ALGORITHMIC DETERMINATION OF STRUCTURE OF INFINITE LIE PSEUDOGROUPS OF SYMMETRIES OF PDES

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Abstract

We describe a method which uses a finite number of differentiations and linear operations to determine the Cartan structure coefficients of a structurally transitive Lie pseudogroup from its infinitesimal defining equations. If the defining system is of first order and the pseudogroup has no scalar invariants, the structure coefficients can be simply extracted from the coefficients of the infinitesimal system. We give an algorithm which reduces the higher order case to the first order case. The reduction process uses only differentiation and linear eliminations, for which several well-known algorithms are available. Our method makes feasible the calculation of the Cartan structure of infinite Lie pseudogroups of symmetries of differential equations. Examples including the KP equation and Liouville's equation are given.

1 INTRODUCTION

This paper is one of a series in which we investigate the determination of structure of infinite Lie pseudogroups. The main results from the preprint [16] are presented in the present paper. Detailed proofs given in [16] will be published elsewhere. The objects we study are *infinite Lie pseudogroups,* which are an infinite-dimensional generalization of the concept of a Lie group. Techniques and applications of Lie symmetry methods for PDEs can be found in [1, 19].

Consider a system of PDEs in the m variables $x = (x^1, x^2, \ldots, x^m)$, each x^i being either an independent or a dependent variable. For example, for the heat equation $u_{yy} = u_t$, we have $x = (y, t, u)$. A Lie (point) symmetry operator for such a system is a differential operator of the form

$$\mathcal{L} = \xi^i(x)\frac{\partial}{\partial x^i} \tag{1}$$

whose associated flow maps every solution of the PDEs to another solution. (We suppress the summation sign over repeated indices.) The *infinitesimals* ξ^i of a symmetry operator are found by solving an associated system of (linear homogeneous partial differential) *defining equations* for $\xi(x)$. Defining equations are derived by an explicit algorithm [1, 19], for which many computer implementations are available [11]. Although heuristic programs for solving infinitesimal defining equations exist [11], there is no algorithm which always succeeds in this solution process.

Symmetry operators (1) span a *Lie algebra* [1, 19], that is, a vector space closed under the skew-symmetric *commutator bracket*

$$[\mathcal{P}, \mathcal{Q}] := \mathcal{P}\mathcal{Q} - \mathcal{Q}\mathcal{P}. \tag{2}$$

If $\mathcal{P} = P^j(x)\frac{\partial}{\partial x^j}$, and $\mathcal{Q} = Q^j(x)\frac{\partial}{\partial x^j}$, their commutator $\mathcal{R} = [\mathcal{P}, \mathcal{Q}] = R^j(x)\frac{\partial}{\partial x^j}$ is thus

$$\mathcal{R} = \left(P^i\frac{\partial Q^j}{\partial x^i} - Q^i\frac{\partial P^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}. \tag{3}$$

If a finite-dimensional Lie algebra is resolved with respect to basis $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_N$, the commutation relations become

$$[\mathcal{L}_i, \mathcal{L}_j] = C^k_{ij}\mathcal{L}_k, \qquad 1 \le i, j \le N\ , \tag{4}$$

with C^k_{ij} being the *structure constants* of the algebra [1, 19]. Symmetry methods for PDEs frequently require structural information provided by C^k_{ij}.

The conventional method for calculating structure constants is to (a) integrate the infinitesimal defining equations and (b) substitute the resulting basis of operators into (2) to find C^k_{ij}. This process is not strictly algorithmic since no method is guaranteed to successfully perform step (a). However, in [25] we showed how C^k_{ij} can be calculated from the infinitesimal defining system without solving it, by a process involving only differentiation and linear substitutions. This method has been implemented in the symbolic language MAPLE [28].

