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Lie symmetry analysis and exact solutions for the short pulse equation[☆]

Hanze Liu[a,b,∗], Jibin Li[a,c]

[a] Center for Nonlinear Science Studies, Kunming University of Science and Technology, Kunming Yunnan 650093, China
[b] Department of Mathematics, Binzhou University, Binzhou Shandong 256603, China
[c] Department of Mathematics, Zhejiang Normal University, Jinhua Zhejiang 321004, China

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ABSTRACT

In this paper, the Lie symmetry analysis and the generalized symmetry method are performed for a short pulse equation (SPE). The symmetries for this equation are given. For the traveling wave solutions, the exact parametric representations are investigated. To guarantee the existence of the above solutions, all parameter conditions are determined. Furthermore, the exact analytic solutions are obtained by using the power series method.

1. Introduction

Nonlinear partial differential equations (PDEs) arising in many physical fields like the condense matter physics, fluid mechanics, plasma physics and optics, etc, exhibit a rich variety of nonlinear phenomena. Recently, many PDEs generated from the systems of impulse and neural networks as well. The investigation of the exact solutions plays an important role in the study of nonlinear physical systems and such neural networks. A wealth of methods have been developed to find these exact solutions of a PDE though it is rather difficult. Some of the most important methods are the inverse scattering method [1], Darboux and Bäcklund transformations [2], Hirota's bilinear method [2–4], Lie symmetry analysis [5–8], CK method [9,10], etc. It is well-known that the Lie group method is a powerful and direct approach to construct exact solutions of nonlinear differential equations. Furthermore, based on the Lie group method, many other type of exact solutions of PDE can be obtained, such as the traveling wave solutions, soliton solutions, fundamental solutions [11,12], and so on.

In this paper, we will consider the short pulse equation (SPE) which has the general form

$$u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx}, \tag{1}$$

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∗ Corresponding author at: Center for Nonlinear Science Studies, Kunming University of Science and Technology, Kunming Yunnan 650093, China.
E-mail addresses: hzliumath@hotmail.com (H. Liu), jibinli@gmail.com (J. Li).

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where $u = u(x, t)$ is the unknown real function and subscripts denote differentiation, $x, t \in R$, α and β are real parameters, $\alpha\beta \neq 0$. In practical, Eq. (1) can be written as the more usual form

$$u_{xt} = \alpha u + 2\beta u u_x^2 + \beta u^2 u_{xx}. \tag{2}$$

This general SPE was derived by T.Schäfer and C.E.Wayne as a model equation describing the propagation of ultra-short light pulses in silica optical fibres (see [13], p.94), and the numerical computations were presented in that paper. In particular, if we let $\alpha = 1$ and $\beta = \frac{1}{2}$, then Eq. (1) will be changed to the special form $u_{xt} = u + \frac{1}{6}(u^3)_{xx}$. In [14–20], many results are obtained about the special SPE. In recent works [21–23], we have investigated the dynamical behavior of loop soliton solutions for several equations.

In the present paper, by using Lie group analysis and the generalized symmetry method, we will investigate the short pulse equation in detail, and the exact explicit traveling wave solutions and analytic solutions will be given.

For the sake of Lie symmetry analysis, we write Eq. (1) as the following another usual form in mathematical physics:

$$u_t = \alpha D^{-1} u + \beta u^2 u_x + p, \tag{3}$$

where $D^{-1} = \int \cdot \mathrm{d}x$, $p = p(t)$ is an arbitrary integral function. Moreover, we have

$$u_{tt} = \alpha^2 D^{-1} v + \frac{4}{3}\alpha\beta u^3 + 4\beta^2 u^3 u_x^2 + 2\alpha\beta v u u_x + \beta^2 u^4 u_{xx} + 2\beta p u u_x + r, \tag{4}$$

where $v = D^{-1}u$, $r = r(x, t)$ is an integral function. We note that Eqs. (3) and (4) are necessary for Lie symmetry analysis in what follows.

