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Lie Symmetries, Conservation Laws and Exact Solutions for Two Rod Equations

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Abstract In this paper, the Lie symmetry analysis are performed for the two rod equations. The infinite number of conservation laws (CLs) for the two equations are derived from the direct method. Furthermore, the all similarity reductions and exact explicit solutions are provided.

Keywords Rod equation · Lie symmetry analysis · Conservation law · Similarity reduction · Exact solution

Mathematics Subject Classification (2000) 17B80 · 22E70 · 35C05

1 Introduction

Partial differential equations (PDEs) arised in many physical fields which exhibit a rich variety of nonlinear phenomena, such as condense matter physics, fluid mechanics, plasma physics and optics, etc. It is known that to find the exact solutions of the PDEs is always one of the central themes in mathematics and physics. In the past few decades, there are noticeable progress in this field. And various methods have been developed, such as the inverse

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scattering transformation (IST) [1], Darboux and Bäcklund transformations [2], Hirota's bilinear method [2–4], Lie symmetry analysis [5–8], CK method [9, 10], and so on.

Conservation laws (CLs) contribute a lot to the study of nonlinear evolution-type equations. They have many potential applications in integrability of equations, linearization of equations, and analysis of solutions by providing conserved quantities. For ordinary differential equations (ODEs), a conservation law is equivalent to the classical notion of a first integral. As a result, how to find the conservation laws of PDEs attracts the attention of most mathematicians and physicists who are dedicated to the research of this area.

In the present paper, we will consider the higher-order rod equation

$$u_{tt} + \lambda u_{xxxx} = 0, \tag{1.1}$$

where $u = u(x, t)$ denotes the unknown function, the parameter $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.

Equation (1.1) is also called the beam equation which describes the vibrations of a rod. In practical, $\lambda = c^2 = \frac{EI}{A\rho}$ is a constant, E denotes the Young's constant (determined by the constitutive material of the beam), I denotes the moment of inertia of a cross section of the beam with respect to an axis through its center of mass and perpendicular to the (x, u)-plane, ρ is the density (mass per unit volume), and A denotes the area of cross section. The $u = u(x, t)$ is the unknown real function which represents the displacement on the beam corresponding to position x at time t. Simultaneously, we consider the following degenerate rod equation [11]:

$$u_{tt} + \mu u_{xxtt} = 0, \tag{1.2}$$

where the parameter $\mu \in \mathbb{R}$ and $\mu \neq 0$.

In [12], (1.1) with the initial conditions are considered by the method of separation variables and some numerical solutions are obtained. In [11], the general rod equation is derived and the solitary waves are given. Both the two equations are important in dynamical systems and applications. In this paper, based on the Lie group method, we will investigate the similarity reductions and exact explicit solutions for the two rod equations.

The outline of this paper is as follows. In Sect. 2, we perform Lie symmetry analysis for (1.1) and (1.2). In Sect. 3, by using the direct method, an infinite number of conservation laws (CLs) are obtained for the two rod equations. In Sect. 4, we consider the symmetry reductions by using Lie group method and provide exact explicit solutions. In Sect. 5, we deal with the solution of the reduced equation by employing the power series method. In Sect. 6, we conclude and make some remarks.

2 Lie Symmetries for (1.1) and (1.2)

In this section, we will perform Lie symmetry analysis for the two rod equations (1.1) and (1.2).

The Lie group method, is also called symmetry analysis sometimes. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones.

First of all, let us consider a one-parameter Lie group of infinitesimal transformation:

$$x \to x + \epsilon\xi(x,t,u), \tag{2.1a}$$

$$t \to t + \epsilon\tau(x,t,u), \tag{2.1b}$$

$$u \to u + \epsilon\phi(x,t,u), \tag{2.1c}$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}, \tag{2.2}$$

where $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ are coefficient functions to be determined.

The symmetry groups of (1.1) and (1.2) will be generated by vector fields of the form (2.2). Applying the fourth prolongation $\mathrm{pr}^{(4)}V = \mathrm{pr}^{(3)}V + \phi^{xxxx}\frac{\partial}{\partial u_{xxxx}} + \phi^{xxxt}\frac{\partial}{\partial u_{xxxt}} + \phi^{xxtt}\frac{\partial}{\partial u_{xxtt}} + \phi^{xttt}\frac{\partial}{\partial u_{xttt}} + \phi^{tttt}\frac{\partial}{\partial u_{tttt}}$ of V to (1.1) and (1.2), we find that the coefficient functions ξ, τ and ϕ must satisfy the symmetry conditions, respectively

$$\phi^{tt} + \lambda\phi^{xxxx} = 0, \tag{2.3}$$

$$\phi^{tt} + \mu\phi^{xxtt} = 0, \tag{2.4}$$

where ϕ^{tt}, ϕ^{xxxx} and ϕ^{xxtt} are all the coefficients of $\mathrm{pr}^{(4)}V$. Furthermore, we have

$$\phi^{tt} = D_t^2(\phi - \xi u_x - \tau u_t) + \xi u_{xtt} + \tau u_{ttt}, \tag{2.5a}$$

$$\phi^{xxxx} = D_x^4(\phi - \xi u_x - \tau u_t) + \xi u_{xxxxx} + \tau u_{xxxxt}, \tag{2.5b}$$

