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LIE TRANSFORMATION GROUPS OF BANACH MANIFOLDS

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Introduction

Let M be a Banach manifold which is not assumed to be Hausdorff, and let D denote the group of diffeomorphisms of M and $\mathbf{V}$ the Lie algebra of vector fields on M. A Lie group $\mathscr{G}$ is called a *Lie transformation group* of M if the underlying group G of $\mathscr{G}$ is a subgroup of D and the natural map α: $(g, p) \mapsto g(p)$ from $\mathscr{G} \times M$ into M is a morphism (of manifolds). In this case, α induces a homomorphism α^+ from the Lie algebra $L(\mathscr{G})$ of $\mathscr{G}$ into $\mathbf{V}$ (cf. § 3). Conversely, we prove that the set of complete vector fields of a finite-dimensional subalgebra of $\mathbf{V}$ is a subalgebra (Proposition 8), and if $\mathbf{L}$ is a complete finite-dimensional subalgebra of $\mathbf{V}$ then there exists a unique connected Lie transformation group $\mathscr{G}$ such that α^+ is an isomorphism from $L(\mathscr{G})$ onto $\mathbf{L}$ (Theorem 9). In case M is finite-dimensional and Hausdorff, this result is due to Palais [4]. For the numerous applications in differential geometry, the reader is referred to [1]. Unfortunately, the proof of the just-mentioned special case given in [1] seems to be incomplete. The proof to be presented here is quite elementary; it relies heavily on the use of one-parameter families of diffeomorphisms, instead of one-parameter groups. To be more precise, we define a *curve in D* to be a morphism φ: $I_\varphi \times M \to M$ such that

(i) I_φ is an open interval in $\boldsymbol{R}$ containing 0;
(ii) the map φ_t: $p \mapsto \varphi(t, p)$ belongs to D, for all $t \in I_\varphi$;
(iii) $\varphi_0 = \mathrm{Id}_M$.

With φ we associate a time-dependent vector field $\delta\varphi$ by

$$\delta\varphi(t, p) = (\delta\varphi)_t(p) = (d/ds)_{s=t}\varphi_s(\varphi_t^{-1}(p)) \ .$$

The map $\varphi \mapsto \delta\varphi$ is injective (Proposition 4). The underlying group G of $\mathscr{G}$ turns out to be the set of diffeomorphisms φ_1 where φ is any curve in D such that $I_\varphi = \boldsymbol{R}$ and $(\delta\varphi)_t \in \mathbf{L}$ for all $t \in \boldsymbol{R}$. Using canonical coordinates of the second kind, G becomes a Lie group with the desired properties. We also prove the following criterion for a subgroup G of D to be a Lie transformation group (Theorem 10): assume there is a set S of curves in D such that $\{\varphi_t$: $\varphi \in S$ and $t \in I_\varphi\}$ generates G and that $\{(\delta\varphi)_t$: $\varphi \in S$ and $t \in I_\varphi\}$ generates a

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finite-dimensional subalgebra $\mathbf{L}$ of $\mathbf{V}$. Then $\mathbf{L}$ is complete and G is the underlying group of the connected Lie transformation group generated by $\mathbf{L}$.

We work throughout in the category of real Banach manifolds of class C^k where $k = \infty$ or $k = \omega$, and a morphism is a map of class C^k. For the basic facts on Banach manifolds we refer to Lang [3].

1. Curves of diffeomorphisms and time-dependent vector fields

Notational convention. If f is a map on a product space, then the partial maps $p \mapsto f(t, p)$ and $t \mapsto f(t, p)$ will be denoted by f_t and f^p, respectively. If t is a real variable, then $\dot{f}^p(t) = \dot{f}_t(p) = \dfrac{d}{dt} f(t, p)$ is the tangent vector of the curve f^p at $f(t, p)$. By I we denote an open interval in $\boldsymbol{R}$ containing 0.

Let $D(I)$ be the set of all curves in D with $I_\varphi = I$. Then with the operations

$$(\varphi\psi)(t, p) = \varphi_t \circ \psi_t(p) ; \qquad \varphi^{-1}(t, p) = \varphi_t^{-1}(p) ,$$

$D(I)$ is a group. Indeed, the only non-obvious fact is that φ^{-1} is a morphism, and this follows from the implicit function theorem.

A time-dependent vector field is a morphism $\xi : I \times M \to T(M)$, the tangent bundle of M, such that $\xi_t \in \mathbf{V}$ for every $t \in I$. Note that ξ^p is a curve in the tangent space $T_p(M)$ for every $p \in M$. Identifying as usual the tangent space of $T_p(M)$ at $\xi^p(t)$ with $T_p(M)$, we define a time-dependent vector field $\dfrac{\partial \xi}{\partial t}$ by $\dfrac{\partial \xi}{\partial t}(t, p) = \dot{\xi}^p(t)$. The set $\mathbf{V}(I)$ of time-dependent vector fields becomes a Lie algebra with

$$[\xi, \eta](t, p) = [\xi_t, \eta_t](p) .$$

Also $\mathbf{V} \subset \mathbf{V}(I)$ by setting $X(t, p) = X(p)$ for $X \in \mathbf{V}$, and then $\xi \in \mathbf{V}$ if and only if $\dfrac{\partial \xi}{\partial t} = 0$, i.e., ξ is time-independent.