The outline of this paper is as follows. In Section 2, we perform Lie group analysis for the short pulse equation. In Section 3, the generalized symmetry method was employed for investigating the symmetries of Eq. (1). In Section 4, we will present the qualitative analysis and provide all the traveling wave solutions for this equation. In Section 5, the exact analytic solutions are obtained by using the power series method. In Section 6, we conclude and make some remarks.

2. Lie symmetry analysis for SPE

In this section, we will perform Lie group method for Eq. (1).

The Lie group method is sometimes also called symmetry analysis. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones.

Firstly, let us consider a one-parameter Lie group of infinitesimal transformation:

$$x \to x + \epsilon\xi(x, t, u),$$
$$t \to t + \epsilon\tau(x, t, u),$$
$$u \to u + \epsilon\phi(x, t, u),$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$V = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u}. \tag{5}$$

The symmetry group of Eq. (1) will be generated by the vector field of the form (5). Applying the second prolongation $\mathrm{pr}^{(2)}V$ of V to Eq. (2), we find that the coefficient functions ξ, τ and ϕ must satisfy the symmetry condition

$$-\alpha\phi - 2\beta u_x^2\phi - 2\beta u u_{xx}\phi - 4\beta u u_x \phi^x - \beta u^2 \phi^{xx} + \phi^{xt} = 0, \tag{6}$$

where ϕ, ϕ^x, ϕ^{xx} and ϕ^{xt} are all coefficients of $\mathrm{pr}^{(2)}V = \mathrm{pr}^{(1)}V + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}$, and furthermore, we have

$$\phi^x = D_x\phi - u_x D_x\xi - u_t D_x\tau, \tag{7}$$

$$\phi^{xx} = D_x^2\phi - u_x D_x^2\xi - u_t D_x^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau, \tag{8}$$

$$\phi^{xt} = D_t D_x\phi - u_x D_t D_x\xi - u_{xt}D_x\xi - u_{xx}D_t\xi - u_t D_t D_x\tau - u_{tt}D_x\tau - u_{xt}D_t\tau, \tag{9}$$

where D_x and D_t are the total derivatives with respect to x and t, respectively.

Substituting (2)–(4) into (7)–(9), respectively, then plugging (7)–(9) into (6), and equating the coefficients of the various monomials in the first, second and the other order partial derivatives with respect to x and various powers of u, we can find the determining equations for the symmetry group of the short pulse equation. Solving these equations, we get the following forms of the coefficient functions

$$\xi = c_1 x + c_3, \qquad \tau = -c_1 t + c_2, \qquad \phi = c_1 u,$$

where c_1, c_2 and c_3 are arbitrary constants. Thus the Lie algebra of infinitesimal symmetries of Eq. (1) is spanned by the three vectors

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

It is easy to verify that $\{V_1, V_2, V_3\}$ is closed under the Lie bracket. So we can see that the generator of invariant group $V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$ of Eq. (1) construct three-dimensional Lie algebra, which is spanned by the basis $\{V_1, V_2, V_3\}$.

Thus, we have the corresponding one-parameter group of symmetries of the short pulse equation:

$$G_1 : (x,t,u) \to (x+\epsilon, t, u),$$
$$G_2 : (x,t,u) \to (x, t+\epsilon, u),$$
$$G_3 : (x,t,u) \to (e^{\epsilon}x, e^{-\epsilon}t, e^{\epsilon}u).$$

We can see that G_1 is a space translation, G_2 is a time translation, and G_3 is a scaling transformation.

3. Generalized symmetries for SPE

In Section 2, we have obtained the symmetry group of Eq. (1). Now we consider the symmetries of Eq. (1) by using the generalized symmetry method. This method is also called the method of undetermined coefficient [24].

