MA3249

GEOMETRY : FROM ANCIENT TO MODERN

Wong Yan Loi
Department of Mathematics
The National University of Singapore
Singapore 119260
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Chapter 1

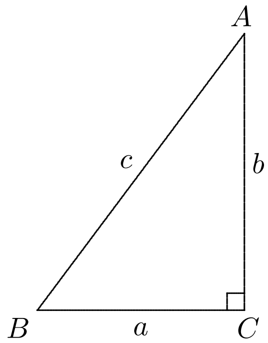
THE GEOMETRY OF THE ANCIENT TIMES

§1.1 PYTHAGORAS' THEOREM

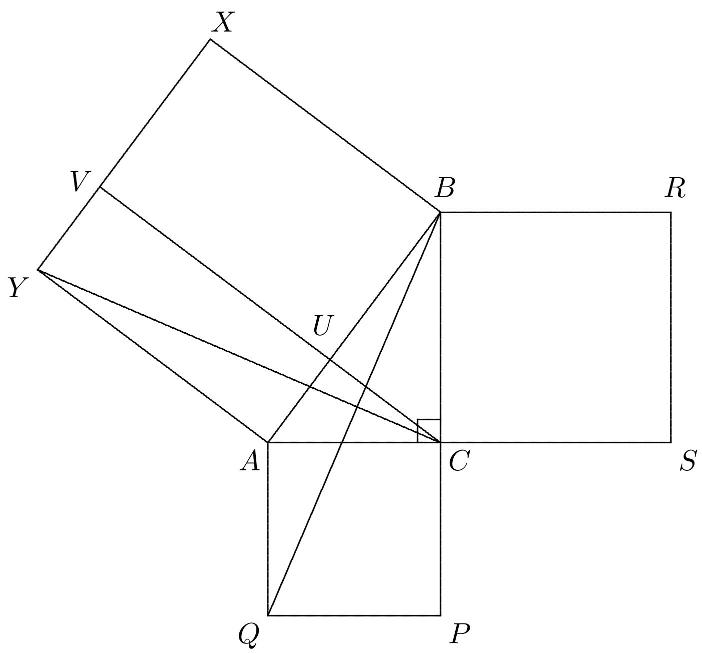
In a right-angled triangle ABC with $\angle C = 90^o$,

$$a^2 + b^2 = c^2,$$

where a = base, b = height and c = hypotenuse. There are more than 400 proofs of this simple result.



The proof appeared in Euclid's "*Elements*" around 300BC is most widely known.



Let's use the notations $(PQAC)$ and (ABQ) to denote the area of the square $PQAC$ and the triangle ABQ etc. In the above diagram, $AC^2 = (PQAC) = 2(ABQ) = 2(ACY) = (AUVY)$. Similarly, $BC^2 = (BUVX)$. Hence, $AC^2 + BC^2 = AB^2$.

In ancient China, mathematicians make use of Pythagoras' theorem in many important scientific calculations. On the starting page of *Zhou-Bi Suan-Jing* around 1100BC, Zhou Kung asks Shang Gao the following question.

"There are no steps by which one may ascend the heavens, and it is impracticable to take a ruler and measure the extent of the earth. Then how does one obtain the measurement of the height of the heaven?"

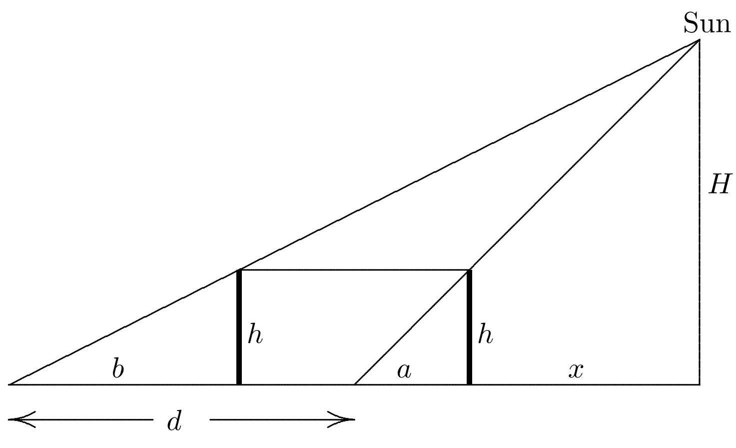
In Shang Gao's reply, he mentions the "3-4-5" right-angled triangle as the basis in such calculations. Also Chen Tsu in *Zhou-Bi Suan-Jing* states the following.

"Square the base and the height. Take the square root of their sum to obtain the hypotenuse."

This is precisely Pythagoras' theorem.

In *Zhou-Bi Suan-Jing*, there is the following calculation.

A stick measures 8ft long. During the day of the summer solstice, i.e. on the 21st of June, it casts a shadow of length 1ft 6in long. Here the stick represents the height of a right-angled triangle, while the shadow is the base. When the stick is moved 1000mi south, the shadow becomes 1ft 5in long. Whereas if the stick is moved 1000mi north, the shadow is 1ft 7in long. As the sun moves southward, the shadow of the stick is longer. When the shadow is exactly 6ft long, a bamboo tube of length 8ft long and diameter 1in wide is used to observe the sun. Its aperture just covers the sun. This gives the ratio of the distance from the earth to the sun and the diameter of the sun, which is 80 to 1. Also, the length of the shadow of the stick decreases by 1in for each additional 1000mi advancement of the stick towards south. It follows from this that the shadow is gone when the stick is moved 60000mi south. Consequently, the height of the sun is 80000mi.



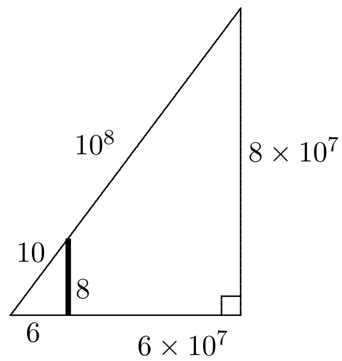
In the above diagram, $h = 8ft, a = 1.5ft, b = 1.6ft$ and $d \sim 10^6 ft$.

By similar triangles, we have $\dfrac{a+x}{H} = \dfrac{a}{h}$ and $\dfrac{d+a+x}{H} = \dfrac{b}{h}$.

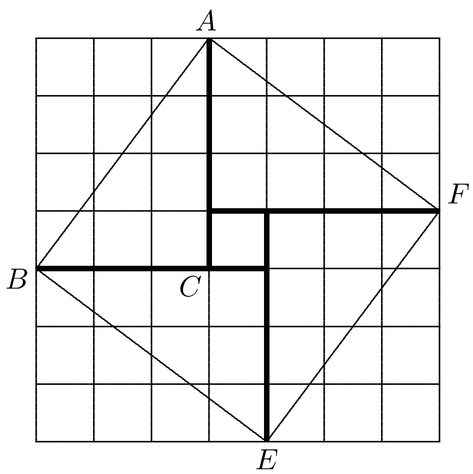
By subtracting the first from the second, we have $\dfrac{d}{H} + \dfrac{a}{h} = \dfrac{b}{h}$.

Hence, $H = \dfrac{dh}{b-a} = 8 \times 10^7 ft$.

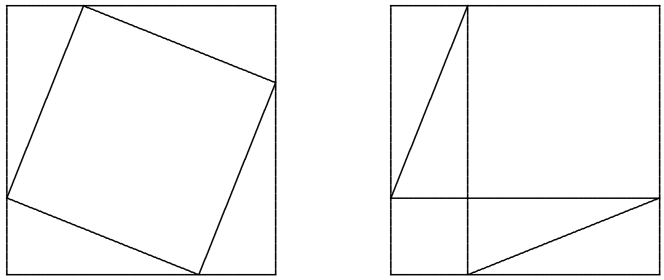
Consequently, the distance from the earth to the sun is $10^8 ft$. Then by similar triangles, the diameter of the sun is $1/8 \times 10^7 ft$. Though the calculation is far from accurate, it gives an illustration on how the ancient Chinese estimate these important astronomical values by means of Pythagoras' theorem. Note that we take $1mi = 1000ft$ and $1ft = 10in$ in our calculations.



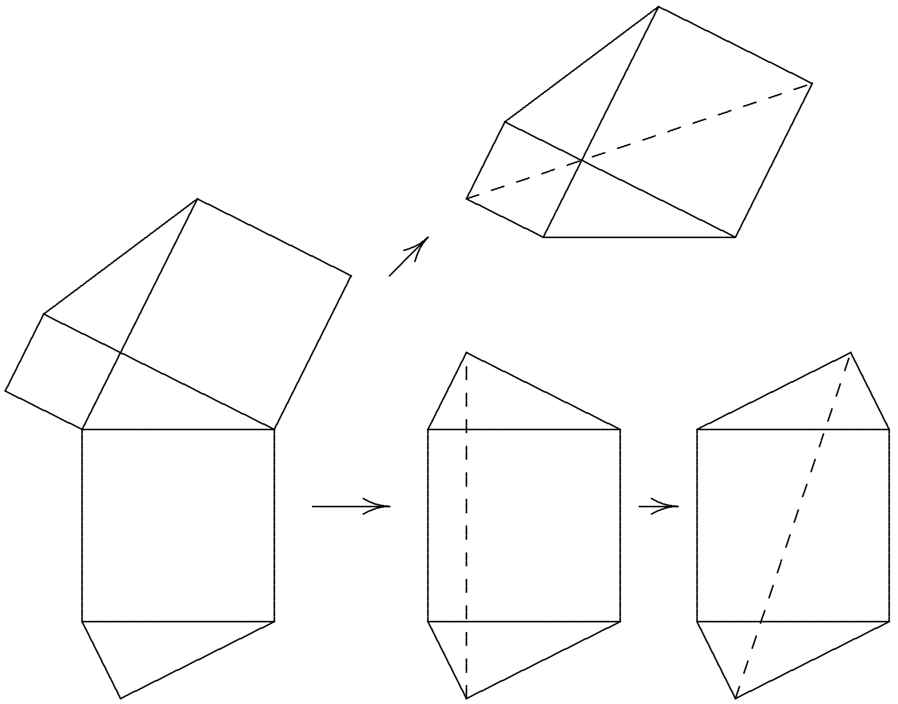
During the Han dynasty of China (206BC-220BC), Zhao Jun Qing gives a proof of Pythagoras' theorem in the form of the following diagram.



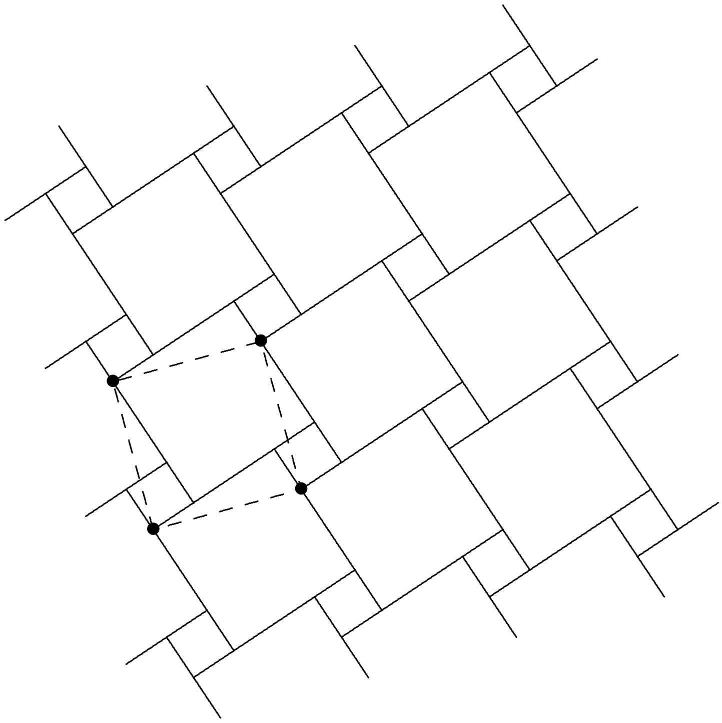
The above diagram illustrates that $3^2 + 4^4 = 5^2$. However, it does admit a general proof of Pythagoras' theorem. This is illustrated in the next diagram. The same proof is given by the Indian mathematician Bhaskara-Acharya (1114AD-1185AD). This is in fact the most easiest proof of Pythagoras' theorem.



Leonardo da Vinci (1452-1519) gives an interesting proof of Pythagoras' theorem. His proof is explained in the following diagram. Can you figure out how it works?



The following tiling of the floor contains a proof of Pythagoras' theorem. See if you can discover how it works.



§1.2 PYTHAGOREAN TRIPLES

Definition 1.1 A triple (x, y, z) of positive integers is called a *Pythagorean triple* if it satisfies the equation $x^2 + y^2 = z^2$.

The triples (3,4,5) and (5,12,13) are Pythagorean triples. In fact there are infinitely many Pythagorean triples. The problem of finding all Pythagorean triples is a little exercise in elementary arithmetic. It is much less trivial than finding all real solutions to the equation $x^2 + y^2 = z^2$. The latter problem is equivalent to finding all the right-angled triangles. Note that the graph of the equation $x^2 + y^2 = z^2$ is a double cone joined at the origin. Each point (x, y, z) on the graph gives a right-angled triangle. To find all the Pythagorean triples, one needs to find those integer points (x, y, z) lying on this double cone. A generalization of this problem is the Fermat Last Theorem.

Fermat Last Theorem Let n be an integer greater than 2. Then $x^n + y^n = z^n$ has no positive integer solutions in x, y and z.

This is conjectured by Fermat (1608-1677) and is finally solved by A.Wile in 1994. Therefore, apart from $n = 2$, there are no positive integer solutions to the equation $x^n + y^n = z^n$. Now, we proceed to find all the Pythagorean triples.

Theorem 1.2 The set of all integer solutions to the equation $x^2 + y^2 = z^2$ is $\{(m^2 - n^2, 2mn, m^2 + n^2)$ or $(2mn, m^2 - n^2, m^2 + n^2) : m > n, m, n \in \mathbb{Z}^+\}$.

Proof Let x, y, z be positive integers satisfying $x^2 + y^2 = z^2$. We may assume that x and y have no common factors. If not, all x, y, z have a common factor, then we can divide the equation $x^2 + y^2 = z^2$ by the square of this common factor, thus obtaining the equation $x'^2 + y'^2 = z'^2$ with x' and y' relatively prime. Then we can work with this new equation. Next we shall prove that either x or y is even. If both x and y are odd, then $z^2 = x^2 + y^2$ is even. Hence z is even. This means that $z^2 \equiv 0 \pmod 4$. But the square of an integer modulo 4 is either 0 or 1. As $z^2 = x^2 + y^2 \equiv 0 \pmod 4$, we must have both x and y are even, contradicting our assumption that x and y are relatively prime. Now we may assume without loss of generality that y is even. Let's write $y = 2k$, where k is a positive integer. Hence, $4k^2 = y^2 = z^2 - x^2 = (z+x)(z-x)$. As y is even, the equation $x^2 + y^2 = z^2$ shows that x and z should have the same parity. Hence both $(z+x)$ and $(z-x)$ are even. Thus, $k^2 = (\frac{z+x}{2})(\frac{z-x}{2})$. That is, k^2 factors into 2 positive integers $\frac{z+x}{2}$ and $\frac{z-x}{2}$. If these 2 integers have a common factor $p > 1$, then p divides both z and x because $z = \frac{z+x}{2} + \frac{z-x}{2}$ and $x = \frac{z+x}{2} - \frac{z-x}{2}$. Thus p divides all x, y, z which again contradicts our assumption that x, y, z have no common factors. Hence, $k^2 = (\frac{z+x}{2})(\frac{z-x}{2})$ and $\frac{z+x}{2}$, $\frac{z-x}{2}$ are relatively prime. This implies that $\frac{z-x}{2} = n^2$ and $\frac{z+x}{2} = m^2$ for some positive integers m and n with $m > n$. Consequently, $z = m^2 + n^2, x = m^2 - n^2$ and $y = 2mn$.

