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Painlevé analysis, Lie symmetries, and exact solutions for the time-dependent coefficients Gardner equations

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Abstract In this paper, the three variable-coefficient Gardner (vc-Gardner) equations are considered. By using the Painlevé analysis and Lie group analysis method, the Painlevé properties and symmetries for the equations are obtained. Then the exact solutions generated from the symmetries and Painlevé analysis are presented.

Keywords Variable-coefficient Gardner equation · Painlevé analysis · Lie symmetry analysis · Exact solution

1 Introduction

The nonlinear evolution equations (NLEEs) arising in many physical fields like the condense matter physics, fluid mechanics, plasma physics, and optics, etc., which exhibit a rich variety of nonlinear phenomena. When the inhomogeneities of media and nonuniformity of boundaries are taken into account in various real physical situations, the variable-coefficient NLEEs often can provide more powerful and realistic models than their constant-coefficient counterparts in describing a large variety of real phenomena. It is known that to find exact solutions of the NLEEs is always one of the central themes in mathematics and physics. In the past few decades, there is noticeable progress in this field, and various methods have been developed, such as the inverse scattering transformation (IST) [1], Darboux and Bäcklund transformations [2], Hirota's bilinear method [2–4], Lie symmetry analysis [5–12], CK method [13, 14], and so on.

Recently, there has been a growing interest in studying variable-coefficient NLEEs [8, 15–17]. In the present paper, we will consider the Gardner equation

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with time-dependent coefficients is of the form

$$u_t + \alpha t^m u u_x + \beta t^n u^2 u_x + \gamma t^p u_{xxx} + \lambda t^q u_x + \mu t^r u = 0, \tag{1.1}$$

where $u = u(x,t)$ denotes the amplitude of the relevant wave model, such as for the internal waves in a stratified ocean, x is the horizontal coordinate, and t is the time. While $\alpha, \beta, \gamma, \lambda$, and ν are arbitrary constant parameters, m, n, p, q, and r are given real numbers.

Equation (1.1) can be used to model such physical situations as the dust-acoustic solitary waves in dusty plasmas, internal solitary waves in stable, stratified shear flows in ocean and atmosphere, ion acoustic waves in plasmas with a negative ion, interfacial solitary waves over slowly varying topographies, and wave motion in a nonlinear elastic structural element with large deflection, etc. When $\mu = 0$, (1.1) is as follows:

$$u_t + \alpha t^m u u_x + \beta t^n u^2 u_x + \gamma t^p u_{xxx} + \lambda t^q u_x = 0. \tag{1.2}$$

In particular, if $m = n = p$, then (1.2) becomes the special form

$$u_t + \alpha t^p u u_x + \beta t^p u^2 u_x + \gamma t^p u_{xxx} + \lambda t^q u_x = 0. \tag{1.3}$$

For a general variable-coefficient NLEE, it is not completely integrable unless the variable coefficients satisfy some specific constraint conditions. Thereby, we will find the conditions for the equations to pass the Painlevé test firstly, then the symmetries and exact solutions are considered.

The rest of this paper is organized as follows. In Sect. 2, we perform Painlevé analysis for (1.1), (1.2), and (1.3). In Sect. 3, the symmetries for the equations are obtained by the Lie group analysis method. In Sect. 4, we investigate the symmetry reductions and exact explicit solutions for the vc-Gardner equations. In Sect. 5, we conclude and make some remarks.

2 Painlevé analysis for the three equations

Firstly, we employ the Kruskal's simplified method to carry out the Painlevé analysis for (1.1).

Thus, we assume that

$$u = \phi^{-\rho} \sum_{j=0}^{\infty} u_j \phi^j, \tag{2.1}$$

where $\phi = x + \psi(t)$, $u_j = u_j(t)$ are analytic functions in a neighborhood of the noncharacteristic singular manifold, $u_0 \neq 0$ and ρ is a positive integer.