$$\phi^{xxtt} = D_x^2 D_t^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxxtt} + \tau u_{xxttt}, \tag{2.5c}$$

where D_x, D_t are the total derivatives with respect to x, t, respectively. Substituting (2.5a) and (2.5b) into (2.3), (2.5a) and (2.5c) into (2.4), respectively, replacing u_{tt} by $-\lambda u_{xxxx}$ whenever it occurs in (2.3), u_{tt} by $-\mu u_{xxtt}$ whenever it occurs in (2.4), respectively. Then standard symmetry group calculations lead to the following forms of the coefficient functions: for (1.1), we have

$$\xi = c_1 x + c_2, \qquad \tau = 2c_1 t + c_3, \qquad \phi = c_4 u + \alpha(x,t),$$

for (1.2), we have

$$\xi = c_1, \qquad \tau = c_2 t + c_3, \qquad \phi = c_4 u + \beta(x,t),$$

where c_i $(i = 1,\dots,4)$ are arbitrary constants, $\alpha(x,t)$ satisfies $\alpha_{tt} + \lambda\alpha_{xxxx} = 0$ for (1.1) and $\beta_{tt} + \mu\beta_{xxtt} = 0$ for (1.2), respectively.

Thus, the Lie algebra of infinitesimal symmetries of (1.1) is spanned by the vector field

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \qquad V_4 = u\frac{\partial}{\partial u}$$

and the infinite-dimensional subalgebra

$$V_\alpha = \alpha(x,t)\frac{\partial}{\partial u},$$

where α is an any arbitrary solution of (1.1).

Similarly, the Lie algebra of infinitesimal symmetries of (1.2) is spanned by the vector field

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = t\frac{\partial}{\partial t}, \qquad V_4 = u\frac{\partial}{\partial u}$$

and the infinite-dimensional subalgebra

$$V_\beta = \beta(x,t)\frac{\partial}{\partial u},$$

where β is an any arbitrary solution of (1.2).

It is easy to check that $\{V_1, V_2, V_3, V_4\}$ is closed under the Lie bracket. In fact, for (1.1), we have

$$\begin{aligned}
&[V_1, V_1] = [V_2, V_2] = [V_3, V_3] = [V_4, V_4] = 0,\\
&[V_1, V_2] = -[V_2, V_1] = [V_1, V_4] = -[V_4, V_1] = [V_2, V_4] = -[V_4, V_2] = [V_3, V_4]\\
&\qquad = -[V_4, V_3] = 0,\\
&[V_1, V_3] = -[V_3, V_1] = -V_1, \qquad [V_2, V_3] = -[V_3, V_2] = -2V_2,
\end{aligned}$$

and

$$\begin{aligned}
&[V_1, V_\alpha] = -[V_\alpha, V_1] = -V_{\alpha_x}, \qquad [V_2, V_\alpha] = -[V_\alpha, V_2] = -V_{\alpha_t},\\
&[V_3, V_\alpha] = -[V_\alpha, V_3] = -V_{\alpha'}, \qquad [V_4, V_\alpha] = -[V_\alpha, V_4] = V_\alpha,
\end{aligned}$$

where $\alpha' = x\alpha_x + 2t\alpha_t$, $\alpha_{tt} + \lambda\alpha_{xxxx} = 0$.

For (1.2), we have

$$\begin{aligned}
&[V_1, V_1] = [V_2, V_2] = [V_3, V_3] = [V_4, V_4] = 0,\\
&[V_1, V_2] = -[V_2, V_1] = [V_1, V_3] = -[V_3, V_1] = [V_1, V_4] = -[V_4, V_1] = [V_2, V_4]\\
&\qquad = -[V_4, V_2] = [V_3, V_4] = -[V_4, V_3] = 0,\\
&[V_2, V_3] = -[V_3, V_2] = -V_2,
\end{aligned}$$

and

$$\begin{aligned}
&[V_1, V_\beta] = -[V_\beta, V_1] = -V_{\beta_x}, \qquad [V_2, V_\beta] = -[V_\beta, V_2] = -V_{\beta_t},\\
&[V_3, V_\beta] = -[V_\beta, V_3] = -V_{\beta'}, \qquad [V_4, V_\beta] = -[V_\beta, V_4] = V_\beta,
\end{aligned}$$

where $\beta' = t\beta_t$, $\beta_{tt} + \mu\beta_{xxtt} = 0$. So we can see that the generator of invariant group $V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$ of (1.1) and (1.2) construct a infinite-dimensional Lie algebra, which include a four-dimensional subalgebra is spanned by the basis $\{V_1, V_2, V_3, V_4\}$, respectively.