Some Pythagorean triples (x, y, z) with x, y, z relatively prime are listed below.

(3,	4,	5)
(5,	12,	13)
(15,	8,	17)
(7,	24,	25)
(21,	20,	29)
(9,	40,	41)
(35,	12,	37)
(11,	60,	61)
(45,	28,	53)
(33,	56,	65)
(13,	84,	85)

§1.3 COMMENSURABLE AND INCOMMENSURABLE QUANTITIES

The Pythagoreans believe that any two quantities can be compared in terms of ratios of whole numbers. Here the term "*quantity*" means either a positive number, the length of a line segment (or sometimes just the segment itself), the area of a plane figure or the volume of a solid object etc. When we talk about two quantities, we assume that they are of the same type.

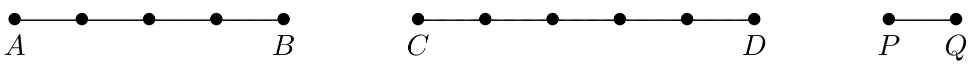
Definition 1.3

(i) Two quantities α and β are said to be *commensurable* if there exist $m, n \in \mathbb{Z}^+$ and a quantity γ of the same type such that $\alpha = m\gamma$ and $\beta = n\gamma$.

(ii) Two positive numbers a and b are said to be *commensurable* if $a/b \in \mathbb{Q}$.

(iii) Two quantities are said to be *incommensurable* if they are not commensurable.

For example, if AB and CD are segments such that $AB = 4PQ$ and $CD = 5PQ$ for some segment PQ, then AB are CD are commensurable.



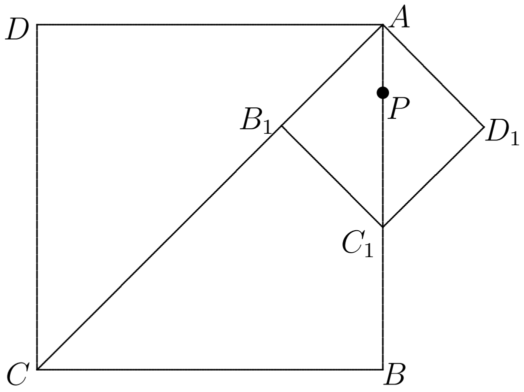
$\sqrt{2}$ and $3\sqrt{2}$ are commensurable, whereas 1 and $\sqrt{2}$ are incommensurable.

If we use the lengths of line segments, or the areas of plane figures etc., to represent positive numbers, then definitions 1 and 2 are in fact equivalent. Take for instance two commensurable positive numbers a and b. Hence, $a/b = p/q$ for some positive integers p and q. Then $a = p(b/q)$ and $b = q(b/q)$ so that a and b are commensurable in the sense of definition (i).

The Pythagoreans claim that all things are "numbers". They mean to imply that all pairs of lengths, areas, volumes are commensurable. In other words, "numbers" means "rational numbers". However, they do discover that there are incommensurable quantities. This puts all the theorems which they proved by means of ratios of whole numbers into shaky grounds.

The Pythagorean Hippasus (470BC) discovers that the side and the diagonal of a square are incommensurable.

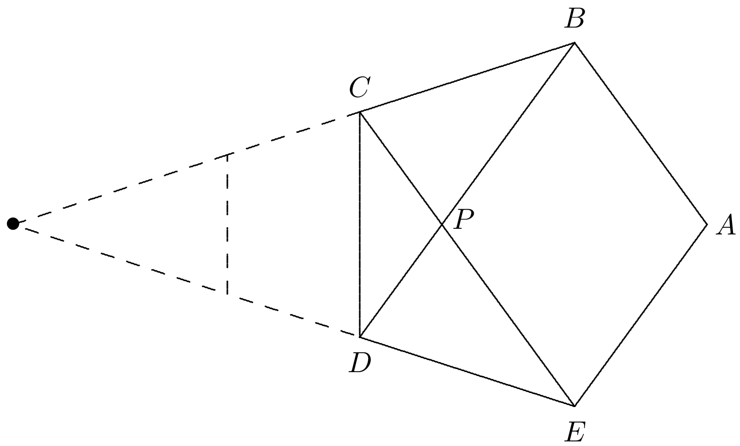
Suppose that AB and AC are commensurable with respect to a segment AP. Construct B_1 on AC such that $CB_1 = AB$. The perpendicular at B_1 meets AB at a point C_1. Then $AB_1 = B_1C_1 = BC_1$. Hence AB_1 and AC_1 are commensurable with respect to AP. But $AB_1 < \frac{1}{2}AB$.



Repeating the argument, we end up with quantities commensurable with respect to AP but less than AP! It is this problem which causes the existentialists crisis in ancient Greek mathematics. This is known as the first crisis in mathematics. In modern mathematics, we understand that $\sqrt{2}$ is irrational. The Pythagoreans are not able to define irrational numbers. However, they know that they exist and they know how to approximate any ratio of incommensurable quantities by commensurable quantities. For the formal construction of irrational numbers, one has to wait until the nineteen century when Peano (1858-1932) formulates his axioms for the natural numbers and Dedekind (1831-1919) uses his "Dedekind cuts" to construct the real numbers.

Proposition 1.4 The side and the diagonal of a regular pentagon are incommensurable.

Proof



Let $CD = a$ and $BD = b$. Let P be the point of intersection of BD and CE. Let $PC = c$. Then $ED = EP = a$ and $c = b - a$. Now suppose a and b are commensurable so that a and b can be measured by some integral multiples of a segment α. As $c = b - a$, c and a can also be measured by integral multiples of the segment α. Hence we have another regular pentagon whose side and diagonal are commensurable and both can be measured by integral multiples of the segment α. By repeating this construction, we obtain a sequence of regular pentagons whose sizes are getting arbitrarily small and their sides and diagonals are commensurable with respect to α. Hence, we will eventually obtain a segment of length less than α. But a segment of length less than α cannot be measured by α. This gives a contradiction. The fact that these regular pentagons are getting arbitrarily small in size can be seen by observing that all of them are bounded by the lines BC and ED which intersect at a point. More precisely, each pentagon decreases in size by a factor of $\tan 36^o < 1$.

Proposition 1.5 $\sqrt{2}$ is irrational.

Proof Suppose that $\sqrt{2} = p/q$ for some $p, q \in \mathbb{Z}^+$. After cancelling common factors, we may assume that p and q are relatively prime. Now $p^2 = 2q^2$, so that p^2 is even. Hence, p is even. Let $p = 2k$. Then $4k^2 = 2q^2$. In other word, $2k^2 = q^2$. As before, we see that q is even. This contradicts the fact that p and q are relatively prime.

Let a and b be two quantities of the same type. If a and b are commensurable, then a/b or $a : b$ has a clear meaning, namely $a/b \in \mathbb{Q}$. If a and b are incommensurable, then the Pythagoreans do not know what a/b means. That is because $a/b \notin \mathbb{Q}$. For the equality of ratios of commensurable quantities, the meaning is also clear.

Let a, b be two commensurable quantities and c, d another two commensurable quantities. Then we may express $a = ms, b = ns$ and $c = pt, d = qt$ for some quantities t, s and $m, n, p, q \in \mathbb{Z}^+$. To the Pythagoreans, $a/b = c/d$ simply means the equality of the two fractions m/n and p/q.

Now how do we define the equality of ratios of incommensurable quantities without knowing what they are?

§1.4 EUDOXUS' THEORY OF PROPORTION

Definition 1.6 Let a, b, c, d be positive numbers. $\dfrac{a}{b} = \dfrac{c}{d}$ if for any $p, q \in \mathbb{Z}^+$,

(i) $pa > qb \iff pc > qd$,

(ii) $pa = qb \iff pc = qd$,

(iii) $pa < qb \iff pc < qd$.

Proposition 1.7 Suppose that a, b are commensurable, and c, d are commensurable. Then the usual meaning of $a/b = c/d$ agrees with the meaning of equality defined by Eudoxus.

Proposition 1.8 Let a, b, c, d be positive numbers. Then $a/b = c/d$ in the usual sense of real numbers iff $a/b = c/d$ in the sense of Eudoxus.

Proof Assume that $a/b = c/d$ in the usual sense of real numbers. Let $p, q \in \mathbb{Z}^+$. Then, $pa > qb$ iff $pa > q(ad/c)$ iff $pc > dq$. This verifies (i) in Eudoxus' definition. The other two cases are similarly verified. Now Assume that $a/b = c/d$ in the sense of Eudoxus. Suppose $a/b > c/d$. By the density theorem for real numbers, insert a rational number q/p between a/b and c/d. Hence, $a/b > q/p > c/d$. Then, we have $pa > qb$, but $pc < qd$, contradicting (i) in Eudoxus's definition. Similarly, if $a/b < c/d$, then we have a contradiction against (iii). Consequently, $a/b = c/d$.

As an application of Eudoxus's definition, consider the following result.

Proposition 1.9 In a triangle ABC, D is a point on AB and E is a point on AC such that DE is parallel to BC. Suppose that AD are AB are commensurable. Then AE and AC are commensurable. Furthermore, $\dfrac{AD}{AB} = \dfrac{AE}{AC}$.

Proof Let $AD = mAP$ and $AB = nAP$. Divide the line segment AB into n equal parts with subdivision points $A = A_0, A_1, \ldots, A_m = D, \ldots, A_n = B$. For $i = 1, \ldots, n-1$, draw a line through A_i parallel to BC. These lines cut AC in $n-1$ equally spaced points. From this we have, $AE = mAQ$ and $AC = nAQ$. Hence, AE and AC are commensurable. Now $AD/AB = AE/AC$ means that $mnAP \cdot AQ = mnAP \cdot AQ$, which is true.

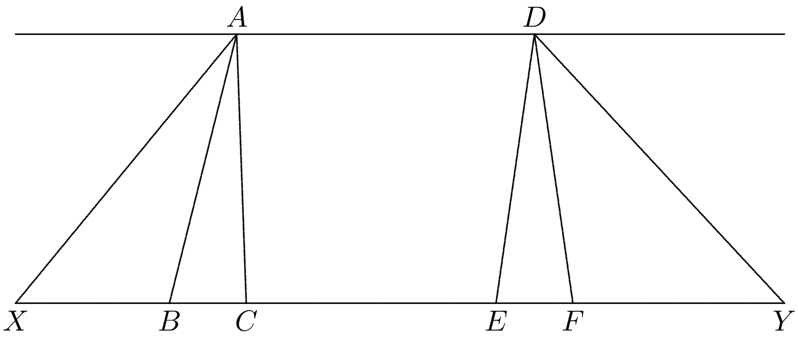
Suppose that AE and AC are incommensurable. We cannot subdivide AB into a finite number of equal parts with D being one of the subdivision point. How do we prove that $AD/AB = AE/AC$? In this situation, Eudouxs makes use of his definition. Take any two positive integers p and q. The same procedure of drawing lines parallel to BC shows that $pAD < qAB$ iff $pAE < qAC$. This verifies (i) in Eudoxus' definition. Similarly, one can verifies (ii) and (iii). Therefore, $AD/AB = AE/AC$.

Proposition 1.10 Let ABC and DEF be two triangles such that the height from A to BC equals to the height from D to EF. Then

$$\frac{BC}{EF} = \frac{(ABC)}{(DEF)},$$

where (ABC) and (DEF) denote the areas of the $\triangle ABC$ and $\triangle DEF$ respectively. (Note that this result is true when BC and EF are commensurable by the usual method of subdivision of line segments.)

Proof



Take any multiple pBC of BC and qEF of EF where $m, n \in \mathbb{Z}^+$. Let XC be a segment of length pBC and EY a segment of length qEF. Then

$$pBC > qEF \iff XC > EY \iff (AXC) > (EDY) \iff p(ABC) > q(DEF).$$

This verifies (i) in Eudoxus' definition. Similarly, (ii) and (iii) can be verified. Hence, $BC/EF = (ABC)/(DEF)$.

Exercises

1. Using the definition of proportion given by Eudoxus, show that

$$\frac{a}{b} = \frac{c}{d} \iff \frac{d}{c} = \frac{b}{a}.$$

2. Using the definition of proportion given by Eudoxus, show that

$$\frac{a}{b} = \frac{c}{d} \iff \frac{a+c}{b+d} = \frac{c}{d}.$$

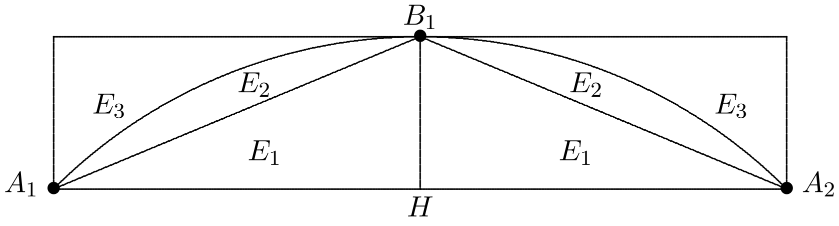
3. Let a, b and c be positive numbers with $a < b$. Prove that there exists a positive number z such that $a < z < b$ and z is commensurable with c.

§1.5 METHOD OF EXHAUSTION

The method of exhaustion is invented by Eudoxus. It is a way of proving certain proportionalities for curved figures by first proving that the proportionalities hold for similar polygons, then approximating the curved line by a polygon. Before we illustrate this method, let's first prove the following result due to Antiphon (425BC).

Proposition 1.11 The area s_n of an inscribed regular 2^n-gon in a circle of area c is greater than $(1 - \frac{1}{2^{n-1}})c$.

Proof



Consider the inscribed regular 2^{n-1}-gon $A_1A_2\cdots A_{2^{n-1}}$. Let $B_1,\ldots,B_{2^{n-1}}$ be the midpoints of the arcs $A_1A_2, A_2A_3,\ldots,A_{2^{n-1}}A_1$. Then an inscribed regular 2^n-gon $A_1B_1A_2\cdots B_{2^{n-1}}$ can be formed by joining A_1 to B_1, B_1 to A_2, ..., $B_{2^{n-1}}$ to A_1. Consider the sector of the circle cut out by the chord A_1A_2. Let H be the foot of the perpendicular from B_1 onto A_1A_2. Construct a rectangle of width A_1A_2 and height B_1H. This rectangle is partitioned into 6 regions whose areas are labelled as E_1, E_2 and E_3 as shown in the diagram. From the diagram, we see that $E_2 < E_2 + E_3 = E_1$ and $2E_2 < (E_1 + E_2) = \frac{1}{2}(2E_1 + 2E_2)$. It follows from this that we have $(c - s_n) < \frac{1}{2}(c - s_{n-1})$. Hence, inductively we have $(c - s_n) < c/2^{n-1}$.

