Jet Groupoids, Natural Bundles and the Vessiot Equivalence Method

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Diplom-Physiker

Arne Lorenz

aus Elmshorn

Berichter: Universitätsprofessor Dr. Wilhelm Plesken gen. Wigger
Universitätsprofessor Dr. Vladimir P. Gerdt

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Introduction

The present thesis takes a work of Vessiot [Ves03] as the starting point to develop a new equivalence method which has theoretical advantages over Cartan's method [Car08]. The development is focused on both theoretical and computational aspects. Equivalence means that two geometric objects on a manifold can locally be mapped to each other by a smooth transformation. The main difference between Vessiot's and Cartan's approach is that Vessiot's method works on arbitrary geometric objects, whereas Cartan has to reduce all problems to a coframe, which is a very special geometric object. A coframe is a basis of the cotangent space.

The Vessiot equivalence method, developed in this thesis, is successfully applied to the example of linear partial differential operators (LPDOs) under gauge transformations. For third and fourth order LPDOs in dimension two, generating sets of invariants have been calculated. They allow to decide equivalence of LPDOs under gauge transformations. Furthermore they are of interest for factorisation and the exact integration of the operators. At order three, this leads to the improvement of several results from Mansfield and Shemyakova [MS08], who use Cartan's moving frame method. The fourth order results are completely new.

In order to treat LPDOs with Cartan's method, the problem must be formulated in terms of coframes and this requires human interaction. Choosing a coframe generally involves unnatural choices that have to be ruled out in the end (see e.g. [Olv95, Ex. 9.2] on Riemannian metrics). In contrast to Cartan's method, as mentioned above, Vessiot's approach works directly with the geometric objects. Their transformation is encoded in natural bundles, which are either given by the problem or constructed automatically. In case of LPDOs the coordinates of the natural bundle are simply the coefficients of the operators.

For Cartan's approach, all problems have to be transformed to first order, since coframes are first order objects. The LPDOs under consideration are of order three and four. Higher order geometric objects are directly supported by Vessiot's approach.

In standard literature on Cartan's method (e.g. [Gar89], [Olv95]), only the transitive case without invariants is thoroughly covered. The intransitive case is described as too complicated for a general treatment [Gar89, p. 37]. For LP-

DOs, invariants occur in every step of computation. A generalisation of Vessiot's approach presented in this thesis allows to treat invariants without extra effort. Only minor adaptations for the special case of LPDOs are necessary.

The Vessiot equivalence method can be developed along four central questions. To answer them, the language of differential geometry is used.

(1) What is a geometric object?

(2) What are the symmetries of a geometric object?

(3) Is it possible to find all invariants for a given class of geometric objects?

(4) Under which conditions are two given geometric objects equivalent?

First of all, Nijenhuis [Nij72] considers geometric objects as sections of a natural bundle. Here, natural bundles are fibre bundles $\mathcal{F} \to X$ over a manifold X with the special property that all local diffeomorphisms $\varphi : X \to X$ can be lifted to morphisms $\tilde{\varphi} : \mathcal{F} \to \mathcal{F}$. Natural bundles are the main tool for the Vessiot equivalence method. They are explained in Chapter 3.

In standard differential geometry, there are many very simple examples of natural bundles and geometric objects. The tangent bundle $T \to X$ is a natural bundle with vector fields as geometric objects. Each diffeomorphism φ is lifted to T by multiplying the tangent vectors with the Jacobian matrix of φ. Other examples are Riemannian metrics and Christoffel symbols. Their behaviour under coordinate changes defines the corresponding natural bundle.

The main motivation for the introduction of natural bundles are the symmetries of geometric objects. In order to answer question (2) it is convenient to follow the historical development.

Symmetries are those diffeomorphisms φ which leave the geometric object unchanged. They are defined by partial differential equations (PDEs). Since symmetries of geometric objects can be locally composed, they have the structure of Lie pseudogroups. Lie himself [Lie91] called them 'infinite groups', because they usually depend on an infinite number of parameters.

In the same article, Lie presented the central idea for the treatment of symmetries of geometric objects. He proved that the defining PDEs for pseudogroups can be written in the so-called *Lie form*

$$\Phi_{\omega(y)}(y, y_q) = \omega(x). \tag{0.1}$$

Here, ω is a geometric object and y_q stands for the derivatives of the diffeomorphism $y(x)$ up to order q. The Lie form separates the variables, since the differential invariants Φ_ω are independent from x.

The first major discovery of Vessiot [Ves03] is that the Lie form (0.1) is nothing but the transformation law of a geometric object, namely ω, being a section of a natural bundle. A good illustration are the symmetry equations of a Riemannian metric. In coordinates, a metric is given by a symmetric matrix $\omega(x) = (g_{ij}(x))$. If (y_i^k) denotes the Jacobian matrix of a diffeomorphism $y = \varphi(x)$, the symmetry equations in Lie form are

$$g_{kl}(y)\, y_i^k\, y_j^l = g_{ij}(x).$$

The left hand sides give a coordinate description of the natural bundle S^2T^* of metrics. With Lie's idea, all PDEs for pseudogroups can be transformed into Lie form and Vessiot constructs a natural bundle from this.

The symmetry pseudogroup of the geometric object consists of the solutions of the Lie form (0.1) regarded as a differential equation. The jet groupoid is simply the set of solutions of the Lie form (0.1) regarded as an algebraic equation on the jet space. In an informal way, Lie used both points of view while Kumpera and Spencer emphasised the latter in their book 'Lie Equations I' [KS72]. Vessiot's work was more or less forgotten until taken up by Pommaret [Pom78]. Following Vessiot, he emphasised the importance of natural bundles for study of symmetries.

To present the connection between jet groupoids and natural bundles, more details on jet groupoids are needed. In contrast to pseudogroups, jet groupoids always allow a finite-dimensional description for the symmetries of a geometric object. The jet bundle $J_q(X \times X)$ provides coordinates (x, y, y_q) for all derivatives of a diffeomorphism $y = \varphi(x)$ up to order q. An element of $J_q(X \times X)$ can be identified with the Taylor coefficients of a smooth map $\varphi : X \to X$ up to order q. The jet groupoid $\Pi_q \subset J_q(X \times X)$ consists of all those elements which correspond to invertible maps φ. Details on jet groupoids are given in Chapter 2.

In the jet groupoid interpretation, the Lie form (0.1) now determines a subgroupoid $\mathcal{R}_q(\omega)$ of Π_q. It contains all combinations of Taylor coefficients which may be continued to a symmetry of ω. Pommaret defines $\mathcal{R}_q(\omega)$ with the exact sequence

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow \Pi_q \underset{\omega}{\overset{\Phi_\omega}{\rightrightarrows}} \mathcal{F},$$

where the maps on the double arrows stand for the Lie form (0.1). This is the sequence which connects jet groupoids and natural bundles. It is the first goal of Chapter 3 to construct and explain this sequence.

In this thesis, a generalisation of natural bundles is introduced. They are called natural Θ_q-bundles, since only a subgroupoid Θ_q of Π_q acts on them. It is remarkable that most proofs from the Π_q-case remain valid if Π_q is replaced by Θ_q. With natural Θ_q-bundles, Vessiot's approach is applicable to far wider class of equivalence problems, such as the LPDOs above.

The second discovery of Vessiot is that natural bundles are very useful to check the integrability of the symmetry equations (0.1). Differentiating the equations and then eliminating the highest order derivatives may produce new equations of lower order. This process is called prolongation and projection. If all new equations can be expressed by the old ones, the PDEs are called integrable.

The prolongation and projection can be performed with natural bundles and the result is a natural bundle $\mathcal{F}_{(1)} \to \mathcal{F}$ which also encodes the new equations. The PDEs (0.1) for a geometric object ω are integrable if there exists an equivariant section

$$c : \mathcal{F} \to \mathcal{F}_{(1)}.$$

The section c shows how to express the new equations on $\mathcal{F}_{(1)}$ by those on $\mathcal{F}$. If the system (0.1) is integrable, it is a convenient description for the symmetries of a geometric object and a complete answer to question (2).

Experience shows that only very few symmetry equations (0.1) are given in an integrable form and it is often necessary to prolong and project several times until they become integrable. In this process, the natural bundles $\mathcal{F}_{(1)}, \mathcal{F}_{(2)}, \ldots, \mathcal{F}_{(k)}$ are constructed. If this is done in a naive way, the bundles grow quickly and the PDE systems for symmetries become redundant. This thesis presents a new way to determine minimal subbundles of the $\mathcal{F}_{(i)}$ such that all redundant equations for geometric objects on $\mathcal{F}$ are removed. Minimal bundles are of both theoretical and computational importance.

When performing the prolongations and projections with minimal bundles, the Θ_q-action on $\mathcal{F}_{(i)}$ eventually becomes intransitive. In this case invariants $\psi : \mathcal{F}_{(i)} \to \mathbb{R}$ occur that are valid for the geometric objects on $\mathcal{F}$. The Lie-Tresse Theorem [Tre94], which was proved by Kumpera [Kum75], states that the algebra of invariants is finitely generated. Using natural bundles, it is possible to compute generating sets of invariants, which gives an answer to question (3).

With the help of symmetries, integrability and invariants, it is possible to decide equivalence of geometric objects. This answers the last question (4). To compare two geometric objects ω and ω' on $\mathcal{F}$, the prolongation and projection is repeated until their symmetry equations become integrable with equivariant sections c and c'. If they coincide and the invariants are compatible, ω and ω' are equivalent. Details are found in Chapter 6.

In order to treat nontrivial examples, the Vessiot equivalence method has been implemented in Maple. The development was started by Barakat with an extension of the package `jets` [Bar01]. The author of this thesis joined in with several efficiency improvements. Furthermore, the add-on package `JetGroupoids` was created. It covers the new contributions of this thesis such as natural Θ_q-bundles and the restriction to minimal bundles. A third package, called `Spencer`, implements the computation of Spencer cohomology groups which are presented in Appendix A. All examples in this thesis can be computed with the packages `jets`, `JetGroupoids` and `Spencer`.

In this thesis, the theoretical foundation of Vessiot's equivalence method are developed. The concepts of jet groupoids and natural bundles are a flexible basis for the theory. The limitations of Vessiot's approach lie in the computational complexity of examples. On the computational side, there is still much space for improvements. In many examples, a clever choice of coordinates for the natural bundles makes the difference.

It would be interesting to compute further examples with both Cartan's and Vessiot's method in order to learn more about advantages and limitations of each approach. At this point it seems that first order problems which have a natural formulation in terms of coframes are more efficiently treated with Cartan's method. On the other hand, for higher order examples, such as LPDOs, Vessiot's approach is more promising.

This thesis starts with three introductory chapters. In Chapter 1, jet bundles and Spencer's formal theory of pdes is presented. It is based on the excellent paper of Goldschmidt [Gol67b]. Although developed by Quillen [Qui64] and recently presented by Malgrange [Mal05], the treatment of Spencer cohomology via the Koszul complex in Appendix A may be new to some readers. It is independent from special, δ-regular coordinates.

They are used both for jet groupoids and natural bundles. The concepts of Lie and jet groupoids are explained in Chapter 2 with some references to jet groups in Appendix B.

Natural bundles are treated in Chapter 3. Starting from the Lie form, it is shown how to construct a natural bundle. Furthermore the prolongation and projection is translated to the language of natural bundles. To check integrability, equivariant sections are introduced.

Chapter 4 presents applications of Vessiot's approach. The results from Chapter 3 are used to complete the symmetry equations (0.1) to formal integrability and to calculate generating sets of invariants. In this chapter, the computation of minimal bundles is presented, too.

The complete Vessiot equivalence method will be developed in Chapter 6. It depends both on symmetries and invariants. For a comparison, Cartan's equivalence method is introduced. Especially Sternberg's structure function [Ste64], which is also called torsion, has an interesting interpretation in Vessiot's context.

Finally, in Chapter 7 the Vessiot equivalence method is applied to determine generating sets of invariants for LPDOs of order three and four. The results are either given explicitly, or in electronic form if they are too large.

There is a quick tour through this thesis. The central questions are answered in Chapters 4 and 6. In more details, the references are:

- What are the symmetries of a geometric object?

  See the beginning of Section 3.3 and Section 4.1.

- Is it possible to find all invariants for a given class of geometric objects?

  See Section 4.2.

- Under which conditions are two given geometric objects equivalent?

  See Chapter 6.

- Invariants for linear partial differential operators under gauge transformations.

  See Chapter 7 (possibly skip Section 7.1.2).

Except for the part on linear partial differential operators, natural bundles are extensively used in the above sections. For an introduction to natural bundles see the following keywords and references.

- Natural bundles and geometric objects: Sections 3.1 (skip 3.1.1) and 3.3.
- Prolongation and projection with natural bundles: Section 3.4.
- Integrability conditions, Vessiot structure equations: Section 3.5 (skip Sections 3.5.1 – 3.5.4).

From the introductory chapters, the following objects are needed.

- Jet bundles: Section 1.2.
- Systems of partial differential equations, prolongation and projection: Section 1.3.
- (Lie) Groupoids: Section 2.1 and the beginning of Section 2.2.
- Jet groupoids: Section 2.3.

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Chapter 1

Geometric Formulation of Partial Differential Equations

The principal aim of this chapter is the introduction of a geometric language for systems of partial differential equations (PDEs). In order to reach this goal we restrict ourselves to definitions and short statements of properties.

The reader is expected to be well aquainted with basic concepts of commutative algebra, differential geometry and categories. Concerning commutative algebra we especially need exact sequences of modules and graduations. For an introductory textbook, see e.g. [Eis95]. The background in differential geometry includes fibre bundles, distributions, Lie groups and algebras. These topics are covered in [KMS93, Chaps. I-III], [Sha97, Chaps. 1-3] and [Ste64, Chaps. I - III,V]. For categories, basic knowledge about sequences, cohomology and functors is needed.

A geometric language for PDEs must contain concepts for functions and their derivatives as well as the differential equations themselves. Foundational work on this topic was done by Spencer [Spe69], Quillen [Qui64] and Goldschmidt [Gol67b]. Roughly speaking, the following translations are needed:

$$\begin{array}{lcl} \text{functions} & \leftrightarrow & \text{fibre bundles, sections,} \\ \text{derivatives} & \leftrightarrow & \text{jet bundles,} \\ \text{equations} & \leftrightarrow & \text{subbundles.} \end{array}$$

Section 1.1 starts with an overview on fibre bundles and exact sequences. Jet bundles and their properties are introduced in Section 1.2. In Section 1.2.1 the exact jet bundle functor is presented.

Systems of PDEs are defined in Section 1.3. The next goal is to manipulate the equations and to obtain all their differential consequences up to a certain order of derivatives. Two fundamental operations are available. Firstly, all equations can be formally differentiated yielding conditions on higher order derivatives. Geometrically this is called prolongation and involves the jet functor. Secondly, the highest order derivatives can be cancelled by appropriate combinations of the equations. This elimination process is called projection and may lead to new conditions on lower order derivatives. The following corresponcence is explained in Section 1.3.1:

formal differentiation	$\leftrightarrow$	prolongation,
elimination	$\leftrightarrow$	projection.

The prolongation and projection procedure is a main tool throughout the thesis.

If prolongation and projection of a PDE system does not produce new equations, it is called integrable. Section 1.3.2 deals with a criterion to decide integrability. It depends on the concepts of symbols and Spencer cohomology introduced in Appendix A. Here, also a MAPLE implementation to compute the Spencer cohomology using techniques of commutative algebra is presented.

1.1 Fibre Bundles

In this section, we settle the notation for fibre bundles, which are assumed to be known to the reader. We mainly follow the conventions used in [Pom78]. Precise definitions and an introduction can be found in [Pom78], [Sha97] or [Ste64]. Concerning exact sequences we closely follow a paper by Goldschmidt [Gol67b] which suits the needs perfectly.

Let X be a smooth n-dimensional manifold. If not stated otherwise, we assume all further structures to be smooth, which means C^∞. A fibre bundle $\mathcal{E}$ over the base X with projection π is denoted by $\pi : \mathcal{E} \to X$. Its local sections are $\Gamma(\mathcal{E}) = \{\omega : U \subseteq X \to \mathcal{E} \,|\, \pi \circ \omega = \mathrm{id}_U\}$. The abstract fibre of $\mathcal{E}$ is denoted by E ($E \cong \mathcal{E}_x \ \forall x \in X$). Examples of fibre bundles are the tangent bundle $T = TX$ and its dual bundle, the cotangent bundle $T^* = T^*X$.

Choose a coordinate system $x = (x^1, \dots, x^n)$ on an open subset $U \subseteq X$. Locally, $\mathcal{E}$ can be trivialised as $U \times E$. Then a coordinate system of $\mathcal{E}$ is $(x, u) = (x^1, \dots, x^n, u^1, \dots, u^m)$ where u is a coordinate system of the fibre E. We call x independent variables and u dependent variables. In these coordinates, a section ω of $\mathcal{E} \to X$ is specified by m functions $u^j = \omega^j(x)$. With sections of a bundle, we have found a geometric model for the functions occurring in a system of PDEs, where the dependent variables are considered as placeholders for the functions.

The transition between two fibred coordinate systems (x, u) and $(\hat{x}, \hat{u})$ has the form

$$\begin{aligned} \hat{x}^i &= \varphi^i(x), \\ \hat{u}^j &= \psi^j(x, u). \end{aligned} \tag{1.1}$$

It can be either seen as a transition between coordinates of the same fibre bundle or as the coordinate expression of a bundle morphism $\phi : \mathcal{E} \to \mathcal{E}$. We will use both points of view in Chapter 3 on natural bundles.

Generally, a *bundle morphism* is a smooth map $\psi : \mathcal{E} \to \mathcal{E}'$ between two fibre bundles $\mathcal{E}$ and $\pi' : \mathcal{E}' \to X'$, such that there is diffeomorphism $\varphi : X \to X'$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\psi} & \mathcal{E}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{\varphi} & X' \end{array}$$

We have the usual notion of mono-, epi- and isomorphisms of fibre bundles.

Bundles with Special Fibres

In this section we will shortly recall bundles whose fibres have additional properties like being vector spaces. See e.g. [KMS93, Ch. III] or [Sha97, Ch. 1 §3] for more details.

If the abstract fibre E of $\mathcal{E}$ is a vector space and all coordinate transformations are linear on the fibre, $\mathcal{E}$ is called a *vector bundle*. Vector bundles are denoted by capital letters E, if it is not possible with the abstract fibres.

Two vector bundles $\mathcal{E}, \mathcal{F} \to X$ over the same base give rise to the tensor bundle $\mathcal{E} \otimes \mathcal{F}$ with abstract fibre $E \otimes F$ with transition function constructed by the Kronecker product. If $\mathcal{E} = \mathcal{F}$, the symmetric q-fold tensor bundle is denoted by $S^q\mathcal{E}$, the skew-symmetric one by $\bigwedge^q \mathcal{E}$. Frequently used examples are products of the tangent or cotangent bundle like S^qT, S^qT^* and $\bigwedge^q T^*$.

The duality between T and T^* defines a pairing on the q-fold products:

$$\langle , \rangle : \bigwedge^q T \times_X \bigwedge^q T^* \to \bigwedge^0 T$$

with $\bigwedge^0 T = X \times \mathbb{R}$. There is also the well-known *interior product*

$$i : T^* \times_X \bigwedge^q T \to \bigwedge^{q-1} T,$$

which is the adjoint to the exterior product $\wedge$.

The tangent bundle $T\mathcal{E}$ of a fibre bundle $\pi : \mathcal{E} \to X$ has an important subbundle, called the *vertical bundle* $V(\mathcal{E})$. It is defined as $V(\mathcal{E}) = \ker(\pi_*)$, containing all tangent vectors whose projection under $\pi_* : T\mathcal{E} \to TX$ is zero.

Slightly more complicated than vector bundles are *affine bundles* $\mathcal{E}$, whose fibre E is an affine space and there is a vector bundle W together with a free and faithful translation action on the fibres:

$$\mathcal{E} \times_X W \to \mathcal{E} : (e, w) \mapsto e + w.$$

In this case $\mathcal{E}$ is *modelled over the vector bundle* W. If ω is a local section over $U \subseteq X$, it is possible to trivialise $\mathcal{E}$ as $\mathcal{E}|_U \cong U \times E$. If there is a global section, $\mathcal{E}$ is isomorphic to a vector bundle.

Principal bundles $\pi : P \to X$ are important examples of fibre bundles, where a Lie group G acts freely on P and the orbits are exactly the fibres.

Let F be another manifold with a left G-action. The associated bundle $P \times_G F$ is the orbit space of the diagonal action of G on $P \times F$. It is a fibre bundle with abstract fibre F (see [KMS93, 10.7]). The coordinate chances on the fibre of $P \times_G F$ are induced by the G-action, so that the bundle is called a G-bundle.

1.1.1 Exact Sequences of Fibre Bundles

For a morphism $\varphi : E \to E'$ of vector bundles with constant rank, the image, kernel and cokernel vector bundles are well-defined. So we can talk of exact sequences

$$0 \longrightarrow E' \xrightarrow{\varphi} E \xrightarrow{\psi} E'' \longrightarrow 0$$

of vector bundles, where 0 stands for the trivial vector bundle $\mathrm{id} : X \times \{0\} \to X$. As usual, the exactness condition is $\mathrm{im}(\varphi) = \ker(\psi)$. Exact sequences can be generalised to arbitrary fibre bundles, if it is possible to define a kernel. Because arbitrary fibres have no distinguished point like the origin of a vector space, it has to be specified separately by a section.

Definition 1.1. [Gol67b, Def. 2.3] A sequence of fibre bundles over X

$$\mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{E}''$$

is called *exact* if there exists a section $s'' : X \to \mathcal{E}''$ such that:

(1) The sequence of sets is exact:

$$\mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \underset{s''\circ\pi}{\overset{\psi}{\rightrightarrows}} \mathcal{E}'',$$

namely

$$\begin{aligned} (\psi \circ \varphi)(e') &= (s'' \circ \pi)(e') \qquad \forall\, e' \in \mathcal{E}', \\ \varphi(\mathcal{E}'|_x) &= \psi^{-1}(s''(x)) \qquad \forall\, x \in X. \end{aligned}$$

(2) For each $e' \in \mathcal{E}'$, the sequence of vector spaces is exact:

$$V(\mathcal{E}')_{e'} \xrightarrow{\varphi_*} V(\mathcal{E})|_{\varphi(e')} \xrightarrow{\psi_*} V(\mathcal{E}'')|_{(\psi\circ\varphi)(e')}.$$

$\diamond$

The section s'' specifies a kind of origin in the fibre $\mathcal{E}_x$ over $x \in X$. Exact sequences of vector bundles are the special case with $s'' = 0$. The definition allows to drop the assumption that all bundles have the same base X, if in $s'' \circ \pi = s'' \circ \mathrm{id}_X \circ \pi$ the identity id_X is replaced suitably. With this idea, the kernel of a bundle morphism can be defined.

Definition 1.2. If $\varphi : \mathcal{E} \to \mathcal{E}'$ is a bundle morphism and $s' : X \to \mathcal{E}$ is a section, we define the kernel $\ker_{s'}(\varphi)$ to be the set $\ker_{s'}(\varphi) = \{e \in \mathcal{E} | \varphi(e) = s(\pi(x))\}$. $\diamond$

The kernel is not necessarily a fibre bundle, but under two regularity assumptions it is a consequence of the implicit function theorem. Again, the zero object is the trivial bundle $0 = X \times \{0\}$.

Proposition 1.3. [Gol67b, Prop. 2.1] Using the notation of Definition 1.2, the image $\mathrm{im}(\varphi)$ is a subbundle of $\mathcal{E}' \to X$ if φ is locally of constant rank. If furthermore $s'(X) \subseteq \varphi(\mathcal{E})$, $\ker_{s'}(\varphi)$ is a subbundle. Then we have the exact sequence:

$$0 \longrightarrow \ker_{s'}(\varphi) \xhookrightarrow{\iota} \mathcal{E} \xrightarrow{\varphi} \mathrm{im}(\varphi) \longrightarrow 0.$$ ◇

If the bundles $\mathcal{E}'$, $\mathcal{E}$ and $\mathcal{E}''$ in Definition 1.1 are affine bundles over X, we denote the fact that $\mathcal{E}$ is modelled over the vector bundle $E = V(\mathcal{E})$ by a dashed arrow (see [Pom78, Def. 1.1.30]):

$$E \dashrightarrow \mathcal{E} \longrightarrow X.$$

Using this notation, both sequences of Definition 1.1 fit into a single diagram:

$$\begin{array}{ccccc} E' & \xrightarrow{\phi} & E & \longrightarrow & E'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}' & \xrightarrow{\varphi} & \mathcal{E} & \rightrightarrows & \mathcal{E}'' \\ \downarrow & & \downarrow & & \downarrow \\ X & = & X & = & X \end{array}$$

This diagram plays the same role as commutative exact diagrams do in homological algebra. If $e, f \in \mathcal{E}$ are in the same fibre $\mathcal{E}_x$ for $x \in X$, their difference $e - f$ is an element of E_x. We will use exact sequences of affine bundles in Section 4.3.1.

For a morphism $\varphi : \mathcal{E}' \to \mathcal{E}$ of affine bundles it is possible to define the cokernel $\mathcal{E}'' = \mathrm{coker}(\varphi)$ by an equivalence relation on the fibres (see [Gol67b, p. 276]). Let $\phi : E' \to E$ be the morphism of vector bundles corresponding to φ. Two elements $a, b \in \mathcal{E}_x$ are equivalent $a \sim b$, if and only if there exists an $e' \in E'$ such that $a + \phi(e') = b$. Define $\mathrm{coker}(\varphi)_x := \mathcal{E}_x / \sim$ and $\mathrm{coker}(\varphi) = \bigcup_{x \in X} \mathrm{coker}(\varphi)_x$. Since the projection of $\varphi(\mathcal{E}'_x)$ to $\mathrm{coker}(\varphi)_x$ is a distinguished element in each fibre, $\mathrm{coker}(\varphi)$ can be identified with $\mathrm{coker}(\phi)$.

In this thesis, exact sequences of fibre bundles are used to define partial differential equations on manifolds. But before doing so, we have to introduce jet bundles in order to have a proper language.

1.2 Jet Bundles

Jet bundles provide a geometric formulation of functions and their derivatives. The basic idea is to add coordinates standing for the derivatives of the dependent variables up to a given order.

We start with a fibre bundle $\mathcal{E} \to X$ with independent variables x and dependent variables u. To handle the first order derivatives of sections $\partial_{x^j}\omega^i(x)$ we add the coordinates u^i_j as placeholders. This process can be continued to higher orders. The result is the jet bundle $J_q(\mathcal{E}) \to \mathcal{E}$, which will be constructed in the first part of the section.

The approach of using variables as placeholders for derivatives was informally used in many calculations during the late 19th century, for example by Lie [Lie91] and Vessiot [Ves03]. Ehresmann [Ehr53] was the first who gave a formal definition of jet bundles. Nowadays, jet bundles are quite common in differential geometry (see e.g. [KMS93, Ch. IV], [Mal05, §4.2], [Olv95, Ch. 4] or [Pom78, Ch. 1.9]).

To complete the model for functions and their derivatives, we need a procedure called prolongation that continues sections of $\mathcal{E} \to X$ to sections of the jet bundle $J_q(\mathcal{E}) \to X$. The prolongation is not limited to sections, but can also be applied to transformations of $\mathcal{E}$ or bundle morphisms $\varphi : \mathcal{E} \to \mathcal{E}'$. In Section 1.2.1, we present the jet bundle functor which is a useful tool for systems of PDEs.

Construction of Jet Bundles

For the construction of jet bundles, we need multi-indices $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ with $|\mu| = \mu_1 + \cdots + \mu_n$ and $\mu! = \mu_1! \cdots \mu_n!$. They allow to write monomials as $x^\mu = (x^1)^{\mu_1} \cdots (x^n)^{\mu_n}$ and derivatives as $\frac{\partial^{|\mu|}}{\partial x^\mu} y(x) = \partial_{x^\mu} y(x) = \partial_\mu y(x)$. There is a partial ordering defined by $\nu < \mu :\Longleftrightarrow \nu_i < \mu_i$ for $i = 1, \dots, n$. We use the summation convention $u_\mu x^\mu = \sum_\mu u_\mu x^\mu$ whenever a multi-index appears twice.

Furthermore, we need an equivalence relation on the sections of $\mathcal{E} \to X$. The jet bundle then consists of all equivalence classes. Consider two sections $f, g \in \Gamma(\mathcal{E})$ and their coordinate expressions $f^i(x)$, $g^i(x)$. They have contact of order q at $x \in X$, if their values and their derivatives up to order q coincide:

$$f^i(x) = g^i(x), \quad \partial_\mu f^i(x) = \partial_\mu g^i(x) \qquad \forall\, i \leq m,\ \mu \in \mathbb{Z}^n_{\geq 0},\ 1 \leq |\mu| \leq q.$$

It is easy to prove, that this definition is independent of the coordinate system. We define an equivalence relation on the germs of sections at $x \in X$ by saying that f and g are *q-eqivalent at x*, if they have contact of order q at x. We call the equivalence class of f at x the *q-jet* of f and denote it by $j_q(f)(x)$.

Remark 1.4. A standard representative for $j_q(f)(x)$ can be given by the truncation of the Taylor series to order q:

$$f^i(x) = a^i + a^i_j x^j + \cdots + \frac{1}{\mu!} a^i_\mu x^\mu, \quad |\mu| \leq q,\ a^i_\nu \in \mathbb{R}.$$

All equivalence classes are parametrised by the finite number of variables a^i_ν. ◇

Inspired by the Taylor expansion, which can be done in any coordinate system, we now define jet bundles.

Definition 1.5. The set of all q-jets of sections at $x \in X$ is denoted by $J_q(\mathcal{E})_x$ and the *q-th jet bundle* is $J_q(\mathcal{E}) = \bigcup_{x\in X} J_q(\mathcal{E})_x$. Identify $J_0(\mathcal{E})$ with $\mathcal{E}$. $\diamond$

If f is a local section of $\mathcal{E} \to XX$, we can define a local section of the jet bundle $J_q(\mathcal{E}) \to X$ by taking the q-jets:

$$j_q : \Gamma(\mathcal{E}) \to \Gamma(J_q(\mathcal{E})) : f \mapsto (x \mapsto j_q(f)(x)).$$

This important map j_q is called *prolongation*.

It remains to show that $J_q(\mathcal{E})$ is a finite-dimensional fibre bundle. We will do this by defining fibre coordinates and transition functions based on coordinate changes of $\mathcal{E}$.

Proposition 1.6. $\pi_0^q : J_q(\mathcal{E}) \to \mathcal{E}$ is a fibre bundle. Furthermore there are projections

$$\begin{aligned} \pi^q_{q-i} &: J_q(\mathcal{E}) \to J_{q-i}(\mathcal{E}), \qquad \forall i \leq q, \\ \pi &: J_q(\mathcal{E}) \to X, \end{aligned}$$

turning $J_q(\mathcal{E})$ into a bundle over lower order jet bundles and over X. The dimension of $J_q(\mathcal{E})$ is $n + m\binom{q+n}{n}$. $\diamond$

Proof (Sketch). A coordinate system (x, u) of $\mathcal{E}$ is extended to $J_q(\mathcal{E})$ by adding the *jet coordinates* u^i_μ for $\mu \in \mathbb{Z}^n_{\geq 0}$, $|\mu| \leq q$ as placeholders for the derivatives of u^i. In coordinates, the q-jet of a germ of a section f of $\mathcal{E} \to X$ has the form:

$$j_q(f)(x) = (x,\, u = f^i(x),\, u^i_j = \partial_j f^i(x),\, \dots ,\, u^i_\mu = \partial_\mu f^i(x)), \quad |\mu| \leq q.$$

Let $(\hat{x}, \hat{u})$ be another coordinate system of $\mathcal{E}$ with transition functions

$$\begin{aligned} \hat{x}^i &= \varphi^i(x^j), \\ \hat{u}^i &= \psi^i(x^j, u^k). \end{aligned}$$

Construct the transition functions for $\hat{u}^i_\mu$ by repeated use of the chain rule:

$$\begin{aligned} \hat{u}^i_j\, \partial_l\varphi^j(x) &= \frac{\partial\psi^i(x,u)}{\partial u^j} u^j_l + \partial_l\psi^i(x,u) \\ \hat{u}^i_{jk}\, \partial_l\varphi^j(x)\, \partial_p\varphi^k(x) + \hat{u}^i_j\, \partial_l\partial_p\varphi^j(x) &= \frac{\partial\psi^i(x,u)}{\partial u^j} u^j_{lp} + \text{ lower order} \qquad (1.2) \\ &\;\;\vdots \end{aligned}$$

This continuation of a coordinate transformation from $\mathcal{E}$ to $J_q(\mathcal{E})$ is also called *prolongation*. Equation (1.2) is given in a form convenient for formal differentiation, but still has to be solved for the new coordinates $\hat{u}^i_\mu$. Since coordinate expressions of f transform like $\hat{f}^i(\hat{x}) = \psi^i(x, f(x))$, it is clear, that the $\hat{u}^i_\mu$ transform like the derivatives of f with respect to $\hat{x}$.

Abbreviate all jet coordinates u^i_μ of strict order $|\mu| = q$ by u_q. The transition functions for jets of order q depend on jets of order $\leq q$ only. It follows that the canonical projections

$$\pi^q_{q-i} : J_q(\mathcal{E}) \to J_{q-i}(\mathcal{E}) : (x, u, \ldots, u_{q-i}, \ldots, u_q) \mapsto (x, u, \ldots, u_{q-i})$$

and

$$\pi : J_q(\mathcal{E}) \to X : (x, u, u_q) \mapsto x$$

are well defined and coordinate independent, thus completing the fibre bundle structures over J_{q-i} and X. The dimension formula is obtained by counting the number of derivatives up to order q. □

Whenever (x, u) is a coordinate system of $\mathcal{E}$, we now have a coordinate system $(x, u, u^i_\mu,\ |\mu| \leq q)$ for its q-th order jet bundle. The coordinates u^i_μ are also called *jet variables* or simply jets. The order of a jet u^i_μ is $|\mu|$, where the dependent variables are included as zero order jets $u^i = u^i_{\mu=0}$. Whenever possible, we abbreviate a coordinate system of $J_q(\mathcal{E})$ by (x, u, u_q), where u_q stands for all jets up to order q.

In the literature on this subject (see e.g. [KMS93, §12]), a different specification of jet spaces can be found. It is easy to see, that they are only a special case of Definition 1.5.

Remark 1.7. Let X, Y be manifolds and $U \subseteq X$ be an open subset. Another possibility to construct a jet space is to consider smooth maps $f : U \to Y$. The resulting space $J_q(X, Y)$ contains all q-jets of maps $X \to Y$. Obviously each smooth map $U \to Y$ can be seen as a local section of the trivial bundle $\mathcal{E} = X \times Y$ and thus $J_q(X, Y) \cong J_q(X \times Y)$. ◇

Properties of Jet Bundles

Having established that $J_q(\mathcal{E})$ is a fibre bundle, we recall well-known properties of $J_q(\mathcal{E})$ that are necessary for the following chapters. At first, we turn to coordinates and sections of $J_q(\mathcal{E}) \to X$.

Remark 1.8. All jet coordinates u^i_μ, including the dependent variables u^i, are treated in the same way as the independent variables. Especially, they are functionally independent:

$$\frac{\partial u^i}{\partial x^j} = 0, \quad \frac{\partial u^i_\mu}{\partial x^j} = 0, \quad \frac{\partial u^i_\mu}{\partial u^j_\nu} = \delta^i_j \delta^\nu_\mu.$$

Unlike a section $u^i = f^i(x)$ and its derivatives, the jet coordinates do not depend on the independent varibles x. To simulate the x-dependence, we introduce the total derivative in Definition 1.11. ◇

Remark 1.9. The coordinate expression of a section f_q of $J_q(\mathcal{E}) \to X$ is specified by functions $u^i_\mu = f^i_\mu(x)$, which may all be chosen independently. An important subset of these sections are the prolongations $j_q(f)$ of sections $f : X \to \mathcal{E}$. They satisfy the differential equations

$$f^i_{\mu+1_j}(x) = \partial_j f^i_\mu(x), \qquad |\mu| < q$$

because $j_q(f)(x)$ has the coordinate expressions $u^i_\mu = \partial_{x^\mu} f^i(x)$. Obviously, most sections of $J_q(\mathcal{E}) \to X$ are not prolonged from a $f \in \Gamma(\mathcal{E})$. The distance between a section f_q and the prolongation of $f = \pi^q_0(f_q)$ is measured by means of the Spencer operator that will be introduced in Proposition C.12. ◇

We now list general properties of the jet bundles.

Proposition 1.10. Let $\mathcal{E} \to X$ and $\mathcal{E}' \to X$ be fibre bundles.

(1) $J_q(\mathcal{E})$ is an affine bundle over $J_{q-1}(\mathcal{E})$, modelled over $S^qT^* \otimes V(\mathcal{E})$.

(2) If $\mathcal{E} \to X$ is a vector bundle, then $J_q(\mathcal{E}) \to X$ is also a vector bundle.

(3) The prolongation of the trivial bundle $0 = X \times \{0\} \to X$ can be identified with 0 again: $J_q(0) \cong 0$. In exact sequences of fibre bundles, 0 plays the role of the zero object.

(4) The jet bundle of fibre products $\mathcal{E} \times_X \mathcal{E}'$ is naturally isomorphic to:

$$J_q(\mathcal{E} \times_X \mathcal{E}') \cong J_q(\mathcal{E}) \times_X J_q(\mathcal{E}').$$

(5) $J_q(V(\mathcal{E}))$ and $V(J_q(\mathcal{E}))$ are naturally isomorphic. ◇

The first three properties follow directly from the coordinate expressions in equation (1.2) and (4) is a consequence of the jet bundle functor which will be defined in Section 1.2.1. For detailed proofs, we refer to [KMS93, §12.11-17] and [Pom78, La. 1.9.12].

The Total Derivative

The total derivative is the link between jet variables and their interpretation as representatives of derivatives. It is defined in coordinates (x, u, u_q) of $J_q(\mathcal{E})$ and treats a jet u^j_μ like the derivative $\partial_\mu u^j$ of an x-dependent function u^j. So the total derivative D_{x^i} must satisfy:

$$D_{x^i}\, u^j_\mu = u^j_{\mu+1_i}.$$

With the help of the total derivative, coordinate changes and morphisms of bundles can be prolonged to the jet bundles.

Definition 1.11. Let (x, u) be a coordinate system of $\mathcal{E}$. The i-th *total derivative* $D_i = D_{x^i}$ is the first order differential operator

$$D_{x^i} : \Gamma(J_q(\mathcal{E}) \times \mathbb{R}) \to \Gamma(J_{q+1}(\mathcal{E}) \times \mathbb{R})$$

defined by:

$$D_{x^i} := \frac{\partial}{\partial x^i} + u^j_{\mu+1_i} \frac{\partial}{\partial u^j_\mu}, \quad 0 \le |\mu| \le q. \tag{1.3}$$

◇

An element of $\Gamma(J_q(\mathcal{E}) \times \mathbb{R})$ is a real-valued function $\Phi(x, u, u_q)$ depending on jets up to order q. Plugging the $q+1$-jet of a section $f \in \Gamma(\mathcal{E})$ into the total derivative of Φ,

$$D_i\Phi(x, u, u_q) = \frac{\partial \Phi}{\partial x^i}(x, u, u_q) + u^j_{\mu+1_i} \frac{\partial \Phi}{\partial u^j_\mu}(x, u, u_q),$$

yields the same as the formal differentiation of $\Phi(x, f(x), \partial_\nu f(x))$ by x^i:

$$\frac{d}{dx^i}\Phi(x, f(x), \partial_\nu f(x)) = \frac{\partial \Phi}{\partial x^i}(x, f(x), \partial_\nu f(x)) + \partial_{\mu+1_i} f^j(x) \frac{\partial \Phi}{\partial f^j_\mu(x)}(x, f, \partial_\nu f(x)).$$

The total derivative D_i implements the chain rule for sections of $\mathcal{E}$. The next lemma states that all D_i are commuting first order differential operators, which can be proved by direct calculation (using equation (1.3)).

Lemma 1.12. [Pom78, §2.1] The total derivative transforms like $D_{x^i} = \frac{\partial \hat{x}^i}{\partial x^j} D_{\hat{x}^j}$ under a coordinate change $\hat{x} = \hat{x}(x)$, $\hat{u} = \hat{u}(x, u)$. For $\Phi_k \in \Gamma(J_q(\mathcal{E}) \times \mathbb{R})$, we have the following properties:

(1) $D_i(\Phi_1 + \Phi_2) = D_i(\Phi_1) + D_i(\Phi_2)$,

(2) $D_i(\Phi_1 \cdot \Phi_2) = D_i(\Phi_1) \cdot \Phi_2 + \Phi_1 \cdot D_i(\Phi_2)$,

(3) $D_i \circ D_j = D_j \circ D_i$. ◇

Using the last property, we define the product $D_\nu := D_1^{\nu_1} \cdots D_n^{\nu_n}$ for a multi-index $\nu \in \mathbb{Z}^n_{\ge 0}$. With the help of the total derivative, the prolongation of transition functions in equation (1.2) can be rewritten as:

$$\begin{aligned} \hat{u}^i_j\, \partial_l \varphi^j(x) &= D_l\, \psi^i(x, u), \\ \hat{u}^i_{jk}\, \partial_l \varphi^j(x)\, \partial_p \varphi^k(x) + \hat{u}^i_j\, \partial_l \partial_p \varphi^j(x) &= D_p D_l\, \psi^i(x, u), \\ &\vdots \end{aligned} \tag{1.4}$$

This prolongation still suffers from the disadvantage that the equations have to be solved for the new coordinates $\hat{u}^i_\mu$ in order to write the coordinate change in the form of equation (1.1). Vessiot's notation avoids this problem.

Remark 1.13. In many applications it is convenient to modify the transition functions of equation (1.1). Inverting the transformations on the fibres of $\mathcal{E}$, we obtain the coordinate changes in Vessiot's notation [Ves03, eq. (36)]:

$$\begin{array}{rcl} \hat{x}^i & = & \varphi^i(x), \\ u^j & = & \psi^j(\hat{x}, \hat{u}). \end{array}$$

In practice, the formulae are often shorter than in the usual notation. The advantage of Vessiot's notation is that the prolongation to $J_q(\mathcal{E})$ is very simple:

$$u^j_\mu \quad = \quad D_{x^\mu}\psi^j(\varphi(x), \hat{u}), \qquad |\mu| \le q.$$

It involves the generalised total derivative:

$$D_{x^i} = \frac{\partial}{\partial x^i} + \hat{u}^k_{\nu+1_l}\, \partial_i \varphi^l(x) \frac{\partial}{\partial \hat{u}^k_\nu} = \frac{\partial}{\partial x^i} + \hat{u}^k_{\nu+1_l}\, \varphi^l_i(x) \frac{\partial}{\partial \hat{u}^k_\nu}.$$

D_{x^i} implements the chain rule for functions $\hat{u}$ depending on $\hat{x} = \varphi(x)$. The prolonged transition functions are automatically written in Vessiot's notation. All examples of natural bundles in Chapter 3 will be presented in this way. $\diamond$

1.2.1 Prolongation and the Jet Bundle Functor

As we have seen, coordinate changes of $\mathcal{E} \to X$ can be prolonged to $J_q(\mathcal{E}) \to X$. Interpreting them as local diffeomorphisms $\mathcal{E} \to \mathcal{E}$, it is natural to ask if morphisms of fibre bundles can be prolonged, too. The current section gives a positive answer. It also allows to define the jet bundle functor $J_q()$ on the category of fibre bundles over X with bundle morphisms. We follow the exposition in [Gol67b, §4] and shortly recapitulate basic properties of J_q including its exactness. Proofs are omitted.

Proposition 1.14. Let $\varphi : J_k(\mathcal{E}) \to \mathcal{E}'$ be a morphism of fibre bundles over X. There exists a unique morphism $p_q(\varphi) : J_{k+q}(\mathcal{E}) \to J_q(\mathcal{E}')$ such that the following diagram on the sections commutes:

$$\begin{array}{ccc} \Gamma(J_{k+q}(\mathcal{E})) & \xrightarrow{p_q(\varphi)} & \Gamma(J_q(\mathcal{E}')) \\ \uparrow{\scriptstyle j_{k+q}} & & \uparrow{\scriptstyle j_q} \\ \Gamma(\mathcal{E}) & \xrightarrow{\varphi \circ j_k} & \Gamma(\mathcal{E}') \end{array}$$

The map $p_q(\varphi)$ is called the *q-th prolongation* of φ. In the special case of $k = 0$, $p_q(\varphi)$ is also denoted by $J_q(\varphi)$. $\diamond$

Whenever we talk of a unique prolongation and refer to this proposition, the uniqueness is provided by the above commutative diagram. Alternatively, we call the prolongation compatible with j_q.

The prolongation is calculated by interpreting equation (1.1) as a morphism of bundles where (x, u) are coordinates of $\mathcal{E}$ and $(\hat{x}, \hat{u})$ are coordinates of $\mathcal{E}'$. The prolongation is computed by equation (1.4).

The next Proposition recalls basic and very useful properties of the prolongation of morphisms.

Proposition 1.15. (1) The prolongation of the identity map $\mathrm{id}_q : J_q(\mathcal{E}) \to J_q(\mathcal{E})$ is the natural embedding

$$p_r(\mathrm{id}_q) : J_{q+r}(\mathcal{E}) \hookrightarrow J_r(J_q(\mathcal{E})).$$

(2) Composition of morphisms commutes with prolongation. Two morphisms $\varphi : J_q(\mathcal{E}) \to \mathcal{E}'$ and $\psi : J_r(\mathcal{E}') \to \mathcal{E}''$ fulfill:

$$p_s(\psi \circ p_r(\varphi)) = p_s(\psi) \circ p_{r+s}(\varphi) \qquad \forall s \in \mathbb{Z}_{\geq 0}.$$

(3) The following diagram commutes for all $r, s \in \mathbb{Z}_{\geq 0}$:

$$\begin{array}{ccc} J_{q+r+s}(\mathcal{E}) & \xrightarrow{p_{r+s}(\varphi)} & J_{r+s}(\mathcal{E}') \\ \downarrow{\scriptstyle p_s(\mathrm{id}_q)} & \searrow{\scriptstyle p_s(p_r(\varphi))} & \downarrow{\scriptstyle p_s(\mathrm{id}_r)} \\ J_s(J_{q+r}(\mathcal{E})) & \xrightarrow{J_s(p_r(\varphi))} & J_s(J_r(\mathcal{E}')) \end{array}$$

◇

The natural embedding (1) is very simple in coordinates. If $(u_\mu)_\nu$ with $|\mu| \leq q$ and $|\nu| \leq r$ is an r-jet in $J_r(J_q(\mathcal{E}))$, the canonical embedding just symmetrises the indices by setting $(u_\mu)_\nu = u_{\mu+\nu}$.

Having defined J_q on objects and morphisms, the functorial property $J_q(\varphi \circ \psi) = J_q(\varphi) \circ J_q(\psi)$ is a special case of the composition (2). Effectively, it is a consequence of the chain rule. As a last step, we state the exactness of J_q.

Proposition 1.16. Let $\varphi : \mathcal{E} \to \mathcal{E}'$ be a morphism of fibre bundles over X, which is locally of constant rank. Then $p_q(\varphi) : J_q(\mathcal{E}_1) \to J_q(\mathcal{E}_2)$ is a morphism of bundles over X of locally constant rank. If φ is a mono- or epimorphism, then $p_q(\varphi)$ is also a mono- or epimorphism.

If $s' : X \to \mathcal{E}'$ is a section with $s(X) \subseteq \varphi(\mathcal{E})$, then $j_q(\varphi) \subseteq p_q(\varphi)(J_q(\mathcal{E}))$ and the sequence

$$0 \longrightarrow J_q(\ker_{s'}(\varphi)) \xrightarrow{p_q(\iota)} J_q(\mathcal{E}) \xrightarrow{p_q(\varphi)} J_q(\mathcal{E}')$$

is exact with $p_q(\varphi) \circ p_q(\iota) = j_q(s') \circ \pi$, where $\iota : \ker_{s'}(\varphi) \to \mathcal{E}$ is the embedding.◇

The main benefit we derive from jet bundles are systems of partial differential equations, which will be defined in Section 1.3. Whenever possible, a system of partial differential equations is described by an exact sequence of bundles. In this case the exactness of J_q implies that the formal differentiation of the system is again specified by an exact sequence.

Prolongation of Vector Fields

Before turning to systems of PDEs, we finish the section on jet bundles by introducing the prolongation of vector fields. It is needed in Appendix C.2 and in Chapter 3. We follow the book of Olver [Olv95, Ch. 4].

Vector fields can be interpreted as infinitesimal local diffeomorphisms $\mathcal{E} \to \mathcal{E}$ and their prolongation is the infinitesimal analogue to the prolongation of maps $\varphi : \mathcal{E} \to \mathcal{E}$. In this thesis, vector fields occur as infinitesimal generators of a Lie group action on a manifold or, as a generalisation, as infinitesimal generators of a Lie groupoid action on a bundle (see Section 2.2). The prolongation of both finite actions becomes too large to compute even in Vessiot's notation. The prolongation of the generating vector fields remains small enought for efficient computations.

Let us first illustrate the prolongation of a vector field with an example.

Example 1.17. Let $\mathcal{E} = \mathbb{R} \times \mathbb{R}$ with global coordinates (x, u) be the trivial bundle. On $\mathcal{E}$, the prolongation of bundle morphisms can be generalised to arbitrary diffeomorphisms:

$$\hat{x} = \varphi(x, u), \qquad \hat{u} = \psi(x, u).$$

It is done by replacing the partial derivatives on the left hand sides of equation (1.4) by total ones:

$$\hat{u}^i_j \, D_{x^l} \varphi^j(x, u) = D_{x^l} \psi^i(x, u), \, \ldots$$

The one-parameter group of transformations (which are no bundle morphism)

$$\begin{aligned} \hat{x} &= x \cos(\varepsilon) - u \sin(\varepsilon), \\ \hat{u} &= x \sin(\varepsilon) + u \cos(\varepsilon) \end{aligned}$$

is generated by the vector field:

$$v = \left.\frac{d\hat{x}}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial x} + \left.\frac{d\hat{u}}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial u} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}.$$

Its prolongation to $J_q(\mathcal{E})$,

$$\begin{aligned} \hat{u}_{\hat{x}} &= \frac{D_x \hat{y}}{D_x \hat{x}} = \frac{\sin(\varepsilon) + u_x \cos(\varepsilon)}{\cos(\varepsilon) - u_x \sin(\varepsilon)}, \\ \hat{u}_{\hat{x}\hat{x}} &= \frac{D_x \hat{u}_{\hat{x}}}{D_x \hat{x}} = \frac{u_{xx}}{(\cos(\varepsilon) - u_x \sin(\varepsilon))^3} \end{aligned}$$

remains a one-parameter group of transformations, but this time on $J_q(\mathcal{E})$. Differentiating again, we obtain its infinitesimal generator $\rho_2(v)$:

$$\begin{aligned} \rho_2(v) &= \left.\frac{d\hat{x}}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial x} + \left.\frac{d\hat{u}}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial u} + \left.\frac{d\hat{u}_{\hat{x}}}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial u_x} + \left.\frac{d\hat{u}_{\hat{x}\hat{x}}}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial u_{xx}} \\ &= -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x} + 3 u_x u_{xx} \frac{\partial}{\partial u_{xx}}. \end{aligned} \tag{1.5}$$

◇

In the example, the prolongation of a vector field was computed by prolonging the corresponding finite transformations. It is possible to prolong a vector field without integrating it, using the following definition.

Definition 1.18. [Olv95, p. 117] The *prolongation* of a vector field v on $\mathcal{E}$, given in local coordinates (x, u) by

$$v = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^j(x,u)\frac{\partial}{\partial u^j},$$

is defined as

$$\rho_q(v) = D_\mu Q^j \frac{\partial}{\partial u^j_\mu} + \xi^i D_i \quad \in \mathfrak{X}J_q(\mathcal{E}),$$

where $Q^j = \eta^j - u^j_i \xi^i$ denotes the *characteristic* of v and $\mu \in \mathbb{Z}^n_{\geq 0}$ cycles through all multi-indices with $|\mu| \leq q$. ◇

The additional term $\xi^i D_i$ cancels the jets of order $q+1$ occurring in $D_\mu Q^j$. According to [Olv95, Thm. 4.16] it is the infinitesimal version of the prolongation of transition functions.

Example 1.19. The vector field $v = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$ from Example 1.17 has the characteristic $Q = x + uu_x$ and the second prolongation:

$$\begin{aligned}\rho_2(v) &= Q\frac{\partial}{\partial u} + D_x Q\frac{\partial}{\partial u_x} + (D_x)^2 Q\frac{\partial}{\partial u_{xx}} - u\,D_x \\ &= (x+uu_x)\frac{\partial}{\partial u} + (1+u_x^2+uu_{xx})\frac{\partial}{\partial u_x} + (3u_xu_{xx}+uu_{xxx})\frac{\partial}{\partial u_{xx}} - uD_x \\ &= -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x} + 3u_xu_{xx}\frac{\partial}{\partial u_{xx}}. \qquad (1.6)\end{aligned}$$

◇

We have seen in Proposition 1.15 (2) that the prolongation of diffeomorphisms respects composition, which is translated in terms of the Lie brackets to vector fields. It will be used in Section C.2.3 to construct a Lie bracket on $J_q(T)$.

Lemma 1.20. Two vector fields v, w of $\mathcal{E}$ satisfy:

$$[\rho_q(v), \rho_q(w)] = \rho_q([v,w]).$$

◇

1.3 Systems of Partial Differential Equations

In this section, we define a system of partial differential equations in a geometric language. Jet bundles provide a geometric model for functions and derivatives and a system of PDEs is simply a subbundle of a jet bundle. This so-called formal theory of PDEs was introduced by Spencer [Spe69], Quillen [Qui64] in the linear and Goldschmidt [Gol67b] in the nonlinear case.