Furthermore, for (1.1), the one-parameter groups G_i generated by the V_i $(i = 1, \ldots, 4)$ are given in the following table:

$$\begin{aligned}
&G_1 : (x,t,u) \to (x+\epsilon, t, u),\\
&G_2 : (x,t,u) \to (x, t+\epsilon, u),
\end{aligned}$$

$$G_3 : (x, t, u) \to (e^{\epsilon} x, e^{2\epsilon} t, u),$$

$$G_4 : (x, t, u) \to (x, t, e^{\epsilon} u),$$

$$G_\alpha : (x, t, u) \to (x, t, u + \epsilon\alpha(x, t)).$$

For (1.2), the one-parameter groups G_i generated by the V_i ($i = 1, \ldots, 4$) are given in the following table:

$$G_1 : (x, t, u) \to (x + \epsilon, t, u),$$

$$G_2 : (x, t, u) \to (x, t + \epsilon, u),$$

$$G_3 : (x, t, u) \to (x, e^{\epsilon} t, u),$$

$$G_4 : (x, t, u) \to (x, t, e^{\epsilon} u),$$

$$G_\beta : (x, t, u) \to (x, t, u + \epsilon\beta(x, t)).$$

From the two tables, we observe that G_1 is a spatial translation, G_2 is a temporal translation and G_3 is a scaling translation. While the symmetry groups G_4, G_α and G_β reflect the linearity of the two equations, so we can add solutions and multiply them by arbitrary constants. This is an important property for the two equations which derived from Lie symmetry analysis above. The last groups G_α and G_β denote also an infinite-dimensional system, but it is trivial for dealing with the exact solutions.

3 Conservation Laws of the Two Rod Equations

As is well known, an infinite number of conservation laws is the intriguing property shared by many integrable systems. In recent years, much attention has been paid to studying such intrinsic features of integrable systems. Many methods for dealing with the conservation laws are derived, such as the method based on the Noether's theorem, the multiplier method, by the relationship between the conserved vector of a PDE and the Lie-Bäcklund symmetry generators of the PDE, the direct method, etc. [2, 5, 6, 13, 14].

Now, we derive the conservation laws from the direct method.

Firstly, we construct the following conservation laws of (1.1):

$$D_t T + D_x X = 0, \tag{3.1}$$

where D_t, D_x are the operator of total differentiation defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j}, \quad i = 1, \ldots, n$$

as

$$u_i = D_i(u), \qquad u_{ij} = D_j D_i(u), \ldots$$

Suppose $u = u(x, t)$ is a solution of (1.1), the conserved form (3.1) along the solutions u of (1.1) separates by fourth derivative terms in u as

$$u_{tttt} : T_{u_{ttt}} = 0, \tag{3.2a}$$

$$u_{xttt} : T_{u_{xtt}} + X_{u_{ttt}} = 0, \tag{3.2b}$$

$$u_{xxtt} : T_{u_{xxt}} + X_{u_{xtt}} = 0, \tag{3.2c}$$

$$u_{xxxt} : T_{u_{xxx}} + X_{u_{xxt}} = 0, \tag{3.2d}$$

$$u_{xxxx} : \lambda T_{u_t} + X_{u_{xxx}} = 0. \tag{3.2e}$$

Solving these equations, we can get that

$$T = -Au_{xtt} - Du_{xxt} + \frac{1}{\lambda}Gu_t + F, \qquad X = Au_{ttt} + Du_{xtt} + Gu_{xxx} + H, \tag{3.3}$$

where A, D and G are arbitrary constants, $F = F(t, x, u, u_i, u_{j_1 j_2})$ and $H = H(t, x, u, u_i, u_{j_1 j_2})$ are functions, u_i denotes the all first-order partial derivatives of u, $u_{j_1 j_2}$ denotes the all second-order partial derivatives of u. At the same time, (3.1) becomes the following form:

$$\begin{aligned} &T_t + u_t T_u + u_{xt} T_{u_x} + u_{xxt} T_{u_{xx}} + u_{xtt} T_{u_{xt}} + u_{ttt} T_{u_{tt}} + X_x + u_x X_u + u_{xx} X_{u_x} \\ &\quad + u_{xt} X_{u_t} + u_{xxx} X_{u_{xx}} + u_{xxt} X_{u_{xt}} + u_{xtt} X_{u_{tt}} = 0. \end{aligned} \tag{3.4}$$

Substituting (3.3) into (3.4), and separating by third derivative terms in u, we can get that

$$\begin{aligned} T &= -Au_{xtt} - Du_{xxt} + Iu_{xx} - I_1 u_{xt} + \frac{1}{\lambda}Gu_t + J, \\ X &= Au_{ttt} + Du_{xtt} + Gu_{xxx} + I_1 u_{tt} - Iu_{xt} + L, \end{aligned} \tag{3.5}$$

where I and I_1 are arbitrary constants, $J = J(t, x, u, u_i)$ and $L = L(t, x, u, u_i)$ are functions, u_i denotes the all first-order partial derivatives of u. In the similar way, we can get the following forms at last:

$$\begin{aligned} T &= -Au_{xtt} - Du_{xxt} + Iu_{xx} - I_1 u_{xt} + \frac{1}{\lambda}Gu_t - Mu_x + \alpha, \\ X &= Au_{ttt} + Du_{xtt} + Gu_{xxx} + I_1 u_{tt} - Iu_{xt} + Mu_t + \beta, \end{aligned} \tag{3.6}$$

where A, D, G, I, I_1 and M are arbitrary constants, $\alpha = \alpha(x, t)$ and $\beta = \beta(x, t)$ are arbitrary functions with $\alpha_t + \beta_x = 0$. Furthermore, we have

$$D_t T + D_x X = \frac{1}{\lambda}G(u_{tt} + \lambda u_{xxxx}) = 0.$$

It shows that (T, X) is the conservation vector for (1.1).