Let $f \in D$ and $X \in \mathbf{V}$, and denote by Tf the induced map on the tangent bundle of M. Then

$$\operatorname{Ad} f \cdot X = Tf \circ X \circ f^{-1}$$

is a vector field on M, and in this way D acts on $\mathbf{V}$ by automorphisms. Similarly, $D(I)$ acts on $\mathbf{V}(I)$ by

$$(\operatorname{Ad} \varphi \cdot \xi)(t, p) = (\operatorname{Ad} \varphi_t \cdot \xi_t)(p) .$$

We define $\delta : D(I) \to \mathbf{V}(I)$ by

$$\delta\varphi(t, p) = \dot{\varphi}_t(\varphi_t^{-1}(p)) .$$

Then we have

$$(1) \qquad \delta(\varphi\psi) = \delta\varphi + \mathrm{Ad}\,\varphi\cdot\delta\psi ,$$

$$(2) \qquad \delta\varphi^{-1} = -\mathrm{Ad}\,\varphi^{-1}\cdot\delta\varphi .$$

Indeed,

$$\begin{aligned}\delta(\varphi\psi)(t,p) &= \frac{d}{dt}(\varphi_t(\psi_t(p)) = \dot\varphi_t(\psi_t(p)) + T\varphi_t(\dot\psi_t(p))\\ &= \delta\varphi(t, \varphi_t\circ\psi_t(p)) + T\varphi_t(\delta\psi(t,\psi_t(p))\\ &= (\delta\varphi + \mathrm{Ad}\,\varphi\cdot\delta\psi)(t,p) ,\end{aligned}$$

and (2) follows by setting $\psi = \varphi^{-1}$. Note that δ is a crossed homomorphism from $D(I)$ into $\mathbf{V}(I)$.

Lemma 1. *For $\varphi\in D(I)$ and $\xi\in\mathbf{V}(I)$ let $\eta = \mathrm{Ad}\,\varphi\cdot\xi$. Then*

$$(3) \qquad \frac{\partial\eta}{\partial t} = [\delta\varphi, \eta] + \mathrm{Ad}\,\varphi\cdot\frac{\partial\xi}{\partial t} .$$

Proof. This is a local result. Let U and V be coordinate neighborhoods of p and $\varphi_{t_0}^{-1}(p)$, and choose $V'\subset V$, $U'\subset U$ and $\varepsilon>0$ such that $\varphi((t_0-\varepsilon, t_0+\varepsilon)\times V')\subset U$ and $\varphi^{-1}((t_0-\varepsilon, t_0+\varepsilon)\times U')\subset V'$. By continuity, this is possible. We may identify U and V with open sets in a Banach space E. Then $T(U) = U\times E$ and $T(V) = V\times E$. For $y\in V$, let $\xi(t,y) = (y, g(t,y))$ where $g\colon (t_0-\varepsilon, t_0+\varepsilon)\times V\to E$. For $x\in U'$ and $|t-t_0|<\varepsilon$ we have $\delta\varphi(t,x) = (x, f(t,x))$ and $\eta(t,x) = (x, h(t,x))$ where $f(t,x) = \dot\varphi_t(\varphi_t^{-1}(x))$ and $h(t,x) = D\varphi_t(\varphi_t^{-1}(x))\cdot g(t, \varphi_t^{-1}(x))$, $D\varphi_t$ denoting the derivative of φ_t; see [3, p. 6 ff.]. Let $\varphi_t^{-1}(x) = y$ for short. Then from $\dot{\widehat{D\varphi_t}} = D\dot\varphi_t$ it follows

$$\begin{aligned}\dot h(t,x) &= \mathrm{D}\dot\varphi_t(y)\cdot g_t(y) + \mathrm{D}^2\varphi_t(y)(\dot\varphi_t^{-1}(x), g_t(y))\\ &\quad + \mathrm{D}\varphi_t(y)\cdot\dot g_t(y) + \mathrm{D}\varphi_t(y)\circ \mathrm{D}g_t(y)\cdot\dot\varphi_t^{-1}(x) ,\end{aligned}$$

$$\begin{aligned}&\mathrm{D}f_t(x)\cdot h_t(x) - \mathrm{D}h_t(x)\cdot f_t(x) + \mathrm{D}\varphi_t(y)\cdot\dot g_t(y)\\ &\quad = \mathrm{D}\dot\varphi_t(y)\circ \mathrm{D}\varphi_t^{-1}(x)\cdot h_t(x) - \mathrm{D}^2\varphi_t(y)(\mathrm{D}\varphi_t^{-1}(x)\cdot\dot\varphi_t(y), g_t(y))\\ &\qquad - \mathrm{D}\varphi_t(y)\circ \mathrm{D}g_t(y)\circ \mathrm{D}\varphi_t^{-1}(x)\cdot\dot\varphi_t(y) + \mathrm{D}\varphi_t(y)\cdot\dot g_t(y) .\end{aligned}$$

From $\varphi_t(\varphi_t^{-1}(x)) = x$ for all $x\in U'$ we get

$$\dot\varphi_t(y) + \mathrm{D}\varphi_t(y)\cdot\dot\varphi_t^{-1}(x) = 0 , \qquad (\mathrm{D}\varphi_t(y))^{-1} = \mathrm{D}\varphi_t^{-1}(x) ,$$

and the assertion of Lemma 1 follows.