Let

$$\sigma(x,t,u) = a(x,t)u_t + b(x,t)u_x + c(x,t)u + d(x,t) \tag{10}$$

be a symmetry of Eq. (1), where $a(x,t)$, $b(x,t)$, $c(x,t)$ and $d(x,t)$ are coefficient functions to be determined. On the other hand, by the definition of generalized symmetry [2,5,24], it is easy to show that $\sigma = \sigma(x,t,u)$ is a symmetry of Eq. (1) if and only if

$$\sigma_{xt} - \alpha\sigma - 2\beta u_x^2\sigma - 4\beta u u_x\sigma_x - 2\beta u u_{xx}\sigma - \beta u^2\sigma_{xx} = 0. \tag{11}$$

Substituting (2)–(4) and (10) into (11). We note that the coefficient of $D^{-1}v$ in the left-hand side of (11) requires that $a_x = 0$, so the coefficient a is a function of t only. The coefficient of u_{xx} implies that $b_t = 0$, so we can suppose $b = b(x)$. The coefficient of u_x^2 requires that $d = 0$. So it is not difficult to show that the coefficient functions $a(x,t)$, $b(x,t)$, $c(x,t)$ and $d(x,t)$ are as follows:

$$a(x,t) = -c_1t + c_2, \quad b(x,t) = c_1x + c_3, \quad c(x,t) = -c_1, \quad d(x,t) = 0, \tag{12}$$

where c_i $(i = 1, 2, 3)$ are arbitrary constants. Substituting (12) into (10), we have

$$\sigma = c_2u_t + c_3u_x + c_1(xu_x - tu_t - u).$$

Hence, we obtain that the symmetries for Eq. (1) are of the forms as follows:

$$\sigma_1 = u_x, \quad \sigma_2 = u_t, \quad \sigma_3 = xu_x - tu_t - u,$$

which coincide precisely with the vector field V_1, V_2 and V_3 are obtained in Section 2.

Since each G_i $(i = 1, 2, 3)$ is a symmetry group, it implies that if $u = f(x,t)$ is a solution of the short pulse equation, then $u^{(1)}$, $u^{(2)}$ and $u^{(3)}$ as follows are solutions of Eq. (1) as well:

$$u^{(1)} = f(x-\epsilon, t), \tag{13}$$
$$u^{(2)} = f(x, t-\epsilon), \tag{14}$$
$$u^{(3)} = e^{\epsilon}f(e^{-\epsilon}x, e^{\epsilon}t), \tag{15}$$

where ϵ is an arbitrary real number.

Next, we reduce Eq. (1) to ordinary differential equations (ODEs) by using the above vector field.

(i) In general, the linear combination of the two generators V_1 and V_2 will generate the traveling wave solutions for a PDE (see Section 4).

(ii) For the generator of the scaling transformation V_3, we have the following similarity variables

$$\xi = xt, \quad \omega = tu,$$

and the group-invariant solution is $\omega = f(\xi)$, that is

$$u = \frac{1}{t}f(xt). \tag{16}$$

Substituting (16) into (1), we reduce the SPE to the following ODE:

$$\xi f'' - \beta f^2 f'' - 2\beta f f'^2 - \alpha f = 0, \tag{17}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$. It implies that if $\omega = f(\xi)$ is a solution of Eq. (17), then (16) is a solution of Eq. (1). Note that Eq. (17) is a nonautonomous and nonlinear ODE, we cannot obtain the exact solution by using the elementary functions.

4. Exact traveling wave solutions of SPE

In this section, we consider the traveling wave solutions of the SPE equation.

In general, the traveling wave solutions to a PDE arise as special group-invariant solutions in which the group under consideration is a translational group on the space of independent variables. In the present case, we consider the translation group $(x, t, u) \mapsto (x + c\epsilon, t \pm \epsilon, u)$ $(\epsilon \in R)$, generated by the generator $c\frac{\partial}{\partial x} \pm \frac{\partial}{\partial t}$, where c is a fixed constant, which will determine the speed of the waves. So we can obtain that the global invariants of this group are as follows:

$$\xi = x \mp ct, \quad \phi = u. \tag{18}$$

In view of (18), let $\xi = x \pm ct$, we have $u(x, t) = \phi(x \pm ct) = \phi(\xi)$, where c is the propagating wave velocity. Substituting it into the original PDE, we obtain a traveling wave equation. Then, solving this equation by the bifurcation theory method of dynamical systems, we will obtain the traveling wave solutions.