Similar to the above argument, we can get the conservation vectors (T, X) for (1.2) are as follows:

$$\begin{aligned} T &= -Au_{xtt} - Du_{xxt} - Gu_{xxx} + Ju_{xx} - Ku_{xt} - Nu_x + \xi, \\ X &= Au_{ttt} + Du_{xtt} + Gu_{xxt} + Ku_{tt} - Ju_{xt} + Nu_t + \eta, \end{aligned} \tag{3.7}$$

and $(T, X) = (u_t, \mu u_{xtt})$, where A, D, G, J, K and N are arbitrary constants, $\xi = \xi(x, t)$ and $\eta = \eta(x, t)$ are arbitrary functions with $\xi_t + \eta_x = 0$.

Remark 3.1 In view of (3.6), letting $G = \lambda$ and $A = D = I = I_1 = M = \alpha = \beta = 0$, then we get the obvious conservation vector $(T, X) = (u_t, \lambda u_{xxx})$ for (1.1). But the obvious conservation vector $(T, X) = (u_t, \mu u_{xtt})$ for (1.2) cannot be obtained by (3.7) directly.

4 Symmetry Reductions and the Exact Solutions

In Sect. 2, we have obtained the symmetry groups of (1.1) and (1.2). Now we deal with the exact solutions for the two equations based on the symmetry analysis. Since each G_i $(i = 1, \ldots, 4, \alpha)$ is a symmetry group, it implies that if $u = f(x,t)$ is a solution of (1.1), then $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$ and $u^{(\alpha)}$ as following are solutions of (1.1) as well:

$$u^{(1)} = f(x-\epsilon, t), \tag{4.1a}$$

$$u^{(2)} = f(x, t-\epsilon), \tag{4.1b}$$

$$u^{(3)} = f(e^{-\epsilon}x, e^{-2\epsilon}t), \tag{4.1c}$$

$$u^{(4)} = e^{\epsilon} f(x,t), \tag{4.1d}$$

$$u^{(\alpha)} = f(x,t) + \epsilon\alpha(x,t), \tag{4.1e}$$

where ϵ is any real number and $\alpha(x,t)$ satisfies $\alpha_{tt} + \lambda\alpha_{xxxx} = 0$.

Similarly, if $u = f(x,t)$ is a solution of (1.2), then $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$ and $u^{(\beta)}$ as following are solutions of (1.2) also:

$$u^{(1)} = f(x-\epsilon, t), \tag{4.2a}$$

$$u^{(2)} = f(x, t-\epsilon), \tag{4.2b}$$

$$u^{(3)} = f(x, e^{-\epsilon}t), \tag{4.2c}$$

$$u^{(4)} = e^{\epsilon} f(x,t), \tag{4.2d}$$

$$u^{(\beta)} = f(x,t) + \epsilon\beta(x,t), \tag{4.2e}$$

where ϵ is any real number and $\beta(x,t)$ satisfies $\beta_{tt} + \mu\beta_{xxtt} = 0$. We note that the property of linearity of (1.1) and (1.2) are reflected clearly also from (4.1d) and (4.1e), (4.2d) and (4.2e), respectively.

Next, we consider the exact and explicit solutions for the two rod equations.

4.1 The Traveling Wave Solutions

In general, the traveling wave solutions to a PDE arise as special group-invariant solutions in which the group under consideration is a translational group on the space of independent variables. In the present case, we consider the translation group $(x,t,u) \mapsto (x + v\epsilon, t + \epsilon, u)$ $(\epsilon \in R)$, generated by the generator $v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$, where v is a fixed constant, which will determine the speed of the waves. So the global invariants of this group are as follows:

$$\xi = x - vt, \qquad \phi = u. \tag{4.3}$$

In view of (4.3), let $\xi = x - vt$, so we have $u(x,t) = \phi(x - vt) = \phi(\xi)$, where v is the propagating wave velocity. Substituting it into (1.1), we have

$$v^2\phi'' + \lambda\phi'''' = 0,$$

where $\phi' = \frac{d\phi}{d\xi}$. Integrating the above equation twice we have

$$v^2\phi + \lambda\phi'' = g_1\xi + g_2, \tag{4.4}$$

where g_1 and g_2 are constants of integration.

In particular, if $g_1 = g_2 = 0$, then (4.4) becomes

$$v^2\phi + \lambda\phi'' = 0. \tag{4.5}$$

Equation (4.5) is equivalent to the planar system

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = -\frac{v^2}{\lambda}\phi, \tag{4.6}$$

which is a Hamiltonian system with the Hamiltonian function

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{v^2}{2\lambda}\phi^2 = h. \tag{4.7}$$

It is easy to see that the system (4.6) has one equilibrium point $O(0,0)$ in the (ϕ, y)-phase plane. Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (4.6) at an equilibrium point (ϕ_e, y_e) and $J(\phi_e, y_e)$ be its Jacobian determinant. Then,we have $J(0,0) = \det M(0,0) = \frac{v^2}{\lambda} > 0$ and $\operatorname{Tr} M(0,0) = 0$. By the theory of planar dynamical system, we know that the equilibrium point $O(0,0)$ is a center point of system (4.6).

