the Riemann surface $\mathcal{X}_1$ which is constructed by pasting countably many domains $\mathcal{D}_n$ as layers in such a way that for any integer n, on the layer $\mathcal{D}_n$ of $\mathcal{X}_1$, if one goes across any of the two branch lines $s_\pm$ from the lower half plane, one gets to the next lower layer $\mathcal{D}_{n-1}$ of the Riemann surface; whereas if one goes across any of the branch lines $s_\pm$ from the upper-half plane, one arrives at the adjacent upper layer $\mathcal{D}_{n+1}$ of the Riemann surface. Furthermore, if each branch line $\{iy;\ \big((2n+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi \le y \le \big((2n+1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi\}$ connecting the branch points $\big((2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,\big)\pi$ of Ξ_1, for $n = 0, \pm1, \pm2, \cdots$, is regarded as a cut with a left side and a right side, then the function Ξ_1 is also continuous up to the boundary of $\mathcal{Y}_1$.

Proof. The proof is similar to that of Theorem 4.5. We begin with the single-valued branch Δ_0 which is shown in *(b)* to be a homeomorphism of $\overline{\mathcal{D}_0}$ onto $\overline{\mathcal{E}_0}$ such that Δ_0

is a one-to-one mapping of F_+ and E_+ onto the left side and the right side of the line segment $\{iy;\ (\sqrt{\rho}-\sqrt{\delta}\,)\pi \le y \le \sqrt{\rho}\,\pi\}$, respectively, and it is also a one-to-one mapping of F_- and E_- onto the left side and the right side of the line segment $\{iy;\ -\sqrt{\rho}\,\pi \le y \le -(\sqrt{\rho}-\sqrt{\delta}\,)\pi\}$, respectively. If we take any path going across either the line s_- or the line s_+ from the upper half domain $\mathcal{D}_0$ to the lower half domain $\mathcal{D}_0$ and extend values of Δ_0 continuously after crossing the line, then we obtain another single-valued branch Δ_1 of the function Δ defined on $\mathcal{D}_0$ such that $\Delta_1 = \Delta_0 + 2\sqrt{\rho}\,\pi i$, Δ_1 is a homeomorphism of $\overline{\mathcal{D}_0}$ onto $\overline{\mathcal{E}_1}$ and its inverse as a function defined on $\overline{\mathcal{E}_1}$ is an extension of $t = \Xi(z)$ from $\overline{\mathcal{E}_0}$ to $\overline{\mathcal{E}_0 \cup \mathcal{E}_1}$, denoted by $\Xi_1(z)$. Noticing that values of $\Delta(t)$ along the path γ_3, which starts from a point t_1 in the upper half plane with $\Delta(t_1) = \Delta_0(t_1)$ and proceeds in clockwise direction for a full circuit, form of a rectangle in the z-plane as illustrated in Figure 12, one concludes that the value of $\Xi_1(z)$ does not change after z goes around the rectangle once. Therefore, if we define $\{iy;\ (\sqrt{\rho}-\sqrt{\delta}\,)\pi \le y \le (\sqrt{\rho}+\sqrt{\delta}\,)\pi\}$ as a branch line, then the extension Ξ_1 of Ξ to the domain $\overline{\mathcal{E}_0 \cup \mathcal{E}_1}$ is a continuous and single-valued function and analytic in the interior of the domain. In a similar way, one may use the single-valued branch Δ_n of Δ, which is a homeomorphism of $\overline{\mathcal{D}_0}$ onto the domain $\overline{\mathcal{E}_n}$, to define the extension $\Xi_1(z)$ to $\mathcal{E}_n$ for any integer n, such that $\Xi_1|_{\overline{\mathcal{E}_n}} = (\Delta_n|_{\overline{\mathcal{D}_0}})^{-1}$, Ξ_1 is a conformal mapping of $\mathcal{Y}_1$ onto $\mathcal{X}_1$ and is continuous up to the boundary of $\mathcal{Y}_1$. $\square$

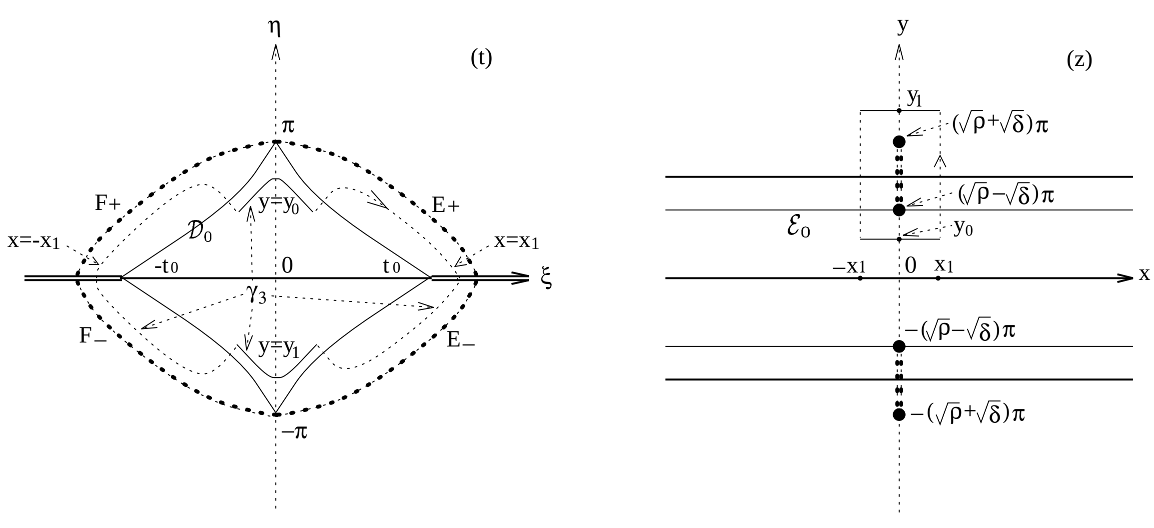

Fig. 12. The path γ_3 and the corresponding rectangle in the z-plane

Another extension of $\Xi(z)$ is also obtained by using single-valued branch of Δ as follows. Let $\mathcal{G}_0$ be the open set bounded by the lines $\{\xi + i\pi;\ \xi \ge 0\}$, $\{\xi - i\pi;\ \xi \ge 0\}$ and $\{t = \xi;\ t_0 \le \xi < \infty\}$, as well as part of the streamline $(\sqrt{\rho}-\sqrt{\delta}\,)\pi = y(\xi,\eta)$ connecting points $t=-t_0,\pi i$ and part of the streamline $-(\sqrt{\rho}-\sqrt{\delta})\pi = y(\xi,\eta)$ connecting points $t=-t_0,-\pi i$ in the left half plane as illustrated in Figure 13.

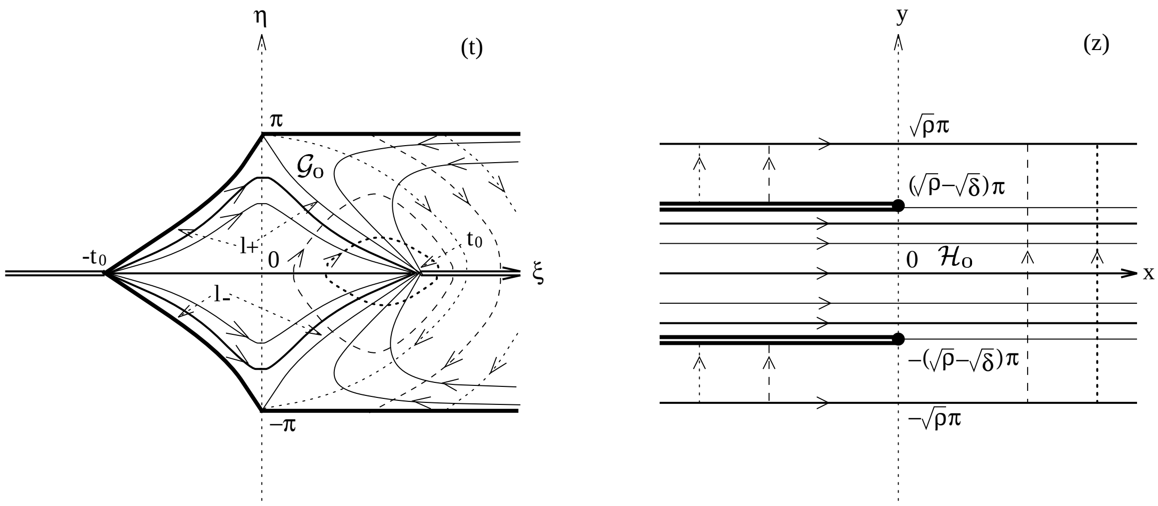

Fig. 13. A sketch of sets $\mathcal{G}_0$ in the t-plane and $\mathcal{H}_0$ in the z-plane

As we have pointed out in Lemma 5.1 that Δ_0 is a conformal mapping of the open set $D\subset\mathcal{G}_0$ onto the strip $\{x+iy;\, x\in\mathbb{R}, |y|<(\sqrt{\rho}-\sqrt{\delta})\pi\}$. In addition, Δ_0 is a one-to-one mapping of the streamlines $l_\pm$ onto the lines $\{x\pm i(\sqrt{\rho}-\sqrt{\delta})\pi;\, x\in\mathbb{R}\}$, respectively, where l_+ is the streamline $(\sqrt{\rho}-\sqrt{\delta})\pi = y(\xi,\eta)$ connecting points $t=-t_0,\pi i$ and t_0, and l_- is the streamline $-(\sqrt{\rho}-\sqrt{\delta})\pi = y(\xi,\eta)$ connecting points $t=-t_0,-\pi i$ and t_0. If extending Δ_0 to the entire open set $\mathcal{G}_0$, one finds that Δ_0 is also a conforming mapping of $\mathcal{G}_0$ onto the open set $\mathcal{H}_0=\{x+iy;\, x\in\mathbb{R}, |y|<\sqrt{\rho}\,\pi\}\setminus\{x\pm(\sqrt{\rho}-\sqrt{\delta})\pi i;\, x\le 0\}$, where $\{x-(\sqrt{\rho}-\sqrt{\delta})\pi i;\, x\le 0\}$ and $\{x+(\sqrt{\rho}-\sqrt{\delta})\pi i;\, x\le 0\}$ are branch lines of the function $\Xi(z)$ and are considered as lines with upper sides and lower sides. In consequence, we extend the function $\Xi(z)$ to the closed set $\overline{\mathcal{H}_0}$, whose inverse is Δ_0 defined on $\overline{\mathcal{G}_0}$ such that the extension Ξ_2 of Ξ is a conformal mapping of $\mathcal{H}_0$ onto $\mathcal{G}_0$ and continuous up to the boundary of $\mathcal{H}_0$.

Using the single-valued branch Δ_0 of the function Δ, one may obtain other single-valued branches of Δ. For any integer n, the value of the single-valued branch Δ_n at any point $\tilde{t}\in\mathcal{G}_0$ is defined by $\Delta_n(\tilde{t})=\Delta_0(\tilde{t})+2\sqrt{\rho}\,n\pi i$. In consequence, the function Δ_n is a conformal mapping of $\mathcal{G}_n=\mathcal{G}_0$ onto the open set

$$\mathcal{H}_n=\{x+iy;\, x\in\mathbb{R}, |y-2\sqrt{\rho}\,n\pi|<\sqrt{\rho}\,\pi\}\setminus\{x+i\big(2n\sqrt{\rho}\pm(\sqrt{\rho}-\sqrt{\delta})\big)\pi;\, x\le 0\},$$

where $\mathcal{G}_n$ is used to specify the domain of Δ_n for $n=0\pm 1,\pm 2,\cdots$. Furthermore, the inverse of Δ_n and the inverse of Δ_{n+1} defined on $\overline{\mathcal{H}_n}$ and $\overline{\mathcal{H}_{n+1}}$, respectively, share the same value on the intersection $\overline{\mathcal{H}_n} \cap \overline{\mathcal{H}_{n+1}}$. This fact and the theory of Riemann covering surface [1] imply a further analytic extension Ξ_2 of the function Ξ to the complex plane specified in the following theorem.

Theorem 5.4. *Let* $\mathcal{Y}_2 = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{x + i\big((2n+1)\sqrt{\rho} \pm \sqrt{\delta}\big)\pi;\, x \le 0\}$. *Then the function* $\Xi(z)$ *defined in Lemma 5.1 has an analytic extension, denoted by* $\Xi_2(z)$*, to the domain* $\mathcal{Y}_2$ *such that* Ξ_2 *is a homeomorphism of the open set* $\mathcal{H}_n$ *onto the open set* $\mathcal{G}_n$ *with* $\Xi_2|_{\mathcal{H}_n} = (\Delta_n|_{\mathcal{G}_n})^{-1}$ *for any integer* n*, and it is a conformal mapping of* $\mathcal{Y}_2$ *onto the Riemann surface* $\mathcal{X}_2$ *which is constructed by pasting countably many sets* $\mathcal{G}_n$ *as layers in such a way that for any integer* n*, the upper side of the line* $l_b = \{t = \xi;\, t_0 \le \xi < \infty\}$ *in* $\mathcal{G}_n$ *is glued to the lower side of the line* l_b *in* $\mathcal{G}_{n+1}$ *which is placed right on the top of* $\mathcal{G}_n$*; while the lower side of the line* l_b *in* $\mathcal{G}_n$ *is glued to the upper side of the line* l_b *in* $\mathcal{G}_{n-1}$ *which is positioned right below* $\mathcal{G}_n$*. Furthermore,* Ξ_2 *also has a continuous extension to the boundary* $\bigcup_{n=-\infty}^{\infty} \{x + i\big((2n+1)\sqrt{\rho} \pm \sqrt{\delta}\big)\pi;\, x \le 0\}$ *of* $\mathcal{Y}_2$*, if for any integer* n *the branch line* $\{x + i\big((2n+1)\sqrt{\rho} - \sqrt{\delta}\big)\pi;\, x \le 0\}$ *or* $\{x + i\big((2n+1)\sqrt{\rho} + \sqrt{\delta}\big)\pi;\, x \le 0\}$ *of* Ξ_2 *is considered as one with an upper side and a lower side.*

We have shown by concrete constructions that analytic extension of any solitary wave solution to Equation (3.7) is not unique, which is a distinct property different from solitary wave solutions of the KdV equation. There are also some other different extensions of solitary wave solutions of (3.7). We leave them to interested reader to find out. What we will focus on in the next section is the convergence of solitary wave solutions of (3.7) as functions extended to the complex plane to a compacton, a peakon or a solitary wave solution of the KdV equation, as well as the explanation of why compactons and peakons are weak solutions of Equation (3.7) and why they have singularities on the real axis.