Through the leading order analysis, it is readily found that $\rho = 1$ and $u_0 = a$, where $a = \pm\sqrt{-\frac{6\gamma}{\beta}} t^{\frac{p-n}{2}}$. Then substituting (2.1) into (1.1), we have

$$j = 0, \quad u_0 = a, \tag{2.2}$$

$$j = 1, \quad u_1 = -\frac{\alpha}{2\beta} t^{m-n}, \tag{2.3}$$

$$j = 2, \quad u_2 = -\frac{1}{a\beta} t^{-n} \phi_t + \frac{\alpha^2}{4a\beta^2} t^{2(m-n)} - \frac{\lambda}{a\beta} t^{q-n}, \tag{2.4}$$

$$j = 3, \quad u_{0t} + \mu t^r u_0 = 0, \tag{2.5}$$

$$j = 4, \quad u_{1t} + u_2 \phi_t + \alpha t^m (u_0 u_3 + u_1 u_2) + \beta t^n \left(2u_0 u_1 u_3 + u_0 u_2^2 + u_1^2 u_2\right) + \lambda t^q u_2 + \mu t^r u_1 = 0. \tag{2.6}$$

By (2.2)–(2.4), we can get u_0, u_1, and u_2 in a unique manner. But from (2.5) and (2.6), we cannot get u_3 and u_4, so $j = 3, 4$ are the resonances. In fact, we have the recursion relations are as follows:

$$\begin{aligned}(j+1)(j-3)(j-4)\gamma t^p u_j &= -u_{j-3,t} - (j-3) u_{j-2} \phi_t \\ &\quad - \alpha t^m \sum_{k=0}^{j-1} (j-k-2) u_k u_{j-k-1} \\ &\quad + \beta t^n \Bigg[u_0 \sum_{k=1}^{j-1} u_k u_{j-k} - \sum_{k=1}^{j-1} \sum_{i=0}^{k} (j-k-1) u_i u_{k-i} u_{j-k} \Bigg] \\ &\quad - \lambda t^q (j-3) u_{j-2} - \mu t^r u_{j-3}.\end{aligned} \tag{2.7}$$

It is found that the resonances occur at $j = -1, 3, 4$. The compatibility conditions at $j = 3, 4$ are satisfied identically for arbitrary chosen u_3 and u_4. Therefore, (1.1) possesses the Painlevé property (PP) under the conditions (2.5) and (2.6). We now specialize (2.1) by setting the resonance functions $u_3 = u_4 = 0$. Furthermore, by requiring $u_2 = 0$, it is easily demonstrated that $u_j = 0$, for all $j \geq 2$.

Then plugging into (2.6), we get

$$u_{1t} + \mu t^r u_1 = 0. \tag{2.8}$$

Solving (2.5) and (2.8), we have

$$t^n = k_1 t^p e^{\frac{2\mu}{r+1} t^{r+1}}, \qquad t^n = k_2 t^m e^{\frac{2\mu}{r+1} t^{r+1}}, \tag{2.9}$$

where k_1, k_2 are arbitrary constants and $r \neq -1$. Therefore, under the condition (2.9), we can say that (1.1) possesses the Painlevé property and becomes integrable.

Remark 2.1 In view of (2.9), if $\mu = 0$, then we get the condition

$$t^n = k_1 t^p, \qquad t^n = k_2 t^m, \tag{2.10}$$

where k_1 and k_2 are arbitrary constants. That is to say, under the condition (2.10), (1.2) becomes integrable.

In particular, if $m = n = p$, then condition (2.10) is trivial. In other words, (1.3) is integrable.

Remark 2.2 For the condition of Painlevé integrable, the Kruskal's simplified method is the same as the general Weiss–Tabor–Carnevale (WTC) procedure [2, 18, 19]. But we cannot get the other integrable properties sometimes, such as the Bäcklund transformation (BT), Lax pair (LP), etc., by using the simplified method. The properties of complete integrability for the equations are not discussed in this paper.

3 Lie symmetry analysis for the vc-Gardner equations

In this section, we make the Lie symmetry analysis under the following conditions:

(C1) The parameters $\alpha\beta\gamma \neq 0$
(C2) $\mu = 0$

Recall that the vector fields of the NLEEs are as follows:

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}, \tag{3.1}$$

where the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ of the vector fields are to be determined later.

If the vector field (3.1) generates a symmetry of (1.1), then V must satisfy the Lie's symmetry condition $\mathrm{pr}^{(3)}V(\Delta)|_{\Delta=0} = 0$, where $\Delta = u_t + \alpha t^m u u_x + \beta t^n u^2 u_x + \gamma t^p u_{xxx} + \lambda t^q u_x + \mu t^r u$. Here, $\mathrm{pr}^{(3)}V$ denotes the third prolongation of V and is given by $\mathrm{pr}^{(3)}V = \mathrm{pr}^{(2)}V + \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxt}\frac{\partial}{\partial u_{xxt}} + \phi^{xtt}\frac{\partial}{\partial u_{xtt}} + \phi^{ttt}\frac{\partial}{\partial u_{ttt}}$.