In view of the Hamiltonian function (4.7), we can see that $h > 0$. In this case, from (4.7), we have

$$y^2 = -\frac{v^2}{\lambda}\phi^2 + 2h.$$

By using this formula and the first equation of (4.6), we have

$$\phi(\xi) = -\frac{1}{v}\sqrt{2h\lambda}\cos\left(\frac{v}{\sqrt{\lambda}}\xi\right).$$

Hence, we obtain the following traveling wave solution of (1.1):

$$u(x,t) = -\frac{\sqrt{\lambda}}{v}\sqrt{2h}\cos\left[\frac{v}{\sqrt{\lambda}}(x - vt)\right]. \tag{4.8}$$

In general, we consider (4.4), that is

$$\frac{d^2\phi}{d\xi^2} + \frac{v^2}{\lambda}\phi = \frac{g_1}{c^2}\xi + \frac{g_2}{\lambda}.$$

It is a second-class linear ordinary differential equation (ODE), solving this equation, we can get

$$\phi(\xi) = c_1\cos\left(\frac{v}{\sqrt{\lambda}}\xi\right) + c_2\sin\left(\frac{v}{\sqrt{\lambda}}\xi\right) + \frac{g_1}{v^2}\xi + \frac{g_2}{v^2}.$$

Thus, we obtain the general solution of (1.1) as following:

$$u(x,t) = c_1\cos\left[\frac{v}{\sqrt{\lambda}}(x - vt)\right] + c_2\sin\left[\frac{v}{\sqrt{\lambda}}(x - vt)\right] + \frac{g_1}{v^2}(x - vt) + \frac{g_2}{v^2}, \tag{4.9}$$

where c_i and g_i $(i = 1, 2)$ are arbitrary constant numbers.

Remark 4.1 Note that the case $g_1 = g_2 = 0$ is an important in both theory and applications. In this case, the system is a Hamiltonian. So it is valuable to consider it especially.

Remark 4.2 If we have obtained the general solution (4.9) for (1.1), then setting $c_1 = -\frac{1}{v}\sqrt{2h\lambda}$ and $c_2 = g_1 = g_2 = 0$ in (4.9), we can get the solution (4.8) for (1.1) at once.

For (1.2), when $\mu > 0$, it is the same as the above discussion for (1.1). Thus, we have the same traveling wave solutions of (1.2) as (4.8) and (4.9) (replacing λ by μ only).

When $\mu < 0$, in view of (4.3), let $\xi = x - vt$, so we have $u(x,t) = \phi(x - vt) = \phi(\xi)$. Substituting it into (1.2), we have

$$v^2\phi'' + \mu\phi'''' = 0,$$

where $\phi' = \frac{d\phi}{d\xi}$. Integrating the above equation twice we have

$$v^2\phi = -\mu\phi'' + g_1\xi + g_2, \quad -\mu > 0, \tag{4.10}$$

where g_1 and g_2 are constants of integration.

In particular, if $g_1 = g_2 = 0$, then (4.10) becomes

$$v^2\phi = -\mu\phi''. \tag{4.11}$$

Equation (4.11) is equivalent to the planar system

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = -\frac{v^2}{\mu}\phi, \tag{4.12}$$

which is a Hamiltonian system with the Hamiltonian function

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{v^2}{2\mu}\phi^2 = h. \tag{4.13}$$

It is easy to see that the system (4.12) has one equilibrium point $O(0,0)$ in the (ϕ, y)-phase plane also. Moreover, we have $J(0,0) = \det M(0,0) = \frac{v^2}{\mu} < 0$ and $\operatorname{Tr} M(0,0) = 0$. By the theory of planar dynamical system, we know that the equilibrium point $O(0,0)$ is a saddle point of system (4.12).

If $h = 0$, in view of (4.13), then we have

$$y^2 = -\frac{v^2}{\mu}\phi^2.$$

By using this formula and the first equation of (4.12), we have

$$\phi(\xi) = \exp\left(\frac{v}{\sqrt{-\mu}}\xi\right).$$

Thus, we obtain the following traveling wave solution of (1.2):

$$u(x,t) = \exp\left[\frac{v}{\sqrt{-\mu}}(x - vt)\right]. \tag{4.14}$$

If $h > 0$, in view of (4.13), then we have

$$y^2 = -\frac{v^2}{\mu}\phi^2 + 2h.$$

By using this formula and the first equation of (4.12), we have

$$\phi(\xi) = \frac{\sqrt{-2h\mu}}{v}\sinh\left(\frac{v}{\sqrt{-\mu}}\xi\right).$$

Hence, we obtain the following traveling wave solution of (1.2):

$$u(x,t) = \frac{\sqrt{-2h\mu}}{v}\sinh\left[\frac{v}{\sqrt{-\mu}}(x - vt)\right]. \tag{4.15}$$

If $h < 0$, in view of (4.13), then we have

$$y^2 = -\frac{v^2}{\mu}\phi^2 + 2h \quad (-h > 0).$$

By using this formula and the first equation of (4.12), we have

$$\phi(\xi) = \frac{\sqrt{2h\mu}}{v}\cosh\left(\frac{v}{\sqrt{-\mu}}\xi\right).$$

Thus, we obtain the following traveling wave solution of (1.2):

$$u(x,t) = \frac{\sqrt{2h\mu}}{v}\cosh\left[\frac{v}{\sqrt{-\mu}}(x - vt)\right]. \tag{4.16}$$

In general, we consider (4.10), that is

$$\frac{d^2\phi}{d\xi^2} + \frac{v^2}{\mu}\phi = \frac{g_1}{c^2}\xi + \frac{g_2}{c^2}.$$

Solving this second-class linear ordinary differential equation, we can get

$$\phi(\xi) = c_1\exp\left(\frac{v}{\sqrt{-\mu}}\xi\right) + c_2\exp\left(-\frac{v}{\sqrt{-\mu}}\xi\right) + \frac{g_1}{v^2}\xi + \frac{g_2}{v^2}.$$

Thus, we obtain the general solution of (1.2) as following:

$$u(x,t) = c_1\exp\left[\frac{v}{\sqrt{-\mu}}(x-vt)\right] + c_2\exp\left[-\frac{v}{\sqrt{-\mu}}(x-vt)\right] + \frac{g_1}{v^2}(x-vt) + \frac{g_2}{v^2}, \tag{4.17}$$

where c_i and g_i $(i = 1, 2)$ are arbitrary constant numbers.