6. Convergence to compactons.

In Section 4, for each $\delta, \epsilon > 0$, we constructed two different analytic extensions $\Theta_{\epsilon,\delta,0}(z)$ and $\Theta_{\epsilon,\delta,1}(z)$, defined on their respective domains

$$Y_{\epsilon,\delta,0} = \mathbb{C} \setminus \{x \pm i\sqrt{\epsilon}\,\pi;\, |x| \ge \sqrt{\delta}\,\pi\},$$
$$Y_{\epsilon,\delta,1} = \mathbb{C} \setminus \bigcup_{n=-\infty}^{\infty} \{x + i(2n+1)\sqrt{\epsilon}\,\pi;\, |x| \le \sqrt{\delta}\,\pi\},$$

for the solution $\theta(x)$ to Equation (4.3). Then $\varphi(x) = \frac{\delta+\epsilon}{2\delta}\sin\theta(x) + \frac{\delta-\epsilon}{2\delta}$ is the corresponding solitary wave solution of (4.1).

Theorem 6.1. *As $\epsilon \to 0$, the functions $\Theta_{\epsilon,\delta,0}(z)$ converge to*

$$\Theta_0(z) = \begin{cases} \dfrac{\pi}{2} - \dfrac{z}{\sqrt{\delta}}, & \text{if } \Im z \neq 0, \text{ or } \Im z = 0 \text{ and } |\Re z| < \sqrt{\delta}\,\pi \\ -\dfrac{\pi}{2}, & \text{if } \Im z = 0 \text{ and } \Re z \geq \sqrt{\delta}\,\pi \\ \dfrac{3\pi}{2}, & \text{if } \Im z = 0 \text{ and } \Re z \leq -\sqrt{\delta}\,\pi \end{cases}$$

Therefore, as functions extended to the domain $\overline{Y_{\epsilon,\delta,0}}$, solitary wave solutions $\Phi_{\epsilon,\delta,0}(z) = \frac{\delta+\epsilon}{2\delta}\sin\Theta_{\epsilon,\delta,0}(z) + \frac{\delta-\epsilon}{2\delta}$ to (4.1) converge to the function

$$\Phi_0(z) = \begin{cases} \cos^2 \dfrac{z}{2\sqrt{\delta}}, & \text{if } \Im z \neq 0, \text{ or } \Im z = 0 \text{ and } |\Re z| < \sqrt{\delta}\,\pi \\ 0, & \text{if } \Im z = 0, \text{ and } |\Re z| \geq \sqrt{\delta}\,\pi \end{cases}$$

when $\epsilon \to 0$. Note that $\Phi_0(z)$ is an analytic function on the set $\mathbb{C} \setminus \{z = x;\ |x| \geq \sqrt{\delta}\,\pi\}$, having line segments $\{z = x;\ x \geq \sqrt{\delta}\,\pi\}$ and $\{z = x;\ x \leq -\sqrt{\delta}\,\pi\}$ as its natural boundary.

Proof. We show that the functions $\{\Theta_{\epsilon,\delta,0}\}$ form a normal family on any compact set $K \subset \mathbb{C} \setminus \{z = x;\ |x| \geq \sqrt{\delta}\,\pi\}$. Let M, N, r and $\tilde{r}$ be any constants such that $0 < \tilde{r} < M$ and $0 < r < \sqrt{\delta}\,\pi < N$, and let K be the compact set whose boundary consists of line segments with vertices $z = -N + iM, -N + i\tilde{r}, -r + i\tilde{r}, -r - i\tilde{r}, -N - i\tilde{r}, -N - iM, N - iM, N - i\tilde{r}, r - i\tilde{r}, r + i\tilde{r}, N + i\tilde{r}$ and $N + iM$ as sketched in Figure 14.

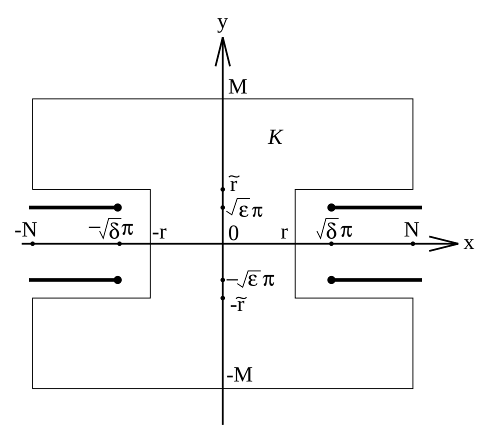

Fig. 14. A sketch of the compact set K

It follows from the discussion in *(ii)* of Section 4 that $\Theta_{\epsilon,\delta,0}$ maps vertical lines on the boundary of K to equipotentials of the function $\Sigma_{\epsilon,\delta,0} = \Sigma_0$ and horizontal lines to streamlines of $\Sigma_{\epsilon,\delta,0}$. The maximum and minimum values of $\eta_{\epsilon,\delta,0}(z) = \Im(\Theta_{\epsilon,\delta,0}(z))$ on

K is attained on the streamlines $\pm M = y(\xi,\eta)$ of $\Sigma_{\epsilon,\delta,0}$ such that $-M = y(\frac{\pi}{2}, \eta_{\epsilon,\delta,0,M})$, where $\eta_{\epsilon,\delta,0,M} = \max_{z\in K} |\eta_{\epsilon,\delta,0}(z)|$. Since $(y,\eta) = (-M, \eta_{\epsilon,\delta,0,M})$ satisfies the equation

$$y = -i\Sigma_{\epsilon,\delta,0}\big(\frac{\pi}{2} + i\eta\big) = -\left(\sqrt{\delta}\,\eta + 2\sqrt{\epsilon}\tan^{-1}\big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta}{2}\big)\right) \tag{6.1}$$

as demonstrated in Lemma 4.1. It follows that

$$\lim_{\epsilon\to 0}\eta_{\epsilon,\delta,0,M} = \lim_{\epsilon\to 0}\eta_{\epsilon,\delta,0}(0,-M) = \frac{M}{\sqrt{\delta}}.$$

On the other hand, the extremum of $\xi_{\epsilon,\delta,0}(z) = \Re(\Theta_{\epsilon,\delta,0}(z))$ on K is attained on the equipotentials given by the equation $\pm N = x(\xi,\eta)$ of $\Sigma_{\epsilon,\delta,0}$. Let n be the smallest integer greater than or equal to $N/(2\sqrt{\delta}\,\pi)$. Then $\xi_{\epsilon,\delta,0,N} = \max_{z\in K}|\xi_{\epsilon,\delta,0}(z)| \le 2(n+1)\pi$ because $\xi_{\epsilon,\delta,0}(x + 2k\sqrt{\delta}\,\pi, y) = \xi_{\epsilon,\delta,0}(x,y) - 2k\pi$ for any integer k. Therefore, the functions $\{\Theta_{\epsilon,\delta,0}(z)\}$ are uniformly bounded on K for any ϵ sufficiently small. Hence, there exists a subsequence of $\{\Theta_{\epsilon,\delta,0}\}$, for the sake of simplicity, still denoted by $\{\Theta_{\epsilon,\delta,0}\}$, analytic in K and uniformly convergent to an analytic function on any compact subset of K as $\epsilon \to 0$. Equation (6.1) leads to the definition of the limiting function on the imaginary axis

$$\lim_{\epsilon\to 0}\Theta_{\epsilon,\delta,0}(iy) = \lim_{\epsilon\to 0}\big(\frac{\pi}{2} + i\eta_{\epsilon,\delta,0}(iy)\big) = \frac{\pi}{2} - \frac{iy}{\sqrt{\delta}}, \qquad |y| < M.$$

The fact that the limiting function is holomorphic in K and its restriction to the imaginary axis is the same as that of the function $\mathit{\Theta}_0(z) = \frac{\pi}{2} - \frac{z}{\sqrt{\delta}}$ leads to the conclusion that the limiting function defined in K is the same as $\mathit{\Theta}_0(z)$. Since any convergent subsequence of $\{\Theta_{\epsilon,\delta,0}\}$ has the same limit, this implies that the sequence $\{\Theta_{\epsilon,\delta,0}(z)\}$ itself converges to $\mathit{\Theta}_0(z)$ in the interior of K when $\epsilon \to 0$.

Next, we need to show the convergence of $\{\Theta_{\epsilon,\delta,0}\}$ on lines $\{z = x;\ |x| \ge \sqrt{\delta}\,\pi\}$ which will be carried out by using Equation (4.4). For any ν with $0 < \nu < \pi$, let θ_1 be a fixed constant such that $0 < \theta_1 + \frac{\pi}{2} < \frac{\nu}{2}$. Then there is an $\epsilon_0 > 0$, such that if $0 < \epsilon < \epsilon_0$,

$$e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta_1+\frac{\pi}{2})} < \frac{1}{3}, \quad \frac{\sin\frac{\theta_1+\theta_0}{2}}{\cos\frac{\theta_1-\theta_0}{2}} > \frac{1}{2} \quad \text{and} \quad -\frac{\pi}{2} < -\theta_0 < \theta_1.$$

Moreover,

$$\frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}} > \frac{\sin\frac{\theta_1+\theta_0}{2}}{\cos\frac{\theta_1-\theta_0}{2}} > e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta_1+\frac{\pi}{2})} > e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta+\frac{\pi}{2})}$$

for any $\theta \in (\theta_1, \pi+\theta_0)$ and any $\epsilon \in (0, \epsilon_0)$. Therefore, the solution $\theta_{\epsilon,\delta,0,\frac{\pi}{2}}$ of the equation

$$\frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}} = e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta+\frac{\pi}{2})}$$

must satisfy the condition $-\theta_0 < \theta_{\epsilon,\delta,0,\frac{\pi}{2}} < \theta_1 < \nu - \frac{\pi}{2}$. It follows that $\theta_{\epsilon,\delta,0,\frac{\pi}{2}} = \Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi)$ and $\lim\limits_{\epsilon\to 0}\Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi) = -\frac{\pi}{2}$. Because for any $x > \sqrt{\delta}\,\pi$,

$$e^{-\frac{1}{\sqrt{\epsilon}}\left(x+\sqrt{\delta}(\theta-\frac{\pi}{2})\right)} < e^{-\sqrt{\frac{\delta}{\epsilon}}(\theta+\frac{\pi}{2})} \quad \text{and} \quad \frac{\sin\frac{\theta+\theta_0}{2}}{\cos\frac{\theta-\theta_0}{2}} > \frac{\sin\frac{\theta_{\epsilon,\delta,0,\frac{\pi}{2}}+\theta_0}{2}}{\cos\frac{\theta_{\epsilon,\delta,0,\frac{\pi}{2}}-\theta_0}{2}}$$

hold for any θ with $\theta_{\epsilon,\delta,0,\frac{\pi}{2}} \le \theta < \pi+\theta_0$, the solution $\theta_{\epsilon,\delta,0,x}$ of Equation (4.4) satisfies $-\theta_0 < \theta_{\epsilon,\delta,0,x} < \Theta_{\epsilon,\delta,0}(\sqrt{\delta}\pi)$ and $\theta_{\epsilon,\delta,0,x} = \Theta_{\epsilon,\delta,0}(x)$. This implies that functions $\Theta_{\epsilon,\delta,0}(x)$ are uniformly convergent to $-\frac{\pi}{2}$ on the interval $[\sqrt{\delta}\,\pi, \infty)$ as $\epsilon \to 0$. Then the equality $\Theta_{\epsilon,\delta,0}(-x) = \pi - \Theta_{\epsilon,\delta,0}(x)$ leads to the conclusion that the sequence of the functions $\Theta_{\epsilon,\delta,0}$ is uniformly convergent to $\frac{3\pi}{2}$ on the interval $(-\infty, -\sqrt{\delta}\,\pi]$. As a result, functions $\Phi_{\epsilon,\delta,0}(z)$ converge to $\varPhi_0(z)$ for any $z \in \mathbb{C}$ when $\epsilon \to 0$. □

Theorem 6.2. *The solutions $\{\Theta_{\epsilon,\delta,1}(z)\}$ of Equation (4.3) converge to the function defined as*

$$\varTheta_1(z) = \begin{cases} \dfrac{\pi}{2} - \dfrac{x}{\sqrt{\delta}}, & \text{if } \Re z = x \text{ and } |x| \le \sqrt{\delta}\,\pi \\ -\dfrac{\pi}{2}, & \text{if } \Re z > \sqrt{\delta}\,\pi \\ \dfrac{3\pi}{2}, & \text{if } \Re z < -\sqrt{\delta}\,\pi \end{cases}$$

when $\epsilon \to 0$. Hence, solitary wave solutions $\Phi_{\epsilon,\delta,1}(z) = \frac{\delta+\epsilon}{2\delta}\sin\Theta_{\epsilon,\delta,1}(z) + \frac{\delta-\epsilon}{2\delta}$ of Equation (4.1) converge to the function given by

$$\varPhi_1(z) = \begin{cases} \cos^2\dfrac{x}{2\sqrt{\delta}}, & \text{if } \Re z = x \text{ and } |x| \le \sqrt{\delta}\,\pi \\ 0, & \text{if } |\Re z| > \sqrt{\delta}\,\pi \end{cases}$$

as $\epsilon \to 0$. The function $\varPhi_1(z)$ is analytic on the set $\mathbb{C} \setminus \{z \in \mathbb{C};\ |\Re z| \le \sqrt{\delta}\,\pi\}$, having the natural boundary $\{\pm\sqrt{\delta}\,\pi + iy;\ y \in \mathbb{R}\}$.