Now, we consider the traveling wave solutions of the general short pulse equation (1). Let $\xi = x + ct$. We have $u(x, t) = \phi(x + ct) = \phi(\xi)$, where $c > 0$ is the propagating wave velocity. Substituting it into Eq. (1), we obtain the following ordinary differential equation (ODE):

$$c\phi'' - \beta\phi^2\phi'' - 2\beta\phi(\phi')^2 - \alpha\phi = 0, \tag{19}$$

where $\phi' = \frac{d\phi}{d\xi}$. Furthermore, Eq. (19) is equivalent to the planar system

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{\phi(\alpha + 2\beta y^2)}{c - \beta\phi^2}. \tag{20}$$

This system has the first integral

$$H(\phi, y) = (c - \beta\phi^2)^2 y^2 - \alpha\phi^2\left(c - \frac{1}{2}\beta\phi^2\right) = h, \tag{21}$$

where h is the integral constant.

When $\beta > 0$, the right hand of the second equation of system (20) is discontinuous. We call such systems singular traveling wave systems. The straight lines $\phi = \pm\sqrt{\frac{c}{\beta}}$ in the (ϕ, y)-phase plane are called singular straight lines. It derives the existence of some non-smooth behavior and breaking properties of traveling wave solutions of system (20). Making the transformation $d\xi = (c - \beta\phi^2)d\zeta$, system (20) becomes its associated regular system

$$\frac{d\phi}{d\zeta} = (c - \beta\phi^2)y, \qquad \frac{dy}{d\zeta} = \phi(\alpha + 2\beta y^2). \tag{22}$$

Clearly, system (22) has the same invariant curve solutions as (20). But when $\beta > 0$, the variables ξ and ζ of Eqs. (20) and (22) have different scales near the straight lines $\phi = \pm\sqrt{\frac{c}{\beta}}$. The variable ξ is a slow variable, while the variable ζ is a fast variable (see [21]).

When $\alpha\beta > 0$ or $\alpha > 0,\ \beta < 0$, system (22) has only one equilibrium point $O(0, 0)$. When $\alpha < 0,\ \beta > 0$, system (22) has five equilibrium points $O(0, 0)$ and $A_j\left(\pm\sqrt{\frac{c}{\beta}}, \pm\sqrt{-\frac{\alpha}{2\beta}}\right)$, $j = 1, 2, 3, 4$. Let $h_0 = H(0, 0) = 0$, $h_s = H\left(\pm\sqrt{\frac{c}{\beta}}, \pm\sqrt{-\frac{\alpha}{2\beta}}\right) = -\frac{\alpha c^2}{2\beta}$ defined by (21).

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (22) at an equilibrium point (ϕ_e, y_e) and $J(\phi_e, y_e)$ be its Jacobian determinant. Then,we have $J(0, 0) = \det M(0, 0) = -\alpha c$ and $\mathrm{Tr}M(0, 0) = 0$, $J(A_j) = 4\alpha c < 0$ $(j = 1, 2, 3, 4)$. By the theory of planar dynamical system, we know that if $\alpha > 0$, then the equilibrium point $O(0, 0)$ is a saddle point; if $\alpha < 0$, then the equilibrium point $O(0, 0)$ is a center point of system (22), while the equilibrium points A_j $(i = 1, \dots, 4)$ are saddle points.

By qualitative analysis, we obtain the following phase portraits of system (22).