Starting with the definition of a PDE system $\mathcal{R}_q$, we turn to the fundamental operations of prolongation and projection in Section 1.3.1. The prolongation of $\mathcal{R}_q$ uses the jet bundle functor and the total derivative. It is equivalent to the formal differentiation of all equations defining $\mathcal{R}_q$. In terms of equations, the projection involves the elimination of highest order derivatives. Geometrically, the projection is more simple as it only involves the jet bundle projections $\pi_q^{q+r} : J_{q+r}(\mathcal{E}) \to J_q(\mathcal{E})$ which forgets the highest order jet coordinates.

Section 1.3.2 describes formally integrable systems where the prolongation and projection will not produce new equations. It depends on Appendix A symbols and Spencer cohomology are introduced.

Definition 1.21. [Gol67b, Def. 7.1] Let $\mathcal{E} \to X$ be a fibre bundle. A *system of partial differential equations* $\mathcal{R}_q \to X$ of order q is a subbundle of $J_q(\mathcal{E}) \to X$. A *solution* of $\mathcal{R}_q$ is a local section $f : U \subseteq X \to \mathcal{E}$ such that $j_q(f)(x) \in \mathcal{R}_q$ for all $x \in U$. If $\mathcal{E}$ and $\mathcal{R}_q$ are vector bundles, the system of PDEs is called *linear* and otherwise *nonlinear*. ◇

Analogous to the notation for vector bundles, a linear system of PDEs is denoted by $R_q \subseteq J_q(E)$. This notation becomes useful when treating a nonlinear system $\mathcal{R}_q$ and its linearisation R_q.

Let us verify that the above definition is nothing else than a geometric formulation of PDEs on manifolds. Choose a coordinate system (x, y, y_q) of $J_q(\mathcal{E})$. The system $\mathcal{R}_q$ is a subbundle of $J_q(\mathcal{E}) \to X$, so there is another coordinate system (x, Φ) of $J_q(\mathcal{E})$ such that $\mathcal{R}_q$ is given by $\Phi^1 = \ldots = \Phi^k = 0$. Expressing Φ^i in the old coordinates (x, y, y_q), we obtain equations

$$\Phi^1(x, y, y_q) = \ldots = \Phi^k(x, y, y_q) = 0,$$

locally defining $\mathcal{R}_q$. So a subbundle introduces relations between the jet variables. The definition of a solution reveals the differential equations. Each solution f of $\mathcal{R}_q$ has to satisfy $j_q(f)(x) \in \mathcal{R}_q$. The conditions are PDEs in the usual sense:

$$\Phi^1(x, f(x), \partial_\mu f(x)) = \ldots = \Phi^k(x, f(x), \partial_\mu f(x)) = 0.$$

Exact sequences of fibre bundles are a very convenient way of specifying a system of PDEs, because the prolongation involves little more than the jet bundle functor. At first we need differential maps and operators (cf. [Pom78, Def. 2.1.1]).

Definition 1.22. A bundle morphism $\Phi : J_q(\mathcal{E}) \to \mathcal{F}$ over X of locally constant rank is called *differential map* of order q. The map

$$\mathcal{D} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$$

defined on the sheaf of sections is called *differential operator*, if there exists a $q \in \mathbb{N}$ and a differential map Φ, such that $\mathcal{D} = \Phi \circ j_q$. If Φ is a vector bundle morphism, $\mathcal{D}$ is called *linear* and otherwise *nonlinear*. ◇

As a direct consequence of Proposition 1.3, a differential map Φ together with a suitable section $\omega : X \to \mathcal{F}$ ($\omega(X) \subseteq \mathrm{im}(\Phi)$) defines a system of PDEs $\mathcal{R}_q(\omega)$:

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow J_q(\mathcal{E}) \underset{\omega\circ\pi}{\overset{\Phi}{\rightrightarrows}} \mathcal{F}. \qquad (1.7)$$

In coordinates (x, y, y_q) of $J_q(\mathcal{E})$ and (x, u) of $\mathcal{F}$, the differential map Φ is expressed as $u^\alpha = \Phi^\alpha(x, y, y_q)$ and the equations defining $\mathcal{R}_q(\omega)$ are

$$\Phi^\alpha(x, y, y_q) = \omega^\alpha(x), \qquad 1 \leq \alpha \leq k. \qquad (1.8)$$

Essentially, $\mathcal{R}_q(\omega) = \ker_\omega(\Phi)$ is the kernel of Φ with respect to ω. Note that another choice of the section ω specifies a PDE system with a different right hand side.

Not all PDE systems $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ have a kernel representation and Goldschmidt [Gol67b, §7] gave a topological restriction. In this thesis, we are only interested in local questions and on a single coordinate neighbourhood it is possible to construct a kernel representation.

Example 1.23. On $\mathcal{E} = X \times X$, $X = \mathbb{R}^n$, with coordinates (x, y), the equations for the second order jets

$$y^i_{jk} = 0, \qquad 1 \leq i, j, k \leq n$$

define the subbundle $\mathcal{R}_2 \subset J_2(\mathcal{E}) \to X$ which is isomorphic to $J_1(\mathcal{E})$. Solutions of $\mathcal{R}_2$ are affine transformations $X \to X$:

$$y^i(x) = a^i_j x^j + b^i, \qquad a^i_j, b^i \in \mathbb{R}.$$

The differential map Φ for a kernel representation of $\mathcal{R}_2$ is:

$$\Phi : J_2(\mathcal{E}) \to X \times \mathbb{R}^d : (x, y, y^i_j, y^i_{jk}) \to (x, y^i_{jk}),$$

where d is the number of second order derivatives. As right hand side for Φ, the zero section $\omega = 0$ has to be used.

In Section 2.3, we restrict ourselves to the invertible jets $\Pi_2(X \times X)$, where all solutions must have nonzero determinant $\det(a^i_j)$. We observe that the composition of two affine transformations is affine and thus also a solution of $\mathcal{R}_2$. ⋄

1.3.1 Prolongation and Projection

In this section we will define the prolongation and projection of PDE systems and give examples. If a system $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ is given by equation (1.8), we use the total derivative to compute the additional equations for higher order jets:

$$D_\mu \Phi^\alpha(x, y, y_{q+r}) = \partial_\mu \omega^\alpha(x), \qquad |\mu| \leq r,\ 1 \leq \alpha \leq k. \qquad (1.9)$$

We expect that the prolongation $\mathcal{R}_{q+r}$ of $\mathcal{R}_q$ satisfies these equations. To project to a lower order, it is necessary to eliminate the highest order jets from the equations. The step to order $q + r - 1$ is easy, because all jets of strict order $q + r$ occur linearly. Further projections may become more complicated.

Definition 1.24. [Gol67b, Def. 7.1] The *r-prolongation* of a PDE system $\mathcal{R}_q$ is the subset

$$\mathcal{R}_{q+r} = \rho_r(\mathcal{R}_q) := J_r(\mathcal{R}_q) \cap J_{q+r}(\mathcal{E}).$$

$\mathcal{R}_{q+r}$ is called *regular* if it is a subbundle of $J_{q+r}(\mathcal{E}) \to X$. The set

$$\mathcal{R}_{q+r}^{(s)} := \pi_{q+r}^{q+r+s}(\mathcal{R}_{q+r+s}) \subseteq \mathcal{R}_{q+r}$$

is called *projection*. ◇

Note that due to the intersection with $J_{q+r}(\mathcal{E})$, both prolongation and projection are only subsets of jet bundles but not necessarily subbundles.

Locally, it is again possible to work with exact sequences. If the system $\mathcal{R}_q = \ker_\omega(\Phi)$ is specified by a kernel, then $\mathcal{R}_{q+r}$ is defined by the kernel of the prolonged map $p_r(\Phi)$ with respect to $j_r(\omega)$ and we have the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{R}_{q+r} & \longrightarrow & J_{q+r}(\mathcal{E}) & \overset{p_r(\Phi)}{\underset{j_r(\omega)\circ\pi}{\rightrightarrows}} & J_r(\mathcal{F}) \\ & & \Big\downarrow \pi_q^{q+r} & & \Big\downarrow \pi_q^{q+r} & & \Big\downarrow \\ 0 & \longrightarrow & \mathcal{R}_q & \longrightarrow & J_q(\mathcal{E}) & \overset{\Phi}{\underset{\omega\circ\pi}{\rightrightarrows}} & \mathcal{F} \end{array} \tag{1.10}$$

Another possibility to obtain this diagram, is applying the jet functor J_q to the sequence for $\mathcal{R}_q$. The intersection with $J_{q+r}(\mathcal{E})$ is done with the canonical embedding $J_{q+r}(\mathcal{E}) \hookrightarrow J_r(J_q(\mathcal{E}))$.

In coordinates, the prolongation $p_r(\Phi)$ is computed by equation (1.4), which simplifies to

$$u_\mu^\alpha = D_\mu \Phi^\alpha(x, y, y_{q+r}),$$

because the projection of Φ to the base is the identity map $\varphi = \mathrm{id}_X$. As expected, the equations (1.9) are added to the system.

The next example (compare [Sei02, Ex. 1.3.3]) shows a PDE system where the prolongation is no longer a bundle.

Example 1.25. On the trivial bundle $\mathcal{E} = X \times \mathbb{R}$ over the base $X = \mathbb{R}^2$ we use the global coordinate system (x, y) of X and (x, y, u) of $\mathcal{E}$. Define the nonlinear second order system $\mathcal{R}_2$ by the equations

$$u_{yy} - \frac{1}{2}u_{xx}^2 = 0, \qquad u_{xy} - u_{xx} = 0.$$

They induce a differential map $\Phi : J_2(\mathcal{E}) \to \mathcal{F} = X \times \mathbb{R}^2$:

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \mapsto (x, y, u_{yy} - \frac{1}{2}u_{xx}^2, u_{xy} - u_{xx}),$$

which is locally of constant rank. The section ω of $\mathcal{F} \to X$ corresponding to $\mathcal{R}_2$ is the zero section. Apply the total derivative for the first prolongation $\mathcal{R}_3$:

$$\begin{aligned} p_1(\Phi) : J_3(\mathcal{E}) &\to J_1(\mathcal{F}) \\ (x, y, u, u_3) &\mapsto (\ldots, u_{xyy} - u_{xx}u_{xxx},\, u_{yyy} - u_{xx}u_{xxy}, \\ &\qquad u_{xxy} - u_{xxx},\, u_{xyy} - u_{xxy}). \end{aligned}$$

Again, the zero section of $J_1(\mathcal{F}) \to X$ defines $\mathcal{R}_3$. From the last two equations of $\mathcal{R}_3$, one deduces $u_{xyy} = u_{xxx}$ and the first equation becomes $(1 - u_{xx})u_{xxx}$. $\mathcal{R}_3$ consists of the union of two submanifolds of $J_3(\mathcal{E})$.

The rank of $p_1(\Phi)$ drops by one for $u_{xx} = 1$ or $u_{xxx} = 0$, so the the prolongation of Φ to $p_1(\Phi) : J_3(\mathcal{E}) \to J_1(\mathcal{F})$ from Proposition 1.15 does not preserve exactness. In contrast to that, Proposition 1.16 shows that the functor J_1 is exact. Applying J_1 yields a different map $p_1(\Phi) : J_1(J_2(\mathcal{E})) \to J_1(\mathcal{F})$. Here jets like $D_x u_{xy} = u_{xy,x}$ and $D_y u_{xx} = u_{xx,y}$ have to be distinguished so that the rank drop is impossible. ◇

Let us turn to the projection. In theory, it is simply the application of π_q^{q+r} to the prolonged system $\mathcal{R}_{q+r}$. In terms of equations, there is more work to do, as it includes the elimination of all jets of order $\geq q$. For linear systems the projection can be done with Gaussian elimination.

Example 1.26. With $\mathcal{E}$, X as in Example 1.25, we consider a second order linear system:

$$\mathcal{R}_2 : \begin{cases} u_{xx} - u_x = 0, \\ u_{xy} - u = 0. \end{cases}$$

Its first prolongation is:

$$\mathcal{R}_3 : \begin{cases} u_{xx} - u_x = 0, \\ u_{xy} - u = 0, \\ u_{xxx} - u_{xx} = 0, \\ u_{xxy} - u_{xy} = 0, \\ u_{xxy} - u_x = 0, \\ u_{xyy} - u_y = 0. \end{cases}$$

The projection back to second order contains one new equation:

$$\mathcal{R}_2^{(1)} : \begin{cases} u_{xx} - u_x = 0, \\ u_{xy} - u = 0, \\ u_x - u = 0. \end{cases}$$

It is obtained by eliminating first u_{xxy} in the third order equations and then u_{xy} among the second order equations. Another prolongation and projection restricts the system further:

$$\mathcal{R}_2^{(2)} : \begin{cases} u_{xx} - u = 0, \\ u_{xy} - u = 0, \\ u_x - u = 0, \\ u_y - u = 0. \end{cases}$$

Now the question arises if the process of prolongation and projection leads to more equations and when to stop. In this example, one last step produces the equation $u_{yy} - u = 0$ for $\mathcal{R}_2^{(3)}$. Any new equation of order ≤ 2 must have zero order. This would imply $u = const$, but $u = e^{x+y}$ is a solution of $\mathcal{R}_2$.

Here, the projection to a first order system $\mathcal{R}_1^{(3)}$ simplifies matters:

$$\mathcal{R}_1^{(3)} : \begin{cases} u_x - u = 0, \\ u_y - u = 0, \end{cases}$$

because all second order equations for $\mathcal{R}_2^{(3)}$ are differential consequences of the two equations of $\mathcal{R}_1^{(3)}$. All jets u_μ of order ≥ 1 can be expressed as $u_\mu - u = 0$, $|\mu| \geq 1$. This means prolongation to arbitrary high orders $q + r + s$ and then projection to order $q + r$ does not yield any new equations. ⋄

In the example we could easily describe all equations that must be satisfyed by a solution of the original system $\mathcal{R}_2$. We call such a system formally integrable. The next step is a method to decide integrability in more complicated cases.

1.3.2 Formal Integrability

In this section we give a definition of formal integrability and state results by Goldschmidt [Gol67b] which allow to test formal integrability. In the analytic context it is possible to prove the existence of solutions.

The test of formal integrability depends on the symbol $\mathcal{M}_q$ of a PDE system $\mathcal{R}_q$ introduced in Appendix A. Here we treat the assumption that the symbol $\mathcal{M}_q$ is 2-acyclic as a technical condition.

Definition 1.27. A system $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ of PDEs is called *formally integrable* if $\mathcal{R}_{q+r}$ is a subbundle of $J_{q+r}(\mathcal{E}) \to X$ for all $r \in \mathbb{N}$ and if the projections $\pi_{q+r}^{q+r+s} : \mathcal{R}_{q+r+s} \to \mathcal{R}_{q+r}$ are epimorphisms for all $r, s \in \mathbb{Z}_{\geq 0}$. ⋄

Formal integrability means that the prolongation to arbitrary high orders produces no new equations of lower order that would prevent the projections π_{q+r}^{q+r+s} from being surjective. To assure integrability for a system $\mathcal{R}_q$, we have to check infinitely many conditions. So before turning to the existence of solutions, it is necessary to provide tools that decide formal integrability in a finite number of steps.

Theorem 1.28. [Gol67b, Thm 8.1] Let $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ be a q-th order PDE system. If the symbol $\mathcal{M}_q$ is 2-acyclic, $\mathcal{M}_{q+1} \to \mathcal{R}_q$ is a vector bundle and if the map $\pi_q^{q+1} : \mathcal{R}_{q+1} \to \mathcal{R}_q$ is surjective, then $\mathcal{R}_q$ is formally integrable. ⋄

See Definition A.1 for symbols and Definition A.5 for 2-acyclicity. The theorem can be read as follows. The conditions on $\mathcal{M}_{q+1}$ assures the regularity of $\mathcal{R}_{q+1}$. If the symbol $\mathcal{M}_q$ is 2-acyclic, a single prolongation and projection is sufficient to check formal integrability. If $\mathcal{M}_q$ is not 2-acyclic, there exists an $r \in \mathbb{N}$

such that $\mathcal{M}_{q+r}$ is 2-acyclic. If $\mathcal{R}_{q+r}$ is regular, we can apply Theorem 1.28 to the prolonged system.

A formally integrable system $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ allows to construct formal power series solutions order by order. First choose coordinates (x, u) of $\mathcal{E}$ and an element $r_q \in \mathcal{R}_q$. By Remark 1.4 there is a taylor series representation for r_q:

$$u^i = r^i + r^i_j\, x^j + \cdots + \frac{1}{\mu!} r^i_\mu\, x^\mu, \quad |\mu| \leq q,\ r^i_\mu \in \mathbb{R}.$$

Because $\pi^{q+1}_q : \mathcal{R}_{q+1} \to \mathcal{R}_q$ is surjective, there exists an element $r_{q+1} \in \mathcal{R}_{q+1}$ that projects down to r_q. In other words, the taylor series representatives of r_{q+1} and r_q coincide up to order q. This can be continued since all projections π^{q+r+1}_{q+r} are surjective.

However the constructed power series are only formal and may not converge. In the smooth context, nothing is said about the existence of solutions. Even more, there exist counterexamples for the existence of smooth solutions [Lew57]. A modification of the counterexample (see [GS67]) also applies to equivalence problems, which are treated in Chapter 6.

In the analytic context, where X, $\mathcal{E}$ and $\mathcal{R}_q$ are real-analytic manifolds, there is an existence theorem for solutions.

Theorem 1.29. [Gol67b, Thm 9.1] Let $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ be a q-th order analytic PDE system on $\mathcal{E}$, which is formally integrable. Then given $p \in \mathcal{R}_{q+r}$ with $\pi^{q+r}_0(p) = x \in X$, there exists an analytic solution s over a neighbourhood of x such that $j_{q+r}(s)(x) = p$. ⋄

We give some examples, where the formal integrability of systems is checked. The occuring symbols are all involutive which is a stronger condition than 2-acyclicity.

Example 1.30. (1) In Example 1.26, all symbols of R_2, $R^{(1)}_2$, $R^{(2)}_2$ and the last system $R^{(3)}_2$ are involutive ($\mathcal{M}^{(3)}_2$ is even zero), but the surjectivity condition of Theorem 1.28 is violated in all but the last system. Solutions of R_2 are $u(x, y) = c\, e^{x+y}$ for $c \in \mathbb{R}$.

(2) The system $\mathcal{R}_2$ with $y^i_{jk} = 0$ from Example 1.23 has a zero symbol $\mathcal{M}_2 = 0$, which is involutive. The prolongation to third order yields:

$$\mathcal{R}_3 :\ y^i_\mu = 0 \qquad |\mu| \leq 3.$$

The projection π^3_2 is surjective and $\mathcal{R}_2$ is formally integrable. Because all higher order jets also vanish, the power series solutions are linear polynomials $y^i(x) = a^i_j x^j + b^i$. ⋄

Chapter 2

Lie Groupoids

In this chapter Lie groupoids are introduced which are a generalisation of Lie groups. The underlying concept of groupoids was first defined by Brandt [Bra27]. A main application for groups are symmetries of mathematical objects. It turns out that geometric objects on a manifold, such as metrics, often have lower symmetries that cannot be described by groups. In this case, the symmetries have the structure of a Lie groupoid, which takes the underlying manifold into account.

Ehresmann developed Lie groupoids in a series of notes (see e.g. [Ehr55]) on q-jets of diffeomorphisms. This connects the geometric theory of PDEs with symmetries. The current chapter is based on the introductory textbooks[Mac05] and [MM03] and repeats basic facts for Lie groupoids. The textbooks also contain a comprehensive list of references on this topic.

To get an intuition for Lie groupoids and their connection to groups, their theory is developed analogous to Lie groups:

$$\begin{array}{ccccc} \text{Groups} & \longrightarrow & \text{Lie groups} & \longrightarrow & \text{Jet groups} \\ \text{Groupoids} & \longrightarrow & \text{Lie groupoids} & \longrightarrow & \text{Jet groupoids} \end{array}$$

In Section 2.1 groupoids are defined and a basic notation is layed down. Adding smooth manifold structures, Lie groupoids are obtained in Section 2.2. In the remaining two sections, we turn to the special case of jet groupoids and algebroids which are connected to jet groups from Appendix B. Jet groupoids are systems of PDEs over the trivial bundle $\mathcal{E} = X \times X$ with additional structure. They are needed in Chapter 3 as symmetry groupoids of geometric objects.

In analogy to the Lie algebra of a Lie group, Lie algebroids are introduced in Appendix C as the infinitesimal versions of Lie groupoids.

2.1 Groupoids

We start with the definition of groupoids and their actions. Roughly speaking, a groupoid behaves like a group, except that the multiplication is only defined for certain pairs of elements.

Definition 2.1. A *groupoid* G is a small category with invertible morphisms. A *subgroupoid* H of G is a subcategory of G with invertible morphisms. A subgroupoid is denoted by $H \leq G$. $\diamond$

A small category consists of a set of objects and morphisms between them. For a groupoid, the set of objects $G^{(0)}$ is called the *base* and the set of morphisms is denoted by $G^{(1)}$. Each *arrow* $g : x \to y \in G^{(1)}$ is a morphism between two elements x and y of the base, which are called source and target of g:

$$\begin{aligned} s : G^{(1)} \to G^{(0)} : &\quad g \mapsto x \quad \text{source}, \\ t : G^{(1)} \to G^{(0)} : &\quad g \mapsto y \quad \text{target}. \end{aligned}$$

The arrows of a groupoid are the analogue to group elements. Whenever we speak of a groupoid element $g \in G$, we think of an arrow $g \in G^{(1)}$. In contrast to groups, the multiplication is not defined for all groupoid elements $g, h \in G^{(1)}$. The composition of morphisms in a category induces only a partial multiplication for elements with matching source and target:

$$\mu : G^{(1)} \underset{G^{(0)}}{{}_s\times_t} G^{(1)} \to G^{(1)} : (g, h) \mapsto gh.$$

It is defined over the pullback

$$G^{(2)} = G^{(1)} \underset{G^{(0)}}{{}_s\times_t} G^{(1)} = \{(g, h) \in G^{(1)} \times G^{(1)} |\, s(g) = t(h)\}.$$

The base $G^{(0)}$ can be embedded in $G^{(1)}$ as the identity arrows:

$$\iota : G^{(0)} \hookrightarrow G^{(1)} : x \mapsto 1_x$$

A groupoid G contains a unit element 1_x for each $x \in G^{(0)}$ satisfying $1_x g = g$ for all $g \in G^{(1)}$ with $s(g) = x$. The multiplication with the inverse g^{-1} of an arrow $g : x \to y$ generally leads to two different unit elements:

$$g^{-1}g = 1_x, \qquad gg^{-1} = 1_y.$$

There are a few useful notations for subsets of a groupoid G. Take $x, y \in G^{(0)}$.

$$\begin{aligned} G(x, y) &= \{g \in G^{(1)} |\, t(g) = y, s(g) = x\}, && \\ G(x, -) &= \{g \in G^{(1)} |\, s(g) = x\}, && \text{source-fibre} \\ G(-, y) &= \{g \in G^{(1)} |\, t(g) = y\}, && \text{target-fibre} \\ G(x, x) &= \{g \in G^{(1)} |\, s(g) = t(g) = x\} && \text{isotropy group of } x, \\ Gx &= t(G(x, -)) = s(G(-, x)) && \text{orbit of } x. \end{aligned}$$

In the case of $Gx = G^{(0)}$ the groupoid is called *transitive*. Otherwise, the base decomposes into orbits similar to the case of a group action on a set. It is easy to see that the isotropy groups for x, y in the same orbit are isomorphic.

A *groupoid morphism* $\phi : G \to H$ is a covariant functor, which implies, that ϕ commutes with source and target:

$$\phi(s(g)) = s(\phi(g)), \quad \phi(t(g)) = t(\phi(g)) \qquad \forall g \in G^{(1)}$$

and preserves multiplication:

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \qquad \forall\, (g_1, g_2) \in G^{(2)}.$$

Furthermore, $\phi(1_x) = 1_{\phi(x)}$ for $x \in G^{(0)}$ and $\phi(g^{-1}) = \phi(g)^{-1}$.

The following examples show that the concept of groupoids is very general, as most mathematical objects can be considered as groupoids. The question is whether the groupoid structure gives new insights. The parameters of groupoids that are of interest to us are the base, the isotropy groups and the orbits.

Example 2.2. (1) Every group H is a groupoid over a single point $\{1\}$, which is usually the identity of H. In this case, the restriction of the multiplication map becomes irrelevant. A group is a groupoid with the smallest possible base and a large isotropy.

(2) Each set M is a groupoid, which only contains identity morphisms 1_m for $m \in M$. M is the exact opposite to a group H, because the base is large and the isotropy is trivial. The groupoid structure does not give new information on the set.

(3) An equivalence relation $\sim$ on a set M defines an groupoid as

$$\begin{aligned} G^{(0)} &= M, \\ G^{(1)} &= \{(a,b) \in M \times M | a \sim b\}, \end{aligned}$$

with $s(a,b) = b$ and $t(a,b) = a$. Symmetry ensures that every arrow is invertible, reflexivity provides the units $1_a = (a,a)$ and the multiplication $(a,b)(b,c) := (a,c)$ is well defined because of transitivity. In this case, the orbit structure of G is interesting and the isotropy groups are trivial.

Every groupoid G defines an equivalence relation on the base $G^{(0)}$ by taking its partition into orbits Gx for $x \in G^{(0)}$. If G is transitive, there is only one equivalence class and the groupoid corresponding to this equivalence relation is called the *pair groupoid* with $G^{(1)} = G^{(0)} \times G^{(0)}$. $\diamond$

These examples illustrate extreme cases of groupoids. The interesting cases of groupoids have a nontrivial base as well as nontrivial isotropy groups. We give two more examples.

Example 2.3. (1) Let H be a group and M be a set. Then $G^{(1)} = M \times H \times M$ is the *trivial groupoid* on base $G^{(0)} = M$ with group H. It has the source and target projections

$$s : G^{(1)} \to M : (m_1, g, m_2) \mapsto m_2, \qquad t : G^{(1)} \to M : (m_1, g, m_2) \mapsto m_1.$$

The product of (m_1, h_1, m_2) and (m_3, h_2, m_4) is defined if and only if $m_2 = m_3$:

$$(m_1, h_1, m_2)(m_2, h_2, m_4) = (m_1, h_1 h_2, m_4).$$

Obviously, the groupoid is transitive and the two copies of M seem like an artificial restriction of the multiplication. In fact, each transitive Lie groupoid is locally isomorphic to a trivial groupoid.

(2) The *action groupoid* of a group H acting on a set M is a little more complicated. Define $G^{(0)} = M$ with arrows $G^{(1)} = H \times M$. Source and target are given as

$$s((h, m)) = m, \quad t((h, m)) = hm.$$

The multiplication is induced by the group action:

$$(h_2, m_2)\,(h_1, m_1) = (h_2\,h_1, m_1) \quad \text{for } h_1 m_1 = m_2.$$

The inverse of (h, m) is (h^{-1}, hm). An action groupoid treats the group H on the same level as the set M on which H acts. ⋄

Groupoid Actions

We now turn to the action of a groupoid G on a set $\mathcal{F}$. Similar to the restricted multiplication on G, not every groupoid element may act on each $f \in \mathcal{F}$.

Definition 2.4. Let G be a groupoid and $\mathcal{F}$ a set with a map $\pi : \mathcal{F} \to G^{(0)}$. A left *groupoid action* of G on $\mathcal{F}$ is a map $G^{(1)}\, {}^{s}\!\underset{G^{(0)}}{\curlywedge}{}^{\pi}\, \mathcal{F} \to \mathcal{F}$ (shortly $G \curlywedge \mathcal{F} \to \mathcal{F}$) with

$$(ab)f = a(bf) \qquad \forall\,(a, b, f) \in G^{(2)}\, {}^{s}\!\underset{G^{(0)}}{\curlywedge}{}^{t}\, \mathcal{F},$$

$$1_x f = f \qquad \forall\, f \in \pi^{-1}(x).$$

The first condition requires $\pi(bf) = t(b)$. There is an analogous definition for a right action $\mathcal{F}\, {}^{\pi}\!\curlywedge^{t}_{G^{(0)}}\, G^{(1)} \to \mathcal{F}$ (abbreviated as $\mathcal{F} \curlywedge G \to \mathcal{F}$). ⋄

We usually assume that $\pi : \mathcal{F} \to G^{(0)}$ is surjective. In the case of $|G^{(0)}| = 1$ this definition coincides with the action of a group. The definitions of groupoid orbits Gf and stabilisers $\mathrm{Stab}_G(f)$ for $f \in \mathcal{F}$ are analogous to the group case. However there are two restrictions for transitive actions, because groupoids may be intransitive themselves.

We give two simple examples of groupoid actions and refer to Example 2.12 and Chapters 3-7 for more elaborate actions.

Example 2.5. (1) Every groupoid G over the base M acts on M by:

$$G\, {}^{s}\!\underset{M}{\curlywedge}{}^{\mathrm{id}}\, M \to M : (g, m) \mapsto t(g).$$

The orbits of the action are identical with the orbits of the groupoid itself.

(2) An important action of G is the multiplicative action on itself:

$$\mu : G^{(1)} {}_{s}\!\underset{G^{(0)}}{\curlywedge}\!{}_{t}\, G^{(1)} \to G^{(1)} : (g,h) \mapsto gh,$$

which is a left action for $(\mathcal{F}, \pi) = (G^{(1)}, t)$ and a right action for $(\mathcal{F}, \pi) = (G^{(1)}, s)$. In this context, we define left and right multiplication:

$$L_g : G(s(g), -) \to G^{(1)} : h \mapsto gh,$$

$$R_h : G(-, t(h)) \to G^{(1)} : g \mapsto gh. \qquad \diamond$$

2.2 Lie Groupoids

We now turn to Lie groupoids and their properties. Similar to a Lie group, a Lie groupoid is essentially a groupoid with smooth manifold structures (cf [MM03, §5.1]). In the last part of this section, we treat actions of transitive Lie groupoids.

Definition 2.6. A groupoid G is called a *Lie groupoid* if $G^{(0)}$ and $G^{(1)}$ are smooth Hausdorff manifolds. Additionally, the source map s is a smooth submersion and all other maps are also smooth.

A morphism of Lie groupoids is a smooth groupoid morphism (both on the base and the arrows). A Lie subgroupoid is a subgroupoid $H \leq G$, which is also a Lie groupoid (and all manifolds are submanifolds). $\diamond$

We also require a Lie groupoid action on a fibre bundle $\pi : \mathcal{F} \to G^{(0)}$ to be a smooth map $G \curlywedge \mathcal{F} \to \mathcal{F}$.

Both Examples 2.2 and 2.3 can be turned into Lie groupoids if H is a Lie group, M a manifold and the equivalence relation defines a foliation of M. In particular, a Lie group is a Lie groupoid over a single point. The next important example shows the connection between Lie groupoids and principal bundles.

Example 2.7. Let H be a Lie group and $\pi : P \to X$ be a principal H-bundle. H acts on the cartesian product $P \times P$ from the right by $(p_1, p_2)h = (p_1 h, p_2 h)$. Then the *gauge groupoid* is the orbit space under the H-action:

$$\mathrm{Gauge}(P) = P \times_H P = P \times P/H.$$

Set pr_1 and pr_2 as the projections to the first and second copy of P and choose source and target as $s = \pi \circ \mathrm{pr}_2$ and $t = \pi \circ \mathrm{pr}_1$. The product of two residue classes is:

$$[(p_1, p_2)] \cdot [(q_1, q_2)] = [(p_1, q_2 h)] \qquad \text{for } \pi(p_2) = \pi(q_1),$$

where $h \in H$ is the unique element with $q_1 h = p_2$. $\diamond$

The following theorem from [MM03] shows that Lie groupoids have far more structure than groupoids. Particularly, the isotropy groups are all Lie groups and the source-fibres are principal bundles. This leads to consequences for transitive Lie groupoids and their actions on bundles.

Theorem 2.8. [MM03, Thm. 5.4] Let G be a Lie groupoid and let $x, y \in G^{(0)}$.

(1) $G(x, y)$ is a closed submanifold of G.

(2) $G(x, x)$ is a Lie group.

(3) Gx is an immersed submanifold of $G^{(0)}$.

(4) $t_x : G(x, -) \to Gx$ is a principal $G(x, x)$-bundle. ⋄

Proof. Since s is a submersion, $G(x, -)$ is a closed submanifold of G. Put $E_g = \ker(s_*)_g \cap \ker(t_*)_g$ for all $g \in G$. We will show that $E|_{G(-,x)}$ is an involutive subbundle of the tangent bundle $TG(x, -) \to X$, therefore defining a foliation of $G(x, -)$.

For a $g \in G(x, -)$, the left translation L_g gives a diffeomorphism. We note that for any $h \in G(x, -)$, E_h is a subspace of $T_hG(x, -) \subset T_hG$. Since $s \circ L_g = s|_{G(x,-)}$, it follows that

$$(L_{g*})_{1_x}(E_{1_x}) = E_g.$$

So, any basis $v_1, \ldots, v_k$ of E_{1_x} can be extended to a global frame $X_1, \ldots, X_k$ of $E|_{G(-,x)}$ via left translation $(X_i)_g = (L_{g*})_{1_x}(v_i)$. This shows, that $E_{G(x,-)}$ is indeed a subbundle of $TG(x, -)$. Involutivity follows from the fact that it is exactly the kernel of t_{x*}. Hence it defines a foliation of $G(x, -)$, parallelisable by $X_1, \ldots, X_k$. The leaves are the connected components of t_x and thus closed manifolds. Particularly, the fibre $t_x^{-1}(x) = G(x, x)$ is a Lie group.

We have a smooth and free action of $G(x, x)$ on $G(x, -)$, that is transitive on the fibres of t_x. As t_x fulfills condition (iii) of [MM03, La 5.5], Gx is a smooth manifold and $t_x : G(x, -) \to Gx$ is a principal bundle. The fact that $G^{(0)}$ is Hausdorff implies, that Gx is also Hausdorff. □

The last property (4) essentially shows that transitive groupoids are all isomorphic to gauge groupoids. This leads to the following proposition, which is a shortened version of [MM03, Prop. 5.14].

Proposition 2.9. For a Lie groupoid G, the following conditions are equivalent:

(1) G is transitive.

(2) $t : G(x_0, -) \to G^{(0)}$ is a surjective submersion for at least one $x_0 \in G^{(0)}$.

(3) G is isomorphic to $\mathrm{Gauge}(P)$ of a principal $G(x_0, x_0)$-bundle P. ⋄

Proof. (1) $\Rightarrow$ (2): As (s,t) is a surjective submersion, also the pullback $t : G(x_0,-) \to G^{(0)}$ along $G^{(0)} \to G^{(0)} \times G^{(0)} : y \mapsto (x_0,y)$ is a submersion.

(2) $\Rightarrow$ (3): According to Theorem 2.8, $t : G(x_0,-) \to G^{(0)}$ is a principal bundle. The map

$$\mathrm{Gauge}(G(x_0,-)) \to G : (g,h) \mapsto gh^{-1}$$

is an isomorphismus of Lie groupoids (easy check).

(3) $\Rightarrow$ (1): If $G \cong \mathrm{Gauge}(P)$ for $\pi : P \to G^{(0)}$, then the map $(s,t) : \mathrm{Gauge}(P) \to G^{(0)} \times G^{(0)}$ is induced by the surjective submersion $(\pi,\pi) : P \times P \to M \times M$. So $\mathrm{Gauge}(P)$ is transitive. □

Lie Groupoid Actions

The last two propositions are now applied to actions of transitive Lie groupoids. This section is based on [Mac05, §1.6], which provides further details. From the fact that each transitive Lie groupoid is isomorphic to a gauge groupoid certain consequences follow for their actions on fibre bundles $\mathcal{F}$. Each groupoid action is uniquely defined by the action of the isotropy group on the abstract fibre F of $\mathcal{F}$. The following proposition becomes important in Chapter 3 where it is applied to the action of jet groupoids on natural bundles. It is the reason why jet groups are introduced in Appendix B.

Proposition 2.10. Let G be a transitive Lie groupoid over the base X. Each fibre bundle $\pi : \mathcal{F} \to X$ with G-action and fibre F is isomorphic to the associated bundle:

$$\mathcal{F} \cong G(x,-) \times_{G(x,x)} F$$

for some $x \in X$. ◇

Proof. Use the isomorphism $F \cong \mathcal{F}_x$. For each $f \in \mathcal{F}_y$, $y \in X$, there is a groupoid element $g \in G(y,x)$ such that $gf \in \mathcal{F}_x$. Two different choices g, g' differ only by an element $h \in G(x,x)$. So the map

$$\mathcal{F} \to G(x,-) \times_{G(x,x)} F : u \mapsto (g^{-1}, gf)$$

is well-defined. The converse is trivial by the G-action on $G(x,-)$. □

The proposition says that each fibre bundle $\mathcal{F}$ with a G-action is uniquely defined by the isotropy group action on the fibre F of $\mathcal{F}$ which will be used in Chapter 3 (especially Section 3.1.1).

In the next proposition we show that also G-equivariant bundle morphisms $\psi : \mathcal{F} \to \mathcal{F}'$ with $\psi(gf) = g\psi(f)$ for $g \in G$ are also defined by maps on the fibres which are equivariant under the isotropy group action. This correponds to Appendix B.5.3 on equivariant maps under jet groups.

Proposition 2.11. Let G be a transitive Lie groupoid over the base X. Denote the principal bundle $G(x,-)$ by P and the isotropy group $G(x,x)$ by H. Let F and F' be two manifolds with an H-action.

(1) If $\varphi : F \to F'$ is an H-equivariant map then

$$\tilde{\varphi} : P \times_H F \to P \times_H F'$$

is a well-defined G-equivariant morphism of fibre bundles.

(2) Conversely if $\psi : \mathcal{F} \to \mathcal{F}'$ is an G-equivariant morphism of bundles over X, then $\psi = \tilde{\varphi}$ for some H-equivariant map $\varphi : F \to F'$ on the fibres. ⋄

Proof. (1) follows directly from Proposition 2.10 and (2) follows from:

$$\psi(g'f) = g'\psi(f) = g'\psi(g^{-1}(gf)) = g'g^{-1}\psi(gf),$$

where $f \in \mathcal{F}_y$, $g \in G(y,x)$ such that $fg \in \mathcal{F}_x \cong F$. Then $\psi(gf) \in \mathcal{F}'_x \cong F'$. □

The prolongation and projection of the isotropy group action in Section B.5 requires jet groupoids and will be treated in Section 3.4. As a last step we give an example for the action of a gauge groupoid.

Example 2.12. Let $\pi : P \to X$ be a principal H-bundle. The gauge groupoid $\mathrm{Gauge}(P)$ acts on $\mathcal{F} = P$:

$$\mathrm{Gauge}(P) \, {}_{s}\!\!\underset{X}{\times}\!\!{}_{\pi} \, P \to P : ([(p_1, p_2)], q) \mapsto p_1 h,$$

where $h \in H$ is the unique element with $p_2 h = q$. The action is transitive.

$\mathrm{Gauge}(P)$ also acts on the quotient bundle P/K where K is a closed subgroup $K \leq H$, because P/K is isomorphic to the associated bundle

$$P/K \cong P \times_H H/K.$$

In Section 3.3, the bundle P/K will be used extensively for the construction of a natural bundle from a groupoid. ⋄

2.3 Jet Groupoids

This section introduces jet groupoids, which are PDE systems (as in Section 1.3) with a Lie groupoid structure. As a PDE system, a jet groupoid is a subbundle of $J_q(X \times X) \to X$ for a manifold X. Using the results of Section 1.2 we construct the full jet groupoid Π_q and show basic examples of subgroupoids. As a last step we show that the jet groups from Appendix B are isotropy groups of jet groupoids.

Jet groupoids were first mentioned by Ehresmann [Ehr54] who already had the action on bundles and their prolongation in mind. These topics will be covered in the following Chapters 3, 4 and 6. Most of the material in this section is based on [Pom78] and [Pom83].

Definition 2.13. Let X be an n-dimensional manifold. The *full jet groupoid* of order $q \in \mathbb{N}$ is the open subset $\Pi_q = \Pi_q(X \times X) \subseteq J_q(X \times X)$ of invertible jets. A *jet groupoid* $\mathcal{R}_q$ is a submanifold of Π_q that is closed with respect to all groupoid operations. ◇

Similar to Definition 1.5, we set the zero order jet groupoid $\Pi_0 = X \times X$ as the pair groupoid. For higher orders we can specify all invertible jets in local coordinates (x, y) of $X \times X$. A jet $(x, y, y_q) \in J_q(X \times X)$ is invertible if and only if the Jacobian matrix (y^i_j) has a nonzero determinant $\det(y^i_j) \neq 0$. If $(x, y, y_q) = j_q(\varphi)(x)$ is the q-jet of a smooth map $\varphi : X \to X$, it is invertible in a neighbourhood of $x \in X$ if and only if (x, y, y_q) is invertible.

It remains to show that Π_q actually is a groupoid, so we check the constructions from Section 2.1, starting with source and target. On $\Pi_0 = X \times X$, source and target are the projections $s = \mathrm{pr}_1$, $t = \mathrm{pr}_2$ onto the different copies of X. They extend to Π_q by composition with the jet projection π^q_0 to zero order:

$$s = \mathrm{pr}_1 \circ \pi^q_0, \qquad t = \mathrm{pr}_2 \circ \pi^q_0.$$

As it is usually clear from the context, we do not distinguish between source and target maps for different jet groupoids. In coordinates the maps are:

$$s : \Pi_q \to X : (x, y, y_q) \mapsto x, \qquad t : \Pi_q \to X : (x, y, y_q) \mapsto y.$$

To construct the multiplication map μ we apply Proposition 1.15 on the composition of two local diffeomorphisms $\varphi, \psi : X \to X$:

$$j_q(\varphi \circ \psi)(x) = j_q(\varphi)(y) \circ j_q(\psi)(x), \qquad y = \psi(x). \tag{2.1}$$

Basically, we apply the chain rule to calculate $j_q(\varphi \circ \psi)(x)$ which yields a formula for the multiplication on Π_q:

$$\mu : \Pi_q \, {}_s\!\underset{X}{\times}\!{}_t \, \Pi_q \to \Pi_q.$$

The units 1_x are the q-jets of the identity map $\mathrm{id}_X : X \to X$ with coordinate expressions $1_x = (x, x, \delta^i_j, 0, \dots, 0)$. Inverse elements for $f_q \in \Pi_q$ exist by inverting the local diffeomorphism from Remark 1.4 and then taking the q-jet. It can be computed by solving the equation

$$f_q f_q^{-1} = 1_{t(f_q)}$$

for increasing orders q. For $q = 1$ the Jacobian matrix (y^i_j) of f_1 must be inverted (uniquely possible for $\det(y^i_j) \neq 0$). Inverse elements of higher orders require solving inhomogenous linear equations, depending on the lower order results. We give an example of the multiplication and the construction of inverse elements.

Example 2.14. On $\Pi_2(\mathbb{R}\times\mathbb{R})$ with coordinates (x,y) for $\mathbb{R}\times\mathbb{R}$, every element is of the form (x,y,y_x,y_{xx}). It represents a 2-jet of a diffeomorphism which maps x to y.

The product of two elements (y,z,z_y,z_{yy}), $(x,y,y_x,y_{xx})\in\Pi_2(\mathbb{R}\times\mathbb{R})$ is:

$$(y,z,z_y,z_{yy})(x,y,y_x,y_{xx}) = (x,z,\ z_y y_x,\ z_{yy}y_x^2 + z_y y_{xx}),$$

now mapping x to z. For first order jets, the Jacobian matrices are multiplied (here, they are only 1×1-matrices). The expression for the second order jets is obtained by formal differentiation of $z_y y_x$.

The inverse element $(y,z,z_y,z_{yy}) = (x,y,y_x,y_{xx})^{-1}$ is the solution of

$$(y,z,z_y,z_{yy})(x,y,y_x,y_{xx}) = (x,x,\ 1,\ 0) = \mathrm{id}_x,$$

which implies:

$$z = x,\quad z_y y_x = 1,\quad z_{yy}y_x^2 + z_y y_{xx} = 0.$$

Once z_y is known, the last equation becomes linear and the inverse is:

$$(x,y,y_x,y_{xx})^{-1} = (y,x,\frac{1}{y_x},-\frac{y_{xx}}{y_x^3}).$$

It exists if and only if $y_x \neq 0$, which is the condition defining $\Pi_2 \subset J_2(\mathbb{R}\times\mathbb{R})$. ⋄

Of more interest than Π_q itself are subgroupoids, which are also systems of PDEs according to Definition 1.21. Writing down differential equations on Π_q does not automatically define a subgroupoid $\mathcal{R}_q$. At least the units 1_x must be elements of $\mathcal{R}_q$ and the product of $r_q, s_q \in \mathcal{R}_q$ must also be an element of $\mathcal{R}_q$. The next example shows two equations that actually define jet groupoids.

Example 2.15. (1) For $X=\mathbb{R}$, the equation $y_{xx}=0$ defines the subgroupoid $\mathcal{R}_2$ of Π_2 of affine transformations. Using Example 2.14 it is easy to verify, that $\mathcal{R}_2$ is closed with respect to all operations.

(2) Example 1.23 with $y^i_{jk}=0$ for all $1\le i,j,k\le n$ generalises (1) to arbitrary dimensions.

(3) The equation $y_{xxx} - \frac{3}{2}\frac{y_{xx}^2}{y_x} = 0$ defines the subgroupoid $\mathcal{R}_3 \subseteq \Pi_3(\mathbb{R}\times\mathbb{R})$ of projective transformations on $\mathbb{R}$ (see e.g. [Pom78, Ex. 7.1.6], [Pom83, Ex. 2.A.2.11]).

We now turn to the isotropy groups of jet groupoids. Because X is locally diffeomorphic to $\mathbb{R}^n$ the isotropy groups of the full groupoid Π_q are all isomorphic to GL_q. Interpreted as PDE systems, we can prolong subgroupoids $\mathcal{R}_q \le \Pi_q$ and consider their isotropy groups. The chains of subgroups $G_q \le \mathrm{GL}_q$ from Appendix B are constructed to describe the isotropy groups of integrable jet groupoids. This will be applied in Section 3.4 when describing the projection of jet groupoids.

Lemma 2.16. Let $\mathcal{R}_{q_0} \leq \Pi_{q_0}$ be a formally integrable jet groupoid. Then at each point $x \in X$, the isotropy groups $\mathcal{R}_q(x,x)$ for $q \in \mathbb{N}$ are isomorphic to subgroups of $G_q \leq \mathrm{GL}_q$ which are compatible with prolongation. $\diamond$

Proof. Use the isomorphism $\Pi_q(x,x) \cong \mathrm{GL}_q$ to define G_q for all $q \in \mathbb{N}$. Check the conditions in Proposition B.7. The projections from condition (1) are epimorphisms by formal integrability of $\mathcal{R}_q$.

As $G_{q+r} \cong \mathcal{R}_{q+r}(x,x)$ is the isotropy group of the r-th prolongation of $\mathcal{R}_q$, condition (2) is also satisfied. $\square$

In Section 2.2, we have considered actions of Lie groupoids on bundles. For jet groupoids, this will be done in Chapter 3. We will show that each jet groupoid of order q is defined by the Π_q-action on a so-called natural bundle.

Chapter 3

Natural Bundles

The current chapter establishes a link between geometric objects on a manifold and jet groupoids. Examples of geometric objects are metrics, differential forms or connections. They appear as sections of natural bundles, which were introduced by Nijenhuis [Nij72] as fibre bundles where each diffeomorphism on the base lifts to the bundle. The symmetries of geometric objects have the structure of jet groupoids.

The connection between natural bundles and jet groupoids goes back to Lie [Lie91] who realised that each jet groupoid can be defined by differential invariants. Vessiot [Ves03] computed what is now called a groupoid action on the differential invariants, intending to write equations for similar groupoids in an unified manner. In modern language, he constructed a natural bundle and the appropriate groupoid action. Vessiot's approach seems to be forgotten until Pommaret [Pom78] reformulated it using natural bundles and the formal (or geometric) theory of PDEs initiated by Spencer [Spe69]. It is worth noting that Ehresmann [Ehr54], [Ehr55] considered jet groupoid actions on fibre bundles and their prolongation.

In Sections 3.1 and 3.3, following the path from Lie and Vessiot to Pommaret it is shown how a jet groupoid $\mathcal{R}_q$ determines a geometric object ω on a natural bundle $\mathcal{F}$. By doing so, the theory of PDE systems from Section 1.3 is combined with the groupoid structure. The result is an exact sequence (1.7) for the groupoid $\mathcal{R}_q$

$$0 \longrightarrow \mathcal{R}_q \longrightarrow \Pi_q \underset{\omega\circ\pi}{\overset{\Phi_\omega}{\rightrightarrows}} \mathcal{F}.$$

with the additional properties that $\mathcal{F}$ is a natural bundle and the differential map Φ_ω is determined by the groupoid action on $\mathcal{F}$. As the main result, a test for formal integrability is presented which works on natural bundles. In many examples this gives a geometric interpretation of the integrability conditions.

As a new contribution, the definition of natural bundles is generalised to the relative situation, where only a subset of all diffeomorphisms of the base lift to the fibre bundle. This implies only minor theoretical changes, but opens a wide range of applications in the following chapters.

Section 3.2 introduces a completely functorial approach to natural bundles presented by Kolář, Michor and Slowák [KMS93, Ch. IV]. It will be useful for the Vessiot equivalence problem in Chapter 6.

The motivation to work with natural bundles is that all operations from the geometric formulation of PDEs in Section 1.3 can be done with natural bundles. At first the prolongation and projection is presented in Section 3.4. The main result is the test of formal integrability in Section 3.5. It generalises the results of Pommaret and Vessiot to intransitive groupoid actions. We also show how Spencer cohomology is computed via natural bundles. A special focus lies on the examples, which illustrate every theoretical step in this chapter. Section 5.1 treats the example of metrics and Christoffel symbols as an introduction to the Maple packages `jets` of Barakat and Hartjen [Bar01] and `JetGroupoids` developed for this thesis.

3.1 Definitions

We start with a definition of natural bundles that depends on jet groupoids treated in Section 2.3. It is suited for the computation of symmetry groupoids. In Section 3.2, we give a more general description via functors and show that both definitions agree.

Definition 3.1. [Pom83, Def. 2.A.2.35] A fibre bundle $\mathcal{F} \xrightarrow{\pi} X$ is called *natural bundle* if there exists a $q \in \mathbb{N}$ and a groupoid action of Π_q on $\mathcal{F}$. A section $\omega \in \Gamma(\mathcal{F})$ is called *geometric object*. A *morphism of natural bundles* is an equivariant bundle morphism $\Phi : \mathcal{F} \to \mathcal{G}$, i.e. Φ commutes with the Π_q-action on the natural bundles $\mathcal{F}$ and $\mathcal{G}$. An *invariant on* $\mathcal{F}$ is a smooth map $\mathcal{F} \to \mathbb{R}$ which is constant on the Π_q-orbits on $\mathcal{F}$. ◇

Interpreting the elements of Π_q as q-jets of local diffeomorphisms on X, each diffeomorphism of the base lifts to the natural bundle $\mathcal{F}$. The same applies to coordinate changes. A special case is a zero order natural bundle, which must be a trivial bundle $\mathcal{F} = X \times F$ for a manifold F. The action of $\Pi_0 = X \times X$ then is:

$$\mathcal{F}\,{}^{\pi}{\curlywedge}^{t}\,\Pi_0 : ((y, f), (x, y)) \mapsto (x, f).$$

All fibre coordinates of $\mathcal{F}$ are invariants on $\mathcal{F}$.

Example 3.2. (1) The tangent bundle $T \to X$ is a natural bundle, since each local diffeomorphism $\varphi : X \to X$ induces a bundle morphism $\varphi_* : T \to T$. For each $x \in X$, $\varphi_*|_x : T_x \to T_{\varphi(x)}$ depends on the first order jet $j_1(\varphi)(x)$ and thus defines a Π_1-action on T. Analogously, the cotangent bundle T^* is a natural bundle.

(2) The bundle $S^2T^* \to X$ is a natural bundle. A geometric object ω defines a symmetric 2-form on X. If ω is positive definite, it defines a Riemannian

metric on X (or a Riemannian structure). The bundle of metrics $\mathcal{F}_g = S^2T^*_{>0}$ is an open subbundle of S^2T^* and we can choose the entries of the metric g_{ij} ($g_{ij} = g_{ji}$) as coordinates for $\mathcal{F}_g$. In Example 3.8, the Π_1-action is explicitly constructed.

(3) A section ω on the natural bundle $\bigwedge^2 T^* \to X$ defines a 2-form on X. If X has even dimension and if ω is closed ($d\omega = 0$), we are dealing with a symplectic structure on X.

(4) Almost complex structures are modelled on the natural bundle $T \otimes T^*$.

(5) Each tensor bundle on X is a natural bundle with the Π_1-action defined by part (1). ◇

So far, we have only given linear first order examples resulting in natural vector bundles. Natural vector bundles were studied and classified by Terng [Ter78]. The definition of natural bundles includes also nonlinear or higher order geometric structures and we show some in the next example.

Example 3.3. (1) An affine connection on X can be defined by its Christoffel symbols. For coordinate systems (x) and (y) of X, a diffeomorphism $y^i = y^i(x)$ changes the Christoffel symbols Γ^i_{jk} to $\hat{\Gamma}^i_{jk}$ so that

$$y^t_i \Gamma^i_{jk} = y^s_j\, y^r_k\, \hat{\Gamma}^t_{sr} + y^t_{jk} \tag{3.1}$$

Plugging in $y^a_b = \frac{\partial y^a}{\partial x^b}$ and inverting y^t_i yields the well-known transition functions for Christoffel symbols. We define the natural bundle $\mathcal{F}_\Gamma$ with coordinates Γ^i_{jk} and the Π_2-action of equation (3.1).

The natural bundle $\mathcal{F}_\Gamma$ automatically arises in the study of Riemannian metrics from Example 3.2 (2). The first prolongation $J_1(\mathcal{F}_g)$ contains $\mathcal{F}_\Gamma$:

$$J_1(\mathcal{F}_g) \cong \mathcal{F}_g \times_X \mathcal{F}_\Gamma.$$

With first order jets $g_{ij,k}$ as fibre coordinates of $J_1(\mathcal{F}_g)$, the isomorphism is given by:

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{ir}\left(g_{rj,k} + g_{rk,j} - g_{jk,r}\right), \tag{3.2}$$

where g^{ij} is the inverse of g_{ij} ($g^{ij} g_{jk} = \delta^i_k$). For more details on affine connections see e.g. [dC92].