Proof. It follows from the definition of $\Theta_{\epsilon,\delta,1}$ that for any integer n, the restriction of $\Theta_{\epsilon,\delta,1}(z)$ to the strip $\{(2n-1)\sqrt{\epsilon}\,\pi \le \Im z \le (2n+1)\sqrt{\epsilon}\,\pi\}$ is the inverse of the single-valued branch Σ_n such that $\Sigma_n(\theta) = \Sigma_0(\theta) + i2n\pi\sqrt{\epsilon}$ for any $\theta \in \overline{\Omega}$, where Σ_0 is the single-valued branch of Σ defined in *(i)*. That means for any $(2n-1)\sqrt{\epsilon}\,\pi \le y \le (2n+1)\sqrt{\epsilon}\,\pi\}$, we have

$\Theta_{\epsilon,\delta,1}(x+iy) = \Theta_{\epsilon,\delta,1}(x+i(y-2n\pi\sqrt{\epsilon})) \in \overline{\Omega}$. Let $\eta_{\epsilon,\delta,1}(x,y) = \Im\big(\Theta_{\epsilon,\delta,1}(z)\big)$ for $z = x+iy$. Then $\eta_\epsilon = \max_{x+iy\in Y_{\epsilon,\delta,1}} |\eta_{\epsilon,\delta,1}(x,y)| = \max_{x\in\mathbb{R},|y|\le\sqrt{\epsilon}\,\pi} |\eta_{\epsilon,\delta,1}(x,y)|$. It follows from lemma 4.1 that $(y,\eta) = (\sqrt{\epsilon}\,\pi, -\eta_\epsilon)$ satisfies Equation (6.1), *i.e.*

$$\sqrt{\epsilon}\,\pi = \sqrt{\delta}\,\eta_\epsilon + 2\sqrt{\epsilon}\tan^{-1}\Big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta_\epsilon}{2}\Big).$$

Therefore, $\lim_{\epsilon\to 0}|\eta_{\epsilon,\delta,1}(x,y)| \le \lim_{\epsilon\to 0}\eta_\epsilon = 0$ for any $x+iy\in\mathbb{C}$, which implies that the sequence $\{\eta_{\epsilon,\delta,1}(x,y)\}$ is uniformly convergent to 0 as $\epsilon\to 0$. Now we only need to show that functions $\xi_{\epsilon,\delta,1}(x,y) = \Re(\Theta_{\epsilon,\delta,1}(x+iy))$ converge to $\Theta_1(x+iy)$ when $\epsilon\to 0$.

For any fixed $x\in[0,\sqrt{\delta}\,\pi]$, Lemma 4.1 implies that $\frac{\pi}{2} - \frac{x}{\sqrt{\delta}} \le \xi_{\epsilon,\delta,1}(x,y) \le \xi_{\epsilon,\delta,1}(x) = \Theta_{\epsilon,\delta,1}(x)$ for all y. Thus $(x,\theta) = (x,\xi_{\epsilon,\delta,1}(x))$ satisfies Equation (4.4) with $-\theta_0 < \xi_{\epsilon,\delta,1}(x) \le \frac{\pi}{2}$. Implicit differentiation of Equation (4.4) with respect to x yields the estimate

$$\frac{d}{dx}\left(\xi_{\epsilon,\delta,1}(x) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2}\right) = \frac{1}{\sqrt{\delta}} - \frac{\sin\xi_{\epsilon,\delta,1}(x) + \sin\theta_0}{\sqrt{\delta}\,(\sin\xi_{\epsilon,\delta,1}(x)+1)} > 0.$$

Therefore $\xi_{\epsilon,\delta,1}(x) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2}$ is an increasing function of x on the interval $[0,\infty)$, and thus

$$0 = \xi_{\epsilon,\delta,1}(0) - \frac{\pi}{2} \le \xi_{\epsilon,\delta,1}(x) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2} \le \xi_{\epsilon,\delta,1}(\sqrt{\delta}\,\pi) + \frac{\pi}{2}$$

for any $\epsilon > 0$. Since $\Theta_{\epsilon,\delta,0}(x) = \Theta_{\epsilon,\delta,1}(x) = \xi_{\epsilon,\delta,1}(x)$, Theorem 6.1 implies that

$$0 \le \lim_{\epsilon\to 0}\left(\xi_{\epsilon,\delta,1}(x,y) + \frac{x}{\sqrt{\delta}} - \frac{\pi}{2}\right) \le \lim_{\epsilon\to 0}\left(\xi_{\epsilon,\delta,1}(\sqrt{\delta}\,\pi) + \frac{\pi}{2}\right) = 0.$$

In other words, the functions $\xi_{\epsilon,\delta,1}(x,y)$ are uniformly convergent to the function $\Theta_1(x+iy)$ on the strip $\{0\le\Re z\le\sqrt{\delta}\,\pi\}$ as $\epsilon\to 0$.

If $x\in[\sqrt{\delta}\,\pi,\infty)$, then $-\frac{\pi}{2} \le \xi_{\epsilon,\delta,1}(x+iy) \le \xi_{\epsilon,\delta,1}(\sqrt{\delta}\,\pi)$. Hence,

$$-\frac{\pi}{2} \le \varliminf_{\epsilon\to 0}\xi_{\epsilon,\delta,1}(x+iy) \le \varlimsup_{\epsilon\to 0}\xi_{\epsilon,\delta,1}(x+iy) \le \lim_{\epsilon\to 0}\xi_{\epsilon,\delta,1}(\sqrt{\delta}\,\pi) = -\frac{\pi}{2},$$

i.e. $\xi_{\epsilon,\delta,1}(x,y)$ are uniformly convergent to $-\frac{\pi}{2}$ on the strip $\{\Re z\ge\sqrt{\delta}\,\pi\}$. Using the identity $\xi_{\epsilon,\delta,1}(x,y) = \pi - \xi_{\epsilon,\delta,1}(-x,y)$ for any $x+iy\in\mathbb{C}$ also leads to the conclusion that $\xi_{\epsilon,\delta,1}(x,y)$ are uniformly convergent to $\Theta_1(x+iy)$ in the half plane $\{x+iy;\ x\le 0, y\in\mathbb{R}\}$. In consequence, the sequence of functions $\{\Phi_{\epsilon,\delta,1}(z)\}$ is uniformly convergent to the function $\Phi_1(z)$ in the complex plane $\mathbb{C}$ as $\epsilon\to 0$. Also, for any integer $n>0$, the n-th order derivatives $\Phi^{(n)}_{\epsilon,\delta,1}(z)$ of $\Phi_{\epsilon,\delta,1}(z)$ are uniformly convergent to the derivative $\Phi_1^{(n)}(z)\equiv 0$ on any compact set contained in the half planes $\{\Re z < -\sqrt{\delta}\,\pi\}$ and $\{\Re z > \sqrt{\delta}\,\pi\}$. $\square$

The fact that solitary wave solutions converge to the compacton is a justification for it to be a weak solution to the limiting equation

$$-\varphi' + 3\varphi\varphi' + \delta(2\varphi'\varphi'' + \varphi\varphi''') = 0 \tag{6.2}$$

of the perturbed equation

$$-\varphi' + \epsilon\varphi''' + 3\varphi\varphi' + \delta(2\varphi'\varphi'' + \varphi\varphi''') = 0 \tag{6.3}$$

when $\epsilon \to 0$ with $\epsilon > 0$ and $\delta > 0$. As we have shown in the last two theorems, the solitary wave solution of (6.3), denoted by $\varphi_{\epsilon,\delta}$, has two different analytic extensions $\Phi_{\epsilon,\delta,0}(z)$ and $\Phi_{\epsilon,\delta,1}(z)$, and $\varphi_{\epsilon,\delta}(x)$ is uniformly convergent to the compacton $\mathit{\Phi}_0(x) = \varphi_0(x)$ on the real axis. Since $\varphi_{\epsilon,\delta}(x)$ also satisfies Equation (4.1),

$$\varphi'_{\epsilon,\delta}(x) = -\frac{\operatorname{sign} x\, \varphi_{\epsilon,\delta}(x)\sqrt{1-\varphi_{\epsilon,\delta}(x)}}{\sqrt{\delta\varphi_{\epsilon,\delta}(x)+\epsilon}}. \tag{6.4}$$

It follows from the inequality $0 \le \varphi_{\epsilon,\delta} \le 1$ that

$$\left|\varphi'_{\epsilon,\delta}(x) + \frac{\operatorname{sign} x}{\sqrt{\delta}}\sqrt{\varphi_{\epsilon,\delta}(x)(1-\varphi_{\epsilon,\delta}(x))}\right| < \frac{\sqrt{\epsilon}}{\delta}, \qquad x \in \mathbb{R}.$$

Therefore, the functions $\varphi'_{\epsilon,\delta}(x)$ are also uniformly convergent, and

$$\lim_{\epsilon\to 0}\varphi'_{\epsilon,\delta}(x) = \varphi'_0(x) = \begin{cases} -\dfrac{1}{2\sqrt{\delta}}\sin\dfrac{x}{\sqrt{\delta}}, & \text{if } |x| \le \sqrt{\delta}\,\pi \\ 0, & \text{if } |x| > \sqrt{\delta}\,\pi. \end{cases}$$

As a matter of fact, one may also obtain L^p-convergence of the functions $\varphi_{\epsilon,\delta}$, $\varphi'_{\epsilon,\delta}$, $\varphi''_{\epsilon,\delta}$, $\varphi_{\epsilon,\delta}\varphi'''_{\epsilon,\delta}$ and $\epsilon\varphi'''_{\epsilon,\delta}$ for any p with $p \ge 1$. Notice that as a distribution φ_0''' is not a L^p-function, but $\varphi_0\varphi_0'''$ belongs to the space L^p as a well-defined distribution in the sense that $\varphi_0\varphi_0'''$ can be expressed as $\varphi_0\varphi_0''' = \frac{1}{2}(\varphi_0^2)''' - 3\varphi_0'\varphi_0''$ which is the difference of the functions

$$\frac{1}{2}\left(\varphi_0^2(x)\right)''' = \begin{cases} \frac{1}{4\delta\sqrt{\delta}}(2\sin\frac{2x}{\sqrt{\delta}} + \sin\frac{x}{\sqrt{\delta}}), & \text{if } |x| \le \sqrt{\delta}\,\pi \\ 0, & \text{otherwise} \end{cases}$$

and

$$3\varphi_0'(x)\varphi_0''(x) = \begin{cases} \frac{3}{8\delta\sqrt{\delta}}\sin\frac{2x}{\sqrt{\delta}}, & \text{if } |x| \le \sqrt{\delta}\,\pi \\ 0, & \text{otherwise} \end{cases}$$

such that $(\varphi_0{}')^2 \in H^2$ and $\varphi_0^2 \in H^4$. These facts indicate that the compacton φ_0 satisfies Equation (6.2) in the way that each term formed by the compacton and its derivatives in (6.2) is a L^p-function, having the limit equal to zero at its singularities $x = \pm\sqrt{\delta}\,\pi$ and continuous everywhere else. Therefore, the compacton falls into our category of "pseudo-classical" solutions. Now we summarize results about uniform convergence and L^p-convergence to the compacton φ_0 and its derivatives in the following theorem.

Theorem 6.3. *Let p be any constant with $p \geq 1$. Then as $\epsilon \to 0$, the compacton φ_0 and its derivatives up to the second order are limits of the functions $\varphi_{\epsilon,\delta}$ and their derivatives $\varphi'_{\epsilon,\delta}$ and $\varphi''_{\epsilon,\delta}$ in L^p-norm, respectively, and the functions $\varphi_{\epsilon,\delta}\varphi'''_{\epsilon,\delta}$ and $\epsilon\varphi'''_{\epsilon,\delta}$ also converge to $\varphi_0\varphi_0'''$ and 0 in L^p, respectively. Furthermore, when $\epsilon \to 0$, $\varphi_{\epsilon,\delta}$ and $\varphi'_{\epsilon,\delta}$ converge to φ_0 and φ_0' uniformly on the real axis, and $\varphi^{(n)}_{\epsilon,\delta}$ converge to $\varphi_0^{(n)}$ uniformly on any compact set contained in the open set $(-\infty, -\sqrt{\delta}\,\pi)\cup(-\sqrt{\delta}\,\pi, \sqrt{\delta}\,\pi)\cup(\sqrt{\delta}\,\pi, \infty)$ for any integer n. The functions $\varphi_0 \in W^{2,p}$ and φ_0' are continuous on the real axis with $\varphi_0(x) = \varphi_0'(x) = 0$ for any x with $|x| \geq \sqrt{\delta}\,\pi$. In consequence, the compactons φ_0 and $\phi_0 = -(\beta + c\nu) + \big(3(\beta + c\nu) - \nu(\alpha + c)\big)\varphi_0$ given by (3.9) satisfy Equations (6.2) and (3.7) everywhere, respectively.*

Proof. For any fixed $x \in \mathbb{R}$, the derivative of $\varphi_{\epsilon,\delta}$ with respect to ϵ takes the form

$$\frac{\partial \varphi_{\epsilon,\delta}(x)}{\partial \epsilon} = \frac{(\delta+\epsilon)\big(x + \sqrt{\delta}(\theta - \frac{\pi}{2})\big)}{4\epsilon\delta\sqrt{\delta}} \, \frac{\cos\theta(\sin\theta + \sin\theta_0)}{\sin\theta + 1}. \tag{6.5}$$

Equation (4.4) shows that $-\theta_0 \leq \theta \leq \frac{\pi}{2}$ and $x + \sqrt{\delta}(\theta - \frac{\pi}{2}) \geq 0$ when $x \geq 0$, and $\frac{\pi}{2} \leq \theta \leq \pi + \theta_0$ and $x + \sqrt{\delta}(\theta - \frac{\pi}{2}) \leq 0$ if $x \leq 0$. Therefore $\frac{\partial}{\partial\epsilon}\varphi_{\epsilon,\delta}(x) \geq 0$ for any $x \in \mathbb{R}$. The estimate

$$\int_{-\infty}^{\infty} |\varphi_{\epsilon,\delta}(x)|^p dx = 2\int_0^1 \frac{y^{p-1}\sqrt{\delta y + \epsilon}}{\sqrt{1-y}} dy < 2\sqrt{\delta+\epsilon}\int_0^1 \frac{y^{p-1}}{\sqrt{1-y}} dy$$

along with the uniform convergence of $\varphi_{\epsilon,\delta}$ to φ_0 implies that the limit

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} |\varphi_{\epsilon,\delta}(x) - \varphi_0(x)|^p dx = 0$$

exists for any $p > 0$. Uniform convergence of $\varphi'_{\epsilon,\delta}$ to φ_0' and the following estimate

$$\int_{-\infty}^{\infty} |\varphi'_{\epsilon,\delta}(x) - \varphi_0'(x)|^p dx \leq 2\int_0^{\sqrt{\delta}\,\pi} |\varphi'_{\epsilon,\delta}(x) - \varphi_0'(x)|^p dx + \frac{2}{\delta^{\frac{p}{2}}}\int_{\sqrt{\delta}\,\pi}^{\infty} |\varphi_{\epsilon,\delta}(x)|^{\frac{p}{2}} dx$$

show that $\lim\limits_{\epsilon\to 0}\int_{-\infty}^{\infty} |\varphi'_{\epsilon,\delta}(x) - \varphi_0'(x)|^p dx = 0$ for any $p > 0$.