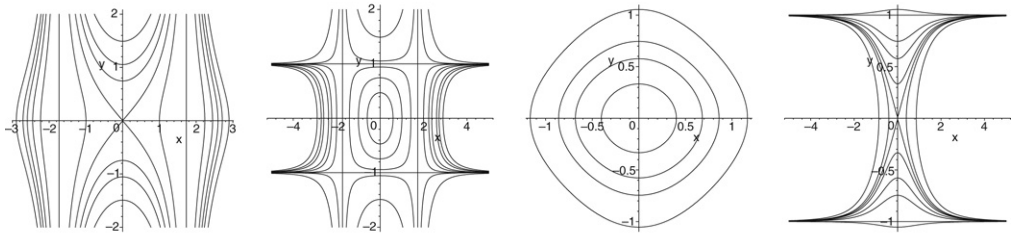
Fig. 1 gives rise to all possible traveling wave solutions of (1) for $c > 0$ in (α, β)-parametric plane.

We next consider the exact parametric representations of the bounded solutions of (20). The case $\alpha > 0$, $\beta > 0$ has been discussed by Ref. [21].

4.1. $\alpha < 0, \beta > 0$ *(see Fig. 1(1-2))*

Corresponding to the closed orbits defined by $H(\phi, y) = h,\ h \in (0, h_s)$, we have

$$y = \frac{\sqrt{\frac{|\alpha|\beta}{2}\left(\frac{2h}{|\alpha|\beta} - \frac{2c}{\beta}\phi^2 + \phi^4\right)}}{c - \beta\phi^2} = \frac{\sqrt{\frac{\alpha\beta}{2}(r_1^2 - \phi^2)(r_2^2 - \phi^2)}}{c - \beta\phi^2}.$$



(1-1) (1-2) (1-3) (1-4)

(1-1) $\alpha > 0,\ \beta > 0$. (1-2) $\alpha < 0,\ \beta > 0$. (1-3) $\alpha < 0,\ \beta < 0$. (1-4) $\alpha > 0,\ \beta < 0$.

Fig. 1. The different phase portraits of (20) when α and β are changed.

By using the first equation of (20), we obtain

$$\sqrt{\frac{|\alpha|\beta}{2}}\xi = \int_0^{\phi} \frac{c\mathrm{d}\phi}{\sqrt{(r_1^2-\phi^2)(r_2^2-\phi^2)}} - \beta\int_0^{\phi} \frac{\phi^2\mathrm{d}\phi}{\sqrt{(r_1^2-\phi^2)(r_2^2-\phi^2)}}.$$

Introducing a new variable χ, the above integral implies the parametric representation of the family of periodic traveling wave solutions of (1):

$$\begin{aligned} &\phi(\chi) = r_2\mathrm{sn}(\chi, k), \\ &\xi(\chi) = x+ct = \sqrt{\frac{2}{|\alpha|\beta}}\left[\left(\frac{c}{r_1} - \beta r_2\right)\chi + \beta r_2 E(\arcsin(\mathrm{sn}(\chi, k)), k)\right], \end{aligned} \tag{23}$$

where $k^2 = \frac{r_2}{r_1}$.

Corresponding to two heteroclinic orbits of (22) given by $y = \pm\sqrt{\frac{|\alpha|}{2\beta}}$, we have two breaking wave solutions of (1):

$$\phi(\xi) = \pm\sqrt{\frac{|\alpha|}{2\beta}}\xi, \qquad -\sqrt{\frac{2c}{|\alpha|}} \le \xi \le \sqrt{\frac{2c}{|\alpha|}}. \tag{24}$$

4.2. $\alpha < 0, \beta < 0$ (see Fig. 1(1-3))

Corresponding to the closed orbits defined by $H(\phi, y) = h$, $h \in (0, \infty)$, we have

$$y = \frac{\sqrt{\frac{\alpha\beta}{2}\left(\frac{2h}{\alpha\beta} + \frac{2c}{\beta}\phi^2 - \phi^4\right)}}{c+|\beta|\phi^2} = \frac{\sqrt{\frac{\alpha\beta}{2}(r_1^2+\phi^2)(r_2^2-\phi^2)}}{c+|\beta|\phi^2}.$$