However not all Lie symmetry algebras are of finite dimension [1, 19, 20]. The generalization of Lie theory to 'infinite Lie groups' was proposed by Lie [15], and brought to fruition by Vessiot [31] and especially by Cartan [2, 3, 5]. It is based on working directly with the defining system. Although Lie advocated infinitesimal methods, he did not succeed in producing a structure theory similar to that provided by the structure constants C^k_{ij} (4) for finite Lie groups. Cartan [2] succeeded in generalizing the C^k_{ij}, but his methods are not suited to symbolic calculation of symmetries, because he does not work with the infinitesimal defining equations. In the present paper we describe methods similar to those in [25] which permit calculation of Cartan's structure coefficients from the infinitesimal defining equations, using only *differentiation* and *linear algebra.* Our methods are thus suitable for symbolic computatation, and we have preliminary working MAPLE code for most of the steps. One part of the process is described in detail here, other results being just stated. For the full suite of methods we refer to the preprint [16].

2 INFINITE-DIMENSIONAL STRUCTURE THEORY

We define a *Lie algebra system* as a set of local vector fields $\zeta^i\partial_{z^i}$ such that (a) the ζ^i are local solutions of a system of linear homogeneous PDEs with analytic coefficients, and (b) the commutator of two such local solutions is also (if it is defined) a solution. The locality of the solutions means that the bracket operation is not defined for all pairs of vector fields, so a Lie algebra system is not (quite) a Lie algebra. A linear homogeneous system of defining equations can be brought to *involutive form* [21, 18] by a finite process involving only differentiations and linear substitutions. (Involutive implies that all integrability conditions are included in the system; see [21] for the precise definition.) Hence we take as our starting point the infinitesimal defining equations in involutive form. A Lie algebra system is *transitive* if under the flows generated by its local vector fields, a point z_0 can be mapped to every point in a neighbourhood of z_0. Transitivity is detected from the involutive defining system by the absence of 0-th order (algebraic) equations.

Each local vector field in a Lie algebra system is associated with a 1-parameter local Lie group of local transformations of z. The set of local transformations formed by finite composition of these 1-parameter local groups constitutes a *Lie pseudogroup* [30]. The transformations in a Lie pseudogroup satisfy a (generally nonlinear) system of defining equations. Cartan [2, 5] took these pseudogroup defining equations in involutive form as his starting point, and exhibited an algorithmic procedure for calculating a number of 1-forms ω^i invariant under the action of the pseudogroup. Exterior differentiation of ω^i then yields

$$d\omega^k = a^k_{i\rho}\pi^\rho \wedge \omega^i - \tfrac{1}{2}c^k_{ij}\omega^i \wedge \omega^j, \qquad (5)$$

where π^ρ are certain additional 1-forms, invariant modulo ω^i. The Cartan structure coefficients are c^k_{ij}, $a^k_{i\rho}$.

If the pseudogroup is of finite type then $a^k_{i\rho}$ are absent, c^k_{ij} are constant, and the Cartan structure equations reduce to the Maurer-Cartan equations, which are dual to Lie's commutation relations (4). In the infinite case, nontrivial $a^k_{i\rho}$ terms appear and the relationship to (4) is less clear. Cartan recognized a fundamental distinction between *structurally transitive* infinite Lie pseudogroups, which are isomorphic to a transitive Lie pseudogroup, and the *structurally intransitive* case, where no such isomorphism exists. Structurally intransitive pseudogroups possess *essential invariants* which are present in every realization of the pseudogroup. For instance, the pseudogroup $X = x$, $Y = y + f(x)$ has the essential invariant x. The structurally intransitive case is much more difficult, and we confine ourselves to the transitive case, where $a^k_{i\rho}$, c^k_{ij} are *constants.*

Cartan obtained many important results using his structure theory. However, the method is difficult to apply to symmetry analysis of differential equations. The difficulty is that Cartan works at the pseudogroup level where the defining system is nonlinear. Although it is possible to generate such a defining system for symmetries of a PDE(see e.g. [26]), there is currently no effective algorithm for reducing the nonlinear defining system to involutive form (although see [27]). This is in strong contrast to the availability of many algorithms for reducing linear infinitesimal defining systems to involutive form [22, 29].