To show convergence of $\{\varphi''_{\epsilon,\delta}\}$ and $\{(\delta\varphi_{\epsilon,\delta} + \epsilon)\varphi'''_{\epsilon,\delta}\}$, we estimate their upper bounds as follows,

$$|\varphi''_{\epsilon,\delta}(x)| \leq \frac{\varphi_{\epsilon,\delta}(x)}{\delta} + \frac{\delta+\epsilon}{\delta}\,\frac{\varphi_{\epsilon,\delta}(x)}{\delta\varphi_{\epsilon,\delta}(x) + \epsilon} \leq \frac{2\delta+\epsilon}{\delta^2}, \tag{6.6}$$

and

$$|\varphi_{\epsilon,\delta}(x)\varphi'''_{\epsilon,\delta}(x)| \leq \frac{3\delta+\epsilon}{\delta^2\sqrt{\delta}}\sqrt{\varphi_{\epsilon,\delta}(x)} \quad \text{and} \quad |\epsilon\varphi'''_{\epsilon,\delta}(x)| \leq \frac{2\epsilon+\delta}{\delta\sqrt{\delta}}\sqrt{\varphi_{\epsilon,\delta}(x)} \tag{6.7}$$

for any $x \in \mathbb{R}$. Noticing that

$$\frac{\varphi_{\epsilon,\delta}}{\delta\varphi_{\epsilon,\delta}+\epsilon}=\frac{\sin\theta+\sin\theta_0}{\delta(\sin\theta+1)}=\frac{-1}{\sqrt{\delta}}\frac{d\theta}{dx}\to 0 \tag{6.8}$$

for any x with $|x|>\sqrt{\delta}\,\pi$ when $\epsilon\to 0$ as shown in Theorem 6.2, where $\theta=\Theta_{\epsilon,\delta,0}(x)$ is the same as defined in (6.5). On the other hand, it follows from (6.5) and uniform convergence of the functions $\theta=\Theta_{\epsilon,\delta,0}(x)$ on the real axis that for any fixed x_0 with $x_0>\sqrt{\delta}\,\pi$, there is an $\epsilon_0>0$, such that whenever $x>x_0$ and $0<\epsilon<\epsilon_0$, we have

$$\frac{\partial}{\partial\epsilon}\left(\frac{\varphi_{\epsilon,\delta}(x)}{\delta\varphi_{\epsilon,\delta}(x)+\epsilon}\right)=\frac{\varphi(x)\cot(\frac{\pi}{4}+\frac{\theta}{2})}{(\delta\varphi_{\epsilon,\delta}(x)+\epsilon)^2}\left(\frac{x+\sqrt{\delta}(\theta-\frac{\pi}{2})}{2\sqrt{\delta}}-\tan(\frac{\pi}{4}+\frac{\theta}{2})\right)\geq 0.$$

Lebesgue's Dominated Convergence Theorem, combined with (6.6), (6.8) and the convergence of $\varphi_{\epsilon,\delta}$ in L^p, shows that the limit $\lim\limits_{\epsilon\to 0}\int_{-\infty}^{\infty}|\varphi''_{\epsilon,\delta}(x)-\varphi''_0(x)|^p dx=0$ exist. In a similar way, using (6.7) and Lebesgue's Dominated Convergence Theorem, one may also show that for any $p>0$, the limits

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}|\varphi_{\epsilon,\delta}(x)\varphi'''_{\epsilon,\delta}(x)-\varphi_0(x)\varphi'''_0(x)|^p dx=0 \quad\text{and}\quad \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}|\epsilon\varphi'''_{\epsilon,\delta}(x)|^p dx=0$$

exist. Therefore, the compacton φ_0 belongs to the Sobolev space $W^{2,p}$ for any p with $p\geq 1$. Because $\Phi_{\epsilon,\delta,0}(z)$ converge to $\Phi_0(z)$ uniformly on any compact set of $\mathbb{C}\setminus\{z\in\mathbb{R};\ |z|\geq\sqrt{\delta}\,\pi\}$, $\Phi_{\epsilon,\delta,1}(z)$ converge to $\Phi_1(z)$ uniformly on any compact set of $\mathbb{C}\setminus\{z\in\mathbb{C};\ |\Re z|\leq\sqrt{\delta}\,\pi\}$ and $\Phi_0(x)=\Phi_1(x)=\varphi_0(x)$ for any $x\in\mathbb{R}$, for any positive integer $n\geq 2$, the n-th order derivatives $\varphi^{(n)}_{\epsilon,\delta}(x)$ converge to $\varphi^{(n)}_0(x)$ everywhere except for the points $x=\pm\sqrt{\delta}\,\pi$. Furthermore, $\varphi_0(\pm\sqrt{\delta}\,\pi)=\varphi'_0(\pm\sqrt{\delta}\,\pi)=0$. In consequence, we come to the conclusion that the compactons φ_0 and ϕ_0 given in (3.9) satisfy Equations (4.1) and (3.7) everywhere, respectively. □

7. Convergence to peakons.

In Section 5, we have constructed two different extensions $\Xi_{\delta,\rho,1}(z)$ and $\Xi_{\delta,\rho,2}(z)$ for the solution $t=\Xi(x)$ of Equation (5.4) to the respective domains

$$\mathcal{Y}_{\delta,\rho,1}=\mathbb{C}\setminus\bigcup_{n=-\infty}^{\infty}\{iy;\ \big((2n+1)\sqrt{\rho}-\sqrt{\delta}\,\big)\pi\leq y\leq\big((2n+1)\sqrt{\rho}+\sqrt{\delta}\,\big)\pi\},$$

$$\mathcal{Y}_{\delta,\rho,2}=\mathbb{C}\setminus\bigcup_{n=-\infty}^{\infty}\{x+i\big((2n+1)\sqrt{\rho}\pm\sqrt{\delta}\,\big)\pi;\ x\leq 0\}.$$

Then $\Phi_{\delta,\rho,i}=\frac{\rho-\delta}{2\delta}\cosh\Xi_{\delta,\rho,i}-\frac{\rho+\delta}{2\delta}$ is an analytic extension of the solitary wave solution $\varphi_{\delta,\rho}=\frac{\rho-\delta}{2\delta}\cosh\Xi-\frac{\rho+\delta}{2\delta}$ of Equation (5.1) to the domain $\mathcal{Y}_i$, and continuous up to the boundary of $\mathcal{Y}_i$ for $i=1,2$. The following two theorems present results on convergence of the functions $\Phi_{\delta,\rho,1}$ and $\Phi_{\delta,\rho,2}$ to the limiting functions Φ_1 and Φ_2, respectively, whose restrictions to the real axis are identical and define a peakon solution.

Theorem 7.1. *When $\rho \to \delta$, the sequence of the functions $\{\Phi_{\delta,\rho,1}\}$ converges to*

$$\Phi_1(z) = \begin{cases} -e^{\frac{-z}{\sqrt{\delta}}}, & \text{if } \Re z > 0 \\ -e^{\frac{z}{\sqrt{\delta}}}, & \text{if } \Re z < 0. \end{cases}$$

For any $z = iy$ on the imaginary axis, $\lim_{\rho\to\delta} \Phi^l_{\delta,\rho,1}(z) = -e^{\frac{z}{\sqrt{\delta}}}$ and $\lim_{\rho\to\delta} \Phi^r_{\delta,\rho,1}(z) = -e^{\frac{-z}{\sqrt{\delta}}}$, where $\Phi^l_{\delta,\rho,1}(iy) = \lim_{\substack{z\to iy \\ Rez<0}} \Phi_{\delta,\rho,1}(z)$ and $\Phi^r_{\delta,\rho,1}(iy) = \lim_{\substack{z\to iy \\ Rez>0}} \Phi_{\delta,\rho,1}(z)$ for any $y \in \mathbb{R}$. Therefore, Φ_1 is a holomorphic function on both the left and right half planes $\{\Re z < 0\}$, $\{\Re z > 0\}$, having the imaginary axis as its natural boundary.

Theorem 7.1 follows from the following result.

Theorem 7.2. *When $\rho \to \delta$, the sequence of the functions $\{\Phi_{\delta,\rho,2}\}$ converges to*

$$\Phi_2(z) = \begin{cases} -e^{\frac{-z}{\sqrt{\delta}}}, & \text{if } \Im z \neq 0, \text{ or } \Im z = 0 \text{ and } \Re z > 0 \\ -e^{\frac{z}{\sqrt{\delta}}}, & \text{if } \Im z = 0 \text{ and } \Re z \leq 0. \end{cases}$$

Therefore, Φ_2 is a holomorphic function on the open set $\mathbb{C} \setminus \{z \in \mathbb{C};\ \Re z \leq 0, \Im z = 0\}$.

Proof. The method to be used here is to show that the functions $(\rho - \delta)e^t$ are uniformly bounded on the half plane $\{\Re z \geq -N\}$ for any constant $N > 0$, where $t = \Xi_{\delta,\rho,2}(z)$, so that as ρ is close to δ, $\{(\rho - \delta)e^t\}$ is a normal family convergent on the compact sets to be defined presently.

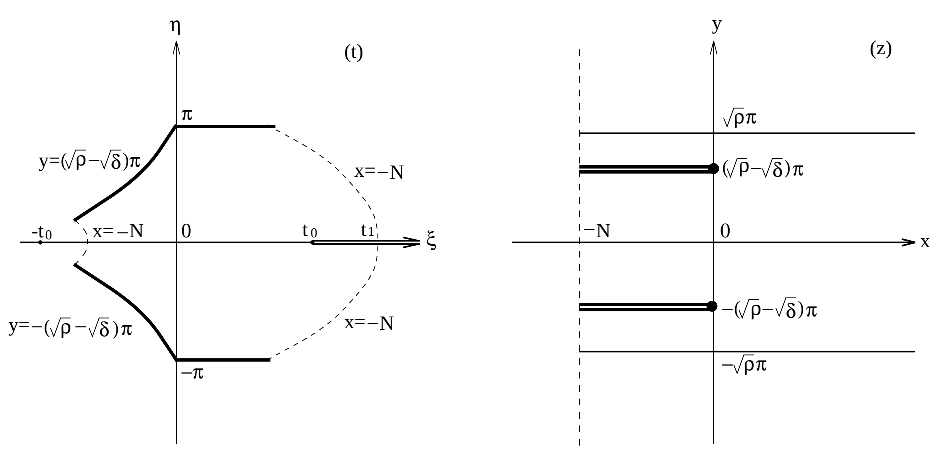

Fig. 15. A sketch of the image of the half plane $\{z \in \mathbb{C};\ \Re z \geq -N\}$

It follows from Lemma 5.1 and Theorem 5.4 that $\Xi_{\delta,\rho,2}$ maps the set $\overline{\mathcal{Y}_{\delta,\rho,2}} \setminus \{z \in \mathbb{C};\ \Re z < -N\}$ to the region bounded by the streamlines $\pm(\sqrt{\rho} - \sqrt{\delta})\pi = y(\xi, \eta)$ and the equipotential $-N = x(\xi,\eta)$ of the single-valued branch Δ_0 as shown in Figure 15. Then $\sup\limits_{\Re z\geq -N} |\Im(\Xi_{\delta,\rho,2}(z))| \leq \pi$ and $t_0 < \sup\limits_{\Re z\geq -N} |\Re(\Xi_{\delta,\rho,2}(z))| \leq t_1 = \Xi_{\delta,\rho,2}(-N+i\sqrt{\rho}\,\pi)$. It follows from (5.4) that

$$\tanh\frac{t_1}{2} = \sqrt{\frac{\delta}{\rho}}\coth\frac{\sqrt{\delta}\,t_1 - N}{2\sqrt{\rho}}.$$

Using the inequality $\tanh\frac{t_1}{2} \geq \tanh\left(\sqrt{\frac{\delta}{\rho}}\,\frac{t_1}{2}\right)$, valid for $0<\delta<\rho$, we obtain

$$(\rho-\delta)e^{t_1} \leq \frac{(\sqrt{\rho}+\sqrt{\delta}\,)\left(\sqrt{\rho}+\sqrt{\delta}+\sqrt{4\sqrt{\delta\rho}+(\sqrt{\rho}-\sqrt{\delta}\,)^2\tanh^2\frac{N}{2\sqrt{\rho}}}\,\right)^{\sqrt{\frac{\rho}{\delta}}}}{(\sqrt{\rho}-\sqrt{\delta}\,)^{\sqrt{\frac{\rho}{\delta}}-1}\left(1-\tanh\frac{N}{2\sqrt{\rho}}\right)^{\sqrt{\frac{\rho}{\delta}}}}.$$

Therefore, $(\rho-\delta)|e^t| = (\rho-\delta)e^{\Re(\Xi_{\delta,\rho,2})} \leq (\rho-\delta)e^{t_1} < \infty$ holds on the set $\overline{\mathcal{Y}_{\delta,\rho,2}}\setminus\{z\in\mathbb{C};\ \Re z < -N\}$ for any $\delta<\rho\leq\rho_0$.