Remark 4.3 Similar to Remark 4.2, if we have obtained the general solution (4.17) for (1.2), setting $c_1 = -c_2 = \frac{\sqrt{-2h\lambda}}{2v}$ and $g_1 = g_2 = 0$, $c_1 = c_2 = \frac{\sqrt{2h\lambda}}{2v}$ and $g_1 = g_2 = 0$ in (4.17), we can get the solutions (4.15) and (4.16) of (1.2), respectively.

4.2 The Similarity Reductions and Exact Solutions

In this subsection, we will consider the similarity reductions and exact solutions for (1.1) only, (1.2) is similar to be dealt with.

(i) For the generator V_1, we have $u = f(\xi)$, where $\xi = t$. Equation (1.1) reduced to the following ODE

$$f'' = 0, \tag{4.18}$$

where $f' = \frac{df}{d\xi}$. Clearly, (4.18) has the solution $f = c_1\xi + c_2$. Thus, we get the solution of (1.1) is

$$u(x,t) = c_1 t + c_2, \tag{4.19}$$

where c_1, c_2 are arbitrary constants.

(ii) Similar to case (i), for the generator V_2, we can get the solution of (1.1) is

$$u(x,t) = c_1 x^3 + c_2 x^2 + c_3 x + c_4, \tag{4.20}$$

where c_i $(i = 1, \dots, 4)$ are arbitrary constants.

For the generator V_4, we can only get the solution of (1.1) is $u(x,t) = c$, c is a constant. We note that the three solutions are trivial solutions for (1.1).

(iii) For the linear combination $V = V_1 + aV_4$, we have $u = e^{ax} f(\xi)$, where $\xi = t$. Equation (1.1) reduced to the following ODE

$$f'' + a^4 \lambda f = 0, \tag{4.21}$$

where $f' = \frac{df}{d\xi}$.

Equation (4.21) is a linear second-order ODE. When $\lambda > 0$, solving this equation, we have $f = c_1 \cos(a^2\sqrt{\lambda}\xi) + c_2 \sin(a^2\sqrt{\lambda}\xi)$. Thus, we obtain the solution of (1.1) is

$$u(x,t) = e^{ax}[c_1 \cos(a^2\sqrt{\lambda}t) + c_2 \sin(a^2\sqrt{\lambda}t)], \tag{4.22}$$

where c_1, c_2 are arbitrary constants. Here and in what follows, we assume $a \neq 0$ is an arbitrary constant.

When $\lambda < 0$, solving this equation, we have $f = c_1 e^{a^2\sqrt{-\lambda}\xi} + c_2 e^{-a^2\sqrt{-\lambda}\xi}$. Thus, we obtain the solution of (1.1) is

$$u(x,t) = c_1 e^{ax + a^2\sqrt{-\lambda}t} + c_2 e^{ax - a^2\sqrt{-\lambda}t}, \tag{4.23}$$

where c_1, c_2 are arbitrary constants.

(iv) For the linear combination $V = V_2 + aV_4$, we have $u = e^{at} f(\xi)$, where $\xi = x$. Equation (1.1) reduced to the following ODE

$$\lambda f'''' + a^2 f = 0, \tag{4.24}$$

where $f' = \frac{df}{d\xi}$.

Equation (4.23) is a linear fourth-order ODE, the characteristic equation is $\lambda r^4 + a^2 = 0$. When $\lambda > 0$, we have $f(\xi) = e^{\sqrt[4]{\frac{a^2}{4\lambda}}\xi}[c_1\cos(\sqrt[4]{\frac{a^2}{4\lambda}}\xi) + c_2\sin(\sqrt[4]{\frac{a^2}{4\lambda}}\xi)] + e^{-\sqrt[4]{\frac{a^2}{4\lambda}}\xi} \times [c_3\cos(\sqrt[4]{\frac{a^2}{4\lambda}}\xi) + c_4\sin(\sqrt[4]{\frac{a^2}{4\lambda}}\xi)]$. Thus, we get the solution of (1.1) is

$$u(x,t) = e^{\sqrt[4]{\frac{a^2}{4\lambda}}x+at}\left[c_1\cos\left(\sqrt[4]{\frac{a^2}{4\lambda}}x\right) + c_2\sin\left(\sqrt[4]{\frac{a^2}{4\lambda}}x\right)\right] + e^{-\sqrt[4]{\frac{a^2}{4\lambda}}x+at}\left[c_3\cos\left(\sqrt[4]{\frac{a^2}{4\lambda}}x\right) + c_4\sin\left(\sqrt[4]{\frac{a^2}{4\lambda}}x\right)\right], \tag{4.25}$$

where c_i $(i = 1, \ldots, 4)$ are arbitrary constants.