Corollary 1.12 If $\alpha < c$, then there exists a positive integer n such that $s_n > \alpha$.

Theorem 1.13 The area of a circle is proportional to its diameter squared.

Proof Let c_1, c_2 and d_1, d_2 be the areas and the diameters of the circles C_1 and C_2 respectively. Let $\dfrac{d_1^2}{d_2^2} = \dfrac{c_1}{\alpha}$ for some α. We shall prove $\alpha = c_2$.
Suppose that $\alpha < c_2$. Inscribe regular 2^n-gons of area p_1 and p_2 respectively in the circle C_1 and C_2. By 1.12, we can pick n large enough so that $p_2 > \alpha$.
Now $\dfrac{d_1^2}{d_2^2} = \dfrac{p_1}{p_2} < \dfrac{c_1}{\alpha}$ which is a contradiction.
Here the first equality holds because the result is true for inscribed regular polygons. The second inequality holds because $c_1 > p_1$ and $p_2 > \alpha$.
Similarly, $\alpha > c_2$ is not possible. Consequently, $\alpha = c_2$.

In the above theorem, we have used the following result also proved by Antiphon.

Proposition 1.14 Let the areas of the regular 2^n-gons inscribed respectively in the circles of diameter 1 and d be p_1 and p_2. Then $p_2 = p_1 d^2$.

Now let's give a more direct proof of theorem 1.14 due to Eudoxus.
Let k be the area of the circle of diameter 1 and c the area of the circle of diameter d. (We know that k is just $\pi/4$.) We wish to prove that $c = kd^2$. First let's suppose that $kd^2 < c$. By 1.12, we can pick an inscribed regular 2^n-gon of area p_2 inside the circle of diameter d so that $c > p_2 > kd^2$. Let p_1 be the area of the

regular 2^n-gon inscribed in the circle of diameter 1. Then, $kd^2 > p_1 d^2 = p_2 > kd^2$ which is a contradiction. On the other hand, suppose $c < kd^2$. Pick n large enough so that $\frac{kd^2}{2^{n-1}} < kd^2 - c$. This implies that $[k(1 - \frac{1}{2^{n-1}})]d^2 > c$. By 1.11, we have $p_1 > [k(1 - \frac{1}{2^{n-1}})]$. Therefore, $c < [k(1 - \frac{1}{2^{n-1}})]d^2 < p_1 d^2 = p_2 < c$, which again is a contradiction. Consequently, $c = kd^2$.

Note that in order to have the exact formula for the area of the circle of diameter d, it is necessary to know the value of k which in turn ask us to find the value of π.

§1.6 CONTINUED FRACTIONS

The Pythagoreans know how to write a rational number as a continued fraction. A procedure found in Proposition 2 of Book VII of the Elements and possibly due to the Pythagorean Archytas is as follow.

I Expand $\frac{a}{b}$ into a continued fraction.

$$
\begin{aligned}
\frac{a}{b} &= a_0 + \frac{r}{b} && (0 < r < b) \\
\frac{b}{r} &= a_1 + \frac{r_1}{r} && (0 < r_1 < r) \\
&\;\;\vdots \\
\frac{r_{n-3}}{r_{n-2}} &= a_{n-1} + \frac{r_{n-1}}{r_{n-2}} && (0 < r_{n-1} < r_{n-2}) \\
\frac{r_{n-2}}{r_{n-1}} &= a_n && (r_n = 0)
\end{aligned}
$$

Note that the r_i's form a decreasing sequence of positive integers. The process stops when $r_n = 0$. Then we have

$$
\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}
$$

We use the notation $[a_0, a_1, a_2, \dots, a_n]$ to denote the above continued fraction. Hence,

$$\frac{a}{b} = [a_0, a_1, a_2, \dots, a_n].$$

For each $i = 1, \dots, n$, $\dfrac{p_i}{q_i} = [a_0, a_1, a_2, \dots, a_i]$ is called the ith partial quotient of the continued fraction $[a_0, a_1, a_2, \dots, a_n]$.

For example, let's express 19/7 as a continued fraction.

$$\begin{aligned} \frac{19}{7} &= 2 + \frac{5}{7} \\ \frac{7}{5} &= 1 + \frac{2}{5} \\ \frac{5}{2} &= 2 + \frac{1}{2} \\ \frac{2}{1} &= 2 \end{aligned}$$

Thus

$$\frac{19}{7} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}$$

or we may write $19/7 = [2, 1, 2, 2]$.

For any real number x, let's use $\lfloor x \rfloor$ to denote the largest integer less than or equal to x. When we carry out the computation of the continued fraction expansion of a/b, we notice that $a_0 = \lfloor \frac{a}{b} \rfloor$, $a_1 = \lfloor \frac{b}{r} \rfloor$, ..., $a_n = \lfloor \frac{r_{n-2}}{r_{n-1}} \rfloor$.

Therefore, instead of carrying out the division, one can simply take the integral part of the reciprocal of the corresponding remaining fraction.

II For an irrational number α, we can also perform the same expansion.

$$a_0 = \lfloor \alpha \rfloor, \quad a - a_0 = \frac{1}{\alpha_1} \qquad \text{(note that } \alpha_1 > 1\text{)}$$

$$a_1 = \lfloor \alpha_1 \rfloor, \quad \alpha_1 - a_1 = \frac{1}{\alpha_2}$$

$$a_2 = \lfloor \alpha_2 \rfloor, \quad \alpha_2 - a_2 = \frac{1}{\alpha_3}$$

$$\vdots$$

endless.

Then

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots\cdots}}}}$$

In other word, $\alpha = [a_0, a_1, a_2, \ldots]$. The following theorem guarantees that the sequence of partial quotients converges to α.

Theorem 1.15

(i) $\{q_n\}$ is an increasing sequence of positive integers.

(ii) For each $n \in \mathbb{Z}^+$, $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$.

As an example, let's find the continued fraction expansion of $\sqrt{2}$.

$$\begin{array}{rclcrcl} 1 & = & \lfloor \sqrt{2} \rfloor, & & \sqrt{2} - 1 & = & \frac{1}{\sqrt{2}+1}, \\ 2 & = & \lfloor \sqrt{2} + 1 \rfloor, & & (\sqrt{2}+1) - 2 & = & \frac{1}{\sqrt{2}+1}, \\ 2 & = & \lfloor \sqrt{2} + 1 \rfloor, & & (\sqrt{2}+1) - 2 & = & \frac{1}{\sqrt{2}+1}, \\ & \vdots & & & & \vdots & \end{array}$$

Therefore, $\sqrt{2} = [1, 2, 2, 2, \cdots]$. The first few partial quotients are $3/2, 7/5, 17/12$ etc.

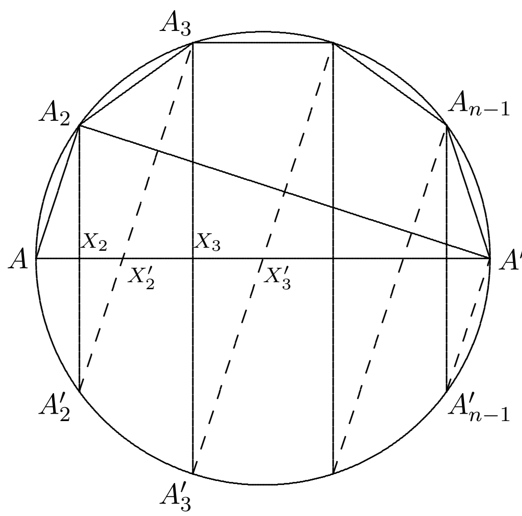
As an exercise, show that $\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, \cdots]$ and find the first few partial quotients of the continued fraction expansion of $\sqrt{3}$.

It is known that the continued fraction expansion of a number is periodic if and only if the number is the surd of a square free integer.

§1.7 THE SURFACE AREA OF A SPHERE

In this section, we shall illustrate how Archimedes uses the Method of Exhaustion to find the surface area of a sphere. First, let's recall that the lateral surface area of a frustum having top diameter a_1, base diameter a_2 and slant height s is given by $\frac{1}{2}\pi(a_1 + a_2)s$.

Consider a circle C of diameter AA'. Let the length of AA' be $2r$. Let $A_1 = A, A_2, \ldots, A_n = A'$ be points equally spaced on the arc AA'. When the arc AA' is rotated about the axis AA', a sphere S of diameter $2r$ is generated. Similarly, when the polygonal arc $A_1A_2 \cdots A_n$ is rotated about AA', a figure in the form of a union of $n-3$ frustums with two cones attached to its ends is generated. Let's denote this object by S_n and its surface area by s_n. S_n is inscribed in the sphere S. Our first goal is to calculate s_n.



For each $i = 2, \ldots, n-1$, let A_i' be the point on C obtained by reflecting A_i along AA' and let X_i be the intersection of A_iA_i' and AA'. Join $A_i'A_{i+1}$, $i = 2, \ldots, n-1$. Also for each $i = 2, \ldots, n-2$, let X_i' be the intersection of $A_i'A_{i+1}$ and AA'.

Then all the right-angled triangles AA_2X_2, $X_2A_2'X_2'$, $X_2'A_3X_3$, ..., $X_{n-2}'A_{n-1}X_{n-1}$, $X_{n-1}A_{n-1}'A'$ are similar. Furthermore, each of these triangles is similar to the triangle AA_2A'. Hence, we have the following relations of ratios of their sides.

$$\frac{X_2A_2}{AX_2} = \frac{X_2A_2'}{X_2X_2'} = \frac{X_3A_3}{X_2'X_3} = \cdots = \frac{X_{n-1}A_{n-1}'}{X_{n-1}A'} = \frac{A'A_2}{A_2A}.$$

Adding all these ratios together, we have

$$\frac{A_2A_2' + A_3A_3' + \cdots + A_{n-1}A_{n-1}'}{AA'} = \frac{A'A_2}{A_2A}.$$

Thus, $A_2A(A_2A_2' + A_3A_3' + \cdots + A_{n-1}A_{n-1}') = A'A_2 \cdot AA'$. Therefore,

$$\begin{aligned}
s_n &= \tfrac{1}{2}\pi A_2A[A_2A_2' + (A_2A_2' + A_3A_3') + \cdots + (A_{n-2}A_{n-2}' + A_{n-1}A_{n-1}') + A_{n-1}A_{n-1}'] \\
&= \pi A_2A(A_2A_2' + A_3A_3' + \cdots + A_{n-1}A_{n-1}') \\
&= \pi A'A_2 \cdot AA' \\
&= 4\pi r^2 \cdot \frac{A'A_2}{2r}.
\end{aligned}$$

Note that $A'A_2$ is less than $2r$ so that s_n is less than $4\pi r^2$ but is arbitrarily close to $4\pi r^2$.

Now we use the Method of Exhaustion. Let s be the surface area of the sphere S. First suppose that $s > 4\pi r^2$. Clearly $s_n < s$. One can show that s_n is arbitrarily close to s. The proof of this fact is similar to 1.11. Pick n large enough so that $s - s_n < s - 4\pi r^2$. Then $s_n > 4\pi r^2$, which is a contradiction.

Next suppose that $s < 4\pi r^2$. Pick n large enough so that $4\pi r^2 - s_n < 4\pi r^2 - s$. Then $s_n > s$, which is again a contradiction. Hence, $s = 4\pi r^2$.

The formula $s = 4\pi r^2$ is proved by inscribing polygonal objects in the sphere in such a way that their areas get not only arbitrarily close to s but also the expression $4\pi r^2$. This latter property usually leads us to discover the formula.

§1.8 THE METHOD

The Method is the last of the 10 treaties written by Archimedes (287-212BC). It describes a principle of calculating the volume of a three-dimensional object. Nowadays, this is known as Cavalieri's Principle. The radical idea is to think of a three-dimensional object as a stack of thin layers. For example, a rectangular box becomes a deck of cards. Then, if this deck is pushed sideways, a second solid having the same volume is obtained. Using this principle, we see that two triangular pyramids with the same base and height have the same volume. This principle proved by Cavalieri (1598-1647) is stated as follow.

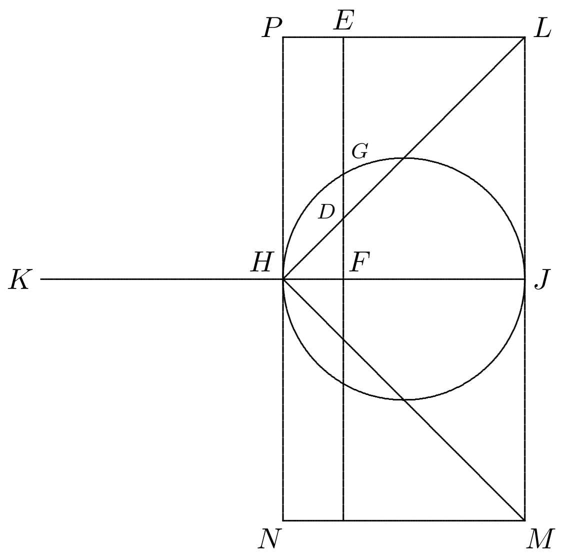
Cavalieri's Principle *If two 3-dimensional solids have equal altitudes and if cross-sections made by planes parallel to their bases and at equal distances from them have their areas always equal, then the two solids have equal volumes.*

In china, this principle of regarding the volume of a three dimensional solid as the "sum" of areas of parallel planar cross-sections is known as Zu Geng's Principle. (Zu Geng is the son of the famous Chinese mathematician Zu Chong Zhi (430-501).)

Archimedes' *Method* is a further refinement of this principle. Archimedes makes use of Cavalieri's Principle and his *Law of the Level* to calculate the volume of a sphere.

The Law of the lever Two weights w_1 and w_2 balance on a lever arm with fulcrum F if $w_1 s_1 = w_2 s_2$, where s_1 is the distance from w_1 to F and s_2 is the distance from w_2 to F.

Archimedes' idea is to compare the volume of the sphere with that of a cylinder and a cone by means of parallel sections. The diagram below shows that a solid sphere of diameter HJ and a solid cone of height HJ and base radius LJ lie inside a solid cylinder of height HJ and base radius LJ. Produce JH to a point K so that $KH = HJ$. Let the radius of the sphere be $2r$. From the diagram, we have $HK = HJ = EF = 2r$.



Using Pythagoras' theorem, we have $FD^2+FG^2 = HF^2+FG^2 = HG^2 = HF \cdot HJ$. Multiplying both sides by π and HK, we obtain

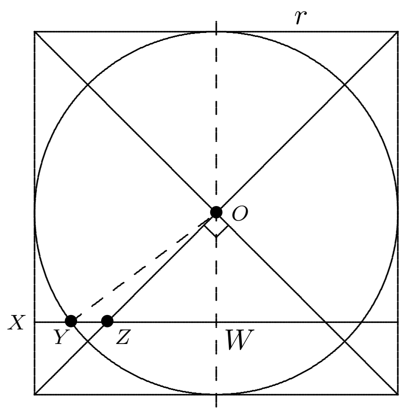
$$(\pi FD^2 + \pi FG^2) \cdot HK = (\pi EF^2) \cdot HF.$$

This equation means the following:

The cone placed at K	$+$	The sphere placed at K	$=$	The cylinder placed at its centre of gravity, which is the midpoint of HJ.