Moreover, there is a $\rho_0>\delta$ such that for each $\rho\in(\delta,\rho_0)$, the function $(\rho-\delta)e^{\Xi_{\delta,\rho,2}}$ is analytic in the set

$$M = \{z\in\mathbb{C};\ \Re z\geq -N_0, |\Im z|\leq (2N_0+1)\sqrt{\delta}\,\pi\}\setminus E,$$

where $E = \bigcup\limits_{k=-N_0}^{N_0}\{z\in\mathbb{C};\ \Re z\leq 0, (2k\sqrt{\delta}-\nu)\pi\leq\Im z\leq(2k\sqrt{\delta}+\nu)\pi\}$ for some integer N_0 sufficiently large. Thus $\{(\rho-\delta)e^{\Xi_{\delta,\rho,2}}\}$ is a normal family in M. Let $\{(\rho_n-\delta)e^{\Xi_{\delta,\rho_n,2}}\}$ be a subsequence uniformly convergent to a holomorphic function, denoted by $g(z)$, in any compact set of M as $n\to\infty$, where $\rho_n>\rho_{n+1}$ and $\lim\limits_{n\to\infty}\rho_n=\delta$. Since for $t=\Xi_{\delta,\rho_n,2}(z)$, it follows from (5.4) that

$$(\rho-\delta)e^t = \frac{(\rho-\delta)\left((\sqrt{\rho}+\sqrt{\delta}\,)e^{\frac{z+\sqrt{\delta}\,t}{\sqrt{\rho}}}+\sqrt{\rho}-\sqrt{\delta}\,\right)}{(\sqrt{\rho}-\sqrt{\delta}\,)e^{\frac{z+\sqrt{\delta}\,t}{\sqrt{\rho}}}+\sqrt{\rho}+\sqrt{\delta}}.$$

Taking the limit on both sides of the above equation, one obtains the relation

$$g(z) = \frac{4\delta g(z)}{g(z)+4\delta e^{-\frac{z}{\sqrt{\delta}}}}. \tag{7.1}$$

It follows from Lemma 5.1 that for any fixed $x_1>0$, when $\Re z\geq x_1$,

$$\Re(\Xi_{\delta,\rho,2}(z)) \geq \Re(\Xi_{\delta,\rho,2}(x_1)) = \xi_1$$

and

$$\tanh\frac{\xi_1}{2} = \tanh\frac{t_0}{2}\tanh\frac{x_1+\sqrt{\delta}\,\xi_1}{2\sqrt{\rho}} \geq \frac{\delta}{\rho}\tanh\left(\frac{\xi_1}{2}+\frac{x_1}{2\sqrt{\delta}}\right). \tag{7.2}$$

This leads to the estimate

$$(\rho-\delta)e^{\xi_1} \geq \frac{(\delta+\rho)\tanh\frac{x_1}{2\sqrt{\delta}} + \sqrt{4\delta\rho\tanh^2\frac{x_1}{2\sqrt{\delta}} + (\rho-\delta)^2}}{\tanh\frac{x_1}{2\sqrt{\delta}} + 1}.$$

Therefore,

$$(\rho-\delta)\left|e^{\Xi_{\delta,\rho,2}(z)}\right| \geq (\rho-\delta)e^{\xi_1} \geq \frac{4\delta\tanh\frac{x_1}{2\sqrt{\delta}}}{\tanh\frac{x_1}{2\sqrt{\delta}}+1},$$

and $g(z) \neq 0$ when $\Re z \geq x_1$. Combined with (7.1) yields $g(z) = 4\delta(1-e^{-\frac{z}{\sqrt{\delta}}})$ for any $z \in M$. Then

$$\lim_{n\to\infty}\Phi_{\delta,\rho_n,2}(z) = \lim_{n\to\infty}\left(\frac{\rho_n-\delta}{2\delta}\cosh\Xi_{\delta,\rho_n,2}(z) - \frac{\rho_n+\delta}{2\delta}\right) = \frac{g(z)}{4\delta} - 1 = -e^{-\frac{z}{\sqrt{\delta}}} = \Phi_2(z).$$

Because any convergent subsequence of the normal family $\{\Phi_{\delta,\rho,2}\}$ has the same limit as $\rho \to \delta$, the sequence $\{\Phi_{\delta,\rho,2}\}$ itself is convergent to $\Phi_2(z)$ in M. In addition, since m and ν are chosen arbitrarily, the set $\mathcal{M} = \mathbb{C} \setminus \bigcup_{k=-\infty}^{\infty}\{z \in \mathbb{C};\ \Re z \leq 0, \Im z = 2k\sqrt{\delta}\pi\}$ is the union of such sets as M defined above, and the sequence $\{\Phi_{\delta,\rho,2}\}$ is convergent to Φ_2 in $\mathcal{M}$.

To show the limit $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(z) = \Phi_2(z)$ exists for $z = x + i2k\sqrt{\delta}\,\pi$ and $x \leq 0$, one may take implicit differentiation with respect to ρ on both sides of the equation

$$\frac{\tanh\frac{t_0}{2} - \tanh\frac{t}{2}}{\tanh\frac{t_0}{2} + \tanh\frac{t}{2}} = e^{-\frac{z+\sqrt{\delta}t}{\sqrt{\rho}}} \tag{7.3}$$

with $t = \Xi_{\delta,\rho,2}(z)$ to obtain

$$(\rho-\delta)\frac{\partial t}{\partial\rho} = -\tanh\frac{t}{2} - \frac{z+\sqrt{\delta}\,t}{2\sqrt{\delta}}\big(\tanh^2\frac{t_0}{2} - \tanh^2\frac{t}{2}\big). \tag{7.4}$$

Then

$$\frac{\partial\Phi_{\delta,\rho,2}}{\partial\rho} = \frac{\rho-\delta}{2\delta}\frac{\partial t}{\partial\rho}\sinh t + \frac{\cosh t - 1}{2\delta} = \frac{z+\sqrt{\delta}\,t}{2\sqrt{\delta}\,\rho}\frac{\sinh t\,\Phi_{\delta,\rho,2}}{\cosh t + 1}. \tag{7.5}$$

When $z = x \leq 0$, $-t_0 < t = \Xi_{\delta,\rho,2}(x) \leq 0$, and $\Phi_{\delta,\rho,2}(x) < 0$. Hence, (7.5) shows that $\frac{\partial}{\partial\rho}\Phi_{\delta,\rho,2}(x) \leq 0$ which implies that both limits $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(x)$ and $\lim_{\rho\to\delta}(\rho-\delta)e^{-\Xi_{\delta,\rho,2}(x)}$ exist. Taking the limit on both side of the equality

$$(\rho-\delta)e^{-t} = \frac{(\rho-\delta)\big(\sqrt{\rho}-\sqrt{\delta}+(\sqrt{\rho}+\sqrt{\delta})e^{-\frac{x+\sqrt{\delta}\,t}{\sqrt{\rho}}}\big)}{\sqrt{\rho}+\sqrt{\delta}+(\sqrt{\rho}-\sqrt{\delta})e^{-\frac{x+\sqrt{\delta}\,t}{\sqrt{\rho}}}}$$

obtained from (7.3) leads to the evaluations $\lim_{\rho\to\delta}(\rho-\delta)e^{-\Xi_{\delta,\rho,2}(x)} = 4\delta(1-e^{\frac{x}{\sqrt{\delta}}})$ and $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(x) = -e^{\frac{x}{\sqrt{\delta}}}$.

If $z = x + 2k\sqrt{\delta}\,\pi i$ with $k > 0$ and $\delta < \rho < \rho_0$, then $(2k-1)\sqrt{\rho} < 2k\sqrt{\delta} < (2k-1)\sqrt{\rho} + \sqrt{\delta}$ and the function $\Xi_{\delta,\rho,2}$ maps the line $\{x + 2k\sqrt{\delta}\,\pi i;\ x \in \mathbb{R}\}$ to the right half plane with $-\pi < \Im(\Xi_{\delta,\rho,2}) = \eta < 0$ and $\frac{\sqrt{\delta}\,\Re(\Xi_{\delta,\rho,2})+x}{\sqrt{\rho}} \geq 0$. Then similar to the derivation of (7.1), one may show that the sequence $\{(\rho-\delta)e^{\Xi_{\delta,\rho,2}(z)}\}$ is bounded on the line $\{z \in \mathbb{C};\ \Im z = 2k\sqrt{\delta}\,\pi i,\ \Re z < 0\}$, and thus $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(x + 2k\sqrt{\delta}\,\pi i) = -e^{-\frac{x}{\sqrt{\delta}}}$. Because $\overline{\Phi_{\delta,\rho,2}(x + 2k\sqrt{\delta}\,\pi i)} = \Phi_{\delta,\rho,2}(x - 2k\sqrt{\delta}\,\pi i)$, $\lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(x - 2k\sqrt{\delta}\,\pi i) = -e^{-\frac{x}{\sqrt{\delta}}}$. □

Proof of Theorem 7.1. It follows from the definition of the function $\Xi_{\delta,\rho,1}(z)$ that

$$\Xi_{\delta,\rho,1}(z) = \Xi_{\delta,\rho,2}(z) \quad \text{and} \quad \lim_{\substack{z\to z_0\\ \Re z>0}} \Xi_{\delta,\rho,1}(z) = \Xi_{\delta,\rho,2}(z_0)$$

for any z and z_0 with $\Re z > 0$ and $\Re z_0 = 0$. Hence, $\lim_{\rho\to\delta}\Phi_{\delta,\rho,1}(z) = \lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(z) = -e^{-\frac{z}{\sqrt{\delta}}}$ for $\Re z > 0$ and $\lim_{\rho\to\delta}\Phi^{r}_{\delta,\rho,1}(z) = \lim_{\rho\to\delta}\Phi_{\delta,\rho,2}(z) = -e^{-\frac{z}{\sqrt{\delta}}}$ for any z with $\Re z = 0$. On the other hand, $\Xi_{\delta,\rho,1}(z) = -\overline{\Xi_{\delta,\rho,1}(-\bar{z})}$ and then $\Phi_{\delta,\rho,1}(z) = \overline{\Phi_{\delta\rho,1}(-\bar{z})}$. Therefore, one may obtain the rest of results in Theorem 7.1 symmetrically. □

Remark. Although there are also other different analytic extensions of the solitary wave solution $\varphi_{\delta,\rho}$ besides $\Phi_{\delta,\rho,1}$ and $\Phi_{\delta,\rho,2}$, all of them are identical and analytic on the strip $\{z \in \mathbb{C};\ |\Im z| < (\sqrt{\rho}-\sqrt{\delta})\pi\}$, having branch points $z = \pm(\sqrt{\rho}-\sqrt{\delta})\pi i$. It is these two singularities approaching the real axis as $\rho \to \delta^+$, which causes the formation of the singularity $z = 0$ of the peakon $\varphi_p(x) = -e^{-\frac{|x|}{\sqrt{\delta}}}$.

Next, we show that the sequence $\{\varphi_{\delta,\rho}\}$ is uniformly convergent to φ_p on the real axis, and $\{\varphi_{\delta,\rho}\}$, $\{\varphi'_{\delta,\rho}\}$ and $\{(\delta\varphi_{\delta,\rho}+\rho)\varphi''_{\delta,\rho}\}$ are L^q-convergent to φ_p, φ'_p and $\delta(\varphi_p+1)\varphi''_p$, respectively, for any $q \geq 1$. Here $(\varphi_p+1)\varphi''_p$ is defined as the difference of the two distributions $\frac{[(\varphi_p+1)^2]''}{2}$ and $(\varphi'_p)^2$ such that $(\varphi_p+1)^2 \in H^3$, $(\varphi'_p)^2 \in H^1$ and

$$\big(\varphi_p(x)+1\big)\varphi''_p(x) = \frac{e^{\frac{-|x|}{\sqrt{\delta}}}\big(e^{\frac{-|x|}{\sqrt{\delta}}}-1\big)}{\delta}$$

almost everywhere. These results help understand that the limiting equation

$$\varphi' - \delta\varphi''' + 3\varphi\varphi' - \delta(2\varphi'\varphi'' + \varphi\varphi''') = 0 \tag{7.6}$$

of the equation

$$\varphi' - \rho\varphi''' + 3\varphi\varphi' - \delta(2\varphi'\varphi'' + \varphi\varphi''') = 0 \tag{7.7}$$

has the peakon solution φ_p as the limit of solitary solutions $\varphi_{\rho,\delta}$ satisfying Equation (7.7) when $\rho \to \delta$ and $\rho > \delta > 0$. Furthermore, the combined terms $-\delta((\varphi_p + 1)\varphi_p'')'$ and $\big(\varphi_p + \frac{3}{2}\varphi_p^2 - \frac{\delta}{2}(\varphi_p')^2\big)'$ are well defined L^p-functions such that they are continuous everywhere except at $x = 0$, but the limit

$$\lim_{x\to 0}\left[\Big(\varphi_p + \frac{3}{2}\varphi_p^2 - \frac{\delta}{2}(\varphi_p')^2\Big)' - \delta((\varphi_p + 1)\varphi_p'')'\right] = 0$$

exits at this point. Thus, the peakon φ_p is another example of pseudo-classical solutions proposed in this paper.

