ARTICLE IN PRESS YJMAA:14877

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lie symmetry analysis, optimal systems and exact solutions to the fifth-order KdV types of equations ☆

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ARTICLE INFO

Article history:
Received 17 August 2009
Available online xxxx
Submitted by J. Shi

Keywords:
Lie symmetry analysis
Optimal system
Power series method
Exact solution
Kawahara equation

ABSTRACT

In this paper, the Lie symmetry analysis is performed on the fifth-order KdV types of equations which arise in modeling many physical phenomena. The similarity reductions and exact solutions are obtained based on the optimal system method. Then the exact analytic solutions are considered by using the power series method.

1. Introduction

The celebrated KdV types of equations are very important in both nonlinear theory and physical application. Such equations have been studied extensively. Especially, the soliton solutions, solitary wave solutions and the periodic wave solutions, etc. to the classical KdV equation $u_t + \alpha u u_x + \beta u_{xxx} = 0$ are considered by many authors (see, e.g., [1–9]). Recently, H. Liu et al. considered the periodic wave solutions of a higher-order KdV equation by using the Hirota's direct method [10]. Moreover, by using Lie symmetry analysis and the dynamical system method, we get the symmetries, bifurcations and exact explicit solutions to other nonlinear evolution equations (NLEEs) [11–16]. In the present paper, we will consider the fifth-order KdV types of equations:

$$u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_{xxxxx} = 0, \tag{1}$$

and

$$u_t + \lambda u u_x + \mu u_{xxxxx} = 0, \tag{2}$$

where $u = u(x,t)$ denotes the unknown function, all the parameters $\alpha, \beta, \gamma, \lambda, \mu \in R$ and $\gamma\mu \neq 0$. Then we will treat the following general modified Kawahara equation

$$u_t + \alpha u^2 u_x + \beta u_{xxx} + \gamma u_{xxxxx} = 0, \tag{3}$$

☆ This work is supported by the Natural Science Foundation of China (No. 10971018), the Natural Science Foundation (Bzxykj0806) and the Doctoral Foundation (2009Y01) of Binzhou University.

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0022-247X/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2010.03.026

simultaneously. In particular, if $\beta = 0$, then Eq. (3) becomes the following equation:

$$u_t + \lambda u^2 u_x + \mu u_{xxxxx} = 0. \tag{4}$$

We know that Eq. (1) is the general Kawahara equation (see [17,18] and references therein), Eq. (2) is the simplified Kawahara equation, and Eq. (4) is called the simplified modified Kawahara equation. These fifth-order KdV types of equations have been derived to model many physical phenomena, such as gravity-capillary waves on a shallow layer and magneto-sound propagation in plasmas, and so on. In [17], the authors proved the existence of traveling wave solutions to a fifth-order partial differential equation, which is a formal asymptotic approximation for water waves with surface tension. The paper [18] is mainly concerned with the local well-posedness of the initial-value problems for the Kawahara and the modified Kawahara equations in Sobolev spaces.

The rest of this paper is organized as follows: in Section 2, the vector fields of Eqs. (1)–(4) are presented by using Lie symmetry analysis method. In Sections 3 and 4, based on the optimal system method, all the similarity reductions and exact solutions to the four Kawahara equations are obtained. In Section 5, the exact analytic solutions to the equations are investigated by means of the power series method. Finally, the conclusions and remarks will be given in Section 6.

2. Lie symmetry analysis for the Kawahara equations

First of all, by using Lie symmetry analysis method, we obtain the vector field of the general Kawahara equation (1) as follows:

$$V_1 = \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = \frac{\partial}{\partial t}.$$

Similarly, we can get the vector fields of Eqs. (2), (3) and (4), respectively.