The Cartan method of equivalence [3, 4, 7] can also yield symmetry pseudogroup structure, but is not suited to symmetry analysis of particular PDEs.

Although Cartan [5, p.1335] was sceptical of the possibility of a structure theory based on infinitesimal methods, Kuranishi [13, 14] and Singer and Sternberg [30] eventually developed an infinitesimal interpretation of Cartan's structure theory for structurally transitive infinite Lie pseudogroups (see also [8, 9, 10, 17]). It is by combining this theory with algorithms for reduction of linear defining systems to involutive form that we are able [16] to achieve a constructive algorithm for calculating Cartan structure coefficients $a^k_{i\rho}$, c^k_{ij} from the infinitesimal defining system in the transitive case. The methods generalize those of [25] for calculating C^k_{ij} in the finite dimensional case.

3 REDUCTION OF INFINITESIMAL DEFINING SYSTEMS TO A FIRST ORDER INVOLUTIVE FORM

Our calculation of Cartan structure utilizes a correspondence between the the structure equations (5) and commutator brackets described by Singer and Sternberg [30]. Exploiting this correspondence requires that the infinitesimal defining system be of *first order*, so our first task is to give a method for converting an infinitesimal defining system of q-th order to one of first order.

Consider an infinitesimal defining system for the infinitesimals $\zeta(z)$ with corresponding vector field $\zeta^1\partial_{z^1}+\cdots+\zeta^p\partial_{z^p}$ which we suppose to be involutive at order q. For the remainder of this section, i, j, k, l will be indices ranging between 1 and p. If $J = (j_1 j_2 \cdots j_k)$ is a symmetric multi-index, we use the notation ζ^i_J to represent the partial derivative $\partial^k\zeta^i/\partial z^{j_1}\cdots\partial z^{j_k}$; the order of J will be denoted by $k = \#(J)$.

Theorem 1 *An analytic infinitesimal defining system can be constructively transformed to an equivalent first order system which is in involutive form.*

This result follows from a method of Pommaret [21, p.109, p.161], which constructively reduces a q-th order system in involution to an equivalent first order system with equivalent symbol. An outline of the process is as follows. First the derivatives ζ^l_L, $1 \le \#(L) \le q-1$ are relabelled as new dependent variables, and the given system is expressed as a first order system with respect to these variables. Certain first order differential relations between the ζ^l_L are then appended to this first order system, and the composite system is shown to be involutive.

We next need to show that we can convert the involutive first order system to a first order *defining* system, i.e. we must arrange that the dependent variables are components of a vector field. Our construction is guided by Cartan's method for determination of the structure of Lie pseudogroups, which proceeds from the pseudogroup defining system in first order involutive form (i.e. as an involutive system of 1-forms). Cartan works with the pseudogroup defining system whose solutions $Z = \tau(z)$ are the pseudogroup transformations. In his process he prolongs the pseudogroup action on z to a pseudogroup action on (z, Z) with trivial action on Z. That is, a transformation $z \mapsto \tau(z)$ prolongs to $(z, Z) \mapsto (\tau(z), Z)$. At the infinitesimal level we prolong the vector field $\zeta^i\partial_{z^i}$ to a vector field on (z, Z) with trivial action on Z, that is to $\zeta^i\partial_{z^i} + \psi^i\partial_{Z^i}$, with $\psi^i = 0$. Cartan then prolongs the pseudogroup transformations to the derivatives of Z^i up to order $q-1$; correspondingly we prolong the vector field to

$$\mathbf{Z}^{(q-1)} = \zeta^i\partial_{z^i} + \psi^i\partial_{Z^i} + \psi^i_J\partial_{Z^i_J} \tag{6}$$

where $\psi^i = 0$, the ψ^i_J are given by the standard extension formula [19, p.113], and there is summation on the repeated index i and the repeated multi-index J, $1 \le \#(J) \le q-1$. Then $\mathbf{Z}^{(q-1)}$ is a vector field on the $(q-1)$-th order jet bundle with coordinates z, Z, Z^l_J. Here the derivatives of the Z's are denoted by $Z^l_J = \partial^n Z^l/\partial z^{j_1}\partial z^{j_2}\cdots\partial z^{j_n}$ where $J = (j_1, ..., j_n)$. We wish to show that:

Theorem 2 *Let $\mathcal{L}$ be a Lie algebra system of vector fields $\mathbf{Z} = \zeta^i\partial_{z^i}$ whose infinitesimal defining system is involutive at order q. Then the prolongation $\mathbf{Z}^{(q-1)}$ of $\mathbf{Z}$ to (z^i, Z^i, Z^i_J)-space is a Lie algebra system with an infinitesimal defining system for ζ^i, ψ^i, ψ^i_J which is involutive at order 1, and which is constructively determined.*

The remainder of this section will be devoted to proving this result. Note that the independent variables in the infinitesimal defining system of the original Lie algebra are z^i. For the prolonged Lie algebra the independent variables in the first order defining system are (z^i, Z^i, Z^i_J), and the corresponding dependent variables ζ^i, ψ^i, ψ^i_J, $1 \le \#(J) \le q-1$.

The ψ^l_J are determined in terms of the ζ^i and ζ^l_L by the standard extension formula [19, p.113] which recursively defines

$$\begin{aligned} \psi^l &= 0 \\ \psi^l_{J,i} &= D_i\psi^l_J - \zeta^k_i Z^l_{J,k}, \qquad 0 \le \#(J) \le q-1, \end{aligned}$$

where D_i is the total derivative operator with respect to z^i:

$$D_i = \partial_{z^i} + \sum_{\#(J)\ge 0} \zeta^l_{J,i}\partial_{\zeta^l_J}. \tag{7}$$

For the vector field (6) the required extensions are

$$\begin{aligned} \psi^l &= 0 \\ \psi^l_i &= D_i\psi^l - \zeta^k_i Z^l_k, \\ \psi^l_{j,i} &= D_i\psi^l_j - \zeta^k_i Z^l_{j,k}, \\ &\vdots \\ \psi^l_{J,i} &= D_i\psi^l_J - \zeta^k_i Z^l_{J,k}, \end{aligned}$$

where J is a symmetric multi-index with $\#(J) = q-2$. Evaluating the total derivative using (7) the above system becomes

$$\begin{aligned} \psi^l_i &= -\zeta^k_i Z^l_k, \\ \psi^l_{j,i} &= -\zeta^k_{j,i} Z^l_k + R^l_{j,i}, \\ &\vdots \\ \psi^l_{J,i} &= -\zeta^k_{J,i} Z^l_k + R^l_{J,i}, \end{aligned} \tag{8}$$

where each of the remainders $R^l_{K,i}$ depends on $\zeta^k_{L,i}$ only for $\#(L) < \#(K)$.

Lemma 3 *In a neighbourhood of $Z^l_k = \delta^l_k$, $Z^l_L = 0$, $2 \le \#(L) \le q-1$ the relations (8) define an invertible linear map from ζ^l_L to ψ^l_L.*

Proof: First note that the system (8), including the remainder terms, is indeed linear in ζ^l_L. Secondly the highest order terms ζ^l_L (i.e. those with maximum $\#(L)$) occur in the explicitly displayed terms in (8), so that the equations have a block triangular structure. When $Z^l_k = \delta^l_k$ the diagonal blocks are identity matrices, and relations (8) reduce to

$$\begin{aligned} \psi^l_i &= -\zeta^l_i, \\ \psi^l_{j,i} &= -\zeta^l_{j,i} + R^l_{j,i}, \\ &\vdots \\ \psi^l_{J,i} &= -\zeta^l_{J,i} + R^l_{J,i}, \end{aligned}$$

which are clearly invertible. Since the coefficients of ζ^l_L in (8) are analytic, invertibility holds in some neighbourhood of $Z^l_k = \delta^l_k$. ■

The main Theorem 2 now follows easily:

Proof: Let S denote the first order involutive system with independent variables z^i obtained by the transformation of Pommaret from the q-th order involutive infinitesimal defining system. Let T denote S augmented with the equations

$$\begin{aligned} &\partial_{Z^l_J}\zeta^i = 0, \qquad \partial_{Z^l_J}\zeta^i_L = 0, \\ &1 \le \#(L) \le q-1, \qquad 0 \le \#(J) \le q-1, \end{aligned} \tag{9}$$

(Thus the system T has independent variables z^i, Z^i_J, and dependent variables ζ^i_J, with $0 \le \#(J) \le q-1$.) Since the system S has no Z^i_J, the integrability conditions between the new equations and S are trivial, and the system T is also first order involutive.

The map

$$(z^i, Z^i, Z^i_J, \zeta^i, \zeta^i_J) \mapsto (z^i, Z^i, Z^i_J, \zeta^i, \psi^i_J), \quad 1 \le \#(J) \le q-1$$

induced by (8) is an analytic invertible change of coordinates on the space of independent and dependent variables of the system T, by virtue of Lemma 3. Both involutivity and the order of a system are geometric properties which are preserved under invertible changes of coordinates. Consequently the system obtained by making the change of coordinates above and adjoining the conditions $\psi^l = 0$ is first order and involutive, and Theorem 2 is proved. ■

Note that the variables Z^i play a trivial role in the defining system for the prolonged vector field $\mathbf{Z}^{(q-1)}$. Their infinitesimals ψ^i vanish, and it is readily confirmed that all ψ^i_J are independent of Z^i. We retain them only for formal convenience.

We illustrate the steps in this reduction to first order defining system for a simple example.

Example 4 Consider the infinitesimal defining system

$$\zeta_{zz} = 0, \tag{10}$$

which is in involutive form. To reduce to Pommaret's first order involutive form we introduce the new dependent variable $\zeta_1 := \zeta_z$. The Pommaret form of the defining system is therefore

$$\zeta_z = \zeta_1, \quad (\zeta_1)_z = 0,$$

(system S above). The augmented system T is S together with

$$\begin{aligned} \zeta_Z &= 0, \qquad (\zeta_1)_Z = 0 \\ \zeta_{Z_1} &= 0 \qquad (\zeta_1)_{Z_1} = 0. \end{aligned}$$

The extension formulae (8) become

$$\psi_1 = -\zeta_1 Z_1$$

which can be inverted in a neighbourhood of $Z_1 = 1$ to yield

$$\zeta_1 = -\psi_1 / Z_1.$$

Hence we have explicitly constructed the map $(z, Z, Z_1, \zeta, \zeta_1) \mapsto (z, Z, Z_1, \zeta, \psi_1)$. Applying this map to system T and after a little simplification we obtain the required first order involutive system

$$\begin{array}{lll} \psi = 0 & & \\ \zeta_z = -\psi_1/Z_1 & \zeta_Z = 0 & \zeta_{Z_1} = 0 \\ (\psi_1)_z = 0 & (\psi_1)_Z = 0 & (\psi_1)_{Z_1} = \psi_1/Z_1 \end{array}$$

which is equivalent to the original defining system (10).

4 DETERMINATION OF CARTAN STRUCTURE FOR FIRST ORDER INVOLUTIVE DEFINING SYSTEMS WITH NO INVARIANTS

With the form of a first order infinitesimal defining system now achieved, we describe the the method for extracting Cartan structure from it. If the system contains 0-th order (i.e. algebraic) equations, then the Lie pseudogroup has scalar invariants. In [16] an infinitesimal method is presented for diagnosing whether these invariants are essential. In the case where they are not essential, the invariants may be assigned constant values, thereby restricting the pseudogroup action to an orbit. An infinitesimal method for doing this is also described in [16]. Here we consider only the simplest case, where there are no invariants, so that the pseudogroup is transitive, and there are no 0-th order defining equations.