(Note that our definition of the bracket of vector fields differs from the usual one by sign; this is the 'good' definition for transformation groups acting on the left.)

Corollary. *Let $Y \in \mathbf{V}$. Then $\eta = \operatorname{Ad}\varphi \cdot Y$ is the unique solution of the partial differential equation*

$$\frac{\partial \eta}{\partial t} = [\delta\varphi, \eta] \tag{4}$$

for the time-dependent vector field η with initial condition $\eta_0 = Y$.

Proof. From (3) it follows that $\operatorname{Ad}\varphi \cdot Y$ is a solution of (4). To prove unicity, let η be any solution of (4), and let $\zeta = \operatorname{Ad}\varphi^{-1} \cdot \eta$. Then, from (2) and (3),

$$\begin{aligned}\frac{\partial \zeta}{\partial t} &= [\delta\varphi^{-1}, \zeta] + \operatorname{Ad}\varphi^{-1} \cdot \frac{\partial \eta}{\partial t} \\ &= [-\operatorname{Ad}\varphi^{-1} \cdot \delta\varphi, \operatorname{Ad}\varphi^{-1} \cdot \eta] + \operatorname{Ad}\varphi^{-1} \cdot [\delta\varphi, \eta] = 0 .\end{aligned}$$

Hence $\operatorname{Ad}\varphi_t^{-1} \cdot \eta_t = \zeta_t = \zeta_0 = \operatorname{Ad}\varphi_0^{-1} \cdot \eta_0 = Y$ and therefore $\eta_t = \operatorname{Ad}\varphi_t \cdot Y$ for all $t \in I$. q.e.d.

A curve $\varphi \in D(R)$ is called a *one-parameter group* if $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in R$.

Lemma 2. a) *If φ is a one-parameter group, then $\delta\varphi$ is time-independent.*

b) *Let $\varphi \in D(I)$ and $\delta\varphi = X$ be time-independent. Then $\operatorname{Ad}\varphi_t \cdot X = X$ for all $t \in I$, and φ can be extended uniquely to a one-parameter group.*

Proof. a) This follows by differentiating the identity $\varphi_{s+t}(\varphi_t^{-1}(p)) = \varphi_s(p)$ with respect to s at $s = 0$.

b) From (2) and (3) we get

$$\frac{\partial}{\partial t}(\operatorname{Ad}\varphi^{-1} \cdot X) = [\delta\varphi^{-1}, \operatorname{Ad}\varphi^{-1} \cdot X] = [-\operatorname{Ad}\varphi^{-1} \cdot X, \operatorname{Ad}\varphi^{-1} \cdot X] = 0 .$$

Hence $\operatorname{Ad}\varphi_t \cdot X = \operatorname{Ad}\varphi_0 \cdot X = X$ for all $t \in I$. Now let $s \in I$, and set $\alpha_t = \varphi_{s+t} \circ \varphi_t^{-1}$ for $t \in J = I \cap I - s$. Then

$$\begin{aligned}\dot{\alpha}_t(p) &= \dot{\varphi}_{s+t}(\varphi_t^{-1}(p)) + T\varphi_{s+t}(\dot{\varphi}_t^{-1}(p)) \\ &= (X - \operatorname{Ad}\varphi_{s+t}\operatorname{Ad}\varphi_t^{-1} \cdot X)(\alpha_t(p)) = 0 .\end{aligned}$$

Since J is connected and $0 \in J$, it follows $\varphi_{s+t} \circ \varphi_t^{-1} = \alpha_t = \alpha_0 = \varphi_s$, i.e., $\varphi_s \circ \varphi_t = \varphi_{s+t}$. Now it is a standard fact that φ can be extended uniquely to a one-parameter group. q.e.d.

The following change of parameter will be useful.

Lemma 3. *There exists a C^∞-diffeomorphism $f\colon R \to I$ such that $f(0) = 0$. The map $f^*\colon D(I) \to D(R)$ defined by $(f^*\varphi)(t, p) = \varphi(f(t), p)$ is a group isomorphism, and $\delta(f^*\varphi)(t, p) = \dfrac{df}{dt} \cdot \delta\varphi(f(t), p)$.*

The proof is left to the reader.