(2) Two metrics $g, h \in \Gamma(\mathcal{F}_g)$ are conformally equivalent if there exists a diffeomorphism φ and a positive function $c \in C^\infty(X)$ such that

$$\varphi^*(g) = c(x)h.$$

Each equivalence class of metrics defines a first order conformal structure. The corresponding natural bundle $\mathcal{F}$ can be constructed from $\mathcal{F}_g$ (see Example 3.2 (2)) by taking the coodinates $\tilde{g}_{ij} = g_{ij}/g_{11}$ with $(i, j) \neq (1, 1)$.

This is the first example, where $\mathcal{F}$ is not a vector bundle. The groupoid action is computed in Example 3.8, equation (3.4).

Analogous to the previous example, the first prolongation $J_1(\mathcal{F})$ defines a second order conformal structure on X.

(3) Consider the real projective space $\mathbb{RP}^n$ and the group of projective transformations

$$\mathrm{PGL}(n+1, \mathbb{R}) = \mathrm{GL}(n+1, \mathbb{R})/\mathbb{R}^*.$$

The stabiliser $G_0 \leq \mathrm{PGL}(n+1, \mathbb{R})$ of $(1 : 0 : \ldots : 0) \in \mathbb{RP}^n$ can be embedded into the jet group $\mathrm{GL}_2(\mathbb{R}^n)$. We thus define the natural bundle of projective structures,

$$\mathcal{F} = \Pi_2(-, y_0)/G_0 = P_2/G_0 \to X,$$

which is a quotient of the second order frame bundle P_2 (see Definition 3.6). A section of $\mathcal{F} \to X$ defines a projective structure, namely a subbundle $P \subseteq P_2 \to X$ which is a principal G_0-bundle (see [Yan92, §VI.3] for more details). ◇

The next definition is new and generalises the definition of a natural bundle $\mathcal{F}$ to subgroupoids $\Theta_q \leq \Pi_q$ acting on $\mathcal{F}$. This is essential if we are dealing with a geometric object where only a subset of all diffeomorphisms $\varphi : X \to X$ lift to transformations of the object. Examples are given in Chapter 7 on linear partial differential operators. Another application is the relative equivalence problem in Section 6.1.2.

Definition 3.4. Let Θ_{q_0} be a formally integrable subgroupoid of Π_{q_0} for some $q_0 \in \mathbb{N}$. A fibre bundle $\mathcal{F} \xrightarrow{\pi} X$ is called *natural* Θ_q*-bundle* if there exists a $q \geq q_0$ and a groupoid action of Θ_q on $\mathcal{F}$. ◇

Geometric objects, invariants and morphisms of Θ_q-bundles are defined analogous to the Π_q-case. The assumption of formal integrability ensures well-defined prolongation and projection properties in Section 3.4. However the lower bound q_0 on the jet order is not strictly necessary (see Remark 3.27).

The most interesting geometric objects on a natural bundle are the generic objects, which are defined as follows.

Definition 3.5. A local section ω of a natural Θ_q-bundle $\mathcal{F} \to X$ is called *generic at* $x \in X$ if the map

$$\Theta_q(-, x) \to \mathcal{F} : \theta \mapsto \omega(x)\theta$$

induced by the Θ_q-action has full rank. If ω is defined on $U \subseteq X$, then ω is *generic* if it is generic at all $x \in U$. ◇

Nearly all constructions of natural bundles can also be done with natural Θ_q-bundles and usually the proofs are identical up to an exchange of Π_q and Θ_q. So we proceed with natural Θ_q-bundles, where Π_q-bundles are just a special case. If the groupoid Θ_q is clear from the context, we will speak of natural bundles.

3.1.1 Structure of Natural Bundles

In this section we apply the results on actions of transitive Lie groupoids from Section 2.2 to natural Θ_q-bundles. Under additional assumptions we also obtain a local result for intransitive jet groupoids.

By Proposition 2.10, each natural bundle $\mathcal{F}$ is associated to a principal bundle and uniquely defined by the isotropy group action on the fibre F of $\mathcal{F}$. The isotropy groups of Π_q are all isomorphic to GL_q, so each natural Π_q-bundle has the structure

$$\mathcal{F} \cong \Pi_q(x_0, -) \times_{\mathrm{GL}_q} F.$$

The principal GL_q-bundle $\Pi_q(x_0, -)$ is isomorphic to the q-th order frame bundle. This is done by identifying an open neighbourhood of $x_0 \in X$ with $\mathbb{R}^n$. Define the frame bundle as follows (see e.g. [KMS93, §12.12], [Yan92, §VI.3]).

Definition 3.6. Let X^n be an n-dimensional manifold. The q-th order *frame bundle* $P^q = P^q(X)$ is the set of all q-jets of diffeomorphisms $\mathbb{R}^n \to X$ with source $0 \in \mathbb{R}^n$. ◇

The first order frame bundle P^1 is the usual bundle of linear frames, where each element $p \in P^1_x$ is a basis of the tangent space T_xX. The isomorphism $P^q \cong \Pi_q(x_0, -)$ shows that Π_q acts on $\mathcal{F}$ on the left. To treat right groupoid actions, we need the *coframe bundle* $P_q = P_q(X)$. Dual to P^q, it is defined as the set of all q-jets $j_q(\varphi)(x)$ of diffeomorphisms $\varphi : X \to \mathbb{R}^n$ with $\varphi(x) = 0$. $P_q \cong \Pi_q(-, y_0)$ is a principal GL_q-bundle with a left GL_q-action. Starting in Section 3.3, we construct all examples of natural bundles with a right groupoid action.

Now turn to natural Θ_q-bundles for transitive subgroupoids $\Theta_q \leq \Pi_q$. Here, the frame bundle $P^q \cong \Pi_q(x_0, -)$ is replaced by the subbundle $P^{\Theta_q} = \Theta_q(x_0, -)$. It is a principal G_q-bundle for the jet group $G_q = \Theta_q(x_0, x_0)$. For right Θ_q actions we use the subbundle $P_{\Theta_q} = \Theta_q(-, y_0)$ of the coframe bundle.

For intransitive groupoids Θ_q, the subbundle P_{Θ_q} of the coframe bundle has to be constructed differently. The standard candidate $\Theta_q(-, y_0)$ would pick out a single orbit, which is not satisfactory. However, a local construction is possible, if the isotropy groups are essentially the same. All examples of Θ_q-bundles with intransitive Θ_q treated in this thesis have isomorphic isotropy groups (see Chapter 7).

Proposition 3.7. Let Θ_q be a intransitive integrable subgroupoid of Π_q with a local trivialisation

$$U \times U' \times G_q, \qquad G_q \leq \mathrm{GL}_q$$

for open subsets $U \subseteq X$, $U' \subseteq \mathbb{R}^{n-k}$. Then all natural Θ_q-bundles are locally associated to $P_{\Theta_q} = U \times G_q$. ◇

Proof. Construct the trivialisation. On coordinates (x, y) of $X \times X$ define Θ_q via differential invariants (see Theorem 3.19) Denote the defining equations of order zero by $\Phi^\tau(y) = \Phi^\tau(x)$ for $1 \leq \tau \leq k$. On an open subset $U \subseteq X$ they define a map

$$U \to \mathbb{R}^k : x \mapsto \Phi^\tau(x).$$

Modify the coordinates x and y such that $(\Phi^\tau, x^{k+1}, \dots, x^n)$ is a coordinate system of X (and analogous for y). On U it is possible to trivialise $\mathcal{R}_q$ as

$$\mathcal{R}_q|_U \;\cong\; U \times_{\mathbb{R}^k} U \times G_q \;\cong\; U \times U' \times G_q$$

with coordinates $(y^{k+1}, \dots, y^n)$ of $\mathbb{R}^{n-k}$ and $U' \subseteq \mathbb{R}^{n-k}$. Then choose $P_{\Theta_q}|_U$ as the preimage of $y^{k+1} = \ldots = y^n = 0$. On the trivialisation, each natural bundle is of the form $U \times F \cong (U \times G_q) \times_{G_q} F$. □

To illustrate the structure of natural bundles, we construct a natural bundle. Starting with the jet group action on a fibre, we associate it to the frame bundle. The example also shows that we intuitively interpret sections of natural bundles as geometric objects.

Example 3.8. Let X be an n-dimensional manifold and $V = \mathbb{R}^n$ a vector space with the standard action of the general linear group $\mathrm{GL}_1 \cong \mathrm{GL}(V)$. The action on V induces a GL_1-action on the space of symmetric bilinear forms $V \times V \to \mathbb{R}$:

$$S^2V^* \times \mathrm{GL}(V) \to S^2V^* : (g, h) \mapsto h^{-1}\, g\, h^{-tr}.$$

We restrict to the positive definite forms $S^2V^*_{>0}$ that define scalar products on V. Using the isomorphism $\mathrm{GL}_1 \to \mathrm{GL}(V)$ that identifies h with the Jacobian matrix y^i_j of a local diffeomorphism, we obtain the associated bundle

$$\mathcal{F} \;=\; P^1 \times_{\mathrm{GL}_1} S^2V^*_{>0} \;\cong\; S^2T^*_{>0}.$$

Sections on $\mathcal{F} \to X$ define a scalar product at each tangent space T_xX. In other words, $\mathcal{F} = S^2T^*_{>0}$ is the natural bundle with metrics as geometric objects.

Interpreting a metric $g_{ij}(x)$ as a symmetric differential form, we can also take the pullback of $g_{ij}(x)$ along a local diffeomorphism $\varphi : X \to X$. This is equivalent to the right Π_1-action on S^2T^*, which is given in coordinates by:

$$\mathcal{F} \underset{X}{{}^{\pi}\!\times^{t}} \Pi_1 \to \mathcal{F} : ((y,\, g_{ij}), (x, y, y^i_j)) \mapsto (x,\, y^k_i\, y^l_j\, g_{kl}).$$

The formulae for the right Π_1-action on $\mathcal{F}$ can be interpreted as the coordinate changes of $\mathcal{F}$ in Vessiot notation:

$$\hat{x}^i = y^i(x), \qquad g_{ij} = y^k_i(x)\, y^l_j(x)\, \hat{g}_{kl}. \tag{3.3}$$

The Π_1-action on the natural bundle for first order conformal structures of Example 3.3 (2) can now be given explicitly:

$$\tilde{g}_{ij} \mapsto \frac{g_{ij}}{g_{11}} = \frac{y^k_i\, y^l_j\, g_{kl}}{y^k_1\, y^l_1\, g_{kl}} = \frac{y^1_i\, y^1_j + y^k_i\, y^l_j\, \tilde{g}_{kl}}{y^1_1\, y^1_1 + y^k_1\, y^l_1\, \tilde{g}_{kl}}. \tag{3.4}$$

◇

3.2 Natural Bundle Functors

In this section we briefly introduce the natural bundle functor $\mathcal{F}$ which turns a manifold X into a natural Π_q-bundle $\mathcal{F}(X)$. We show how to construct a natural bundle functor and the connection to natural bundles of Definition 3.1. As a last step we turn to operators between natural bundle functors which are related to differential operators. The exposition follows [KMS93, §14], except for the distinction between a natural bundle and the functor. In contrast to the remaining sections, we are here working with left actions on the natural bundles.

Natural bundle functors are of interest in Chapter 6 because they allow a simple definition for equivalence of geometric objects.

Definition 3.9. [KMS93, 14.1] A *natural bundle functor* $\mathcal{F}$ on the category $\mathcal{M}f_n$ of n-dimensional manifolds is a covariant functor $\mathcal{F} : \mathcal{M}f_n \to \mathcal{FM}$ into the category of fibre bundles satisfying the following conditions:

(1) (Prolongation) $\mathcal{B} \circ \mathcal{F} = \mathrm{id}_{\mathcal{M}f_n}$, where $\mathcal{B} : \mathcal{FM} \to \mathcal{M}f_n$ is the base functor which projects a bundle to its base.

(2) (Locality) If $\iota : U \to X$ is an inclusion of an open submanifold, then $\mathcal{F}(U) = \pi_X^{-1}(U)$ and $\mathcal{F}(\iota) : \pi_X^{-1}(U) \to \mathcal{F}(X)$ is the inclusion (with the projection $\pi_X : \mathcal{F}(X) \to X$).

(3) (Regularity) Let $X, Y \in \mathcal{M}f_n$ and P be manifolds. If $f : P \times X \to Y$ is a smooth map such that for all $p \in P$ the maps $f_p = f(p, -) : X \to Y$ are local diffeomorphisms then $\tilde{\mathcal{F}}(f) : P \times \mathcal{F}(X) \to \mathcal{F}(Y)$, defined by $\tilde{\mathcal{F}}(f) = \mathcal{F}(f_p)$, is smooth, i. e. smoothly parametrised systems of local diffeomorphisms are transformed into smoothly fibred local isomorphisms. ◇

With extra effort (see [ET79]) it is possible to omit the last condition. A well-known example of a natural bundle functor is the tangent bundle functor T which maps each X to TX. In order to construct all natural bundle functors, we need the frame bundle functor $\mathcal{P}^q$ that maps each manifold X to its q-th order frame bundle $P^q \to X$. On the morphisms $\varphi : X \to Y$, the composition of jets induces a morphism

$$\mathcal{P}^q(\varphi) \;:\; \mathcal{P}^q(X) \to \mathcal{P}^q(Y) \;:\; j_q(\psi) \mapsto j_q(\psi) \circ j_q(\varphi), \qquad \psi : \mathbb{R}^n \to X.$$

The following useful Proposition has been proved by Palais and Terng [PT77].

Proposition 3.10. A natural bundle functor $\mathcal{F}$ has a finite order, i. e. there exists a $q \in \mathbb{Z}_{\geq 0}$ depending on $\mathcal{F}$ such that for all local diffeomorphisms $f, g : X \to Y$ and every $x \in X$ the equality $j_q(f)(x) = j_q(g)(x)$ implies $\mathcal{F}(f)|_{\mathcal{F}(M)_x} = \mathcal{F}(g)|_{\mathcal{F}(M)_x}$. ◇

Both examples T and $\mathcal{P}^q$ of natural bundle functors obviously yield the natural bundles TX and P^q if applied to a manifold X. A special case of the next proposition helps to recover the Π_q-action on the value $\mathcal{F}(X)$ of a natural bundle functor. It involves the bundle $\Pi_q(X, Y) \subseteq J_q(X \times Y)$ that consists of all q-jets of local diffeomorphisms $X \to Y$.

Lemma 3.11. [KMS93, 14.4] Let $\mathcal{F}$ be a natural bundle functor of order q and X, Y be n-dimensional manifolds. There are smooth maps

$$\mathcal{F}_{X,Y} : \Pi_q(X, Y) \times_X \mathcal{F}(X) \to \mathcal{F}(Y) : (j_q(f)(x), u) \mapsto \mathcal{F}(f)(u),$$

called associated maps. ◇

Proof. Proposition 3.10 implies that $\mathcal{F}_{X,Y}$ is well-defined. Because smoothness is a local local property, it is possible to restrict X and Y to open subsets which are isomorphic to $\mathbb{R}^n$. The polynomial representation of each q-jet gives a smoothly parametrised system of local diffeomorphisms such that $\mathcal{F}_{\mathbb{R}^n,\mathbb{R}^n}$ coincides with $\tilde{\mathcal{F}}$ from the regularity property of $\mathcal{F}$. □

It follows directly that the bundle $\mathcal{F}(X)$ is a natural bundle in the sense of Definition 3.1. We can again apply Proposition 2.10 to $\mathcal{F}(X)$ and see that it is associated to the frame bundle P^q. This carries over to natural bundle functors, which are also completely determined by the GL_q-action on the fibre F.

Theorem 3.12. [KMS93, 14.6] Each natural bundle functor $\mathcal{F}$ of order q is naturally equivalent to the functor $\mathcal{G} = L_F \circ \mathcal{P}^q$. L_F is the functor that associates a manifold F with GL_q-action to a principal GL_q-bundle:

$$L_F(P) = P \times_{\mathrm{GL}_q} F$$

(On morphisms, we have $L_F(f) = (f, \mathrm{id}_F)$). In other words, there is a bijection between q-th order natural bundle functors and GL_q-actions on smooth manifolds. ◇

A consequence of this theorem is the bijection between natural bundles and natural bundle functors, as they are both uniquely defined the fibre.

We now turn to natural operators Φ between natural bundle functors $\mathcal{F}$ and $\mathcal{G}$ which turn sections of $\mathcal{F}(X) \to X$ into sections of $\mathcal{G}(X) \to X$. For each manifold X, a natural operator determines a differential operator according to Definition 1.22 which is induced by a smooth morphism of natural bundles. Compare the next definition with [KMS93, §14.13-14].

Definition 3.13. A *natural operator* $\Phi : \mathcal{F} \rightsquigarrow \mathcal{G}$ between two natural bundle functors $\mathcal{F}$ and $\mathcal{G}$ is a system of differential operators $\Phi_X : \Gamma(\mathcal{F}(X)) \to \Gamma(\mathcal{G}(X))$ satisfying the following conditions for all sections s of$\mathcal{F}(X) \to X$:

(1) For all diffeomorphisms $f : X \to Y$ we have:

$$\Phi_Y(\mathcal{F}(f) \circ s \circ f^{-1}) = \mathcal{G}(f) \circ \Phi_X(s) \circ f^{-1}.$$

(2) $\Phi_U(s|_U) = \Phi_X(s)|_U$ for all open submanifolds $U \subseteq X$.

A natural operator is of order $r \in \mathbb{N}$ if all Φ_X are of order r. ◇

In Chapter 6, natural operators help to decide the equivalence of geometric objects. Similar to natural bundle functors, all natural operators are uniquely defined by equivariant maps on certain fibres. For zero order operators, this is connected to Proposition 2.11, and the generalisation to higher order operators uses the following proposition.

Proposition 3.14. [KMS93, 14.16] For every q-th order natural bundle functor $\mathcal{F}$ on $\mathcal{M}f_n$ its composition with the r-th order jet bundle functor J_r is a natural bundle functor $J_r \circ \mathcal{F}$ of order $q + r$. ◇

A proof of this proposition for natural bundles will be given in Proposition 3.26. It reduces the question of r-th order natural operators $\mathcal{F} \rightsquigarrow \mathcal{G}$ to zero order operators $J_r \circ \mathcal{F} \rightsquigarrow \mathcal{G}$. By Definition B.14, the fibre determining $J_r \circ \mathcal{F}$ is the algebraic prolongation $F^{(r)}$ of F (called $T^r_n F$ in [KMS93]). As in Proposition B.17, natural operators are determined by equivariant maps.

Theorem 3.15. [KMS93, 14.18] There is a canonical bijective correspondence between the set of r-th order natural operators $\Phi : \mathcal{F} \rightsquigarrow \mathcal{G}$ and the set of all smooth GL_{q+r}-equivariant maps between the GL_{q+r}-spaces $F^{(r)}$ and G. ◇

In this section, we restricted to the full jet groupoid Π_q, because subgroupoids $\Theta_q \leq \Pi_q$ are only defined for a specific base manifold X. A possible generalisation requires equivalent geometric objects on all n-dimensional manifolds X. We refer to Chapter 6 on the equivalence problem and especially to Section 6.1.2 on the relative equivalence problem.

3.3 Natural Bundles and Jet Groupoids

This section deals with the correspondence between natural Θ_q-bundles $\mathcal{F}$ and subgroupoids $\mathcal{R}_q \leq \Theta_q$. The main goal is to show that each jet groupoid can locally be defined by an exact sequence (1.7):

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow \Theta_q \underset{\omega \circ s}{\overset{\Phi_\omega}{\rightrightarrows}} \mathcal{F}, \tag{3.5}$$

where $\mathcal{F}$ is a natural Θ_q-bundle and ω is a geometric object. The differential map Φ_ω is constructed by the Θ_q-action on $\mathcal{F}$. Conversely, each geometric object ω on $\mathcal{F}$ determines a symmetry groupoid $\mathcal{R}_q(\omega)$.

In this section we generalise results of Pommaret [Pom78, Ch. 7] who constructed the above sequence for transitive subgroupoids $\mathcal{R}_q \leq \Pi_q$. His results are based on the work of Lie [Lie91] and Vessiot [Ves03]. Lie showed that each groupoid can be defined by differential invariants and constructed a coordinate version of the map Φ_ω. Vessiot realised that Π_q acts on the differential invariants, which leads to the natural bundle $\mathcal{F}$.

All bundles and maps are explicitly computed for a small example in Section 3.3.1. A MAPLE version of this example follows in Section 5.1.

Throughout the section, $\Theta_q \leq \Pi_q(X \times X)$ is a jet groupoid which is formally integrable as a system of PDEs. Denote the algebroid of Θ_q by $\mathfrak{g}_{\Theta_q}$ and let $\mathcal{R}_q \leq \Theta_q$ be an arbitrary jet subgroupoid with algebroid R_q. We start with the definition of differential invariants.

Definition 3.16. [Pom78, Def. 7.1.44] A smooth map $\Phi : \Theta_q \to \mathbb{R}$ is called *differential invariant* on Θ_q under the action of $\mathcal{R}_q$ if

$$\Phi(r_q f_q) = \Phi(f_q) \qquad \forall\, r_q \in \mathcal{R}_q,\ f_q \in \Theta_q,\ t(f_q) = s(r_q).$$

A set $\{\Phi^\tau | 1 \leq \tau \leq k\}$ of differential invariants is called *complete* if it is a maximal set of invariants such that all Φ^τ are functionally independent. ⋄

In coordinates (x, y) of $X \times X$, the left $\mathcal{R}_q$-action on Θ_q does not change the source coordinates x. They are trivial differential invariants, which are often omitted. All remaining differential invariants are of the form $\Phi(y, y_q)$.

Differential invariants are effectively computed by integrating an involutive distribution as the next proposition shows. For distributions and their integral manifolds we refer to [MM03], [Sha97, Ch. 1, §2] and [Ste64, III.5]. Due to the use of Frobenius' theorem, the results in this section are local.

Lemma 3.17. Let R_q be the algebroid of $\mathcal{R}_q$. Then $\Phi : \Theta_q \to \mathbb{R}$ is a differential invariant under the left $\mathcal{R}_q$-action if and only if its Lie derivative

$$L_{\eta_q}\Phi = L_{\sharp(\eta_q)}\Phi = 0$$

vanishes for all sections η_q of $R_q \to X$. ⋄

Proof. $R_q \leq \mathfrak{g}_{\Theta_q}$ is not only a subbundle of $\mathfrak{g}_{\Theta_q} \to X$ but also a Lie subalgebroid. So $\sharp$ induces an involutive distribution on Θ_q. Integrating the flows of $\sharp(\eta_q)$ for $\eta_q \in \Gamma(R_q)$ is equivalent to the left $\mathcal{R}_q$-action on Θ_q. □

Example 3.18. On $X = \mathbb{R}$, the groupoid of affine transformations $\mathcal{R}_2 \leq \Pi_2$ is defined by $y_{xx} = 0$. Constructing the algebroid R_2 of $\mathcal{R}_2$ via right invariant vector fields, we obtain the equation

$$R_2 \ :\ \eta_{yy} = 0$$

defining a subbundle of $J_2(T) \to X$. The involutive distribution on Π_2 is calculated with equation (C.13):

$$\sharp((\eta, \eta_y, \eta_{yy} = 0)) = \eta\frac{\partial}{\partial y} + \eta_y(y_x\frac{\partial}{\partial y_x} + y_{xx}\frac{\partial}{\partial y_{xx}}),$$

where (η, η_y) is an arbitrary section of $R_2 \to X$. So the distribution is generated by $C^\infty(X)$-linear combinations of the vector fields

$$\frac{\partial}{\partial y}, \quad y_x \frac{\partial}{\partial y_x} + y_{xx} \frac{\partial}{\partial y_{xx}}.$$

According to Lemma 3.17, a differential invariant $\Phi(y, y_x, y_{xx}) : \Pi_2 \to \mathbb{R}$ under the left $\mathcal{R}_2$-action must be of the form

$$\Phi(y, y_x, y_{xx}) = \Phi(\frac{y_{xx}}{y_x}).$$

$\mathcal{R}_2$ can also be defined by the equation $\frac{y_{xx}}{y_x} = 0$ depending on differential invariants only. ◇

The main result of two papers by Lie [Lie91] is that the defining equations for an infinite continuous group of transformations can be written in terms of differential invariants. A continuous group of transformations is a subgroupoid $\mathcal{R}_q \leq \Pi_q$. His proof uses infinitesimal transformations, which are translated into the algebroid action of R_q on Π_q. We generalise Lie's result to subgroupoids $\Theta_q \leq \Pi_q$ in a straightforward way.

Theorem 3.19. Each subgroupoid $\mathcal{R}_q \leq \Theta_q$ may locally be defined by differential invariants on Θ_q. ◇

Proof. $\mathcal{R}_q$ is a subbundle of $\Theta_q \to X$ with algebroid $R_q \leq \mathfrak{g}_{\Theta_q}$. The restriction of the involutive distribution $\sharp(R_q)$ to $\mathcal{R}_q$ is $T^s(\mathcal{R}_q)$ so for each $x \in X$, $\mathcal{R}_q(x, -)$ is an integral submanifold of the distribution $\sharp(R_q)$ through 1_x.

Let $\{\Phi^\tau | \tau = 1, \ldots, k\}$ be a complete set of differential invariants, which exists by Frobenius' theorem. Then $\Phi^\tau(y, y_q) = \Phi^\tau(1_x)$ determines $\mathcal{R}_q$ in a neighbourhood of 1_x. □

Definition 3.20. [Pom78, Def. 1.4.4] The equations $\Phi^\tau(y, y_q) = \Phi^\tau(1_x)$ defining $\mathcal{R}_q$ in terms of differential invariants Φ^τ are called *Lie form.* ◇

Writing the equations for a groupoid $\mathcal{R}_q$ in Lie form separates the source coordinates from target and higher order jet variables, because all equations are of the form $\Phi(y, y_q) = \omega(x)$. This is convenient for a kernel representation of $\mathcal{R}_q = \ker_{\omega_0}(\Phi)$ as in equation (1.7).

Example 3.21. The Lie form for the groupoid of projective transformations on $X = \mathbb{R}$ (see Example 2.15 (3)) is:

$$\frac{y_{xxx}}{y_x} - \frac{3}{2}\frac{y_{xx}^2}{y_x^2} = 0.$$

◇

In the exact sequence (3.5), we have constructed the differential map Φ_{ω_0} for the section $\omega_0^\tau(x) = \Phi^\tau(1_x)$ and it remains to construct the natural bundle $\mathcal{F}$. Define a right Θ_q-action on smooth maps $\Phi : \Theta_q \to \mathbb{R}$ by right multiplication:

$$\Phi(f_q)g_q := \Phi(f_q g_q), \qquad \forall f_q, g_q \in \Theta_q,\ t(g_q) = s(f_q).$$

If Φ is a differential invariant, the associative law on Θ_q implies that the map $\Phi(f_q)g_q$ is also a differential invariant:

$$\Phi(f_q)g_q = \Phi(f_q g_q) = \Phi(r_q f_q g_q), \qquad \forall r_q \in \mathcal{R}_q,\ t(f_q) = s(r_q). \tag{3.6}$$

In other words, Θ_q acts on a complete set of differential invariants under the left $\mathcal{R}_q$-action. The differential invariants are local coordinates of a natural bundle. Vessiot [Ves03] proved this for Π_q, using the infinitesimal algebroid action of $J_q(T)$ on Π_q. The following theorem takes up his ideas to construct natural Θ_q-bundles. It is based on Appendix C.2, where jet algebroids are constructed both for left and right invariant vector fields.

Theorem 3.22. A complete set $\{\Phi^\tau | \tau = 1, \ldots, k\}$ of differential invariants under the left $\mathcal{R}_q$-action on Θ_q defines a natural Θ_q-bundle $\mathcal{F} \to X$. ⋄

Proof. Construct the natural bundle $\mathcal{F}$ by a morphism of natural Θ_q-bundles

$$\Theta_q \to \mathcal{F} : (x, y, y_q) \mapsto (x, u^\tau = \Phi^\tau(y, y_q)), \tag{3.7}$$

where the differential invariants Φ^τ are taken as coordinates u^τ of $\mathcal{F}$. Specifying the infinitesimal right $\mathfrak{g}_{\Theta_q}$-action is equivalent to the definition of infinitesimal coordinate changes on $\mathcal{F}$.

The infinitesimal left and right $J_q(T)$-actions $\sharp$ and $\flat$ on Π_q are defined in equations (C.13) and (C.15). Since Θ_q is a subgroupoid of Π_q and R_q is a subalgebroid of $\mathfrak{g}_{\Theta_q}$ they can be restricted to the actions:

$$\sharp \ :\ \Gamma(R_q) \to \mathfrak{X}^s(\Theta_q) : \eta_q \mapsto \eta^j(y)\,\frac{\partial}{\partial y^j} + \eta^j_\nu(y)\, B^\nu_j(q),$$
$$\flat \ :\ \Gamma(\mathfrak{g}_{\Theta_q}) \to \mathfrak{X}^t(\Theta_q) : \xi_q \mapsto \xi^i(x)\,\frac{\partial}{\partial x^i} + \xi^i_\mu(x)\, A^\mu_i(q).$$

This follows directly from the defintion of $\sharp$ and $\flat$ in equations (C.7) and (C.10). Here $\eta_q = (\eta(y), \eta_q(y))$ stands for a section of $R_q \to X$ and $\xi_q = (\xi(x), \xi_q(x))$ is a section of $\mathfrak{g}_{\Theta_q} \to X$. The Lie derivatives L_{η_q} and $L_{\xi_q} = L_{\flat(\xi_q)}$ commute by Proposition C.9 and thus $L_{\xi_q}\Phi^\tau$ of a differential invariant is an invariant again:

$$L_{\eta_q} L_{\xi_q} \Phi^\tau = L_{\xi_q} L_{\eta_q} \Phi^\tau = L_{\xi_q} 0 = 0.$$

So there are functions

$$L_{\xi_q}\Phi^\tau = L^\tau(\xi_q, x, \Phi) = \xi^i_\mu L^{\tau,\mu}_i(x, \Phi) \tag{3.8}$$

expressing the Lie derivative in terms of all differential invariants (including the trivial ones). The trivial invariants fulfill $L_{\xi_q} x^i = \xi^i$. It follows that the infinitesimal $\mathfrak{g}_{\Theta_q}$-action on $\mathcal{F}$ is given by:

$$L_{\xi_q} : \Gamma(\mathfrak{g}_{\Theta_q}) \to \mathfrak{X}\mathcal{F} : \xi_q \mapsto \xi^i(x)\frac{\partial}{\partial x^i} + \xi^i_\mu(x) L_i^{\tau,\mu}(x,u)\frac{\partial}{\partial u^\tau}. \tag{3.9}$$

□

Effectively, we have computed orbit space of the left $\mathcal{R}_q$-action on Θ_q, so denote the natural bundle by $\mathcal{F} = \mathcal{R}_q \backslash \Theta_q$. Due to the R_q-action, the result is only local. To compute the finite Θ_q-action on $\mathcal{F}$, take equation (3.6) and express the groupoid action on the differential invariants $\Phi^\tau(f_q g_q) = \chi^\tau(g_q, \Phi(f_q))$ in terms of differential invariants. In practise, the infinitesimal action is more efficient and for large examples it is even impossible to compute the groupoid action.

In the case of transitive subgroupoids $\mathcal{R}_q \le \Pi_q$, it is possible to simplify the calculation of $\mathcal{F}$. This connects Theorem 3.22 with the results of Pommaret [Pom78, §7.2]. The full $\mathcal{R}_q$-action can be replaced by the action of the isotropy group $\mathcal{R}_q(y_0, y_0)$.

Proposition 3.23. If $\mathcal{R}_q$ and Θ_q are transitive, the bundle $\mathcal{F} = \mathcal{R}_q \backslash \Theta_q$ is isomorphic to the orbit space $\mathcal{R}_q(y_0, y_0)\backslash \Theta_q(-, y_0)$ of the isotropy group action. ⋄

Proof. If $\mathcal{R}_q$ is transitive, each orbit of the $\mathcal{R}_q$-action on Θ_q contains an element of $\Theta_q(-, y_0)$ for an arbitrary $y_0 \in X$. So the orbit space is isomorphic to the orbit space of $\mathcal{R}_q(y_0, y_0)$ on $\Theta_q(-, y_0)$. □

By Proposition 2.10, the natural bundle $\mathcal{F} = \mathcal{R}_q(y_0, y_0)\backslash \Theta_q(-, y_0)$ is isomorphic to the associated bundle

$$\mathcal{F} \cong P_{\Theta_q} \times_{G_q} G_q / \mathcal{R}_q(y_0, y_0),$$

where $G_q \cong \Theta_q(y_0, y_0)$ is the isotropy group of Θ_q.

For a single groupoid $\mathcal{R}_q \le \Theta_q$ we have constructed the exact sequence (3.5) defining $\mathcal{R}_q$ as the kernel $\ker_{\omega_0}(\Phi_{\omega_0})$. The main result of this section shows that the differential map Φ_{ω_0} is induced by the Θ_q-action on the section ω_0 of $\mathcal{F} \to X$. Furthermore, every section ω of $\mathcal{F} \to X$ determines a jet groupoid $\mathcal{R}_q(\omega)$. We call $\mathcal{R}_q(\omega)$ the *symmetry groupoid* of the geometric object ω. The section ω_0 which was used to construct $\mathcal{F}$ is called special section.

Theorem 3.24. Each section ω of a natural Θ_q-bundle $\mathcal{F} \to X$ locally defines a jet groupoid $\mathcal{R}_q(\omega) = \mathrm{Stab}^q_{\mathcal{F}}(\omega) \le \Theta_q$. Conversely, each jet groupoid $\mathcal{R}_q \le \Theta_q$ defines a natural Θ_q-bundle $\mathcal{F} \to X$ with section ω_0, such that $\mathcal{R}_q = \mathrm{Stab}^q_{\mathcal{F}}(\omega_0)$ is the full symmetry groupoid of ω_0 in Θ_q. ⋄

Proof. Define the symmetry groupoid $\mathrm{Stab}^q_{\mathcal{F}}(\omega)$ via the Θ_q-action on $\mathcal{F}$

$$\Phi_\omega : \Theta_q \to \mathcal{F} : f_q \mapsto \omega(y) f_q, \qquad y = t(f_q)$$

as the kernel $\ker_\omega(\Phi_\omega)$ of the differential map Φ_ω or equivalently by the exact sequence

$$0 \longrightarrow \mathrm{Stab}^q_{\mathcal{F}}(\omega) \longrightarrow \Theta_q \overset{\Phi_\omega}{\underset{\omega \circ s}{\rightrightarrows}} \mathcal{F} \tag{3.10}$$

as in equations (1.7), (3.5). The symmetry equations are:

$$\omega(y) \circ f_q = \omega(x), \qquad x = s(f_q),\ y = t(f_q). \tag{3.11}$$

The Θ_q-action implies that $\mathrm{Stab}^q_{\mathcal{F}}(\omega)$ is a groupoid since it is closed under μ and inversion. Furthermore all id_x are symmetries such that $\iota(X) \subseteq \mathrm{Stab}^q_{\mathcal{F}}$. As ω is smooth, there is an open submanifold $Y \subseteq X$ where Φ_ω has constant rank. We restrict X to this open subset. If the fibre F of $\mathcal{F}$ is homogeneous, Φ_ω is automatically of constant rank. By the implicit function theorem, $\mathrm{Stab}^q_{\mathcal{F}}(\omega)$ is a Lie groupoid.

The converse follows from Theorem 3.22 or Proposition 3.23 in the transitive case. The section ω_0 is constructed by the map

$$\mathcal{R}_q \to \mathcal{F} = \mathcal{R}_q \backslash \Theta_q : r_q \mapsto \mathcal{R}_q r_q.$$

For each $x \in X$, all $r_q \in \mathcal{R}_q(x, -)$ are in the same $\mathcal{R}_q$-orbit such that the map

$$\omega_0 : X \to \mathcal{F} : x \mapsto \mathcal{R}_q(x, -)$$

is a well-defined section of $\mathcal{F} \to X$. In the transitive case replace $\mathcal{R}_q$ by $\mathcal{R}_q(-, y_0)$.

An element $f_q \in \Theta_q$ stabilises the section ω_0 if and only if $r_q f_q \in \mathcal{R}_q$ □

Remark 3.25. The coordinate expressions of the symmetry equations (3.11) for $\mathcal{R}_q(\omega)$,

$$\omega(y) \circ f_q = \omega(x), \qquad x = s(f_q),\ y = t(f_q),$$

are in Lie form. If the section is not specified, they are called *general Lie form*. The corresponding equations for the algebroid $R_q(\omega)$ are called *Medolaghi form* (cf. [Pom83, p. 294]). They are obtained by linearising the general Lie form and then pulling them back with the unit embedding ι. Alternatively, we can use the infinitesimal $\mathfrak{g}_{\Theta_q}$-action:

$$L_{\xi_q} u^\tau|_{u=\omega(x)} = L_{\xi_q} \omega^\tau(x).$$

This is equivalent to setting the characteristic of the vector fields (3.9) to zero

$$Q^\tau = \xi^i_\mu L^{\tau,\mu}_i(u) - \xi^i u^\tau_i = 0$$

and plugging in the section $u^\tau = \omega^\tau(x)$:

$$\xi^i_\mu L^{\tau,\mu}_i(\omega(x)) - \xi^i \partial_{x^i} \omega^\tau(x) = 0. \tag{3.12}$$

If ω is not specified, the equations are called *general Medolaghi form*. For the characteristic of a vector field see Definition 1.18 and for further reading on characteristics and evolutionary vector fields see [Olv93, §5.1]. ⋄

Theorem 3.22 shows that PDE systems for jet groupoids always have an interpretation as symmetries of geometric objects. The remaining part of this chapter shows how to reach formal integrable jet groupoids using natural bundles. On natural bundles it is possible to compute prolongations and projections for a generic section which is not specified in advance. This leads to a classification of geometric objects, which will be applied in Chapter 6 to decide the equivalence of geometric objects.

3.3.1 Example

To illustrate the construction of a natural bundle, we follow all the steps of this section for the flat metric on a two-dimensional base X. This is a special case of Example 3.8. We will repeat this calculation in Section 5.1 as an introduction to the computations in MAPLE.

Denote the independent variables by (x^1, x^2) and the dependent ones by (y^1, y^2). They are base and fibre coordinates of the bundle $X \times X$. The equations

$$(y_1^1)^2 + (y_1^2)^2 = 1, \quad y_2^1 y_1^2 + y_1^1 y_2^2 = 0, \quad (y_2^1)^2 + (y_2^2)^2 = 1 \tag{3.13}$$

define a subgroupoid $\mathcal{R}_1 \leq \Pi_1$. It is the groupoid of isometries of the flat metric on X and we construct the bundle $\mathcal{F} = S^2 T^*$ of metrics. To define $\mathcal{R}_1$ by differential invariants, we apply Lemma 3.17. First compute the algebroid $R_1 = \iota^* V(\mathcal{R}_1)$ of $\mathcal{R}_1$:

$$(\iota^* \circ \delta)\left((y_1^1)^2 + (y_1^2)^2\right) = \iota^*(2y_1^1\,\eta_1^1 + 2y_1^2\,\eta_1^2) = 2\eta_1^1,$$

where (η^1, η^2) are the fibre coordinates of T and δ ist the vertical derivative. All defining equations for R_1 are:

$$2\eta_1^1 = 0, \quad \eta_2^1 + \eta_1^2 = 0, \quad 2\eta_2^2 = 0. \tag{3.14}$$

Involutive Distribution on Π_1 and Differential Invariants

The distribution $\sharp(R_1)$ involves the map (C.13) which is constructed by prolongation:

$$\rho_1(\eta^i \partial_{y^i}) = \eta^i \partial_{y^i} + \eta^i_j B^j_i(1).$$

Using the defining equations for R_1, the involutive distribution $\sharp(\eta, \eta_1)$ on $V(\Pi_1)$ is generated by:

$$\partial_{y^1}, \quad \partial_{y^2}, \quad B_2^1(1) - B_1^2(1) = -y_1^2\,\partial_{y_1^1} - y_2^2\,\partial_{y_2^1} + y_1^1\,\partial_{y_1^2} + y_2^1\,\partial_{y_2^2}. \tag{3.15}$$

With these vector fields, it is easy to see that the equations for $\mathcal{R}_1$ are already in Lie form and the left hand sides of equation (3.13) are the differential invariants. The special section ω_0 of $\mathcal{F} \to X$ is given by the right hand sides of equation (3.13).

If the differential invariants were unknown, the linear system of PDEs

$$\begin{aligned} \partial_{y^1}\,\Phi(y,y_1) &= 0 \\ \partial_{y^2}\,\Phi(y,y_1) &= 0 \\ (B_2^1(1)-B_1^2(1))\,\Phi(y,y_1) &= 0 \end{aligned}$$

has to be solved. The solution

$$\Phi(y,y_1)=\Phi(\,(y_1^1)^2+(y_1^2)^2,\ y_2^1y_1^2+y_1^1y_2^2,\ (y_2^1)^2+(y_2^2)^2\,)$$

depends on the differential invariants. Each differential invariant corresponds to a fibre coordinate of $\mathcal{F}$. We choose them as $(u^{ij})=(u^{11},u^{12},u^{22})$ according to their interpretation as entries of a metric. The projection $\Pi_1\to\mathcal{F}$ computed in Theorem 3.22 therefore is:

$$u^{11}=(y_1^1)^2+(y_1^2)^2,\quad u^{12}=y_2^1y_1^2+y_1^1y_2^2,\quad u^{22}=(y_2^1)^2+(y_2^2)^2. \tag{3.16}$$

The special section ω_0 in coordinates determines the flat metric:

$$\omega_0^{11}(x)=1,\quad \omega_0^{12}(x)=0,\quad \omega_0^{22}=1.$$

Natural Bundle $\mathcal{F}$ – Infinitesimal $J_1(T)$-Action

The infinitesimal $J_1(T)$-action on $\mathcal{F}$ is computed according to Theorem 3.22 using the map $\flat$:

$$\flat(\xi,\xi_1)=\xi^i\,\partial_{x^i}+\xi_j^i\,A_i^j(q).$$

By Proposition C.8, the vector fields $A_i^j(1)$ are:

$$A_i^j(1)=-y_i^1\,\partial_{y_j^1}-y_i^2\,\partial_{y_j^2},\qquad 1\le i,k\le 2.$$

Apply the infinitesimal $J_1(T)$-action to equation (3.16) and express the result in the coordinates u^{ij} again:

$$\begin{aligned} \xi_j^iA_i^j(1)u^{11} &= -2\xi_1^1u^{11}-2\xi_1^2u^{12} \\ \xi_j^iA_i^j(1)u^{12} &= -\xi_2^1u^{11}-(\xi_1^1+\xi_2^2)u^{12}-\xi_1^2u^{22} \\ \xi_j^iA_i^j(1)u^{22} &= -2\xi_2^2u^{22}-2\xi_2^1u^{12} \end{aligned} \tag{3.17}$$

Collecting for ξ_j^i, we obtain the vector fields $L_i^j(u)$ corresponding to $A_i^j(1)$:

$$\begin{aligned} L_1^1(u) &= -2u^{11}\,\partial_{u^{11}} \quad -u^{12}\,\partial_{u^{12}}, \\ L_2^1(u) &= \quad -u^{11}\,\partial_{u^{12}} \quad -2u^{12}\,\partial_{u^{22}}, \\ L_1^2(u) &= -2u^{12}\,\partial_{u^{11}} \quad -u^{22}\,\partial_{u^{12}}, \\ L_2^2(u) &= \quad -u^{12}\,\partial_{u^{12}} \quad -2u^{22}\,\partial_{u^{22}}. \end{aligned}$$

Finally, we obtain the infinitesimal $J_1(T)$-action on $\mathcal{F}$:

$$\Gamma(J_1(T))\to\mathfrak{X}\mathcal{F}:\xi_1\mapsto\xi^i(x)\,\partial_{x^i}+\xi_j^i(x)\,L_i^j(u).$$

Natural Bundle $\mathcal{F} - \Pi_1$-Action

The example is small enough to compute the Π_1-action on $\mathcal{F}$. For coordinate systems x, y and z of X, the composition of first order jets is done by multiplication of Jacobian matrices:

$$z^i_{x^j} = z^i_{y^k} y^k_j,$$

where both x- and y-jets of z are present. Plug this formula into the differential invariant

$$u^{11} = (z^1_{x^1})^2 + (z^2_{x^1})^2$$

and express everything by the differential invariants $\hat{u}^{ij}(z_y)$:

$$\begin{aligned} u^{11} &= (z^1_{x^1})^2 + (z^2_{x^1})^2 \\ &= (z^1_{y^1} y^1_1 + z^1_{y^2} y^2_1)^2 + (z^2_{y^1} y^1_1 + z^2_{y^2} y^2_2)^2 \\ &= (y^1_1)^2 \left[(z^1_{y^1})^2 + (z^2_{y^1})^2 \right] + 2\, y^1_1 y^2_1 \left[z^1_{y^1} z^2_{y^2} + z^1_{y^2} z^2_{y^1} \right] \\ &\quad + (y^2_1)^2 \left[(z^1_{y^2})^2 + (z^2_{y^2})^2 \right] \\ &= (y^1_1)^2\, \hat{u}^{11} + 2\, y^1_1 y^2_1\, \hat{u}^{12} + (y^2_1)^2\, \hat{u}^{22}. \end{aligned} \tag{3.18}$$

The remaining two transformations are obtained analogously, such that the action on the fibre of $\mathcal{F}$ is:

$$\begin{aligned} u^{11} &= (y^1_1)^2\, \hat{u}^{11} + 2\, y^1_1 y^2_1\, \hat{u}^{12} + (y^2_1)^2\, \hat{u}^{22}, \\ u^{12} &= y^1_1 y^1_2\, \hat{u}^{11} + \left(y^1_1 y^2_2 + y^1_2 y^2_1 \right) \hat{u}^{12} + y^2_1 y^2_2\, \hat{u}^{22}, \\ u^{22} &= (y^1_2)^2\, \hat{u}^{11} + 2\, y^1_2 y^2_2\, \hat{u}^{12} + (y^2_2)^2\, \hat{u}^{22}. \end{aligned} \tag{3.19}$$

The action is linear in the fibre coordinates $\hat{u}$ and we have explicitly computed the natural bundle $\mathcal{F} \cong S^2T^*$ of Example 3.8 in the 2-dimensional case. The above equations coincide with the coordinate changes of S^2T^* in Vessiot notation (3.3).

General Lie and Medolaghi Form

The general Lie form for a symmetry groupoid $\mathcal{R}_1(\omega)$ follows from the symmetry equations (3.11) by plugging an arbitrary section ω of $\mathcal{F} \to X$into the Π_1-action (3.19) on $\mathcal{F}$:

$$\begin{array}{ccccccc} (y^1_1)^2\, \omega^{11}(y) & + & 2\, y^1_1 y^2_1\, \omega^{12}(y) & + & (y^2_1)^2\, \omega^{22}(y) & = & \omega^{11}(x), \\ y^1_1 y^1_2\, \omega^{11}(y) & + & \left(y^1_1 y^2_2 + y^1_2 y^2_1 \right) \omega^{12}(y) & + & y^2_1 y^2_2\, \omega^{22}(y) & = & \omega^{12}(x), \\ (y^1_2)^2\, \omega^{11}(y) & + & 2\, y^1_2 y^2_2\, \omega^{12}(y) & + & (y^2_2)^2\, \omega^{22}(y) & = & \omega^{22}(x). \end{array}$$

For the special section ω_0, the Lie form coincides with equation (3.13). We compute the general Medolaghi form for the symmetry algebroid $R_1(\omega)$ using

Remark 3.25:

$$\begin{array}{rcrcrcrcl} 2\,\xi_1^1\,\omega^{11}(x) & & & + & 2\,\xi_1^2\,\omega^{12}(x) & & & = & -\xi^i\partial_{x^i}\omega^{11}(x), \\ \xi_1^1\omega^{12}(x) & + & \xi_2^1\omega^{22}(x) & + & \xi_1^2\omega^{11}(x) & + & \xi_2^2\omega^{12}(x) & = & -\xi^i\partial_{x^i}\omega^{12}(x), \\ & & & & 2\,\xi_1^2\omega^{12}(x) & + & 2\,\xi_2^2\omega^{22}(x) & = & -\xi^i\partial_{x^i}\omega^{22}(x). \end{array}$$

We continue this example with the prolongation and projection in Section 5.1.

3.4 Prolongation and Projection

Whenever there is a geometric object ω on a natural bundle $\mathcal{F}$, the symmetry groupoid $\mathcal{R}_q(\omega) = \mathrm{Stab}^q_{\mathcal{F}}(\omega)$ is a system of PDEs. For each $\mathcal{R}_q(\omega)$, the formal integrability may be checked individually, which may lead to large computations. With natural bundles, only one calculation is necessary. It yields integrability conditions which can be applied to each geometric object of interest.

In this section, we prepare the integrability conditions and show how the prolongation and projection of symmetry groupoids is done in terms of natural bundles and their sections. In the last part, we present an example and explicitly show all necessary calculations.

3.4.1 Prolongation

To compute the prolongation, it is very convenient to work with exact sequencees. As already seen in the proof of Theorem 3.24, each symmetry groupoid $\mathcal{R}_q(\omega)$ is defined by the exact sequence (3.10)

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow \Theta_q \underset{\omega\circ s}{\overset{\Phi_\omega}{\rightrightarrows}} \mathcal{F}.$$

Analogous to equation (1.10), we expect that $\mathcal{R}_{q+r}(\omega)$ is defined by the following exact sequence projecting down to order q:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{R}_{q+r}(\omega) & \longrightarrow & \Theta_{q+r} & \underset{j_r(\omega)\circ s}{\overset{p_r(\Phi_\omega)}{\rightrightarrows}} & J_r(\mathcal{F}) \\ & & \downarrow{\scriptstyle \pi^{q+r}_q} & & \downarrow{\scriptstyle \pi^{q+r}_q} & & \downarrow \\ 0 & \longrightarrow & \mathcal{R}_q(\omega) & \longrightarrow & \Theta_q & \underset{\omega\circ s}{\overset{\Phi_\omega}{\rightrightarrows}} & \mathcal{F} \end{array} \tag{3.20}$$

The next proposition shows that the expectations are correct and that $\mathcal{R}_{q+r}(\omega)$ is again a symmetry groupoid. In the Π_q-case, the fact that $J_r(\mathcal{F})$ is a natural bundle is equivalent to Proposition 3.14 (or [KMS93, §14.16]), otherwise we generalise [Pom83, Thm. 2.A.2.52] to Θ_q.

Proposition 3.26. Let $\mathcal{F} \to X$ be a natural Θ_q-bundle. Then $J_r(\mathcal{F})$ is a Θ_{q+r}-bundle for all $r \in \mathbb{N}$. If ω is a section of $\mathcal{F}$ with symmetry groupoid $\mathcal{R}_q(\omega) =$

$\mathrm{Stab}^q_{\mathcal{F}}(\omega)$ then the symmetry groupoid $\mathrm{Stab}^{q+r}_{J_r(\mathcal{F})}(j_r(\omega))$ is the r-th prolongation $\mathcal{R}_{q+r}(\omega)$ of $\mathcal{R}_q(\omega)$. ◇

Proof. Apply the exact functor J_r to the action

$$\mathcal{F}\ {}^{\pi}\!\curlywedge^{t}\ \Theta_q \longrightarrow \mathcal{F}$$

and use the embedding $\Theta_{q+r} \hookrightarrow J_r(\Theta_q)$ to obtain the commutative diagram:

$$\begin{array}{ccccc} J_r(\mathcal{F}) & {}^{\pi}\!\curlywedge^{t} & J_r(\Theta_q) & \longrightarrow & J_r(\mathcal{F}) \\ \| & & \uparrow & & \| \\ J_r(\mathcal{F}) & {}^{\pi}\!\curlywedge^{t} & \Theta_{q+r} & \longrightarrow & J_r(\mathcal{F}) \end{array}$$

By abuse of language, $p_r(\pi)$ and $p_r(t)$ are again called π and t, because they map to $J_r(X) = X$. The property $j_r(\omega \circ f_q) = j_r(\omega) \circ j_r(f_q)$ for sections ω of $\mathcal{F} \to X$ and f_q of $\Theta_q \to X$ establishes all properties of Θ_{q+r}-action on $J_r(\mathcal{F})$.

According to Definition 1.24, the prolongation of $\mathcal{R}_q(\omega)$ is:

$$\mathcal{R}_{q+r}(\omega) = J_r(\mathcal{R}_q(\omega)) \cap \Pi_{q+r} = J_r(\mathcal{R}_q(\omega)) \cap \Theta_{q+r}. \tag{3.21}$$

The last equality follows from the fact that $\mathcal{R}_q(\omega)$ is a subbundle of $\Theta_q \to X$ and the exactness of J_r.

Having established the Θ_{q+r}-action on $J_r(\mathcal{F})$, we can apply the functor J_r to equation (3.10). Use canonical embedding $\Theta_{q+r} \hookrightarrow J_r(\Theta_q)$ to obtain the commutative and exact diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & J_r(\mathcal{R}_q(\omega)) & \longrightarrow & J_r(\Theta_q) & \overset{j_r(\Phi_\omega)}{\underset{j_r(\omega)\circ s}{\rightrightarrows}} & J_r(\mathcal{F}) \\ & & \uparrow & & \uparrow & & \| \\ 0 & \longrightarrow & \mathrm{Stab}^{q+r}_{J_r(\mathcal{F})}(j_r(\omega)) & \longrightarrow & \Theta_{q+r} & \rightrightarrows & J_r(\mathcal{F}) \end{array}$$

The intersection $J_r(\mathcal{R}_q(\omega)) \cap \Theta_{q+r}$ is actually the symmetry groupoid of $j_r(\omega)$. □

Remark 3.27. In Definition 3.4, we restricted to actions of formally integrable groupoids above their geometric order q_0. Otherwise, the last equality of equation (3.21) would not hold for $q \leq q_0$.