For the simplified Kawahara equation (2), we have

$$V_1 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 4u\frac{\partial}{\partial u}, \qquad V_2 = \lambda t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad V_3 = \frac{\partial}{\partial x}, \qquad V_4 = \frac{\partial}{\partial t}.$$

For the general modified Kawahara equation (3), we have

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}.$$

For the simplified modified Kawahara equation (4), we have

$$V_1 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = \frac{\partial}{\partial t}.$$

It is easy to verify that the vector fields are closed under the Lie bracket, respectively. Take the vector field of Eq. (2) for an example, we have

$$[V_1, V_1] = [V_2, V_2] = [V_3, V_3] = [V_4, V_4] = 0, \qquad [V_3, V_2] = -[V_2, V_3] = [V_3, V_4] = -[V_4, V_3] = 0,$$
$$[V_1, V_2] = -[V_2, V_1] = -4V_1, \qquad [V_1, V_3] = -[V_3, V_1] = V_3, \qquad [V_1, V_4] = -[V_4, V_1] = 5V_4$$

and

$$[V_2, V_4] = -[V_4, V_2] = \lambda V_3.$$

Furthermore, we can compute the adjoint representations of the vector fields. For the general Kawahara equation (1), we have

$$\mathrm{Ad}\big(\exp(\varepsilon V_i)\big)V_i = V_i, \quad i = 1, 2, 3,$$

and

$$\mathrm{Ad}\big(\exp(\varepsilon V_1)\big)V_2 = V_2, \qquad \mathrm{Ad}\big(\exp(\varepsilon V_1)\big)V_3 = V_3 - \alpha\varepsilon V_2, \qquad \mathrm{Ad}\big(\exp(\varepsilon V_2)\big)V_1 = V_1,$$
$$\mathrm{Ad}\big(\exp(\varepsilon V_2)\big)V_3 = V_3, \qquad \mathrm{Ad}\big(\exp(\varepsilon V_3)\big)V_1 = V_1 + \alpha\varepsilon V_2, \qquad \mathrm{Ad}\big(\exp(\varepsilon V_3)\big)V_2 = V_2,$$

for any $\varepsilon \in$R.

The adjoint representations of the vector fields of Eqs. (2), (3) and (4) can be obtained in the similar way. Based on the adjoint representations of the vector fields, we obtain the optimal systems of the four Kawahara equations as follows:

For Eq. (1), we have

$$\{V_1,\ V_2,\ V_3,\ V_1 + \nu V_3\},$$

where ν is an arbitrary constant.

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For Eq. (2), we have

$$\{V_1,\ V_2,\ V_3,\ V_4,\ V_2 + vV_4\},$$

where v is an arbitrary constant.

For Eq. (3), we have

$$\{V_1,\ V_2 + vV_1\},$$

where v is an arbitrary constant.

For Eq. (4), we have

$$\{V_1,\ V_2,\ V_3 + vV_2\},$$

where v is an arbitrary constant.

3. Similarity reductions and exact solutions to Eqs. (1) and (2)

In the preceding section, we obtained the vector fields and the optimal systems of the four Kawahara equations. Now, we deal with the symmetry reductions and exact solutions to the equations. We will consider the following similarity reductions and group-invariant solutions based on the optimal system method. From an optimal system of group-invariant solutions to an equation, every other such solution to the equation can be derived.

3.1. Reductions and exact solutions to Eq. (1)

(i) For the generator V_1, we have

$$u = f(\xi) + \frac{1}{\alpha} x t^{-1}, \tag{5}$$

where $\xi = t$. Substituting (5) into Eq. (1), we reduce it to the following ODE

$$\xi f' + f = 0, \tag{6}$$

where $f' = \frac{df}{d\xi}$.

Solving Eq. (6), we have $f(\xi) = c\xi^{-1}$. Thus, we obtain the solution to Eq. (1) is

$$u(x,t) = \frac{1}{\alpha} x t^{-1} + c t^{-1}, \tag{7}$$

where c is an arbitrary constant.

(ii) For the generator V_2, we get the trivial solution to Eq. (1) is $u(x,t) = c$, where c is an arbitrary constant.