Proposition 4. $\delta\colon D(I) \to \mathbf{V}(I)$ *is injective.*

Proof. By Lemma 3, we may assume $I = \boldsymbol{R}$. For $\varphi \in D(\boldsymbol{R})$, define

$$\tilde{\varphi}_t(s, p) = (t + s, \varphi_{t+s} \circ \varphi_s^{-1}(p)) \qquad (t \in \boldsymbol{R}, (s, p) \in \boldsymbol{R} \times M) .$$

An immediate verification shows that $\tilde{\varphi}$ is a one-parameter group on $\boldsymbol{R} \times M$. As usual, $T(\boldsymbol{R})$ is identified with $\boldsymbol{R} \times \boldsymbol{R}$ and $T(\boldsymbol{R} \times M)$ with $T(\boldsymbol{R}) \times T(M)$. Then by Lemma 2 the (time-independent) vector field $X = \delta\tilde{\varphi}$ on $\boldsymbol{R} \times M$ is given by

$$X(s, p) = \frac{d}{dt}\bigg|_{t=0} (t + s, \varphi_{t+s} \circ \varphi_s^{-1}(p)) = ((s, 1), \delta\varphi(s, p)) .$$

Let $\varphi, \psi \in D(\boldsymbol{R})$. Clearly, $\delta\varphi = \delta\psi$ implies $\delta\tilde{\varphi} = \delta\tilde{\psi}$, and $\tilde{\varphi} = \tilde{\psi}$ implies $\varphi = \psi$. Hence it suffices to prove the proposition for one-parameter groups. Finally, let φ and ψ be one-parameter groups such that $X = \delta\varphi = \delta\psi$. Then from Lemma 2 and (1) and (2) we have $\delta(\varphi\psi^{-1}) = \delta\varphi + \operatorname{Ad}\varphi \cdot \delta\psi^{-1} = X - \operatorname{Ad}\varphi \operatorname{Ad}\psi^{-1} \cdot X = X - X = 0$. Setting $\alpha = \varphi\psi^{-1}$, this implies that $\dot{\alpha}^p(t) = 0$ for all $p \in M$, $t \in \boldsymbol{R}$. Therefore the map $\alpha^p\colon \boldsymbol{R} \to M$ is constant for all $p \in M$, and it follows $\alpha_t = Id_M$, i.e., $\varphi = \psi$.

Note that $\varphi^p\colon t \mapsto \varphi(t, p)$ is a solution of the differential equation $\dfrac{dx}{dt} = \delta\varphi(t, x)$ with initial condition $x(0) = p$. In case M is Hausdorff, this solution is unique which gives a simpler proof of Proposition 4. q.e.d.

A vector field X such that $X = \delta\varphi$ for some (uniquely determined) $\varphi \in D(\boldsymbol{R})$ is called *complete*. It is well known that on a compact manifold every vector field is complete. It can be shown that this is still true for time-dependent vector fields, so that $\delta\colon D(I) \to V(I)$ is a bijection for compact M.

2. Lie algebras of vector fields

In this section, $\mathbf{L}$ will denote an arbitrary *finite-dimensional* subalgebra of $\mathbf{V}$. Let

$$(5) \qquad \mathbf{L}(\boldsymbol{R}) = \{\xi \in \mathbf{V}(\boldsymbol{R})\colon \xi_t \in \mathbf{L} \text{ for all } t \in \boldsymbol{R}\} .$$

As a finite-dimensional vector space, $\mathbf{L}$ is a manifold in a natural way. Then we have

Lemma 5. $\mathbf{L}(\boldsymbol{R})$ *is naturally isomorphic to the set of morphisms from* $\boldsymbol{R}$ *into* $\mathbf{L}$.

Proof. Let $p \in M$. Since $\mathbf{L}$ is finite-dimensional, the subspace $\{X(p)\colon X \in \mathbf{L}\}$ of the Banach space $T_p(M)$ is closed and admits a closed complementary subspace. Hence, again by finite-dimensionality of $\mathbf{L}$, there exist $p_i \in M$ and continuous linear forms λ_i on $T_{p_i}(M)$ $(i = 1, \cdots, r)$ such that the map $F\colon X \mapsto$

$(\lambda_1(X(p_1)), \cdots, \lambda_r(X(p_r)))$ is a linear isomorphism from $\mathbf{L}$ onto $\boldsymbol{R}^r$. Let $e_1, \cdots, e_r$ be a basis of $\boldsymbol{R}^r$ and set $X_i = F^{-1}(e_i)$. For any $\xi \in \mathbf{L}(\boldsymbol{R})$, the map $\xi^p: \boldsymbol{R} \to T_p(M)$ is a morphism. Hence $f_i = \lambda_i \circ \xi^{p_i}$ is a morphism from $\boldsymbol{R}$ into $\boldsymbol{R}$, and $\xi_t = \sum f_i(t) X_i$ shows that $t \mapsto \xi_t$ is a morphism from $\boldsymbol{R}$ into $\mathbf{L}$. If conversely $\eta: \boldsymbol{R} \mapsto \mathbf{L}$ is a morphism, then $\eta(t) = \sum g_i(t) X_i$ with morphisms $g_i: \boldsymbol{R} \to \boldsymbol{R}$, and this shows that the map $(t, p) \mapsto \eta(t)(p)$ belongs to $\mathbf{L}(\boldsymbol{R})$. q.e.d.

In view of Lemma 5, we will identify $\mathbf{L}(\boldsymbol{R})$ with the set of morphisms from $\boldsymbol{R}$ into $\mathbf{L}$. Then $\dfrac{\partial\xi}{\partial t} = \dfrac{d\xi}{dt}$, where $\dfrac{d\xi}{dt}$ denotes the usual derivative of a curve in a vector space.