However it is possible to omit the assumption $q \geq q_0$ if the restriction to Θ_{q+r} is done separately, because a Θ_q-bundle is always a Θ'_q-bundle for any integrable subgroupoid $\Theta'_q \leq \Theta_q$. The projection back to order $q + r - 1$ in Propositon 3.30 is not affected by the restriction. ◇

There is also a connection between the prolongation of natural bundles and the algebraic prolongation of their fibres (see Section B.5.1).

Corollary 3.28. The fibre F_r of $J_r(\mathcal{F})$ is isomorphic to the algebraic prolongation $F^{(r)}$ of the fibre F of $\mathcal{F}$ as a manifold with G_{q+r}-action. ⋄

Proof. By Proposition 1.16, the prolongation of a local trivialisation $U \times F$ of $\mathcal{F}$ embeds into $J_r(\mathcal{F})|_U$. □

3.4.2 Projection

The projection of symmetry groupoids can be expressed by natural bundles. To construct the natural bundle for the projection, we work on the fibres and use the jet groups introduced in Appendix B. The approach applied in this section is new and the idea based on jet groups is due to Barakat. The results of Pommaret [Pom83] are only applicable for natural Π_q-bundles under the assumption that integrable symmetry groupoids exist. In this case, both approaches coincide.

If F is the fibre of a natural bundle $\mathcal{F}$, the algebraic prolongation $F^{(r)}$ has already been identified as the fibre of $J_r(\mathcal{F})$. Following the idea of Barakat, we compute the projection $F^{(r)} \to F^{(r)}/K_{q+r}$ defined in Section B.5.2. $F_1 = F^{(r)}/K_{q+r}$ is the fibre of the natural bundle for the projection to order $q+r-1$. Projections to lower orders are computed analogously. In order to prove this, we need a preparational Lemma.

Lemma 3.29. Identify the kernel of the projection $\ker(\pi^{q+r}_q) \leq \Theta_{q+r}(x,x)$ to order q with K^{q+r}_q for each $x \in X$. For each $f_{q+r} \in \Theta_{q+r}(x,y)$ we have the equality of sets:

$$K^{q+r}_q f_{q+r} = f_{q+r} K^{q+r}_q \subseteq \Theta_{q+r}(x,y).$$ ⋄

Proof. Each element $f_{q+r}k_{q+r}$ with $k_{q+r} \in K^{q+r}_q \leq \Theta_{q+r}(x,x)$ can be written as $k'_{q+r}f_{q+r}$ for $k'_{q+r} = f_{q+r}k_{q+r}f^{-1}_{q+r} \in K^{q+r}_q \leq \Theta_{q+r}(y,y)$, because π^{q+r}_q is a groupoid morphism. □

In the following, this will be used like a commutation law without further notice. We construct the natural bundle for the projection from order $q+r$ to $q+r-1$.

Proposition 3.30. Let $\mathcal{F}$ be a natural Θ_q bundle with fibre F, such that $J_r(\mathcal{F})$ has the fibre $F^{(r)}$. Define $F_1 = F^{(r)}/K^{q+r}_{q+r-1}$. The bundle

$$\mathcal{F}_{(1)} = P_{\Theta_{q+r}} \times_{G_{q+r}} F_1$$

is a natural bundle of order $q+r-1$ and the projection $I : J_r(\mathcal{F}) \to \mathcal{F}_{(1)}$ is a morphism of natural Θ_q-bundles. ⋄

In the following chapters, use the notation

$$J_r(\mathcal{F})/K^{q+r}_{q+s} := P_{\Theta_{q+r}} \times_{G_{q+r}} F^{(r)}/K^{q+r}_{q+s}$$

for the projection of natural bundles. Note that the bundle $\mathcal{F}_{(1)}$ does not always coincide with the Janet bundle $\mathcal{F}_1$ used by Pommaret (see below Remark 2.A.3.8 in [Pom83]), since $\mathcal{F}_{(1)}$ is not necessarily a vector bundle. The construction of $\mathcal{F}_{(1)}$ presented here does not depend on an integrable symmetry groupoid. It thus avoids the problems which are adressed in [Pom83, Ex. 3.11]. In Section 3.5.3, the Janet bundles are defined and the distinction between $\mathcal{F}_{(1)}$ and $\mathcal{F}_1$ becomes apparent.

Proof. The isotropy group G_{q+r} acts on the fibre $F^{(r)}$ and by Proposition B.16, G_{q+r-1} acts on F_1 by taking arbitrary preimages in G_{q+r}. Since $P_{\Theta_{q+r}}/K_{q+r} \cong P_{\Theta_{q+r-1}}$, the bundle $\mathcal{F}_{(1)}$ is isomorphic to:

$$\mathcal{F}_{(1)} = P_{\Theta_{q+r}} \times_{G_{q+r}} F_1 \cong P_{\Theta_{q+r-1}} \times_{G_{q+r-1}} F_1.$$

The projection $F^{(r)} \to F_1$ is G_{q+r}-equivariant by Proposition B.17 (2) and the projection $\pi : \Theta_{q+r} \to \Theta_{q+r-1}$ is a morphism of groupoids which restricts to P_{Θ_q}, such that

$$I : P_{\Theta_{q+r}} \times_{G_{q+r}} F^{(r)} \to P_{\Theta_{q+r-1}} \times_{G_{q+r-1}} F_1 \ : \ (p, f) \mapsto (\pi(p), fK_{q+r})$$

is morphism of natural Θ_q-bundles. □

In the Π_q-case, it follows from Theorem 3.15 (cf. [KMS93, 14.18]) that I is a morphism of natural bundles. Essentially, we have used the same idea to construct the bundle morphism. Using the same notation as above, we show the connection between $\mathcal{F}_{(1)}$ and the projection $\mathcal{R}_{q+r}(\omega) \to \mathcal{R}^{(1)}_{q+r-1}(\omega)$.

Proposition 3.31. Let $\mathcal{R}_q(\omega)$ be the symmetry groupoid of the section ω of $\mathcal{F} \to X$. The exact sequence for the prolongation $\mathcal{R}_{q+r}(\omega)$ projects down to:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{R}_{q+r}(\omega) & \longrightarrow & \Theta_{q+r} & \rightrightarrows & J_r(\mathcal{F}) \\ & & \downarrow{\scriptstyle \pi^{q+r}_{q+r-1}} & & \downarrow{\scriptstyle \pi^{q+r}_{q+r-1}} & & \downarrow{\scriptstyle I} \\ 0 & \longrightarrow & \mathcal{R}^{(1)}_{q+r-1}(\omega) & \longrightarrow & \Theta_{q+r-1} & \rightrightarrows & \mathcal{F}_{(1)} \end{array} \tag{3.22}$$

The symmetry groupoid

$$\mathcal{R}^{(1)}_{q+r-1}(\omega) = \mathrm{Stab}^{q+r-1}_{\mathcal{F}_{(1)}}((I \circ j_r)(\omega))$$

is the image of $\mathcal{R}_{q+r}(\omega)$ under π^{q+r}_{q+r-1} and thus the projection in the sense of Definition 1.24. ◇

Proof. Since $I(j_r(\omega)(x)) = j_r(\omega)(x)K_{q+r}$, the diagram (3.22) commutes by Lemma 3.29. We show that $f_{q+r-1} \in \Theta_{q+r-1}(x, y)$ satisfies the symmetry equations (3.11),

$$I(j_1(\omega))(y) \circ f_{q+r-1} = I(j_1(\omega))(x),$$

on $\mathcal{F}_{(1)}$ if and only if there exists an element $f_{q+r} \in \Theta_{q+r}$ projecting down to f_{q+r-1}. Each element $v \in \mathcal{F}_{(1)}$ is the K_{q+r}-orbit $v = uK_{q+r}$ for a suitable $u \in J_r(\mathcal{F})$. Θ_{q+r-1} acts on $\mathcal{F}_{(1)}$ by taking arbitrary preimages $f_{q+r} \in \pi^{-1}(f_{q+r-1})$:

$$v \circ f_{q+r-1} = uK_{q+r}f_{q+r} = uf_{q+r}K_{q+r}.$$

Here, we applied Lemma 3.29. Rewrite the symmetry equations

$$\begin{aligned} I(j_1(\omega))(y) \circ f_{q+r-1} &= j_r(\omega)(y)K_{q+r}f_{q+r} \\ &= j_r(\omega)(y)f_{q+r}K_{q+r} \\ &\stackrel{!}{=} j_r(\omega)(x)K_{q+r}. \end{aligned}$$

They are satisfied if and only if there exists a modified element $f'_{q+r} \in \Theta_{q+r}$ with $j_r(\omega)(y)f'_{q+r} = j_r(\omega)(x)$ that projects to f_{q+r-1}. □

Remark 3.32. If the projection $\pi^{q+r}_{q+r-s} : \mathcal{R}_{q+r}(\omega) \to \mathcal{R}^{(s)}_{q+r-s}(\omega)$ is needed, the fibre F_1 has to be replaced by $F' = F^{(r)}/K^{q+r}_{q+r-s}$. The proofs of Propositions 3.30 and 3.31 are the same except for different indices. ◇

The last two propositons provide the theoretical background for the projection of symmetry groupoids. For practical applications, we have to compute the projection $J_r(\mathcal{F}) \to \mathcal{F}_{(1)}$ and the Θ_{q+r-1}-action on $\mathcal{F}_{(1)}$. Locally, the projection can be calculated with the help of Proposition B.16 on the fibres. The infinitesimal $\mathfrak{k}_{q+r}$-action on $F^{(r)}$ defines an involutive distribution on $F^{(r)}$ which can be integrated by using Frobenius' theorem. The resulting coordinates of the orbit space F_1 are the fibre coordinates of $\mathcal{F}_{(1)}$.

Integrating a distribution usually involves solving linear PDE systems, which cannot be avoided for most projections. However the next proposition shows that a projection $J_r(\mathcal{F}) \to \mathcal{F}_{(1)}$ from order $q+r$ to $q+r-1$ only involves linear algebra. For simplicity, we start with natural Π_q-bundles and generalise later (see also [Pom78, §7.3] for the case of Π_q).

Proposition 3.33. The coordinates for the Π_{q+r-1}-bundle $\mathcal{F}_{(1)} = J_r(\mathcal{F})/K^{q+r}_{q+r-1}$ over $J_{r-1}(\mathcal{F})$ can be calculated by solving linear equations over the quotient field $K(u)$, where (x, u) are the coordinates of $\mathcal{F}$. ◇

Proof. According to Theorem 3.22, the infinitesimal $J_q(T)$-action on $\mathcal{F}$ is given by the vector field

$$L_\xi = \xi^i \partial_{x^i} + \xi^i_\mu L^{\mu,\tau}_i(u)\partial_{u^\tau}, \qquad |\mu| \le q. \tag{3.23}$$

The prolongation according to Definition 1.18 provides the $J_{q+r}(T)$-action on $J_r(\mathcal{F})$:

$$\rho_r(L_\xi) = D_\nu Q^\tau \partial_{u^\tau_\nu} + \xi^i D_i, \qquad Q^\tau = \xi^i_\mu L^{\mu,\tau}_i(u) - u^\tau_i \xi^i.$$

The generators of the $\mathfrak{k}_{q+r}$-action on $F^{(r)}$ are the coefficients of ξ^i_σ in $\rho_r(L_\xi)$ with $|\sigma| = q + r$. They are computed by differentiating the first summand of Q^τ:

$$L_{\xi_{q+r}} = \xi^i_{\mu+\nu} L^{\mu,\tau}_i(u)\partial_{u^\tau_\nu}, \qquad |\mu| = q,\, |\nu| = r. \tag{3.24}$$

It follows that K_{q+r} acts on the fibres of $F^{(r)} \to F^{(r-1)}$ by u-dependent translations. The generators of the distribution are:

$$L_{\xi^i_\rho} = \sum_{\substack{\mu+\nu=\rho \\ |\mu|=q, |\nu|=r}} L^{\mu,\tau}_i(u)\partial_{u^\tau_\nu}.$$

Obviously all jets $(u, u_1, \ldots, u_{r-1})$ up to order $r - 1$ are coordinates of the factor space and we only have to care about r-th order jets. The linear ansatz

$$v = A(u)u_r = A^\nu_\tau(u)u^\tau_\nu,$$

turns $L_{\xi^i_\rho} v = 0$ into linear equations for $A(u)$. If d is the fibre dimension of $F^{(r)} \to F^{(r-1)}$, its solution space has the correct dimension $d - k$. A basis $(v^\beta = A^{\beta,\mu}_\alpha)$ for the solution space thus provides the fibre coordinates of $\mathcal{F}_{(1)} \to J_{r-1}(\mathcal{F})$. In coordinates, the map $I : J_r(\mathcal{F}) \to \mathcal{F}_{(1)}$ is therefore given by:

$$I^\beta(u, u_r) = (u, u_{r-1}, A^{\beta,\mu}_\alpha(u)u^\alpha_\mu), \qquad |\mu| = r.$$

Usually I is written as $I^\beta = A^{\beta,\mu}_\alpha(u)u^\alpha_\mu$. □

Remark 3.34. If $\mathcal{F}$ is a Θ_q-bundle, the algebroid $\mathfrak{g}_{\Theta_q}$ can locally be defined by equations in Medolaghi form (3.12) for a section θ of a natural Π_q-bundle $\mathcal{F}' \to X$:

$$\xi^i_\mu M^{\alpha,\mu}_i(\theta(x)) - \xi^i\, \partial_{x^i}\theta^\alpha(x) = 0, \qquad 1 \le \alpha \le k,\, |\mu| \le q.$$

Locally, we can choose a subset of the coordinates ξ^i_μ of $J_q(T)$ as coordinates for $\mathfrak{g}_{\Theta_q}$. A choice of coordinates ξ^i_μ for the algebroid makes both equations (3.23) and (3.24) x-dependent. However the equations for coordinates remain linear and fibre coordinates for $\mathcal{F}$ are of the form

$$v^\beta = A^{\beta,\mu}_\alpha(u, x)u^\alpha_\mu.$$ ◇

3.4.3 Example

We give an example which is small enough to follow all steps by hand. It was also calculated by Vessiot [Ves03, §17].

On a 2-dimensional manifold X with coordinates (x^1, x^2) the natural bundle $\mathcal{F} = T^* \times_X \bigwedge^2 T^*$ with fibre coordinates (u^1, u^2, u) such that they are the coefficients of the differential forms

$$\omega = u^1dx^1 + u^2dx^2, \qquad \Omega = udx^1 \wedge dx^2.$$

Sections of $\mathcal{F} \to X$ specify a differential 1-form ω and a 2-form Ω by:

$$\begin{aligned} \omega &= \omega^1(x)dx^1 + \omega^2(x)dx^2, \\ \Omega &= \Omega(x)dx^1 \wedge dx^2. \end{aligned}$$

The Π_1-action on $\mathcal{F}$ is in coordinates given by:

$$\begin{aligned} u^1 &= y_1^1\,\hat{u}^1 + y_1^2\,\hat{u}^2, \\ u^2 &= y_2^1\,\hat{u}^1 + y_2^2\,\hat{u}^2, \\ u &= (y_1^1\,y_2^2 - y_2^1\,y_1^2)\,\hat{u}. \end{aligned} \tag{3.25}$$

It is equivalent to the pullback of the forms ω and Ω via the diffeomorphism

$$y^1 = y^1(x^1, x^2), \qquad y^2 = y^2(x^1, x^2).$$

Prolongation

The prolongation to $J_1(\mathcal{F}) \cong J_1(T^*) \times_X J_1(\bigwedge^2 T^*)$ with coordinates

$$(u^1, u^2, u, u_1^1, u_2^1, u_1^2, u_2^2, u_1, u_2)$$

is done according to Remark 1.13. Effectively it is formal differentiation having in mind that the u-coordinates depend on x, and the $\hat{u}$-coordinates depend on y:

$$\begin{aligned} u_1^1 &= y_{11}^1\,\hat{u}^1 + y_{11}^2\,\hat{u}^2 + y_1^1\,y_1^1\,\hat{u}_1^1 + y_1^1\,y_1^2\,\hat{u}_2^1 + y_1^2\,y_1^1\,\hat{u}_1^2 + y_1^2\,y_1^2\,\hat{u}_2^2, \\ u_2^1 &= y_{12}^1\,\hat{u}^1 + y_{12}^2\,\hat{u}^2 + y_1^1\,y_2^1\,\hat{u}_1^1 + y_1^1\,y_2^2\,\hat{u}_2^1 + y_1^2\,y_2^1\,\hat{u}_1^2 + y_1^2\,y_2^2\,\hat{u}_2^2, \\ u_1^2 &= y_{12}^1\,\hat{u}^1 + y_{12}^2\,\hat{u}^2 + y_1^1\,y_2^1\,\hat{u}_1^1 + y_2^1\,y_1^2\,\hat{u}_2^1 + y_2^2\,y_1^1\,\hat{u}_1^2 + y_1^2\,y_2^2\,\hat{u}_2^2, \\ &\vdots \end{aligned} \tag{3.26}$$

Even in this small example the prolongation becomes rather large, for bigger examples it is often impossible to compute the finite transformations. Note that the second order jets in the expressions of u_2^1 and u_1^2 coincide. It is recommended to work with the vector fields of the infinitesimal $J_1(T)$-action on $\mathcal{F}$. It is given by:

$$L_\xi = \xi^1\partial_{x^1} + \xi^2\partial_{x^2} - (\xi_1^1\,u^1 + \xi_1^2\,u^2)\partial_{u^1} - (\xi_1^1\,u^1 + \xi_1^2\,u^2)\partial_{u^1} - (\xi_1^1 + \xi_2^2)\,u\,\partial_u \tag{3.27}$$

The first prolongation of v for the $J_2(T)$-action on $J_1(\mathcal{F})$ is at least shorther than the Π_2-action:

$$\begin{aligned} \rho_1(L_\xi) &= L_\xi \\ &\quad -(u^1\xi_{11}^1 + u^2\xi_{11}^2 + 2u_1^1\xi_1^1 + u_1^2\xi_1^2 + u_2^1\xi_1^2)\partial_{u_1^1} \\ &\quad -(u^1\xi_{12}^1 + u^2\xi_{12}^2 + u_2^1\xi_1^1 + u_2^2\xi_1^2 + u_1^1\xi_2^1 + u_2^1\xi_2^2)\partial_{u_2^1} \\ &\quad -(u^1\xi_{12}^1 + u^2\xi_{12}^2 + u_1^1\xi_2^1 + u_1^2\xi_2^2 + u_1^2\xi_1^1 + u_2^2\xi_1^2)\partial_{u_1^2} \\ &\quad -(u^1\xi_{22}^1 + u^2\xi_{22}^2 + u_2^1\xi_2^1 + 2u_2^2\xi_2^2 + u_1^2\xi_2^1)\partial_{u_2^2} \\ &\quad -(u\xi_{11}^1 + u\xi_{12}^2 + 2u_1\xi_1^1 + u_1\xi_2^2 + u_2\xi_1^2)\partial_{u_1} \\ &\quad -(u\xi_{12}^1 + u\xi_{22}^2 + u_2\xi_1^1 + 2u_2\xi_2^2 + u_1\xi_2^1)\partial_{u_2} \end{aligned} \tag{3.28}$$

A Prolonged Section which is not a Lie Groupoid

We give an example of a jet groupoid $\mathcal{R}_1$ whose prolongation is not a Lie groupoid anymore. It is due to the fact that the prolongation is defined as the intersection of two manifolds. Unlike in Example A.3, the symbol of $\mathcal{R}_2$ is a vector bundle and the problems occur in lower order equations.

Choose the section

$$\begin{aligned}\omega_0 &= (1+(x^2)^2)dx^1,\\ \Omega_0 &= dx^1 \wedge dx^2.\end{aligned}$$

It specifies the symmetry groupoid $\mathcal{R}_1(\omega_0)$ by the equations

$$\begin{aligned}1+(x^2)^2 &= y^1_1\,(1+(y^2)^2)\\ 0 &= y^1_2\,(1+(y^2)^2),\\ 1 &= y^1_1\,y^2_2 - y^1_2\,y^2_1,\end{aligned}$$

which are obviously of constant rank. The section is constructed such that the first prolongation

$$\begin{aligned}0 &= y^1_{11}\,(1+(y^2)^2) + 2y^1_1\,y^2_1\,y^2,\\ 2x^2 &= y^1_{12}\,(1+(y^2)^2) + 2y^1_1\,y^2_2\,y^2,\\ 0 &= y^1_{12}\,(1+(y^2)^2) + 2y^1_1\,y^2_1\,y^2,\\ &\vdots\end{aligned}$$

contains the equation

$$2x^2 = 2(y^1_1\,y^2_2 - y^1_2\,y^2_1)y^2$$

as difference between the second and third equation. The map $j_1(\Phi_{\omega_0})$ is not of constant rank around $y^2 = 0$. In contrast to Example A.3, the nonconstant rank is due to the lower order equations.

Projection

In equation (3.28), the first prolongation $\rho_1(L_\xi)$ of the infinitesimal $J_1(T)$-action on $\mathcal{F}$ was computed. The infinitesimal $\mathfrak{k}^2_1$-action on the fibre $F^{(1)}$ can be read off $\rho_1(L_\xi)$ by setting all zero and first order jets of ξ to zero:

$$\begin{aligned}\rho_1(L_\xi)|_{\xi_0=\xi_1=0} &= -(u^1\xi^1_{11} + u^2\xi^2_{11})\partial_{u^1_1} - (u^1\xi^1_{12} + u^2\xi^2_{12})\partial_{u^1_2}\\ &\quad -(u^1\xi^1_{12} + u^2\xi^2_{12})\partial_{u^2_1} - (u^1\xi^1_{22} + u^2\xi^2_{22})\partial_{u^2_2}\\ &\quad -(u\xi^1_{11} + u\xi^2_{12})\partial_{u_1} - (u\xi^1_{12} + u\xi^2_{22})\partial_{u_2}.\end{aligned}$$

Collecting for the ξ^i_μ, the distribution is generated by the vector fields $L_{\xi^i_\rho}$:

$$u^1\partial_{u^1_1} + u\partial_{u_1}, \quad u^1\partial_{u^1_2} + u^1\partial_{u^2_1} + u\partial_{u_2}, \quad u^1\partial_{u^2_2},$$

$$u^2\partial_{u^1_1}, \quad u^2\partial_{u^1_2} + u^2\partial_{u^2_1} + u\partial_{u_1}, \quad u^2\partial_{u^2_2} + u\partial_{u_2}.$$

To integrate the distribution naively, the equations $L_{\xi^i_\rho} c(u^i, u, u^i_j, u_j) = 0$ must be solved. Instead, we use Proposition 3.33 and solve the system of linear equations. Taking the coefficients of the derivatives in $L_{\xi^i_\rho}$ in the order

$$\partial_{u^1_1}, \partial_{u^1_2}, \partial_{u^2_1}, \partial_{u^2_2}, \partial_{u_1}, \partial_{u_2},$$

we obtain the matrix for $L_{\xi^i_\rho} v = 0$:

$$\begin{pmatrix} u^1 & 0 & 0 & 0 & u & 0 \\ 0 & u^1 & u^1 & 0 & 0 & u \\ 0 & 0 & 0 & u^1 & 0 & 0 \\ u^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & u^2 & u^2 & 0 & u & 0 \\ 0 & 0 & 0 & u^2 & 0 & u \end{pmatrix}$$

It has the kernel generated by $(0, -1, 1, 0, 0, 0)^{\mathrm{tr}}$, which is translated into the coordinate $v = u^2_1 - u^1_2$ completing the coordinates for $\mathcal{F}_{(1)}$ to (u^1, u^2, u, v).

The Natural Bundle $\mathcal{F}_{(1)}$

$\mathcal{F}_{(1)}$ is again a natural bundle, so we are interested in the Π_1-action. For the finite version compute

$$u^2_1 - u^1_2 = (y^1_1\, y^2_2 - y^1_2\, y^2_1)(\hat{u}^2_1 - \hat{u}^1_2) \tag{3.29}$$

using equation (3.26) and express the result in terms of v:

$$v = (y^1_1\, y^2_2 - y^1_2\, y^2_1)\, \hat{v}.$$

A comparison with equation (3.25) shows that $\mathcal{F}_{(1)} \cong \mathcal{F} \times_X \bigwedge^2 T^*$ is the bundle modelling a single 1-form and two 2-forms. The differential map I of order one is nothing else than the exterior derivative $d : T^* \to \bigwedge^2 T^*$. It is also possible to compute the infinitesimal action on $\mathcal{F}_{(1)}$ by using $\rho_1(L_\xi)$:

$$\rho_1(L_\xi)(u^2_1 - u^1_2) = (\xi^1_1 + \xi^2_2)(u^2_1 - u^1_2).$$

The infinitesimal $J_1(T)$-action on $\mathcal{F}_{(1)}$ is given by the vector field

$$L_{\xi,1} = L_\xi + (\xi^1_1 + \xi^2_2)\, v\, \partial_v. \tag{3.30}$$

The Π_1-action on the fibre $F_{(1)}$ of $\mathcal{F}_{(1)}$ is no longer transitive and we find an invariant on $\mathcal{F}_{(1)}$:

$$\psi : \mathcal{F}_{(1)} \to \mathbb{R} : (x, u^1, u^2, u, v) \mapsto \frac{v}{u},$$

which is only defined on the open subbundle $u \neq 0$. With equations (3.25) and (3.29) it is easy to check that v/u is an invariant.

Now the question arises, for which sections (ω, Ω) of $\mathcal{F} \to X$ the symmetry groupoid $\mathcal{R}_1(\omega, \Omega)$ is integrable. By plugging the derivatives of ω into equation (3.29), there is a possibly new equation. It might be a multiple of the last line in equation (3.25). The question will be answered in the next section.

3.5 Integrability Conditions and Vessiot Structure Equations

In this section, the main result of this chapter is presented. It is a test for formal integrability of symmetry groupoids of geometric objects ω on $\mathcal{F}$ that works directly on natural bundles. The results are equations, called Vessiot structure equations, on a bundle $\mathcal{F}_{(1)}$ where each section on $\mathcal{F} \to X$ can be tested for integrability. In the case of Π_q-bundles $\mathcal{F} = P_q/G_q$ for a subgroup $G_q \leq \mathrm{GL}_q$, this result goes back to Vessiot [Ves03] and was taken up by Pommaret [Pom78, §7.3] [Pom83, §2.A.3]. For a recent proof using groupoids see [Lor08b].

The new contribution in this thesis is a generalisation to natural Θ_q-bundles and, most importantly, to the case of intransitive Θ_q-actions on $\mathcal{F}$. These extensions are crucial for the applications in Chapter 4 and the Vessiot equivalence method in Chapter 6.

In Section 3.5.1 we consider the question whether the bundle $\mathcal{F}_{(1)}$ has a vector bundle structure, which simplifies the Θ_q-action notably. Section 3.5.2, briefly shows how to compute the symbols of symmetry groupoids efficiently. The integrability conditions give rise to a sequence of differential operators, the nonlinear Janet sequence. It will be introduced in Section 3.5.3.

The test for formal integrability of symmetry groupoids $\mathcal{R}_q(\omega)$ is based on Theorem 1.28. We prolong $\mathcal{R}_q(\omega)$ to $\mathcal{R}_{q+r}(\omega)$ until it has 2-acyclic symbol and then test a single projection. Translated to the language of natural bundles, it means computing the section $j_r(\omega)$ of $J_r(\mathcal{F}) \to X$ and then projecting to the bundle $\mathcal{F}_{(1)} = J_r(\mathcal{F})/K^{q+r}_{q+r-1}$. The surjectivity of the corresponding projection of jet groupoids is treated in the next theorem.

Theorem 3.35. Let $\mathcal{F} \to X$ be a natural Θ_q-bundle of order q with section ω and let $G_{q+s} \leq \mathrm{GL}_{q+s}$ be the isotropy group of Θ_{q+s} (where $G_{q+s} \cong \Theta_{q+s}(x, x)$ $\forall x \in X$, $s \geq 0$). Define the natural bundle $\mathcal{F}_{(1)} = J_r(\mathcal{F})/K^{q+r}_{q+r-1}$ with projection $I : J_r(\mathcal{F}) \to \mathcal{F}_{(1)}$ for an $r \in \mathbb{N}$. Then the projection of symmetry groupoids,

$$\pi = \pi^{q+r}_{q+r-1} : \mathcal{R}_{q+r}(\omega) \to \mathcal{R}^{(1)}_{q+r-1}(\omega),$$

is surjective, if and only if there is an equivariant section $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$, i.e. $c(u\, f_{q+r-1}) = c(u) f_{q+r-1}$ for all $f_{q+r-1} \in \Theta_{q+r-1}$, satisfying

$$I(j_r(\omega)) = c(j_{r-1}(\omega)). \tag{3.31}$$

◇

Before proving the theorem, it is helpful to have an idea of its content. The defining equations for $\mathcal{R}_q(\omega)$ are given by coordinate expressions of the symmetry condition $\omega(y) r_q = \omega(x)$. Taking the r-th order jet bundle $J_r(\mathcal{F})$ means differentiating them r times. The projection I, standing for *integrability conditions*, eliminates all jets of order $q + r$ from the equations of order $q + r$. Surjectivity of π means that all new equations of order $\leq q + r - 1$ can be expressed by the equations from $J_{r-1}(\mathcal{F})$. The equivariant section c shows how the new equations are expressed in terms of $J_{r-1}(\mathcal{F})$.

Proof. If not stated otherwise, we use the convention that a quantity a_{q+r} denotes an arbitrary preimage of a_{q+r-1} under the appropriate projection.

If there exists an equivariant section $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$, then for each $r_{q+r-1} \in \mathcal{R}_{q+r-1}(\omega)(x, y)$ we have $j_{r-1}(\omega)(y) r_{q+r-1} = j_{r-1}(\omega)(x)$ on $J_{r-1}(\mathcal{F})$. By the virtue of the equivariant section c, r_{q+r-1} is also an element of $\mathcal{R}^{(1)}_{q+r-1}$:

$$\begin{aligned} I(j_r(\omega)(y)) r_{q+r-1} &= c(j_{r-1}(\omega)(y)) r_{q+r-1} \\ &= c(j_{r-1}(\omega)(y) r_{q+r-1}) \\ &= c(j_{r-1}(\omega)(x)) \\ &= I(j_r(\omega)(x)). \end{aligned}$$

Reading the first and last part of the equations as

$$j_r(\omega)(y) K_{q+r} r_{q+r-1} = j_r(\omega)(x) K_{q+r},$$

it follows that each lift $r_{q+r} \in \pi^{-1}(r_{q+r-1})$ can be modified by an element of K_{q+r} such that $r_{q+r} \in \mathcal{R}_{q+r}(\omega)$.

The converse direction is more complicated and we distinguish several cases to give a local definition of c. Let F be the abstract fibre of $\mathcal{F}$.

(1) If G_{q+r-1} acts transitively on $F^{(r-1)}$, each $u \in J_{r-1}(\mathcal{F})_y$ can be written as $u = j_{r-1}(\omega)(y) g_{q+r-1}$ with $g_{q+r-1} \in \mathrm{GL}_{q+r-1} \cong \Pi_{q+r-1}(y, y)$. Then define

$$c(u) = I(j_r(\omega)(y)) g_{q+r-1} = j_r(\omega)(y) g_{q+r} K_{q+r},$$

which is well-define due to $g_{q+r} K_{q+r}$ being the whole preimage in G_{q+r}. For each $f_{q+r-1} \in \Theta_{q+r-1}$, we can find $h_{q+r-1} \in G_{q+r-1}$ with

$$\begin{aligned} j_{r-1}(\omega)(x) h_{q+r-1} &= u\, f_{q+r-1} \\ &= j_{r-1}(\omega)(y) g_{q+r-1} f_{q+r-1} \\ &= j_{r-1}(\omega)(y) r_{q+r-1} h_{q+r-1} \\ &\ \text{and} \quad f_{q+r-1} = g^{-1}_{q+r-1} r_{q+r-1} h_{q+r-1}. \end{aligned}$$

Surjectivity of π implies the existence of r_{q+r} over r and we can choose an arbitrary lift of f_{q+r}. The lifts g_{q+r} and h_{q+r} can be chosen such that

$$r_{q+r} = g_{q+r} f_{q+r} h_{q+r}^{-1}.$$

This is sufficient to prove that c is equivariant:

$$\begin{aligned} c(u\, f_{q+r-1}) &= j_r(\omega)(x) h_{q+r} K_{q+r} \\ &= j_r(\omega)(y) g_{q+r} (g_{q+r}^{-1} r_{q+r} h_{q+r}) K_{q+r} \\ &= c(u) f_{q+r} K_{q+r} \\ &= c(u) f_{q+r-1}. \end{aligned}$$

(2) Assume that the G_{q+r-1}-action on $F^{(r-1)}$ is intransitive with invariants

$$\psi = (\psi_1, \ldots, \psi_k) : F^{(r-1)} \to \mathbb{R}^k$$

and that Φ_ω has maximal rank. Then $\psi(\omega(y))$ also has full rank. It implies that we can find an open neighbourhood U of $\mathcal{F}$ over the open neighbourhood $Y \subseteq X$ such that for each $u \in U$ there is a $z \in Y$ with:

$$\psi(u) = \psi(j_{r-1}(\omega)(z)).$$

It is constructed by $y \in X$ and the preimage under ψ of an open neighbourhood of $\psi(\omega(y))$ in $\mathbb{R}^k$.

In this case, u can be written as $u = j_{r-1}(\omega)(z) g_{q+r-1}$ for some $g_{q+r-1} \in \Theta_{q+r-1}(y, z)$. The equivariant section c will then be defined by:

$$c(u) = I(j_r(\omega)(z)) g_{q+r-1} = j_r(\omega)(z) g_{q+r} K_{q+r}.$$

If there exists another $w \in X$ with $\psi(j_{r-1}(\omega)(z)) = \psi(j_{r-1}(\omega)(w))$, we can find $h_{q+r-1} \in \Theta_{q+r-1}(x, w)$ with

$$u\, f_{q+r-1} = j_{r-1}(\omega)(w) h_{q+r-1}.$$

But then

$$j_{r-1}(\omega)(z) g_{q+r-1} f_{q+r-1} = j_{r-1}(\omega)(w) h_{q+r-1}$$

gives rise to an element $r = gfh^{-1} \in \mathcal{R}_{q+r-1}(\omega)$ of the symmetry groupoid. As in case (1), lifts can be chosen such that

$$r_{q+r} = g_{q+r} f_{q+r} h_{q+r}^{-1}.$$

This is sufficient to prove that c is equivariant and well-defined:

$$\begin{aligned} c(u\, f_{q+r-1}) &= c(j_{r-1}(\omega)(w) h_{q+r-1}) \\ &= j_r(\omega)(w)\, h_{q+r}\, K_{q+r} \\ &= j_r(\omega)(z)\, r_{q+r}\, h_{q+r}\, K_{q+r} \\ &= j_r(\omega)(z)\, g_{q+r}\, f_{q+r}\, K_{q+r} \\ &= (j_r(\omega)(z)\, g_{q+r}\, K_{q+r})\, f_{q+r-1} \\ &= c(u) f_{q+r-1}. \end{aligned}$$

(3) To complete the cases, assume that $j_{r-1}(\omega)$ restricts to a closed subbundle $\mathcal{F}' \to X$ of $J_{r-1}(\mathcal{F})$. This case occurs if the equations for $\mathcal{R}_{q+r-1}(\omega)$ on $J_{r-1}(\mathcal{F})$ are redundant. Using case (1) or (2) on $I' : J_1(\mathcal{F}') \to \mathcal{F}'_{(1)}$, we obtain an equivariant section $c' : \mathcal{F}' \to \mathcal{F}'_{(1)}$, which has to be continued to a section $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$.

Using Proposition B.17, there is an embedding $\iota : \mathcal{F}'_1 \hookrightarrow \mathcal{F}_{(1)}$, such that $\iota \circ c'$ has the correct image. Around $x \in X$ we can find a coordinate system u_{r-1} of $J_{r-1}(\mathcal{F})$ such that $\mathcal{F}'$ is given by $u^{s+1}_{r-1} = \ldots = u^d_{r-1} = 0$. As $\mathcal{F}_{(1)}$ is a bundle over $J_{r-1}(\mathcal{F})$, we can find a coordinate system (u_{r-1}, v) of $\mathcal{F}_{(1)}$ over u_{r-1}. In these coordinates, $\iota \circ c'$ is given by:

$$v^\alpha = c^\alpha(u^1_{r-1}, \ldots, u^s_{r-1}).$$

Then there exists an open neighbourhood where the c^α define a section $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$. The equivariance is proved as in case (2) with $u^{s+1}_{r-1}, \ldots, u^d_{r-1}$ playing the role of the invariants. □

To use Theorem 3.35 effectively, we compute all possible equivariant sections $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$.

Remark 3.36. With the notation of Theorem 3.35 and the coordinates $v = v^\alpha$ for the fibre $\mathcal{F}_{(1)} \to J_{r-1}(\mathcal{F})$ the infinitesimal version of the equivariance condition $c(ur_{q+r-1}) = c(u)r_{q+r-1}$ is:

$$L_\xi c^\alpha(u) = L_\xi v^\alpha|_{v=c(u)}, \tag{3.32}$$

where L_ξ denotes the vector field (3.9) of the algebroid action on $\mathcal{F}$. To compute the equivariant sections, a linear inhomogenous system of PDEs must be solved.◇

For the next definition we need to assume that the natural bundles $\mathcal{F}$ and $\mathcal{F}_{(1)}$ are analytic, i.e. the coordinate changes and the Θ_q-action can be represented by analytic functions. Since $\mathcal{F}_{(1)} \to J_{r-1}(\mathcal{F})$ is an affine bundle (see Section 3.5.1), equation (3.32) is an inhomogenous linear PDE system. Its solutions split into a particular solution and the vector space of homogenous solutions.

Definition 3.37. Let $\mathcal{F}$ and $\mathcal{F}_{(1)}$ and I from Theorem 3.35 be analytic. The integrability conditions of equation (3.31),

$$I(j_r(\omega)) = c(j_{r-1}(\omega)), \tag{3.33}$$

where c parametrises all analytic equivariant sections, are called *Vessiot structure equations* and the natural bundle $\mathcal{F}_{(1)}$ is called *bundle of integrability conditions*.◇

In the examples treated by Pommaret [Pom78] and Vessiot [Ves03], the equivariant sections can be parametrised by a finite set of constants. As we are dealing with a more general situation, they may depend on arbitrary functions of the invariants on $J_{r-1}(\mathcal{F})$. In some cases there may be no equivariant sections at all. We will show an example, where first constants and then functions of invariants occur.

Example 3.38. Continue the example in Section 3.4.3. The $J_1(T)$-action on $\mathcal{F}_{(1)}$ is computed in Equation 3.30. With only a single coordinate v for the fibre of $\mathcal{F}_{(1)} \to \mathcal{F}$, the equivariance condition is according to equation (3.32):

$$L_{\xi,1}c(u^1, u^2, u) = (\xi_1^1 + \xi_2^2)c(u^1, u^2, u).$$

Writing $c(u) = c$ in jet notation and collecting for jets in ξ, the equivariance equations are:

$$u^1\, c_{u^1} + u\, c_u = c, \quad u^1\, c_{u^2} = 0, \quad u^2\, c_{u^1} = 0, \quad u^2\, c_{u^2} + u\, c_u = c.$$

Any equivariant section $c : \mathcal{F} \to mcF_{(1)}$ must be of the form

$$c(u^1, u^2, u) = C_1 u$$

for an arbitrary constant C_1. This means for a section (ω, Ω) of $\mathcal{F} \to X$, the symmetry groupoid $\mathcal{R}_1(\omega, \Omega)$ is integrable if and only if there is a constant C_1 such that the Vessiot structure equations are satisfied:

$$\partial_{x^1}\omega^2(x) - \partial_{x^2}\omega^1(x) = C_1\Omega(x).$$

The geometric interpretation is that the derivative of the 1-form ω must be a constant multiple of the 2-form:

$$d\omega = C_1\Omega.$$

If $\mathcal{R}_1(\omega, \Omega)$ is not formally integrable, we can continue with the bundle $\mathcal{F}_{(1)}$ and apply Theorem 3.35 again. Another prolongation and projection yields the bundle $\mathcal{F}_{(2)}$ with new coordinates

$$v^1 = \frac{vu_1 + uv_1}{(u)^2}, \qquad v^2 = \frac{vu_2 + uv_2}{(u)^2},$$

which are the total derivatives of the invariant $\frac{v}{u}$ on $\mathcal{F}_{(1)}$. It follows from Lemma 1.12 (or by direct computation) that $\mathcal{F}_{(2)} = \mathcal{F}_{(1)} \times_X T^*$. The Vessiot structure equations in this case are

$$v^1 = H_1(\frac{v}{u})\, u^1, \qquad v^2 = H_1(\frac{v}{u})\, u^2,$$

with an arbitrary function H_1 that depends on the invariant. Geometrically, this means that the newly found 1-form $\gamma = v^1 dx^1 + v^2 dx^2$ is a (nonconstant) multiple of ω.

Note that the bundle $\mathcal{F}_{(2)}$ is only a subbundle of $J_1(\mathcal{F}_{(1)})/K_1^2 \to \mathcal{F}_{(1)}$ and that we have applied the results of Section 4.3.1. ◇

More examples will follow in Chapter 5, where all calculations are done with the MAPLE packages `jets` [Bar01] and `JetGroupoids`.

3.5.1 Bundle Structure of $\mathcal{F}_{(1)}$

In this section we have a closer look at the structure of the first bundle of integrability conditions

$$\mathcal{F}_{(1)} = J_r(\mathcal{F})/K^{q+r}_{q+r-1}$$

and ask the question whether it can be turned into a vector bundle by a suitable change of coordinates. This was the case in all examples treated by Pommaret and Vessiot ([Pom78], [Pom83], [Ves03]), but especially when dealing with natural Θ_q-bundles, $\mathcal{F}_{(1)}$ is not necessarily a vector bundle anymore.

The advantage of vector bundle coordinates for $\mathcal{F}_{(1)}$ is that further prolongations and projections can be done more efficiently. If $\mathcal{F}_{(1)}$ is a vector bundle, some parts of the Vessiot structure equations can be predicted, which will be used in Section 4.1.

We choose fibre coordinates (u) for $\mathcal{F}$ and (u, u_r) for $J_r(\mathcal{F})$ such that $\mathcal{F}_{(1)}$ has coordinates (u, u_{r-1}, v). Computing the coordinates v^β with the help of Proposition 3.33 yields

$$v^\beta = A^{\beta,\mu}_\alpha(u, u_{r-1})u^\alpha_\mu.$$

The example on metrics and Christoffel symbols in Section 5.1 shows that the coordinates of $\mathcal{F}_{(1)}$ do not automatically show that $\mathcal{F}_{(1)}$ is a vector bundle,especially when dealing with second order structures. The coordinates usually have to be modified with an affine term

$$v^\beta = A^{\beta,\mu}_\alpha(u, u_{r-1})u^\alpha_\mu + B^\beta(u, u_{r-1}), \qquad |\mu| = r, \tag{3.34}$$

in order that $\mathcal{F}_{(1)}$ is a vector bundle. For examples of natural Θ_q-bundles where the intermediate bundles do not possess a vector bundle structure, see Section 4.3.3 and Chapter 7. In the following, we determine conditions for the existence of an affine term $B(u, u_{r-1})$ and show how to construct it from an equivariant section.

An immediate consequence of Proposition 1.10 (1) and Proposition 3.33 is the following Lemma.

Lemma 3.39. $\mathcal{F}_{(1)} \to J_{r-1}(\mathcal{F})$ is an affine bundle. ⋄

An equivariant section c of $\mathcal{F}_{(1)} \to J_{r-1}(\mathcal{F})$ specifies an origin in each fibre of $F_1 \to F^{(r-1)}$. The point is that this specification is G_{q+r-1}-invariant, which proves that $\mathcal{F}_{(1)}$ is a vector bundle.

Lemma 3.40. $\mathcal{F}_{(1)}$ is a natural vector bundle if and only if there exists an equivariant section $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$ with $c(u_{r-1}f_{q+r-1}) = c(u_{r-1})f_{q+r-1}$ for all $f_{q+r-1} \in \Theta_{q+r-1}$ ⋄

Proof. If $\mathcal{F}_{(1)} \to J_{r-1}(\mathcal{F})$ is a natural vector bundle, then the zero section is equivariant.

If $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$ is equivariant, we use the fact that $\mathcal{F}_{(1)} \to J_{r-1}(\mathcal{F})$ is an affine bundle with coordinates (u, v), modelled over a natural vector bundle $\mathcal{F}_1$. The map

$$\mathcal{F}_{(1)} \to \mathcal{F}_1 : (u, u_{r-1}, v) \mapsto (u, u_{r-1}, v - c(u, u_{r-1}))$$

is an isomorphism of natural bundles by the equivariance of c:

$$(u, u_{r-1}, v - c(u_{r-1}))f = (uf, u_{r-1}f, vf - c(u_{r-1})f) = (uf, vf - c(u_{r-1}f))$$

for all $f \in \Theta_{q+r-1}$. □

We can thus choose the additional term $B(u, u_{r-1})$ in equation (3.34) as

$$B(u, u_{r-1}) = -c(u, u_{r-1})$$

to turn $\mathcal{F}_{(1)}$ into a vector bundle. If the natural bundle $\mathcal{F}$ was constructed by a formally integrable groupoid $\mathcal{R}_q(\omega_0)$, one usually takes the equivariant section $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$ corresponding to ω_0.

The bundles $\mathcal{F}_{(1)}$ constructed by Pommaret [Pom83, §2.A.3] (called $\mathcal{F}_1$ there), are all vector bundles (see [Pom83, p. 296]). Additionally the existence of a formally integrable symmetry groupoid is explicitly assumed. There is another special case where all bundles $\mathcal{F}_{(1)}$ are automatically vector bundles.

Lemma 3.41. If $\mathcal{F}$ is a natural Π_1-bundle and $\mathcal{F}_{(1)} = J_1(\mathcal{F})/K_q^{q+1}$ is constructed by a single prolongation, then $\mathcal{F}_{(1)}$ is a vector bundle. ⋄

Proof. Compute the first prolongation of L_{ξ_1} from equation (3.9) for $\mu = j$:

$$\rho_1(L_{\xi_1}) = \xi^i \partial_{x^i} + \xi^i_j L_i^{\tau,j}(u)\partial_{u^\tau} + \left(\xi^i_j\, u^\alpha_k\, \partial_{u^\alpha} L_i^{\tau,j}(u) - \xi^i_k u^\tau_j\right) \partial_{u^\tau_k} + \xi^i_{jk}\partial_{u^\tau_k}.$$

The last summand is the infinitesimal $\mathfrak{k}_1^2$-action that vanishes on $\mathcal{F}_{(1)}$. As the projection to $\mathcal{F}_{(1)}$,

$$v^\beta = A_\alpha^{\beta,\mu}(u, u_{r-1})u_\mu, \qquad |\mu| = r,$$

is quasilinear in u^τ_k, the coefficient of $\partial_{u^\tau_k}$ show that $\rho_1(L_{\xi_1})v^\beta$ can be expressed by a linear combination of the coordinates v^β of $\mathcal{F}_{(1)}$. □

The vector bundle structure of $\mathcal{F}_{(1)}$ has very practical consequences if there are invariants on $J_{r-1}(\mathcal{F})$, because the equivariant sections depend on them. In Section 4.1, we will use this fact to predict the Vessiot structure equations in special situations.

Proposition 3.42. Let $\mathcal{F}$ be a natural Θ_q-bundle and assume that $\mathcal{F}_{(1)} = J_r(\mathcal{F})/K_{q+r-1}^{q+r}$ is a vector bundle. If there exist invariants $\psi^1, \dots, \psi^k$ on $J_{r-1}(\mathcal{F})$ and a nonzero equivariant section $c : J_{r-1}(\mathcal{F}) \to \mathcal{F}_{(1)}$ then also

$$f(\psi^1(u), \dots, \psi^k(u))\, c(u)$$

is an equivariant section for an arbitrary smooth function $f : \mathbb{R}^k \to \mathbb{R}$. ⋄

Proof. Use equation (3.32) and the fact that c is equivariant:

$$L_\xi(f(\psi)\, c^\alpha(u)) = f(\psi) L_\xi c^\alpha(u) = f(\psi) L_\xi v^\alpha|_{v=c(u)} = L_\xi v^\alpha|_{v=f(\psi)c(u)},$$

since $L_\xi v^\alpha$ is linear in v ($\mathcal{F}_{(1)}$ is a vector bundle). □

3.5.2 Symbols of Jet Groupoids

The Projection Theorem 3.35 provides a test for formal integrability of a symmetry groupoid $\mathcal{R}_q(\omega)$ of a section ω of $\mathcal{F} \to X$ if its symbol is 2-acyclic. In this section, we adapt a theorem of Pommaret [Pom78, Thm. 7.2.32] to natural Θ_q-bundles which reduces the Spencer cohomology computation to the algebroid $R_q(\omega)$. Furthermore, the Medolaghi form allows to compute the Spencer cohomology for an open subset of all sections of $\mathcal{F}$ in a single computation.

Throughout the section, let $\mathcal{F}$ be a natural Θ_q-bundle and ω be a section of $\mathcal{F} \to X$. Further let $\mathcal{R}_q(\omega)$ be the symmetry groupoid of ω with corresponding algebroid $R_q(\omega)$.

As a preparation, the symbols are equipped with a Lie algebra structure which corresponds to the graduation of the Lie algebra $\mathfrak{gl}_q$ introduced in Section B.1.

Lemma 3.43. Let $G_{q+r} \cong \mathcal{R}_{q+r}(x, x)$ be the isotropy groups of $\mathcal{R}_{q+r}(\omega)$ for $x \in X$ and $r \geq 0$ with graded Lie algebras

$$\mathfrak{g}_{q+r} = \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^{q+r}$$

according to Proposition B.12. Then the symbol $\mathcal{M}_{q+r,x}$ is isomorphic to the higest order component $\mathfrak{g}^{q+r} \cong \mathfrak{k}^{q+r}_{q+r-1}$ where $\mathfrak{k}$ is defined in Proposition B.4. ◇

Proof. Since the tangent bundle functor is a natural bundle functor, the fibre of $J_{q,0}(T) = \ker(\mathrm{an}) = \ker(\pi^q_0)$ over each point $x \in X$ is isomorphic to the Lie algebra $\mathfrak{gl}_q$ of Proposition B.3, which is the fibre of $J_{q,0}(T\mathbb{R}^n)$ at the origin. Define the bundle of subalgebras $R_{q+r,0}(\omega) = \ker(\pi^{q+r}_0|_{R_{q+r}(\omega)})$. The isotropy algebra $R_{q+r,0}(\omega)|_x \cong \mathfrak{g}_{q+r}$ is defined by setting $\xi^i = 0$ in the Medolaghi form (3.12):

$$\xi^i_{\mu+\nu} L^{\tau,\mu}_i(\omega(x)) = 0, \qquad |\nu| \leq r.$$

Restricting to $|\mu| = q$ and $|\nu| = r$ yields $\mathcal{M}_{q+r,x} \cong \mathfrak{g}^{q+r}$. □

We show that the symbol of a groupoid and its corresponding algebroid are essentially the same. In the transitive case, this result is due to Pommaret [Pom78, Thm. 7.2.32f]. We silently assume that the order q of the groupoid is ≥ 1.

Theorem 3.44. Let $\mathcal{M}_{q,r_q}$ be the symbol of $\mathcal{R}_q(\omega)$ at the point $r_q \in \mathcal{R}_q(\omega)$ and $\mathcal{M}_{q+r,r_q}$ for $r \in \mathbb{N}$ be the higher order symbols defined in equation (A.2). Further let $\mathcal{M}_{q+r,y}$ be the corresponding symbols of $R_q(\omega)$ at $y \in X$. For $y = t(r_q)$ the symbols are isomorphic:

$$\mathcal{M}_{q+r,r_q} \cong \mathcal{M}_{q+r,\mathrm{id}_y} = \mathcal{M}_{q+r,y}, \qquad \forall r \geq 0.$$

Additionally, the Spencer cohomology groups are isomorphic:

$$H^i_j(\mathcal{M}_{q,r_q}) \cong H^i_j(\mathcal{M}_{q,y}), \qquad y = t(r_q).$$

If $\mathcal{R}_q(\omega)$ is transitive, all $\mathcal{M}_{q+r}$ are vector bundles and the Spencer cohomology groups for all $r_q \in \mathcal{R}_q(\omega)$ are isomorphic. ◇

Proof. Drop the ω-dependence of all groupoids during the proof and set $\mathcal{R}_{q-1} = \pi^q_{q-1}(\mathcal{R}_q)$. Then $\mathcal{R}_q$ acts on $V(\mathcal{R}_q) = T^s(\mathcal{R}_q)$ by right multiplication and the action is commutes with the projection to jet order $q-1$:

$$\begin{array}{ccc} T^s(\mathcal{R}_q) \, {}_{s}{\times}_{t} \, \mathcal{R}_q & \longrightarrow & T^s(\mathcal{R}_q) \\ \downarrow \qquad\quad \downarrow & \circlearrowleft & \downarrow \\ T^s(\mathcal{R}_{q-1}) \, {}_{s}{\times}_{t} \, \mathcal{R}_{q-1} & \longrightarrow & T^s(\mathcal{R}_{q-1}) \end{array}$$

As the symbol $\mathcal{M}_{q,r_q}$ is the kernel of the projection $T^s(\mathcal{R}_q) \to T^s(\mathcal{R}_{q-1})$ and the $\mathcal{R}_{q-1}$-action is linear on the fibres of $T^s(\mathcal{R}_{q-1})$, the multiplication with r_q^{-1} induces an isomorphism between $\mathcal{M}_{q,r_q}$ and $\mathcal{M}_{q,\mathrm{id}_y}$. By definition of the algebroid, we have $\mathcal{M}_{q,\mathrm{id}_y} = \mathcal{M}_{q,y}$. Pulling back the higher order symbols $\mathcal{M}_{q+r}$ over $\mathcal{R}_q$, we also obtain isomorphisms $\mathcal{M}_{q+r,r_q} \cong \mathcal{M}_{q+r,\mathrm{id}_y}$ for $r \in \mathbb{N}$.