Applying the symmetry condition to (1.1), we find that the coefficient functions ξ, τ, and ϕ must satisfy the following condition:

$$\begin{aligned}&\phi^t + \gamma t^p \phi^{xxx} + \lambda t^q \phi^x + \beta t^n u^2 \phi^x + \alpha t^m u \phi^x \\ &\quad + \mu t^r \phi + 2\beta t^n u u_x \phi + \alpha t^m u_x \phi + m\alpha t^{m-1}\tau u u_x \\ &\quad + n\beta t^{n-1}\tau u^2 u_x + p\gamma t^{p-1}\tau u_{xxx} \\ &\quad + q\lambda t^{q-1}\tau u_x + r\mu t^{r-1}\tau u = 0,\end{aligned} \tag{3.2}$$

where ϕ^t, ϕ^x, and ϕ^{xxx} are the coefficients of $\mathrm{pr}^{(3)}V$. Furthermore, we have

$$\phi^x = D_x\phi - u_x D_x\xi - u_t D_x\tau, \tag{3.3a}$$

$$\phi^t = D_t\phi - u_x D_t\xi - u_t D_t\tau, \tag{3.3b}$$

$$\phi^{xxx} = D_x^3(\phi - \xi u_x - \tau u_t) + \xi u_{xxxx} + \tau u_{xxxt}, \tag{3.3c}$$

where D_x and D_t are the total derivatives.

Suppose that (1.1) admits a symmetry of the form (3.1). Application of the symmetry condition (3.2) to this equation yields the following determining equations:

$$\tau_x = \tau_u = 0, \qquad \tau = \tau(t), \tag{3.4a}$$

$$\xi_u = 0, \qquad \xi = \xi(x,t), \tag{3.4b}$$

$$\phi_{uu} = 0, \qquad \phi = a(x,t)u + b(x,t), \tag{3.4c}$$

$$p\tau + t\tau_t - 3t\xi_x = 0, \tag{3.4d}$$

$$a_x - \xi_{xx} = 0, \tag{3.4e}$$

$$n\tau + 2ta - t\xi_x + t\tau_t = 0, \tag{3.4f}$$

$$m\alpha t^{m-1}\tau + \alpha t^m a + 2\beta t^n b - \alpha t^m \xi_x + \alpha t^m \tau_t = 0, \tag{3.4g}$$

$$q\lambda t^{q-1}\tau + \alpha t^m b - \lambda t^q \xi_x + \lambda t^q \tau_t - \xi_t + 3\gamma t^p a_{xx} - \xi_{xxx} = 0, \tag{3.4h}$$

$$a_x = 0, \tag{3.4i}$$

$$b_x = 0, \tag{3.4j}$$

$$a_t = 0, \tag{3.4k}$$

$$b_t = 0. \tag{3.4l}$$

Then by the Lie symmetry analysis method, we obtain the following results:

When $m = n = p$, the vector field of (1.3) is

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \qquad V_2 = \lambda t^{q-p}\frac{\partial}{\partial x} + t^{-p}\frac{\partial}{\partial t}, \\ V_3 &= \big[2\beta(p+1)(q+1)x \\ &\quad + 2\beta\lambda(3q-p+2)t^{q+1} \\ &\quad - \alpha^2(q+1)t^{p+1}\big]\frac{\partial}{\partial x} + 6\beta(q+1)t\frac{\partial}{\partial t} \\ &\quad - \big[2\beta(p+1)(q+1)u + \alpha(p+1)(q+1)\big]\frac{\partial}{\partial u}. \end{aligned} \tag{3.5}$$

When $m \neq n$, $m \neq n+p+1$, and $m \neq p$ or $n \neq p$, the vector field of (1.2) is

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \big[2(p+1)(q+1)x + 2\lambda(3q-p+2)t^{q+1}\big]\frac{\partial}{\partial x} \\ &\quad + 6(q+1)t\frac{\partial}{\partial t} \\ &\quad - (q+1)(3n-p+2)u\frac{\partial}{\partial u}, \end{aligned} \tag{3.6}$$

where m, n, and p satisfy $6m - 3n - p + 2 = 0$.

Clearly, the vector fields (3.5) and (3.6) are closed under the Lie bracket $[V_i, V_j]$, respectively. Meanwhile, we note that (1.1) has a trivial symmetry $V = \frac{\partial}{\partial x}$, also.

Summarizing the discussion in Sects. 2 and 3, we have the following conclusion.