The solution $t = \Xi_{\delta,\rho}(x) = \Xi_{\delta,\rho,i}(x)$ of Equation (7.3) satisfies the equivalent equation $\tanh\frac{t}{2} = \tanh\frac{t_0}{2}\tanh\frac{x+\sqrt{\delta}\,t}{2\sqrt{\rho}}$, which leads to the estimates

$$\tanh\frac{\sqrt{\delta}\,|t|}{2\sqrt{\rho}} \geq \frac{\delta}{\rho}\tanh\frac{|x+\sqrt{\delta}\,t|}{2\sqrt{\rho}}, \quad \text{and} \quad (\rho-\delta)\cosh\frac{x+\sqrt{\delta}\,t}{\sqrt{\rho}} \geq \frac{\delta\tanh\frac{|x|}{2\sqrt{\rho}}}{1-\tanh\frac{|x|}{2\sqrt{\rho}}}. \tag{7.8}$$

The identity

$$\Phi_{\delta,\rho,1}(z) = \Phi_{\delta,\rho,2}(z) = \frac{-2\rho}{(\rho-\delta)\cosh\frac{z+\sqrt{\delta}\,t}{\sqrt{\rho}} + \rho + \delta} \tag{7.9}$$

and (7.8) yields the inequality

$$|\varphi_{\delta,\rho}(x)| \leq \frac{2\rho}{\frac{\delta\tanh\frac{|x|}{2\sqrt{\rho}}}{1-\tanh\frac{|x|}{2\sqrt{\rho}}} + 2\delta} \leq \frac{2\rho}{\delta}\Big(1 - \tanh\frac{|x|}{2\sqrt{\rho}}\Big).$$

This implies that the functions $\varphi_{\delta,\rho}(x)$ decay exponentially to zero at infinity. Then for any $\epsilon > 0$ and for a fixed $\rho_0 > \delta$, there is an $N > 0$, when $|x| > N$ and $\delta < \rho < \rho_0$,

$$|\varphi_{\delta,\rho}(x)| \leq \epsilon. \tag{7.10}$$

Since

$$\frac{d}{dx}\frac{\varphi_{\delta,\rho}}{\varphi_p} = \frac{\varphi_{\delta,\rho}(\rho-\delta)\,\mathrm{sign}\,x}{\sqrt{\delta}\,\varphi_p\sqrt{\delta\varphi_{\delta,\rho}+\rho}\big(\sqrt{\delta\varphi_{\delta,\rho}+\rho} + \sqrt{\delta\varphi_{\delta,\rho}+\delta}\big)} \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases}$$

and $\frac{\varphi_{\delta,\rho}(0)}{\varphi_p(0)} = 1$, one has

$$0 \leq \frac{\varphi_{\delta,\rho}(x)}{\varphi_p(x)} - 1 \leq \frac{\varphi_{\delta,\rho}(N) - \varphi_p(N)}{\varphi_p(N)}$$

for any x with $|x| \leq N$. Because $\varphi_{\delta,\rho}(N) \to \varphi_p(N)$ as $\rho \to \delta$ and $|\varphi_p(x)| \leq 1$, the inequality $0 \leq \varphi_p(x) - \varphi_{\delta,\rho}(x) \leq \frac{\varphi_{\delta,\rho}(N)-\varphi_p(N)}{\varphi_p(N)} < \epsilon$ holds when ρ is sufficiently close to δ. Combined with (7.10), this shows the uniform convergence of the sequence $\{\varphi_{\delta,\rho}\}$. It follows from (7.5) that $\frac{\partial \varphi_{\delta,\rho}}{\partial \rho} < 0$, *i.e.*

$$0 \geq \varphi_p \geq \varphi_{\delta,\rho_1} \geq \varphi_{\delta,\rho_2} \tag{7.11}$$

if $\delta < \rho_1 < \rho_2$. In addition, the integral transformation $x = \varphi_{\delta,\rho}^{-1}(y)$ yields the estimate

$$\int_{-\infty}^{\infty} |\varphi_{\delta,\rho}(x)|^q dx = 2\int_0^{-1} \frac{|y|^{q-1}\sqrt{\delta y + \rho}}{\sqrt{1+y}} dy < \infty$$

for any $q > 0$. These results together with the other estimates $|\varphi'_{\delta,\rho}| \leq \frac{|\varphi_{\delta,\rho}|}{\sqrt{\delta}}$,

$$|(\delta\varphi_{\delta,\rho} + \rho)\varphi''_{\delta,\rho}| = \left| \frac{\varphi^2_{\delta,\rho}(\rho - \delta)}{2(\delta\varphi_{\delta,\rho} + \rho)} + \varphi_{\delta,\rho}(1 + \varphi_{\delta,\rho}) \right| \leq 3|\varphi_{\delta,\rho}|$$

and Lebesgue's Dominated Convergence Theorem leads to the conclusion that the limits

$$\lim_{\rho\to\delta} \int_{-\infty}^{\infty} |\varphi_p(x) - \varphi_{\delta,\rho}(x)|^q dx = 0, \quad \lim_{\rho\to\delta} \int_{-\infty}^{\infty} |\varphi'_p(x) - \varphi'_{\delta,\rho}(x)|^q dx = 0,$$
$$\lim_{\rho\to\delta} \int_{-\infty}^{\infty} |\delta(\varphi_p(x) + 1)\varphi''_p(x) - (\delta\varphi_{\delta,\rho}(x) + \rho)\varphi''_{\delta,\rho}(x)|^q dx = 0.$$

exist. We summarize the above results in the following theorem.

Theorem 7.3. *Let q be any constant with $q \geq 1$. Then as $\rho \to \delta$, the sequences of functions $\{\varphi_{\delta,\rho}\}$, $\{\varphi'_{\delta,\rho}\}$ and $\{(\delta\varphi_{\delta,\rho} + \rho)\varphi''_{\delta,\rho}\}$ converge to the peakon φ_p, its derivative φ'_p and $\delta(\varphi_p + 1)\varphi''_p$ in the Banach space L^q, respectively. Therefore, $\varphi_p \in W^{1,q}$ for any $q \geq 1$. In addition, $\{\varphi_{\delta,\rho}\}$ converges to φ_p uniformly on the real axis and for any positive integer n, the derivatives $\varphi^{(n)}_{\delta,\rho}$ converge to $\varphi^{(n)}_p$ uniformly on any compact set contained in $(-\infty, 0) \cup (0, \infty)$, and*

$$\begin{aligned}\varphi_p(0) + 1 &= \lim_{x\to 0^-} \left(\varphi'_p(x) + 3\varphi_p(x)\varphi'_p(x) - 2\delta\varphi'_p(x)\varphi''_p(x)\right) \\ &= \lim_{x\to 0^+} \left(\varphi'_p(x) + 3\varphi_p(x)\varphi'_p(x) - 2\delta\varphi'_p(x)\varphi''_p(x)\right) = 0.\end{aligned}$$

In consequence, the peakon solutions φ_p and $\phi_p = \frac{\beta - \nu\alpha}{2} - [\frac{3}{2}(\beta + c\nu) - \frac{\nu}{2}(\alpha + c)]\varphi_p$ satisfy Equations (7.6) and (3.7) on the intervals $(-\infty, 0) \cup (0, \infty)$, having limits at $x = 0$, respectively.

8. Convergence to solitary wave solutions of the KdV equation.

In this section, we examine behavior of solitary wave solutions of Equation (4.1) or (5.1) as functions defined in the complex plane when nonlinear dispersion terms are vanishing or the parameter $\delta \to 0$. We shall demonstrate that these functions converge to solitary wave solutions of the KdV equation on the real axis, but not necessarily on the entire complex plane, due to the different definition of their branch lines.

Theorem 8.1. *Let $\Phi_{\epsilon,\delta,0}(z)$ be the solitary wave solution of Equation (4.1) as defined in Theorem 6.1, and for any $z_0 \in \{z \in \mathbb{C};\ |\Im z| = \sqrt{\epsilon}\,\pi\}$, let*

$$\Phi^{+}_{\epsilon,\delta,0}(z_0) = \lim_{\substack{z\to z_0\\ |\Im z|<\sqrt{\epsilon}\,\pi}} \Phi_{\epsilon,\delta,0}(z) \quad \text{and} \quad \Phi^{-}_{\epsilon,\delta,0}(z_0) = \lim_{\substack{z\to z_0\\ |\Im z|>\sqrt{\epsilon}\,\pi}} \Phi_{\epsilon,\delta,0}(z).$$

Then the following limit

$$\lim_{\delta\to 0} \Phi_{\epsilon,\delta,0}(z) = \begin{cases} \operatorname{sech}^2 \dfrac{z}{2\sqrt{\epsilon}}, & |\Im z| < \sqrt{\epsilon}\,\pi \\ \infty, & |\Im z| > \sqrt{\epsilon}\,\pi, \end{cases}$$

exists, and the limits

$$\lim_{\delta\to 0} \Phi^{+}_{\epsilon,\delta,0}(z) = \operatorname{sech}^2 \frac{z}{2\sqrt{\epsilon}} \quad \text{and} \quad \lim_{\delta\to 0} \Phi^{-}_{\epsilon,\delta,0}(z) = \infty$$

hold for any z with $|\Im z| = \sqrt{\epsilon}\,\pi$.

Proof. Let $\nu > 0$ be any fixed constant. It follows from Lemma 4.1 that

$$\inf_{|\Im z|\geq(\sqrt{\epsilon}+\nu)\pi} |\Im(\Theta_{\epsilon,\delta,0}(z))| = |\Im(\Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi + i(\sqrt{\epsilon}+\nu)\pi))| = \eta_\nu,$$

and $\Theta_{\epsilon,\delta,0}(\sqrt{\delta}\,\pi + i(\sqrt{\epsilon}+\nu)\pi) = -\frac{\pi}{2} - i\eta_\nu$, such that

$$\frac{\cos\theta_0 \sinh\eta_\nu}{\sin\theta_0\cosh\eta_\nu - 1} = \tan\frac{(\sqrt{\epsilon}+\nu)\pi - \sqrt{\delta}\,\eta_\nu}{\sqrt{\epsilon}}$$

with $0 \leq (\sqrt{\epsilon}+\nu)\pi - \sqrt{\delta}\,\eta_\nu \leq \sqrt{\epsilon}\,\pi$. Equivalently, $\frac{\nu\pi}{\sqrt{\delta}} \leq \eta_\nu \leq \frac{(\nu+\sqrt{\epsilon})\pi}{\sqrt{\delta}}$, which implies that $\lim\limits_{\delta\to 0}\eta_\nu = \infty$. Since $\Phi_{\epsilon,\delta,0} = \frac{\delta+\epsilon}{2\delta}(\sin\Theta_{\epsilon,\delta,0} + \sin\theta_0)$,

$$|\Phi_{\epsilon,\delta,0}(z)| = \frac{\delta+\epsilon}{2\delta}\sqrt{(\cosh\eta + \cos(\xi-\theta_0))(\cosh\eta - \cos(\xi+\theta_0))} \geq \frac{\delta+\epsilon}{2\delta}(\cosh\eta - 1),$$

where $\eta = \Im(\Theta_{\epsilon,\delta,0}(z))$ and $\xi = \Re(\Theta_{\epsilon,\delta,0}(z))$ for any $z \in \mathbb{C}$. When $|\Im z| \geq (\sqrt{\epsilon}+\nu)\pi$, $|\Phi_{\epsilon,\delta,0}(z)| \geq \frac{\delta+\epsilon}{2\delta}(\cosh\eta_\nu - 1) \to \infty$ as $\delta \to 0$. Because $\nu > 0$ is chosen arbitrarily, the limit $\lim\limits_{\delta\to 0}\Phi_{\epsilon,\delta,0}(z) = \infty$ holds for any z with $|\Im z| > \sqrt{\epsilon}\,\pi$.

To show $\lim\limits_{\delta\to 0}\Phi^-_{\epsilon,\delta,0}(z) = \infty$ for any z with $|z| = \sqrt{\epsilon}\,\pi$, one uses the streamline (4.8)

$$\sin\xi = -\sin\theta_0\cosh\eta - \cos\theta_0\sinh\eta\cot\frac{\sqrt{\delta}\,\eta}{\sqrt{\epsilon}}.$$

Since $|\eta| \leq \eta_\delta$, where η_δ satisfies the equation

$$\sqrt{\epsilon}\,\pi = \sqrt{\delta}\,\eta_\delta + 2\sqrt{\epsilon}\tan^{-1}\Big(\sqrt{\frac{\epsilon}{\delta}}\tanh\frac{\eta_\delta}{2}\Big)$$

and has the limit $\lim\limits_{\delta\to 0}\eta_\delta = \eta_0$ with $\frac{\eta_0}{2}\tanh\frac{\eta_0}{2} = 1$, $|\Im(\Theta_{\epsilon,\delta,0}(z))|$ are uniformly bounded for all z with $|\Im z| = \sqrt{\epsilon}\,\pi$. Equation (4.7) then implies $\lim\limits_{\delta\to 0}(x + \sqrt{\delta}\,(\xi - \frac{\pi}{2})) = 0$. Taking the limit on both sides of the equality

$$\Phi_{\epsilon,\delta,l}(z) = \frac{2\epsilon}{(\delta+\epsilon)\cosh\frac{z+\sqrt{\delta}\,(\Theta_{\epsilon,\delta,l}(z)-\frac{\pi}{2})}{\sqrt{\epsilon}} - \delta + \epsilon}, \quad l = 0, 1 \tag{8.1}$$

for $l = 0$ as $\delta \to 0$, one obtains

$$\lim_{\delta\to 0}|\Phi_{\epsilon,\delta,0}(z)| = \lim_{\delta\to 0}\frac{2\epsilon}{\left|-(\delta+\epsilon)\cosh\frac{x+\sqrt{\delta}\,(\xi-\frac{\pi}{2})+\sqrt{\delta}\,\eta i}{\sqrt{\epsilon}} - \delta + \epsilon\right|} = \infty.$$