Suppose the partial derivatives of a system are ranked by total order of derivative. Gauss reduction of such systems with respect to this ranking yields a solved-form: the LHS derivatives (called *principal derivatives*) are expressed as functions of non-principal (or *parametric*) derivatives. Of particular interest to us will be the set of parametric and principal derivatives of k-th order, which we denote by $\mathcal{P}_k$ and $\overline{\mathcal{P}}_k$ respectively.

To make the connection with Cartan structure we introduce new variables ϕ^μ defined by

$$\frac{\partial \xi^k}{\partial x^i} = \phi^\mu, \qquad \mu = 1, \ldots, \#(\mathcal{P}_1), \tag{11}$$

where $\frac{\partial \xi^k}{\partial x^i} \in \mathcal{P}_1$ (i.e. the $\frac{\partial \xi^k}{\partial x^i}$ are first order parametric derivatives). We then eliminate the first order parametrics from the involutive form of the infinitesimal defining system by using (11), and append equations (11) to the Gauss reduced involutive form to obtain the infinitesimal defining system as

$$\frac{\partial \xi^k}{\partial x^i} = \sum_{j=1}^{n} b_{ij}^k(x)\xi^j + \sum_{\rho=1}^{\#(\mathcal{P}_1)} A_{i\rho}^k(x)\phi^\rho, \quad \text{for } i, k = 1, \ldots, n. \tag{12}$$

Note that cases $\frac{\partial \xi^k}{\partial x^i} \in \mathcal{P}_1$, and $\frac{\partial \xi^k}{\partial x^i} \in \overline{\mathcal{P}}_1$ (=all nonparametric first order derivatives), are covered. In particular when $\frac{\partial \xi^k}{\partial x^i} \in \mathcal{P}_1$, then $b_{ij}^k = 0$, for $j = 1, \ldots, n$ and $A_{i\mu}^k = \delta_\mu^\rho$ so that (12) yields (11). The main result of this section is

Proposition 5 *Let x_0 be a nonsingular point for the infinitesimal defining system (12) in involutive form. Let $a_{i\rho}^k = A_{i\rho}^k(x_0)$, and*

$$c_{ij}^k = b_{ij}^k(x_0) - b_{ji}^k(x_0). \tag{13}$$

Then $a_{i\rho}^k$, c_{ij}^k can be identified with those in the Cartan structure equations (5).

The proof is presented in [16]. It relies on a correspondence between Cartan structure coefficients and Taylor series expansions of vector fields described in [30].

Example 6 Consider the Lie algebra system of vector fields $\mathbf{X} = \xi\partial_x + \eta\partial_y$ on $\mathbb{R}^2 \setminus \{y = 0\}$ with first order involutive infinitesimal defining system

$$\begin{array}{ll} \xi_x = \frac{1}{y}\eta & \eta_x = \phi \\ \xi_y = 0 & \eta_y = \frac{1}{y}\eta. \end{array} \tag{14}$$

(where $x \equiv x^1$, $y \equiv x^2$, $\xi \equiv \xi^1$, $\eta \equiv \xi^2$). The parametric derivatives of order 0 are ξ, η. The only parametric first order derivative is η_x so we have introduced the new variable ϕ as described by (11). We construct c_{ij}^k and $a_{i\rho}^k$ according to Proposition 5.

For system (14) we have $n = 2$, $\rho = 1$, and comparing (12) with (14) we see that

$$b_{ij}^k = \left\{ \begin{array}{ll} \frac{1}{y} & \text{for } (i,j,k) = (1,2,1) \text{ or } (2,2,2) \\ 0 & \text{otherwise} \end{array} \right.$$

$$A_{i\rho}^k = \left\{ \begin{array}{ll} 1 & \text{for } (i,\rho,k) = (1,1,2) \\ 0 & \text{otherwise.} \end{array} \right.$$