When $\lambda < 0$, we have $f = c_1e^{\sqrt[4]{\frac{a^2}{-\lambda}}\xi} + c_2e^{-\sqrt[4]{\frac{a^2}{-\lambda}}\xi} + c_3\cos(\sqrt[4]{\frac{a^2}{-\lambda}}\xi) + c_4\sin(\sqrt[4]{\frac{a^2}{-\lambda}}\xi)$. Thus, we get the solution of (1.1) is

$$u(x,t) = c_1e^{\sqrt[4]{\frac{a^2}{-\lambda}}x+at} + c_2e^{-\sqrt[4]{\frac{a^2}{-\lambda}}x+at} + e^{at}\left[c_3\cos\left(\sqrt[4]{\frac{a^2}{-\lambda}}x\right) + c_4\sin\left(\sqrt[4]{\frac{a^2}{-\lambda}}x\right)\right], \tag{4.26}$$

where c_i $(i = 1, \ldots, 4)$ are arbitrary constants.

(v) For the linear combination $V = V_3 + aV_4$, we have $u = t^{\frac{a}{2}}f(\xi)$, where $\xi = xt^{-\frac{1}{2}}$. Equation (1.1) reduced to the following ODE

$$\lambda f'''' + \frac{1}{4}\xi^2 f'' - \frac{2a-3}{4}\xi f' + \frac{a(a-2)}{4}f = 0, \tag{4.27}$$

where $f' = \frac{df}{d\xi}$.

(vi) For the generator V_3, we have $u = f(\xi)$, where $\xi = xt^{-\frac{1}{2}}$. Equation (1.1) reduced to the following ODE

$$4\lambda f'''' + \xi^2 f'' + 3\xi f' = 0, \tag{4.28}$$

where $f' = \frac{df}{d\xi}$. Note that (4.27) and (4.28) are higher-order nonlinear ODEs, we will deal with the equations in the next section.

5 The Exact Power Series Solution

In general, we cannot obtain the exact explicit solutions for the higher-order nonlinear ODEs by using the elementary functions and integrals. Now, we deal with the reduced (4.28) of (1.1) by using the power series method [12, 15]. Clearly, if $f' = 0$, then (4.28) has a trivial solution $f = \eta$ (η is an arbitrary constant number). Assume $f' \neq 0$, let $f' = y(\xi)$, from (4.28), we have

$$4\lambda y''' + \xi^2 y' + 3\xi y = 0. \tag{5.1}$$

Next, we will seek a solution of (5.1) in a power series of the form

$$y(\xi) = \sum_{n=0}^{\infty} y_n\xi^n. \tag{5.2}$$

In view of (5.2), we can get

$$y'(\xi) = \sum_{n=0}^{\infty}(n+1)y_{n+1}\xi^n \tag{5.3}$$

and

$$y'''(\xi) = \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)y_{n+3}\xi^n. \tag{5.4}$$

Substituting (5.2), (5.3) and (5.4) into (5.1), we have

$$4\lambda(6y_3 + 24y_4\xi) + 4\lambda\sum_{n=2}^{\infty}(n+1)(n+2)(n+3)y_{n+3}\xi^n + \sum_{n=2}^{\infty}(n-1)y_{n-1}\xi^n + 3y_0\xi + 3\sum_{n=2}^{\infty}y_{n-1}\xi^n = 0. \tag{5.5}$$

From (5.5), comparing coefficients, we obtain

$$y_3 = 0, \qquad y_4 = -\frac{y_0}{32\lambda}. \tag{5.6}$$

Generally, for $n \geq 2$, from (5.5), we have

$$y_{n+3} = -\frac{y_{n-1}}{4\lambda(n+1)(n+3)}, \quad n = 2, 3, \ldots. \tag{5.7}$$

Note that from (5.7), let $n = 1$, we can get $y_4 = -\frac{y_0}{32\lambda}$ also.

In view of (5.7), we can get all the coefficients y_n of power series (5.2), e.g.,

$$y_5 = -\frac{y_1}{60\lambda} \quad (n=2); \qquad y_6 = -\frac{y_2}{96\lambda} \quad (n=3);$$

$$y_7 = -\frac{y_3}{140\lambda} = 0 \quad (n=4); \qquad y_8 = -\frac{y_4}{168\lambda} = -\frac{1}{168\lambda}\left(-\frac{y_0}{32\lambda}\right) = \frac{y_0}{5376\lambda^2} \quad (n=5);$$

$$y_9 = -\frac{y_5}{252\lambda} = \frac{y_1}{15120\lambda^2} \quad (n=6); \qquad y_{10} = -\frac{y_6}{320\lambda} = \frac{y_2}{30720\lambda^2} \quad (n=7);$$

$$y_{11} = -\frac{y_7}{396\lambda} = \frac{y_3}{55440\lambda^2} = 0 \quad (n=8); \qquad y_{12} = -\frac{y_8}{480\lambda} = -\frac{y_0}{2580480\lambda^3} \quad (n=9)$$

and so on. Thus, for arbitrary chosen y_0, y_1 and y_2, then the other terms of the sequence $\{y_n\}_{n=3}^{\infty}$ can be determined successively from (5.6) and (5.7) in a unique manner. Furthermore, in view of (5.6), we have $y_3 = 0$, that is $y_n = y_{3+4k} = 0$ $(k = 0, 1, 2, \ldots)$. This implies that for (5.1), there exists a power series solution (5.2).