The limits $\lim\limits_{\delta\to 0}\Phi_{\epsilon,\delta,0}(z) = \frac{1}{\cosh^2\frac{z}{2\sqrt{\epsilon}}}$ for $|\Im z| < \sqrt{\epsilon}\,\pi$ and $\lim\limits_{\delta\to 0}\Phi^+_{\epsilon,\delta,0}(z) = \frac{1}{\cosh^2\frac{z}{2\sqrt{\epsilon}}}$ are direct consequences of the identities $\Phi_{\epsilon,\delta,0}(z) = \Phi_{\epsilon,\delta,1}(z)$ for any z with $|\Im z| < \sqrt{\epsilon}\,\pi$ and $\Phi^+_{\epsilon,\delta,0}(z) = \Phi_{\epsilon,\delta,1}(z)$ for any z with $|\Im z| = \sqrt{\epsilon}\,\pi$, as well as the limit $\lim\limits_{\delta\to 0}\Phi_{\epsilon,\delta,1}(z) = \frac{1}{\cosh^2\frac{z}{2\sqrt{\epsilon}}}$ for any $z \in \mathbb{C}$, which will be verified in the next theorem. □

Theorem 8.2. *Let $\Phi_{\epsilon,\delta,1}(z)$ be the solitary wave solution as defined in Theorem 6.2, satisfying Equation (4.1) on the domain $Y_{\epsilon,\delta,1}$. Then $\lim\limits_{\delta\to 0}\Phi_{\epsilon,\delta,1}(z) = \operatorname{sech}^2\frac{z}{2\sqrt{\epsilon}}$ for any $z \in \mathbb{C}$.*

Proof. Let μ be any fixed constant with $0 < \mu < \frac{\sqrt{\epsilon}\,\pi}{2}$, and let

$$\mathcal{M}_\mu = \mathbb{C} \setminus \bigcup_{k=-\infty}^{\infty}\{z \in \mathbb{C};\ |\Im z - (2k+1)\sqrt{\epsilon}\,\pi| < \mu, |\Re z| < \mu\}.$$

Since $\Theta_{\epsilon,\delta,1}$ is a periodic function with the period $T = 2\sqrt{\epsilon}\,\pi i$, $\Theta_{\epsilon,\delta,1}$ maps $\mathcal{M}_\mu$ to the set as shown in Figure 9, which is bounded by the streamlines $y_1 = \sqrt{\epsilon}\,\pi - \mu = y(\xi,\eta)$ and

$y_2 = \sqrt{\epsilon}\,\pi + \mu = y(\xi,\eta)$, and the equipotentials $x_1 = \mu = x(\xi,\eta)$ and $-x_1 = -\mu = x(\xi,\eta)$ of the single-valued branch Σ_0. Because

$$\sup_{z\in\mathcal{M}_\mu} \left|\Im\big(\Theta_{\epsilon,\delta,1}(z)\big)\right| = \left|\Im\big(\Theta_{\epsilon,\delta,1}(i(\sqrt{\epsilon}\,\pi-\mu))\big)\right| = \eta_\mu,$$

and $\Theta_{\epsilon,\delta,1}\big(i(\sqrt{\epsilon}\,\pi-\mu)\big) = \frac{\pi}{2} - i\eta_\mu$, it follows from (4.5) that

$$\sqrt{\epsilon}\,\pi - \mu = \sqrt{\delta}\,\eta_\mu + 2\sqrt{\epsilon}\,\tan^{-1}\Big(\sqrt{\frac{\epsilon}{\delta}}\,\tanh\frac{\eta_\mu}{2}\Big).$$

In other words, $\sqrt{\frac{\delta}{\epsilon}}\,\tan\frac{\sqrt{\epsilon}\,\pi-\mu-\sqrt{\delta}\,\eta_\mu}{2\sqrt{\epsilon}} = \tanh\frac{\eta_\mu}{2}$ with $0 < \sqrt{\delta}\,\eta_\mu < \sqrt{\epsilon}\,\pi-\mu$, which implies $\lim\limits_{\delta\to 0}\eta_\mu = 0$. In addition, $\left|\Re\big(\Theta_{\epsilon,\delta,1}(z)\big)\right| \le \frac{3\pi}{2}$. Therefore, taking the limit on both sides of Equality (8.1) for $l=1$ as $\delta\to 0$, one obtains

$$\lim_{\delta\to 0}\Phi_{\epsilon,\delta,1}(z) = \frac{2\epsilon}{\epsilon(\cosh\frac{z}{\sqrt{\epsilon}}+1)} = \operatorname{sech}^2\frac{z}{2\sqrt{\epsilon}},$$

for any $z\in\mathcal{M}_\mu$. Since $\mu>0$ is chosen arbitrarily small, the above limit holds for any $z\in\mathbb{C}\setminus\{(2k+1)\sqrt{\epsilon}\,\pi i;\ k=0,\pm1,\pm2,\cdots\}$. To show $\lim\limits_{\delta\to 0}\Phi_{\epsilon,\delta,1}((2k+1)\sqrt{\epsilon}\,\pi i) = \infty$, one uses the identities $\Re\big(\Theta_{\epsilon,\delta,1}((2k+1)\sqrt{\epsilon}\,\pi i)\big) = \frac{\pi}{2}$, $\left|\Im\big(\Theta_{\epsilon,\delta,1}((2k+1)\sqrt{\epsilon}\,\pi i)\big)\right| = \eta_\epsilon$, and $\sqrt{\frac{\delta}{\epsilon}}\,\cot\sqrt{\frac{\delta}{\epsilon}}\,\frac{\eta_\epsilon}{2} = \tanh\frac{\eta_\epsilon}{2}$ with $0\le\sqrt{\delta}\,\eta_\epsilon\le\sqrt{\epsilon}\,\pi$. It follows that $\lim\limits_{\delta\to 0}\eta_\epsilon = \eta_0$ exists and $\frac{\eta_0}{2}\tanh\frac{\eta_0}{2} = 1$. Then taking the limit on both sides of Equality (8.1) for $l=1$ yields

$$\lim_{\delta\to 0}\left|\Phi_{\epsilon,\delta,1}((2k+1)\sqrt{\epsilon}\,\pi i)\right| = \lim_{\delta\to 0}\frac{2\epsilon}{\left|(\delta+\epsilon)\cosh\frac{(2k+1)\sqrt{\epsilon}\,\pi i\pm\sqrt{\delta}\,\eta_\epsilon i}{\sqrt{\epsilon}} - \delta+\epsilon\right|} = \infty$$

for any integer k. □

Theorem 8.3. *Let $\Phi_{\delta,\rho,1}$ be the solitary wave solution as defined in Section 5.2, satisfying Equation (5.1) on the domain $\mathcal{Y}_{\delta,\rho,1}$. Then $\lim\limits_{\delta\to 0}\Phi_{\delta,\rho,1}(z) = -\operatorname{sech}^2\frac{z}{2\sqrt{\rho}}$ for any $z\in\mathbb{C}$.*

Proof. Let ν be any fixed constant with $0<\nu<\sqrt{\rho}\,\pi$, and let

$$\mathcal{M}_\nu = \mathbb{C}\setminus\bigcup_{k=-\infty}^{\infty}\{z\in\mathbb{C};\ |\Im z-(2k+1)\sqrt{\rho}\,\pi|<\nu, |\Re z|<\nu\}.$$

Then $\Xi_{\delta,\rho,1}$ maps $\mathcal{M}_\nu$ to the region bounded by the streamlines $y_0 = \sqrt{\rho}\,\pi-\nu = y(\xi,\eta)$ and $y_1 = \sqrt{\rho}\,\pi+\nu = y(\xi,\eta)$, and the equipotentials $x_1 = \nu = x(\xi,\eta)$ and $-x_1 = -\nu = x(\xi,\eta)$ of the single-valued branch Δ_0, as illustrated in Figure 12. It follows from Lemma 5.1 and (5.3) that

$$\sup_{z\in\mathcal{M}_\nu}\left|\Re\big(\Xi_{\delta,\rho,1}(z)\big)\right| = \Xi_{\delta,\rho,1}(\nu+\sqrt{\rho}\,\pi i) = \xi_\nu,$$

and

$$-\nu-\sqrt{\rho}\,\pi i=\sqrt{\delta}\,\xi_\nu+\sqrt{\rho}\,\log\frac{\tanh\frac{t_0}{2}-\tanh\frac{\xi_\nu}{2}}{\tanh\frac{t_0}{2}+\tanh\frac{\xi_\nu}{2}}.$$

Thus $\tanh\frac{\xi_\nu}{2}=\tanh\frac{t_0}{2}\coth\frac{\sqrt{\delta}\,\xi_\nu+\nu}{2\sqrt{\rho}}$, which leads to the estimate

$$\begin{aligned}e^{\xi_\nu}&\le\frac{1}{\big(1-\sqrt{\frac{\delta}{\rho}}\big)^{\sqrt{\frac{\rho}{\delta}}}}\left(1+\frac{\sqrt{\frac{\delta}{\rho}}+\sqrt{4\sqrt{\frac{\delta}{\rho}}+\big(1-\sqrt{\frac{\delta}{\rho}}\big)^2\tanh^2\frac{\nu}{2\sqrt{\rho}}}-\tanh\frac{\nu}{2\sqrt{\rho}}}{1+\tanh\frac{\nu}{2\sqrt{\rho}}}\right)^{\sqrt{\frac{\rho}{\delta}}}\\&\xrightarrow{\delta\to 0} e^{\frac{2}{\tanh\frac{\nu}{2\sqrt{\rho}}}}.\end{aligned}$$

Therefore, there is a $\delta_0>0$ such that

$$\sup_{z\in\mathcal{M}_\nu}|\Xi_{\delta,\rho,1}(z)|\le\sup_{0<\delta\le\delta_0}(\xi_\nu+\pi)<\infty.$$

Then taking the limit on both sides of Equality (7.9) for $j=1$ leads to the convergence

$$\lim_{\delta\to 0}\Phi_{\delta,\rho,1}(z)=\lim_{\delta\to 0}\frac{-2\rho}{(\rho-\delta)\cosh\frac{z+\sqrt{\delta}\,\Xi_{\delta,\rho,1}(z)}{\sqrt{\rho}}+\rho+\delta}=-\operatorname{sech}^2\frac{z}{2\sqrt{\rho}}.$$

for any $z\in\mathcal{M}_\nu$. Since $\nu>0$ is arbitrary, the above limit exists for any $z\in\mathbb{C}\setminus\{z=(2k+1)\sqrt{\rho}\,\pi i;\ k=0,\pm1,\pm2,\cdots\}$.

When $z=(2k+1)\sqrt{\rho}\,\pi i$ for some integer k, $|\Xi_{\delta,\rho,1}(z)|=|\Re(\Xi_{\delta,\rho,1}(z))|=\xi_\delta$, where ξ_δ is the intersection point of the equipotential $0=x(\xi,\eta)$ of Δ_0 and the real axis. It follows from (5.3) that $\tanh\frac{\xi_\delta}{2}=\sqrt{\frac{\delta}{\rho}}\coth\sqrt{\frac{\delta}{\rho}}\frac{\xi_\delta}{2}$, and thus $\lim_{\delta\to 0}\xi_\delta=\xi_0$ exists with $\frac{\xi_0}{2}\tanh\frac{\xi_0}{2}=1$. Taking the limit on both sides of Equality (7.9) for $j=1$ and $z=(2k+1)\sqrt{\rho}\,\pi i$ again, one obtains

$$\lim_{\delta\to 0}|\Phi_{\delta,\rho,1}(z)|=\lim_{\delta\to 0}\frac{2\rho}{\left|(\rho-\delta)\cosh\frac{(2k+1)\sqrt{\rho}\,\pi i\pm\sqrt{\delta}\,\xi_\delta}{\sqrt{\rho}}+\rho+\delta\right|}=\infty.$$

This completes the proof. □

Remark. The nonlinear integrable Equation (1.1) has provided an interesting example to show how singularities of solitary wave solutions change their nature when the nonlinear dispersion terms vanish. Both functions $\Phi_{\epsilon,\delta,1}$ and $\Phi_{\delta,\rho,1}$ have movable branch points of order three. When $\delta\to 0$, each pair of the singularities $\pm\sqrt{\delta}\,\pi+i(2n+1)\sqrt{\epsilon}\,\pi$, for $n=$

$0, \pm 1, \pm 2, \ldots$, of the function $\Phi_{\epsilon,\delta,1}$ becomes closer and closer, and the points eventually collide with each other to form a movable pole of a solitary wave solution to the KdV equation. The same situation also happens to each pair of singularities $i[(2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,]\pi$ of the function $\Phi_{\delta,\rho,1}$ as $\delta \to 0$. Whereas the branch points $i[(2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,]\pi$ and the branch lines $\{z = x + i[(2n+1)\sqrt{\rho} \pm \sqrt{\delta}\,]\pi;\ x \le 0\}$ of the function $\Phi_{\delta,\rho,2}$ collide such that the limiting function becomes infinity on the line $\{z = x + i(2n+1)\sqrt{\rho}\,\pi;\ x \le 0\}$, for $n = 0, \pm 1, \pm 2, \ldots$, as δ reaches zero.

Theorem 8.4. *Let $\Phi_{\delta,\rho,2}$ be the solitary wave solution satisfying Equation (5.1) on the domain $\mathcal{Y}_{\delta,\rho,2}$ as defined in Section 7. Then*

$$\lim_{\delta\to 0} \Phi_{\delta,\rho,2}(z) = \begin{cases} \infty, & \textit{if } \Im z = (2n+1)\sqrt{\rho}\,\pi \textit{ and } \Re z \le 0 \\ -\operatorname{sech}^2 \dfrac{z}{2\sqrt{\rho}}, & \textit{otherwise} \end{cases}$$

for integers $n = 0, \pm 1, \pm 2 \cdots$.