To show that the Spencer cohomologies are also isomorphic, we use Lemma 3.43. The symbols $\mathcal{M}_{q+r-1,x} \cong \mathfrak{g}_{q+r-1}$ at $x \in X$ and $\mathcal{M}_{q+r,x} \cong \mathfrak{g}_{q+r}$ satisfy the condition $\mathfrak{g}_{q+r} \subseteq \mathfrak{g}^{(1)}_{q+r-1}$ of Proposition B.12. The embedding

$$\iota : S^{q+r}T^* \otimes T \hookrightarrow T^* \otimes S^{q+r-1}T^* \otimes T$$

from the construction of the Spencer δ-map in equation A.3 restricts to the map

$$\mathcal{M}_{q+r} \hookrightarrow T^* \otimes \mathcal{M}_{q+r-1}$$

by Definition B.10 for the case of $\mathfrak{g}_{q+r} \leq \mathfrak{gl}_{q+r}$. Since the $\mathcal{R}_q$-action on $\mathcal{M}_{q+r}$ is the restriction on the Π_q-action on $S^{q+r}T^*$, it commutes with the full Spencer δ-map.

By right multiplication, $\mathcal{R}_q$ acts on $T^t\mathcal{R}_q$, $T^t\mathcal{R}_q \cap T^s\mathcal{R}_q$ and on $\mathcal{M}_{q+r}$, since $\mathcal{M}_{q+r,r_q}$ is a subspace of $T^t\mathcal{R}_q \cap T^s\mathcal{R}_q$ for each $r_q \in \mathcal{R}_q$. Analogous to the left action, the right $\mathcal{R}_q$-action induces an isomorphism

$$\mathcal{M}_{q+r,r_q} \cong \mathcal{M}_{q+r,\mathrm{id}_x}.$$

for $x = s(r_q)$. If $\mathcal{R}_q$ is transitive, the conjugation of $\mathcal{M}_{q+r,\mathrm{id}_x}$ with $r_q \in \mathcal{R}_q(x,y)$ induces an isomorphism between $\mathcal{M}_{q+r,\mathrm{id}_x}$ and $\mathcal{M}_{q+r,\mathrm{id}_y} = r_q \mathcal{M}_{q+r,\mathrm{id}_x} r_q^{-1}$. □

Having established the link between the symbols of groupoids and their corresponding algebroids, we turn to the Medolaghi form for the symmetry algebroids. It has a simple consequence that connects the points on the natural bundle with the symbols.

Proposition 3.45. The higher order symbols $\mathcal{M}_{q+r,x}$, $r \geq 0$, of $R_q(\omega)$ at $x \in X$ depend only on the value of the section ω at x. ◇

Proof. The general Medolaghi form (3.12) determines the algebroid $R_q(\omega)$ and its symbol is the intersection with $S^qT^* \otimes T$. Setting $\xi^i = \xi^i_\mu = 0$ for $|\mu| < q$ in equation (3.12),

$$\xi^i_\mu L_i^{\tau,\mu}(\omega(x)) = 0 \qquad |\mu| = 0,$$

shows that $\mathcal{M}_q$ depends on the point $\omega(x) \in \mathcal{F}$ only. Differentiating the equations and eliminating lower order terms,

$$\xi^i_{\mu+\nu} L_i^{\tau,\mu}(\omega(x)) = 0 \qquad |\mu| = 0, |\nu| = r,$$

shows that it is also valid for $\mathcal{M}_{q+r}$. □

Since the calculation of Spencer cohomology using Theorem A.11 can be done with the coordinates of x and $\omega(x)$ as parameters, we have the following corollary.

Corollary 3.46. The computation of Spencer cohomology via Koszul complex introduced in Appendix A is valid on an open subset of all sections of a natural bundle $\mathcal{F}$. ◇

Remark 3.47. In the following sections we abbreviate the notation for Spencer cohomology. If we say that 'generic sections of $\mathcal{F}$ have 2-acyclic symbols', this stands for the fact that 'on an open subset of the generic sections ω of $\mathcal{F} \to X$, the symbols for symmetry groupoids $\mathcal{R}_q(\omega)$ have 2-acyclic symbol'. ◇

For a sample computation of Spencer cohomology for symmetry groupoids see Section 5.1.2.

3.5.3 Janet Sequence

In this section, the nonlinear Janet sequence is constructed, which was introduced by Pommaret (see [Pom78, §7.4] or [Pom83, §2.A.3]). This section is not necessary to understand the applications in Chapter 4 or the Vessiot equivalence method in Chapter 6. However the affine bundle version of the Janet sequence and the curvature map in Section 3.5.4 are essential tools to prove the Embedding Theorem 4.22 in Section 4.3.2.

We follow [Pom83, §2.A.3] and adapt the construction to the case of Θ_q-bundles. Each jet groupoid can be defined by an exact sequence (3.10)

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow \Theta_q \underset{\omega \circ s}{\overset{\Phi_\omega}{\rightrightarrows}} \mathcal{F},$$

involving a natural Θ_q-bundle $\mathcal{F}$. Assuming that the generic sections ω of $\mathcal{F} \to X$ have 2-acyclic symbol and that all symbols are vector bundles, we can continue the sequence to a sequence of differential operators

$$\{\mathrm{id}_X\} \longrightarrow \Gamma_\omega \longrightarrow \Gamma_{\Theta_q} \underset{\omega \circ s}{\overset{\Phi_\omega \circ j_q}{\rightrightarrows}} \mathcal{F} \underset{c}{\overset{I \circ j_1}{\rightrightarrows}} \mathcal{F}_{(1)} \xrightarrow{J \circ j_1} \mathcal{F}_2. \tag{3.35}$$

It is called the *nonlinear Janet sequence*. Here Γ_ω and Γ_{Θ_q} stand for the local solutions of $\mathcal{R}_q(\omega)$ and Θ_q. The bundles $\mathcal{F}$, $\mathcal{F}_{(1)}$ and $\mathcal{F}_2$ are abbreviations for the corresponding sections $\Gamma(\mathcal{F})$ of $\mathcal{F} \to X$ etc. The first two differential operators are defined using the differential maps Φ_ω from the sequence (3.10) and the projection I from Proposition 3.30. As they are not necessary in the following, we will only sketch the construction of the Jacobian conditions J and the bundle $\mathcal{F}_2$ in this section and refer to [Pom78, §7.4] or [Pom83, §2.A.3] for details.

If we drop the assumption that there exists a section ω of $\mathcal{F} \to X$ with integrable symmetry groupoid $\mathcal{R}_q(\omega)$, there is no equivariant section $c : \mathcal{F} \to \mathcal{F}_{(1)}$ and the Jacobian conditions J and $\mathcal{F}_2$ do not make sense anymore. Nevertheless, there is still an exact sequence of affine bundles which becomes useful for the Embedding Theorem 4.22 in Sections 4.3.1 and 4.3.2.

Preparations

With the assumption that there exists a section ω_0 of $\mathcal{F} \to X$ such that $\mathcal{R}_q(\omega_0)$ is formally integrable, we construct a differential equation $\mathcal{B}_1(c) \subseteq J_1(\mathcal{F})$ using equation (3.34) for the case of $r = 1$:

$$I = A(u)u_x + B(u) = c(u) \tag{3.36}$$

Here $B(u)$ was chosen such that ω_0 is integrable with the zero section $c = 0$ of $\mathcal{F}_{(1)} \to \mathcal{F}$ according to Theorem 3.35. We study the formal integrability of the system $\mathcal{B}_1(c)$ and at the same time derive the Jacobian conditions.

Since $\mathcal{R}_q(\omega_0)$ is integrable, we know that $\mathcal{B}_1 = \mathcal{B}_1(0)$ is also integrable. Denote the symbol of $\mathcal{B}_1(c)$ by $\mathcal{N}_{q+1}$. In [Pom83], it is called $\mathcal{N}_1$, but we have chosen the indices according to sequence (3.39) below. Let $\mathcal{F}_0 = V(\mathcal{F})$ be the zeroth Janet bundle. For higher indices, the Janet bundles are defined with the help of the Spencer δ-map:

$$\mathcal{F}_r = \bigwedge^r T^* \otimes \mathcal{F}_0 / \delta(\bigwedge^{r-1} T^* \otimes \mathcal{N}_{q+1}) \tag{3.37}$$

All Janet bundles are vector bundles. Construct the natural bundle $\mathcal{F}_{(1)} = J_1(\mathcal{F})/K_q^{q+1}$ by the exact sequence of affine bundles:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N}_{q+1} & \longrightarrow & T^* \otimes \mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{B}_1 & \longrightarrow & J_1(\mathcal{F}) & \longrightarrow & \mathcal{F}_{(1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{F} & = & \mathcal{F} & = & \mathcal{F} & &
\end{array} \tag{3.38}$$

As $\mathcal{F}_{(1)}$ is the cokernel of the inclusion map $\mathcal{B}_1 \hookrightarrow J_1(\mathcal{F})$, it is isomorphic to the first Janet bundle $\mathcal{F}_{(1)} \cong \mathcal{F}_1$. The symbol $\mathcal{N}_{q+1}$ is a special case of the symbols $\mathcal{N}_{q+r}$ defined by the exact sequences

$$0 \longrightarrow \mathcal{M}_{q+r} \longrightarrow \mathcal{M}_{\Theta_{q+r}} \longrightarrow \mathcal{N}_{q+r} \longrightarrow 0 \tag{3.39}$$

of vector bundles over $\mathcal{R}_q(\omega)$. Setting $\mathcal{M}_{q-s} = \mathcal{M}_{\Theta_{q-s}}$ for $s \le q$ as in Lemma A.4, the symbols $\mathcal{N}_{q-s} = 0$ are zero. The symbols $\mathcal{N}_{q+r}$ also give rise to restricted Spencer δ-sequences as in Lemma A.4.

Lemma 3.48. The Spencer δ-sequences restrict to sequences

$$0 \longrightarrow \mathcal{N}_{q+r} \xrightarrow{\delta} T^* \otimes \mathcal{N}_{q+r-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigwedge^n T^* \otimes \mathcal{N}_{q-n} \longrightarrow 0.$$

If the module $M_\Theta = \oplus_k \mathcal{M}^*_{\Theta_k}$ is r-involutive then $H^{i+1}_{j-1}(\mathcal{M}_q) \cong H^i_j(\mathcal{N}_q)$ for $j > r$.⋄

Proof. Set $M = \oplus \mathcal{M}^*_k$ and $N = \oplus \mathcal{N}^*_k$. Taking the direct sum over the duals of equation (3.39) yields the sequence in Proposition A.9 (4) with A^r replaced by the A-module M_Θ. It follows that N is also an A-module. Tensoring the sequence with the Koszul complex $K_\bullet(\underline{\xi}, A)$ and taking the long exact cohomology sequence as in Proposition A.9 (5) proves $H^{i+1}_{j-1}(\mathcal{M}_q) \cong H^i_j(\mathcal{N}_q)$ for $j > r$. □

In most applications, we assume $\Theta_q = \Pi_q$ and thus $\mathcal{M}_{\Theta_k} = S^k T^* \otimes T$. Then the isomorphism of Spencer cohomology groups follows directly from Proposition A.9 (5) and is valid for $(i, j) \neq (0, 0)$.

As last preparational point, we will verify that the map $\mathcal{N}_{q+1} \to T^* \otimes \mathcal{F}_0$ in diagram (3.38) is indeed a Spencer δ-map. The defining equations (A.2) for $\mathcal{M}_{q+1}$ imply that $\mathcal{N}_{q+1}$ is generated by the elements

$$\frac{\partial \Phi_\omega}{\partial y^i_\mu} \xi^i_{\mu+1_j}, \qquad |\mu| = q.$$

Using coordinates δu^τ for $\mathcal{F}_0$, each generator of $\mathcal{N}_{q+1}$ is mapped to

$$\delta u^\tau dx^j = \frac{\partial \Phi_\omega}{\partial y^i_\mu} \xi^i_{\mu+1_j} dx^j \in T^* \otimes \mathcal{F}_0.$$

This coincides with the Spencer δ-map in equation (A.4). Applying Lemma 3.43 to $\mathcal{M}_{\Theta_{q+1}}$ shows that the definition of $\mathcal{F}_{(1)}$ and the Janet bundle $\mathcal{F}_1$ coincide if $\mathcal{F}_{(1)}$ is a vector bundle.

Jacobian Condition and the Differential Janet Sequence[1]

We now sketch the construction of the Jacobi conditions J, which is a first order differential map between $J_1(\mathcal{F})$ and $\mathcal{F}_2$ (for details see [Pom83, Thm. 2.A.3.22]). Assume that $\mathcal{M}_q$ is 3-acyclic (which implies that $\mathcal{N}_{q+1}$ is 2-acyclic) and check the integrability of $\mathcal{B}_1(c)$ by differentiating the equations (3.36):

$$D_x I = A(u) u_{xx} + \partial_u A(u) u_x u_x + D_x B(u).$$

[1] The results of this section will not be used later on, but they are included for a complete treatment of the nonlinear Janet sequence.

The system $\mathcal{B}_1(0)$ is formally integrable since $\mathcal{R}_q(\omega_0)$ is. It follows that each linear combination $\gamma(u)D_xI$, which is independent from u_{xx}, can be expressed in terms of I:

$$\gamma(u)D_xI + \alpha(u)u_xI + \beta(u)I = 0.$$

In other words, there exist functions α, β, such that the equation is valid. Denote the coordinates of $\mathcal{F}_2$ by (w). Then the morphism of natural bundles J for the Jacobian conditions is

$$J : J_1(\mathcal{F}_1) \to \mathcal{F}_2 : (x, u, v, u_x, v_x) \mapsto (x, u, w = \gamma(u)v_x + \alpha(u)u_xv + \beta(u)v).$$

Since $\mathcal{F}_2$ is only a natural bundle of order q, the first order jets u_x may only occur in combinations $A(u)u_x + B(u)$. Modifying α and β, we obtain

$$J : J_1(\mathcal{F}_1) \to \mathcal{F}_2 : (x, u, v, u_x, v_x) \mapsto (x, u, w = \gamma(u)v_x + \alpha(u)vv + \beta(u)v).$$

We can now plug in the prolongation of the equivariant section c to check the integrability of $\mathcal{B}_1(c)$. It is integrable if and only if the *Jacobian conditions* are satisfied:

$$\gamma(u)\frac{\partial c(u)}{\partial u}c + \alpha(u)cc + \beta(u)c = 0. \tag{3.40}$$

The functions α, β and γ were chosen such that all dependence on jets of u vanishes when plugging in D_xI. This implies, that the u-dependence in the Jacobian conditions can be replaced by a dependence on the invariants on $\mathcal{F}$. Only equivariant sections that satisfy the Jacobian conditions correspond to integrable symmetry groupoids $\mathcal{R}_q(\omega)$. Before interpreting the differential Janet sequence, we given an example.

Example 3.49. Let X be a 3-dimensional manifold with coordinates (x^1, x^2, x^3) and $\mathcal{F} = T^* \times \bigwedge^2 T^*$. This is similar to the example in Section 3.4.3 for a three-dimensional base. Choose the coordinates $(u^1, \ldots, u^6)$ of $\mathcal{F}$ such that they are the coefficients of the differential forms:

$$\begin{aligned}
\omega &= u^1\,dx^1 + u^2\,dx^2 + u^3\,dx^3, \\
\Omega &= u^4\,dx^2 \wedge dx^3 - u^5\,dx^1 \wedge dx^3 + u^6\,dx^1 \wedge dx^2.
\end{aligned}$$

Note the reversed sign for u^5. The symbols for generic sections of $\mathcal{F} \to X$ are 2-acyclic and we prolong to $J_1(\mathcal{F})$ with coordinates (u, u^i_j). The projection to the first natural bundle of the series

$$\mathcal{F}_{(1)} = J_1(\mathcal{F})/K^2_1 \cong \mathcal{F} \times \bigwedge^2 T^* \times \bigwedge^3 T^*$$

that has the fibre coordinates (v^1, v^2, v^3, w). The projection map is given by:

$$\begin{aligned}
v^1 &= u^3_2 - u^2_3, \\
v^2 &= u^1_3 - u^3_1, \\
v^3 &= u^2_1 - u^1_2, \\
w &= u^4_1 + u^5_2 + u^6_3.
\end{aligned}$$

The Vessiot structure equations can be written shortly as:

$$d\omega = C_1\,\Omega, \qquad d\Omega = C_2\,\omega \wedge \Omega,$$

with two arbitrary constants $C_1, C_2 \in \mathbb{R}$. In coordinates, the equations (3.36) for the system $\mathcal{B}_1(c)$ are obtained by plugging in the expressions v^i and w into:

$$v^i = C_1 u^{i+3}, \qquad w = C_2(u^1u^4 + u^2u^5 + u^3u^6), \qquad 1 \le i \le 3.$$

Differentiate and eliminate the second order jets of u to calculate the single nontrivial Jacobian condition

$$C_1\,C_2 = 0.$$

An integrable system $\mathcal{R}_1(\omega, \Omega)$ determined by a 1-form and a 2-form implies either $C_1 = 0$ or $C_2 = 0$. ◇

The Jacobian conditions finish the nonlinear Janet sequence

$$\{\mathrm{id}_X\} \longrightarrow \Gamma_\omega \longrightarrow \Gamma_{\Theta_q} \underset{\omega\circ s}{\overset{\Phi_\omega\circ j_q}{\rightrightarrows}} \mathcal{F} \underset{c}{\overset{I\circ j_1}{\rightrightarrows}} \mathcal{F}_{(1)} \xrightarrow{J\circ j_1} \mathcal{F}_2.$$

It can be interpreted as follows. A local diffeomorphism, which satisfies the equations for Θ_q is a solution of $\mathcal{R}_q(\omega)$ if and only if it is in the kernel of the differential operator $\Phi_\omega \circ j_q$ with respect to the section ω. Assuming 2-acyclic symbols, a section ω of $\mathcal{F} \to X$ is integrable if and only if it is in the kernel of $I \circ j_1$ with respect to a suitable equivariant section $c : \mathcal{F} \to \mathcal{F}_{(1)}$. The bundles $\mathcal{F}_{(1)} \cong \mathcal{F}_1$ and $\mathcal{F}_2$ are vector bundles such that the Jacobian conditions do not depend on a double arrow. An equivariant section c corresponds to a section ω if and only if it fulfills the Jacobian conditions.

Sequence of Affine Bundles

We now drop the assumption that there exists a section ω of $\mathcal{F} \to X$ with integrable symmetry groupoid $\mathcal{R}_q(\omega)$. This case frequently occurs when dealing with Θ_q-bundles $\mathcal{F}$ where $\Theta_q \le \Pi_q$ is a subgroupoid. The nonlinear Janet sequence relies on the existence of equivariant sections, but in the non-integrable case, we can still construct an exact sequence of natural bundles completely analogous to [Pom83, p. 301].

To do this, we inspect Definition 1.1 closely. Condition (2) for the vertical bundles means that on the fibres over e', $e = \varphi(e')$ and $e'' = \psi(e)$ the maps φ_* and ψ_* define an exact sequence of vector spaces. We turn this into a sequence of vector bundles by pulling back $V(\mathcal{E})$ and $V(\mathcal{E}'')$ over $\mathcal{E}''$.

$$\begin{array}{ccc} \widetilde{V(\mathcal{E})} & \longrightarrow & V(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{E}' & \xrightarrow{\varphi} & \mathcal{E} \end{array} \qquad \begin{array}{ccc} \widetilde{V(\mathcal{E}'')} & \longrightarrow & V(\mathcal{E}'') \\ \downarrow & & \downarrow \\ \mathcal{E}' & \xrightarrow{\psi\circ\varphi} & \mathcal{E}'' \end{array}$$

For simpicity, we assume that the pullback is implicitly done and omit the tilde. The pullback induces an exact sequence of vector bundles:

$$V(\mathcal{E}') \longrightarrow V(\mathcal{E}) \longrightarrow V(\mathcal{E}'').$$

We will use a similar construction to obtain the exact sequence affine bundles:

$$0 \longrightarrow \mathcal{R}_{q+1}(\omega) \longrightarrow \Theta_{q+1} \rightrightarrows J_1(\mathcal{F}) \rightrightarrows \mathcal{F}_{(1)} \longrightarrow 0. \tag{3.41}$$

Start with the exact sequence (3.5) for $\mathcal{R}_q(\omega)$ and define the vector bundles $\mathcal{F}_0 = V(\mathcal{F})$ and $\mathcal{F}_1 = V(\mathcal{F}_{(1)})$ as substitutes for the Janet bundles. The sequence of vertical bundles in Definition 1.1 (2) gives an exact sequence which may be restricted to the symbols:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_q & \longrightarrow & \mathcal{M}_{\Theta_q} & \longrightarrow & \mathcal{F}_0 \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & V(\mathcal{R}_q(\omega)) & \longrightarrow & V(\Theta_q) & \longrightarrow & \mathcal{F}_0
\end{array} \tag{3.42}$$

Lemma 3.50. If $\mathcal{F} = \mathcal{R}_q(\omega)\backslash\Theta_q$ is the natural bundle constructed in Theorem 3.22, the morphism $V(\Theta_q) \to \mathcal{F}_0$ in diagram (3.42) is surjective. If furthermore all equations for $\mathcal{R}_q(\omega)$ are of order q (i.e. $\pi^q_{q-1}(\mathcal{R}_q) = \Theta_{q-1}$) then $\mathcal{F}_0 \cong \mathcal{N}_q$. $\diamond$

Proof. Since $\Theta_{q-1} = \pi^q_{q-1}(\mathcal{R}_q(\omega))$, we have $\mathcal{F}/K^q_{q-1} = 0$ and the following exact and commutative diagram of vertical bundles.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_q & \longrightarrow & \mathcal{M}_{\Theta_q} & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & V(\mathcal{R}_q(\omega)) & \longrightarrow & V(\Theta_q) & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V(\Theta_{q-1}) & \longrightarrow & V(\Theta_{q-1}) & \longrightarrow & 0 & &
\end{array}$$

Comparing with the sequence (3.39) yields $\mathcal{F}_0 \cong \mathcal{N}_q$. $\square$

Prolong the sequence (3.42) to the symbol $\mathcal{M}_{q+1}$:

$$0 \longrightarrow \mathcal{M}_{q+1} \longrightarrow \mathcal{M}_{\Theta_{q+1}} \longrightarrow T^* \otimes \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow 0.$$

It is the sequence of model vector bundles for the exact sequence of affine bundles

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{M}_{q+1} & \longrightarrow & \mathcal{M}_{\Theta_{q+1}} & \longrightarrow & T^* \otimes \mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}_{q+1} & \longrightarrow & \Theta_{q+1} & \longrightarrow & J_1(\mathcal{F}) & \longrightarrow & \mathcal{F}_{(1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}_q & \longrightarrow & \Theta_q & \longrightarrow & \mathcal{F} & & \mathcal{F} & &
\end{array} \tag{3.43}$$

As in [Pom83, p. 301], all double arrows are omitted for brevity. If there exists no equivariant section $c : \mathcal{F} \to \mathcal{F}_{(1)}$, it makes no sense to speak of an exact sequence at $J_1(\mathcal{F})$. The problem is that we are not dealing with an exact sequence of bundles over a fixed base. An exact sequence should have the following property. Two elements of Θ_{q+1} can be mapped to $J_1(\mathcal{F})$. If their images are in the same fibre over $\mathcal{F}$, we expect that both are projected to the same point of $\mathcal{F}_{(1)}$. This is not automatically the case, as the next lemma shows.

Lemma 3.51. If in the diagram (3.43), there are two elements

$$a = p_1(\Phi_\omega)(f), \qquad b = p_1(\Phi_\omega)(g)$$

for $f, g \in \Theta_{q+1}$ in the same fibre over $\mathcal{F}$ ($\pi_0^1(a) = \pi_0^1(b)$), their projections $I(a)$ and $I(b)$ to $\mathcal{F}_{(1)}$ coincide if and only if $\mathcal{R}_q(\omega)$ is integrable with an equivariant section $c : \mathcal{F} \to \mathcal{F}_{(1)}$, $I(j_1(\omega)) = c(\omega)$. ◇

Proof. The prolonged map $p_1(\Phi_\omega)$ is the Θ_{q+1}-action on the section $j_1(\omega)$:

$$\begin{aligned} a &= j_1(\omega)(y_f)f, \qquad y_f = t(f) \\ b &= j_1(\omega)(y_g)g, \qquad y_g = t(g) \end{aligned}$$

Define $h := fg^{-1}$ and the elements f_q, g_q, h_q by projection to order q. Since a and b are in the same fibre over $\mathcal{F}$, we have $h_q \in \mathcal{R}_q(\omega)$. If $I(a) = I(b)$, the fact that I is a morphism of Θ_q-bundles implies that

$$j_1(\omega)(y_f)fg^{-1}gK_q^{q+1} = j_1(\omega)(y_g)gK_q^{q+1}.$$

Hence h can be modified by K_q^{q+1} to an element of $\mathcal{R}_{q+1}(\omega)$ projecting to h_q. Since f and g are arbitrary, $\mathcal{R}_q(\omega)$ is integrable and we can find an equivariant section by Theorem 3.35.

Assume that there exists an equivariant section c. Then

$$I(a) = I(j_1(\omega)(y_f))f_q = c(\omega(y_f))f_q = c(\omega(y_f)f_q) = c(\omega(y_g)g_q) = I(b),$$

since a and b are in the same fibre over $\mathcal{F}$. □

To turn diagram (3.43) into an exact sequence of affine bundles, we pull back $J_1(\mathcal{F})$ and $\mathcal{F}_{(1)}$ over Θ_q,

$$\begin{array}{ccc} \widetilde{J_1(\mathcal{F})} & \longrightarrow & J_1(\mathcal{F}) \\ \downarrow & & \downarrow \\ \Theta_q & \xrightarrow{p_1(\Phi_\omega)} & \mathcal{F} \end{array} \qquad \begin{array}{ccc} \widetilde{\mathcal{F}_{(1)}} & \longrightarrow & \mathcal{F}_{(1)} \\ \downarrow & & \downarrow \\ \Theta_q & \xrightarrow{I \circ p_1(\Phi_\omega)} & \mathcal{F}_{(1)} \end{array}$$

Again, we will not explicitly write down the pullbacks in the diagram (3.43). The pullback repairs the problems with the missing equivariant section. If there are two elements a, b as in Lemma 3.51, which are in the same fibre of the pullback bundle $J_1(\mathcal{F}) \to \Theta_q$, they differ only by K_q^{q+1}. By construction of $\mathcal{F}_{(1)}$, a and b project to the same element $I(a) = I(b)$ in the pullback bundle $\mathcal{F}_{(1)} \to \Theta_q$.

3.5.4 The Curvature Map

The exact sequence (3.43) of affine bundles gives rise to a curvature map κ, which was constructed by Goldschmidt [Gol67b, Prop. 8.3]. We follow the work of Pommaret [Pom83, §1.A.3, p. 301]. Similar to the Janet sequence, it is only necessary for the proof of the Embedding Theorem 4.22 and in Section 6.2.3.

Another very instructive reference for the curvature map is the work of Malgrange [Mal05, §II.3], where a dual approach is used. Here, the curvature is called torsion.

Remark 3.52. The vector bundle $\mathcal{F}_1$ and diagram (3.43) can be used to define the curvature map $\kappa : \mathcal{R}_q(\omega) \to F_1$, where $F_1 = \omega^*(\mathcal{F}_1)$ is the pullback of $\mathcal{F}_1$ along the section $\omega : X \to \mathcal{F}$. The restriction of

$$\mathcal{R}_{q+1}(\omega) \longrightarrow \mathcal{R}_q(\omega) \overset{\kappa}{\underset{0}{\rightrightarrows}} F_1$$

to

$$0 \longrightarrow \mathcal{R}_q^{(1)}(\omega) \longrightarrow \mathcal{R}_q(\omega) \overset{\kappa}{\underset{0}{\rightrightarrows}} F_1 \tag{3.44}$$

defines $\mathcal{R}_q^{(1)}(\omega)$ as the kernel of the curvature map. The construction of κ is quite instructive for the work with diagram (3.43). At the same time we prove $\mathcal{R}_q^{(1)}(\omega) = \ker_0(\kappa)$. It is done by a diagram chase.

Take $r_q \in \mathcal{R}_q(\omega)$ and consider it as an element of Θ_q. An arbitrary preimage $\bar{r}_{q+1} \in \Theta_{q+1}$ can be mapped by $\rho_1(\Phi_\omega)$ to $J_1(\mathcal{F})$, where the difference

$$\rho_1(\Phi_\omega)(\bar{r}_{q+1}) - j_1(\omega)(x) = j_1(\omega)(y)\bar{r}_{q+1} - j_1(\omega)(x)$$

can be lifted to $T^* \otimes \mathcal{F}_0$, since $\omega(y) r_q = \omega(x)$. Its image in $\mathcal{F}_{(1)}$ is equal to

$$I(j_1(\omega)(y)\bar{r}_{q+1}) - I(j_1(\omega)(x)),$$

since $I : J_1(\mathcal{F}) \to \mathcal{F}_{(1)}$ is a morphism of affine bundles. The definition

$$\kappa(r_q) = I(j_1(\omega)(y)\bar{r}_{q+1}) - I(j_1(\omega)(x)) \tag{3.45}$$

does not depend on the choice of $\bar{r}_{q+1}$, because the difference between two choices is an element of $\mathcal{M}_{\Theta_{q+1}}$. There exists an element $r_{q+1} \in \mathcal{R}_{q+1}(\omega)$ projecting onto r_q if and only if $\kappa(r_q) = 0$. Equivalently there exists an element of $f_{q+1} \in \mathcal{M}_{\Theta_{q+1}}$ such that

$$I(j_1(\omega)(y)(\bar{r}_{q+1} + f_{q+1})) - I(j_1(\omega)(x)) = 0$$

and $r_{q+1} = \bar{r}_{q+1} + f_{q+1} \in \mathcal{R}_{q+1}(\omega)$. ◇

The next theorem shows a connection between the fibre of the natural bundle $\mathcal{F}_1$ and a Spencer cohomology group, which is implicitly present in the work of Pommaret [Pom83, p. 44]. For the curvature map, this fact is well-known (see [Mal05, §II.3] and [IL03, §5.5] for the case of exterior differential systems).

Theorem 3.53. If all equations defining $\mathcal{R}_q(\omega)$ are of order q and M_Θ of Lemma 3.48 is $q-1$-involutive, the bundle $\mathcal{F}_1$ is isomorphic to the Spencer cohomology group $H^2_{q-1}(\mathcal{M}_q)$ and the curvature κ is a map

$$\kappa : \mathcal{R}_q(\omega) \to H^2_{q-1}(\mathcal{M}_q).$$ ◇

In the theorem, we have omitted all pullbacks. To construct $H^2_{q-1}(\mathcal{M}_q) = \mathcal{F}_1 \to \mathcal{R}_q(\omega)$, we have to pull back $\mathcal{F}_1$ in diagram (3.43) over $\mathcal{R}_q(\omega)$. Then κ is a section of $\mathcal{F}_1 \to \mathcal{R}_q(\omega)$.

Proof. By Lemma 3.50, we have $\mathcal{F}_0 \cong \mathcal{N}_q$. The sequence of model vector bundles in diagram (3.43) splits into

$$0 \longrightarrow \mathcal{N}_{q+1} \xrightarrow{\delta} T^* \otimes \mathcal{N}_q \longrightarrow \mathcal{F}_1 \longrightarrow 0$$

using equation (3.39). Since $\mathcal{N}_{q-1} = 0$ we have

$$\mathcal{F}_1 \cong H^1_q(\mathcal{N}_q) = T^* \otimes \mathcal{N}_q / \delta(\mathcal{N}_{q+1})$$

By Lemma 3.48, this is isomorphic to $H^2_{q-1}(\mathcal{M}_q)$. □

Remark 3.54. The condition that all equations are of order q in Theorem 3.53 can be weakened to the condition that for all equations $\Phi^\tau_\omega = \omega^\tau$ of order $< q$ for $\mathcal{R}_q(\omega)$ also their total derivatives $D_i\Phi^\tau_\omega = \partial_i\omega^\tau$ are present. In this case, the Embedding Theorem 4.22 can be used to find the minimal subbundle of $J_1(\mathcal{F}) \to \mathcal{F}$. This is setting of [Mal05, §II.3,4]. ◇

Chapter 4

Applications of Natural Bundles

In the last chapter, the connection between jet groupoids $\mathcal{R}_q(\omega)$ and geometric objects on natural bundles has been presented. Furthermore, the prolongation and projection of the jet groupoids, this time considered as systems of PDEs, has been translated into the language of natural bundles. This is called Vessiot's approach to geometric objects and jet groupoids. In the present chapter, problems are stated that can be solved with Vessiot's approach. Examples are shown in Chapter 5. To solve these tasks, new and crucial optimisations are developed.

The problems are the same as in the introduction of this thesis:

- Complete the equations for a groupoid $\mathcal{R}_q(\omega)$ to formal integrability.
- Classify the symmetry groupoids $\mathcal{R}_q(\omega)$ of geometric objects ω on $\mathcal{F}$.
- Find a generating system for the invariants of the Θ_{q+r}-action on $J_r(\mathcal{F})$.
- Decide equivalence for geometric objects on $\mathcal{F}$ under the action of Θ_q.

The first and second question are focused on the groupoids and their formal integrability. In the first case, a single groupoid is chosen and in the second case all sections of $\mathcal{F} \to X$ are considered. Then the process of prolongation and projection is applied until formal integrability is reached. The symmetry groupoids $\mathcal{R}_q(\omega)$ are classified by the number of steps that were necessary to reach integrability and the equivariant sections. Both questions will be treated in Section 4.1 and the examples in Chapter 5 show several steps of the classification.

To find a generating set of invariants, one proceeds similarly to the classification of symmetry groupoids. A theorem of Lie and Tresse [LSE93], [Tre94] states that the set of invariants of a pseudogroup action on a manifold is finitely generated with respect to invariant differentiation. It was proved by Kumpera [Kum75]. With a trivial translation into the language of natural bundles and jet groupoids, a generating set of invariants can be computed with the Vessiot approach. The general case is presented in Section 4.2 and an example is found in Section 5.2.

Natural bundles also give an alternative approach to Cartan's well-known equivalence problem [Car08, Car10], which will be presented in detail in Chapter 6 on the Vessiot equivalence problem.

All of the above problems start with a natural Θ_q-bundle $\mathcal{F}$ and one or more sections ω of $\mathcal{F} \to X$. To solve them, natural bundles

$$\mathcal{F}_{(1)}, \quad \mathcal{F}_{(2)}, \ldots$$

are computed by prolongation and projection and formal integrability is being checked. For an effective treatment, it is necessary to answer three important questions:

- Which set of geometric objects is relevant?
- Is it possible to shrink Θ_q or $\mathcal{F}$?
- Which is the *minimal* natural bundle $\mathcal{F}_{(i)}$ to decide formal integrability?

It depends on the careful answer to these questions whether it is possible to perform the necessary calculations in nontrivial examples. None of the above questions arise in Pommaret's [Pom78] or Vessiot's [Ves03] work, because in their examples only the first bundle of integrability conditions $\mathcal{F}_{(1)}$ is computed.

In most cases, all sections of $\mathcal{F} \to X$ are interesting and no optimisation for Θ_q and $\mathcal{F}$ is necessary before starting the prolongation and projection.

But if only a subset of the sections of $\mathcal{F} \to X$ are relevant and all symmetry groupoids $\mathcal{R}_q(\omega)$ restrict to $\Theta_q' \leq \Theta_q$, it is possible to shrink both Θ_q and $\mathcal{F}$ in the exact sequence

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow \Theta_q \overset{\Phi_\omega}{\underset{\omega \circ s}{\rightrightarrows}} \mathcal{F}$$

without losing information. Details will be presented in Section 4.3.3.

The first bundle $\mathcal{F}_{(1)}$ of integrability conditions is the minimal bundle to check formal integrability. However, not all sections of $\mathcal{F}_{(1)} \to X$ are relevant for the original problem which leads to redundancies for the next step of prolongation and projection. Based on the Embedding Theorem 4.22 minimal bundles $\mathcal{F}_{(i)}$ of integrability conditions can be computed. The approach using minimal bundles is new. If nonminimal bundles are chosen, the calculations quickly become too large to compute. Theorem 4.22 is also necessary to compute generating sets of invariants with the Vessiot approach.

4.1 Towards Formal Integrability

In this section, a flowchart is presented that summarises the necessary steps to complete a specific jet groupoid $\mathcal{R}_q(\omega)$ defined by a section ω on the natural

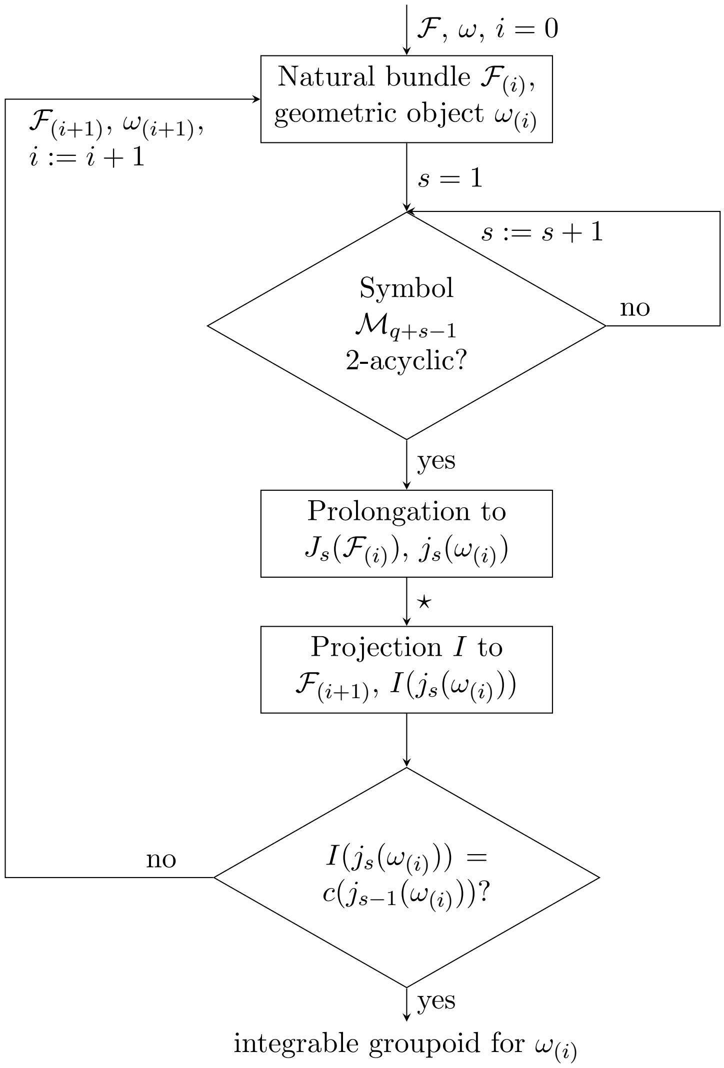
Θ_q-bundle $\mathcal{F} \to X$ to formal integrability. The flowchart is inspired by Gardner [Gar89], who used similar diagrams to summarise Cartan's method of equivalence. We will take up the flowcharts in Chapter 6 on the Vessiot equivalence method.

The classification of symmetry groupoids is very similar and we modify the diagram for this case.

Completing $\mathcal{R}_q(\omega)$ to Formal Integrability

We comment the flowchart in Figure 4.1 and refer to the relevant sections for detailed information in each step.

Figure 4.1: Completing $\mathcal{R}_q(\omega)$ to formal integrability



- Enter the flowchart at the top with the natural Θ_q-bundle $\mathcal{F}$ and the section $\omega : X \to \mathcal{F}$ defining $\mathcal{R}_q(\omega)$. We set:

$$\mathcal{F}_{(0)} = \mathcal{F}, \qquad \omega_{(0)} = \omega \qquad i = 0.$$

  The variable i counts the number of times, we have gone through the main loop, $\mathcal{F}_{(i)}$ and $\omega_{(i)} : X \to \mathcal{F}_{(i)}$ are the natural bundle and the section at step i. At each step q denotes the order of $\mathcal{F}_{(i)}$.

- Test if the symbols on $\mathcal{F}_{(i)}$ are 2-acyclic using the results of Appendix A.If not, choose $s \in \mathbb{N}$ such that the symbol $\mathcal{M}_{q+s-1}$ for the symmetry groupoid of $\omega_{(i)}$ becomes 2-acyclic.

- Prolong $\mathcal{F}_{(i)}$ and $\omega_{(i)}$ according to Proposition 3.26. At the position $\star$, it is highly recommended to restrict to the minimal subbundle of $J_s(\mathcal{F}_{(i)}) \to \mathcal{F}_{(i)}$ according to the Embedding Theorem 4.22.

- Compute the projection I defining $\mathcal{F}_{(i+1)}$ and the section

$$\omega_{(i+1)} = j_s(\omega_{(i)}) K^{q+r}_{q+r-1}$$

  with Propositions 3.30 and 3.33.

- Compute the Vessiot structure equations and check if the projection is surjective for $\omega_{(i)}$ using the Projection Theorem 3.35.

- If not, replace $\mathcal{F}_{(i)}$ by $\mathcal{F}_{(i+1)}$, the section $\omega_{(i)}$ by $\omega_{(i+1)}$, increment i and start another loop at the beginning.

Classification of Symmetry Groupoids

Figure 4.2 shows the flowchart for the classification of symmetry groupoid on $\mathcal{F}$. We leave out the construction of the sections $\omega_{(i)}$ which is analogous to the previous case.

- Start with $\mathcal{F}$ and compute the Spencer cohomology for generic sections with Corollary 3.46 and find s such that the symbols become 2-acyclic.

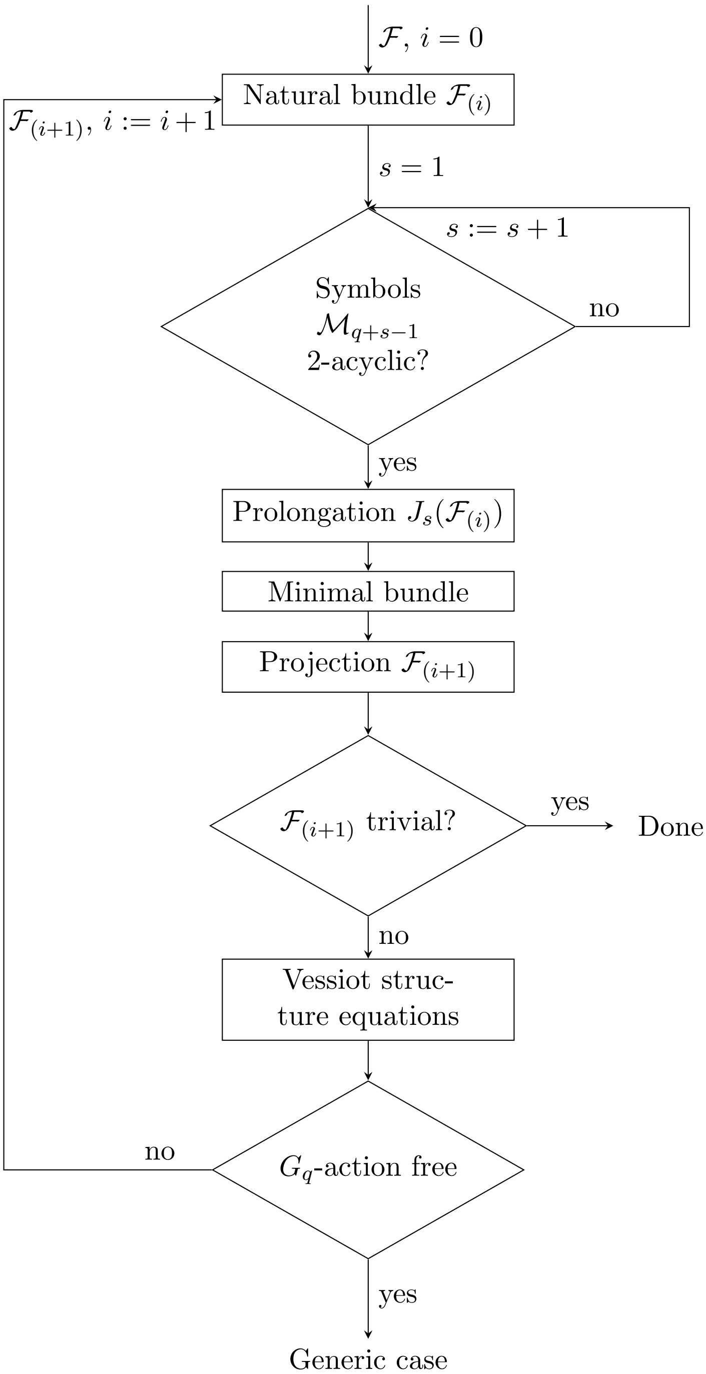
- Prolong to $J_s(\mathcal{F}_{(i)})$ with Proposition 3.26 and construct the minimal subbundle of $J_s(\mathcal{F}_{(i)}) \to \mathcal{F}_{(i)}$ to which all sections from $\mathcal{F} \to X$ restrict (using the Embedding Theorem 4.22).

- Use Propositions 3.30 and 3.33 to project to $\mathcal{F}_{(i+1)}$. Check whether $\mathcal{F}_{(i+1)}$ is trivial, namely

$$\mathcal{F}_{(i+1)} = J_{s-1}(\mathcal{F}_{(i)}).$$

  Due to this test, the construction of the minimal subbundle in the last step is mandatory. If it is trivial, all symmetry groupoids on $J_{s-1}(\mathcal{F}_{(i)})$ that are obtained by prolongation and projection from $\mathcal{F}$ are integrable.

Figure 4.2: Classification of symmetry groupoids



- Compute the Vessiot structure equations to classify the sections of $\mathcal{F} \to X$ which are generic on $\mathcal{F}_{(i)}$ by their equivariant sections $c : J_{s-1}(\mathcal{F}_{(i)}) \to \mathcal{F}_{(i+1)}$ (see Theorem 3.35).
- Let G_q be the isotropy group of Θ_q. If the G_q-action on the fibre $F_{(i+1)}$ of $\mathcal{F}_{(i+1)}$ is locally free, then exit the classification at the generic case.

The classification computes a pair (i, c) for each section ω of $\mathcal{F} \to X$ that does not run into the generic case. The integer i is the minimal step in the process of prolongation and projection where $\mathcal{R}_q(\omega)$ becomes integrable and

$$c : J_{s-1}(\mathcal{F}_{(i)}) \to \mathcal{F}_{(i+1)}$$

is the equivariant section where the symmetry groupoid $\mathrm{Stab}_{\mathcal{F}_{(i)}}(\omega_{(i)})$ becomes integrable. We will see in Chapter 6 that we only need to add the invariants on $\mathcal{F}_{(i)}$ to decide equivalence of sections ω of $\mathcal{F} \to X$.

Example 4.1. For the example of $\mathcal{F} = T^* \times_X \bigwedge^2 T^*$ with $\dim(X) = 2$, the classification was effectively done in in Section 3.4.3 and Example 3.38. At each step of the computation, we display the bundle $\mathcal{F}_{(i)}$ and the section $\omega_{(i)}$ which comes from a section ω of $\mathcal{F} \to X$. The classification leads to the following tree:

$$\begin{array}{lcl}
\mathcal{F} = T^* \times_X \bigwedge^2 T^*, & & \\
\omega_{(0)} = (\omega, \Omega) & & \\
\quad\quad\downarrow & & \\
\mathcal{F}_{(1)} = \mathcal{F} \times \mathbb{R}, & & \\
\omega_{(1)} = (\omega, \Omega, \psi = d\omega/\Omega) & \longrightarrow & d\omega = C_1 \Omega \\
\quad\quad\downarrow & & \\
\mathcal{F}_{(2)} = \mathcal{F}_{(1)} \times_X T^*, & & \\
\omega_{(2)} = (\omega, \Omega, \psi, d\psi) & \longrightarrow & d\psi = F_1(\psi)\omega \\
\quad\quad\downarrow & & \\
\text{generic case} & &
\end{array}$$

If the symmetry groupoid is integrable at $\mathcal{F}_{(i)}$, follow the arrow to the right. If not, prolong and project another time by going down. All the integrable groupoids are classified by the constant C_1 in the first step and by the function $F_1(\psi)$ in the second step. ◇

Remark 4.2. In nearly all cases, the classification ends in the generic case. Assume that this happens for the bundle $\mathcal{F}_{(j)}$. Then the symbols for generic sections of $\mathcal{F}_{(j)} \to X$ are trivial and thus involutive. We have to prolong only once to check formal integrability. There may still be non-integrable symmetry groupoids, but we can trivially describe all integrability conditions using Proposition 3.42.

Since the G_q-action on the fibre $F_{(j)}$ is already free, the fibre coordinates $(v^1, \dots v^k)$ of the bundle $\mathcal{F}_{(j+1)} \to \mathcal{F}_{(j)}$ can all be chosen as invariants. Denote the invariants on $\mathcal{F}_{(j)}$ by ψ. Then the Vessiot structure equations are

$$v^1 = c_1(\psi), \quad \dots, \quad v^k = c_k(\psi)$$

for arbitrary functions c_i of the invariants, because they are obviously equivariant. For further integrability conditions, it is recommended to work with a generating set of invariants computed in Section 4.2. ◇

To illustrate the case where the bundles of integrability conditions become trivial after several steps of prolongation and projection, we give one of the few examples.

Example 4.3. Let X be a manifold of dimension $n > 2$ and $\mathcal{F} = T^*$ be the natural bundle of 1-forms. The symbols of generic sections $\omega : X \to T^*$ are 2-acyclic and we compute the first bundle of integrability conditions

$$\mathcal{F}_{(1)} = J_1(\mathcal{F})/K_1^2 = T^* \times_X \bigwedge^2 T^*.$$

The differential operator $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}_{(1)})$ is the exterior derivative

$$d : \Gamma(T^*) \to \Gamma(\bigwedge^2 T^*).$$

Another prolongation and projection yields the bundle

$$\mathcal{F}'_{(2)} = \mathcal{F}_{(1)} \times_X \bigwedge^2 T^* \times_X \bigwedge^3 T^*,$$

but the Embedding Theorem 4.22 implies that all sections from $\mathcal{F} \to X$ restrict to the subbundle

$$\mathcal{F}_{(2)} = \mathcal{F}_{(1)} \subseteq \mathcal{F}'_{(2)}.$$

This can be seen as follows. Let ω be a section of $\mathcal{F} \to X$. Then all sections of $\mathcal{F}_{(1)} \to X$ that come from $\mathcal{F}$ are of the form $(\omega, d\omega)$. The differential operator $\mathcal{F}_{(1)} \to \mathcal{F}'_{(2)}$ is constructed from two copies of the exterior derivative d. It follows that a section of $\mathcal{F}'_{(2)} \to X$ that comes from $\mathcal{F}$ is of the form

$$(\omega,\, d\omega,\, d\omega,\, d^2\omega),$$

but $d^2 = 0$ and both 2-forms – and the corresponding symmetry equations coincide. Effectively there are no integrability conditions on $\mathcal{F}_{(2)}$ and all 1-forms have nontrivial symmetry groupoids. For generic sections, there are only two cases:

- $d\omega = 0$: The symmetry groupoid $\mathcal{R}_1(\omega)$ is integrable.

- $d\omega \neq 0$, such that $(\omega, d\omega)$ is generic on $\mathcal{F}_{(1)}$. The symmetry groupoid $\mathcal{R}_1^{(1)}(\omega)$ is integrable.

However this classification does not apply to all 1-forms, as the well-known Darboux Theorem [Dar82] shows (see also [IL03, Thm. 1,9,17], [Olv95, p. 30]). Vessiot's approach only covers the generic case, which means sections $(\omega, d\omega)$ that do not restrict to a subbundle of $\mathcal{F}_{(1)} \to \mathcal{F}$. $\diamond$

4.2 Invariants on Natural Bundles

In this section it is described how a generating set for the invariants under the Θ_{q+r}-action on the prolonged natural bundle $J_r(\mathcal{F})$ is determined. With certain regularity assumption, the Lie-Tresse Theorem [LSE93], [Tre94] implies that the algebra of invariants on the infinite jet bundle $J_\infty(\mathcal{F})$ is finitely generated with respect to invariant differentiation. It was proved rigorously by Kumpera [Kum75] and extended to PDE systems on $J_r(\mathcal{F})$ by Kruglikov and Lychagin [KL06].

The section is based on the classical approach to compute the invariants on $J_\infty(\mathcal{F})$, as it is presented in [KL06], [Olv95, Ch. 5] and [OP]. There, a generating set of invariants $\psi_1, \dots, \psi_k$ and an invariant coframe is constructed. The invariant differential operators $\mathcal{D}_i$ needed to generate all invariants are dual to the coframe. We translate the necessary definitions and results to the case of natural bundles and modify the classification of symmetry groupoids in Figure 4.2 to the computation of invariants. The advantage of Vessiot's approach is that the projection of natural bundles avoids large computations on $J_r(\mathcal{F})$ and that it is possible to work with finitely many prolongations.

Throughout this section, we fix an n-dimensional base manifold X, a jet groupoid Θ_q and a natural Θ_q-bundle $\pi : \mathcal{F} \to X$. All other occurring natural bundles are interpreted as Θ_{q+r}-bundles for a suitable r. The analogue statement holds for morphisms of natural bundles.

Invariant Coframes and Invariant Differential Operators

An invariant on $\mathcal{F}$ is a smooth map $\psi : \mathcal{F} \to \mathbb{R}$, i.e. $\psi \in C^\infty(\mathcal{F})$, which is constant on the Θ_q-orbits. We adopt the following point of view for functions on $\mathcal{F}$. Each $\varphi \in C^\infty(\mathcal{F})$ defines a morphism of fibre bundles

$$\varphi : \mathcal{F} \to X \times \mathbb{R}.$$

An invariant ψ is a morphism of *natural bundles*, where $X \times \mathbb{R}$ is a natural bundle of order zero. We extend this point of view to horizontal differential forms $\omega \in \Omega_h^k(\mathcal{F}) = \Gamma(\mathcal{F} \times_X \bigwedge^k T^*)$, now interpreting them as fibre bundle morphisms

$$\omega : \mathcal{F} \to \bigwedge^k T^*.$$

With this point of view, it is easy to define invariant k-forms and coframes. They are an adaption of the contact-invariant coframe [OP, §6] to natural bundles.