Hence, $\frac{1}{3}\pi(2r)^3$ + Volume of the sphere $= \frac{1}{2}\pi(2r)^3$, where V is the volume of the sphere. Consequently, the volume of the sphere is $\frac{4}{3}\pi r^3$.

There is a direct method in computing the volume of the sphere by means of Cavalieri's principle. To illustrate this idea, take a circle of diameter $2r$ inscribing in a square. The two diagonals of the square meet at the center O. Let the whole plane figure revolve about the vertical axis passing through O. Then a solid sphere of diameter $2r$ and a solid double cone are generated and they both sit inside a solid cylinder of height $2r$ and base radius r.



From the diagram, we have $OY^2 = OW^2 + WY^2$. As $OY = XW$ and $OW = WZ$, we have $XW^2 = WZ^2 + WY^2$. Now applying Cavalieri's principle, we have the following conclusion.

Volume of the cylinder	$=$	Volume of the sphere	$+$	Volume of the cone

From this, the volume of the sphere is $\frac{4}{3}\pi r^3$.

Exercises

1. Using Cavalieri's Principle, find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. Using Cavalieri's Principle, find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
3. Using the Method of Exhaustion, show that for pyramids of the same triangular base, the volume is proportional to the height.

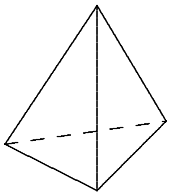
Area and volume can both be computed by means of Cavalieri's principle. However, there are some differences between the two concepts essentially due to different properties of two and three dimensions. In the plane the following theorem can be proved.

Theorem 1.16 Two polygons have the same area if and only if one can be dissected into a finite number of pieces and then reassembled to form the other.

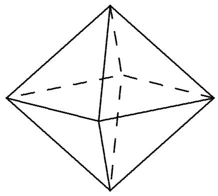
The famous Hilbert's third problem asks whether or not the same assertion is true for 3-dimensional polyhedra. Max Dehn gives a counterexample. He exhibits two polyhedra of equal volume which are not congruent by dissection. Another difference between two and three dimensions is suggested by the famous paradox of Banach-Tarski (1942) in which a sphere of radius 1 cm can be dissected into a finite number of pieces which are then reassembled to form a sphere having the size of the earth.

§1.9 REGULAR POLYHEDRA

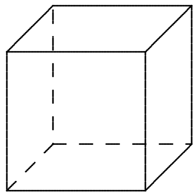
A *polyhedron* is a 3-dimensional solid whose surfaces consists of polygonal faces. A polyhedron is *regular* or *Platonic* if its faces are congruent regular polygons. The Pythagoreans discover that there are only five regular polyhedra. They are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. Almost everyone who encounters the regular polyhedra finds something appealing about them. That the Greeks make such detailed and sustained studies of them is probably connected with their unexpected finiteness: there are only five of them, in contrast to the unlimited number of regular polygons. Fascination with these polyhedra leads both Kepler (1571-1630) and Plato (427-347BC) to use them in their theories of the cosmos. In his dialogue, *Timaeus*, Plato discusses the four "elements" of which everything is composed: earth, air, fire and water. Earth particles have the form of cubes which stand solidly on their bases. Air particles have the form of regular octahedra which are light and rotate freely when held by opposite vertices. Fire particles have the form of regular tetrahedra which have sharp corners. Lastly, water particles have the form of regular icosahedra which are almost spherical and roll around like liquid. Kepler adds to Plato's cosmology by giving the entire universe the shape of the dodecahedron. He incorporates the regular polyhedra into Copernicus' system of planets and builds a model of his universe.



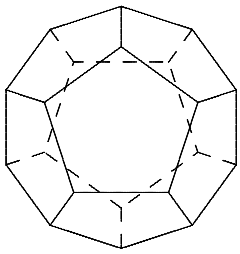
Tetrahedron



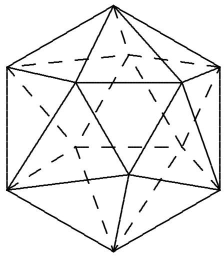
Octahedron



Cube



Dodecahedron



Icosahedron

Theorem 1.17 There are only 5 regular polyhedra.

Proof Consider a regular polyhedron in which m regular n-gons meet at a vertex. The angle sum around a vertex must be less than 2π. Hence, $m(\pi - 2\pi/n) < 2\pi$. This is equivalent to the inequality $1/m + 1/n > 1/2$. As $m, n \geq 3$, we have the following solutions.

m	n	Solid
3	3	Tetrahedron
3	4	Cube
3	5	Dodecahedron
4	3	Octahedron
5	3	Icosahedron

Theorem 1.18 Let V, E and F be the number of vertices, edges and faces of a convex polyhedron respectively. Then $V - E + F = 2$. This called Euler's formula.

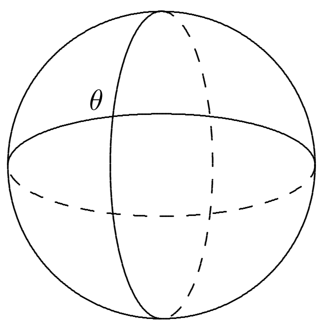
Proof Each face of the polyhedron contributes an "one" in the expression $V - E + F$. If a face $\triangle$ is an n-gon, then we may join the vertices of $\triangle$ to a point

inside $\triangle$. Hence, $\triangle$ gives rise to 1 new vertex, n new edges and n triangular faces. But the contribution of $V - E + F$ is still 1 $(= 1 - n + n)$. Consequently, the expression $V - E + F$ remains unchanged under subdivision of the faces into triangles. Therefore, we may assume that each face of the polyhedron is triangular. Now consider the polyhedron being built by adding a triangle one at a time. Initially, there is only one triangle. The expression $V - E + F$ is clearly 1. Since there are F faces, it will take F steps to build the polyhedron. Let P_n be the resulting object obtained by attaching n faces together. Therefore, P_1 is just the first triangle and P_F is the polyhedron. Now examine how each triangle is being added to P_n. A triangle $\triangle$ is attached to P_n so that either (i) a vertex of $\triangle$ is attached to a vertex of P_n. (ii) one edge of $\triangle$ is matched with one edge of P_n, (iii) two edges of $\triangle$ are matched with two edges of P_n or (iv) it is the final stage where all the three edges of $\triangle$ are matched with three edges of P_n. Let's check the change of the expression $V - E + F$ in each case.
In (i),(ii) and (iii), $V - E + F$ remains to be 1, just as the case for one triangle. In the last stage, this last triangle contributes an "one" in the term F, thus making $V - E + F = 2$.
(What we have above is only a sketch of the proof. For a rigorous proof, one needs to be more careful in attaching the faces. There are essentially four different proofs of Euler's formula due to Euler, Legendre, Cauchy and Von Staudt independently.)

Let's list out the number of vertices, edges and faces of the 5 regular polyhedra.

	V	E	F
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Legendre's proof of Euler's formula is by means of spherical geometry. To understand his proof, we need to find out the area of a spherical triangle on a sphere S of radius 1. Two great circles on S always intersect at two antipodal points and the intersecting region consists of two lunes. If they intersect at an angle θ radians, then the area of one of the lune is $(\theta/2\pi) \cdot (4\pi) = 2\theta$.



Let ABC be a spherical triangle formed by the intersection of three great circles a, b and c. We shall use the convention that the arc BC of the spherical triangle lies on the great circle a. Similarly, AC lies on b and AB lies on c. The great circles a, b and c intersect not only at A, B and C but also at their antipodal points A', B' and C'. The great circles a and b bound two lunes at an angle equal to $\angle C$. Similarly for the other two pairs. These six lunes together cover the sphere S and they cover the triangle ABC and $A'B'C'$ three times. Thus $2(2\angle A + 2\angle B + 2\angle C) = 4\pi + 4(ABC)$, where (ABC) is the area of the spherical triangle ABC. Hence, $(ABC) = (\angle A + \angle B + \angle C) - \pi$. This formula is due to Thomas Harriot (1560-1621).

Consider a small convex polyhedron P placed inside the unit sphere S so that the centre of S is inside P. As in Euler's proof, we may assume that P has only triangular faces. The radial projection from the centre of S maps P onto a polyhedral network P' on S in which the edges are great circular arcs and the faces are spherical triangles on S. Clearly the expression $V - E + F$ is same for both P and P'. Let $\mathcal{F}$ be the set of all the triangular faces of P'.

The sum of the areas of all the spherical triangles is equal to

$$\sum_{\triangle \in \mathcal{F}} (\text{angle sum of } \triangle - \pi)$$

which is equal to $2\pi V - \pi F$. As all the faces are triangles, we have $2E = 3F$. Hence, $2\pi V - \pi F = 2\pi V - 2\pi E + 2\pi F$. Now the sum of the areas of all the spherical triangles is just the surface area of the unit sphere S which is equal to 4π. Therefore, $2\pi V - 2\pi E + 2\pi F = 4\pi$. Consequently, we have proved the Euler's formula $V - E + F = 2$.

Euler's formula may be used to deduce that there are only five regular polyhedra.

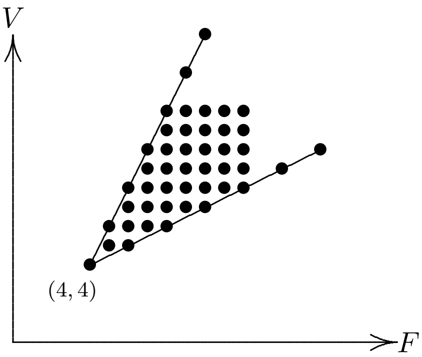
In a regular polyhedra, suppose that m regular n-gons meet at a vertex. Then $nF = 2E$ and $mV = 2E$. Substituting $E = nF/2$ and $V = nF/m$ into Euler's formula, we have $nF/m - nF/2 + F = 2$. That is $nm - 2n - 2m = -4m/F$. Hence, $nm - 2n - 2m < 0$. This inequality is equivalent to $(n-2)(m-2) < 4$. Since $m, n \geq 3$, this gives all the five pairs of solutions of (m, n), and there are only 5 regular polyhedra.

Note that in any convex polyhedron, we always have $3F \leq 2E$ and $3V \leq 2E$. This is because that each face must be at least a triangle and there are at least 3 faces meeting at a vertex. Substituting these inequalities into Euler's theorem, we deduce the following corollary.

Corollary 1.19 In any convex polyhedron with V vertices, E edges and F faces, we have

$$\frac{F}{2} + 2 \leq V \leq 2F - 4.$$

In fact all pairs (F, V) with $F, V \geq 4$ inside the region defined by the above inequalities can be realized by a convex polyhedron.



Let F_n be the number of n-gons of a convex polyhedron. Thus F_3 is the number of triangular faces of the polyhedron. Now, the total number of faces is

$$F = F_3 + F_4 + F_5 + F_6 + \cdots + F_n + \cdots$$

and the total number of edges of all the faces is

$$2E = 3F_3 + 4F_4 + 5F_5 + 6F_6 + \cdots + nF_n + \cdots .$$

The inequality $V \leq 2F - 4$ together with Euler's formula gives $E \leq 3F - 6$. Substituting the above relations into this inequality gives the following inequality.

$$3F_3 + 2F_4 + F_5 - F_7 - \cdots - (n-6)F_n - \cdots \geq 12.$$

Theorem 1.20 Every convex polyhedron contains a 3-sided, 4-sided or 5-sided face.

This result is the key to prove the 5-colour theorem.

§1.10 SYMMETRIES

Definition 1.21 A group is a pair $(G, \cdot)$, where G is a nonempty set and $\cdot$ is a binary operation on G satisfying the following axioms.

(i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$.

(ii) There exists $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$.

(iii) For each $a \in G$, there exists an unique element in G, denoted by a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

We shall simply denote a group by its underlying set G when the binary operation is understood. Very often, we shall omit the "dot" in writing $a \cdot b$. The element e is called the identity of G. a^{-1} is called the inverse of a. A group G is said to be *Abelian* if $ab = ba$ for all $a, b \in G$. A group which is not Abelian is called *non-Abelian*.

Examples

1. $\mathbb{Z}$ equipped with the addition $+$ as the binary operation is a group. Similarly $\mathbb{Q}$ and $\mathbb{R}$ are groups under addition. $\mathbb{Z}$ is called the infinite cyclic group. The identity element in $\mathbb{Z}$ is 0. Every integer n can be written as sum of n ones. If we use the multiplicative notation and denote the identity of $\mathbb{Z}$ by e and the integer 1 by a, then an element of the infinite cyclic group is denoted by a^n. The group operation applied to two elements a^n and a^m is simply $a^n a^m = a^{m+n}$.

2. $\mathbb{Q}^+$ together with multiplication is a group. Similarly $\mathbb{R}^+$ is a group under multiplication.

3. For each positive integer m, the set $\mathbb{Z}_m$ of all residue classes modulo m is a group under addition. $\mathbb{Z}_m$ is also called the cyclic group of order m. Again we shall use the multiplicative notation for $\mathbb{Z}_m$ like the infinite cyclic group. For instance, $a^m = e$. Note that $\mathbb{Z}_m$ has m elements.

4. Let S_n be the set of all permutations of n distinct objects. Then S_n under the composition of permutations is a group. One may regard an element of S_n as a bijection σ from the set $\{1, 2, \ldots, n\}$ onto itself. Such a bijection can be represented by a $2 \times n$ array, where the first row consists of $1, 2, \ldots, n$, and the second row consists of the images of σ, namely, $\sigma(1), \sigma(2), \ldots, \sigma(n)$.

   For instance, $e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ is the identity of S_n.

   In S_4, let $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$.

Then $\sigma_1\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ and $\sigma_2\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$.

Note that S_n is non-Abelian for $n \geq 3$. S_n is called the symmetric group of degree n. It has $n!$ elements. Let's describe some of the low degree symmetric groups. S_1 has only 1 element. S_2 has 2 elements and is essentially $\mathbb{Z}_2$. S_3 has 6 elements. The elements of S_3 are listed as follow.

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

To describe the binary operation of the group S_3, one can compile a group multiplication table.

$\cdot$	e	τ_1	τ_2	τ_3	σ_1	σ_2
e	e	τ_1	τ_2	τ_3	σ_1	σ_2
τ_1	τ_1	e	σ_2	σ_1	τ_3	τ_2
τ_2	τ_2	σ_1	e	σ_2	τ_1	τ_3
τ_3	τ_3	σ_2	σ_1	e	τ_2	τ_1
σ_1	σ_1	τ_2	τ_3	τ_1	σ_2	e
σ_2	σ_2	τ_3	τ_1	τ_2	e	σ_1

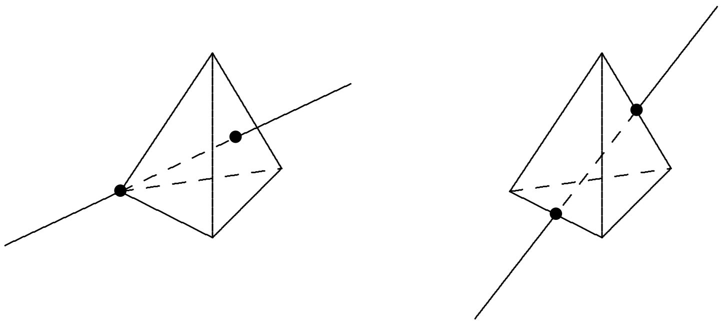
Definition 1.22 A symmetry of a 2- (or 3-) dimensional geometric figure is a motion in the plane (or the 3-dimensional space) which moves the figure to a position occupying the same space as before.