(iii) For the generator V_3, we have

$$u = f(\xi), \tag{8}$$

where $\xi = x$. Substituting (8) into Eq. (1), we reduce it to the following ODE

$$\gamma f^{(5)} + \beta f''' + \alpha f f' = 0, \tag{9}$$

where $f' = \frac{df}{d\xi}$.

(iv) For the linear combination $V = V_1 + vV_3$, we have

$$u = f(\xi) + \frac{1}{v} t, \tag{10}$$

where $\xi = x - \frac{\alpha}{2v} t^2$. Substituting (10) into Eq. (1), we reduce it to the following ODE

$$\gamma f^{(5)} + \beta f''' + \alpha f f' + \frac{1}{v} = 0, \tag{11}$$

where $f' = \frac{df}{d\xi}$, $v \neq 0$ is an arbitrary constant.

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3.2. Reductions and exact solutions to Eq. (2)

(i) For the generator V_1, we have

$$u = t^{-\frac{4}{5}} f(\xi), \tag{12}$$

where $\xi = xt^{-\frac{1}{5}}$. Substituting (12) into Eq. (2), we reduce this equation to the following ODE

$$\mu f^{(5)} + \lambda f f' - \frac{1}{5}\xi f' - \frac{4}{5} f = 0, \tag{13}$$

where $f' = \frac{df}{d\xi}$.

(ii) For the generator V_2, we have

$$u = f(\xi) + \frac{1}{\lambda} x t^{-1}, \tag{14}$$

where $\xi = t$. Substituting (14) into Eq. (2), we reduce this equation to the following ODE

$$\xi f' + f = 0, \tag{15}$$

where $f' = \frac{df}{d\xi}$.

Eq. (15) has the solution $f(\xi) = c\xi^{-1}$. Thus, we obtain the solution of Eq. (2) is

$$u(x,t) = \frac{1}{\lambda} x t^{-1} + c t^{-1}, \tag{16}$$

where c is an arbitrary constant.

(iii) For the generator V_3, we get the trivial solution of Eq. (2) is $u(x,t) = c$, where c is an arbitrary constant.

(iv) For the generator V_4, we have

$$u = f(\xi), \tag{17}$$

where $\xi = x$. Substituting (17) into Eq. (2), we reduce this equation to the following ODE

$$\mu f^{(5)} + \lambda f f' = 0, \tag{18}$$

where $f' = \frac{df}{d\xi}$.

(v) For the linear combination $V = V_2 + v V_4$, we have

$$u = f(\xi) + \frac{1}{v} t, \tag{19}$$

where $\xi = x - \frac{\lambda}{2v} t^2$. Substituting (19) into Eq. (2), we reduce this equation to the following ODE

$$\mu f^{(5)} + \lambda f f' + \frac{1}{v} = 0, \tag{20}$$

where $f' = \frac{df}{d\xi}$, $v \neq 0$ is an arbitrary constant.

4. Similarity reductions and exact solutions to Eqs. (3) and (4)

4.1. Reductions and exact solutions to Eq. (3)

(i) For the generator V_1, we get the trivial solution of Eq. (3) is $u(x,t) = c$, where c is an arbitrary constant.

(ii) For the linear combination $V = V_2 + v V_1$, we have

$$u = f(\xi), \tag{21}$$

where $\xi = x - vt$. Substituting (21) into Eq. (3), we reduce this equation to the following ODE

$$\gamma f^{(5)} + \beta f''' + \alpha f^2 f' - v f' = 0, \tag{22}$$

where $f' = \frac{df}{d\xi}$, $v \neq 0$ is an arbitrary constant.

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4.2. Reductions and exact solutions to Eq. (4)

(i) For the generator V_1, we have

$$u = t^{-\frac{2}{5}} f(\xi), \tag{23}$$

where $\xi = xt^{-\frac{1}{5}}$. Substituting (23) into Eq. (4), we reduce this equation to the following ODE

$$5\gamma f^{(5)} + 5\alpha f^2 f' - \xi f' - 2f = 0, \tag{24}$$

where $f' = \frac{df}{d\xi}$.