In fact, the power series solution of (5.1) can be written as follows:

$$y(\xi) = y_0 + y_1\xi + y_2\xi^2 + \sum_{n=1}^{\infty}y_{n+3}\xi^{n+3} = y_0 + y_1\xi + y_2\xi^2 + \sum_{n=1}^{\infty}\frac{-y_{n-1}}{4\lambda(n+1)(n+3)}\xi^{n+3}, \tag{5.8}$$

where $y_{3+4k}=0$ ($k=0,1,2,\ldots$). Note that in terms of the above example, we can write the approximate form of (5.8) as follows:

$$y(\xi)=y_0+y_1\xi+y_2\xi^2-\frac{y_0}{32\lambda}\xi^4-\frac{y_1}{60\lambda}\xi^5-\frac{y_2}{96\lambda}\xi^6+\frac{y_0}{5376\lambda^2}\xi^8+\frac{y_1}{15120\lambda^2}\xi^9 +\frac{y_2}{30720\lambda^2}\xi^{10}-\frac{y_0}{2580480\lambda^3}\xi^{12}-\cdots.$$

Thus, from $f'=y(\xi)$, we obtain the power series solution of (4.28):

$$\begin{aligned} f(\xi)&=g+y_0\xi+\frac{1}{2}y_1\xi^2+\frac{1}{3}y_2\xi^3+\sum_{n=1}^{\infty}\frac{y_{n+3}}{n+4}\xi^{n+4}\\ &=g+y_0\xi+\frac{1}{2}y_1\xi^2+\frac{1}{3}y_2\xi^3+\sum_{n=1}^{\infty}\frac{-y_{n-1}}{4\lambda^2(n+1)(n+3)(n+4)}\xi^{n+4}, \end{aligned} \tag{5.9}$$

where g is an arbitrary constant number, which is equivalent to the trivial solution $f=\eta$ of (4.28). Thus, we get the power series solution of (1.1) as following:

$$\begin{aligned} u(x,t)&=g+y_0(xt^{-\frac{1}{2}})+\frac{1}{2}y_1(xt^{-\frac{1}{2}})^2+\frac{1}{3}y_2(xt^{-\frac{1}{2}})^3+\sum_{n=1}^{\infty}\frac{y_{n+3}}{n+4}(xt^{-\frac{1}{2}})^{n+4}\\ &=g+y_0(xt^{-\frac{1}{2}})+\frac{1}{2}y_1(xt^{-\frac{1}{2}})^2+\frac{1}{3}y_2(xt^{-\frac{1}{2}})^3\\ &\quad+\sum_{n=1}^{\infty}\frac{-y_{n-1}}{4\lambda(n+1)(n+3)(n+4)}(xt^{-\frac{1}{2}})^{n+4}, \end{aligned} \tag{5.10}$$

where g and y_i ($i=0,1,2$) are arbitrary constant numbers, y_n ($n=3,4,\ldots$) can be obtained successively from (5.6) and (5.7). In particular, we have $y_n=0$, for all $n=3+4k, k=0,1,2,\ldots$.

Corresponding to the approximate form of (5.8), we have the approximate form of (5.10) as following:

$$\begin{aligned} u(x,t)=g&+y_0(xt^{-\frac{1}{2}})+\frac{1}{2}y_1(xt^{-\frac{1}{2}})^2+\frac{1}{3}y_2(xt^{-\frac{1}{2}})^3-\frac{y_0}{160\lambda}(xt^{-\frac{1}{2}})^5-\frac{y_1}{360\lambda}(xt^{-\frac{1}{2}})^6\\ &-\frac{y_2}{672\lambda}(xt^{-\frac{1}{2}})^7+\frac{y_0}{48384\lambda^2}(xt^{-\frac{1}{2}})^9+\frac{y_1}{151200\lambda^2}(xt^{-\frac{1}{2}})^{10}+\frac{y_2}{337920\lambda^2}(xt^{-\frac{1}{2}})^{11}\\ &-\frac{y_0}{33546240\lambda^3}(xt^{-\frac{1}{2}})^{13}-\cdots. \end{aligned}$$

6 Conclusion and Remarks

In this paper, we obtain the symmetries for the two rod equations. The conservation laws are derived from the direct method. The traveling wave solutions for the two equations are given. Furthermore, the all similarity reductions and exact explicit solutions based on the Lie symmetry analysis are presented.

We know that the conservation laws can be derived also from the symmetries and Lie point symmetry generator can be used to derive new conservation laws from known conservation laws. Moreover, based on the conservation laws, we may consider the exact solutions for PDEs, and study the properties of solutions of the systems. We hope to investigate in the future.

Remark 6.1 In fact, we can prove the convergence of the power series solution (5.2) (see [7, 8]). So the power series solution is an exact analytic solution. Moreover, we can get the exact power series solution of (4.27) in the similar way. Thus, the another exact analytic solution for (1.1) could be obtained.

Remark 6.2 Note that (1.2) is the degenerate case for the general rod equation (see [11]), and it possesses the physical significance also. The similarity solutions based on the Lie symmetry analysis and exact power series solution for (1.2) can be obtained also in view of our previous discussion. For the sake of succinctness, we omitted in this paper.

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