The set of all symmetries of the geometric figure forms a group under the composition of motions. This group is called the *symmetry group* of the geometric figure. Here the motion I which does not move the figure at all is the identity of the group.

Examples

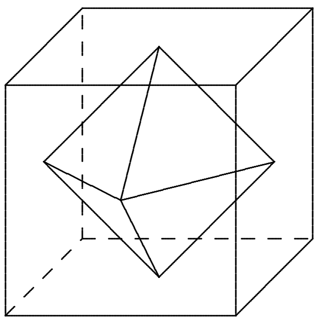
1. The symmetry group of the regular n-gon in the plane is the cyclic group $\mathbb{Z}_n$ of order n. Let R be the anticlockwise rotation of the regular n-gon about the centre through an angle $2\pi/n$. Then the symmetry group is $\{I, R, R^2, \ldots, R^{n-1}\}$.
2. If we allow the motion of the regular n-gon to be carried out in 3-dimensional space, then one can also rotate the regular n-gon through an angle π about an axis joining two opposite vertices of the regular n-gon. This gives a larger group than just the symmetry group of the regular n-gon in the plane.

3. Let's find the symmetry group of the tetrahedron. The tetrahedron has eight axes of symmetries. For each vertex A_i of a tetrahedron $A_1A_2A_3A_4$, there is an axis ℓ_i joining this vertex A_i of the tetrahedron to the centre of its opposite face. Denote the anticlockwise rotation about this axis through an angle $2\pi/3$ by R_i. Also there is an axis joining the centres of each pair of opposite edges of the tetrahedron. Since there are three pairs of opposite edges, there are three such axes. Denote the rotation through an angle π about each of these axes by T_1, T_2 and T_3 respectively. Then the symmetry group of the tetrahedron is $\{I, R_1, R_2, R_3, R_4, R_1^2, R_2^2, R_3^2, R_4^2, T_1, T_2, T_3\}$.



This group is the so called alternating group of degree 4 which is a "subgroup" of S_4. It has 12 elements. In fact there are $4 \times 3 = 12$ ways of putting the tetrahedron on the table.

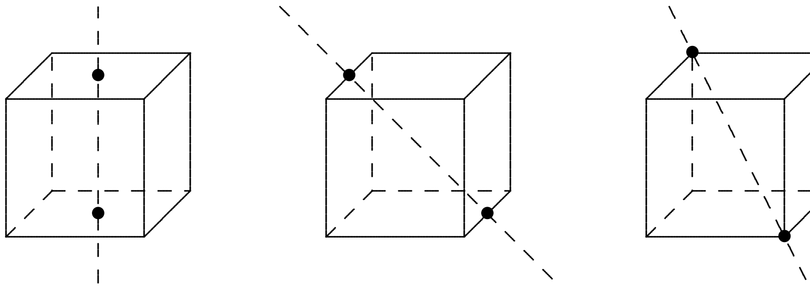
4. For each regular polyhedron, one can form the so called *dual* polyhedron by joining the centres of each pairs of adjacent faces.



The dual of the tetrahedron is the tetrahedron itself. The dual of the cube is the octahedron and the dual of the dodecahedron is the icosahedron. As such, the symmetry group of the octahedron is the same as the symmetry group of the cube. Also, the symmetry group of the dodecahedron is the same as the symmetry group of the icosahedron.

Theorem 1.22 The symmetry group of the cube is S_4.

Proof First let's count the number of elements in the symmetry group of the cube. Since there are 6 faces of the cube, there are 6 possible ways of putting the cube on a table so that one of the face is in contact with the table. Next there are 4 ways of putting the 4 vertical faces facing north. Hence, there are 24 elements in this group. Note that S_4 also has $4! = 24$ elements.



The cube has 3 pairs of opposite faces. Also there is an axis joining the centres of each pair of opposite faces. Denote these 3 axes by ℓ_1, ℓ_2, ℓ_3. For $i = 1, 2, 3$, let R_i be the anticlockwise rotation through an angle $\pi/2$ about the axis ℓ_i. This gives rise to 9 symmetries of the cube, namely, $R_i, R_i^2, R_i^3,\ i = 1, 2, 3$.
Next, the cube has 6 pairs of opposite edges. For each such pair of opposite edges, there is an axis joining the midpoints of these two edges. Let's label these 6 axes by $\ell'_i,\ i = 1, \dots, 6$. Now, for $i = 1, \dots, 6$, let S_i be the rotation through an angle π about the axis ℓ'_i. This gives rise to 6 symmetries of the cube, namely, $S_i,\ i = 1, \dots, 6$.
Let $\overline{\ell}_1, \overline{\ell}_2, \overline{\ell}_3, \overline{\ell}_4$ be the four "long" diagonals of the cube. For $i = 1, \dots, 4$, let T_i be the anticlockwise rotation through an angle $2\pi/3$ about the axis $\overline{\ell}_i$. This gives rise to another 8 symmetries of the cube, namely, $T_i, T_i^2,\ i = 1, \dots, 4$.
Therefore, if we include the identity I, then there are 24 symmetries of the cube. One can compile a group table of the symmetry group of the cube and check that it is identical to the group table of S_4 up to renaming of the elements.

Since the symmetry group of the cube is S_4, we expect there are 4 objects inside the cube that are permuted by this group. What are these 4 objects? The symmetry group of the dodecahedron is a subgroup of S_5. Can you find out the number of elements in this group?

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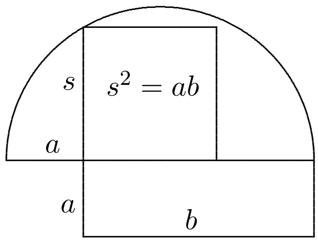
§1.11 RULER AND COMPASS CONSTRUCTIONS

The ancient Greeks know how to construct a square equal in area to a given rectangle, thus giving a way of finding the geometric mean of two positive numbers.



This task can be achieved by using a straightedge and a compass only. Hence, one can construct a square having the same area of any given triangle. Then, using Pythagoras' theorem, which makes it possible to construct a square having an area equal to the sum of areas of two squares, one can easily see how to construct a square having an area equal to any figure that can be triangulated. Since any polygon can be triangulated, we see that it is possible to construct a square equal in area to any polygon. All this theory known as *quadrature*, meaning squaring, must have been known to the Pythagoreans. Naturally, there are attempts to extend these results further. In particular, the following three problems, best known as *the three Greek problems of antiquity* arise.

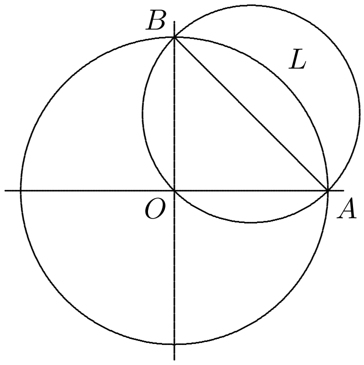
(A) **Squaring the circle** Construct a square having an area equal to the area of a given circle.

(B) **Doubling the cube** Construct a cube having a volume equal to twice the volume of a given cube.

(C) **Trisecting an angle** Can one always trisect a given angle?

The solutions if exist, to these problems should be done using a straightedge and a compass only.

Instead of squaring the circle, Hippocrates of Chios (around 5BC) is successful in squaring certain regions between overlapping circles. Such regions resemble crescent moon and are therefore called *lunes*. The following diagram shows the quadrature of a lune L, which is the region inside the small circle but outside the larger circle. The area of the lune can easily be shown to be the area of triangle OAB.



Hippocrates is able to work out the quadrature of many simple lunes. It is interesting to note that some mysterious complication in the circle is cancelled out when part of one circle is subtracted from another, so that the difference of two circles can be squared, but not a single circle. The problem of squaring the circle has not been solved until 1882, when C. Lindemann (1852-1939) finally proves that π is *transcendental* and hence is not constructible.

The problem of doubling the volume of a cube is equivalent to the construction of the number $\sqrt[3]{2}$ using a straightedge and a compass. There is an interesting legend in ancient Greeks which goes as follow.

Ancient Athens, being faced by a serious plague, sent a delegation to the oracle of Apollo at Delos for advice in their difficulty. The delegation was told to double the cubical altar to Apollo. Unfortunately, they doubled the length of each edge, thereby increasing the volume by a factor of 8 *rather than* 2*; and the plague only got worse!*

The problem of doubling the volume of a cube is known to be impossible using results in Field theory.

Though some angles can be trisected, it is not always possible to trisect an arbitrary angle. There are devices capable of trisecting an arbitrary angle, but the principle is beyond just using a straightedge and a compass.

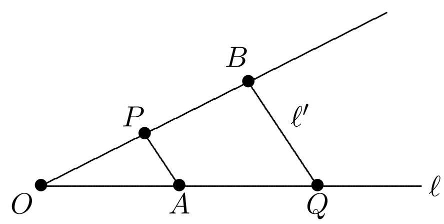
A closely related problem is to find out which regular polygons can be constructed using a straightedge and a compass. The Greeks struggle to find straightedge and compass constructions for regular polygons with 7,9,11,13 and 17 sides. In all this, they fail, but it is not proved until the nineteenth century that the reason for their failure is that all these problems are impossible - except one. In 1796, Gauss discovers a straightedge and compass construction for the regular 17-sided polygon. It is this first advance on Greek construction problems in 2000 years that motivates Gauss to devote himself to mathematics.

§1.12 CONSTRUCTIBLE QUANTITIES

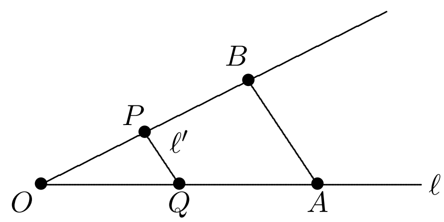
Definition 1.23 Fix a line segment of unit length. A real number α is *constructible* if one can construct a line segment of length $|\alpha|$ in a finite number of steps from this given segment of unit length by using a straightedge and a compass.

Theorem 1.24 If α, β are constructible real numbers, then so are $\alpha+\beta, \alpha-\beta, \alpha\beta$ and α/β, provided $\beta \neq 0$.

Proof We are given that α and β are constructible, so there are line segments of lengths $|\alpha|$ and $|\beta|$ available to us. It is easy to see that $\alpha+\beta$ and $\alpha-\beta$ are constructible. The construction of a segment of length $\alpha\beta$ is indicated in the following diagram.



Mark off a point A on a line ℓ such that OA is of length $|\alpha|$. Draw another line through O and mark off two points P and B on this line so that OP is of unit length and OB is of length $|\beta|$. Then construct a line ℓ' passing through B parallel to PA. ℓ' intersects ℓ at Q and OQ is of length $|\alpha\beta|$.



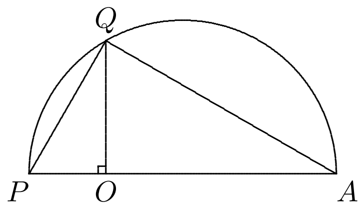
The construction of $|\alpha/\beta|$ is similar to this procedure. To do so, construct a line ℓ' passing through P parallel to AB. Then, ℓ' intersects ℓ at Q and OQ is of length $|\alpha/\beta|$.

Corollary 1.25 All rational numbers are constructible.

However there are irrational numbers such as $\sqrt{2}$ which are also constructible.

Theorem 1.26 If α is a constructible real numbers, then so is $\sqrt{\alpha}$.

Proof



Take a segment POA so that PO is of unit length and OA is of length $|\alpha|$. Draw a semicircle with diameter PA. The perpendicular at O cuts this semicircle at a point Q. Then the length of OQ is $\sqrt{\alpha}$.

The problem of deciding which real number is constructible can also be analyzed using coordinate geometry. First, we know that any rational number is constructible. By regarding the fixed unit segment as the basic unit on the x- and y-axes, we can locate any point (q_1, q_2) in the plane with both coordinates rational. Any further point in the plane that we can locate by using a straightedge and a compass can be found in one of the following three ways.

(I) It is a point obtained as an intersection of two lines, each of which passes through two known points having rational coordinates.

(II) It is a point obtained as an intersection of a line that passes through two points having rational coordinates and a circle whose centre has rational coordinates and the radius is rational.

(III) It is a point obtained as an intersection of two circles whose centers have rational coordinates and the radii are rational.

In all three cases, the point can be obtained by solving the simultaneous equations $kx^2+ky^2+dx+ey+f=0$ and $ax+by+c=0$, where all the coefficients are rational. After substituting one equation into the other and eliminating the unknown y, it reduces to solve either a linear equation or a quadratic equation in x with rational coefficients. For the case of a linear equation, the solution for x and hence y is rational. For the quadratic case, one can solve for x by means of the quadratic formula, and it may have solutions involving square roots of numbers that are not squares in $\mathbb{Q}$. Consequently, we have proved the following theorem.

Theorem 1.27 The set of all constructible numbers consists precisely of all real numbers that can be obtained from rational numbers by taking square roots, and by applying the operations of addition, subtraction, multiplication and division a finite number of times.

If α is a constructible number, then α can be obtained from rational numbers by taking square roots, and by applying the operations of addition, subtraction, multiplication and division a finite number of times. If we successively eliminate the square roots in this expression of α by a combination of rearranging and squaring the expression, we see that α is a root of a polynomial with rational coefficients. By clearing the common denominator, the coefficients can be assumed to be all integers. The important point is that this polynomial must be of degree a power of 2. For example, if $\alpha = 1 - \sqrt{2+\sqrt{2}}$, then $((1-\alpha)^2 - 2)^2 = 2$. Hence, α is a root of the polynomial $x^4 - 4x^3 + 2x^2 + 4x - 1$. More precisely, the so called irreducible polynomial satisfied by α is also of degree equal to a power of 2.

Definition 1.28

(i) A polynomial of degree n is called a *monic polynomial* if the coefficient of x^n is 1.

(ii) A polynomial $p(x)$ with integer coefficients is said to be *irreducible* over $\mathbb{Z}$ ($\mathbb{Q}$) if it cannot be factored into a product of two polynomials with integer (rational) coefficients of lower degrees (≥ 1).

It can be shown that a polynomial $p(x)$ with integer coefficients is irreducible over $\mathbb{Q}$ if and only if it is irreducible over $\mathbb{Z}$. This is the content of the so called Gauss lemma. Suppose that a real number α is a root of a polynomial with integer coefficients. By division algorithm, there exists a monic irreducible polynomial $p(x)$ with rational coefficients satisfied by α. This polynomial is called the *irreducible polynomial* for α over $\mathbb{Z}$. This irreducible polynomial for α is unique. For example, the irreducible polynomial for $\sqrt{2}$ is $x^2 - 2$, and the irreducible polynomial for $\sqrt[3]{2}$ is $x^3 - 2$.

Theorem 1.29 Let α be a constructible number. Then the degree of the irreducible polynomial $p(x)$ for α is a power of 2.

Corollary 1.30 Doubling the cube is impossible.

Proof This is equivalent to show that $\sqrt[3]{2}$ is not constructible. The irreducible polynomial for $\sqrt[3]{2}$ is $x^3 - 2$. It's degree is not a power of 2. Hence, $\sqrt[3]{2}$ is not constructible.