Choosing the initial point $x_0 = (x_0, y_0) = (0, -\frac{1}{2})$ for convenience, Proposition 5 yields

$$\begin{array}{rcl} c_{12}^1 & = & b_{12}^1(x_0) - b_{21}^1(x_0) = 1/y_0 = -2 \\ c_{12}^2 & = & b_{12}^2(x_0) - b_{21}^2(x_0) = 0 \\ a_{i\rho}^k & = & A_{i\rho}^k(x_0) = \left\{ \begin{array}{ll} 1 & \text{for } (i,\rho,k) = (1,1,2) \\ 0 & \text{otherwise.} \end{array} \right. \end{array}$$

Hence from (5) the Cartan structure equations are

$$\begin{array}{rcl} d\omega^1 & = & \omega^1 \wedge \omega^2 \\ d\omega^2 & = & \pi^1 \wedge \omega^1. \end{array} \tag{15}$$

Although we have only presented the simplest case here, the additional results described in [16] permit us to state the following

Theorem 7 *Given an analytic infinitesimal defining system of the Lie pseudogroup of symmetries of a system of PDEs then it is possible to constructively determine*

(a) whether the Lie pseudogroup is structurally transitive,

(b) the Cartan structure of the Lie pseudogroup, if it is structurally transitive.

5 EXAMPLES OF LIE SYMMETRY PSEUDOGROUP STRUCTURE

For symmetry analysis of PDEs, the implication of the above results is that we can pass from a given PDE to its infinitesimal defining system, thence to involutive form, thence to first order involutive form, and finally to the Cartan structure coefficients, using only differentiation and linear algebra at each step. Our procedure is therefore suitable for computer algebra implementation, and we have preliminary MAPLE code for the algorithms in this paper.

We now give the result of applying our algorithms to some PDEs of physical interest known to possess infinite symmetry pseudogroups. First the infinitesimal defining system for the point symmetry vector fields is derived by the usual method [1, 19, 20]; for this we used the Maple program [12]. Next the defining system is brought to involutive form. We used the program [28] for this; an involutivity check showed each system to be involutive at second order. Next a test for structural transitivity is applied. For the examples below, the defining systems contained no 0-th order equations so transitivity is automatic. A program implementing the process of §3 is then applied to transform the systems to first order involutive defining systems. Finally code for the methods of §4 automatically yielded their Cartan structure.

Example 8 (Liouville's equation) Liouville's equation

$$u_{xy} = e^u$$

admits an infinite Lie pseudogroup of symmetries with 2 arbitrary functions of 1 variable.

If we seek symmetry vector fields of the form

$$\xi\partial_x + \tau\partial_y + \eta\partial_u$$

then we obtain the infinitesimal defining system

$$\begin{array}{ll} \eta_{xy} = 0 & \tau_{yy} = -\eta_y \\ \xi_x = -\tau_y - \eta & \tau_x = 0 \\ \xi_y = 0 & \tau_u = 0 \\ \xi_u = 0 & \eta_u = 0 \end{array}$$

The above system is in involution once all the first derivatives of the first order equations are adjoined. After reducing to first order as described in §3 the method of §4 yields Cartan structure equations

$$\begin{aligned} d\omega^1 &= -\omega^1\wedge\omega^6 \\ d\omega^2 &= -\omega^2\wedge\omega^3 + \omega^2\wedge\omega^6 \\ d\omega^3 &= -\omega^1\wedge\omega^4 - \omega^2\wedge\omega^5 \\ d\omega^4 &= \pi^1\wedge\omega^1 + \omega^4\wedge\omega^6 \\ d\omega^5 &= \pi^2\wedge\omega^2 - \omega^3\wedge\omega^5 - \omega^5\wedge\omega^6 \\ d\omega^6 &= -\omega^1\wedge\omega^4 \end{aligned}$$

In [24] we also derive the above structure equations using a different method.