Now we define

$$G(\boldsymbol{R}) = \{\varphi \in D(\boldsymbol{R}) : \delta\varphi \in \mathbf{L}(\boldsymbol{R})\} . \tag{6}$$

The fact that we consider only curves of diffeomorphisms defined on $\boldsymbol{R}$ is convenient but not essential in view of Lemma 3.

Lemma 6. *Let $\varphi \in G(\boldsymbol{R})$ and $\delta\varphi = \xi: \boldsymbol{R} \to \mathbf{L}$. Then $\mathbf{L}$ is invariant under* $\mathrm{Ad}\,\varphi_t (t \in \boldsymbol{R})$, *and the map $t \mapsto \mathrm{Ad}\,\varphi_t | \mathbf{L}$ is the unique solution of the matrix differential equation $\dfrac{dA}{dt} = \mathrm{ad}\,\xi(t) \circ A$ with initial condition $A(0) = \mathrm{Id}_{\mathbf{L}}$. In particular, it is a morphism from $\boldsymbol{R}$ into* $\mathbf{GL}(\mathbf{L})$.

Proof. For $Y \in \mathbf{L}$ let $\eta: \boldsymbol{R} \to \mathbf{L}$ be the unique solution of the ordinary linear differential equation $\dfrac{dX}{dt} = [\xi(t), X]$ in $\mathbf{L}$ with initial condition $\eta(0) = Y$. Then by the remark above, η considered as an element of $\mathbf{L}(\boldsymbol{R})$ is a solution of (4), and $\eta(t) = \mathrm{Ad}\,\varphi_t \cdot Y \in \mathbf{L}$ by the corollary of Lemma 1. Hence the lemma follows from the standard facts on ordinary linear differential equations.

From (1) and (2) we get

Corollary. *$G(\boldsymbol{R})$ is a subgroup of $D(\boldsymbol{R})$.*

We define

$$G = \{\varphi_1 : \varphi \in G(\boldsymbol{R})\} , \qquad \mathbf{L}_0 = \{(\delta\varphi)_0 : \varphi \in G(\boldsymbol{R})\} . \tag{7}$$

Lemma 7. a) *G is a subgroup of D, and $\varphi_s \in G$ for all $\varphi \in G(\boldsymbol{R}), s \in \boldsymbol{R}$.*

b) *$\mathbf{L}_0$ is a subalgebra of $\mathbf{L}$ and $(\delta\varphi)_s \in \mathbf{L}_0$ for all $\varphi \in G(\boldsymbol{R}), s \in \boldsymbol{R}$. Also, $\mathbf{L}_0$ is invariant under $\mathrm{Ad}\,g$ for all $g \in G$.*

Proof. By the above corollary, G is a subgroup of D. Let $s \in \boldsymbol{R}, \varphi \in G(\boldsymbol{R})$, and set $\psi_t = \varphi_{st}$. Then $\varphi_s = \psi_1 \in G$ and also $(\delta\psi)_0 = s \cdot (\delta\varphi)_0$. Thus it follows from (1) that $\mathbf{L}_0$ is a subspace of $\mathbf{L}$. For $\varphi, \psi \in G(\boldsymbol{R})$ and a fixed $s \in \boldsymbol{R}$ set $\alpha_t = \varphi_s \circ \psi_t \circ \varphi_s^{-1}$. Then $(\delta\alpha)_t = \mathrm{Ad}\,\varphi_s \cdot (\delta\psi)_t \in \mathbf{L}$ by Lemma 6. Hence $\alpha \in G(\boldsymbol{R})$, and it follows $\eta(s) = (\delta\alpha)_0 = \mathrm{Ad}\,\varphi_s \cdot (\delta\psi)_0 \in \mathbf{L}_0$. This shows that $\mathbf{L}_0$ is invariant under $\mathrm{Ad}\,G$. Furthermore, by differentiating with respect to s at $s = 0$ we get $\dfrac{d\eta}{ds}(0) = [(\delta\varphi)_0, (\delta\psi)_0] \in \mathbf{L}_0$. Thus $\mathbf{L}_0$ is a subalgebra of $\mathbf{L}$. Finally, let $\beta_t =$

$\varphi_{s+t} \circ \varphi_s^{-1}$. Then $(\delta\beta)_t = (\delta\varphi)_{s+t}$ shows $\beta \in G(\boldsymbol{R})$, and it follows $(\delta\varphi)_s = (\delta\beta)_0 \in \mathbf{L}_0$.

Proposition 8. $\mathbf{L}_0$ *is the set of complete vector fields in* $\mathbf{L}$.