Proof. It follows from the definition of $\Phi_{\delta,\rho,1}$ and $\Phi_{\delta,\rho,2}$ in Section 4 that $\Phi_{\delta,\rho,1}(z) = \Phi_{\delta,\rho,2}(z)$ for any

$$z \in \mathbb{C} \setminus \bigcup_{k=-\infty}^{\infty} \{z \in \mathbb{C};\ \Re z < 0, \big((2k+1)\sqrt{\rho} - \sqrt{\delta}\,\big)\pi < \Im z < \big((2k+1)\sqrt{\rho} + \sqrt{\delta}\,\big)\pi\}.$$

Therefore, for any z with $\Re z \ge 0$, or $z \ne (2k+1)\sqrt{\rho}\,\pi i$ and $\Re z \le 0$, whenever δ is sufficiently small, the equality $\Phi_{\delta,\rho,1}(z) = \Phi_{\delta,\rho,2}(z)$ holds. Hence, $\lim\limits_{\delta\to 0} \Phi_{\delta,\rho,2}(z) = \lim\limits_{\delta\to 0} \Phi_{\delta,\rho,1}(z) = -\operatorname{sech}^2 \frac{z}{2\sqrt{\rho}}$.

If $z = x + (2n+1)\sqrt{\rho}\,\pi i$ for some integer n and $x < 0$, then $\Xi_{\delta,\rho,2}(z) = \xi_x > 0$ and $\tanh \frac{\xi_x}{2} = \tanh \frac{t_0}{2} \coth \frac{\sqrt{\delta}\,\xi_x + x}{2\sqrt{\rho}}$. It follows that $\xi_x \xrightarrow{\delta\to 0} \infty$, and $\sqrt{\delta}\,\xi_x + x \xrightarrow{\delta\to 0} 0$. Taking the limit on both sides of Equality (7.9) for $j = 2$ and $z = x + (2n+1)\sqrt{\rho}\,\pi i$, one obtains

$$\lim_{\delta\to 0} |\Phi_{\delta,\rho,2}(z)| = \lim_{\delta\to 0} \frac{2\rho}{\left|(\rho - \delta)\cosh \frac{(2n+1)\sqrt{\rho}\,\pi i + \sqrt{\delta}\,\xi_x + x}{\sqrt{\rho}} + \rho + \delta\right|} = \infty.$$

□

Remark. It is worth observing how the mass of the solitary wave solution changes as $\delta \to 0$, when their definition is restricted to the real axis . The computation on partial derivatives of $\varphi_{\epsilon,\delta}$ and $\varphi_{\delta,\rho}$ with respect to δ provides the following expressions,

$$\frac{\partial \varphi_{\epsilon,\delta}(x)}{\partial \delta} = \frac{(\delta + \epsilon)(\frac{\pi}{2} - \theta)\cos\theta(\sin\theta + \sin\theta_0)}{4\delta^2(1 + \sin\theta)}$$

and

$$\frac{\partial \varphi_{\delta,\rho}(x)}{\partial \delta} = \frac{\rho\, t \sinh t}{4\delta^2}\big(\tanh^2 \frac{t_0}{2} - \tanh^2 \frac{t}{2}\big),$$

where $\theta = \Theta_{\epsilon,\delta,0}(x) = \Theta_{\epsilon,\delta,1}(x)$, $t = \Xi_{\delta,\rho,1}(x) = \Xi_{\delta,\rho,2}(x)$ for any $x \in \mathbb{R}$. Since $-\frac{\pi}{2} < -\theta_0 \le \theta \le \frac{\pi}{2}$ when $x \le 0$ and $\frac{\pi}{2} \le \theta \le \pi + \theta_0 < \frac{3\pi}{2}$ if $x \ge 0$, $\frac{\partial \varphi_{\epsilon,\delta}(x)}{\partial \delta} \ge 0$ for any $x \in \mathbb{R}$. While $|t| \le t_0$ for all $x \in \mathbb{R}$, and thus $\frac{\partial \varphi_{\delta,\rho}(x)}{\partial \delta} \ge 0$. In consequence, for any $x \in \mathbb{R}$, the following inequalities

$$\operatorname{sech}^2 \frac{x}{2\sqrt{\epsilon}} \le \varphi_{\epsilon,\delta_1}(x) \le \varphi_{\epsilon,\delta_2}(x) \quad \text{and} \quad -\operatorname{sech}^2 \frac{x}{2\sqrt{\rho}} \le \varphi_{\delta_1,\rho}(x) \le \varphi_{\delta_2,\rho}(x) < 0$$

hold if $\delta_1 < \delta_2$. Therefore, the solitary wave solutions of the KdV equation attains the minimum mass $\inf\limits_{\delta}\{\int_{-\infty}^{\infty} \varphi_{\epsilon,\delta}(x)dx\}$ among those of Equation (4.1); whereas it attains the maximum mass $\sup\limits_{\delta}\{\int_{-\infty}^{\infty} |\varphi_{\delta,\rho}(x)|dx\}$ among solitary wave solutions of Equation (5.1).

9. Conclusion. It is worth comparing solitary wave solutions of Equation (1.1) with those of the perturbed evolution equation

$$u_t + \epsilon u_{xxx} + (u^2)_x + \delta(u^2)_{xxx} = 0 \tag{9.1}$$

whose travelling wave solutions, including the compacton

$$u(x-\lambda t) = \begin{cases} \frac{4\lambda}{3}\cos^2 \frac{x-\lambda t}{4\sqrt{\delta}}, & |x-\lambda t| \le 2\sqrt{\delta}\,\pi \\ 0, & |x-\lambda t| > 2\sqrt{\delta}\,\pi \end{cases}$$

have been studied by Rosenau and Hyman [2] when $\epsilon = 0$ and $\delta = 1$. Substituting a travelling wave solution $u(x,t) = \phi(x-ct)$ into Equation (9.1) for some constant $c > 0$, one obtains the equation

$$-c\phi' + \epsilon\phi''' + (\phi^2)' + \delta(\phi^2)''' = 0. \tag{9.2}$$

For $\epsilon, \delta >$, this is a counterpart to Equation (4.1), and has an analytic solitary wave solution decaying exponentially to zero at infinity. Equation (9.2) may be studied in a way similar to that we have used to deal with Equation (3.7), *i.e.* reducing Equation (9.2) by integration to obtain the equation

$$(\phi')^2 = \frac{\delta\phi^2(y_2-\phi)(\phi-y_1)}{(2\delta\phi+\epsilon)^2}. \tag{9.3}$$

We then apply the transformation $\phi(x) = \frac{y_2 - y_1}{2}(\sin\theta(x) + \sin\theta_0)$, where

$$y_1 = \frac{2c\delta - \epsilon - \sqrt{(2c\delta - \epsilon)^2 + 9\delta c\epsilon}}{3\delta}, \qquad y_2 = \frac{2c\delta - \epsilon + \sqrt{(2c\delta - \epsilon)^2 + 9\delta c\epsilon}}{3\delta},$$

and $\sin\theta_0 = \frac{y_2 + y_1}{y_2 - y_1}$ and $|\theta_0| < \pi/2$. Integrating the resulting equation to derive the expression

$$-x = 2\sqrt{\delta}\left(\theta - \frac{\pi}{2}\right) + \sqrt{\frac{\epsilon}{c}} \log \frac{\tan\frac{\theta}{2} + \tan\frac{\theta_0}{2}}{1 + \tan\frac{\theta_0}{2}\tan\frac{\theta}{2}}, \tag{9.4}$$

which implicitly determines θ as a function of x.

Although Equations (9.4) and (4.3) look quite similar, there are still differences between corresponding solitary wave solutions in these two systems. Extending the solution $\theta(x)$ of (9.4) to the complex plane, one finds that the extension has singularities as movable branch points of order two, taking the form

$$z_n^{\pm} = \pm\sqrt{\delta}\left((4k+1)\pi + 2\tilde{\theta} - \sqrt{\frac{\epsilon}{\delta c}} \log \frac{\sin\frac{\tilde{\theta} - \theta_0}{2}}{\cos\frac{\tilde{\theta} + \theta_0}{2}}\right) + \sqrt{\frac{\epsilon}{c}}(2n+1)\pi i$$

for some integers k and n. Here $\sin\tilde{\theta} = \sin\theta_0 + \frac{\sqrt{\epsilon}}{2\sqrt{\delta c}}\cos\theta_0$ and $0 < \tilde{\theta} < \pi/2$. The solitary wave solution ϕ of Equation (9.2), when extended to the complex plane, has a Puiseux series expansion near each branch point z_n, taking the form

$$\phi(z) = -\frac{\epsilon}{2\delta} + \sum_{k=1}^{\infty} a_k (z - z_n)^{\frac{k}{2}} = -\frac{\epsilon}{2\delta} + \frac{\sqrt{\epsilon}\,\sqrt[4]{\epsilon(\epsilon + 4\delta c)}}{2\delta\sqrt[4]{3\delta}}(z - z_n)^{\frac{1}{2}} + \cdots.$$

In contrast, any extension of the solution $\theta(x)$ to Equation (4.3), and in consequence, the corresponding solitary wave solution Φ of (4.1) have branch points of order three, taking simpler expressions as $(2k+1)\sqrt{\delta}\,\pi + \sqrt{\epsilon}\,(2n+1)\pi i$ for some integers k and n. Near each movable branch point z_0, the Puiseux series of the function Φ takes the form

$$\Phi(z) = -\frac{\epsilon}{\delta} + \sum_{k=1}^{\infty} b_k (z - z_0)^{\frac{2k}{3}} = -\frac{\epsilon}{\delta} + \frac{1}{\delta}\left(\frac{9\epsilon^2(\epsilon + \delta)}{4\delta}\right)^{\frac{1}{3}} (z - z_0)^{\frac{2}{3}} + \cdots.$$

Another difference is that the derivative of solitary wave solution of Equation (9.2) has more zeros than that of the solitary wave solution to Equation (4.1). It follows from (9.3), (9.4) and the transformation $\phi = \frac{y_2 - y_1}{2}(\sin\theta + \sin\theta_0)$ that when $\phi(x) = y_2$, $\phi'(x) = 0$, $\theta(x) = 2k\pi + \pi/2$ and $x = 4k\pi\sqrt{\delta} + 2n\sqrt{\frac{\epsilon}{c}}i$; whereas if $\phi(x) = y_1$, then $\phi'(x) = 0$, $\sin\theta(x) = -1$ and $x = 2\sqrt{\delta}\,(2k+1)\pi + \sqrt{\frac{\epsilon}{c}}(2n+1)\pi i$ for some integers k and n, which are located on the same horizontal lines as branch points $z_n^{\pm}$ of the function ϕ. Nevertheless, solitary wave solutions to either Equation (9.2) or Equation (4.1) converge to the compacton of the corresponding limiting equation as $\epsilon \to 0$, and they converge to the solitary wave solution of the corresponding limiting KdV equation as $\delta \to 0$.

Included in our further studies are also singularities of solitary wave solutions to both Equation (5.1) and its counterpart equation

$$\phi' - \rho\,\phi''' + (\phi^2)' - \delta(\phi^2)''' = 0 \tag{9.5}$$

whose solitary wave solution exists when $\rho > 4\delta > 0$ and converges to the peakon $\phi_p = -2e^{\frac{-|x|}{2\sqrt{\delta}}}$ as $\rho \to 4\delta$. Extensions of solitary wave solutions of Equation (9.5) to the complex plane have branch points of order two, having different structure from that of branch points of solitary wave solutions to Equation (5.1). Our inquiry is to understand how different nonlinear dispersion terms in these evolution equations influence behavior of their solitary wave solutions and other travelling wave solutions. As we have also mentioned in *Case II* of Section 3 that when $a \in (-\infty, -\beta - c\nu) \cup (\frac{\beta-\nu\alpha}{2}, \infty)$, System (3.7) has a cuspon at the fixed point $(a, 0, 0)$. That means there exists a function of the form $a + \phi_a(x)$ such that $\phi_a(x)$ is an even, continuous function, approaching zero at both $-\infty$ and ∞, and infinitely differentiable everywhere except at $x = 0$ at which the graph of ϕ_a has a cusp, *i.e.* $\lim\limits_{|x|\to 0} |\phi_a'(x)| = \infty$. In addition, $\phi_a(x)$ satisfies Equation (3.11) everywhere except at $x = 0$ and the trajectory $(\phi_a(x), \phi_a'(x))$ is an orbit in the stable manifold of System (3.11) at the origin when $x > 0$, while the trajectory $(\phi_a(x), \phi_a'(x))$ is an orbit in the unstable manifold of System (3.11) at the origin for $x < 0$. The function ϕ_a may be obtained implicitly in the following way. If $a \in (\frac{\beta-\nu\alpha}{2}, \infty)$, one may adopt a procedure similar to that used in the beginning of Section 4.2 to reduce (3.11) to (5.1). Since $0 < \rho < \delta$ when $a > \frac{\beta-\nu\alpha}{2}$, one needs to use the transformation $\varphi = \frac{\delta-\rho}{2\delta}\cosh t - \frac{\rho+\delta}{2\delta}$ to Equation (5.1) to derive the expression (5.3) with the parameter t_0 defined as $\cosh t_0 = \frac{\delta+\rho}{\delta-\rho}$ and the derivative expressed by $\frac{dx}{dt} = \frac{\sqrt{\delta}\,(1-\cosh t)}{\cosh t - \cosh t_0}$. Therefore, Equation (5.3) still determines $t = t(x)$ as a continuous function of x defined on the real axis implicitly, but having a singularity at $x = 0$. In consequence, $\phi_a(x) = [3a + \nu(\alpha + c)]\big(\frac{\delta-\rho}{2\delta}\cosh t(x) - \frac{\rho+\delta}{2\delta}\big)$ is the function we have described above. In a similar way, one may obtain the continuous function ϕ_a implicitly for $a < -\beta - c\nu$ in *Case II* or for $a \neq -\beta - c\nu$ in *Case IV* as discussed in Section 3, such that ϕ_a also has the character of possessing a singularity at $x = 0$ in the form of a cusp in its graph and thus called a cuspon. Since these cuspons also have Puiseux series expansion of the form $\sum\limits_{k=0}^{\infty} a_k x^{2k/3}$ near their singularity $x = 0$

and ϕ_a exponentially to zero at infinity, they are weak solutions of Equation (3.7) in the sense as defined in Definition 3.1. The singularities we have discussed above are caused by the nonlinear dispersion terms in Equation (1.1), a different feature from travelling wave solutions of the KdV equation and worth being studied further.

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