(ii) For the generator V_2, we get the trivial solution of Eq. (4) is $u(x,t) = c$, where c is an arbitrary constant.

(iii) For the linear combination $V = V_3 + vV_2$, we have

$$u = f(\xi), \tag{25}$$

where $\xi = x - vt$. Substituting (25) into Eq. (4), we reduce this equation to the following ODE

$$\gamma f^{(5)} + \alpha f^2 f' - vf' = 0, \tag{26}$$

where $f' = \frac{df}{d\xi}$, $v \neq 0$ is an arbitrary constant.

Remark 4.1. Note that the reduced equations such as (9), (13) are all higher-order nonlinear or nonautonomous ODEs, we will deal with such equations in the next section.

5. The exact power series solutions

By exact solutions, we mean those that can be obtained from some ODEs or, in general, from PDEs of lower order than the original PDE [19]. In terms of this definition, the exact solutions to the Kawahara equations (1)–(4) are obtained actually in both of the preceding Sections 3 and 4.

In spite of this, we still want to detect the explicit solutions expressed in terms of elementary or, at least, known functions of mathematical physics, in terms of quadratures, and so on. But this is not always the case, even for simple semilinear PDEs. However, we know that the power series can be used to solve differential equations, including many complicated differential equations with nonconstant coefficients, [20–23]. In this section, we will consider the exact analytic solutions to the reduced equations by using the power series method. Once we get the exact analytic solutions of the reduced equations (ODEs), the exact power series solutions to the original PDEs are obtained. As examples, we consider Eqs. (9), (22) and (24), the other equations can be tackled in the similar way.

5.1. Exact analytic solutions to Eq. (9)

Firstly, in view of (9), we have

$$2\gamma f^{(4)} + 2\beta f'' + \alpha f^2 + c = 0, \tag{27}$$

where c is an integration constant.

Now, we seek a solution of Eq. (27) in a power series of the form

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{28}$$

Substituting (28) into (27), we have

$$48\gamma c_4 + 2\gamma \sum_{n=1}^{\infty} (n+1)(n+2)(n+3)(n+4) c_{n+4} \xi^n + 4\beta c_2 + 2\beta \sum_{n=1}^{\infty} (n+1)(n+2) c_{n+2} \xi^n$$
$$+ \alpha c_0^2 + \alpha \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} c_k c_{n-k} \right) \xi^n + c = 0. \tag{29}$$

From (29), comparing coefficients, for $n = 0$, we obtain

$$c_4 = \frac{-1}{48\gamma} \left(\alpha c_0^2 + 4\beta c_2 + c \right). \tag{30}$$

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Generally, for $n \geqslant 1$, we have

$$c_{n+4} = \frac{-1}{2\gamma(n+1)(n+2)(n+3)(n+4)}\left[2\beta(n+1)(n+2)c_{n+2} + \alpha\sum_{k=0}^{n} c_k c_{n-k}\right]. \tag{31}$$

From (30) and (31), we can get all the coefficients c_n $(n \geqslant 4)$ of the power series (28), e.g.,

$$c_5 = \frac{-1}{120\gamma}(\alpha c_0 c_1 + 6\beta c_3), \qquad c_6 = \frac{-1}{720\gamma}\left[\alpha\left(2c_0c_2 + c_1^2\right) + 24\beta c_4\right],$$

and so on.

Thus, for arbitrary chosen constant numbers c_0, c_1, c_2 and c_3, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (30) and (31) in a unique manner. This implies that for Eq. (27), there exists a power series solution (28) with the coefficients given by (30) and (31). Furthermore, it is easy to prove the convergence of the power series (28) with the coefficients given by (30) and (31) [11,12,20–22]. Therefore, this power series solution (28) to Eq. (27) is an exact analytic solution.