By using the first equation of (20), we obtain

$$\sqrt{\frac{\alpha\beta}{2}}\xi = \int_{\phi}^{r_2} \frac{c\mathrm{d}\phi}{\sqrt{(r_2^2-\phi^2)(r_1^2+\phi^2)}} - \beta\int_0^{\phi} \frac{\phi^2\mathrm{d}\phi}{\sqrt{(r_1^2+\phi^2)(r_2^2-\phi^2)}}.$$

Introducing a new variable χ, the above integral implies the parametric representation of the family of periodic traveling wave solutions of (1):

$$\begin{aligned} &\phi(\chi) = r_2\mathrm{cn}(\chi, k), \\ &\xi(\chi) = x+ct = \sqrt{\frac{2}{\alpha\beta}}\left[\left(\frac{c+|\beta|r_2}{\sqrt{r_1^2+r_2^2}} - |\beta|r_2\right)\chi + |\beta| E(\arccos(\mathrm{cn}(\chi, k_1)), k_1)\right], \end{aligned} \tag{25}$$

where $k_1^2 = \frac{r_2}{\sqrt{r_1^2+r_2^2}}$.

5. Exact analytic solutions for SPE

In general, we cannot obtain the exact and explicit solutions for the nonlinear ordinary differential equations (ODEs) such as Eq. (17) by using the elementary functions. But we know that the power series can be used to solve ODEs, including many complicated nonlinear differential equations with nonconstant coefficients, [25–28]. Next, we will consider the power series solution for this reduced equation.

Now, we seek a solution of Eq. (17) in a power series of the form

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n, \tag{26}$$

where $c_0 = f(0) \neq 0$ (see Remark 3). Substituting (26) into (17), we have

$$\begin{aligned} &2\beta c_0^2 c_2 + \beta(6c_0^2 c_3 + 4c_0 c_1 c_2)\xi + \beta c_0^2 \sum_{n=2}^{\infty}(n+1)(n+2)c_{n+2}\xi^n \\ &+ \beta \sum_{n=2}^{\infty}\left[\sum_{k=1}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_i c_{k-i} c_{n+2-k}\right]\xi^n + 2\beta c_0 c_1^2 \\ &+ 2\alpha(c_1^3 + 4c_0 c_1 c_2)\xi + 2\beta \sum_{n=2}^{\infty}\left[\sum_{k=0}^{n}\sum_{i=1}^{k+1} i(k+2-i)c_i c_{k+2-i} c_{n-k}\right]\xi^n \\ &+ \alpha c_0 + \alpha c_1 \xi + \alpha \sum_{n=2}^{\infty} c_n \xi^n - 2c_2\xi - \sum_{n=2}^{\infty} n(n+1)c_{n+1}\xi^n = 0. \end{aligned} \tag{27}$$

From (27), comparing coefficients, we obtain (for $n = 0$)

$$c_0(2\beta c_0 c_2 + 2\beta c_1^2 + \alpha) = 0, \tag{28}$$

and (for $n = 1$)

$$6\beta c_0^2 c_3 + 12\beta c_0 c_1 c_2 + 2\beta c_1^3 + \alpha c_1 - 2c_2 = 0. \tag{29}$$

Generally, for $n \geq 2$, in view of (27), we have

$$\begin{aligned} c_{n+2} = \frac{1}{\beta c_0^2 (n+1)(n+2)} &\left[n(n+1)c_{n+1} - \alpha c_n - 2\beta \sum_{k=0}^{n}\sum_{i=1}^{k+1} i(k+2-i)c_i c_{k+2-i} c_{n-k} \right. \\ &\left. - \beta \sum_{k=1}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_i c_{k-i} c_{n+2-k} \right], \quad n = 2, 3, \ldots. \end{aligned} \tag{30}$$

Thus, from (28), for arbitrary chosen $c_0 = \eta \neq 0$ and $c_1 = \lambda$, we have $c_2 = \frac{-1}{2\beta\eta}(2\beta\lambda^2 + \alpha)$. Furthermore, in view of (29) and (30), we have $c_3 = \frac{-1}{6\beta\eta^2}(12\beta\eta\lambda c_2 + 2\beta\lambda^3 + \alpha\lambda - 2c_2)$, $c_4 = \frac{-1}{12\beta\eta^2}(24\beta\eta\lambda c_3 + 12\beta\eta c_2^2 + 8\beta\lambda^2 c_2 + \alpha c_2 - 6c_3)$, and so on.