Proposition *The result of Lie symmetry analysis coincide with the Painlevé-integrable condition for the two equations. That is to say, under the Painlevé-integrable condition, the equation possesses nontrivial symmetry.*

4 Symmetry reductions and exact solutions

In the preceding section, we obtained the symmetries for the two equations (1.2) and (1.3). Now, we deal with the symmetry reductions and exact solutions based on the Lie group analysis method.

Firstly, we deal with (1.3).

(i) For the generator V_2, we have the following similarity variables:

$$\xi = x - \frac{\lambda}{q+1}t^{q+1}, \qquad \omega = u,$$

and the group-invariant solution is $\omega = f(\xi)$, that is,

$$u = f\left(x - \frac{\lambda}{q+1}t^{q+1}\right). \tag{4.1}$$

Substituting (4.1) into (1.3), we reduce the equation to the following ordinary differential equation (ODE):

$$\gamma f''' + \beta f^2 f' + \alpha f f' = 0, \tag{4.2}$$

where $f' = \frac{df}{d\xi}$. It implies that if $\omega = f(\xi)$ is a solution of (4.2), then (4.1) is a solution of (1.3).

From (4.2), we have

$$6\gamma f'' + 2\beta f^3 + 3\alpha f^2 + k_1 = 0. \tag{4.3}$$

Clearly, this equation has a generator $V = \frac{\partial}{\partial \xi}$. By the symmetry reduction method for ODEs, let $y = f$, $w = \xi$, then we have $f_\xi = \frac{1}{w_y}$, $f_{\xi\xi} = -\frac{w_{yy}}{w_y^3}$. Plugging into (4.3), we get

$$-6\gamma w_{yy} + \left(2\beta y^3 + 3\alpha y^2 + k_1\right)w_y^3 = 0. \tag{4.4}$$

Setting $w_y = z$, we obtain

$$6\gamma\frac{dz}{dy} = \left(2\beta y^3 + 3\alpha y^2 + k_1\right)z^3. \tag{4.5}$$

Thus, we reduce (4.3) to a first-order ODE. Solving (4.5), we have $z = \pm\sqrt{\frac{-6\gamma}{\beta y^4 + 2\alpha y^3 + k_1 y + k_2}}$. That is, $w = \int z(y)dy + k_3$, where k_i $(i = 1, 2, 3)$ are arbitrary constants.

Plugging $y = f$, $w = \xi$ into this equation, we get the solution of (4.2). Accordingly, the solution of (1.3) is obtained. For the concrete parameters, we can get the exact explicit solutions by using the Jacobian elliptic functions.

(ii) For the generator V_1, we get the trivial solution of (1.3) is $u = c$, where c is an arbitrary constant.

Secondly, (1.2) has a generator V_1 also, it is easily seen that this equation has a trivial solution $u = c$, where c is an arbitrary constant.

Thirdly, we consider the solution of (1.1).

Since (1.1) has a generator V_1, we have $u = f(\xi)$, where $\xi = t$. Substituting it into (1.1), we get

$$f' + \mu \xi^r f = 0, \tag{4.6}$$

where $f' = \frac{df}{d\xi}$.

Solving (4.6), we have $f(\xi) = ce^{-\frac{\mu}{r+1}\xi^{r+1}}$. Thus, we obtain the solution of (1.1) is

$$u(x,t) = ce^{-\frac{\mu}{r+1}t^{r+1}}, \tag{4.7}$$

where c is an arbitrary constant.

On the other hand, based on the Painlevé analysis in Sect. 2, we have

$$u = \frac{a}{\phi} - \frac{\alpha}{2\beta}t^{m-n}. \tag{4.8}$$

In view of (2.4), we can get

$$\phi = x + \frac{\alpha^2}{4(2m-n+1)\beta}t^{2m-n+1} - \frac{\lambda}{q+1}t^{q+1} + c, \tag{4.9}$$

where $2m - n \neq -1$, $q \neq -1$, and c is an arbitrary constant.

Substituting (4.9) into (4.8), we obtain the solution of (1.1) under the condition (2.9). In other words, under the integrable condition (2.9), (1.1) has the solution (4.8), where ϕ is given by (4.9).

5 Conclusion and remarks

In this paper, we made the Painlevé analysis for the three time-dependent coefficients Gardner equations. The symmetries of the equations are obtained by using Lie group analysis method. The reduced equations are presented, and the exact explicit solutions are investigated simultaneously. Moreover, the relationship between Lie symmetry analysis and Painlevé property is considered. The Lie symmetry analysis is a very powerful approach for dealing with exact solutions for PDEs. In addition, the relationship between the Lie symmetry analysis and Painlevé properties is an interesting problem; both of them are worthy of studying further.