Definition 4.4. An *invariant (horizontal) coframe* on $\mathcal{F}$ is a morphism of natural bundles $\theta : \mathcal{F} \to P_1(X)$. An *invariant (horizontal) k-form* on $\mathcal{F}$ is a morphism of natural bundles $\mathcal{F} \to \bigwedge^k T^*$. ⋄

Since $P_1 \subset (T^*)^n$ is the bundle of coframes, $\theta(f)$ is a basis of $T^*_{\pi(f)}$ and we can split θ into n invariant 1-forms $\theta^i : \mathcal{F} \to T^*$. The reason why θ is called invariant coframe is the equivariance under the Θ_q-action:

$$\theta(f g_q) = \theta(f) g_q \qquad \forall\, g_q \in \Theta_q.$$

It implies that θ stays invariant under pullback with solutions of Θ_q. We now come to an interpretation of the horizontal differential

$$\hat{d} : C^\infty(J_r(\mathcal{F})) \to \Omega^1_h(J_{r+1}(\mathcal{F})) : \psi \mapsto (D_i \psi) dx^i$$

that turns smooth functions into horizontal 1-forms (see e.g. [KL06, App. A.3] and [OP, §3]). The next lemma shows how to construct an invariant 1-form from an invariant, which was used by Tresse [Tre94] to compute an invariant coframe.

Lemma 4.5. If $\varphi \in C^\infty(J_r(\mathcal{F}))$, the first prolongation $p_1(\varphi)$ can be restricted to a morphism of fibre bundles

$$\hat{d}\varphi : J_{r+1}(\mathcal{F}) \to T^*.$$

If ψ is an invariant on $J_r(\mathcal{F})$ then $\hat{d}(\psi)$ is an invariant 1-form on $J_{r+1}(\mathcal{F})$. ⋄

Proof. Interpret φ as $\varphi : J_r(\mathcal{F}) \to X \times \mathbb{R}$ and apply Proposition 1.14. By the proof of Proposition 1.6, we have $J_1(X \times \mathbb{R}) = T^* \times \mathbb{R}$ and we can restrict to T^*. If ψ is an invariant, all maps are morphisms of natural bundles. □

Each invariant coframe defines a collection of dual vector fields, which can be interpreted as differential operators. They are also called coframe derivatives (see [Olv95, Ch. 8]).

Definition 4.6. Let θ be an invariant coframe on $\mathcal{F}$ consisting of the invariant 1-forms $\theta^1, \dots, \theta^n$. Define the *invariant differential operators* $\mathcal{D}_i : C^\infty(\mathcal{F}) \to C^\infty(\mathcal{F})$ *dual to* θ as

$$\hat{d}\psi = (\mathcal{D}_i \psi)\theta^i, \qquad \psi \in C^\infty(\mathcal{F}).$$ ⋄

Remark 4.7. Analogous to the total derivative (see Definition 1.11), each invariant differential operator defines first order differential operators

$$\mathcal{D}_i : \Gamma(J_r(\mathcal{F}) \times \mathbb{R}) \to \Gamma(J_{r+1}(\mathcal{F}) \times \mathbb{R}).$$

In fact, the invariant differential operators can be represented using total derivatives. If the invariant coframe θ on the bundle $\mathcal{F}$ with coordinates (x, u) has the form

$$\theta^i = A^i_j(u)dx^j,$$

with the matrix $A(u)$, the invariant differential operators $\mathcal{D}_i$ can be represented as

$$\mathcal{D}_i = (A^{-1}(u))^j_i D_j.$$

Unlike total derivatives, the $\mathcal{D}_i$ do not commute in general. ◇

The value of invariant differential operators is that they produce new, higher order differential invariants, as the following, classical proposition shows.

Proposition 4.8. Let θ be an invariant coframe on $J_r(\mathcal{F})$ and $\mathcal{D}_i$ the dual invariant differential operators. If ψ is an invariant on $J_r(\mathcal{F})$ then $\mathcal{D}_i\psi$ is an invariant on $J_{r+1}(\mathcal{F})$. ◇

Proof. Since $\hat{d}\psi$ is an invariant 1-form, we have

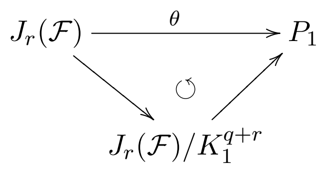
$$\begin{aligned}(\mathcal{D}_i\psi)(fg)\theta^i(fg) &= \hat{d}\psi(fg) = \hat{d}\psi(f)g = \left[(\mathcal{D}_i\psi)(f)\theta^i(f)\right] g \\ &= (\mathcal{D}_i\psi)(f)\theta^i(fg).\end{aligned}$$

for all $f \in J_{r+1}(\mathcal{F})$ and $g \in \Theta_{q+r+1}$. Thus $\mathcal{D}_i\psi$ is also an invariant. □

Existence of Invariant Coframes on $\mathcal{F}$

Invariant coframes do not exist on every natural bundle $\mathcal{F}$. Usually, we have to prolong to $J_r(\mathcal{F})$ in order to compute an invariant coframe. Here we will give conditions for the existence of invariant coframes. The following preparational lemma shows that an invariant coframe is a first order object and that we can lift coframes to $J_{r+s}(\mathcal{F})$.

Lemma 4.9. Let θ be an invariant coframe on $J_r(\mathcal{F})$. Then the map θ factors over the bundle $J_r(\mathcal{F})/K_1^{q+r}$:



If $\pi : J_{r+s}(\mathcal{F}) \to J_r(\mathcal{F})$ is the canonical projection, then $\theta \circ \pi$ is an invariant coframe on $J_{r+s}(\mathcal{F})$. ◇

Proof. Since P_1 is a first order bundle, such that $\theta(fk) = \theta(f)$ for $f \in J_r(\mathcal{F})$ and all $k \in K_1^{q+r}$. Both π and $\theta \circ \pi$ are morphisms of natural bundles. □

Proposition 4.10. There locally exists an invariant coframe on $\mathcal{F}$ if and only if the G_1-action on the fibre of $\mathcal{F}/K_1^q$ is locally free. $G_1 = \pi_1^q(G_q)$ is the projection of the isotropy group G_q of Θ_q. ⋄

Proof. Without loss of generality let $\mathcal{F} = \mathcal{F}/K_1^q$. If the G_1-action is locally free, we can trivialise the fibre $F \cong \mathcal{F}_x$ around $f \in F$ as an open subset of $\mathbb{R}^k \times G_1$, where k denotes the number of invariants. Assume that Θ_q is transitive. Then choose $g_0 \in G_1$ such that $(r, g_0) \in \mathbb{R}^k \times G_1$ and set the image $\theta(x, r, g_0) = \theta_0 \in P_{1,x}$. On an open subset around f, define θ by equivariant continuation. For $h \in \mathcal{F}_y$, there exists an element $g \in \Theta_1(x, y)$ such that $h = (r, g_0)g$ for some $r \in \mathbb{R}^k$. Then $\theta(h) = \theta_0 g$. If Θ_q is intransitive, we have to repeat the same construction with a trivialisation $\mathbb{R}^{n-k_0} \times \mathbb{R}^{k_0} \times F$ including k_0 invariants on X.

If there exists an invariant coframe θ, the G_1-orbit of $\theta(f)$ for $f \in \mathcal{F}$ is isomorphic to G_1, since G_1 acts freely on the fibre GL_1 of P_1. Since θ is an equivariant map, the G_1-action on the fibres of $\mathcal{F}$ is also free. □

Remark 4.11. There are several ways to construct an invariant coframe and the corresponding dual invariant differential operators on $J_r(\mathcal{F})$. One possibility is given by the proof of Proposition 4.10, which is inspired by moving frames (see e.g. [OP] or [Man08]).

Another method was used by Tresse [Tre94]. If there are n functionally independent invariants ψ^i on $J_r(\mathcal{F})$, it is possible to construct an invariant coframe on $J_{r+1}(\mathcal{F})$ by taking $\hat{d}\psi^i$. The condition of functional independence is equivalent to $\hat{d}\psi^1 \wedge \cdots \wedge \hat{d}\psi^n \neq 0$. ⋄

Lie-Tresse Theorem

The invariant differential operators are necessary for the Lie-Tresse Theorem. It was proved by Kumpera [Kum75] for a Lie sheaf of vector fields and we state it in the form presented by Kruglikov and Lychagin [KL06, Thm. 16], modified for the case of groupoid actions on natural bundles. It depends on the infinite jet space $J_\infty(\mathcal{F})$, which is the projective limit

$$J_\infty(\mathcal{F}) = \lim\operatorname{proj}(J_r(\mathcal{F}), \pi^r_{r-1}).$$

The formal pseudogroup Θ_∞ is also constructed by a projective limit.

Theorem 4.12. Let $\mathcal{F}$ be a natural Θ_q-bundle over an n-dimensional base X. Then the infinite jet bundle $J_\infty(\mathcal{F})$ contains a countable collection of open Θ_∞-invariant sets U_α such that the union $U = \bigcup_\alpha U_\alpha$ is dense in $J_\infty(\mathcal{F})$ with the following properties.

On each U_α, there are n independent differential invariants $\psi^1, \dots, \psi^n$ with corresponding invariant differential operators $\mathcal{D}_1, \dots, \mathcal{D}_n$ and another set of invariants $g^1, \dots, g^m$ such that all invariants in U_α can be expressed via g_j and their invariant derivatives $\mathcal{D}^J(g_j)$, $(J \in n^k,\ \mathcal{D}^J := \mathcal{D}_{J_1} \dots \mathcal{D}_{J_k})$. ⋄

The Lie-Tresse theorem constructs the invariant differential operators from invariants, but we are free to choose a different set.

Construction of a Generating Set of Invariants

The invariants on arbitrary prolongations $J_r(\mathcal{F})$ can be computed with Vessiot's approach, if we work with minimal bundles using the results from Section 4.3.1. For examples see Chapter 5. The computation of a generating set chooses one of the open subsets U_α on $J_\infty(\mathcal{F})$ mentioned in the Lie-Tresse Theorem 4.12. Basically, we proceed as in Figure 4.2 with the following modifications:

- Compute the invariants on each $\mathcal{F}_{(i)}$ by projecting to $\mathcal{F}_{(i)}/G_{q_i}$, where q_i is the order of $\mathcal{F}_{(i)}$.
- If the G_{q_i}-action on the fibre of $\mathcal{F}_{(i)}$ (or the G_1-action on $\mathcal{F}_{(i)}/K_1^{q_i}$) is locally free, compute an invariant coframe.
- Prolong and project until all new coordinates of $\mathcal{F}_{(i+1)}$ can be chosen as invariants. This means one additional run through the loop when the generic case is reached.

The additional run through the loop ensures that we have found all invariants for the generating set. Assume that we have completed j cycles until the G_{q_j}-action on the fibre of $\mathcal{F}_{(j)}$ is locally free. We have to prolong only a single time and since we can choose invariants $\psi^1, \ldots, \psi^k$ as fibre coordinates, we have

$$\mathcal{F}_{(j+1)} \cong \mathcal{F}_{(j)} \times \mathbb{R}^k.$$

In the proof of Proposition 4.25 we see that each coordinate for the next bundle $\mathcal{F}_{(j+2)}$ must contain a first order jet of the invariants ψ^l. By using the invariant differential operators, we see that all nk first order jets of the ψ^l can be turned into invariants. It follows that the invariants on $\mathcal{F}_{(j+1)}$ are a generating set, which can be nonminimal.

Denote the number of prolongations necessary for the step from $\mathcal{F}_{(i-1)}$ to $\mathcal{F}_{(i)}$ by s_i and the total number of prolongations by $r_i = s_1 + \ldots s_i$. By the Embedding Theorem 4.22, there is the isomorphism

$$J_{r_i}(\mathcal{F})/K_{q_i}^{q_0+r_i} \cong \mathcal{F}_{(i)}$$

such that each invariant on $\mathcal{F}_{(i)}$ is an invariant on $J_{r_i}(\mathcal{F})$.

Remark 4.13. If for some $i \in \mathbb{N}$, the bundle $\mathcal{F}_{(i+1)}$ is trivial, the invariants on $\mathcal{F}_{(i)}$ are a complete set of invariants on $J_\infty(\mathcal{F})$. It follows from the Embedding Theorem 4.22. There is no need for invariant differential operators.

Example 4.3 shows that there are no invariants on $J_r(T^*)$, since the second bundle $\mathcal{F}_{(2)}$ of integrability conditions is trivial. This implies that there are no invariants on $J_2(T^*)$. We can inductively show that there are no invariants on $J_{2+i}(T^*)$ by prolonging and projecting further. ◇

For examples on the construction of generating sets of invariants and invariant derivatives, we refer to Chapter 5, especially to Section 5.1.7 for Riemannian geometry and Section 5.2 for an example treated by Olver and Pohjanpelto (see e.g. [OP]).

4.3 Extending and Simplifying Vessiot's Method

Vessiot's approach as it is presented in Chapter 3 is not directly applicable to problems that involve multiple prolongations or large examples. In this section we present two new methods of optimisation, which were developed for this thesis. Without these methods, nearly all computations in the following chapters become far too large to compute. If not stated otherwise, the definitions and results cannot be found in the literature (e.g. [Pom78], [Pom83] or [Ves03]).

The first method of optimisation is concerned with the bundles of integrability conditions $\mathcal{F}_{(i)}$ and shows how to choose them minimally for a given problem. We have already seen in Sections 4.1 and 4.2 that a minimal choice of $\mathcal{F}_{(i)}$ is a necessary condition for the classification of symmetry groupoids and the computation of invariants.

The second method of optimisation concerns the proper choice of natural bundle and jet groupoid for a given problem. Though not always applicable, it is possible to simplify computations considerably.

4.3.1 Optimisation I: Minimal Bundles

Multiple prolongations and projections of a natural bundle tend to bureaucratic effects, because the description of the symmetry groupoid becomes more and more redundant. In this section we show how to avoid these redundancies, resulting in smaller natural bundles and faster computations. The main result is the Embedding Theorem 4.22 that implies an embedding

$$\varphi_{i,s} : J_r(\mathcal{F})/K_{q+r-t}^{q+r} \hookrightarrow J_s(\mathcal{F}_{(i)})$$

for suitable choices of r and t if all steps of prolongation and projection were done with 2-acyclic symbols. In the following parts of this thesis we will silently apply the results of this section. The proof of the Embedding Theorem 4.22 is rather long and technical and will be given in Section 4.3.2.

We consider the generic sections ω of a natural bundle $\mathcal{F} \to X$. Each coordinate of $\mathcal{F}$ corresponds to an equation for the symmetry groupoid $\mathcal{R}_q(\omega)$. Several subsequent prolongations and projections lead to natural bundles

$$\mathcal{F},\ \mathcal{F}_{(1)},\ \mathcal{F}_{(2)},\ \mathcal{F}_{(3)}, \dots$$

The goal of this section is to find minimal bundles $\mathcal{F}_{(i)}$ to which the sections from $\mathcal{F}$ restrict. This means removing equations that are redundant for sections of $\mathcal{F} \to X$.

Two Basic Optimisation Strategies

The first and rather obvious possibility for optimisation is that two subsequent prolongations can be replaced by a single one. In this pure form, it only occurs in Section 5.1 to show how the natural bundles grow, but in practice it does play a role. The flowcharts in Figure 4.1 and 4.2 are designed to avoid subsequent prolongations which might occur if the symbols are not 2-acyclic.

Lemma 4.14. Let $\mathcal{F}$ be a natural Θ_q-bundle. All sections ω of $\mathcal{F} \to X$ restrict to the natural Θ_q-subbundle $J_{r+s}(\mathcal{F})$ of $J_r(J_s(\mathcal{F})) \to \mathcal{F}$ for all $r, s \in \mathbb{N}$. ⋄

Proof. The canonical embedding $J_{r+s}(\mathcal{F}) \hookrightarrow J_r(J_s(\mathcal{F}))$ from Proposition 1.15 (1) is a morphism of natural bundles. If (x, u) are coordinates of $\mathcal{F}$ then the fibre coordinates of $J_r(J_s(\mathcal{F}))$ are of the form $u^i_{\mu,\nu}$ with $|\mu| \leq s$ and $|\nu| \leq r$. The prolongation of a section ω, given by $u^i = \omega^i(x)$ to $J_s(\mathcal{F})$, is $u^i_\mu = \partial_\mu \omega^i(x)$. Prolonging to $J_r(J_s(\mathcal{F}))$ yields $u^i_{\mu,\nu} = \partial_{\mu+\nu}\omega^i(x)$, which obviously restricts to the image of the canonical embedding. □

The interpretation in terms of equations for the symmetry groupoid $\mathcal{R}_q(\omega)$ are as follows. $\mathcal{R}_q(\omega)$ is defined by the exact sequence (3.10) as the kernel $\ker_\omega(\Phi_\omega)$. In local coordinates this means

$$\Phi^\alpha_\omega(y, y_q) = \omega^\alpha(x), \qquad 1 \leq \alpha \leq d.$$

The prolongation to $J_r(J_s(\mathcal{F}))$ defines $\mathcal{R}_{q+r+s}(\omega)$ by the equations

$$D_\nu\, D_\mu\, \Phi^\alpha_\omega(y, y_{q+r+s}) = \partial_{\mu+\nu}\omega^\alpha(x), \qquad |\mu| \leq s,\ |\nu| \leq r,$$

where all combinations $\mu' + \nu' = \mu + \nu$ yield identical equations. These redundancies are removed by restricting to $J_{r+s}(\mathcal{F})$.

Example 4.15. In Section 5.1 dealing with Riemannian metrics it is shown how redundancies due to multiple prolongations. They are hidden, because the bundles are computed with the help of geometric interpretation.

The redundancies are even worse when working with Example 3.3 (2) on conformal structures in the special case of a three-dimensional base. Let $u_{ij} = \tilde{g}_{ij}$ be the coordinates

$$(u_{ij}) = (u_{12}, u_{13}, u_{22}, u_{23}, u_{33})$$

of $\mathcal{F}$ containing the first order conformal structure. Like in the case of Riemannian metrics, the symbols of generic sections are not 2-acyclic.

We thus prolong to the bundle $J_1(\mathcal{F})$ of second order conformal structures with fibre coordinates $(u_{ij}, u_{ijk} \,|\, k \leq 3)$. Here $u_{ijk} = u_{ij,k}$ stands for the first order jets. The fibre dimension of $\mathcal{G}$ is 15. To check formal integrability, we compute $\mathcal{F}_{(1)} = J_1(J_1(\mathcal{F}))/K^3_2$ with fibre coordinates (v, w). The indices are chosen such that the double prolongation is still visible:

$$v_{ijk} = u_{ij,k}, \qquad w_{ijkl} = u_{ijk,l} - u_{ijl,k}, \qquad k, l \leq n.$$

In total, the fibre of $\mathcal{F}_{(1)}$ is 30-dimensional. The Vessiot structure equations are:

$$u_{ij,k} = u_{ijk}, \qquad u_{ijk,l} - u_{ijl,k} = 0, \qquad k, l \leq n.$$

Plugging in $u_{ijk} = u_{ij,k}$ and $u_{ijk,l} = u_{ij,kl}$ shows that all integrability conditions are automatically satisfied for every first order conformal structure.

For conformal structure on a three-dimensional base, the bundle $J_2(\mathcal{F})/K_2^3$ is isomorphic to $J_1(\mathcal{F})$ since there were only trivial integrability conditions on $\mathcal{F}_{(1)}$. All equations on $\mathcal{F}_{(1)}$ are the effect of two succeeding prolongations. In higher dimensions, there will be nontrivial conditions. $\diamond$

The second possible optimisation concerns integrability conditions. Prolonging once and then projecting back yields several new equations. The next proposition says that these equations will reappear in the next step of prolongation and projection. It is safe to omit them by restriction to a subbundle.

Proposition 4.16. For a natural bundle $\mathcal{F} \to X$ and the projection $I : J_1(\mathcal{F}) \to \mathcal{F}_{(1)}$ according to Proposition 3.30, there is a projection $\pi : J_1(\mathcal{F}_{(1)}) \to \mathcal{F}_{(1)} \times_{\mathcal{F}} \mathcal{F}_{(1)}$. For each section ω of $\mathcal{F} \to X$ the section $(j_1 \circ I \circ j_1)(\omega)$ restricts to the preimage $\pi^{-1}(\Delta(\mathcal{F}_{(1)}))$ of the diagonal embedding

$$\Delta : \mathcal{F}_{(1)} \to \mathcal{F}_{(1)} \times_{\mathcal{F}} \mathcal{F}_{(1)} : f_1 \mapsto (f_1, f_1). \qquad \diamond$$

Proof. Let (x, u, v) be a coordinate system of $\mathcal{F}_{(1)}$ such that (x, u) is a coordinate system of $\mathcal{F}$. Then there is a projection $J_1(\mathcal{F}_{(1)}) \to J_1(\mathcal{F}) \times_{\mathcal{F}} \mathcal{F}_{(1)}$. It is defined in coordinates by:

$$(x, u, v, u_x, v_x) \mapsto (x, u, v, u_x).$$

It is easy to prove that this map is coordinate independent since $\mathcal{F}_{(1)}$ is a bundle over $\mathcal{F}$. The abstract fibre of $J_1(\mathcal{F}) \times_{\mathcal{F}} \mathcal{F}_{(1)}$ is $F^{(1)} \times_F F_1$ and the K_{q+1}-action leaves F_1 invariant. So the projection to order q has the fibre $F_1 \times_F F_1$. By equation (3.34), the projection $I' : J_1(\mathcal{F}_{(1)}) \to \mathcal{F}_{(1)} \times_{\mathcal{F}} \mathcal{F}_{(1)}$ is given by:

$$v'^{\beta} = A^{\beta}(u)u_x + B^{\beta}(u)$$

such that for sections ω of $\mathcal{F} \to X$ the coordinates $v = v'$ coincide. $\square$

Example 4.17. Continue Example 3.49 with $\mathcal{F} = T^* \times \bigwedge^2 T^*$ on a three-dimensional base X. The projection map $J_1(\mathcal{F}) \to \mathcal{F}_{(1)}$ is given by:

$$\begin{aligned} v^1 &= u_2^3 - u_3^2, \\ v^2 &= u_3^1 - u_1^3, \\ v^3 &= u_1^2 - u_2^1, \\ w &= u_1^4 + u_2^5 + u_3^6. \end{aligned}$$

Essentially, the first order differential operator from $\mathcal{F}$ to $\mathcal{F}_{(1)}$ is the exterior derivative d. As the symbols of generic sections on $\mathcal{F}_{(1)} \to X$ remain 2-acyclic, we compute $J_1(\mathcal{F}_{(1)})$ with coordinates $(u, v, w, u_j^i, v_j^i, w_j)$. The projection

$$J_1(\mathcal{F}_{(1)}) \to J_1(\mathcal{F}) \times_{\mathcal{F}} \mathcal{F}_{(1)} : (u, v, w, u_j^i, v_j^i, w_j) \mapsto (u, v, w, u_j^i)$$

eliminates the last two types of coordinates and the action on the remaining jet coordinates (u^i_j) is the same as on the fibre of $J_1(\mathcal{F}) \to \mathcal{F}$. Projecting back to order 1, we obtain a copy of the above equations:

$$\begin{aligned} v'^1 &= u^3_2 - u^2_3, \\ v'^2 &= u^1_3 - u^3_1, \\ v'^3 &= u^2_1 - u^1_2, \\ w' &= u^4_1 + u^5_2 + u^6_3. \end{aligned}$$

Restricting to the subbundle of $J_1(\mathcal{F}_{(1)}) \to \mathcal{F}_{(1)}$ defined by

$$v^1 - u^3_2 - u^2_3 \;=\; v^2 - u^1_3 - u^3_1 \;=\; v^3 - u^2_1 - u^1_2 \;=\; w - u^4_1 + u^5_2 + u^6_3 \;= 0,$$

obviously eliminates four redundant coordinates on $\mathcal{F}_{(2)}$. However, it is not yet the minimal subbundle. ◇

In the two cases described in Lemma 4.14 and Proposition 4.16, the sections ω of $\mathcal{F} \to X$ restrict to the image of a suitable prolongation $J_r(\mathcal{F})$. The general situation is a mixture of both cases.

Prolongation and Projection with Minimal Bundles

In the applications in Sections 4.1 and 4.2, we computed natural bundles $\mathcal{F}_{(i)}$ by prolonging s times and projecting once in each step. In this section, we describe the minimal subbundles of $\mathcal{F}_{(i)} \to \mathcal{F}_{(i-1)}$ as the image of a morphism of natural bundles. At first we like to specify all possibilities to create the bundles $\mathcal{F}_{(i)}$ by prolongation and projection.

Definition 4.18. Let $\mathcal{F} = \mathcal{F}_{(0)}, \mathcal{F}_{(1)}, \dots, \mathcal{F}_{(s)}, s \in \mathbb{N}$ be natural bundles. It is called a *series of prolongations and projections of* $\mathcal{F}$ if each $\mathcal{F}_{(i)}$ is defined as one of the following:

$$\mathcal{F}_{(i)} = \begin{cases} J_{s_i}(\mathcal{F}_{(i-1)}) & s_i \in \mathbb{N}, \quad t_i = t_{i-1}, \\ \mathcal{F}_{(i-1)}/K^{q+r_i-t_{i-1}}_{q+r_i-t_i}, & s_i = 0, \quad q + r_i > t_i > t_{i-1}, \\ J_{s_i}(\mathcal{F}_{(i-1)})/K^{q+r_i-t_{i-1}}_{q+r_i-t_i}, & s_i \in \mathbb{N}, \quad q + r_i > t_i > t_{i-1}, \end{cases}$$

with $r_i = r_{i-1} + s_i$ and $r_0 = t_0 = 0$. ◇

The numbers r, s and t have the following meaning:

s_i : number of prolongations in step i,
r_i : total number of prolongations up to step i,
t_i : total number of projections up to step i.

In each step, the number of prolongations s_i and projections $t_i - t_{i-1}$ and the bundle $\mathcal{F}_{(i)}$ is computed accordingly. In In Figures 4.1 and 4.2, we prolonged until the symbols became 2-acyclic and projected once, which is the most common, third case in the definition.

For a series of prolongations and projections, a section ω of $\mathcal{F} \to X$ gives rise to sections $\omega_{(i)}$ of $\mathcal{F}_{(i)} \to X$ for $i \leq s$. They are recursively defined by:

$$\omega_{(i)}(x) := j_{s_i}(\omega_{(i-1)})(x) K^{q+r_i-t_{i-1}}_{q+r_i-t_i} \tag{4.1}$$

with $K^t_t = \{\mathrm{id}\}$ and $\omega_{(0)} = \omega$. We are mainly interested in sections γ of $\mathcal{F}_{(i)} \to X$ which are *coming from* $\mathcal{F}$, which means that there exists a section ω of $\mathcal{F} \to X$ such that $\gamma = \omega_{(i)}$ according to Equation (4.1).

With this preparation, we can describe the minimal subbundles of $\mathcal{F}_{(i)} \to \mathcal{F}_{(i-1)}$ as the image of a map φ_i. The following Proposition includes both Lemma 4.14 and Proposition 4.16 as special cases.

Proposition 4.19. Let $\mathcal{F}$ be a natural Θ_q-bundle. For each series of prolongations and projections $\mathcal{F}, \mathcal{F}_{(1)}, \ldots, \mathcal{F}_{(s)}$ of $\mathcal{F}$, the canonical embedding from Proposition 1.15 (1) induces morphism of natural bundles,

$$\varphi_i : J_{r_i}(\mathcal{F}) \to \mathcal{F}_{(i)}, \qquad 1 \leq i \leq s,$$

such that the image of φ_i is the minimal subbundle of $\mathcal{F}_{(i)} \to \mathcal{F}_{(i-1)}$ to which all sections coming from $\mathcal{F}$ restrict. $\diamond$

Here, minimal subbundle means that there is no closed subbundle of $\mathrm{im}(\varphi_s)$ to which all sections coming from $\mathcal{F}$ restrict. There is an immediate but important corollary.

Corollary 4.20. The prolongation of φ_i from Proposition 4.19,

$$p_s(\varphi_i) : J_{r_i+s}(\mathcal{F}) \to J_s(\mathcal{F}_{(i)})$$

factors over $J_{r_i+s}(\mathcal{F})/K^{q+r_i+s}_{q+r_i-t_i+s}$. In other words it induces morphisms of natural bundles

$$\varphi_{i,s} : J_{r_i+s}(\mathcal{F})/K^{q+r_i+s}_{q+r_i-t_i+s} \to J_s(\mathcal{F}_{(i)}), \tag{4.2}$$

such that $\mathrm{im}(\varphi_{i,s})$ is the minimal subbundle to which sections from $\mathcal{F}$ restrict. $\diamond$

Proof (of Proposition 4.19). To obtain all maps φ_i, we have to follow the series of prolongations and projections, starting with $\varphi_0 = \mathrm{id}_{\mathcal{F}}$.

1) Let $\mathcal{F}_{(i)} = J_{s_i}(\mathcal{F}_{(i-1)})$ and $\varphi_{i-1} : J_{r_{i-1}}(\mathcal{F}) \to \mathcal{F}_{(i-1)}$ be the morphism of the previous step. Then apply J_{s_i} to φ_{i-1} and use the canonical embedding:

$$J_{r_i}(\mathcal{F}) \xhookrightarrow{\iota} J_{s_i}(J_{r_{i-1}}(\mathcal{F})) \xrightarrow{J_{s_i}(\varphi_{i-1})} J_{s_i}(\mathcal{F}_{(i-1)}).$$
$$\varphi_i = p_{s_i}(\varphi_{i-1}) : J_{r_i}(\mathcal{F}) \to J_{s_i}(\mathcal{F}_{(i-1)})$$

Set $\varphi_i = p_{s_i}(\varphi_{i-1}) = \iota \circ p_{s_i}(\varphi_{i-1})$ according to Proposition 1.15 (3). All sections of $\mathcal{F}_{(i)} \to X$ coming from $\mathcal{F}$ are of the form

$$(j_{s_i}(\varphi_{i-1}) \circ j_{s_i} \circ j_{r_{i-1}})(\omega)$$

and they restrict to the image of φ_i, since partial derivatives commute.

If all sections coming from $\mathcal{F}$ restrict to a smaller subbundle $\mathcal{F}'_{(i)}$ of $\mathrm{im}(\varphi_i)$, its preimage defines a subbundle of $J_{r_i}(\mathcal{F}) \to \mathcal{F}$. But each r_i-jet $u_{r_i} \in J_{r_i}(\mathcal{F})$ defines a germ of a section of $\mathcal{F} \to X$. It follows that $\mathcal{F}'_{(i)} = \mathrm{im}(\varphi_i)$.

2) Let $\mathcal{F}_{(i)} = \mathcal{F}_{(i-1)}/K^{q+r_i-t_{i-1}}_{q+r_i-t_i}$. The projection to $\mathcal{F}_{(i)}$ gives $\varphi_{i+1} = \pi \circ \varphi_i$:

$$J_{r_{i-1}}(\mathcal{F}) \xrightarrow{\varphi_{i-1}} \mathcal{F}_{(i-1)} \xrightarrow{\pi} \mathcal{F}_{(i)}.$$

As $\mathrm{im}(\varphi_i)$ is already the smallest subbundle for sections from $\mathcal{F}$, it follows for $\mathrm{im}(\varphi_{i+1})$. The case of simultaneous prolongation and projection is a combination of 1) and 2). □

In the following, we are interested in the images of φ_i and $\varphi_{i,s}$, since they are minimal bundles. Obviously, each bundle $\mathcal{F}_{(i)}$ can be replaced by $\mathrm{im}(\varphi_i)$, eliminating redundant equations for sections from $\mathcal{F}$. Continuing the prolongation and projection will be much more efficient with the minimal bundles.

Definition 4.21. Let $\mathcal{F}_{(i)}$, $1 \leq i \leq s$ be a series of prolongations and projections of $\mathcal{F}$. Then $\mathcal{F}_{(i)}$ is called *minimal* if $\mathrm{im}(\varphi_i) = \mathcal{F}_{(i)}$ for φ_i as in Proposition 4.19. The series is called *minimal* if all $\mathcal{F}_{(i)}$ are minimal. ◇

We have theoretically constructed a series of prolongations and projections of $\mathcal{F}$ with minimal bundles described as images $\mathrm{im}(\varphi_i)$. As in the diagrams 4.1 and 4.2 it is most efficient to replace $J_{s_i}(\mathcal{F}_{(i-1)}$ directly by the image of φ_{i-1,s_i}. Three simple operations can shorten the series itself, without affecting tests of integrability.

- Replace subsequent prolongations ($t_i = t_{i+1} = t_{i+2} = \ldots$) by a single one.
- Since we have $K^{q+r}_q \cong K^{q+r+s}_q / K^{q+r+s}_{q+r}$, the same is possible for successive projections
$$(\mathcal{F}/K^{q+r+s}_{q+r})/K^{q+r}_q \cong \mathcal{F}/K^{q+r+s}_q.$$
- Combine each prolongation with following projection to a single step.

The Embedding Theorem

We come to the main result of this section, which has both theoretical and computational consequences. In Section 4.1 it was applied to classify the symmetry groupoids and in Section 4.2 it was a necessary from the theoretical point of view to find a generating set of invariants on $J_r(\mathcal{F})$ by prolongation and projection.

The computational value is that it reduces the computation of the minimal bundles $\mathrm{im}(\varphi_{i,s})$ to linear algebra, which is presented in the next subsection.

Theorem 4.22 (Embedding Theorem). Let $\mathcal{F} \to X$ be a natural Θ_q-bundle for a groupoid Θ_q with involutive symbol. Choose $r \in \mathbb{N}$ such that the symbols of $\mathcal{R}_{q+r-1}(\omega)$ for generic sections ω of $\mathcal{F}$ are 2-acyclic and define $\mathcal{F}_{(1)} = J_r(\mathcal{F})/K^{q+r}_{q+r-1}$. Then the maps from equation (4.2),

$$\varphi_{1,s} : J_{r+s}(\mathcal{F})/K^{q+r+s}_{q+r+s-1} \hookrightarrow J_s(\mathcal{F}_{(1)}),$$

are embeddings for all $s \in \mathbb{Z}_{\geq 0}$. ◇

The proof is rather long, so we first to computational aspects before giving the proof in Section 4.3.2. Applying the Embedding Theorem several times extends it to all maps $\varphi_{i,s}$.

Corollary 4.23. Let $\mathcal{F}_{(i)}$, $1 \leq i \leq s$ be a series of prolongations of $\mathcal{F}$ such that the s_i are chosen such that the symbols for generic sections of $J_{s_i-1}(\mathcal{F}_{(i-1)}) \to X$ are 2-acyclic, then the maps from equation (4.2),

$$\varphi_{i,s} : J_{r_i+s}(\mathcal{F})/K^{q+r_i+s}_{q+r_i-t_i+s} \hookrightarrow J_s(\mathcal{F}_{(i)}),$$

are embeddings for all $s \in \mathbb{Z}_{\geq 0}$. ◇

Proposition 4.19 shows that there are minimal subbundles of $\mathcal{F}_{(i)}$ for all sections from $\mathcal{F}$ and the Embedding Theorem implies that we obtain the maximal number of equations that can be computed with r_i prolongations if we take care of the symbols. This combines the efficiency of several smaller steps of prolongation and projection with the knowledge, that all symmetry equations up to a certain order are present on each $\mathcal{F}_{(i)}$.

Computing Minimal Bundles

So far, only the existence and theoretical properties of minimal bundles $\mathcal{F}_{(i)}$ have been assured. We now construct these bundles under the assumption that we prolong until the symbols of generic sections are 2-acyclic and project only once in each step. This case is needed for all applications in Sections 4.1, 4.2 and the following chapters.

We start with an example to illustrate the computation and the effects of using minimal bundles.

Example 4.24. Continue Examples 3.49 and 4.17. The bundle $\mathcal{F}_{(1)}$ and its prolongation $J_1(\mathcal{F}_{(1)})$ with coordinates $(u, v, w, u^i_j, v^i_j, w_j)$ are already calculated. It remains to find the minimal subbundle $\operatorname{im}(\varphi_{1,1})$. To compute the map $\varphi_{1,1}$ we use the total derivative:

$$v^1_1 = D_{x^1} v^1 = u^3_{12} + u^2_{13}, \quad v^1_2 = u^3_{22} + u^2_{23}, \ldots$$

We observe that the map is linear in second order derivatives u_{ij}, such that the image $\mathrm{im}(\varphi_{1,1})$ can be determined by linear algebra. Eliminating second order derivatives, we obtain only a single equation

$$0 = v_1^1 + v_2^2 + v_3^3,$$

which is due to the fact that the exterior derivative d satisfies $d^2 = 0$:

$$0 = d^2\omega = (v_1^1 + v_2^2 + v_3^3)dx^1 \wedge dx^2 \wedge dx^3.$$

The equations involving only first order jets are obvious. We repeat the results of Example 4.17:

$$v^1 - u_2^3 - u_3^2 \quad = \quad v^2 - u_3^1 - u_1^3 \quad = \quad v^3 - u_1^2 - u_2^1 \quad = \quad w - u_1^4 + u_2^5 + u_3^6 \quad = 0,$$

These are the complete equations defining the subbundle $\mathrm{im}(\varphi_{1,1})$ on $J_1(\mathcal{F}_{(1)}) \to \mathcal{F}_{(1)}$. We go on with prolongation and projection, but only show the fibre dimensions of the bundles which are involved. They are collected in the following table:

i	0	1	2	3
$\mathcal{F}_{(i)}$	6	4	8	15
$J_1(\mathcal{F}_{(i)})/K_1^2$	4	13	36	
$J_1(\mathcal{F}_{(i)})$	18	30	54	
$\mathrm{im}(\varphi_{1,i})$	18	25	33	
$J_1(\mathcal{G}_{(i)})/K_1^2$	4	13	51	

For example, the difference

$$\dim(J_1(\mathcal{F}_{(1)})/K_1^2) - \dim(\mathcal{F}_{(2)}) = 13 - 8 = 5$$

is due to the five equations we have found above and we have

$$J_1(\mathcal{F}_{(1)})/K_1^2 \cong \mathcal{F}_{(2)} \times_X \bigwedge^2 T^* \times_X \bigwedge^3 T^* \times_X \bigwedge^3 T^*,$$

where the 2-form and the first 3-form are repetitions of $d\omega$ and $d\Omega$ and the last form is $d^2\omega$. In the next step, it is possible to eliminate $36 - 15 = 21$ redundant coordinates. This also shows that the unnecessary equations even produce integrability conditions which can be removed.

For comparison the computation without any optimisation is displayed in the last row. Starting with $\mathcal{G}_{(0)} = \mathcal{F}$, we set $\mathcal{G}_{(i)} = J_1(\mathcal{G}_{(i-1)})/K_1^2$. In the last step, this approach becomes very inefficient with a 51-dimensional fibre. ◇

The proof of the next proposition shows how to compute the minimal subbundles in the general case. It depends heavily on the Embedding Theorem 4.22, as it assumes that all maps $\varphi_{i,s}$ are injective.

Proposition 4.25. Let $\mathcal{F} = \mathcal{F}_{(0)}, \mathcal{F}_{(1)}, \ldots \mathcal{F}_{(s)}$ be a minimal series of prolongations and projections with 2-acyclic symbols and assume $t_i = i$ in each step. Then the image of the embedding φ_{i,s_i} is a subbundle of $J_{s_i}(\mathcal{F}_{(i-1)}) \to \mathcal{F}_{(i-1)}$, which can locally be computed by linear algebra. ◇

Proof. We compute the images of φ_{i,s_i} by induction on i and omit indices wherever possible. For $i = 1$,

$$\varphi_{1,s_2} : J_{r_2}(\mathcal{F})/K^{q+r_2}_{q+r_2-1} \to J_{s_2}(\mathcal{F}_{(1)})$$

is an embedding. If (u) are the coordinates of $\mathcal{F}$, then $\mathcal{F}_{(1)}$ has coordinates $(u^{(1)}, v^{(1)})$ and the map $\varphi_1 : J_{r_1}(\mathcal{F}) \to \mathcal{F}_{(1)}$ is given in coordinates by:

$$\begin{array}{rcll} u^{(1)} & = & u_\mu, & 0 \le |\mu| < r_1, \\ v^{(1)} & = & A(u)u_{r_1} + B(u, u_{r_1-1}) & \end{array} \tag{4.3}$$

according to Propositon 3.33 or equation (3.34). The formulae for $v^{(1)}$ are quasilinear in u_{r_1} with coefficients in u. Use the total derivative to compute the prolongation $p_{s_2}(\varphi_1)$:

$$\begin{array}{rcll} u^{(1)}_\nu & = & u_{\mu+\nu}, & |\mu| < r_1, |\nu| \le s_2, \\ v^{(1)}_\nu & = & D_\nu(A(u)u_{r_1} + B(u, u_{r_1-1})) & |\nu| \le s_2. \end{array} \tag{4.4}$$

All formulae are quasilinear in highest order jets $u_{\mu+\nu}$ or $u_{r_i+|\nu|}$ with coefficients in u. Define sets $\mathcal{U}_j$, $0 \le j \le r_2$, that contain all of the above equations with highest occuring jet order j.

Start with $\mathcal{U}_{r_2}$. We are searching for equations containing the coordinates $v^{(1)}_{s_2}$ occurring in $\mathcal{U}_{r_2}$ that define the subbundle $\mathrm{im}(\varphi_{1,s_2})$. A necessary condition is that the highest order jets u_{r_2} on the right hand side are eliminated. We show that this condition is already sufficient. Since the equations are quasilinear, we eliminate u_{r_2} by linear algebra over $k(u)$:

$$\sum C(u)\, v^{(1)}_{s_2} = \underbrace{\sum C(u)A(u)u_{r_2}}_{=0} + \text{lower order.}$$

This is an equation of order $q + r_2 - 1$ which depends on u-jets of order $\le r_2 - 1$ only. Since φ_{i,s_2-1} is injective, there exists a $D(u^{(1)}_\nu, v^{(1)}_\nu)$ that depends only on coordinates occurring in $\mathcal{U}_j$ with $j < r_2$ such that

$$\sum C(u)\, v^{(1)}_{s_2} + D(u^{(1)}_\nu, v^{(1)}_\nu) = 0$$

on the image of $p_{s_2}(\varphi_1)$. Proceed with $\mathcal{U}_{r_2-1}, \ldots$ analogously. As each necessary condition was already sufficient, we have produced all defining equations for $\mathrm{im}(\varphi_1)$.

Turn to the projection $\mathrm{im}(\varphi_1) \to \mathcal{F}_{(2)}$. It can be computed with the methods of Proposition 3.33 if the coordinates for $\mathrm{im}(\varphi_1)$ are chosen carefully. $\mathcal{F}_{(1)}$ was minmal and there are no equations on $u^{(1)}, v^{(1)}$ alone and it is possible to find a subset

$$U \subseteq \{u_\mu^{(1)}, v_\mu^{(1)}, \,|\, |\mu| \geq 1\}$$

such that $(u^{(1)}, v^{(1)}, U)$ are coordinates of $\mathrm{im}(\varphi_1)$. The vector field of the algebroid action on $\mathcal{F}_{(1)}$ is given as in equation (3.23) and the action on $\mathrm{im}(\varphi_1)$ is computed analogous to equation (3.8) by applying the vector field to the coordinates of $\mathrm{im}(\varphi_1)$. As we have left $(u^{(1)}, v^{(1)})$ untouched, the $K_{q+r_2-2}^{q+r_2-1}$-distribution on the fibre of $\mathrm{im}(\varphi_1)$ is of the same form as in equation (3.24). The projection down to $\mathcal{F}_{(2)}$ is therefore given by

$$v^{(2)} = A(u,v)v_{s_2}^{(1)} + A'(u,v)u_{s_2}^{(1)} + B(u_\nu^{(1)}, v_\nu^{(1)}), \quad |\nu| < s_2, \tag{4.5}$$

where not all coefficients $A(u,v)$ are zero. Otherwise we would have found a new equation of order $q + r_2 - 2$ that depends on u-jets of maximal order $r_2 - 1$ only. But since φ_{i-1,s_2-1} is injective, these equations are already present.

The recursive step from $i-1$ to i is analogous to the case of $i = 1$. For coordinates $(u^{(i-1)}, v^{(i-1)})$ of $\mathcal{F}_{(i-1)}$, we compute the map $p_{s_i}(\varphi_{i-1})$ according to equation (4.4). Equation (4.5) with $A \neq 0$ ensures that the formulae stay quasilinear in highest order jets of u, this time with coefficients in $(u^{(i-1)}, v^{(i-1)})$. The remaining proof is completely parallel to the case of $i = 1$. □

4.3.2 Proof of the Embedding Theorem

In this section, we prove the Embedding Theorem 4.22. It combines the nonlinear Janet sequence from Section 3.5.3 with a theorem of Pommaret [Pom78, Thm. 2.4.5] for arbitrary PDE systems. Starting with the affine bundle version of the Janet sequence and the curvature map from Remark 3.52, we construct a three-dimensional exact diagram (Figure 4.6) which finally proves the Embedding Theorem. The reader may compare the diagram and the techniques used to [Pom78, §2.4] and [Pom83, §I.A.3]. See also [Gol68] in the linear case.

Prolongation and Projection with 2-acyclic Symbols

The key idea for the computation of minimal bundles in a series of prolongations and projections is the following theorem. It shows that we can swap prolongations and projections of PDE systems as long as the symbol is 2-acyclic. We will present a pictorial way to use this theorem and then translate it into the language of natural bundles.

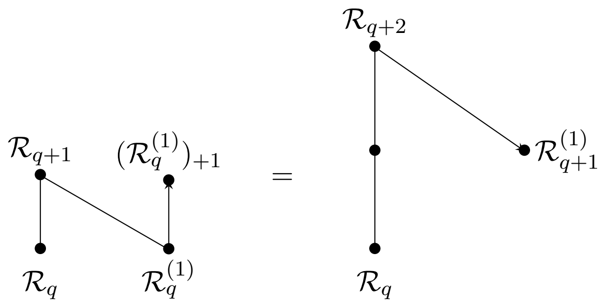
Theorem 4.26. [Pom78, Thm. 2.4.5] Let $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ be a system of PDEs of order q on a bundle $\mathcal{E} \to X$ such that $\mathcal{R}_{q+1}$ is a subbundle of $J_{q+1}(\mathcal{E}) \to X$. If the symbol $\mathfrak{g}_{q+1}$ is a vector bundle and $\mathfrak{g}_q$ is 2-acyclic, then $\mathcal{R}_{q+1}^{(1)} = (\mathcal{R}_q^{(1)})_{+1}$. ⋄

Here $(\mathcal{R}_q^{(1)})_{+1}$ denotes the first prolongation of the system $\mathcal{R}_q^{(1)}$. In Figure 4.3 we visualise the effects of Theorem 4.26 as a path through a grid. A prolongation from order q to order $q+r$ goes upwards r steps and a projection down to $q+r-s$ takes s diagonal steps down. The path on the left hand side,

$$\mathcal{R}_q \quad \rightsquigarrow \quad \mathcal{R}_{q+1} \quad \rightsquigarrow \quad \mathcal{R}_q^{(1)} \quad \rightsquigarrow \quad (\mathcal{R}_q^{(1)})_{+1},$$

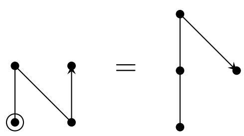
is equivalent to the computation of $\mathcal{R}_{q+1}^{(1)}$ on the right hand side.

Figure 4.3: Visualisation of Theorem 4.26.



For the translation into the language of natural bundles, it is convenient to omit the labels like $\mathcal{R}_q$, which is done in Figure 4.4. Each dot stands for a system of PDEs. If we *use* the fact that $\mathcal{R}_q$ has a 2-acyclic symbol, we indicate it by an extra circle around the dot for $\mathcal{R}_q$. On the right hand side, $\mathcal{R}_q$ still has a 2-acyclic symbol, but we do not apply this piece of information and thus omit the circle.

Figure 4.4: Abstract visualisation of Theorem 4.26.



By equation (A.2), the symbol of $\mathcal{R}_{q+r}$ is also 2-acyclic if the symbol of $\mathcal{R}_q$ is 2-acyclic and we have the following corollary:

Corollary 4.27. Let $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ be a system of PDEs over $\mathcal{E}$ such that $\mathcal{R}_{q+r}$ is regular for all $r \leq s \in \mathbb{N}$. If $\mathfrak{g}_q$ is 2-acyclic and $\mathfrak{g}_{q+1}$ is a vector bundle, then $\mathcal{R}_{q+r}^{(1)} = (\mathcal{R}_q^{(1)})_{+r}$ for all $r \leq s$. $\diamond$

It is not necessary to check if all symbols $\mathfrak{g}_{q+r}$ are vector bundles to satisfy the conditions of Theorem 4.26. Proposition A.6 implies that all $\mathfrak{g}_{q+r}$ are vector bundles if $\mathfrak{g}_q$ is 2-acyclic and $\mathfrak{g}_{q+1}$ is a vector bundle. Theorem 4.26 can be extended to multiple prolongations and projections and the next corollary gives an example.

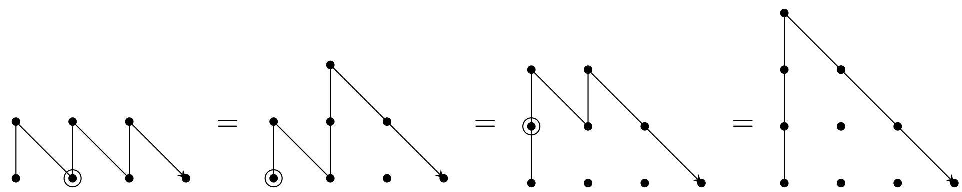
Corollary 4.28. Let $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ be a system of PDEs over $\mathcal{E}$. If all intermediate PDE systems satisfy the conditions of Corollary 4.27 up to order $q+s$, then

$$\pi_q^{q+s}(\mathcal{R}_{q+s}) = \mathcal{R}_q^{(s)} = \mathcal{R}_q^{(1)\ldots(1)}$$

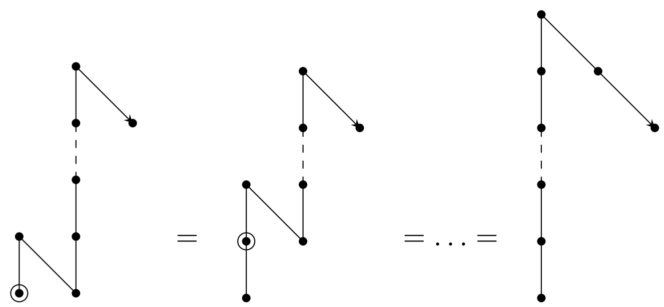
where we prolonged and projected s times on the right hand side. ◇

The visualisation in Figure 4.4 provides a pictorial way to prove corollaries from Theorem 4.26.

Proof. We work with a recursion for $r \in \mathbb{N}$ and apply Theorem 4.26 in each of the following steps. We begin with:



In each step, we have used the assumption that the system denoted with the extra circle has a 2-acyclic symbol. The circle can be seen as anchor points for Figure 4.4. The recursive step also proves Corollary 4.27:



where the dashed line stands for an arbitrary number of prolongations. □

Obviously we can extend Theorem 4.26 to various other ways to compute $\mathcal{R}_q^{(r)}$, like $\mathcal{R}_q^{(r)} = \mathcal{R}_q^{(r-s)(s)}$ if all necessary symbols are 2-acyclic and the involved systems are regular. This has very useful consequences for natural bundles.

Translation into the Language of Natural Bundles

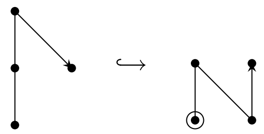
We will now give a translation of the previous section into the language of natural bundles. Throughout this section we silently assume that all occurring PDE systems of order $q+r$ a regular and their symbols $\mathfrak{g}_{q+r+1}$ are vector bundles. We start with a direct interpretation of Theorem 4.26.

Theorem 4.29. Let $\mathcal{F} \to X$ be a natural Π_q-bundle such that symbols for generic sections ω are 2-acyclic. Let $\mathcal{F}_{(1)} = J_1(\mathcal{F})/K_q^{q+1}$. Then the map

$$\varphi_{1,1} : J_2(\mathcal{F})/K_{q+1}^{q+2} \hookrightarrow J_1(\mathcal{F}_{(1)})$$

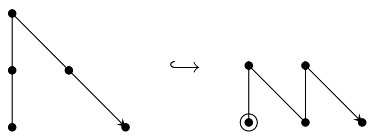
from equation (4.2) is an embedding. ◇

This is already a special case of the Embedding Theorem 4.22. We reinterpret the visualisation of Figure 4.4 in terms of natural bundles. The equality $\mathcal{R}_{q+1}^{(1)}(\omega) = (\mathcal{R}_q^{(1)}(\omega))_{+1}$ implies that $J_1(\mathcal{F}_{(1)})$ contains the equations from $J_2(\mathcal{F})$ which are of order $q+1$, but there may be redundancies. This means $J_2(\mathcal{F})/K_{q+1}^{q+2}$ only embeds into $J_1(\mathcal{F}_{(1)})$. In terms of pictures we say:



Each dot now stands for a natural bundle. An extra circle indicates that we have used that the symbols for generic sections of the corresponding bundle are 2-acyclic.

Projecting down to $\mathcal{F}'_{(2)} = J_1(\mathcal{F}_{(1)})/K_q^{q+1}$, we obtain the picture:



It suggests that the image of the embedding $J_2(\mathcal{F})/K_q^{q+2} \hookrightarrow \mathcal{F}'_{(2)}$ is the minimal subbundle $\mathcal{F}_{(2)}$ of $\mathcal{F}'_{(2)} \to \mathcal{F}_{(1)}$ to which all sections from $\mathcal{F}$ restrict. The embedding follows from Proposition B.17 (3) and minimality analogous to the proof of Proposition 4.19.