Therefore, the other terms of the sequence $\{c_n\}_{n=4}^{\infty}$ can be determined successively from (30) in a unique manner. This implies that for Eq. (17), there exists a power series solution (26) with the coefficients given by (28)–(30).

Now we show that the convergence of the power series solution (26) of Eq. (17). In fact, from (30), we have

$$|c_{n+2}| \leq M\left[|c_n| + |c_{n+1}| + \sum_{k=0}^{n}\sum_{i=1}^{k+1}|c_i||c_{k+2-i}||c_{n-k}| + \sum_{k=1}^{n}\sum_{i=0}^{k}|c_i||c_{k-i}||c_{n+2-k}|\right], \quad n = 2, 3, \ldots,$$

where $M = \max\{\frac{2}{c_0^2}, \frac{1}{|\beta|c_0^2}, \frac{|\alpha|}{|\beta|c_0^2}\}$. If we define a power series $\mu = P(\xi) = \sum_{n=0}^{\infty} p_n \xi^n$ by

$$p_0 = |c_0| = |\eta|, \qquad p_1 = |c_1| = |\lambda|, \qquad p_2 = |c_2|, \qquad p_3 = |c_3|$$

and

$$p_{n+2} = M\left(p_n + p_{n+1} + \sum_{k=0}^{n}\sum_{i=1}^{k+1} p_i p_{k+2-i} p_{n-k} + \sum_{k=1}^{n}\sum_{i=0}^{k} p_i p_{k-i} p_{n+2-k}\right), \quad n = 2, 3, \ldots,$$

then it is easily seen that

$$|c_n| \leq p_n, \quad n = 1, 2, \ldots.$$

In other words, the series $\mu = P(\xi) = \sum_{n=0}^{\infty} p_n\xi^n$ is a majorant series of (26). Next, we show that this series $\mu = P(\xi)$ has a positive radius of convergence. Indeed, note that by formal calculation, we have

$$\begin{aligned} P(\xi) &= p_0 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \sum_{n=2}^{\infty} p_{n+2}\xi^{n+2} \\ &= p_0 + p_1\xi + p_2\xi^2 + p_3\xi^3 + M\left[\sum_{n=2}^{\infty} p_n\xi^{n+2} + \sum_{n=2}^{\infty} p_{n+1}\xi^{n+2}\right. \\ &\quad \left. + \sum_{n=2}^{\infty}\sum_{k=0}^{n}\sum_{i=1}^{k+1} p_i p_{k+2-i} p_{n-k}\xi^{n+2} + \sum_{n=2}^{\infty}\sum_{k=1}^{n}\sum_{i=0}^{k} p_i p_{k-i} p_{n+2-k}\xi^{n+2}\right] \\ &= p_0 + p_1\xi + p_2\xi^2 + p_3\xi^3 + M[2P^3(\xi) - (3p_0 + p_1\xi)P^2(\xi) + (\xi^2 + \xi - 2p_0p_1\xi - 2p_0^2)P(\xi) + 3p_0^3 \\ &\quad + (11p_0^2p_1 - p_0)\xi + (6p_0^2p_2 + 6p_0p_1^2 - p_1)\xi^2 + (10p_0p_1p_2 + 6p_0^2p_3 + 2p_1^3 - p_2)\xi^3]. \end{aligned}$$