Hence, the power series solution of Eq. (27) can be written as following:

$$\begin{aligned} f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \sum_{n=1}^{\infty} c_{n+4}\xi^{n+4} \\ &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 - \frac{1}{48\gamma}\left(\alpha c_0^2 + 4\beta c_2 + c\right)\xi^4 \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{2\gamma(n+1)(n+2)(n+3)(n+4)}\left[2\beta(n+1)(n+2)c_{n+2} + \alpha\sum_{k=0}^{n} c_k c_{n-k}\right]\xi^{n+4}. \end{aligned} \tag{32}$$

Thus, the exact power series solution of Eq. (1) is

$$\begin{aligned} u(x,t) &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \sum_{n=1}^{\infty} c_{n+4}x^{n+4} \\ &= c_0 + c_1x + c_2x^2 + c_3x^3 - \frac{1}{48\gamma}\left(\alpha c_0^2 + 4\beta c_2 + c\right)x^4 \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{2\gamma(n+1)(n+2)(n+3)(n+4)}\left[2\beta(n+1)(n+2)c_{n+2} + \alpha\sum_{k=0}^{n} c_k c_{n-k}\right]x^{n+4}, \end{aligned} \tag{33}$$

where c_i $(i = 0, 1, 2, 3)$ and $v \neq 0$ are arbitrary constants, the other coefficients c_n $(n \geqslant 4)$ can be determined successively from (30) and (31).

In physical applications, it will be convenient to write the solution of Eq. (1) in the approximate form

$$\begin{aligned} u(x,t) &= c_0 + c_1x + c_2x^2 + c_3x^3 - \frac{1}{48\gamma}\left(\alpha c_0^2 + 4\beta c_2 + c\right)x^4 \\ &\quad - \frac{1}{120\gamma}(\alpha c_0c_1 + 6\beta c_3)x^5 - \frac{1}{720\gamma}\left[\alpha\left(2c_0c_2 + c_1^2\right) + 24\beta c_4\right]x^6 + \cdots, \end{aligned} \tag{34}$$

in terms of the above computation.

5.2. Exact analytic solutions to Eq. (22)

In view of (22), we have

$$3\gamma f^{(4)} + 3\beta f'' + \alpha f^3 - 3vf + g = 0, \tag{35}$$

where g is an integration constant.

Now, we seek a solution of Eq. (35) in a power series of the form (28). Substituting (28) into (35), and comparing coefficients, we obtain

$$c_{n+4} = \frac{1}{3\gamma(n+1)(n+2)(n+3)(n+4)}\left[3vc_n - 3\beta(n+1)(n+2)c_{n+2} - \alpha\sum_{k=0}^{n}\sum_{j=0}^{k} c_j c_{k-j} c_{n-k} - g\right],$$
$$n = 0, 1, 2, \ldots. \tag{36}$$

In view of (36), we can get all the coefficients c_n $(n \geqslant 1)$ of the power series (28), e.g.,

$$c_4 = \frac{1}{72\gamma}\left(3\nu c_0 - \alpha c_0^3 - 6\beta c_2 - g\right), \qquad c_5 = \frac{1}{360\gamma}\left(3\nu c_1 - 3\alpha c_0^2 c_1 - 18\beta c_3 - g\right),$$

and so on. Thus, for arbitrary chosen constant numbers c_0, c_1, c_2 and c_3, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (36) in a unique manner. This implies that for Eq. (35), there exists a power series solution (28) with the coefficients given by (36).