Corollary 1.31 Trisecting 60^o is impossible.

Proof We know that 60^o is a constructible angle. We wish to show that 20^o is not constructible. Note that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Let $\alpha = \cos 20^o$. Then the above formula shows that α is a root of the polynomial $p(x) = 8x^3 - 6x - 1$. Now, let's show that this polynomial is irreducible over $\mathbb{Z}$. A factorization of $p(x)$

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would entail a linear factor of the form $(8x \pm 1), (4x \pm 1), (2x \pm 1)$ or $(x \pm 1)$. One can easily check that none of the numbers $\pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}$ and ± 1 is a root of $p(x)$. Thus $p(x)$ is the irreducible polynomial for α. But it's degree is not a power of 2. Therefore, 60^o cannot be trisected.

The problem of squaring the circle is equivalent to whether π is constructible. As π is not even a root of a polynomial with integer coefficients, it is not constructible. Lastly, let's close this chapter by quoting the following result on the constructibility of regular polygons.

Theorem 1.32 Let p be an odd prime. A regular polygon of p sides is constructible if and only if $p = 2^{2^k} + 1$. (A prime of the form $p = 2^{2^k} + 1$ is called a Fermat prime.)

Chapter 2 ABSOLUTE GEOMETRY

§2.1 INCIDENCE GEOMETRIES

Definition 2.1 An *incidence geometry* is a pair $(\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ is a nonempty set and $\mathcal{L}$ is a collection of nonempty subsets of $\mathcal{P}$ such that

(i) For any two distinct elements $A, B \in \mathcal{P}$, there exists a **unique** $\ell \in \mathcal{L}$ containing A and B.

(ii) Any $\ell \in \mathcal{L}$ has at least two elements.

(iii) There exist 3 elements of $\mathcal{P}$ not all in any element of $\mathcal{L}$.

Definition 2.2

(i) An element of $\mathcal{P}$ is called a *point*. An element of $\mathcal{L}$ is called a *line*.

(ii) If $P \in \ell$, then we say that P is *on* ℓ, ℓ *passes through* P, or P and ℓ are *incident*.

(iii) We say that ℓ_1 and ℓ_2 intersect if $\ell_1 \cap \ell_2 \neq \emptyset$.

(iv) A set S of points is said to be *collinear* if $S \subseteq \ell$ for some line ℓ.

(v) If two or more lines intersect at one point, then the lines are said to be *concurrent*.

(vi) Two lines ℓ_1, ℓ_2 are said to be *parallel* if $\ell_1 \cap \ell_2 = \emptyset$. We shall use the symbol $\ell_1 \parallel \ell_2$ to denote that ℓ_1 is parallel to ℓ_2.

Remarks The condition that there exist 3 elements of $\mathcal{P}$ not all in any element of $\mathcal{L}$ guarantees the nontriviality of the incidence geometry. As without this, there might be just exactly one line. In an incidence geometry, condition (iii) is equivalent to the condition that there exist at least two lines. Next we shall introduce some examples of incidence geometries. An example of an incidence geometry is often called a *model* of an incidence geometry.

Examples

1. Let $\mathcal{P} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ and $\mathcal{L} = \{\{A, B\}, \{A, C\}, \{B, C\}\}$. Then $(\mathcal{P}, \mathcal{L})$ is an incidence geometry. This is the most simplest incidence geometry. Note that each incidence geometry should have at least three points.
2. *The Real Cartesian Plane.* Let $\mathcal{P} = \mathbb{R}^2$. $\ell \subseteq \mathbb{R}^2$ is a line iff $\ell = \{(x, y) : ax + by + c = 0\}$ for some $a, b, c \in \mathbb{R}$ and not both a and b are zero. We shall denote the Real Cartesian Plane by $\mathbb{E} = (\mathbb{R}^2, \mathcal{L}_E)$.
3. *The Rational Cartesian Plane.* Let $\mathcal{P} = \mathbb{Q}^2$. $\ell \subseteq \mathbb{Q}^2$ is a line iff $\ell = \{(x, y) : ax + by + c = 0\}$ for some $a, b, c \in \mathbb{Q}$ and not both a and b are zero.

4. *The Complex Cartesian Plane.* Let $\mathcal{P} = \mathbb{C}^2$. $\ell \subseteq \mathbb{C}^2$ is a line iff $\ell = \{(x, y) : ax + by + c = 0\}$ for some $a, b, c \in \mathbb{C}$ and not both a and b are zero. In fact, for any field $\mathbb{F}$, an incidence geometry can be defined in this way. Describe such an incidence geometry when $\mathbb{F}$ is the field of two elements.

5. *The Real Cartesian Space.* Let $\mathcal{P} = \mathbb{R}^3$. $\ell \subseteq \mathbb{R}^3$ is a line iff $\ell = \{(x, y, z) : (x, y, z) - (x_o, y_o, z_o)$ is a multiple of $(a, b, c)\}$ for some $x_o, y_o, z_o,\ a, b, c \in \mathbb{R}$ and not all a, b and c are zero.

6. *The Quadrant Incidence Plane.* Let $\mathcal{P} = \mathbb{R}^2_+$ be the set of all ordered pairs of positive real numbers. $\ell \subseteq \mathbb{R}^2_+$ is a line iff $\ell = \{(x, y) \in \mathbb{R}^2_+ :\ ax + by + c = 0\}$ for some $a, b, c \in \mathbb{R}$ and not both a and b are zero.

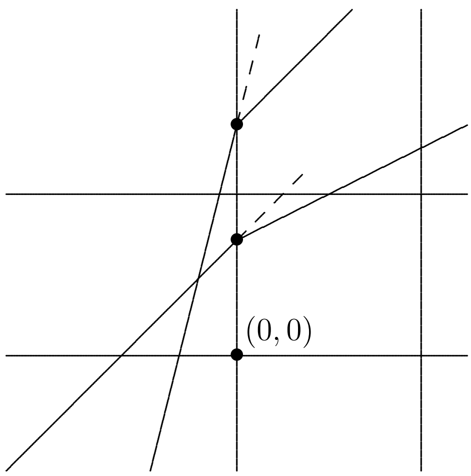
7. *The Halfplane Incidence Plane.* Let $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 \ :\ y > 0\}$. $\ell \subseteq \mathcal{P}$ is a line iff $\ell = \{(x, y) \in \mathcal{P} \ :\ ax + by + c = 0\}$ for some $a, b, c \in \mathbb{R}$ and not both a and b are zero.

8. *The Missing-Quadrant Incidence Plane.* Let $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 \ :\ x \text{ or } y > 0\}$. $\ell \subseteq \mathcal{P}$ is a line iff $\ell = \{(x, y) \in \mathcal{P} \ :\ ax + by + c = 0\}$ for some $a, b, c \in \mathbb{R}$ and not both a and b are zero.

9. *The Missing-Strip Incidence Plane.* Let $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 \ :\ x \leq 1 \text{ or } x > 2\}$. $\ell \subseteq \mathcal{P}$ is a line iff $\ell = \{(x, y) \in \mathcal{P} \ :\ ax + by + c = 0\}$ for some $a, b, c \in \mathbb{R}$ and not both a and b are zero. In general, if $\mathcal{L}$ is the set of all lines of the Real Incidence Plane and $\mathcal{A}$ is a non-collinear subset of $\mathbb{R}^2$ having at least three points, then $(\mathcal{A}, \mathcal{L} \cap \mathcal{A})$ is an incidence geometry.

10. *The Cubic Incidence Plane.* Let $\mathcal{P} = \mathbb{R}^2$. $\ell \subseteq \mathcal{P}$ is a line iff either $\ell = \{(x, y) \in \mathbb{R}^2 \ :\ x = c\}$ for some $c \in \mathbb{R}$ or $\ell = \{(x, y) \in \mathbb{R}^2 \ :\ y = (ax + b)^3\}$ for some $a, b, c \in \mathbb{R}$. Note that two lines in the Cubic Incidence Plane either intersect at one point or do not intersect at all.

11. *The Moulton Plane.*

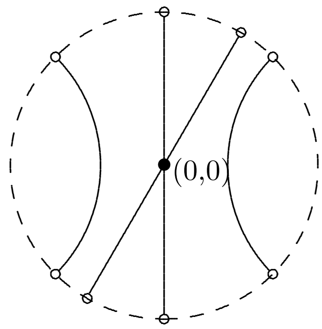


Let $\mathcal{P} = \mathbb{R}^2$. $\ell \subseteq \mathcal{P}$ is a line iff it consists of precisely points (x, y) satisfying one of the following three types of equations:

$$\begin{array}{ll} x = a, & \\ y = mx + b & \text{with } m \leq 0, \\ y = \begin{cases} mx + b & \text{if } x \leq 0 \\ \frac{1}{2}mx + b & \text{if } x > 0 \end{cases} & \text{with } m > 0 \ . \end{array}$$

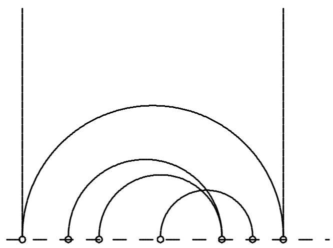
In the Moulton Plane, two distinct points lie on a unique line.

12. *The Poincaré Disk.* Let $\mathcal{P} = \mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. ℓ is a line iff it consists of points $(x, y) \in \mathcal{P}$ satisfying either the equation $(x-a)^2 + (y-b)^2 = a^2 + b^2 - 1$ with $a, b \in \mathbb{R}$ such that $a^2 + b^2 > 1$ or the equation $ax + by = 0$ with $a, b \in \mathbb{R}$ such that $a^2 + b^2 \neq 0$.



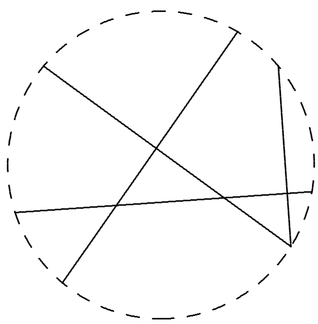
Thus a line in the Poincaré disk is either a circular arc orthogonal to the unit circle or a straight line passing through the origin. We shall denote the Poincaré disk by $\mathbb{P} = (\mathbb{D}, \mathcal{L}_D)$.

13. *The Poincaré Halfplane.* Let $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. ℓ is a line iff it consists of points $(x, y) \in \mathcal{P}$ satisfying either the equation $(x - a)^2 + y^2 = r^2$ with $a \in \mathbb{R}$ and $r \in \mathbb{R}^+$ or the equation $x = a$ with $a \in \mathbb{R}$. Thus a line in the Poincaré Halfplane is either a semi-circular arc orthogonal to the x-axis or a vertical straight line.

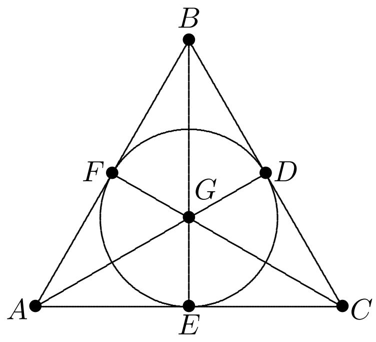


We shall denote the Poincaré Halfplane by $\mathbb{H} = (\mathbb{R}^2_+, \mathcal{L}_H)$. $\mathbb{H}$ is also called the Hyperbolic Plane.

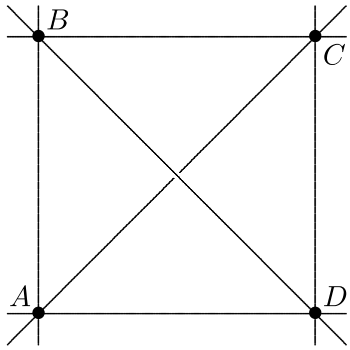
14. *The Cayley-Klein Incidence Plane.* Let $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. ℓ is a line iff it consists of points $(x, y) \in \mathcal{P}$ satisfying an equation $ax + by = c$ with $a, b, c \in \mathbb{R}$ but not both a and b zero. Therefore, a line in the Cayley-Klein Incidence Plane is a usual line segment with endpoints on the unit circle. Note that the points on the unit circle are not included in $\mathcal{P}$.



15. *A 7-Point Incidence Geometry.* Let $\mathcal{P}$ be the set $\{A, B, C, D, E, F, G\}$. $\mathcal{L} = \{\{A, F, B\}, \{A, G, D\}, \{A, E, C\}, \{C, G, F\}, \{B, D, C\}, \{B, G, E\}, \{D, E, F\}\}$. In this incidence geometry, each line contains 3 points and any 3 lines concur at a point. Also any two points intersect at a point. Hence, there are no parallel lines in this incidence geometry.



16. *A 4-Point Incidence Geometry.* Let $\mathcal{P} = \{A, B, C, D\}$. $\mathcal{L} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$. In this incidence geometry, any two distinct points lie on a unique line. The lines $\{A, C\}$ and $\{B, D\}$ do not intersect. Hence, they are parallel. Are there other parallel lines?



17. *The Sphere Incidence Plane.* The set $\mathcal{P}$ of points is the unit sphere in $\mathbb{R}^3$. ℓ is a line iff it is a great circle. (A great circle is a circle on the sphere whose centre is the centre of the sphere.) Note that there are infinitely many lines passing through the north and south poles. Therefore, the north and south poles on the sphere do not determine a unique line. This means that the Sphere Incidence Plane is **not** an incidence geometry.

18. *The Riemann Incidence Plane.* The set $\mathcal{P}$ of points is the set of pairs of antipodal points on unit sphere in $\mathbb{R}^3$. More precisely, let $S^3 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$. Then $\{(x, y, z), (-x, -y, -z)\}$ is called a pair of antipodal points. ℓ is a line in $\mathcal{P}$ iff it consists of all pairs of antipodal points which lie on a great circle. This model is in fact the Real Projective Plane. Again the Riemann Incidence Plane is **not** an incidence geometry.

In the definition of an incidence geometry, condition (i) is called the *straightedge axiom.*

The Straightedge Axiom: For any two distinct elements $A, B \in \mathcal{P}$, there exists a **unique** $\ell \in \mathcal{L}$ containing A and B.

It allows us to draw a unique straight line between two distinct points.

Proposition 2.3 Let ℓ_1 and ℓ_2 be two lines in an incidence geometry. If $\ell_1 \cap \ell_2$ has more than one point, then $\ell_1 = \ell_2$.

Corollary 2.4 In an incidence geometry, two lines are either parallel or they intersect at exactly one point.

§2.2 METRIC GEOMETRIES

Definition 2.5 A *distance function* on a set $\mathcal{P}$ is a function $d : \mathcal{P} \times \mathcal{P} \longrightarrow \mathbb{R}$ such that for two points P and Q in $\mathcal{P}$,

(i) $d(P, Q) \geq 0$,

(ii) $d(P, Q) = 0$ if and only if $P = Q$,

(iii) $d(P, Q) = d(Q, P)$.