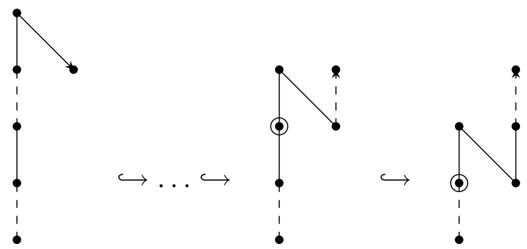
For convenience, we repeat Theorem 4.22.

Theorem 4.22 (Embedding Theorem). Let $\mathcal{F} \to X$ be a natural Θ_q-bundle for a groupoid Θ_q with involutive symbol. Choose $r \in \mathbb{N}$ such that the symbols of $\mathcal{R}_{q+r-1}(\omega)$ for generic sections ω of $\mathcal{F}$ are 2-acyclic and define $\mathcal{F}_{(1)} = J_r(\mathcal{F})/K_{q+r-1}^{q+r}$. Then the maps from equation (4.2),

$$\varphi_{1,s} : J_{r+s}(\mathcal{F})/K_{q+r+s-1}^{q+r+s} \hookrightarrow J_s(\mathcal{F}_{(1)}),$$

are embeddings for all $s \in \mathbb{Z}_{\geq 0}$. ◇

It is possible to construct the embedding $\varphi_{1,s}$ step by step

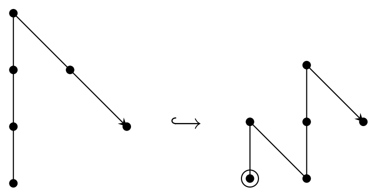


analogous to Corollary 4.28, but we will give a proof that provides it in a single step. The generalisation from $\mathcal{F}_{(1)}$ to all bundles $\mathcal{F}_{(i)}$ in Corollary 4.23 can be done in a pictorial way just as Corollaries 4.27 and 4.28. We give an example.

Example 4.30. Let $\mathcal{F}$ be a natural Θ_q-bundle and assume that

$$\begin{aligned} \mathcal{F}_{(1)} &= J_1(\mathcal{F})/K_q^{q+1}, \\ \mathcal{F}_{(2)} &\subseteq J_2(\mathcal{F}_{(1)})/K_{q+1}^{q+2}, \end{aligned}$$

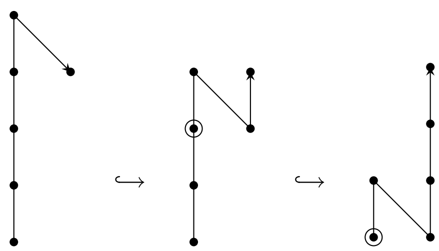
since the generic sections of $\mathcal{F}_{(1)} \to X$ only have 2-acyclic symbols after prolonging once. To construct $\mathcal{F}_{(2)}$, we apply Theorem 4.22 on $\mathcal{F}$ and obtain $\mathcal{F}_{(2)}$ as the image of the embedding



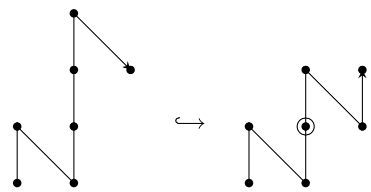
We like to compute the first prolongation $J_1(\mathcal{F}_{(2)})$ and the embedding

$$\varphi_{2,1} : J_4(\mathcal{F})/K_{q+2}^{q+4} \hookrightarrow J_1(\mathcal{F}_{(2)}).$$

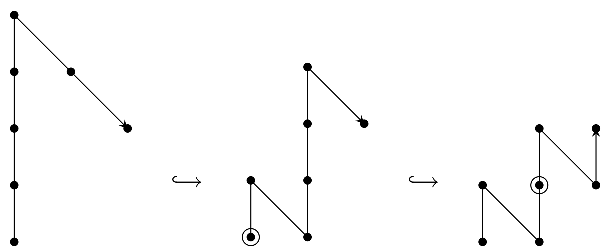
Using Theorem 4.22 on $\mathcal{F}$ and $\mathcal{F}_{(1)}$ yields the two diagrams:



and



Combining both diagrams, we obtain the map $\varphi_{2,1}$.



If the symbols for generic sections of $\mathcal{F}_{(2)} \to X$ are 2-acyclic, we only have to project once to define the next bundle $\mathcal{F}_{(3)}$. ◇

Prolonging the Janet Sequence

We will now adapt the affine bundle version of the Janet sequence (3.43) to the case of $\mathcal{R}_{q+r+1}(\omega)$ and construct a curvature map κ_r as in Remark 3.52. Throughout this section we fix the integrable jet groupoid $\Theta_q \leq \Pi_q$ and assume that the symbols $\mathcal{M}_q$ for generic sections ω of $\mathcal{F} \to X$ are 2-acyclic and vector bundles. The generalisation to arbitrary symbols is rather simple. The idea for the proof of Theorem 4.26 and the Embedding Theorem 4.22 is to embed the exact sequence for the curvature map κ_r into the prolongation of sequence (3.44) for κ:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{R}^{(1)}_{q+r}(\omega) & \longrightarrow & \mathcal{R}_{q+r}(\omega) & \overset{\kappa_r}{\underset{0}{\rightrightarrows}} & S^rT^* \otimes F_1 \\
 & & \big\downarrow & & \big\| & & \big\downarrow \\
0 & \longrightarrow & (\mathcal{R}^{(1)}_{q})_{+r}(\omega) & \longrightarrow & \mathcal{R}_{q+r}(\omega) & \overset{p_r(\kappa)}{\underset{0}{\rightrightarrows}} & J_r(F_1)
\end{array} \tag{4.6}$$

Constructing κ_r relies on the affine version of the Janet sequence (3.43) for $\mathcal{R}_{q+r+1}(\omega)$. If $\mathcal{M}_q$ is 2-acyclic, the above embedding (4.6) is possible. The first step is to prolong the sequence of model vector bundles, where we adapt Proposition A.6 for symbols of jet groupoids.

Proposition 4.31. Let Θ_q be a groupoid with involutive symbol $\mathcal{M}_{\Theta_q}$ such that all $\mathcal{M}_{\Theta_{q+r}}$ for $r \in \mathbb{Z}_{\geq 0}$ are vector bundles. Let $\mathcal{R}_q \subseteq \Theta_q$ be a jet subgroupoid with symbol $\mathcal{M}_q$. If $\mathcal{M}_{q+1}$ is a vector bundle over $\mathcal{R}_q$ and $\mathcal{M}_q$ is 2-acyclic then for all $r \in \mathbb{N}$,

$$0 \longrightarrow \mathcal{M}_{q+r+1} \longrightarrow \mathcal{M}_{\Theta_{q+r+1}} \longrightarrow S^{r+1}T^* \otimes \mathcal{F}_0 \longrightarrow S^rT^* \otimes \mathcal{F}_1 \tag{4.7}$$

is an exact sequence of vector bundles (pulled back over $\mathcal{R}_q$). Additionally, $\mathcal{M}_{q+r}$ is a vector bundle over $\mathcal{R}_q$. ◇

Proof. Begin with the top row of diagram (3.43) and proceed by induction on r. If the sequence is exact for r, a diagram chase in Figure 4.5 shows that the sequence is also exact for $r+1$. The diagram in Figure 4.5 resembles the diagram below equation (1.6.13) in [Spe69].

To see that $\mathcal{M}_{q+r}$ is a vector bundle, set Q_r as the cokernel of the last map in the first row. As the map $J_1(\mathcal{F}) \to \mathcal{F}_{(1)}$ is of constant rank, $\dim(Q_r)$ is also constant. Counting dimensions, we see that

$$\dim(\mathcal{M}_{q+r+1}) + \dim(Q_r) = \dim(\mathcal{M}_{\Theta_{q+r+1}}) - \dim(S^{r+1}T^* \otimes \mathcal{F}_0) + \dim(S^rT^* \otimes \mathcal{F}_1)$$

is constant and thus $\mathcal{M}_{q+r+1}$ is a vector bundle. □

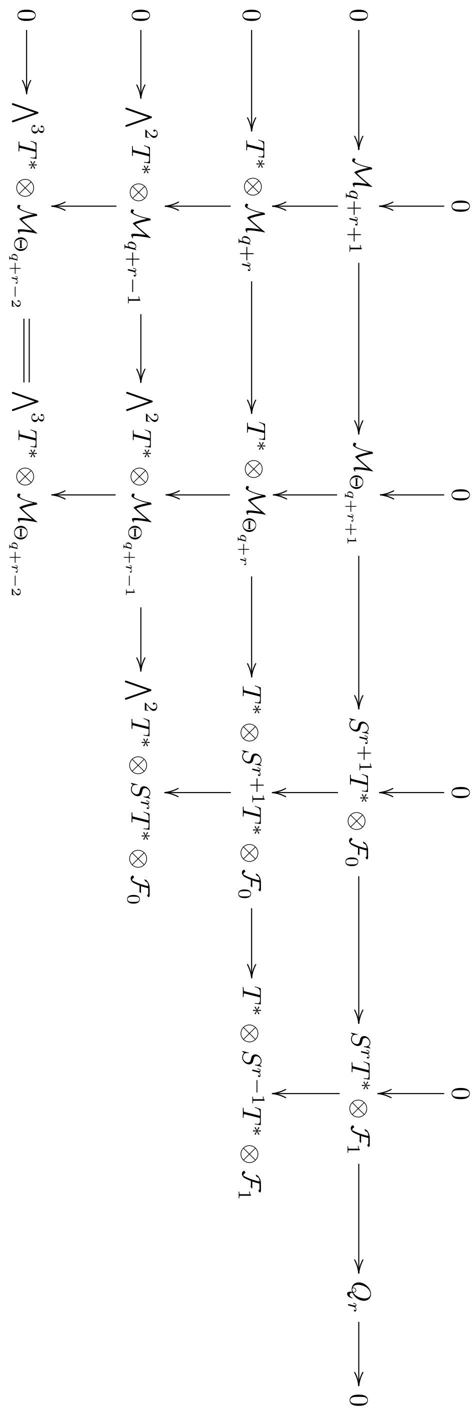


Figure 4.5: Commutative diagram for the proof of Proposition A.6.

The last proposition gives rise to the exact sequence of affine bundles for $\mathcal{R}_{q+r+1}(\omega)$, which starts with the prolongation of equation (3.5):

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{M}_{q+r+1} & \longrightarrow & \mathcal{M}_{\Theta_{q+r+1}} & \longrightarrow & S^{r+1}T^*\otimes\mathcal{F}_0 & \longrightarrow & S^rT^*\otimes\mathcal{F}_1 & & \\
 & & \vdots & & \vdots & & \vdots & & & & \\
0 & \longrightarrow & \mathcal{R}_{q+r+1} & \longrightarrow & \Theta_{q+r+1} & \longrightarrow & J_{r+1}(\mathcal{F}) & \longrightarrow & J_{r+1}(\mathcal{F})/K^{q+r+1}_{q+r} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}_{q+r} & \longrightarrow & \Theta_{q+r} & \longrightarrow & J_r(\mathcal{F}) & & J_r(\mathcal{F}) & &
\end{array} \tag{4.8}$$

As in Section 3.5.3, we omit the ω-dependence, double arrows and possible pullbacks over Θ_{q+r} (if there is no integrable groupoid $\mathcal{R}_{q+r}(\omega)$.

By Lemma 3.43, the fibre of the bundle $\mathcal{M}_{\Theta_{q+r+1}}$ is isomorphic to $\mathfrak{k}^{q+r+1}_{q+r}$ and thus the translational $\mathcal{M}_{\Theta_{q+r+1}}$-action and the K^{q+r+1}_{q+r}-action on the fibre of $J_{r+1}(\mathcal{F})$ coincide. It follows that $J_{r+1}(\mathcal{F})/K^{q+r+1}_{q+r}$ is the cokernel of

$$p_{r+1}(\Phi_\omega) : \Theta_{q+r+1} \to J_{r+1}(\mathcal{F}).$$

Restricting $S^rT^*\otimes\mathcal{F}_{(1)}$ to the cokernel $\mathcal{H} = \mathrm{coker}(\mathcal{M}_{\Theta_{q+r+1}} \to S^{r+1}T^*\otimes\mathcal{F}_0)$ yields the exact sequence of affine bundles and finishes the preparations for the proof of the Embedding Theorem.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{M}_{q+r+1} & \longrightarrow & \mathcal{M}_{\Theta_{q+r+1}} & \longrightarrow & S^{r+1}T^*\otimes\mathcal{F}_0 & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \mathcal{R}_{q+r+1} & \longrightarrow & \Theta_{q+r+1} & \longrightarrow & J_{r+1}(\mathcal{F}) & \longrightarrow & J_{r+1}(\mathcal{F})/K^{q+r+1}_{q+r} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R}_{q+r} & \longrightarrow & \Theta_{q+r} & \longrightarrow & J_r(\mathcal{F}) & & J_r(\mathcal{F}) & &
\end{array} \tag{4.9}$$

Proof of Theorem 4.22

The case, where the symbols for generic sections ω of $\mathcal{F} \to X$ are not 2-acyclic can be reduced to the 2-acyclic case. If the symbols become 2-acyclic after r prolongations, replace $\mathcal{F}$ by $J_r(\mathcal{F})$ and apply Theorem 4.22 to construct the embedding

$$J_{s+1}(J_r(\mathcal{F}))/K^{q+r+s+1}_{q+r+s} \hookrightarrow J_s(\mathcal{F}_{(1)})$$

for $\mathcal{F}_{(1)} = J_1(J_r(\mathcal{F}))/K^{q+r+1}_{q+r}$. Apply Propositions B.17 (3) and 2.10 (or 3.7) on the canonical map $J_{r+s+1}(\mathcal{F}) \hookrightarrow J_{s+1}(J_r(\mathcal{F}))$ to obtain the desired embedding

$$J_{r+s+1}(\mathcal{F})/K^{q+r+s+1}_{q+r+s} \hookrightarrow J_s(\mathcal{F}_{(1)}).$$

We now assume 2-acyclic symbols. Roughly speaking, we apply the functor J_r to the sequence (3.43) and then embed the sequence (4.8):

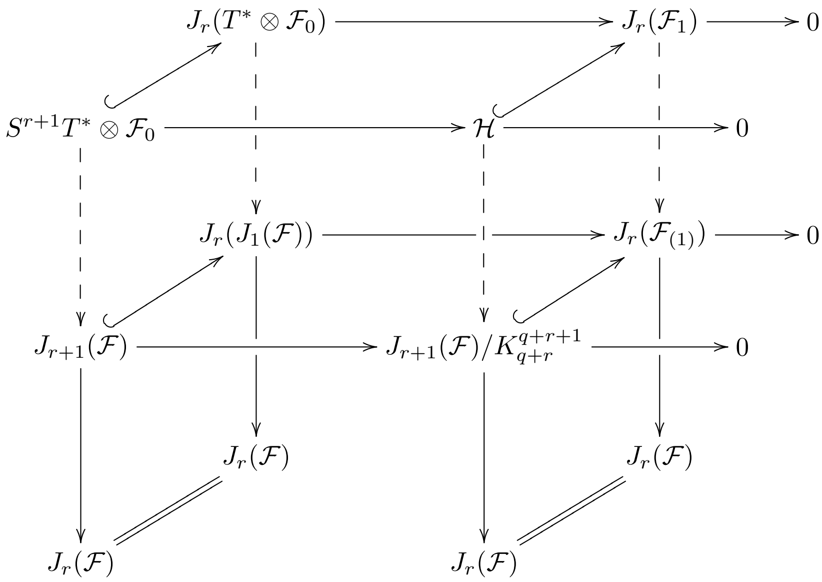
$$\text{eq. (4.8)} \hookrightarrow J_r(\text{eq. (3.43)})$$

The result is the commutative and exact diagram in Figure 4.6, where the leading zeros are omitted to simplify the diagram. The first three columns of diagram (4.8) embed into J_r(eq. (3.43)) because of Definition 1.24 for the groupoids and of Proposition 1.15 (1) for the natural bundles. Since a morphism of affine bundles is injective if and only if the corresponding morphism on the model vector bundles is injective, the embedding follows for the top rows, too. By Proposition 1.15 (3), the diagram commutes.

The embedding $S^rT^* \otimes \mathcal{F}_1 \hookrightarrow J_r(\mathcal{F}_1)$ is a consequence of Proposition 1.10 (1) and (2). It remains to show that the map

$$\varphi_{i,r} : J_{r+1}(\mathcal{F})/K_{q+r}^{q+r+1} \to J_r(\mathcal{F}_{(1)})$$

from equation (4.2) is also an embedding. For this, we restrict $S^rT^* \otimes \mathcal{F}_1$ to the cokernel $\mathcal{H}$ according to diagram (4.9):



As the morphism of model vector bundles $\mathcal{H} \hookrightarrow J_r(\mathcal{F}_1)$ is injective, it also follows for $\varphi_{i,r}$.

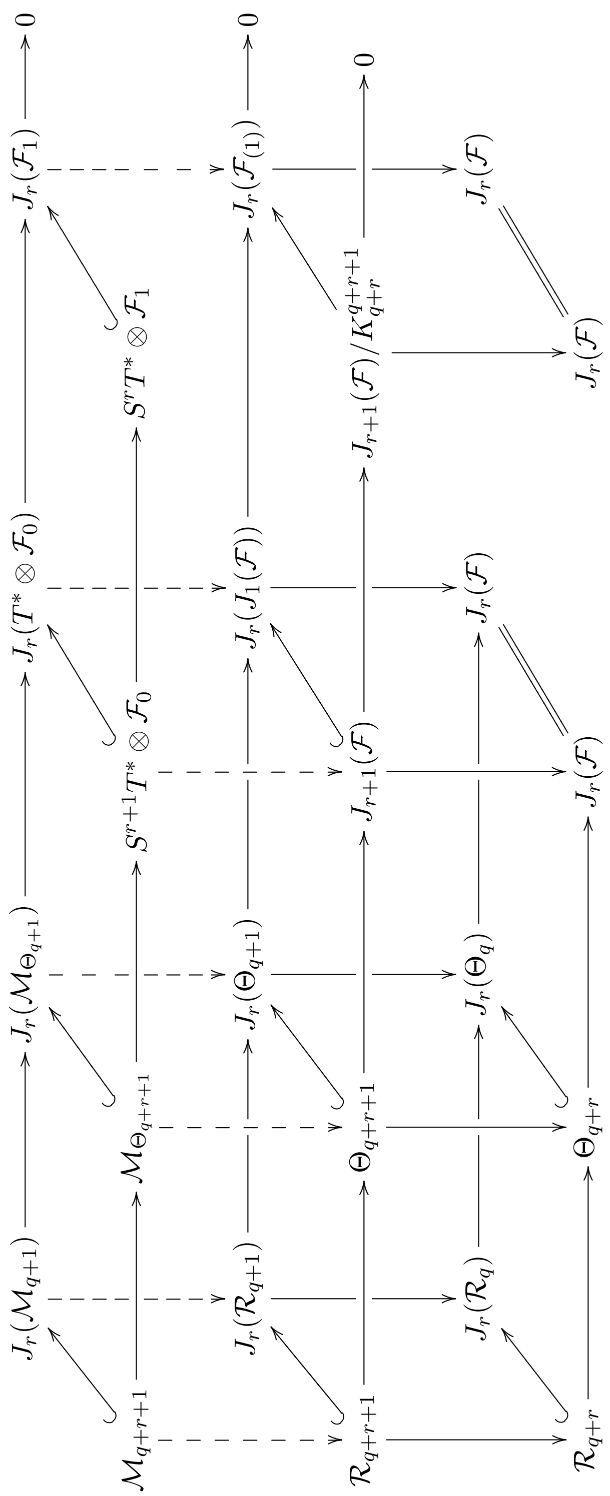


Figure 4.6: Commutative diagram for the proof of Theorem 4.22.

4.3.3 Optimisation II: Subgroupoids

In this section, we present a second possibility to optimise Vessiot's approach. The Embedding Theorem applies to the bundles $\mathcal{F}_{(i)}$ of integrability conditions and here we will shrink the original groupoid Θ_q and the bundle $\mathcal{F}$ before starting the process of prolongation and projection. This is possible if we are only interested in a closed subset of all sections of $\mathcal{F} \to X$. In many examples it is even necessary to shrink $\mathcal{F}$ and Θ_q since the sections of interest will not be generic after a few prolongations and projections. A strong motivation for this optimisation is the relative equivalence problem in Section 6.1.2.

At first we show an example, where the restriction to a subset of all sections is naturally given by the problem itself. With a slight change of notation, it was taken from the thesis of Neut [Neu03, §2.3.1] (see also the references therein).

Example 4.32. Each second order ordinary differential equation (ODE) defines a submanifold X of $J_2(\mathbb{R} \times \mathbb{R})$ by setting

$$y'' = f(x, y, p = y').$$

Choose the coordinates (x, y, p) for X. The pullback of the standard contact forms on $J_2(\mathbb{R} \times \mathbb{R})$ to X is given by

$$\omega^2 = dy - p\,dx, \qquad \omega^3 = dp - f(x, y, p)dx.$$

Adding $\omega^1 = dx$ completes the forms to a coframe $\omega = (\omega^1, \omega^2, \omega^3)^{tr}$ on X. A diffeomorphism φ is a contact transformation on X if the pullbacks $\varphi^*(\omega^2)$, $\varphi^*(\omega^3)$ are multiples of the contact forms. This is the case if there are functions $a_i(x, y, p)$ such that

$$\varphi^*(\omega) = \begin{bmatrix} a_4 & a_5 & a_6 \\ 0 & a_1 & 0 \\ 0 & a_2 & a_3 \end{bmatrix} \omega.$$

We further restrict to transformations with $\varphi^*(dx) = dx$.

$$\varphi^*(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & a_2 & a_3 \end{bmatrix} \omega. \tag{4.10}$$

To obtain a natural bundle, we see that the matrix in equation (4.10) defines a subgroup $G_1 \leq \mathrm{GL}_1$ (with parameters a_1, a_2, a_3). Each coframe ω is a section of the coframe bundle bundle $P_1 \to X$ and thus also of the natural Π_1-bundle $\mathcal{F} = P_1/G_1$. By construction of $\mathcal{F}$, the condition (4.10) is satisfied if and only if φ stabilises the projection of the coframe to $\mathcal{F}$.

In Section 5.3, we compute the projection $P_1 \to \mathcal{F}$, such that $\mathcal{F}$ has fibre coordinates $(u^1, \ldots u^6)$ and the projected coframe corresponds to the section

$$u^1 = 1, \quad u^2 = u^3 = u^4 = 0, \quad u^5 = -p, \quad u^6 = f(x, y, p). \tag{4.11}$$

It is clearly visible that only a small subset of the sections of $\mathcal{F} \to X$ corresponds to second order ODEs. It is parametrised by the single function $f(x, y, p)$. ⋄

The goal is to find a more efficient description of those problems where only a subset $\Omega \subset \Gamma(\mathcal{F})$ of all sections is of interest. The next proposition shows a correspondence between factor bundles and subgroupoids $\Theta'_q \leq \Theta_q$.

Proposition 4.33. Let $\mathcal{F} \to X$ be a natural Θ_q-bundle and $\Omega \subset \Gamma(\mathcal{F})$ be a subset of the sections of $\mathcal{F}$. The following two statements are equivalent.

(1) There exists a natural Θ_q-bundle $\mathcal{F}'$ and a projection

$$\pi : \mathcal{F} \to \mathcal{F}',$$

which is also a morphism of natural Θ_q-bundles such that all sections $\omega \in \Omega$ project to a single section ω' of $\mathcal{F}' \to X$.

(2) There exists a subgroupoid $\Theta'_q \leq \Theta_q$ such that the subgroupoids satisfy

$$\mathcal{R}_q(\omega) \leq \Theta'_q, \qquad \forall \omega \in \Omega. \qquad \diamond$$

We usually assume $\mathcal{F}' \neq X$, since otherwise the statements are trivial.

Proof. If there exists a projection, set $\Theta'_q = \mathcal{R}_q(\omega')$ as the symmetry groupoid of ω'. Since π is a projection, each $r_q \in \mathcal{R}_q(\omega)(x, y)$ also satisfies

$$\omega'(y) r_q = \pi(\omega)(y) r_q = \pi(\omega)(x) = \omega'(x).$$

If all $\mathcal{R}_q(\omega) \subseteq \Theta'_q$ for $\omega \in \Omega$, then the equations for $\mathcal{R}_q(\omega)$ in Lie form (3.11) include those for Θ'_q. Then there are coordinates (x, u) of $\mathcal{F}$ such that the first k coordinates $(u^1, \dots, u^k)$ correspond to the equations for Θ'_q (independent from the chosen section ω). Locally define the bundle $\mathcal{F}'$ by the projection

$$\pi : \mathcal{F} \to \mathcal{F}' : (x, u) \mapsto (x, u^1, \dots, u^k).$$

Since $\omega' = \pi(\omega)$ is a well-defined section defining the jet groupoid Θ'_q, $\mathcal{F}' \cong P_{\Theta_q}/\Theta'_q$ is a natural bundle. $\square$

If we can find a factor bundle $\mathcal{F}'$ where all sections from Ω are effectively represented by a single section ω', then it is possible to shrink Θ_q. At the same time the Θ_q-bundle $\mathcal{F}$ can be replaced by a smaller Θ'_q-bundle, since several equations are automatically satisfied. The following lemma shows how to proceed.

Lemma 4.34. Let $\pi : \mathcal{F} \to \mathcal{F}'$ be a projection of Θ_q-bundles and let ω' be a section of $\mathcal{F}' \to X$. Assume that the symmetry groupoid $\Theta'_q = \mathcal{R}_q(\omega')$ is integrable. Then the bundle $\mathcal{F}''$ defined by the pullback diagram

$$\begin{array}{ccc} \mathcal{F}'' & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{\omega'} & \mathcal{F}' \end{array}$$

is a natural Θ'_q-bundle. $\diamond$

Proof. Since each $f \in \Theta'_q(x, y)$ satisfies the symmetry equations $\omega'(y)f = \omega'(x)$, the fibre $\pi^{-1}(\omega'(y))$ is mapped to the fibre $\pi^{-1}(\omega'(x))$. □

If the symmetry groupoid $\mathcal{R}_q(\omega')$ is not integrable, we prolong and project until we obtain an integrable groupoid $\Theta'_q \subseteq \mathcal{R}_q(\omega')$ and continue as before. Usually $\mathcal{R}_q(\omega')$ is not integrable.

To shrink Θ_q and $\mathcal{F}$ for a subset $\Omega \subset \Gamma(\mathcal{F})$, the easiest choice is to find a projection $\mathcal{F} \to \mathcal{F}'$. In all cases treated in this thesis, the projection is directly given by the problem itself.

Remark 4.35. Not all subsets $\Omega \subset \Gamma(\mathcal{F})$ are suited to shrink Θ_q. Consider the Π_1-bundle $\mathcal{F} = T^* \times_X T^*$ of two 1-forms on a three-dimensional base. Then all sections of the form

$$\omega^1 = dx^1, \qquad \omega^2 = u^1(x)dx^1 + u^2(x)dx^2 + u^3dx^3$$

induce the projection $\pi : \mathcal{F} \to T^*$ and the section $\omega' = dx^1$. Here it is possible to shrink Π_q and $\mathcal{F}$ to T^*.

However choosing the forms ω^1 and ω^2 with the single condition that the coefficient of dx^1 coincides for both forms, does not lead to a subgroupoid of Π_1. The reason is the GL_1-action on the fibre $F \cong \mathcal{F}_x$. In the first case, GL_1 acts on the values $\mathcal{U} = \{\omega(x)|\omega \in \Omega\}$ as a block ($g\mathcal{U} = \mathcal{U}$ or $g\mathcal{U} \cap \mathcal{U} = \emptyset$ for all $g \in \mathrm{GL}_1$). In the second case, the sections do not form a block. ◇

For an example which shows how to restrict the natural bundle $\mathcal{F}$ and the original groupoid Θ_q, we refer to Section 5.3, where Example 4.32 is continued. All computations are done with the MAPLE packages `jets` and `JetGroupoids` (see Chapter 5 and Appendix D).

Chapter 5

Maple Examples

The current chapter is intended to give examples that illustrate the theoretical parts in Chapters 3 and 4. The second purpose is to give an introduction to the Maple packages `jets`, `JetGroupoids` and `Spencer`. They provide procedures for natural bundles, jet groupoids and Spencer cohomology.

In Section 5.1, the example of a Riemannian metric on a two-dimensional base manifold is treated in order to show basic commands of the Maple packages. The first part of the example repeats all computations of Section 3.3.1 with Maple. Furthermore, all mayor steps from Chapters 3 and 4 are explicitly computed. This includes the prolongation and projection of natural bundles as well as the Vessiot structure equations.

The example in Section 5.2 is from Olver and Pohjanpelto (see e.g. [OP07a]). It is the first example of a natural Θ_q-bundle. Here, a generating set of invariants and invariant differential operators are constructed to illustrate Section 4.2.

The last example in Section 5.3 deals with second order ODEs under point transformations. It was calculated by Neut [Neu03] with the Cartan equivalence method (see Section 6.2). Using Vessiot's approach, this example shows how the natural Π_q-bundle $\mathcal{F}$ can be reduced to a natural Θ_q-bundle using the optimisation from Section 4.3.3. It also serves as an example of the Vessiot equivalence method that will be introduced in Chapter 6.

There are three Maple packages used in this chapter. The `jets` package by Barakat and Hartjen [Bar01] contains routines for formal differential geometry calculations and basic commands for jet groupoids and natural bundles. It was extended by the author of this thesis, which led to the add-on package `JetGroupoids` for natural bundle commands and the `Spencer` package for the computation of Spencer cohomology.

See Appendix D for a reference of relevant commands of `jets`, `JetGroupoids`, `Spencer` and a sample worksheet that can be adapted to specific problems.

5.1 Riemannian Geometry

This example deals with a Riemannian metric on a two-dimensional base manifold which were already introduced in Examples 3.8 and 3.3.1. To illustrate the theoretical results from Chapters 3 and 4, the relevant computations are performed with the MAPLE packages `jets`, `JetGroupoids` and `Spencer`. The topics covered are:

- Construction of a natural bundle $\mathcal{F}$ via differential invariants (this repeats Example 3.3.1 from Section 3.3),
- Spencer cohomology (Appendix A),
- Prolongation and projection of natural bundles and integrability conditions (Sections 3.4 and 3.5),
- Minimal bundles (Section 4.3.1),
- Invariants on natural bundles (Section 4.2).

In each section, references to the theoretical part are given.

Calculations in MAPLE

The first part is completely analogous to Example 3.3.1, except that all computations are done with MAPLE.

Load the packages:

```
> with(jets): with(JetGroupoids): with(Spencer):
```

Define the independent and dependent variables, as well as coordinates for the algebroid $J_1(T)$:

```
> ivar := [x1,x2]:    dvar := [y1,y2]:
> Dvar := [xi1,xi2]:  Tvar := [eta1,eta2]:
```

The equations defining the symmetry groupoid $\mathcal{R}_1 = \mathcal{R}_1(\omega_0)$ of the flat metric:

```
> GR_g := [y1[x1]^2+y2[x1]^2 = 1, y1[x1]*y1[x2]+y2[x1]*y2[x2] = 0,
> y1[x2]^2+y2[x2]^2 = 1];
```

$$GR\_g := [{y1_{x1}}^2 + {y2_{x1}}^2 = 1,\ y1_{x1}\, y1_{x2} + y2_{x1}\, y2_{x2} = 0,\ {y1_{x2}}^2 + {y2_{x2}}^2 = 1]$$

The command `grp2alg` converts the equations for the groupoid $\mathcal{R}_1$ into those for the algebroid R_1 by linearisation and pullback to $J_1(T)$. The result is identical to equation (3.14):

```
> T_g := grp2alg(GR_g,ivar,dvar,Tvar,"");
```

$$T\_g := [2\,\eta 1_{y1} = 0,\ \eta 1_{y2} + \eta 2_{y1} = 0,\ 2\,\eta 2_{y2} = 0]$$

Involutive Distribution on Π_1 and Differential Invariants

To compute the involutive distribution $\sharp(R_1)$ on Π_1, we choose η_1^2 as coordinate for the subbundle R_1 of $J_1(T)$. The other coordinates are substituted according to the algebroid equations:

```
> T_g := nrsolve(T_g,[eta1[y1],eta1[y2],eta2[y2]])[1];
```

$$T\_g := [\eta1_{y1} = 0, \, \eta1_{y2} = -\eta2_{y1}, \, \eta2_{y2} = 0]$$

Compute the involutive distribution $\sharp(R_1)$ on $V(\Pi_1)$ using the following command:

```
> iso := isoalg(T_g,ivar,dvar,Tvar):
> iso := subs(iso[3],iso[1]):
```

The output of `isoalg` contains vector fields generating the distribution $\sharp(R_1)$. The `jets` notation for a vector field

$$\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$$

is as follows. A `jets` vector field is a list containing the summands of a vector field. For ξ, the summands are $a\partial_x$ and $b\partial_y$. Each summand $a\partial_x$ is written as a list $[a, [x]]$ containing the coefficient a and the variable x for the directional derivative.

The involutive distribution is generated by the vector fields:

```
> for a in iso do print(a) od;
```

$$[[1, [y1]]]$$

$$[[1, [y2]]]$$

$$[[-y2_{x1}, [y1_{x1}]], [-y2_{x2}, [y1_{x2}]], [y1_{x1}, [y2_{x1}]], [y1_{x2}, [y2_{x2}]]]$$

In the usual notation for vector fields, they coincide with equation (3.15):

$$\partial_{y^1}, \quad \partial_{y^2}, \quad -y_1^2\partial_{y_1^1} - y_2^2\partial_{y_2^1} + y_1^1\partial_{y_1^2} + y_2^1\partial_{y_2^2}.$$

It is already known from Section 3.3.1, that $\mathcal{R}_1$ is defined by differential invariants. Otherwise the following command computes them:

```
> #invtarget(T_g,ivar,dvar,Tvar,"");
```

We define coordinates for the natural bundle $\mathcal{F}_g$ of metrics as well as variables for sections. The projection $\Phi_g : \Pi_1 \to \mathcal{F}_g$ is the same as equation (3.16):

```
> uvar_g := [u11,u12,u22]:
> wvar_g := [omega11, omega12, omega22]:
> Phi_g  := ezip(uvar_g,map(lhs,GR_g));
```

$$Phi\_g := [u11 = {y1_{x1}}^2 + {y2_{x1}}^2, \, u12 = y1_{x1}\, y1_{x2} + y2_{x1}\, y2_{x2}, \, u22 = {y1_{x2}}^2 + {y2_{x2}}^2]$$

It is possible to check that `inv_g` contains only differential invariants by applying the generators of $\sharp(R_1)$ to `inv_g`. According to Lemma 3.17, all results must be zero:

```
> map(b->map(a->lieapp(a,rhs(b),ivar,dvar),iso),Phi_g);
```

$$[[0, 0, 0], [0, 0, 0], [0, 0, 0]]$$

The special section ω_0 defining the flat metric on X:

```
> omega0 := ezip(wvar_g,map(rhs,GR_g));
```

$$\omega 0 := [\omega 11 = 1, \omega 12 = 0, \omega 22 = 1]$$

5.1.1 The Natural Bundle

Having obtained the differential invariants, we now determine the natural bundle $\mathcal{F}_g$ of metrics. This illustrates Section 3.3. It is still covered by Example 3.3.1.

Computing the finite Π_1-action on the natural bundle $\mathcal{F}_g$ works exactly as in equation (3.18)) by transforming the invariants and expressing the output by the coordinates of $\mathcal{F}_g$. The result is the same as equation (3.19):

```
> nat_g  := natfin(Phi_g,ivar,dvar,uvar_g,dvar):
> nat_gi := eqn2ind(nat_g,ivar,dvar);
```

$$\begin{aligned} nat\_gi := [&x1 = y1,\, x2 = y2,\, u11 = {y1_{x1}}^2\, u11 + 2\, y1_{x1}\, y2_{x1}\, u12 + u22\, {y2_{x1}}^2, \\ &u12 = (y2_{x2}\, y1_{x1} + y2_{x1}\, y1_{x2})\, u12 + y2_{x1}\, y2_{x2}\, u22 + y1_{x1}\, y1_{x2}\, u11, \\ &u22 = {y1_{x2}}^2\, u11 + 2\, y1_{x2}\, y2_{x2}\, u12 + {y2_{x2}}^2\, u22] \end{aligned}$$

The corresponding infinitesimal $J_1(T)$-action on $\mathcal{F}_g$ from equation (3.17):

```
> vec_g := natinf(Phi_g,ivar,dvar,uvar_g,Dvar,""):
> vec_g := simplify(vec_g,symbolic);
```

$$\begin{aligned} vec\_g := [&[\xi 1, [x1]], [\xi 2, [x2]], [-2\, u11\, \xi 1_{x1} - 2\, u12\, \xi 2_{x1}, [u11]], \\ &[-u12\, \xi 1_{x1} - u11\, \xi 1_{x2} - \xi 2_{x1}\, u22 - u12\, \xi 2_{x2}, [u12]], \\ &[-2\, \xi 2_{x2}\, u22 - 2\, u12\, \xi 1_{x2}, [u22]]] \end{aligned}$$

If the Π_q-action on a natural bundle is given, the infinititesimal action may alternatively be computed by linearisation. The results are the same:

```
> natfin2inf(nat_g,ivar,dvar,Dvar,"");
```

$$\begin{aligned} [&[\xi 1, [x1]], [\xi 2, [x2]], [-2\, u11\, \xi 1_{x1} - 2\, u12\, \xi 2_{x1}, [u11]], \\ &[-u12\, \xi 1_{x1} - u11\, \xi 1_{x2} - \xi 2_{x1}\, u22 - u12\, \xi 2_{x2}, [u12]], \\ &[-2\, \xi 2_{x2}\, u22 - 2\, u12\, \xi 1_{x2}, [u22]]] \end{aligned}$$

General Lie and Medolaghi Form

The procedure to compute the general Lie form for a section ω of $\mathcal{F}_g$ basically plugs in the section into the equations for the Π_1-action on $\mathcal{F}_g$:

```
> GLF := LieFormG(nat_g,ivar,dvar,dvar,wvar_g):
> eqn2ind(GLF,ivar,dvar);
```

$$\begin{aligned} [&{y1_{x1}}^2\, \omega 11(y1, y2) + 2\, y1_{x1}\, y2_{x1}\, \omega 12(y1, y2) + \omega 22(y1, y2)\, {y2_{x1}}^2 = \omega 11(x1, x2), \\ &(y2_{x2}\, y1_{x1} + y2_{x1}\, y1_{x2})\, \omega 12(y1, y2) + y2_{x1}\, y2_{x2}\, \omega 22(y1, y2) + y1_{x1}\, y1_{x2}\, \omega 11(y1, y2) \\ &= \omega 12(x1, x2), \\ &{y1_{x2}}^2\, \omega 11(y1, y2) + 2\, y1_{x2}\, y2_{x2}\, \omega 12(y1, y2) + {y2_{x2}}^2\, \omega 22(y1, y2) = \omega 22(x1, x2)] \end{aligned}$$

Use the infinitesimal action on $\mathcal{F}_g$ to compute the general Medolaghi form according to Remark 3.25:

```
> GMF := inf2MF(vec_g,ivar,uvar_g,wvar_g,"");
```

$$GMF := [2\,\omega 11\,\xi 1_{x1} + 2\,\omega 12\,\xi 2_{x1} + \omega 11_{x1}\,\xi 1 + \omega 11_{x2}\,\xi 2 = 0,$$
$$\omega 12\,\xi 1_{x1} + \omega 11\,\xi 1_{x2} + \xi 2_{x1}\,\omega 22 + \omega 12\,\xi 2_{x2} + \omega 12_{x1}\,\xi 1 + \omega 12_{x2}\,\xi 2 = 0,$$
$$2\,\xi 2_{x2}\,\omega 22 + 2\,\omega 12\,\xi 1_{x2} + \omega 22_{x1}\,\xi 1 + \omega 22_{x2}\,\xi 2 = 0]$$

Prolonged Actions

We given an overview over the Π_{r+1}-action on the prolongations $J_r(\mathcal{F}_g)$. Since Π_{r+1} is transitive, we only need the GL_{r+1}-action on the fibre F and its algebraic prolongations $F^{(r)}$. For a generic point $f \in F^{(r)}$, we determine the dimensions of

$$\text{orbit space on } F^{(r)}, \quad F^{(r)}, \quad \mathrm{GL}_{r+1}\,f, \quad \mathrm{GL}_{r+1}, \quad \mathrm{Stab}_{\mathrm{GL}_{r+1}}(f)$$

and display them in a matrix:

```
> map(i->[codim_of_action(vec_g,i,ivar,uvar_g,Dvar,"")],[$0..3]):
> matrix(%);
```

$$\begin{bmatrix} 0 & 3 & 3 & 4 & 1 \\ 0 & 9 & 9 & 10 & 1 \\ 1 & 18 & 17 & 18 & 1 \\ 2 & 30 & 28 & 28 & 0 \end{bmatrix}$$

The first line shows the action on $\mathcal{F}_g$ itself. There are no invariants (first number) and the action on the three-dimensional fibre F is transitive. However $\dim(\mathrm{GL}_1) = 4$ such that the stabiliser of a generic point is one-dimensional. We observe that there is an single invariant on $J_2(\mathcal{F}_g)$ and the GL_3-action on the third prolongation $F^{(3)}$ becomes free. We will explicitly compute both invariant.

5.1.2 Spencer Cohomology

Before computing prolongations and projections of symmetry groupoids $\mathcal{R}_1(\omega)$, we first determine the Spencer cohomology groups and see that the symbols of generic sections of $\mathcal{F}_g$ are not 2-acyclic. Symbols and Spencer cohomology are introduced in Appendix A. See also Section 3.5.2 which treats the computation of Spencer cohomology using natural bundles.

```
> IZS  :='InvolutiveZeroSets':
> Scg0 := SpencerCohomology(GMF,ivar,Dvar,Tvar,IZS):
> SCohomDim(Scg0,Tvar,IZS);
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The calculation is invalid on the union of zero sets of the following expressions:

```
> SCZeroSets(Scg0);
```

$$[-\omega 12^2 + \omega 11\,\omega 22,\ \omega 11,\ \omega 12,\ \omega 22]$$

The only interesting case is $\omega_{12} = 0$, since all other assumptions are valid for nondegenerate metrics. We repeat the computation for metrics with diagonal entries and see that they also have a non-2-acyclic symbol.

```
> SpencerCohomology(subs(omega12=0,GMF),ivar,Dvar,Tvar,IZS):
> SCohomDim(%,Tvar,IZS);
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

5.1.3 Prolongation and Projection

We now turn to the prolongation and projection of natural bundles which was introduced in Section 3.4. The package `JetGroupoids` provides a data structure for natural bundles which is used to compute prolongations and projections. We create it with the following command, that basically stores the arguments in a table:

```
> Fg := CreateNaturalBundle(vec_g,ivar,dvar,uvar_g,Dvar):
```

To determine the first prolongation $J_1(\mathcal{F}_g)$, we only have to give the order for the prolongation and a name for the fibre coordinates of $J_1(\mathcal{F}_g) \to \mathcal{F}_g$:

```
> J1Fg := ProlongNaturalBundle(Fg,1,uu):
```

Projecting back to first order shows that $J_1(\mathcal{F}_g)/K_1^2$ and $\mathcal{F}_g$ are identical, because there are no new coordinates:

```
> Fg1 := ProjectNaturalBundle(J1Fg,v1):
> Fg1["inv"];
```

$$[\,]$$

This effect is due to the Spencer cohomology which was not 2-acyclic for any metric. In order to check integrability, we have to prolong twice.

Adding the Christoffel Symbols

Instead of computing $J_2(\mathcal{F}_g)$ directly, we first use the isomorphism of Example 3.3 (1) between $J_1(\mathcal{F}_g)$ and the bundle $\mathcal{F}_g \times_X \mathcal{F}_\Gamma$ for metrics and Christoffel symbols. The advantage of this approach is that the computations are much simpler and we have an immediate geometric interpretation of the integrability conditions.

Define the coordinates for the Christoffel symbols, where `uijk` stands for Γ^i_{jk}:

```
> uvar_Gamma := [u111, u112, u211, u212, u122, u222]:
> uvar := [op(uvar_g),op(uvar_Gamma)]:
```

The bundle $\mathcal{F}_\Gamma$ is constructed by the groupoid $\mathcal{R}_{2,\Gamma}$ of affine transformations on X. We skip the calculation of $\mathcal{F}_\Gamma$ and only give the infinitesimal $J_2(T)$-action on it:

```
> GR_Gamma:=map(a->a=0,jetcoor(2,ivar,dvar));
```

$$GR\_Gamma := [y1_{x1,x1} = 0,\ y1_{x1,x2} = 0,\ y1_{x2,x2} = 0,\ y2_{x1,x1} = 0,\ y2_{x1,x2} = 0,\ y2_{x2,x2} = 0]$$

```
> vec_Gamma;
```

$$[[\xi 1, [x1]], [\xi 2, [x2]], [-u111\,\xi 1_{x1} + \xi 1_{x2}\, u211 - 2\,\xi 2_{x1}\, u112 - \xi 1_{x1,x1}, [u111]],$$
$$[-\xi 1_{x2}\,(-u212 + u111) - \xi 2_{x1}\, u122 - u112\,\xi 2_{x2} - \xi 1_{x1,x2}, [u112]],$$
$$[-2\,u211\,\xi 1_{x1} + \xi 2_{x1}\,(-2\,u212 + u111) + u211\,\xi 2_{x2} - \xi 2_{x1,x1}, [u211]],$$
$$[-u212\,\xi 1_{x1} - \xi 1_{x2}\, u211 + \xi 2_{x1}\,(u112 - u222) - \xi 2_{x1,x2}, [u212]],$$
$$[u122\,\xi 1_{x1} - \xi 1_{x2}\,(-u222 + 2\,u112) - 2\,u122\,\xi 2_{x2} - \xi 1_{x2,x2}, [u122]],$$
$$[-2\,\xi 1_{x2}\, u212 + \xi 2_{x1}\, u122 - u222\,\xi 2_{x2} - \xi 2_{x2,x2}, [u222]]]$$

The action on the fibre product $\mathcal{F} = \mathcal{F}_g \times_X \mathcal{F}_\Gamma$ is computed by concaternation of vector fields:

```
> vec := [op(vec_g),op(vec_Gamma[3..-1])]:
> F := CreateNaturalBundle(vec,ivar,dvar,uvar,Dvar):
```

The Spencer cohomology can be computed directly for the natural bundle $\mathcal{F}$ and the result shows that the symbols for all sections are involutive.

```
> Sc0 := SpencerCohomology(F,ivar,Dvar,Tvar,IZS):
> SCohomDim(Sc0,Tvar,IZS);
> SCZeroSets(Sc0);
```

$$\begin{bmatrix} . & . & . \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\,]$$

Naive Prolongation and Projection of $\mathcal{F}$

Having determined $\mathcal{F} = \mathcal{F}_g \times \mathcal{F}_\Gamma$, we can start with the prolongation and projection procedure. However, prolonging to $J_1(\mathcal{F})$ means computing $J_1(J_1(\mathcal{F}_g))$, which is considerably larger than $J_2(\mathcal{F}_g)$. We can still check integrability on $J_1(\mathcal{F})$, but there are many conditions which are automatically satisfied for metrics. In the next part of this example we show how to avoid these redundancies with minimal bundles from Section 4.3.1. But at first, we naively prolong and project the bundle $\mathcal{F}$.

```
> J1F := ProlongNaturalBundle(F,1,uu):
> F1  := ProjectNaturalBundle(J1F,v,kernelD):
```

The data structure for $\mathcal{F}_{(1)}$ contains the fibre coordinates and the projection $J_1(\mathcal{F}) \to \mathcal{F}_{(1)}$ as $v^\beta = A^\beta(u)u_x$:

```
> F1["inv"][1..6]; F1["inv"][7..10];
```

$$[v1 = u11_{x1},\, v2 = u11_{x2},\, v3 = u12_{x1},\, v4 = u12_{x2},\, v5 = u22_{x1},\, v6 = u22_{x2}]$$

$$[v7 = -u111_{x2} + u112_{x1},\, v8 = -u211_{x2} + u212_{x1},\, v9 = -u112_{x2} + u122_{x1},$$
$$v10 = -u212_{x2} + u222_{x1}]$$

5.1.4 Integrability Conditions

On the natural bundle $\mathcal{F}$, generic sections have a 2-acyclic symbol and it is possible to determine integrability conditions as in Section 3.5. We compute the Vessiot structure equations on $\mathcal{F}_{(1)}$ according to Theorem 3.35:

```
> VSE1 := VessiotStructureEquations(F1):
```

The first six integrability conditions are the explicit form of equation (3.2)

$$\Gamma^i_{jk} = \frac{1}{2} g^{ir} \left(g_{rj,k} + g_{rk,j} - g_{jk,r} \right)$$

and they are automatically satisfied if the Christoffel symbols correspond to the chosen metric.

```
> SUBS_F1 := nrsolve(VSE1[1..6],uvar_Gamma)[1]:
> for a in SUBS_F1 do print(a) od;
```

$$u111 = \frac{1}{2} \frac{-u22\, u11_{x1} - u11_{x2}\, u12 + 2\, u12_{x1}\, u12}{u12^2 - u11\, u22}$$

$$u112 = \frac{1}{2} \frac{-u22\, u11_{x2} + u22_{x1}\, u12}{u12^2 - u11\, u22}$$

$$u211 = \frac{1}{2} \frac{u12\, u11_{x1} - 2\, u11\, u12_{x1} + u11\, u11_{x2}}{u12^2 - u11\, u22}$$

$$u212 = \frac{1}{2} \frac{-u11\, u22_{x1} + u11_{x2}\, u12}{u12^2 - u11\, u22}$$

$$u122 = \frac{1}{2} \frac{u22\, u22_{x1} - 2\, u22\, u12_{x2} + u22_{x2}\, u12}{u12^2 - u11\, u22}$$

$$u222 = -\frac{1}{2} \frac{-2\, u12\, u12_{x2} + u22_{x1}\, u12 + u11\, u22_{x2}}{u12^2 - u11\, u22}$$

The last four integrability conditions deal with the derivatives of the Christoffel symbols and depend on two arbitrary constants:

```
> VSE1[7..10];
```

$$[-u111_{x2} + u112_{x1} = u122\, u211 - u212\, u112 + \_C1\, u12 + \sqrt{u11\, u22 - u12^2}\, \_C2,$$
$$-u211_{x2} + u212_{x1} = -u11\, \_C1 + (-u112 + u222)\, u211 + u212\, u111 - u212^2,$$
$$-u112_{x2} + u122_{x1} = \_C1\, u22 + u112^2 - u112\, u222 + (u212 - u111)\, u122,$$
$$-u212_{x2} + u222_{x1} = u212\, u112 - u122\, u211 - \_C1\, u12 + \sqrt{u11\, u22 - u12^2}\, \_C2]$$

Computing the Jacobi conditions from Section 3.5.3 shows that all equivariant sections corresponding to an integrable groupoid satisfy $C_2 = 0$.

```
> JacobiCond(VSE1,ivar,F1["uvar"],"");
```

$$[2\, \_C2,\ 2\, \_C2,\ 2\, \_C2]$$

With $C_2 = 0$, the above equations are the constant scalar curvature condition

$$R^k_{lij} = \partial_i \Gamma^k_{lj} - \partial_j \Gamma^k_{li} + \Gamma^r_{lj}\Gamma^k_{ri} - \Gamma^r_{li}\Gamma^k_{rj} = C_1(\delta^k_j g_{li} - \delta^k_i g_{lj}),$$

where R^k_{lij} stands for the Riemann curvature tensor.

5.1.5 Vector Bundle Structure of $\mathcal{F}_{(1)}$

We will now change the coordinates of $\mathcal{F}_{(1)}$ to demonstrate that $\mathcal{F}_{(1)}$ is a vector bundle. By the results of Section 3.5.1, it is possible since there exist equivariant sections on $\mathcal{F}_{(1)}$. The vector bundle structure allows to interpret the first four coordinates of $\mathcal{F}_{(1)}$ as the entries of the Riemann curvature tensor.

So far, the coordinates v^β of $\mathcal{F}_{(1)}$ are of the form $v^\beta = A^\beta(u)u_x$ and we will add an affine term $B^\beta(u)$ according to equation (3.34). The coordinate change

$$v^\beta = A^\beta(u)u_x \mapsto A^\beta(u)u_x + B^\beta(u)$$

transforms $\mathcal{F}_{(1)}$ into a vector bundle. We choose $B(u) = -c_0(u)$ where c_0 is the equivariant section with $C_1 = C_2 = 0$. A closer look at the definition of the first four coordinates $v^1, \dots v^4$ of $\mathcal{F}_{(1)}$ shows that they are of the form $\partial_i \Gamma^k_{lj} - \partial_j \Gamma^k_{li}$ and the section c_0 consists of terms $-\Gamma^r_{lj}\Gamma^k_{ri} + \Gamma^r_{li}\Gamma^k_{rj}$. Their difference is exactly the curvature tensor.