Proof. By a) of Lemma 2, a complete vector field in $\mathbf{L}$ belongs to $\mathbf{L}_0$. Conversely, choose $\varphi^{(i)}$ in $G(\boldsymbol{R})$ such that $(\delta\varphi^{(i)})_0$ $(i = 1, \cdots, n)$ form a basis of $\mathbf{L}_0$, and define $\Phi\colon \boldsymbol{R}^n \to G$ by

$$\Phi(x) = \varphi^{(1)}_{x_1} \circ \cdots \circ \varphi^{(n)}_{x_n} .$$

Clearly, $(x, p) \mapsto \Phi(x)(p)$ is a morphism from $\boldsymbol{R}^n \times M$ into M. Also define $F\colon \boldsymbol{R}^n \to \operatorname{Hom}(\boldsymbol{R}^n, \mathbf{L}_0)$ by

$$F_x(v) = \sum_{i=1}^{n} v_i \cdot (\operatorname{Ad} \varphi^{(1)}_{x_1} \circ \cdots \circ \operatorname{Ad} \varphi^{(i-1)}_{x_{i-1}} \cdot \xi_i(x_i)) , \tag{8}$$

where $\xi_i = \delta\varphi^{(i)}\colon \boldsymbol{R} \to \mathbf{L}_0$. By Lemma 6, F is a morphism. Also, F_0 is a vector space isomorphism, since $F_0(v) = \sum v_i \xi_i(0)$ and the $\xi_i(0) = (\delta\varphi^{(i)})_0$ form a basis of $\mathbf{L}_0$.

Let $\gamma\colon I \to \boldsymbol{R}^n$ be a morphism such that $\gamma(0) = 0$. Then $\varphi_t = \Phi(\gamma(t))$ defines a curve in D, and a computation shows

$$(\delta\varphi)_t = F_{\gamma(t)}(\dot\gamma(t)) . \tag{9}$$

Since F_0 is an isomorphism, there exists $r > 0$ such that F_z is an isomorphism for $\|z\| \leq r$. Let $X \in \mathbf{L}_0$ be given, and consider the ordinary differential equation

$$\frac{dz}{dt} = F_z^{-1}(X) \quad (\|z\| \leq r) .$$

Let $\gamma\colon I \to \boldsymbol{R}^n$ be a solution with $\gamma(0) = 0$, and define φ as above. Then $(\delta\varphi)_t = F_{\gamma(t)} F^{-1}_{\gamma(t)}(X) = X$, and X is complete by Lemma 2.

For any $X \in \mathbf{L}_0$ we denote the corresponding one-parameter group by $\operatorname{Exp} tX$. Then we have

$$\operatorname{Ad} \operatorname{Exp} tX \cdot Y = e^{\operatorname{ad} tX} \cdot Y \qquad \text{for } X \in \mathbf{L}_0, Y \in \mathbf{L}. \tag{10}$$

Indeed, by Lemma 6, $\operatorname{Ad} \operatorname{Exp} tX | \mathbf{L}$ is the solution of $\dfrac{dA}{dt} = \operatorname{ad} X \circ A$ with initial condition $A(0) = \operatorname{Id}_{\mathbf{L}}$ which is given by $e^{\operatorname{ad} tX}$.

3. Connected Lie transformation groups

We first recall some facts about group actions. Let $\mathscr{G}$ be a Lie group. A morphism $\alpha\colon (g, p) \mapsto g \cdot p$ from $\mathscr{G} \times M$ into M is called an *action of* $\mathscr{G}$ *on* M *on the left* if

(i) $g \cdot (h \cdot p) = (gh) \cdot p$,

(ii) $e \cdot p = p$,

for $g, h \in \mathscr{G}$ and $p \in M$ (e is the neutral element of G). The Lie algebra $L(\mathscr{G})$ of $\mathscr{G}$ is the tangent space $T_e(\mathscr{G})$ with the bracket $[X, Y] = [\bar{X}, \bar{Y}](e)$, where $\bar{X}$ is the right-invariant vector field on $\mathscr{G}$ such that $\bar{X}(e) = X$ (this coincides with the usual definition in terms of left-invariant vector fields since our bracket of vector fields differs from the usual one by sign). Then α induces a homomorphism $\alpha^+: L(\mathscr{G}) \to \mathbf{V}$ by

$$\alpha^+(X)(p) = T\alpha^p(X) ,$$

(see [4, p. 35]). The proof is a straightforward computation in local charts by using (i) and (ii) and is omitted here.

In case the underlying group G of $\mathscr{G}$ is a subgroup of D and $\alpha(g, p) = g(p)$ is the natural map, we say $\mathscr{G}$ is a *Lie transformation group* of M.

Theorem 9. *Let* $\mathbf{L}$ *be a finite-dimensional complete subalgebra of* $\mathbf{V}$. *Then there exists a unique connected Lie transformation group* $\mathscr{G}$ *of* M *such that* α^+ *is an isomorphism from* $L(\mathscr{G})$ *onto* $\mathbf{L}$, *and for every* $\varphi \in D(I)$ *such that* $\varphi_t \in \mathscr{G}$ *for all* $t \in I$ *the map* $t \mapsto \varphi_t$ *is a morphism from* I *into* $\mathscr{G}$.

Proof. Let G be the subgroup of D defined by (7), choose a basis $X_1, \cdots, X_n$ of $\mathbf{L}$, and define $\Phi: \mathbf{R}^n \to G$ by

$$\Phi(x) = \operatorname{Exp} x_1X_1 \circ \cdots \circ \operatorname{Exp} x_nX_n .$$

We will show that in the canonical coordinates of the second kind given by Φ, G becomes a Lie group with the desired properties.