Examples

1. In the Euclidean Plane (or the Real Cartesian Plane) $\mathbb{E}$, the *Euclidean distance* d_E is given by

$$d_E((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

2. In the Hyperbolic Plane $\mathbb{H}$, the *Hyperbolic distance* of two points $P = (x_1, y_1)$ and $P = (x_2, y_2)$ is given by

$$d_H(P, Q) = \left|\ln(\frac{y_2}{y_1})\right| \text{ if } x_1 = x_2,$$

$$d_H(P, Q) = \left|\ln\left(\frac{\frac{x_1-c+r}{y_1}}{\frac{x_2-c+r}{y_2}}\right)\right| \text{ if } P \text{ and } Q \text{ lie on a semi-circle centred at } (c, 0) \text{ with}$$

radius $r > 0$ in $\mathbb{H}$.

3. Let $P = (x_1, y_1)$ and $P = (x_2, y_2)$ be two points in $\mathbb{R}^2$. The *Taxicab distance* between P and Q is defined by

$$d_T(P, Q) = |x_1 - x_2| + |y_1 - y_2|.$$

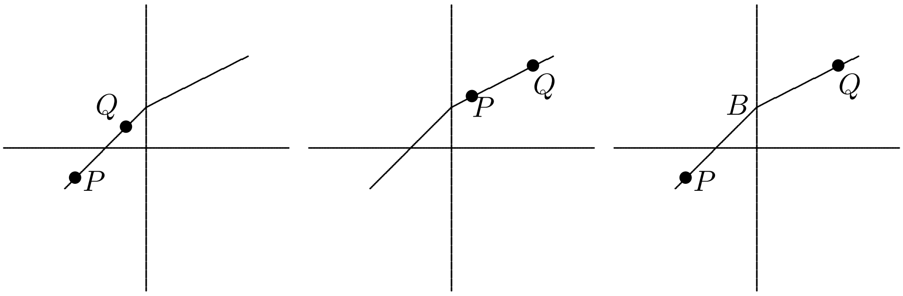
4. Let $\mathcal{P}$ be any nonempty set. Then

$$d(P, Q) = \begin{cases} 0 & \text{if } P = Q, \\ 1 & \text{if } P \neq Q \end{cases}$$

is a distance function on $\mathcal{P}$.

5. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points in the Moulton Plane. The *Moulton distance* between P and Q is defined by

$$d_M(P,Q) = \begin{cases} d_E(P,B) + d_E(B,Q) & \text{if } P, Q \text{ lie on a “bent line” intersecting the } y\text{-axis at a point } B \text{ and } x_1x_2 < 0 \\ d_E(P,Q) & \text{otherwise.} \end{cases}$$



Definition 2.6 Let ℓ be a line in an incidence geometry $(\mathcal{P}, \mathcal{L})$. Assume that there is a distance function d on $\mathcal{P}$. A function $f : \ell \longrightarrow \mathbb{R}$ is a *ruler* for ℓ if f is a bijection and for any points P and Q on ℓ, $|f(P) - f(Q)| = d(P,Q)$.

Let f be a ruler for the line ℓ. Fix a point P_o on ℓ such that $f(P_o) = 0$ under the bijection f. Then the distance from any point P on ℓ to the point P_o is simply given by $|f(P)|$. The real number $f(P)$ is called the *coordinate* of P with respect to f.

Example Let ℓ be the line $y = 2x + 3$ in the Euclidean Plane $\mathbb{E}$. Then $f : \ell \longrightarrow \mathbb{R}$ given by $f((x,y)) = \sqrt{5}x$ is a ruler for ℓ. The coordinate of $Q = (1,5)$ with respect to f is $f(Q) = \sqrt{5}$.

Note that since a point may lie on more than one line, it may have different coordinates with respect to the various lines or rulers used.

Definition 2.7 An incidence geometry $(\mathcal{P}, \mathcal{L})$ together with a distance function d satisfies the **Ruler Axiom** if every line $\ell \in \mathcal{L}$ has a ruler. In this case, we say that $(\mathcal{P}, \mathcal{L}, d)$ is a *metric geometry*.

Examples

1. The Euclidean Plane $\mathbb{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$ is a metric geometry.
2. The Hyperbolic Plane $\mathbb{H} = (\mathbb{R}^2_+, \mathcal{L}_H, d_H)$ is a metric geometry.
3. $(\mathbb{R}^2, \mathcal{L}_E, d_T)$, where d_T is the Taxicab distance, is a metric geometry.
4. $(\mathbb{R}^2, \mathcal{L}_M, d_M)$, where d_M is the Moulton distance, is a metric geometry.

From now on, we use $\mathbb{E}$ to denote the Euclidean Plane, equipped with the Euclidean distance d_E as a metric geometry. For a slightly more precise notation, we sometimes write $\mathbb{E} = (\mathbb{R}^2, d_E)$, but omitting $\mathcal{L}_E$ as it is usually understood, to mean that $\mathbb{E}$ is a metric geometry with distance function d_E. Similarly, we shall use either $\mathbb{H}$ or $(\mathbb{R}^2_+, d_H)$ to denote the Hyperbolic Plane as a metric geometry. Finally, $(\mathbb{R}^2, d_T)$ denotes the Taxicab Plane with the Taxicab distance d_T and $(\mathbb{R}^2, d_M)$ denotes the Moulton Plane with the Moulton distance d_M.

Proposition 2.8 Any line in a metric geometry has infinitely many points.

Theorem 2.9 Let $(\mathcal{P}, \mathcal{L})$ be an incidence geometry. Suppose that for each line $\ell \in \mathcal{L}$, there exists a bijection $f_\ell : \ell \to \mathbb{R}$. Then there is a distance d such that $(\mathcal{P}, \mathcal{L}, d)$ is a metric geometry and each f_ℓ is a ruler for ℓ.

Proof Let P, Q be two points in $\mathcal{P}$. If $P = Q$, define $d(P, Q) = 0$. For any two distinct points P and Q, let ℓ be the unique line through P and Q, and $f_\ell : \ell \longrightarrow \mathbb{R}$ the given bijection. Define $d(P, Q) = |f_\ell(P) - f_\ell(Q)|$. Then d is a distance on $\mathcal{P}$ and each f_ℓ is a ruler for the line ℓ. Hence, $(\mathcal{P}, \mathcal{L}, d)$ is a metric geometry.

Theorem 2.10 (Ruler Replacement Theorem) Let A and B be two distinct points on a line ℓ in a metric geometry. Then there exists a ruler $g : \ell \longrightarrow \mathbb{R}$ such that $g(A) = 0$ and $g(B) > 0$.

Proof Let $f : \ell \longrightarrow \mathbb{R}$ be a ruler for ℓ. Then $h : \ell \longrightarrow \mathbb{R}$ given by $h(P) = f(P) - f(A)$ is also a ruler for ℓ. For this ruler h, we have $h(A) = 0$. If $h(B) > 0$, then this ruler satisfies the required conditions. If $h(B) < 0$, then set $g(P) = -h(P)$. g is also a ruler for ℓ and it satisfies the required conditions.

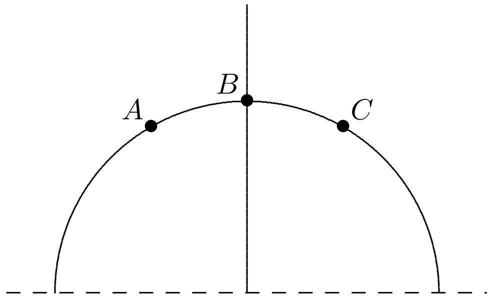
Definition 2.11 A distance function d on a set $\mathcal{P}$ satisfies the **triangle inequality** if $d(A, C) \leq d(A, B) + d(B, C)$ for all $A, B, C \in \mathcal{P}$.

Proposition 2.12 The Euclidean distance function d_E satisfies the triangle inequality.

We shall see later that the triangle inequality is a consequence of certain other axioms that we would like our geometries to satisfy. In particular, it holds for the Hyperbolic Plane. But a direct proof of this fact is quite clumsy.

Definition 2.13 Let A, B and C be three collinear points in a metric geometry $(\mathcal{P}, \mathcal{L}, d)$. B is said to be **between** A and C if $d(A, B) + d(B, C) = d(A, C)$.

Example Let $A = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $B = (0, 1)$ and $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ be three points in the Hyperbolic Plane. Then B is between A and C.



In this example, A, B and C lie on the line $\{(x, y) \in \mathbb{H} : x^2 + y^2 = 1\}$. $d_H(A, B) = d_H(B, C) = \ln\sqrt{3}$ and $d_H(A, C) = \ln 3$.

Proposition 2.14 Let A, B and C be three distinct points lying on a line ℓ in a metric geometry. Then exactly one of these points is between the other two.

Proof Let $f : \ell \longrightarrow \mathbb{R}$ be a ruler for ℓ. Since f is a bijection and A, B, C are distinct, the numbers $f(A), f(B)$ and $f(C)$ are all distinct. Therefore exactly one of these three numbers is between the other two. Let's suppose that $f(A) < f(B) < f(C)$. Then $d(A, B) + d(B, C) = |f(B) - f(A)| + |f(C) - f(B)| = (f(B) - f(A)) + (f(C) - f(B)) = f(C) - f(A) = |f(C) - f(A)| = d(A, C)$. Therefore, B is between A and C.

Proposition 2.15 Let A and B be two distinct points in a metric geometry. Then the following statements hold.
(i) There is a point C such that B is between A and C.

(ii) There is a point D such that D is between A and B.

Proof Let ℓ be a line passing through A and B and $f : \ell \longrightarrow \mathbb{R}$ a ruler for ℓ with $f(A) < f(B)$. To prove (i), take $C = f^{-1}[f(A) + 1]$. Then B is between A and C. To prove (ii), take $D = f^{-1}[\frac{1}{2}(f(A) + f(B))]$. Then D is between A and B.

Definition 2.16 Let A and B be two distinct points in a metric geometry $(\mathcal{P}, \mathcal{L}, d)$.
(i) The *line segment* from A to B is the set

$$\overline{AB} = \{P \in \mathcal{P} : P \text{ is between } A \text{ and } B\} \cup \{A, B\}.$$

(ii) A and B are called the *endpoints* of the segment $\overline{AB}$.

(iii) The *length* of the segment $\overline{AB}$ is $AB = d(A, B)$.

(iv) The *ray* from A towards B is $\overrightarrow{AB} = \overline{AB} \cup \{P \in \mathcal{P} : B \text{ is between } A \text{ and } P\}$.

Note that if ℓ is a line passing through A and B, then $\overline{AB}$ and $\overrightarrow{AB}$ are subsets of ℓ.

Definition 2.17 Two segments $\overline{AB}$ and $\overline{CD}$ are said to be *congruent* if $AB = CD$. We shall use the notation $\overline{AB} \equiv \overline{CD}$ to denote that $\overline{AB}$ and $\overline{CD}$ are congruent.

For example, if $A = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $B = (0, 1)$ and $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ are three points in the Hyperbolic Plane, then the $\overline{AB} \equiv \overline{BC}$.

Theorem 2.18 (Segment Construction Theorem) Let $\overrightarrow{AB}$ be a ray and $\overline{PQ}$ be a segment in a metric geometry. Then there exists a unique point C in $\overrightarrow{AB}$ with $\overline{PQ} \equiv \overline{AC}$.

Proof Let f be a ruler for the line ℓ passing through A and B such that $f(A) = 0$ and $f(B) > 0$. It is easy to show that $\overrightarrow{AB} = \{P \in \ell : f(P) > 0\}$. Let $r = PQ$ and set $C = f^{-1}(r)$. Since $r = PQ > 0$, we have $C \in \overrightarrow{AB}$. Then $AC = |f(A) - f(C)| = |0 - r| = r = PQ$. Hence, $\overline{PQ} \equiv \overline{AC}$.
Now suppose that C' is a point in $\overrightarrow{AB}$ with $\overline{PQ} \equiv \overline{AC'}$. As $C' \in \overrightarrow{AB}$, we have $f(C') > 0$. Then $f(C') = |f(C') - f(A)| = AC' = PQ = f(C)$. Since f is a bijection, we have $C = C'$. This shows that the point C is unique.

Definition 2.19 Let $S \subseteq \mathcal{P}$ be a subset in a metric geometry. S is said to be *convex* if for any two distinct points $A, B \in S$, $\overline{AB} \subseteq S$.

Examples

1. A circular region $\{(x, y) : x^2 + y^2 \leq 1\}$ in the Euclidean Plane $\mathbb{E}$ is convex.
2. The set $\{(x, y) : y > 1\}$ in the Hyperbolic Plane $\mathbb{H}$ is convex.
3. The set $S = \{(x, y) : y \leq 1\}$ in the Hyperbolic Plane $\mathbb{H}$ is **not** convex. The segment joining $(-1, 1)$ and $(1, 1)$ is not inside S.
4. In the Moulton Plane $(\mathbb{R}^2, d_M)$, the set $\{(x, y) : y > x\}$ is convex. Is the set $\{(x, y) : y \leq x\}$ convex?

Definition 2.20 A metric geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfies the **Scissors Axiom**, or the **Plane Separation Axiom**, if for any line ℓ, there are two subsets H_1 and H_2 of $\mathcal{P}$ such that

(i) $\mathcal{P}$ is the disjoint union of ℓ, H_1 and H_2,

(ii) H_1 and H_2 are convex,

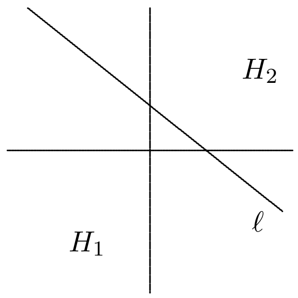
(iii) for any $A \in H_1, B \in H_2$, $\overline{AB} \cap \ell \neq \emptyset$.

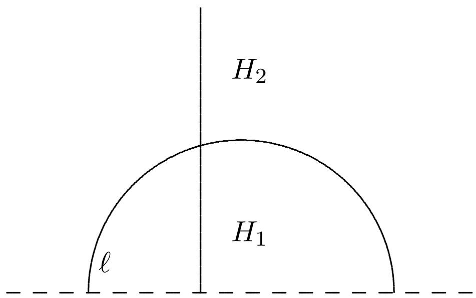
Definition 2.20 Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry which satisfies the scissors axiom. Let H_1 and H_2 be the two halfplanes determined by a line ℓ. Two points A and B are *on the same side of* ℓ if $A, B \in H_1$ or $A, B \in H_2$. A and B are *on opposite sides of* ℓ if $A \in H_1, B \in H_2$ or $A \in H_2, B \in H_1$.

Proposition 2.21 Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry which satisfies the scissors axiom. Let A and B be two distinct points **not** on a given line ℓ. Then

(i) A and B are on opposite sides of ℓ iff $\overline{AB} \cap \ell \neq \emptyset$.

(ii) A and B are on the same side of ℓ iff $\overline{AB} \cap \ell = \emptyset$.





Examples

1. The Euclidean Plane $\mathbb{E} = (\mathbb{R}^2, d_E)$ satisfies the scissors axiom.
2. The Hyperbolic Plane $\mathbb{H} = (\mathbb{R}^2_+, d_H)$ satisfies the scissors axiom.
3. The Taxicab Plane $(\mathbb{R}^2, d_T)$ satisfies the scissors axiom.
4. The Moulton Plane $(\mathbb{R}^2, d_M)$ satisfies the scissors axiom.
5. The Missing-Strip Plane can be given the structure of a metric geometry. It does *not* satisfy the scissor axiom. This is because the condition (iii) in 2.20 is not usually satisfied.