Example 9 (KP equation) The Kadomtsev-Petviashvili equation

$$u_{yy} + \left(u_t + u_{xxx} + 2uu_x\right)_x = 0$$

has an infinite Lie pseudogroup of symmetries depending on 3 arbitrary functions of 1 variable. Using our methods we find structure equations

$$\begin{aligned} d\omega^1 &= \omega^1\wedge\omega^8 + 2\,\omega^3\wedge\omega^9 \\ d\omega^2 &= -\omega^1\wedge\omega^9 + \frac{\omega^2\wedge\omega^8}{2} + 2\,\omega^3\wedge\omega^4 \\ d\omega^3 &= \frac{3\,\omega^3\wedge\omega^8}{2} \\ d\omega^4 &= -\omega^1\wedge\omega^5 - \omega^2\wedge\omega^6 - \omega^3\wedge\omega^7 - \omega^4\wedge\omega^8 \\ d\omega^5 &= \pi^1\wedge\omega^1 + \pi^2\wedge\omega^3 - 2\,\omega^5\wedge\omega^8 + \omega^6\wedge\omega^9 \\ d\omega^6 &= -\pi^1\wedge\omega^3 - \frac{3\,\omega^6\wedge\omega^8}{2} \\ d\omega^7 &= \pi^2\wedge\omega^1 - \pi^1\wedge\omega^2 + \pi^3\wedge\omega^3 - 2\,\omega^4\wedge\omega^6 \\ &\quad - 2\,\omega^5\wedge\omega^9 - \frac{5\,\omega^7\wedge\omega^8}{2} \\ d\omega^8 &= -4\,\omega^3\wedge\omega^6 \\ d\omega^9 &= -2\,\omega^1\wedge\omega^6 + 2\,\omega^3\wedge\omega^5 + \frac{\omega^8\wedge\omega^9}{2} \end{aligned}$$

In [6] the explicit form of the infinitesimal generators of symmetries of the KP equation is given. The generators depend on arbitrary functions and these are used to parametrize the commutation relations of the algebra. Laurent expansion of the generators is used to show that the symmetry algebra has a Kac-Moody-Virasoro structure. It would be interesting to see whether this structure could be determined from the Cartan structure given above.

Example 10 (Steady boundary layer equations) The steady state boundary layer equations [20]

$$\begin{aligned} uu_x + vu_y + p_x &= u_{yy} \\ p_y &= 0 \\ u_x + v_y &= 0 \end{aligned}$$

have an infinite Lie pseudogroup of symmetries depending on one arbitrary function of one variable. Its Cartan structure equations are

$$\begin{aligned} d\omega^1 &= \omega^1\wedge\omega^3 - 2\,\omega^1\wedge\omega^7 \\ d\omega^2 &= \omega^1\wedge\omega^4 - \omega^2\wedge\omega^7 \\ d\omega^3 &= 0 \\ d\omega^4 &= -\omega^1\wedge\omega^4 - \omega^1\wedge\omega^6 + \omega^3\wedge\omega^4 + \omega^4\wedge\omega^7 \\ d\omega^5 &= -2\,\omega^3\wedge\omega^5 \\ d\omega^6 &= \pi^1\wedge\omega^1 + \omega^3\wedge\omega^4 + 2\,\omega^3\wedge\omega^6 + 2\,\omega^4\wedge\omega^7 \\ &\quad + 3\,\omega^6\wedge\omega^7 \\ d\omega^7 &= 0 \end{aligned}$$

It is instructive to compare this result with a much more complicated result in [25] where the commutation relations of the infinite Lie symmetry algebra are parametrized using arbitrary functions.

6 DISCUSSION

There is a well-developed methodology [1, 19, 20] for using finite-parameter Lie symmetry groups of differential equations. Many of these methods exploit the structure of the symmetry Lie algebra to simplify calculations. Methods for using infinite Lie symmetry pseudogroups also exist [20]. However the Cartan structure (5) of such pseudogroups has scarcely been utilized, partly because it has been virtually impossible to calculate the coefficients $a^k_{i\rho}$, c^k_{ij}. We anticipate that the heuristic-free methods described here should make possible the systematic use of Cartan structure of infinite symmetry pseudogroups.

H. Goldschmidt (private communication) has suggested the possibility of bypassing the reduction to first order stage of our process.

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