Consider now the implicit functional equation

$$\begin{aligned} F(\xi,\mu) &= \mu - p_0 - p_1\xi - p_2\xi^2 - p_3\xi^3 - M[2\mu^3 - (3p_0 + p_1\xi)\mu^2 + (\xi^2 + \xi - 2p_0p_1\xi - 2p_0^2)\mu + 3p_0^3 \\ &\quad + (11p_0^2p_1 - p_0)\xi + (6p_0^2p_2 + 6p_0p_1^2 - p_1)\xi^2 + (10p_0p_1p_2 + 6p_0^2p_3 + 2p_1^3 - p_2)\xi^3] = 0. \end{aligned}$$

Since F is analytic in the (ξ,μ)-plane and $F(0, p_0) = 0, F'_\mu(0, p_0) = 1+2Mp_0^2 \neq 0$, by the implicit function theorem [29,30], we see that $\mu = P(\xi)$ is analytic in a neighborhood of the point $(0, p_0)$ of the plane and with a positive radius. This implies that the power series (26) converges in a neighborhood of the point $(0, p_0)$ of the plane. This completes the proof.

Hence, the power series solution of Eq. (17) can be written as follows:

$$f(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \sum_{n=2}^{\infty} c_{n+2}\xi^{n+2}. \tag{31}$$

Note that in terms of the above computation, we can write the approximate form of (31) as follows:

$$\begin{aligned} f(\xi) &= \eta + \lambda\xi - \frac{1}{2\beta\eta}(2\beta\lambda^2 + \alpha)\xi^2 - \frac{1}{6\beta\eta^2}(12\beta\eta\lambda c_2 + 2\beta\lambda^3 + \alpha\lambda - 2c_2)\xi^3 \\ &\quad - \frac{1}{12\beta\eta^2}(24\beta\eta\lambda c_3 + 12\beta\eta c_2^2 + 8\beta\lambda^2 c_2 + \alpha c_2 - 6c_3)\xi^4 + \cdots. \end{aligned}$$

Thus, we can obtain that the power series solution of Eq. (1) is as follows

$$u(x,t) = c_0t^{-1} + c_1x + c_2x^2t + c_3x^3t^2 + \sum_{n=2}^{\infty} c_{n+2}x^{n+2}t^{n+1}, \tag{32}$$

where c_{n+2} $(n = 2, 3, \ldots)$ can be determined successively by (30) in a unique manner.

Corresponding to the approximate form of (31), we have the approximate form of (32) as follows:

$$\begin{aligned} u(x,t) &= \eta t^{-1} + \lambda x - \frac{1}{2\beta\eta}(2\beta\lambda^2 + \alpha)x^2t - \frac{1}{6\beta\eta^2}(12\beta\eta\lambda c_2 + 2\beta\lambda^3 + \alpha\lambda - 2c_2)x^3t^2 \\ &\quad - \frac{1}{12\beta\eta^2}(24\beta\eta\lambda c_3 + 12\beta\eta c_2^2 + 8\beta\lambda^2 c_2 + \alpha c_2 - 6c_3)x^4t^3 + \cdots. \end{aligned}$$

6. Summary and remarks

We have performed Lie symmetry analysis for the short pulse equation and investigated the algebraic structure of the symmetry groups for this equation. Furthermore, by the generalized symmetry method, we also get the symmetries for the SPE, it is precisely the same as the former. In addition, by using the bifurcation theory method of dynamical system, we have obtained the traveling wave solutions of the equation. It is a geometric consideration actually. Moreover, the power series solution of the reduced equation are given simultaneously. These are new solutions for the SPE.

Remark 1. σ_1 and σ_2 are obvious symmetries for Eq. (1). Substituting $\sigma_3 = xu_x - tu_t - u$ into (11), it is easy to show that σ_3 is also a symmetry of Eq. (1).

Remark 2. We reiterate that the power series solution (26) for the short pulse equation is an exact analytic solution. Moreover, the solution of the power series converges quickly, so it is convenient for computations in both application and physical systems.

Remark 3. Note that we assumed $c_0 = f(0) \neq 0$ in the power series solution (26). It is necessary for our arguments. Otherwise, we cannot get the exact analytic solution for this equation.

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