Therefore, the power series solution of Eq. (35) can be written as follows:

$$\begin{aligned} f(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \sum_{n=0}^{\infty} \frac{1}{3\gamma(n+1)(n+2)(n+3)(n+4)}\Bigg[3\nu c_n - 3\beta(n+1)(n+2)c_{n+2} \\ - \alpha\sum_{k=0}^{n}\sum_{j=0}^{k} c_j c_{k-j} c_{n-k} - g\Bigg]\xi^{n+4}. \end{aligned} \tag{37}$$

Accordingly we have the exact traveling wave solution to Eq. (3) is

$$\begin{aligned} u(x,t) = {} & c_0 + c_1(x-\nu t) + c_2(x-\nu t)^2 + c_3(x-\nu t)^3 \\ & + \sum_{n=0}^{\infty} \frac{1}{3\gamma(n+1)(n+2)(n+3)(n+4)}\Bigg[3\nu c_n - 3\beta(n+1)(n+2)c_{n+2} \\ & - \alpha\sum_{k=0}^{n}\sum_{j=0}^{k} c_j c_{k-j} c_{n-k} - g\Bigg](x-\nu t)^{n+4}, \end{aligned} \tag{38}$$

where c_i $(i = 0, 1, 2, 3)$ are arbitrary constants, the other terms c_{n+4} $(n = 0, 1, 2, \ldots)$ are given by (36) successively.

5.3. Exact analytic solutions to Eq. (24)

Similarly, we seek a solution of Eq. (24) in a power series of the form (28). Substituting it into (24), and comparing coefficients, we obtain

$$\begin{aligned} c_{n+5} = \frac{1}{5\gamma(n+1)(n+2)(n+3)(n+4)(n+5)}\Bigg[(n+2)c_n - 5\alpha\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)c_j c_{k-j} c_{n+1-k}\Bigg], \\ n = 0, 1, 2, \ldots. \end{aligned} \tag{39}$$

In view of (39), we can get all the coefficients c_n $(n \geqslant 5)$ of the power series (28) such as

$$c_5 = \frac{1}{600\gamma}\left(2c_0 - 5\alpha c_0^2 c_1\right), \qquad c_6 = \frac{1}{3600\gamma}\left[3c_1 - 10\alpha\left(c_0^2 c_2 + c_0 c_1^2\right)\right],$$

$$c_7 = \frac{1}{12600\gamma}\left[4c_2 - 5\alpha\left(3c_0^2 c_3 + 6c_0c_1c_2 + c_1^3\right)\right], \qquad c_8 = \frac{1}{6720\gamma}\left[c_3 - 4\alpha\left(c_0^2 c_4 + 2c_0c_1c_3 + c_0c_2^2 + c_1^2 c_2\right)\right],$$

and so on. Thus, for arbitrary chosen constant numbers c_0, c_1, c_2, c_3 and c_4, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (39) in a unique manner. This implies that for Eq. (24), there exists a power series solution (28) with the coefficients given by (39).

The exact solution of Eq. (4) and the solution in the approximate form can be written in terms of the above computation. The details are omitted here.

Remark 5.1. By using the integration of ordinary differential equations (ODEs), we know that if we get a one-parameter symmetry group of an ODE, then we can reduce the order of the equation by one. But we note that such reduced ODEs are more complicated than the original equation in addition to some special cases. In view of this, we can see that the power series method is a useful tool of solving such higher-order nonlinear or nonautonomous ODEs.

6. Conclusion and remarks

In this paper, we have obtained the symmetries and similarity reductions of the four fifth-order KdV types of equations by using Lie symmetry analysis method. All the group-invariant solutions to the equations are considered based on the optimal system method for the first time. Then the exact analytic solutions are investigated by using the power series method. Furthermore, how to get the other forms of exact solutions to these Kawahara equations? We hope to investigate this in the near future.

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Remark 6.1. We would like to reiterate that the power series solutions which have been obtained in Section 5 are exact analytic solutions. Moreover, from the above examples, we can see that these power series solutions converge quickly, so it is convenient for computations in both numerical analysis and physical applications.

Acknowledgments

The authors are grateful to the referees and editors for their precious comments and suggestions, which helped to improve the paper greatly. We also express our sincere thanks to Professor C. Tian for his enthusiastic guidance.