Remark 5.1 As is well known, symmetries and first integrals are two fundamental structures of ordinary differential equations. In terms of the integration theory based on the invariants of the group, we know that if we get a one-parameter symmetry group of an ODE, then we can reduce the order of the equation by one. In Sect. 4, we reduced (4.3) to a first-order ODE by the symmetry method and obtained its exact explicit solution.

Generally speaking, for tackling the exact solutions of differential equations, only the simple symmetries are available. For example, we can reduce (1.3) to an ODE by using V_3 also, but the reduced equation is a nonlinear and nonautonomous higher-order ODE. To our knowledge, there is no any effective method for dealing with such complicated equations.

Remark 5.2 From our previous discussion, we see that the Painlevé analysis are performed under the condition $r \neq -1$, and the symmetries are obtained under the condition $\mu = 0$ as well. How can we obtain the Painlevé properties and Lie group classifications under the condition $r = -1$ or $\mu \neq 0$ in general? Are there any other forms of exact explicit solutions to the vc-Gardner equations, etc.? We hope to investigate this in the future.

References

1. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Method for solving the Korteweg-de Vries equation. Phys. Rev. Lett. **19**, 1095–1097 (1967)
2. Li, Y.S.: Soliton and integrable systems. In: Advanced Series in Nonlinear Science. Shanghai Scientific and Technological Education Publishing House, Shanghai (1999) (in Chinese)
3. Hirota, R., Satsuma, J.: A variety of nonlinear network equations generated from the Bäcklund transformation for the Tota lattice. Suppl. Prog. Theor. Phys. **59**, 64–100 (1976)
4. Liu, H., Li, J., Chen, F.: Exact periodic wave solutions for the hKdV equation. Nonlinear Anal. **70**, 2376–2381 (2009)
5. Olver, P.J.: Applications of Lie groups to differential equations. In: Grauate Texts in Mathematics, vol. 107. Springer, New York (1993)
6. Bluman, G.W., Kumei, S.: Symmetries and Differential Equations. Springer/World Publishing Corp., Berlin (1989)
7. Cantwell, B.J.: Introduction to Symmetry Analysis. Cambridge University Press, Cambridge (2002)
8. Sinkala, W., Leach, P.G.L., O'Hara, J.G.: Invariance properties of a general-pricing equation. J. Differ. Equ. **244**, 2820–2835 (2008)
9. Liu, H., Li, J., Zhang, Q.: Lie symmetry analysis and exact explicit solutions for general Burgers' equation. J. Comput. Appl. Math. **228**, 1–9 (2009)

10. Liu, H., Li, J.: Lie symmetry analysis and exact solutions for the short pulse equation. Nonlinear Anal. **71**, 2126–2133 (2009)
11. Liu, H., Li, J.: Lie symmetry analysis and exact solutions for the extended mKdV equation. Acta Appl. Math. (2008). doi:10.1007/s10440-008-9362-8
12. Liu, H., Li, J.: Lie symmetries, conservation laws and exact solutions for two rod equations. Acta Appl. Math. (2009). doi:10.1007/s10440-009-9462-0
13. Clarkson, P., Kruskal, M.: New similarity reductions of the Boussinesq equation. J. Math. Phys. **30**(10), 2201–2213 (1989)
14. Clarkson, P.: New similarity reductions for the modified Boussinesq equation. J. Phys. A: Gen. **22**, 2355–2367 (1989)
15. Nirmala, N., Vedan, M.J., Baby, B.V.: A variable coefficient Korteweg-de Vires equation: Similarity analysis and exact solution. II. J. Math. Phys. **27**(11), 2644–2646 (1986)
16. Li, J., Xu, T., Meng, X.-H., Zhang, Y.-X., Zhang, H.-Q., Tian, B.: Lax pair, Bäcklund transformation and N-soliton-like solution for a variable-coefficient Gardner equation from nonlinear lattice plasma physics and ocean dynamics with symbolic computation. J. Math. Anal. Appl. **336**, 1443–1455 (2007)
17. Zhang, Y., Li, J., Lv, Y.-N.: The exact solution and integrable properties to the variable-coefficient modified Korteweg-de Vires equation. Ann. Phys. **323**, 3059–3064 (2008)
18. Wiss, J.: The Painlevé property for partial differential equations, II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative. J. Math. Phys. **24**, 1405–1413 (1983)
19. Ramani, A., Grammaticos, B.: The Painlevé property and singularity analysis of integrable and non-integrable systems. Phys. Rep. **180**, 159–245 (1989)