First we prove

(11) *Φ is injective in a neighborhood of* 0.

Since $\mathbf{L}$ is finite-dimensional there exist $p_1, \cdots, p_r \in M$ such that the map $X \mapsto (X(p_1), \cdots, X(p_r))$ from $\mathbf{L}$ into $E = T_{p_1}(M) \times \cdots \times T_{p_r}(M)$ is injective. Define $f: \mathbf{R}^n \to M^r$ by $f(x) = (\Phi(x)(p_1), \cdots, \Phi(x)(p_r))$. Then $T_0f(v) = (\sum v_iX_i(p_1), \cdots, \sum v_iX_i(p_r))$, and T_0f is injective since $X_1, \cdots, X_n$ is a basis of $\mathbf{L}$. Thus the image of T_0f in the Banach space E, being finite-dimensional, is closed and admits a closed complementary subspace. Hence by the implicit function theorem, f is injective in a neighborhood of 0 in $\mathbf{R}^n$ which proves (11).

Next we show

(12) *there exists a neighborhood* N *of* 0 *in* $\mathbf{R}^n$ *and a real analytic map* $\mu: N \times N \to \mathbf{R}^n$ *such that* $\mu(0, 0) = 0$ *and* $\Phi(\mu(x, y)) = \Phi(x) \circ \Phi(y)$.

Defining $F: \mathbf{R}^n \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{L})$ in analogy with (8), we obtain, from (10),

$$F_x(v) = \sum_{i=1}^{n} v_i \cdot (e^{\operatorname{ad} x_1X_1} \circ \cdots \circ e^{\operatorname{ad} x_{i-1}X_{i-1}} \cdot X_i) .$$

Thus F is real analytic. As in the proof of Proposition 8, F_0 is a vector space isomorphism, and we choose $r > 0$ such that F_z is an isomorphism for $z \in B_r = \{x \in \boldsymbol{R}^n : \|x\| \leq r\}$. Set

$$A(t, z; x, y) = F_z^{-1}(F_{tx}(x) + e^{\mathrm{ad}\, tx_1X_1} \circ \cdots \circ e^{\mathrm{ad}\, tx_nX_n} \cdot F_{ty}(y)) .$$

Then $A: \boldsymbol{R} \times B_r \times \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is real analytic, and $A(t, z; 0, 0) = 0$. Thus there exists an open neighborhood N of 0 in $\boldsymbol{R}^n$ such that

$$\|A(t, z; x, y)\| \leq 2r/3 \qquad \text{for } |t| \leq 3/2, z \in B_r, \text{ and } x, y \in N .$$

By standard theorems on differential equations, the equation

$$\frac{dz}{dt} = A(t, z; x, y)$$

has a unique solution $\gamma(t; x, y)$ such that $\gamma(0; x, y) = 0$, defined for $|t| \leq 3/2$ and depending real analytically on the parameters $x, y \in N$. We define $\mu(x, y) = \gamma(1; x, y)$, and show that $\Phi(\mu(x, y)) = \Phi(x) \circ \Phi(y)$. Indeed, let $\varphi_t = \Phi(\gamma(t; x, y))$ and $\psi_t = \Phi(tx) \circ \Phi(ty)$. Then, by (1), (6) and (7),

$$\begin{aligned}(\delta\psi)_t &= F_{tx}(x) + \mathrm{Ad}\, \Phi(tx) \cdot F_{ty}(y) = F_{tx}(x) + e^{\mathrm{ad}\, tx_1X_1} \circ \cdots \circ e^{\mathrm{ad}\, tx_nX_n} \cdot F_{ty}(y) \\ &= F_{\gamma(t; x, y)}(\dot{\gamma}(t; x, y)) = (\delta\varphi)_t .\end{aligned}$$

Thus by Proposition 4, $\varphi_t = \psi_t$ for $|t| < 3/2$, and for $t = 1$ the assertion follows.

In a similar fashion, we can prove, with details omitted:

(13) *there exist a neighborhood N of 0 in $\boldsymbol{R}^n$ and a real analytic map $\iota: N \to \boldsymbol{R}^n$ such that $\iota(0) = 0$ and $\Phi(\iota(x)) = \iota(x)^{-1}$;*

(14) *for every $g \in G$ there exist a neighborhood N of 0 in $\boldsymbol{R}^n$ and a real analytic map $\theta: N \to \boldsymbol{R}^n$ such that $\theta(0) = 0$ and $\Phi(\theta(x)) = g \circ \theta(x) \cdot g^{-1}$,*

by considering the differential equations

$$\frac{dz}{dt} = -F_z^{-1}(e^{-\mathrm{ad}\, tx_nX_n} \circ \cdots \circ e^{-\mathrm{ad}\, tx_1X_1} \cdot F_{tx}(x)) ,$$

$$\frac{dz}{dt} = F_z^{-1}(\mathrm{Ad}\, g \cdot F_{tx}(x))$$

depending on the parameter x.