Theorem 2.22 (Pasch's Theorem) Let ABC be a triangle in a metric geometry which satisfies the scissors axiom. Let D be a point between A and B and ℓ a line through D. Then either $\ell \cap \overline{AC} \neq \emptyset$ or $\ell \cap \overline{BC} \neq \emptyset$.

Proof If A or B lies in ℓ, then we are done. Otherwise, A and B are on opposite sides of ℓ. Suppose that $\ell \cap \overline{AC} = \emptyset$. Then A and C are on the same side of ℓ. Hence, Then B and C are on opposite sides of ℓ. Therefore, $\ell \cap \overline{BC} \neq \emptyset$.

The converse of Pasch's Theorem is also true.

§2.3 ANGLE MEASURE

Definition 2.23 Let A, B and C be three noncollinear points in a metric geometry $(\mathcal{P}, \mathcal{L}, d)$. The *angle* $\angle ABC$ is the subset of $\mathcal{P}$ given by $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$. The point B is called the *vertex* of the angle $\angle ABC$.

Note that a straightline is not permitted to be an angle nor is a ray since A, B, C must be noncollinear.

Definition 2.24 The *interior* of an angle $\angle ABC$, denoted by $\text{int}\angle ABC$, is the intersection of the halfplane of $\overleftrightarrow{AB}$ containing C and the halfplane of $\overleftrightarrow{BC}$ containing A.

Definition 2.25 Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry satisfying the Scissor Axiom. A *protractor* (or an *angle measure*) is a function m from the set $\mathcal{A}$ of all angles to $\mathbb{R}$ satisfying the following conditions.

(i) If $\angle ABC \in \mathcal{A}$, then $0 < m(\angle ABC) < 180$.

(ii) If $\overrightarrow{BC}$ lies on the edge of the halfplane H_1, and θ a real number with $0 < \theta < 180$, then there is a unique ray $\overrightarrow{BA}$ with $A \in H_1$ and $m(\angle ABC) = \theta$.

(iii) If $D \in \text{int}\angle ABC$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$.

Here we are using the usual *degree measure*. One could replace 180 by π to get the radian measure.

Definition 2.26 A *Protractor Geometry* $(\mathcal{P}, \mathcal{L}, d, m)$ is a metric geometry satisfying the scissor axiom together with an angle measure.

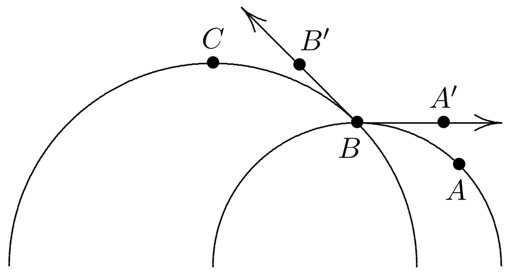
Examples

1. In the Euclidean Plane, the *Euclidean angle measure* of $\angle ABC$ is

$$m_E(\angle ABC) = \cos^{-1}\left(\frac{< A - B, C - B >}{\|A - B\| \|C - B\|}\right).$$

   m_E is an angle measure on $\mathbb{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$. Hence, $\mathbb{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E, m_E)$ is a protractor geometry.

2. Let $\overrightarrow{PQ}$ be a ray in the Hyperbolic Plane $\mathbb{H}$. It has a unique Euclidean ray T_{PQ} tangent to $\overrightarrow{PQ}$ at P. Let $\angle ABC$ be an angle in $\mathbb{H}$. Pick a point A' in T_{BA} and a point C' in T_{BC} with both A' and C' not equal to B. Then the *Hyperbolic angle measure* of $\angle ABC$ is $m_H(\angle ABC) = m_E(\angle A'BC')$. With this angle measure, $\mathbb{H} = (\mathbb{R}^2_+, \mathcal{L}_H, d_H, m_H)$ is a protractor geometry.



$m_H(\angle ABC) = m_E(\angle A'BC')$

3. The taxicab plane equipped with the Euclidean angle measure m_E is a protractor geometry.
4. Let $\angle ABC$ be an angle in the Moulton Plane. If B does not lie on the y-axis, define $m_M(\angle ABC) = m_E(\angle ABC)$. Suppose B lies on the y-axis. Consider the case where $\overleftrightarrow{AB}$ has a positive slope and $\overleftrightarrow{BC}$ has a nonpositive slope. Extend $\overline{AB}$ to a point B'. Then extend the "Euclidean segment" $B'B$ to a point A_b. Define $m_M(\angle ABC) = m_E(\angle A_bBC)$. The other cases are similarly defined. Then the Moulton Plane equipped with this angle measure m_M is a protractor geometry.

Definition 2.27
(i) Two angles in a protractor geometry are said to be *congruent* if they have the same angle measure.
(ii) Two lines, (rays or segments) are said to be *perpendicular* if they intersect at an angle with angle measure 90.

§2.4 THE SAS AXIOM

In a triangle $\triangle ABC$, we will denote $\angle CAB, \angle ABC$ and $\angle BCA$ by $\angle A, \angle B$ and $\angle C$ respectively.

Definition 2.28 Two triangles $\triangle ABC$ and $\triangle DEF$ in a protractor geometry are said to be *congruent* if there exists a bijection $f : \{A, B, C\} \longrightarrow \{D, E, F\}$ such that the corresponding segments and angles are congruent under f. We use the notation $\triangle ABC \equiv \triangle DEF$ to denote that $\triangle ABC$ is congruent to $\triangle DEF$.

Definition 2.29 A protractor geometry satisfies the *Side-Angle-Side Axiom* (SAS Axiom) if whenever $\triangle ABC$ and $\triangle DEF$ are two triangles with $AB = DE, \angle B = \angle E$ and $BC = EF$, then $\triangle ABC \equiv \triangle DEF$.

Definition 2.30 An *absolute geometry* is a protractor geometry which satisfies the SAS Axiom.

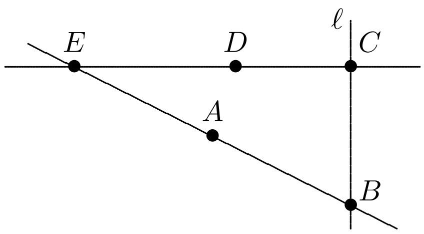
The Euclidean Plane and the Hyperbolic Plane are absolute geometries, but the Taxicab Plane and the Moulton Plane do not satisfy the SAS axiom so that they are not absolute geometries.

Theorem 2.31 In an absolute geometry, there is exactly one line through a given point P perpendicular to a given line ℓ.

Corollary 2.32 In an absolute geometry, two distinct lines having a common perpendicular are parallel.

§2.5 PARALLEL LINES

Euclid's Fifth Axiom Suppose that a line ℓ intersects two lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{CD}$ at B and C respectively and that A, D belong to the same side of ℓ. If $m(\angle ABC) + m(\angle DCB) < 180$, then $\overrightarrow{BA} \cap \overrightarrow{CD} \neq \emptyset$.



Euclidean Parallel Axiom (Playfair's Axiom) For any line ℓ and any point P not on ℓ, there is a unique line ℓ' through P which is parallel to ℓ.

Euclid's Fifth Axiom has to be formulated in an absolute geometry, but the Euclidean Parallel Axiom makes sense in any incidence geometry.

Theorem 2.33 For any line ℓ and any point $P \notin \ell$ in an absolute geometry, there exists a line ℓ' through P which is parallel to ℓ.

Proof Let P be a point not on a line ℓ. By 2.31, there exists a line ℓ_1 through P perpendicular to ℓ. By 2.31 again, there exists a line ℓ' through P perpendicular to ℓ_1. By 2.32, ℓ' is parallel to ℓ.

Theorem 2.34 (Playfair) In an absolute geometry, Euclid's Fifth Axiom is equivalent to the Euclidean Parallel Axiom.

The Euclidean Plane $\mathbb{E}$ satisfies the Euclidean Parallel Axiom, but the Hyperbolic Plane $\mathbb{H}$ does not. Therefore, the Euclidean Parallel Axiom or Euclid's Fifth Axiom cannot be proved in an absolute geometry. It is independent of the defining axioms of an absolute geometry.

Theorem 2.35 (Saccheri) The angle sum of a triangle in an absolute geometry is less than or equal to 180.

Theorem 2.36 (Legendre) In an absolute geometry, the angle sum of any triangle is equal to 180 iff the geometry satisfies the Euclidean Parallel Axiom.

Appendix A EQUIVALENCE RELATIONS

§A1 RELATIONS

Definition A.1 Let A and B be sets. A *relation* from A to B is a subset $\sim$ of $A \times B$. A and B are called the *domain* and *codomain* of the relation $\sim$ respectively. Let $a \in A$ and $b \in B$. We say that a is related to b, written as $a \sim b$, iff $(a, b) \in \sim$. If $A = B = S$, then we say that $\sim$ is a relation on S rather than a relation from S to S.

Examples

1. Let $A = \{1, 2, 3\}$ and $B = \{a, x, t\}$. Then $\sim = \{(1, a), (1, x), (3, t)\}$ is a relation from A to B. In other word, $1 \sim a$, $1 \sim x$ and $3 \sim t$.
2. Let A be the set of all points in a plane and B the set of all lines in the plane. Then $\sim = \{(P, \ell) : P$ is a point on the line $\ell\}$ is a relation from A to B. In other word, $P \sim \ell$ iff P is a point on the line ℓ. $\sim$ is called the incidence relation between the points and the lines of the plane.
3. For any set S, $\emptyset$ and $S \times S$ are relations on S.
4. Let $\mathbb{R}$ be the set of all real numbers. For $a, b \in \mathbb{R}$, define $a \sim b$ iff $a > b$. Then $\sim$ is a relation on $\mathbb{R}$.
5. Let S be a set. For any $A, B \subseteq S$, define $A \sim B$ iff $A \cap B = \emptyset$. Then $\sim$ is a relation on 2^S.

§A2 EQUIVALENCE RELATIONS

Definition A.2 A relation $\sim$ on a set S is called an *equivalence relation* if the following axioms are satisfied.

I (Reflexivity) For any $a \in S$, $a \sim a$.

II (Symmetry) For any $a, b \in S$, $a \sim b \implies b \sim a$.

III (Transitivity) For any $a, b, c \in S$, $a \sim b, b \sim c \implies a \sim c$.

Examples

1. Let S be a set and $\sim = \{(x, x) : x \in S\}$. Then $\sim$ is an equivalence relation. In other word, $x \sim y$ iff $x = y$. Thus $\sim$ is the equality relation.
2. For any set S, $S \times S$ is an equivalence relation on S.
3. Let $\mathbb{Z}$ be the set of all integers. For $a, b \in \mathbb{Z}$, define $a \sim b$ iff $a = b = 0$ or $ab > 0$. Check that $\sim$ is an equivalence relation on $\mathbb{Z}$.
4. Let n be a fixed positive integer. Define a relation $\equiv$ on $\mathbb{Z}$ by $x \equiv y$ iff $x - y$ is a multiple of n. Show that $\equiv$ is an equivalence relation on $\mathbb{Z}$. $\equiv$ is called

the congruence relation modulo n. When two integers x and y are congruent modulo n, we write $x \equiv y \bmod n$.

5. Parallelism is an equivalence relation on the set of all lines in the Euclidean plane. How about in a non-Euclidean plane?
6. Congruence and similarity are equivalence relations on the set of all triangles in the plane.

§A3 EQUIVALENCE CLASSES

Definition A3 Let $\sim$ be an equivalence relation on a set S. For any $a \in S$, the *equivalence class* of a, is the set $[a] = \{x \in S : x \sim a\}$. The set of all equivalence classes of $\sim$ on S is denoted by $S/\sim$.

Theorem A.4 The set of all equivalence classes of an equivalence relation on a nonempty set S is a partition of S into disjoint nonempty subsets. Every element of S is in exactly one equivalence class.

Examples

1. The equivalence relation $S \times S$ on a set S has only one equivalence class. In other word, $[x] = S$ for any $x \in S$.
2. Consider the equivalence relation $\sim$ on $\mathbb{Z}$ given by $a \sim b$ iff $a = b = 0$ or $ab > 0$. Then $\sim$ has 3 equivalence classes, $\{0\}, \mathbb{Z}^+$ and $\mathbb{Z}^-$.
3. Let S be the set of all lines in the plane and $\sim$ the equivalence relation of parallelism. For any line ℓ, $[\ell]$ consists of all lines parallel to ℓ. $[\ell]$ is called a parallel pencil.
4. Given any partition $\mathcal{P}$ of a nonempty set S. That is $\mathcal{P}$ is a collection of nonempty disjoint subsets of S and the union of all these subsets is S. Let $\sim$ be the relation on S defined by $a \sim b$ iff $a, b \in P$ for some $P \in \mathcal{P}$. Then
   (i) $\sim$ is an equivalence relation on S,
   (ii) $\mathcal{P}$ is the set of all equivalence classes of $\sim$.

Theorem A.6 There is a one-to-one correspondence between partitions of S and equivalence relations on S.

Exercises

1. Let S be the set of all squares on a chess board. Define a relation $\sim$ on S by $x \sim y$ iff both the squares x and y have the same colour. Is $\sim$ an equivalence relation on S? How many equivalence classes are there?

2. Let $f : S \longrightarrow S$ be a function and $G = \{(x, f(x)) \ : \ x \in S\}$ be the graph of f. Prove that G is an equivalence relation on S iff f is the identity function.

3. Let A be a subset of a set S. Let $\sim \, = \ \{(x, x) : x \in S\} \cup \{(x, y) : \ x, y \in A\}$. Is $\sim$ an equivalence relation on S?

4. Let $\sim$ be the relation on [0,1] defined by $x \sim y$ iff either (i) $x = y$, (ii) $x = 0$ and $y = 1$ or (iii) $x = 1$ and $y = 0$. Write down $\sim$ as a subset of $[0,1]\times[0,1]$. Show that $\sim$ is an equivalence relation on [0,1]. Find a bijection between $[0,1]/\sim$ and the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

5. Let R and R' be equivalence relations on a set S. Is $R \cap R'$ an equivalence relation on S? How about $R \cup R'$?

6. How many equivalence relations are there on a set of 5 elements?

7. Let S be the set of all nonzero complex numbers. Define a relation $\sim$ on S by $z_1 \sim z_2$ iff $z_1 z_2^{-1}$ is real. Prove that $\sim$ is an equivalence relation on S. Describe the elements of $S/\sim$.

8. Let S be the set of all lines in the Euclidean plane. Define a relation $\sim$ on S by $\ell \sim \ell'$ iff $\ell \parallel \ell'$ or $\ell \perp \ell'$. Is $\sim$ an equivalence relation on S?

9. Let A be a subset of a set S. Define a relation $\sim$ on 2^S by $X \sim Y$ iff $(X \setminus Y) \cup (Y \setminus X) \subseteq A$. Prove that

   (i) $\sim$ is an equivalence relation on 2^S,

   (ii) for any $X \subseteq A$, $X \sim \emptyset$,

   (iii) for any $X \subset S$, $X \sim X \setminus A$.

   Find a bijection between $2^S/\sim$ and $2^{S\setminus A}$.

10. Let S be the set of all triangles in the Euclidean plane. Define an equivalence relation on S by $X \sim Y$ iff X and Y have the same perimeter, the same circumradius and the same inradius. Describe the elements of $S/\sim$.