References

[1] P.J. Olver, Applications of Lie Groups to Differential Equations, Grad. Texts in Math., vol. 107, Springer, New York, 1993.
[2] C. Tian, Lie Groups and Its Applications to Differential Equations, Science Press, Beijing, 2001 (in Chinese).
[3] D.Y. Chen, Introduction to Solitons, Science Press, Beijing, 2006 (in Chinese).
[4] R. Hirota, J. Satsuma, A variety of nonlinear network equations generated from the Bäcklund transformation for the Tota lattice, Suppl. Prog. Theor. Phys. 59 (1976) 64–100.
[5] A. Nakamura, A direct method of calculating periodic wave solutions to nonlinear evolution equations, II, J. Phys. Soc. Japan 48 (1980) 1365–1370.
[6] Y.S. Li, Soliton and Integrable Systems, Adv. Ser. Nonlinear Sci., Shanghai Scientific and Technological Education Publishing House, Shanghai, 1999 (in Chinese).
[7] C.H. Gu, Soliton Theory and Its Applications, Springer-Verlag, Zhejiang Science and Technology Publishing House, Hangzhou, 1995.
[8] Z. Zhu, The soliton solutions of generalized KdV equation, Acta Phys. Sinica 41 (1992) 1057–1062 (in Chinese).
[9] W. Ma, D. Zhou, On solitary wave solutions to a generalized KdV equation, Acta Phys. Sinica 42 (1993) 1731–1734 (in Chinese).
[10] H. Liu, J. Li, F. Chen, Exact periodic wave solutions for the hKdV equation, Nonlinear Anal. 70 (2009) 2376–2381.
[11] H. Liu, J. Li, Lie symmetry analysis and exact solutions for the short pulse equation, Nonlinear Anal. 71 (2009) 2126–2133.
[12] H. Liu, J. Li, Q. Zhang, Lie symmetry analysis and exact explicit solutions for general Burgers' equation, J. Comput. Appl. Math. 228 (2009) 1–9.
[13] H. Liu, J. Li, L. Liu, Lie group classifications and exact solutions for two variable-coefficient equations, Appl. Math. Comput. 215 (2009) 2927–2935.
[14] H. Liu, J. Li, L. Liu, Painlevé analysis, Lie symmetries, and exact solutions for the time-dependent coefficients Gardner equations, Nonlinear Dynam. 59 (2010) 497–502.
[15] H. Liu, J. Li, Lie symmetry analysis and exact solutions for the extended mKdV equation, Acta Appl. Math. 109 (2010) 1107–1119.
[16] H. Liu, J. Li, Lie symmetries, conservation laws and exact solutions for two rod equations, Acta Appl. Math. (2009), doi:10.1007/s10440-009-9462-0.
[17] J. Hunter, J. Scheurle, Existence of perturbed solitary wave solutions to a model equation for water waves, Phys. D 32 (1988) 253–268.
[18] W. Chen, J. Li, C. Miao, J. Wu, Low regularity solutions of two fifth-order KdV type equations, J. Anal. Math. 107 (2009) 221–238.
[19] V.A. Galaktionov, S.R. Svirshchevskii, Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics, Chapman and Hall/CRC, 2006.
[20] H. Liu, W. Li, The exact analytic solutions of a nonlinear differential iterative equation, Nonlinear Anal. 69 (2008) 2466–2478.
[21] H. Liu, F. Qiu, Analytic solutions of an iterative equation with first order derivative, Ann. Differential Equations 21 (2005) 337–342.
[22] H. Liu, W. Li, Discussion on the analytic solutions of the second-order iterative differential equation, Bull. Korean Math. Soc. 43 (2006) 791–804.
[23] N.H. Asmar, Partial Differential Equations with Fourier Series and Boundary Value Problems, second edn., China Machine Press, Beijing, 2005.

Please cite this article in press as: H. Liu et al., Lie symmetry analysis, optimal systems and exact solutions to the fifth-order KdV types of equations, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.03.026