Now let $V \subset W \subset N$ be open neighborhoods of 0 in $\boldsymbol{R}^n$ such that (11), (12) and (13) hold for N, and furthermore $\mu(V, \iota(V)) \subset W$ and $\mu(W, W) \subset N$. For every $a \in G$, let $U_a = a \cdot \Phi(V)$ and define $f_a: U_a \to V$ by $f_a(g) = \Phi^{-1}(a^{-1}g)$. Thus $c_a = (U_a, f_a)$ is a chart at a. Assume $U_a \cap U_b \neq \emptyset$. Then $a^{-1}b = \Phi(x_0) \in \Phi(W)$, and $f_a f_b^{-1}(x) = f_a(b \cdot \Phi(x)) = \Phi^{-1}(a^{-1}b \cdot \Phi(x)) = \Phi^{-1}(\Phi(x_0)\Phi(x)) = \Phi^{-1}(\Phi(\mu(x_0, x))) = \mu(x_0, x)$.

Therefore any two such charts are C^ω-compatible, and the atlas $\mathscr{A} = \{c_a : a \in G\}$ defines on G the structure of an n-dimensional real analytic manifold. From the definition of $\mathscr{A}$ it is obvious that all left-translations of G are real analytic, and by (12), (13) and (14), multiplication, inversion and inner automorphisms are real analytic at $e = \mathrm{Id}_M$. Hence it follows easily that $\mathscr{G} = (G, \mathscr{A})$ is a Lie group.

Since the map $(x, p) \mapsto \Phi(x)(p)$ is a morphism, it is clear that α is a morphism at (e, p) for all $p \in M$, and hence everywhere. Let $X \in L(G)$ be represented by $v \in \boldsymbol{R}^n$ in the chart c_e. Then $\alpha^+(X) = F_0(v)$ shows that α^+ is an isomorphism of $L(\mathscr{G})$ onto $\mathbf{L}$.

To prove the second statement, let $Y_t = (\alpha^+)^{-1}((\delta\varphi)_t)$. This is a curve in $L(\mathscr{G})$, and the differential equation $\dot{a}_t = Y_t a_t$ with initial condition $a_0 = e$ in $\mathscr{G}$ has a unique solution defined for all $t \in I$, [2, Lemma, p. 69]. Then $\psi(t, p) = a_t(p)$ defines a curve in D such that $\delta\psi = \delta\varphi$. By Proposition 4, $a_t = \varphi_t$, and the assertion follows; this also proves that $\mathscr{G}$ is connected.

To prove unicity, let $\mathscr{H}$ be a Lie group with the same properties as $\mathscr{G}$, H be the underlying group of $\mathscr{H}$, and $\beta: \mathscr{H} \times M \to M$ be the map $(h, p) \mapsto h(p)$. Then we have $\exp tX = \operatorname{Exp} t\beta^+(X)$ where $\exp: L(\mathscr{H}) \to \mathscr{H}$ is the usual exponential map. Indeed, $\varphi(t, p) = \beta(\exp tX, p)$ defines a one-parameter group on M, and since $\delta\varphi(0, p) = (d/dt)_{t=0}\beta(\exp tX, p) = T\beta^p(X) = \beta^+(X)(p)$, the assertion follows from Proposition 4. Since $\mathscr{H}$ is connected, it is generated by $\exp L(\mathscr{H})$ and therefore $H = G$. Now the commutative diagram

$$\begin{array}{ccc} L(\mathscr{G}) & \xrightarrow{(\beta^+)^{-1}\circ\alpha^+} & L(\mathscr{H}) \\ \exp\downarrow & & \downarrow\exp \\ \mathscr{G} & \xrightarrow[\mathrm{Id}_G]{} & \mathscr{H} \end{array}$$

shows that Id_G is a Lie group isomorphism.

Theorem 10. *Let G be a subgroup of D, and assume that there is a set S of curves in D such that $\{\varphi_t : \varphi \in S$ and $t \in I_\varphi\}$ and $\{(\delta\varphi)_t : \varphi \in S$ and $t \in I_\varphi\}$ generates G and a finite-dimensional subalgebra $\mathbf{L}$ of V respectively. Then $\mathbf{L}$ is complete and G is the underlying group of the connected Lie transformation group generated by $\mathbf{L}$.*

Proof. After a change of parameter (Lemma 3) we may assume that $I_\varphi = \boldsymbol{R}$ for all $\varphi \in S$. From Lemma 7 and Proposition 8 it follows that $\mathbf{L}$ is complete. Let $\mathscr{G}'$ be the connected Lie transformation group generated by $\mathbf{L}$, with underlying group G'. By Theorem 9, G is a subgroup of G' such that every element of G can be joined to e by a differentiable curve contained in G. Thus by [2, Appendix 4], G is the underlying group of a connected Lie subgroup $\mathscr{G}$ of $\mathscr{G}'$ and $t \mapsto \varphi_t$ is a morphism from $\mathbf{R}$ into G for all $\varphi \in S$. It follows that the vectors $(\alpha^+)^{-1}((\delta\varphi)_t)$ belong to $L(G)$. Since these vectors generate $L(\mathscr{G}')$, we must have $L(\mathscr{G}) = L(\mathscr{G}')$ and hence $G = G'$.

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