At first we show that the $J_2(T)$ action on $\mathcal{F}_{(1)}$ is affine, since the second order terms depend on `uijk` only:

```
> F1["vec"][-1];
```

$$[-\xi 1_{x2}\, v8 + \xi 2_{x1}\, v9 - \xi 1_{x1}\, v10 - \xi 2_{x2}\, v10$$
$$- u212\, \xi 1_{x1,\,x2} + \xi 2_{x1,\,x1}\, u122 + \xi 1_{x2,\,x2}\, u211 - \xi 2_{x1,\,x2}\, u112,\ [v10]]$$

Compute the affine term $B(u)$ and print the last coordinate $v^{10} = A^{10}(u)u_x + B^{10}(u)$, which is the component R^2_{212} of the Riemann tensor:

```
> B := subs([_C1=0,_C2=0],map(-rhs,VSE1)):
> R2_212 := F1["F"][-1] + B[-1];
```

$$R2\_212 := -u212_{x2} + u222_{x1} - u212\, u112 + u122\, u211$$

Perform the coordinate change to obtain a vector bundle structure. The $J_2(T)$-action becomes linear:

```
> F1v := ChangeFibreCoordinates(F1,F1["vvar"]+B):
> F1v["vec"][-1];
```

$$[-\xi 1_{x2}\, v8 + \xi 2_{x1}\, v9 - \xi 1_{x1}\, v10 - \xi 2_{x2}\, v10,\ [v10]]$$

We compute the action of GL_2 on the fibre of $\mathcal{F}_{(1)}$ to see that there must be nine invariants. On the bundle $J_2(\mathcal{F}_g)$, there is a single invariant. It follows that the canonical embeding $J_2(\mathcal{F}_g) \hookrightarrow J_1(\mathcal{F})$ has at least codimension eight and that at least eight integrability conditions on $\mathcal{F}_{(1)}$ are automatically satisfied for metrics.

```
> CodimOfAction(F1v);
```

$$9,\ 19,\ 10,\ 10,\ 0$$

5.1.6 Optimisation: Minimal Bundles

We have seen that the bundle $\mathcal{F}_{(1)}$ was larger than necessary to check integrability for symmetry groupoids of a metric. For the Vessiot structure equations on $\mathcal{F}_{(1)}$ the section for the metric and the bundle could be chosen independently, but we are interested in the case where the Christoffel symbols are derived from a given metric. This is a classical application of minimal bundles from Section 4.3.1. The

Embedding Theorem 4.22 states, that the integrability conditions can be checked on a minimal subbundle $\mathcal{F}'_{(1)} \subseteq \mathcal{F}_{(1)}$ which is isomorphic to $J_2(\mathcal{F}_g)/K_2^3$.

By adding the Christoffel symbols, we have used geometric insight, but the bundle $J_1(\mathcal{F})$ is isomorphic to $J_1(J_1(\mathcal{F}_g))$, where the bundle of metrics has been prolonged twice. This is the case described in Lemma 4.14.

The procedures of the package `JetGroupoids` can compute the minimal bundle $\mathcal{F}'_{(1)}$ if we tell them that the Christoffel symbols actually depend on the first order jets of the metric. Use the internal variable `SUBSvec` to store how the coordinates `uijk` depend on the first order jets of `uij`:

```
> SUBS_F := [[SUBS_F1,[],VSE1[1..6],uvar_g]]:
> Fa := copy(F):
> Fa["SUBSvec"] := SUBS_F:
```

With the extra information, the command `ProlongNaturalBundle` automatically computes the image of the embedding $J_2(\mathcal{F}_g) \hookrightarrow J_1(\mathcal{F})$ and the following projection returns the minimal bundle $\mathcal{F}'_{(1)}$ of integrability conditions for metrics. In this example only a single coordinate is left:

```
> J1Fa := ProlongNaturalBundle(Fa,1,uu):
> F1a  := ProjectNaturalBundle(J1Fa,v):
> F1a["inv"];
```

$$[v = -u111_{x2} + u112_{x1}]$$

The Vessiot structure equations are still equivalent to the constant scalar curvature conditions. On a two-dimensional manifold, the various symmetries of the Riemann curvature tensor imply that there is only a single independent component. For the integrability conditions, it was chosen as R^1_{112}:

```
> VSE1a := VessiotStructureEquations(F1a);
```

$$VSE1a := [-u111_{x2} + u112_{x1} = u122\ u211 - u212\ u112 + \_C1\ u12]$$

Compared with the Vessiot structure equations on $\mathcal{F}_{(1)}$, the Jacobi conditions are automatically taken into account and the spurious constant C_2 does not occur any longer.

```
> member(lhs(VSE1a[1]),map(lhs,VSE1),'pos'):
> VSE1[pos];
```

$$-u111_{x2} + u112_{x1} = u122\ u211 - u212\ u112 + \_C1\ u12 + \sqrt{u11\ u22 - u12^2}\ \_C2$$

5.1.7 Invariants on Natural Bundles

To illustrate the results of Section 4.2, we compute a generating set of invariants on $J_r(\mathcal{F})$ using Vessiot's approach. At first, the invariants are computed and then we use geometric insights to construct the invariant differential operators. This section is also an example for the classification of symmetry groupoids from Section 4.1.

The GL_2-action on the fibre of $\mathcal{F}'_{(1)}$ shows that there is a single invariant. Since $\mathcal{F}'_{(1)}$ is isomorphic to $J_2(\mathcal{F}_g)/K_2^3$, the invariant is identical to the single invariant on $J_2(\mathcal{F}_g)$.

```
> CodimOfAction(F1a);
> Inv1a := InvariantsOnNaturalBundle(F1a);
```

$$1,\ 10,\ 9,\ 10,\ 1$$

$$\mathit{Inv1a} := [\frac{v + \mathit{u212}\ \mathit{u112} - \mathit{u122}\ \mathit{u211}}{\mathit{u12}}]$$

We can pull back the invariant to $J_2(\mathcal{F})$ and realise that it is the scalar curvature of the metric. Due to the length of the result, the output is not printed.

```
> PullbackToF(Inv1a,F1a):
```

The groupoid action on the natural bundle $\mathcal{F}'_{(1)}$ is not yet locally free, so another prolongation and projection is necessary. We first take the invariant on $\mathcal{F}'_{(1)}$ as fibre coordinate. On the bundle $\mathcal{F}_{(2)}$, there is a second invariant which was predicted for $J_3(\mathcal{F}_g)$:

```
> F1c   := ChangeFibreCoordinates(F1a,Inv1a):
> J1F1c := ProlongNaturalBundle(F1c,1,uu):
> F2    := ProjectNaturalBundle(J1F1c,v2):
> F2["inv"];
> Inv2  := InvariantsOnNaturalBundle(F2);
```

$$[\mathit{v21} = v_{\mathit{x2}},\ \mathit{v22} = v_{\mathit{x1}}]$$

$$\mathit{Inv2} := [v,\ \frac{\mathit{v21}^2\ \mathit{u11} + \mathit{v22}^2\ \mathit{u22} - 2\ \mathit{v22}\ \mathit{u12}\ \mathit{v21}}{\mathit{u11}\ \mathit{u22} - \mathit{u12}^2}]$$

The second invariant is constructed by contracting the indices of the first order jets v_i of the scalar curvature with the inverse of the metric as $v_i v_j g^{ij}$.

For the classification of symmetry groupoids in Section 4.1, we compute the Vessiot structure equations on $\mathcal{F}_{(2)}$ and show that the GL_2-action on the fibre of $\mathcal{F}_{(2)}$ is locally free. So we have reached the generic case.

```
> VessiotStructureEquations(F2);
> CodimOfAction(F2);
```

$$[v_{\mathit{x2}} = 0,\ v_{\mathit{x1}} = 0]$$

$$2,\ 12,\ 10,\ 10,\ 0$$

To compute a generating set of invariants, we prolong and project a last time to obtain a three-dimensional bundle:

```
> J1F2 := ProlongNaturalBundle(F2,1,uu):
> F3 := ProjectNaturalBundle(J1F2,v3,kernelD):
> F3["inv"];
```

$$[\mathit{v31} = \mathit{v21}_{\mathit{x1}},\ \mathit{v32} = \mathit{v21}_{\mathit{x2}},\ \mathit{v33} = \mathit{v22}_{\mathit{x2}}]$$

The bundle $\mathcal{F}_{(3)}$ is not yet a vector bundle over $\mathcal{F}_{(2)}$ and we add an affine term to obtain a vector bundle atlas. Comparing the infinitesimal action on the fibre with the action on $\mathcal{F}_g$, we see that the fibre is isomorphic to S^2T^* with new coordinates h_{ij}.

```
> [h11 = v31-(v21*u111+v22*u211), h12 = v32-(v21*u112+v22*u212),
> h22 = v33-(v21*u122+v22*u222)]:
> F3a := ChangeFibreCoordinates(F3,%):
> F3a["vec"][-3..-1];
```

$$[[-2\,h11\,\xi 1_{x1} - 2\,\xi 2_{x1}\,h12,\,[h11]],$$
$$[-h11\,\xi 1_{x2} - h12\,\xi 1_{x1} - h12\,\xi 2_{x2} - h22\,\xi 2_{x1},\,[h12]],$$
$$[-2\,\xi 1_{x2}\,h12 - 2\,h22\,\xi 2_{x2},\,[h22]]]$$

Invariant Differential Operators

In this particular example, it is very convenient to determine the invariant differential operators by geometry. Denote the invariants on $\mathcal{F}_{(2)}$ by (v, w). The contraction $w = v_i v_j g^{ij}$ of the indices of first order jets in v with the inverse of the metric yields an invariant. Analogously, this follows for $w_i\, w_j\, g^{ij}$, since we have again produced a scalar. From this we build two invariant differential operators by setting

$$\mathcal{D}_1 = v_i\, g^{ij}\, D_j, \qquad \mathcal{D}_2 = w_i\, g^{ij}\, D_j.$$

Applying them to v and w produces only two of the three invariants on $\mathcal{F}_{(3)}$, namely $v_i\, w_j\, g^{ij}$ and $w_i\, w_j\, g^{ij}$. However the above choice of coordinates h_{ij} of $\mathcal{F}_{(3)}$ makes it easy to compute the last invariant as $g^{ij} h_{ji}$:

```
> gg := matrix([[u11,u12],[u12,u22]]):
> hh := matrix([[h11,h12],[h12,h22]]):
> Inv31 := simplify(trace(inverse(gg)&*hh));
```

$$Inv31 := -\frac{-u22\,h11 + 2\,u12\,h12 - u11\,h22}{-u12^2 + u11\,u22}$$

We check if $g^{ij} w_{ij}$ is an invariant.

```
> simplify(ldjet(F3a["vec"],Inv31,ivar,F3a["uvar"],Dvar));
```

$$0$$

It follows that v and $g^{ij} h_{ij}$ are a generating set of invariants with respect to the invariant differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$.

5.2 Invariants for Lie Pseudogroup Actions on a Manifold

In this section, we treat the first example of a natural bundle $\mathcal{F}$, on which a subgroupoid $\Theta_1 \leq \Pi_1$ acts. The goal is to find a generating set for the invariants on $J_r(\mathcal{F})$ for all $r \in \mathbb{N}$ as presented in Section 4.2. The example is a simple yet instructive example from a series of papers by Olver and Pohjanpelto [OP07a], [OP07b], [OP] and [OP08].

Olver and Pohjanpelto primarily deal with a Lie pseudogroup action on a manifold M. A Lie pseudogroup $\mathcal{G}$ can be defined as the local solutions of a suitable jet groupoid $\mathcal{G}^{(q)} \leq \Pi_q(M \times M)$ (see [OP07b, Def. 3.1]). The pseudogroup $\mathcal{G}$ under consideration contains the transformations $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ of the following form:

$$y^1 = f(x^1), \quad y^2 = e(x^1, x^2) = f'(x^1)x^2 + g(x^1), \quad U = u + \frac{\partial_{x^1} e(x^1, x^2)}{f'(x^1)}.$$

It consists of the solutions of the groupoid $\mathcal{G}^{(1)} \subseteq \Pi_1(M,M)$ for $M = \mathbb{R}^3$ defined by the equations

$$y^1_{x^2} = y^1_u = 0, \quad y^2_{x^2} - y^1_{x^1} = 0, \quad y^2_u = 0, \quad y^2_{x^1} = (U-u)y^1_{x^1}, \quad U_u = 1.$$

The $\mathcal{G}$-action on M extends to an action on the jet bundle $J_r(M,n)$, which consists of equivalence classes of n-dimensional submanifolds of M under r-th order contact. In coordinates (x,u) of M, one can introduce a splitting of variables into (x) and (u) such that $\mathcal{G}$ acts on submanifolds which are parametrised by functions $u(x)$. The goal in [OP07b] is to find a generating set for the differential invariants $J_r(M,n) \to \mathbb{R}$ and to characterise the algebra of invariants. They apply the method of moving frames developed in [OP] for Lie pseudogroups.

We are interested in the computation of invariants using natural bundles. The extra value of this approach is that we can not only prolong the action to $J_r(M,n)$, but also project to lower orders again, which leads to smaller bundles. The Embedding Theorem 4.22 implies that we compute invariants on $J_r(M,n)$ on the bundles obtained.

The setting of Olver and Pohjanpelto can easily be translated into the language of natural bundles. Locally define the manifold X and the bundle

$$\pi : \mathcal{F} = M \to X : (x,u) \mapsto (x).$$

An n-dimensional submanifold of M now corresponds to a section ω of $\mathcal{F} \to X$. The groupoid $\mathcal{G}^{(1)}$ projects to the groupoid $\Theta_1 \subseteq \Pi_1(X \times X)$ defined by:

$$y^1_2 = 0, \qquad y^2_2 = y^1_1. \tag{5.1}$$

Now Θ_1 acts on $\mathcal{F}$ by setting

$$u = \hat{u} + \frac{y^2_1}{y^1_1} \tag{5.2}$$

and we regain the groupoid $\mathcal{G}^{(1)}$ as the action groupoid for the Θ_1-action on $\mathcal{F}$, which is constructed analogous to Example 2.3 (2).

Calculations in Maple

Load the packages and declare the variables:

```
>  with(jets):  with(JetGroupoids):  with(Spencer):
>  ivar := [x1,x2]:       dvar := [y1,y2]:
>  Ivar := vn(phi,2):  Dvar := vn(xi,2):  Tvar := vn(eta,2):
```

The Groupoid Θ_1

Define the jet groupoid Θ_1 over the base X according to equation (5.1):

```
>  Theta1 := [y1[x2]=0,y2[x2]=y1[x1]];
```

$$\Theta 1 := [y1_{x2} = 0,\ y2_{x2} = y1_{x1}]$$

Calculate the solutions of Θ_1 to ensure that we are dealing with the correct pseudogroup:

```
> jsolve(ind2eqn(Theta1,ivar,dvar),ivar,dvar,"");
```

$$[\mathrm{y1}(x1,\, x2) = \_\mathrm{F1}(x1),\, \mathrm{y2}(x1,\, x2) = (\tfrac{d}{dx1}\,\_\mathrm{F1}(x1))\, x2 + \_\mathrm{F2}(x1)]$$

We determine the equations for the algebroid R_1 of Θ_1. For later prolongations and projections, we also compute their first prolongation R_2.

```
> R1 := grp2alg(Theta1,ivar,dvar,Dvar);
> R2 := PrepareAlgebroidRelations(R1,2,ivar,Dvar):
```

$$R1 := [\xi 1_{x2} = 0,\, \xi 2_{x2} = \xi 1_{x1}]$$

The Natural Bundle $\mathcal{F} = M$

Define the natural Θ_1-bundle $\mathcal{F}$ by using equation (5.2):

```
> uvar := [u]:
> nat  := [x1=y1,x2=y2, u = u + y2[x1]/y1[x1]];
```

$$nat := [x1 = y1,\, x2 = y2,\, u = u + \frac{y2_{x1}}{y1_{x1}}]$$

Linearise to obtain the infinitesimal R_1-action and create the data structure for the natural Θ_1-bundle $\mathcal{F}$:

```
> vec:=natfin2inf(nat,ivar,dvar,Dvar,"");
> F:=CreateNaturalBundle(vec,ivar,dvar,uvar,Dvar, "algebroid"=R2):
```

$$vec := [[\xi 1,\, [x1]],\, [\xi 2,\, [x2]],\, [-\xi 2_{x1},\, [u]]]$$

Prolongation and Projection

To compute the invariants, Olver and Pohjanpelto prolong $\mathcal{F}$ to $J_3(\mathcal{F})$ and then apply the method of moving frames. This means dealing with fourth order differential equations. We first check the Spencer cohomology to realise that the symbols of generic sections on $\mathcal{F}$ are not 2-acyclic and we have to deal with second order equations to compute the invariants.

```
> IZS := `InvolutiveZeroSets/homalg`:
> Sc0 := SpencerCohomology(F,ivar,Dvar,Tvar,IZS):
> SCohomDim(Sc0,Tvar,IZS);
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The computation is valid for all sections of $\mathcal{F}$.

```
> SCZeroSets(Sc0);
```

$$[\,]$$

Due to the non-2-acyclic symbol, we prolong twice before projecting down to second order equations.

```
> J2F := ProlongNaturalBundle(F,2,uu):
> J2F["SUBSvec"][2,1];
> F1  := ProjectNaturalBundle(J2F,v1):
```

$$[uu1 = u_{x1},\ uu2 = u_{x2}]$$

Compute the Vessiot structure equations and pull them back to $J_2(\mathcal{F})$. They show that there are only very few submanifolds which have symmetries at all, as the Θ_2-action on an open subset of $\mathcal{F}_{(1)}$ is free.

```
> VSE1 := VessiotStructureEquations(F1);
> PullbackToF(VSE1,F1);
```

$$VSE1 := [uu2_{x2} = 0]$$

$$[u_{x2,\,x2} = 0]$$

For a generic point $f \in F_{(1)}$ of the fibre of $\mathcal{F}_{(1)}$, we determine the dimensions of the following manifolds:

$$\text{orbit space on } F_{(1)}, \quad F_{(1)}, \quad \text{orbit } G_1 f, \quad G_1, \quad \mathrm{Stab}_{G_1}(f)$$

```
> CodimOfAction(F1);
```

$$0,\ 4,\ 4,\ 4,\ 0$$

Since the stabiliser of a generic point is zero-dimensional (last number), the Θ_2-action is free on an open subset of $\mathcal{F}_{(1)}$. So the invariant differential operators may be computed at this step, even if no invariants (first number) are present. Since an invariant coframe is a first order object, and the Embedding Theorem 4.22 provides the isomorphism of natural bundles

$$\varphi_1 : J_2(\mathcal{F})/K_2^3 \to \mathcal{F}_{(1)},$$

the pullback of an invariant coframe on $\mathcal{F}_{(1)}$ is invariant on $J_{r\geq 2}(\mathcal{F})$ as well. The computation is done by projecting $\mathcal{F}_{(1)}$ to first order and then applying the moving frames approach of [OP]. Note that the projection to first order is two-dimensional with coordinates (u, v^1).

```
> gP := [u=0, v1=1]:
> ID := InvariantDifferentialOperators(F1,gP,nat,Theta1,dvar);
```

$$ID := [[[\frac{1}{\sqrt{v1}},\ [x1]],\ [-\frac{u}{\sqrt{v1}},\ [x2]]],\ [[0,\ [x1]],\ [\frac{1}{\sqrt{v1}},\ [x2]]]]$$

The output is a list of two differential operators in `jets` notation. It is similar to the notation for vector fields explained in Section 5.1, except that partial derivatives are replaced by total derivatives. So the entry $[a, [b]]$ stands for aD_b. In the example, the invariant differential operators are:

$$\mathcal{D}_1 = \frac{1}{\sqrt{v^1}}(D_{x^1} - uD_{x^2}), \qquad \mathcal{D}_2 = \frac{1}{\sqrt{v^1}}D_{x^2}.$$

Their pullback to $J_2(\mathcal{F})$ yields the same operators as [OP07b, eq. (4.21)]:

```
> PullbackToF(ID,F1);
```

$$[[[\frac{1}{\sqrt{u_{x2,\,x2}}},\ [x1]],\ [-\frac{u}{\sqrt{u_{x2,\,x2}}},\ [x2]]],\ [[0,\ [x1]],\ [\frac{1}{\sqrt{u_{x2,\,x2}}},\ [x2]]]]$$

Generating set of Invariants

We prolong and project another time to obtain a generating set of invariants on the original natural bundle $\mathcal{F}$.

```
> J1F1 := ProlongNaturalBundle(F1,1,uuu):
> F2   := ProjectNaturalBundle(J1F1,v2):
> F2["inv"];
```

$$[v21 = v1_{x1},\ v22 = v1_{x2}]$$

The action on $\mathcal{F}_{(1)}$ was already free, so all new coordinates correspond to invariants. Since the number of invariants on $\mathcal{F}_{(2)}$ is equal to the dimension of the base manifold, we have obtained a generating set for $J_r(\mathcal{F})$.

```
> Inv2 := InvariantsOnNaturalBundle(F2);
```

$$Inv2 := [\frac{v22}{v1^{(3/2)}},\ \frac{v21 - 2\,v1\;uu2 - v22\,u}{v1^{(3/2)}}]$$

Using the Embedding Theorem 4.22 again, the invariants can be pulled back to $J_3(\mathcal{F})$, where they match exactly [OP07b, eq. (4.20)]:

```
> PullbackToF(Inv2,F2);
```

$$[\frac{u_{x2,\,x2,\,x2}}{u_{x2,\,x2}{}^{(3/2)}},\ -\frac{-u_{x1,\,x2,\,x2} + 2\,u_{x2,\,x2}\,u_{x2} + u_{x2,\,x2,\,x2}\,u}{u_{x2,\,x2}{}^{(3/2)}}]$$

Change the fibre coordinates of $\mathcal{F}_{(2)}$ to obtain a vector bundle. The Vessiot structure equations on $\mathcal{F}_{(2)}$ state that the symmetry groupoids are integrable if and only if the invariants are constant.

```
> F2a   := ChangeFibreCoordinates(F2,Inv2):
> VSE2a := VessiotStructureEquations(F2a);
```

$$VSE2a := [\frac{v1_{x2}}{v1^{(3/2)}} = \_C2,\ \frac{v1_{x1} - 2\,v1\;uu2 - v1_{x2}\,u}{v1^{(3/2)}} = \_C1]$$

To check the invariant differential operators, the invariants on the bundle $\mathcal{F}_{(3)}$ are computed. We obtain three new invariants, where the last one does not seem to be an invariant derivative of the generating set above.

```
> J1F2a := ProlongNaturalBundle(F2a,1,uuu):
> F3    := ProjectNaturalBundle(J1F2a,v3):
> Inv3  := InvariantsOnNaturalBundle(F3):
> subs(F3["inv"],Inv3);
```

$$[v21,\ v22,\ \frac{v21_{x2}}{\sqrt{v1}},\ \frac{v21_{x1} - u\,v21_{x2}}{\sqrt{v1}},\ \frac{v22_{x1} - u\,v21_{x1} + v21_{x2}\,u^2 + 2\,u\sqrt{v1}}{\sqrt{v1}}]$$

We verify that the last invariant is also obtained by invariant differentiation. Each of the invariant differential operators $\mathcal{D}_i$ is a map

$$\mathcal{D}_i : C^\infty(\mathcal{F}_{(2)}) \to C^\infty(J_1(\mathcal{F}_{(2)})).$$

The procedure `ProlongNaturalBundle` does not compute the full bundle $J_1(\mathcal{F}_{(2)})$ but the image of the embedding

$$\varphi_{2,1} : J_4(\mathcal{F})/K_2^5 \hookrightarrow J_1(\mathcal{F}_{(2)}),$$

since it is the minimal subbundle where all sections coming from $\mathcal{F}$ restrict to. See Section 4.3.1 for more details. The equations defining the subbundle $\mathrm{im}(\varphi_{2,1})$ include:

```
> F3["SUBSvec"][1,2,1];
```

$$v22_{x2} = v21_{x1} - u\, v21_{x2} - 2\sqrt{v1}$$

Applying $\mathcal{D}_1$ to $v22$ yields an invariant on $J_1(\mathcal{F}_{(2)})$ and with the help ot the above equation, we restrict it to $\mathrm{im}(\varphi_{2,1})$. The result coincides with the last invariant on $\mathcal{F}_{(3)}$.

```
> appmt(ID[1],v22,ivar,F3["uvar"]);
> subs(F3["SUBSvec"][1,2,1],%);
```

$$\frac{v22_{x1} - u\, v22_{x2}}{\sqrt{v1}}$$

$$\frac{v22_{x1} - u\,(v21_{x1} - u\, v21_{x2} - 2\sqrt{v1})}{\sqrt{v1}}$$

The approach via natural bundles improves the calculation of invariant differential operators and generating sets of invariants. Due to the Embedding Theorem, the invariant differential operators can be computed on a two-dimensional factor bundle on $\mathcal{F}_{(1)}$ and the generating set of invariants on $\mathcal{F}_{(2)}$ which has a six-dimensional fibre. The method of Olver and Pohjanpelto involves the ten-dimensional bundle $J_3(\mathcal{F})$. The possibilty to project the bundles to lower orders does not restrict the methods to compute the invariants. The reader is free to use the package `JetGroupoids` which integrates an involutive distribution or to apply the method of moving frames, since the Θ_2-action on $\mathcal{F}_{(2)}$ is small enough to be computed explicitly.

5.3 Second Order ODEs under Point Transformations

We continue Example 4.32 with MAPLE and compare it with Neut's approach via Cartan's equivalence method [Neu03, §2]. We slightly change the notation of Proposition 4.33 and Lemma 4.34. For brevity, the original Π_1-bundle will be called $\mathcal{F}_{\Pi_1}$, whereas we denote the restricted bundle by $\mathcal{F}$. The example is nearly small enough to proceed with $\mathcal{F}_{\Pi_1}$, but there are problems:

- The sections representing second order ODEs become non-generic on the first bundle of integrability conditions. All further prolongations and projections must be done with extreme care, since the simple coordinate change $(v^1, v^2, \ldots) \mapsto (u^i v^1 + v^2, v^2, \ldots)$ on the fibre of $\mathcal{F}_{(i)}$ may hide a nontrivial equation – if $i \in \{2,3,4\}$ and $v^2 = 0$ for second order ODEs. Another possibility is to restrict the bundles $\mathcal{F}_{(i)}$ with Proposition 4.33 is presented in Example 6.14.

- Due to non-2-acyclic symbols, we have to prolong twice to check integrability in the last step. The computation of all equivariant sections involves solving an inhomogenous linear PDE system with 350 equations. It was not possible to solve it with MAPLE 11. On the restricted bundle, the equivariant sections are easily computed.

Calculations in MAPLE

```
> with(JetGroupoids): with(jets): with(Spencer):
> ivar := [x,y,p]:    dvar := [X,Y,P]: divar := [dx, dy, dp]:
> Ivar := vn(phi,3):  Dvar := vn(xi,3):
```

Construct the Natural Bundle

Define the coframe $(\omega^1, \omega^2, \omega^3)$ containing the contact forms:

```
> w0 := [dx, dy - p*dx, dp - f(x,y,p)*dx];
```

$$w0 := [dx,\ dy - p\,dx,\ dp - \mathrm{f}(x,\ y,\ p)\,dx]$$

To obtain the bundle $\mathcal{F}_{\Pi_1} = P_1/G_1$, we define a groupoid with structure group G_1 and check if it coincides with the matrix in equation (4.10).

```
> GR := [X[x]=1, X[y]=0, X[p]=0, Y[x]=0, Y[p]=0, P[x]=0];
> Jac := matrix(3,3,jetcoor(1, ivar, dvar)):
> subs(GR,evalm(Jac));
```

$$GR := [X_x = 1,\ X_y = 0,\ X_p = 0,\ Y_x = 0,\ Y_p = 0,\ P_x = 0]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & Y_y & 0 \\ 0 & P_y & P_p \end{bmatrix}$$

We construct the map $\Phi : P_1 \to \mathcal{F}_{\Pi_1}$ for coordinates $(u^1, \ldots u^5, f)$ of $\mathcal{F}_{\Pi_1}$.

```
> T    := grp2alg(GR,ivar,dvar,Dvar,""):
> Phi  := invtarget(T,ivar,dvar,Dvar,""):
> uvar := [u1, u2, u3, u4, u5, f]:
> Phi  := ezip(uvar,Phi);
```

$$\Phi := [u1 = X_x,\ u2 = X_y,\ u3 = X_p,\ u4 = \frac{Y_p}{Y_y},\ u5 = \frac{Y_x}{Y_y},\ f = \frac{P_x\,Y_y - Y_x\,P_y}{Y_p\,P_y - Y_y\,P_p}]$$

The above coframe w is a section of P_1, and we determine the sections $\omega = \Phi(w)$ of $\mathcal{F}_{\Pi_1}$ representing second order odes. The bundle $\mathcal{F}_{\Pi_1}$ was chosen such that the last coordinate of it represents the differential equation $f(x, y, p)$.

```
> ww := map(a->map(b->coeff(a,b),divar),w0):
> ww := evalm(Jac) = matrix(ww);
> omega := subs(ww,Phi);
```

$$ww := \begin{bmatrix} X_x & X_y & X_p \\ Y_x & Y_y & Y_p \\ P_x & P_y & P_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ -\mathrm{f}(x,\ y,\ p) & 0 & 1 \end{bmatrix}$$

$$\omega := [u1 = 1,\ u2 = 0,\ u3 = 0,\ u4 = 0,\ u5 = -p,\ f = \mathrm{f}(x,\ y,\ p)]$$

Prepare the Π_1-action on $\mathcal{F}_{\Pi_1}$ for the invariant coframes. Due to the size, we surpress the output.

```
> nat := natfin(Phi,ivar,dvar,uvar,dvar,""):
> nat := eqn2ind(nat,ivar,dvar):
```

Compute the infinitesimal $J_1(T)$-action and set up the data structure for the natural bundle $\mathcal{F}_{\Pi_1}$:

```
> vec   := natinf(Phi,ivar,dvar,uvar,Dvar,"");
> F_Pi1 := CreateNaturalBundle(vec,ivar,dvar,uvar,Dvar):
```

$$vec := [[\xi 1, [x]], [\xi 2, [y]], [\xi 3, [p]], [-u1\,\xi 1_x - u2\,\xi 2_x - u3\,\xi 3_x, [u1]], [-u1\,\xi 1_y - u2\,\xi 2_y - u3\,\xi 3_y, [u2]], [-u1\,\xi 1_p - u2\,\xi 2_p - u3\,\xi 3_p, [u3]], [u4\,u5\,\xi 1_y + u4\,\xi 2_y + u4^2\,\xi 3_y - u5\,\xi 1_p - \xi 2_p - u4\,\xi 3_p, [u4]], [u5^2\,\xi 1_y + u5\,\xi 2_y + u5\,u4\,\xi 3_y - u5\,\xi 1_x - \xi 2_x - u4\,\xi 3_x, [u5]], [-\xi 1_x\,f + f\,(f\,u4 + u5)\,\xi 1_y - f^2\,\xi 1_p + \xi 3_x - \xi 3_y\,(f\,u4 + u5) + f\,\xi 3_p, [f]]]$$

Restrict the Groupoid and the Natural Bundle

The first five components of the section are the same for all second order odes and the map

$$\pi' : \mathcal{F}_{\Pi_1} \to \mathcal{F}' : (x, u^1, \ldots, u^5, f) \to (x, u^1, \ldots, u^5)$$

defines a new natural bundle $\mathcal{F}'$ where all sections ω defining second order odes are identical. To prove that π' is well-defined, we observe that the algebroid action on the first five coordinates does not depend on f:

```
> getinds(vec[1..-2],uvar,"");
```

$$[u1,\ u2,\ u3,\ u4,\ u5]$$

All sections for second order odes project to the section ω' on $\mathcal{F}'$, such that Proposition 4.33 is applicable:

```
> omega[1..-2];
```

$$[u1 = 1,\ u2 = 0,\ u3 = 0,\ u4 = 0,\ u5 = -p]$$

The equations for the symmetry algebroid $R_1(\omega')$ are not yet integrable. The command `PrepareAlgebroidRelations` computes a Janet basis for $R_1(\omega')$ and completes it to an integrable algebroid $\mathfrak{g}_{\Theta_q}$. Additionally, it chooses coordinates for $R_3(\omega')$ and computes all equations determining the subbundle $R_3(\omega') \subseteq J_3(T)$.

```
> GMF := inf2MF(vec,ivar,uvar,uvar,""):
> R1  := jsubs(omega,GMF[1..-2],ivar,uvar);
> R3 := PrepareAlgebroidRelations(R1,3,ivar,Dvar):
```

$$R1 := [\xi 1_x = 0,\ \xi 1_y = 0,\ \xi 1_p = 0,\ -p\,\xi 1_p + \xi 2_p = 0,\ -p^2\,\xi 1_y + p\,\xi 2_y - p\,\xi 1_x + \xi 2_x - \xi 3 = 0]$$

For the later calculation of invariant differential operators, also the integrable symmetry groupoid $\Theta_3 = \mathcal{R}_3(\omega')$ is necesssary:

```
> GLF := LieFormG(nat,ivar,dvar,Ivar,uvar)[1..-2]:
> Theta3 := PrepareGroupoidRelations(GLF,omega,3,ivar,dvar,uvar):
```

We restrict the infinitesimal $J_1(T)$-action to $\mathfrak{g}_{\Theta_q}$ according to Proposition 4.33 and then pull back $\mathcal{F}_{\Pi_1}$ to the one-dimensional bundle $\mathcal{F}$ according to Lemma 4.34. The infinitesimal action becomes quite simple:

```
> F := RestrictNaturalBundle(F_Pi1,omega[1..-2],uvar[-1..-1],
> "algebroid"=R3):
> F["vec"];
```

$$[[\xi 1,\ [x]],\ [\xi 2,\ [y]],\ [\xi 3,\ [p]],\ [\xi 3_x + p\,\xi 3_y + f\,\xi 3_p,\ [f]]]$$

The finite Θ_1-action on $\mathcal{F}$ is also short enough to be displayed:

```
> nat1 := [op(nat[1..3]),nat[-1]]:
> nat1 := subs(subs(ezip(ivar,dvar),omega[1..-2]),nat1):
> nat1 := subs(Theta3,nat1);
```

$$\mathit{nat1} := [x = X,\ y = Y,\ p = P,\ f = \frac{f\,P_p - P_x\,P_p - P\,P_y + (P - p\,P_p)\,P_y}{{P_p}^2}]$$

Prolongation and Projection

The first step of prolongation and projection yields a one-dimensional bundle $\mathcal{F}_{(1)} \to \mathcal{F}$, buth the Vessiot structure equations show that there cannot be integrable symmetry groupoids. Lemma 3.40 implies that $\mathcal{F}_{(1)}$ is no vector bundle.

```
> J1F := ProlongNaturalBundle(F,1,uu):
> F1  := ProjectNaturalBundle(J1F,v1):
> F1["inv"];
```

$$[\mathit{v1} = f_p]$$

```
> VessiotStructureEquations(F1);

Warning, system is inconsistent
```

It turns out that the symbols for generic sections on $\mathcal{F}_{(1)}$ are not 2-acyclic. Nevertheless we prolong only once to obtain the same case distinction as in [Neu03, §2.6.1]. If we would have prolonged twice directly, we would have missed this case distinction, which sorts out nongeneric sections in the Vessiot equivalence method (see Chapter 6 and especially Figure 6.2).

```
> J1F1 := ProlongNaturalBundle(F1,1,uu):
> F2   := ProjectNaturalBundle(J1F1,v2):
```

The only coordinate of $\mathcal{F}_{(2)}$ can be chosen as an invariant. If the invariant is constant for a choice of $f(x, y, p)$, the projection is surjective, but the symmetry groupoid is not necessarily integrable.

```
> Inv2 := InvariantsOnNaturalBundle(F2):
> F2a := ChangeFibreCoordinates(F2,Inv2):
> F2a["inv"];
> VSE2a :=VessiotStructureEquations(F2a,"");
```

$$[\mathit{v2} = -2\,f_y + \mathit{v1}_x + p\,\mathit{v1}_y + f\,\mathit{v1}_p - \frac{\mathit{v1}^2}{2}]$$

$$\mathit{VSE2a} := [\mathit{v2} = \_C1]$$

Pulling back the Vessiot structure equations to $J_2(\mathcal{F})$ yields exactly the same invariant as the torsion coefficient $T^1_{2,3}$ in [Neu03, eq. (2.59)].

```
> PullbackToF(VSE2a,F2a);
```

$$[-2\,f_y + f_{x,p} + p\,f_{y,p} + f\,f_{p,p} - \frac{1}{2}\,{f_p}^2 = \_C1]$$

The Case of Constant v^2:

First consider the case where $v^2 = c$ is constant. We restrict $\mathcal{F}_2$ to a bundle which is isomorphic to $\mathcal{F}_{(1)}$ and prolong twice. Before projecting again, we show the first order jets occurring as coordinates of the subbundle of $J_2(\mathcal{F}_{(2),z})$.

```
> F2z   := RestrictNaturalBundle(F2a,[v2=c]):
> J2F2z := ProlongNaturalBundle(F2z,2,w):
> J2F2z["SUBSvec"][2,1];
```

$$[w1 = f_x,\ w2 = f_y,\ w3 = v1_y,\ w4 = v1_p]$$

Again, there is only a single coordinate v^3 of $\mathcal{F}_{(3),z}$ and we compute the Vessiot structure equations. Using Theorem 6.10, we recover [Neu03, Thm 10], which states that a second order equation $y_{xx} = f(x, y, p = y_x)$ is equivalent to y_{xx} if and only if $v^2 = 0$ and $f_{ppp} = 0$:

```
> F3z := ProjectNaturalBundle(J2F2z,v3):
> VSE3z := VessiotStructureEquations(F3z);
> PullbackToF(VSE3z,F3z);
```

$$VSE3z := [w4_p = 0]$$

$$[f_{p,p,p} = 0]$$

The G_2-action on $\mathcal{F}_{(3),z}$ is locally free and the next step of prolongation and projection yields a generating set of invariants for the system of pdes on $J_2(\mathcal{F})$ defined by $v^2 = const$. In this case, Theorem 4.12 has to be used in the generalised version [KL06, Thm 16].

```
> CodimOfAction(F3z);
```

$$0,\ 7,\ 7,\ 7,\ 0$$

```
> J1F3z := ProlongNaturalBundle(F3z,1,uu):
> F4z   := ProjectNaturalBundle(J1F3z,v4):
> Inv4z := InvariantsOnNaturalBundle(F4z):
> F4za  := ChangeFibreCoordinates(F4z,Inv4z):
> F4za["inv"];
> VSE4z := VessiotStructureEquations(F4za,"");
```

$$[v41 = \frac{v3_p}{v3^{(3/2)}},\ v42 = \frac{1}{2}\,\frac{2\,v3_y + 2\,v3\,w4 + v3_p\,v1}{v3^{(3/2)}}]$$

$$VSE4z := [v41 = \_C1,\ v42 = \_C2]$$

Compute invariant differential operators $\mathcal{D}_i$ on $\mathcal{F}_{(3),z}$:

```
> gp := [f=0, v1=0, v3=1, P=0]:
> IDz := InvariantDifferentialOperators(F3z,gp,nat1,Theta3,dvar);
```

$$IDz := [[[1,\ [x]],\ [p,\ [y]],\ [f,\ [p]]],\ [[\frac{1}{\sqrt{v3}},\ [y]],\ [\frac{v1}{2\,\sqrt{v3}},\ [p]]],\ [[\frac{1}{\sqrt{v3}},\ [p]]]]$$

Applying $\mathcal{D}_1 = D_x + pD_y + fD_p$ to the first invariant $v41$ on $\mathcal{F}_{(3),z}$, we obtain $-v42$. It shows that the algebra of invariants is generated by $v41$.

```
> appmt(IDz[1], v41, ivar, F4za["uvar"]);
> PushToNB(%, F4za);
```

$$v41_x + p\,v41_y + f\,v41_p$$

$$-v4\mathit{2}$$

The Case of Nonconstant v^2:

In the case of a nonconstant invariant v^2, we prolong $\mathcal{F}_{(1)}$ twice. Using the usual prolongation procedure, the coordinate v^2 would be replaced by v^1_x. We skip the computations containing the coordinate change.

```
> J2F1 := ProlongNaturalBundle(F1,2,w):
> J2F1["SUBSvec"][2,1];
```

$$[v2 = -2\, f_y + v1_x + p\, v1_y + f\, v1_p - \frac{v1^2}{2},\ w1 = f_x,\ w2 = f_y,\ w3 = v1_y,\ w4 = v1_p]$$

Project down and compute the Vessiot structure equations:

```
> F3 := ProjectNaturalBundle(J2F1,v3,kernelD):
> F3["inv"];
> VSE3 := VessiotStructureEquations(F3);
```

$$[v31 = v2_x,\ v32 = v2_y,\ v33 = v2_p,\ v34 = w4_p]$$

$$VSE3 := [v2_x = \_F5(v2),\ v2_y = 0,\ v2_p = 0,\ w4_p = 0]$$

The invariants on $\mathcal{F}_{(3)}$:

```
> Inv3 := InvariantsOnNaturalBundle(F3);
```

$$Inv3 := [v2,\ v31 + v32\, p + f\, v33,\ \frac{v33\, v1 + 2\, v32}{2\,\sqrt{v34}},\ \frac{v33}{\sqrt{v34}}]$$

The invariant differential operators for the case of $v^2 \neq 0$ coincides with the one computed before, up to renaming the variables.

```
> gP := [f=0, v1=0, v34=1, P=0]:
> ID := InvariantDifferentialOperators(F3,gP,nat1,Theta3,dvar);
```

$$ID := [[[1,\ [x]],\ [p,\ [y]],\ [f,\ [p]]],\ [[\frac{1}{\sqrt{v34}},\ [y]],\ [\frac{v1}{2\,\sqrt{v34}},\ [p]]],\ [[\frac{1}{\sqrt{v34}},\ [p]]]]$$

It is easy to check that the invariants on $\mathcal{F}_{(3)}$ are v^2 and $\mathcal{D}_i v^2$. Another prolongation and projection yields eight new invariants, of which six are invariant derivatives from those on $\mathcal{F}_{(3)}$. The pullback of the remaining two invariants coincides with the case $v^2 = const$. Here v^2 and

$$\frac{v34_p}{v34^{\frac{3}{2}}} = \frac{f_{pppp}{}^{\frac{3}{2}}}{f_{ppp}}$$

generate the algebra of invariants. They are not directly comparable to [Neu03, §2.10.1] since the invariants there were computed on a prolonged G-structure, such that the base manifold is no longer X.

Chapter 6

The Vessiot Equivalence Method

> ‘The goal of the method of equivalence is to find necessary and sufficient conditions in order that “geometric objects” be “equivalent”. The word *equivalent* here usually ends up meaning that the geometric objects are mapped onto each other by a class of diffeomorphisms characterized as the set of solutions of a system of differential equations.’
>
> — R. Gardner [Gar89, Lecture 1]

Intended for Cartan’s successful equivalence method, this introduction exactly describes the goal of the Vessiot equivalence method to be developed in this chapter. Both methods are closely connected and for comparison, also Cartan’s method is introduced. In Vessiot’s context, a geometric object has a well-defined meaning as a section of a natural Θ_q-bundle $\mathcal{F} \to X$. The jet groupoid Θ_q specifies the class of diffeomorphisms to be used. Reading Vessiot’s texts shows that he also had the question of equivalence in mind.

> ‘Les systèmes [...] s’offrent d’eux-mêmes quand on cherche à reconnaître si deux groupes (G) et (G') donnés par leurs équations de définition, sont semblantes, et à déterminer les transformations qui changent ces deux groupes l’un par l’autre.’
>
> — E. Vessiot [Ves03, §IX]

Here (G) and (G') stand for the symmetry groupoids $\mathcal{R}_q(\omega)$ and $\mathcal{R}_q(\omega')$ of two geometric objects. For simplicity, Vessiot restricts to transitive groupoids. A modernised version of Vessiot’s equivalence problem has been formulated by Pommaret [Pom78, §7.5], again for the transitive case only. However it is not solved: “As the study of such a problem is out of our scope [...]” [Pom78, above Ex. 7.5.3]. In a later book, the introduction to the equivalence problem reads as follows.

> ‘The purpose of this part is to give a complete treatment of the formal aspect of the famous equivalence problem stated by Cartan at the beginning of this century. The solution we give of this problem is so simple that it is difficult to understand why it was never given before. [...] because otherwise the mathematical content can be found almost completely in [Pom83, Ch. 2] and even in [Pom78].’
>
> — J. F. Pommaret [Pom83, Ch. 4.C]

Based on the work of Pommaret and Vessiot, the Vessiot equivalence method will be presented in this chapter. As it depends on the Projection Theorem 3.35, the results are new and cover more cases than Pommaret [Pom83]. The Vessiot equivalence method is presented in detail in Section 6.1. An introduction of Cartan's equivalence method and a comparison of both methods follows in Section 6.2. Finally, examples are presented in Section 6.3.

6.1 The Equivalence Problem

In this section, necessary and sufficient conditions for the equivalence of geometric objects on natural bundles are given and the Vessiot equivalence method is developed. The basic idea is to modify Vessiot's approach for the symmetries of geometric objects such that equivalence can be decided. This section is based on Chapters 3 and 4. The criteria for equivalence are new contributions in this thesis as they do not occur in the work of Pommaret or Vessiot ([Pom78], [Pom83] and [Ves03]).

At first we consider the equivalence of geometric objects under all diffeomorphisms in Section 6.1.1, which corresponds to Π_q-bundles. We use natural bundle functors for an efficient formulation. A typical example is the equivalence of metrics, where the calculations in Section 5.1 only have to be interpreted properly.

Based on these results, it is possible to treat the relative equivalence problem in Section 6.1.2. Here geometric objects are compared under a certain subset of all diffeomorphisms and it is possible to work with natural Θ_q-bundles. Typical applications are ODEs under point or contact transformations as presented in Sections 5.3 and 6.3.

As the Vessiot equivalence method takes same steps for both the full and the relative problem, we will summarise the practical work with the Vessiot equivalence method in Section 6.1.3.

6.1.1 The Full Equivalence Problem

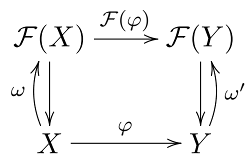
We start with an approach to the equivalence problem that uses natural bundle functors defined in Section 3.2 (see [KMS93]). It allows to compare geometric objects over different base manifolds X and Y under all diffeomorphisms $\varphi : X \to Y$. In Section 6.2 we show that this definition coincides with the definition of equivalence in the context of Cartan's equivalence method (see e.g. [Gar89]).

Definition 6.1. Let $\mathcal{F}$ be a natural bundle functor and X, Y be two n-dimensional manifolds. Two geometric objects ω on $\mathcal{F}(X)$ and ω' on $\mathcal{F}(Y)$ are *equivalent* if there exists a diffeomorphism $\varphi : X \to Y$ such that:

$$\mathcal{F}(\varphi^{-1})(\varphi^*(\omega')) = \omega. \tag{6.1}$$

Two objects ω and ω' are *locally equivalent*, if there are open neighbourhoods $U \subseteq X$ and $V \subseteq Y$ and a local diffeomorphism $\varphi : U \to V$ satisfying equation (6.1). In both cases φ is called a (local) *equivalence* between ω and ω'. ◇

The following commutative diagram illustrates the equivalence condition (6.1).



We follow Vessiot's suggestion [Ves03, §9.1] for the obvious way to check equivalence between ω and ω' and modify the symmetry equations (3.11) for ω.

Assume for simplification that $X = Y$. Instead of asking for the symmetry groupoid $\mathcal{R}_q(\omega)$ under the Π_q-action on $\mathcal{F}(X)$, we reformulate the equivalence condition (6.1) for $f_q \in \Pi_q(x, y)$ as

$$\omega'(y) f_q = \omega(x). \tag{6.2}$$

Dropping the assumption $X = Y$, we need a right multiplication with elements of the bundle $\Pi_q(X, Y) \subseteq J_q(X \times Y)$ of q-jets of diffeomorphisms between X and Y. It is obtained by modifying the associated maps from Lemma 3.11:

$$\mathcal{F}(Y) \times_Y \Pi_q(X, Y) \to \mathcal{F}(X) : (u, f_q = j_q(\varphi)(x)) \mapsto u f_q = \mathcal{F}(\varphi^{-1})(u).$$

Interpreting (6.2) for elements $f_q \in \Pi_q(X, Y)$, it determines a system of PDEs $\mathcal{S}(\omega, \omega')$ on $\Pi_q(X, Y)$. Analogous to the sequence (3.5) for $\mathcal{R}_q(\omega)$, the system $\mathcal{S}_q(\omega, \omega')$ is defined by the sequence

$$0 \longrightarrow \mathcal{S}_q(\omega, \omega') \longrightarrow \Pi_q(X, Y) \underset{\omega \circ s}{\overset{\Phi_{\omega'}}{\rightrightarrows}} \mathcal{F}(X). \tag{6.3}$$

Setting the source and target map as the projections $s = \mathrm{pr}_X$ and $t = \mathrm{pr}_Y$, the map $\Phi_{\omega'}$ is:

$$\Phi_{\omega'} : \Pi_q(X, Y) \to \mathcal{F}(X) : f_q \mapsto \omega'(y) f_q, \qquad y = t(f_q).$$

The condition $\Phi_{\omega'}(f_q) = \omega(s(f_q))$ is identical to the equivalence condition (6.1) by the definition of the right multiplication and the fact that $y = t(f_q) = \varphi(x)$ for $f_q = j_q(\varphi)(x)$. In the case of $\omega = \omega'$ we recover the symmetry equations (3.11).

Formal Equivalence Criterion

Just as formal integrability implies the existence of a formal power series solution, we call geometric objects *formally equivalent* if $\mathcal{S}_q(\omega, \omega')$ has a formal power series solution. If it converges, we have established local equivalence between the objects. The following criterion tests integrability of $\mathcal{S}_q(\omega, \omega')$ in the first condition and then checks if there are solutions with the second condition.

Theorem 6.2. Two geometric objects ω on $\mathcal{F}(X)$ and ω' on $\mathcal{F}(Y)$ are formally equivalent if $\omega : X \to \mathcal{F}(X)$ and $\omega' : Y \to \mathcal{F}(Y)$ are generic sections and the following conditions hold:

(1) Both symmetry groupoids $\mathcal{R}_q(\omega)$ and $\mathcal{R}_q(\omega')$ have 2-acyclic symbols and are formally integrable with the same equivariant section $c : \mathcal{F} \to \mathcal{F}_{(1)}$. Here $\mathcal{F}_{(1)}$ is constructed according to Proposition 3.30.

(2) There exist points $x \in X$ and $y \in Y$ such that all invariants on $\mathcal{F}$ coincide

$$\psi^i(\omega(x)) = \psi^i(\omega(y)), \qquad i = 1, \ldots, k. \qquad \diamond$$

Here, the invariants and the equivariant section are considered as zero order natural operators according to Definition 3.13. Using Theorem 3.15, we can speak of invariants on $\mathcal{F}$ and the *same* equivariant section on both $\mathcal{F}_{(1)}(X) \to \mathcal{F}(X)$ and $\mathcal{F}_{(1)}(Y) \to \mathcal{F}(Y)$, because it is completely determined by the GL_q-equivariant map on the fibres

$$F \to F_{(1)} = F^{(1)}/K_q^{q+1}.$$

Analogously, the invariants ψ are determined by GL_q-equivariant maps $F \to \mathbb{R}$.

Proof. Since ω and ω' are generic, there is a coordinate system of $\mathcal{F}(X)_x$ around $\omega(x)$ where the first k coordinates are the invariants ψ^i and the GL_q-action on the remaining coordinates is transitive on a neighbourhood of $\omega(x)$. The analogous statement is valid for $\mathcal{F}(Y)_y$ around $\omega(y)$.

Condition (2) implies that there exists an $f_q \in \Pi_q(X, Y)$ with

$$\omega'(y) f_q = \omega(x). \tag{6.4}$$

The integrability conditions for $\mathcal{R}_q(\omega)$ and $\mathcal{R}_q(\omega')$ together with the equivariance of c lead to:

$$\begin{aligned} I(j_1(\omega')(y)) f_q &= c(\omega'(y)) f_q \\ &= c(\omega'(y) f_q) \\ &= c(\omega(x)) \\ &= I(j_1(\omega)(x)). \end{aligned}$$

Because $I(j_1(\omega')(y)) = j_1(\omega')(y) K_q^{q+1}$, it follows that there exists an element $f_{q+1} \in \Pi_{q+1}(X, Y)$ with $\pi_q^{q+1}(f_{q+1}) = f_q$ that satisfies

$$j_1(\omega')(y) f_{q+1} = j_1(\omega)(x). \tag{6.5}$$

The groupoid $\mathcal{R}_q(\omega')$ has a 2-acyclic symbol, so $\mathcal{S}_q(\omega, \omega')$ defined by equation (6.4) also has a 2-acyclic symbol (in coordinates, only the right hand sides of equation (6.4) change). Then formal integrability follows from Equation (6.5). □

Remark 6.3. The condition that ω and ω' are generic sections ensures that both conditions of Theorem 6.2 can decide equivalence properly. We construct a simple example where this condition is violated. Choose $\mathcal{F} = T^* \times_X T^*$ and two sets of 1-forms as sections of $\mathcal{F} \to X$:

$$\omega^1 = dx^1, \;\; \omega^2 = dx^1, \qquad \tilde{\omega}^1 = dx^1, \;\; \tilde{\omega}^2 = dx^2.$$

The Vessiot structure equations on $\mathcal{F}_{(1)} = \mathcal{F} \times_X \bigwedge^2 T^* \times_X \bigwedge^2 T^*$ are

$$d\omega^1 = C_1 \, \omega^1 \wedge \omega^2, \qquad d\omega^2 = C_2 \, \omega^1 \wedge \omega^2.$$

Both symmetry groupoids are integrable with the equivariant section $\mathcal{F} \to \mathcal{F}_{(1)}$ corresponding to $C_1 = C_2 = 0$. However, the sets of 1-forms cannot be equivalent, since (ω^1, ω^2) restricts to the diagonal subbundle T^* of $\mathcal{F} \to X$, while $(\tilde{\omega}^1, \tilde{\omega}^2)$ is generic. ◇

Remark 6.4. In examples with invariants on $\mathcal{F}$, it is sometimes possible to choose different equivariant sections to decide the integrability of $\mathcal{R}_q(\omega)$. The condition for equivalence is satisfied, if we can find an equivariant section $c : \mathcal{F} \to \mathcal{F}_{(1)}$ that works for both ω and ω'. ◇

Example 6.5. To demonstrate the use of Theorem 6.2, we take up the example of a Riemannian metric on a two-dimensional base from Section 5.1. The action on the bundle $J_1(\mathcal{F}_g)$ of metrics and Christoffel symbols is transitive, so condition (2) of Theorem 6.2 is trivially satisfied. Two metrics (with corresponding Christoffel symbols) are equivalent if they have the same constant scalar curvature, since this is the only integrability condition on the minimal bundle $\mathcal{F}'_{(1)} \cong J_2(\mathcal{F}_g)/K_2^3$.

If the scalar curvature is constant, but different for two metrics, they cannot be equivalent. If the curvature is nonconstant, Theorem 6.2 does not say anything about equivalence and we have to prolong and project. ◇

The Nonintegrable Situation

Theorem 6.2 assumes that both symmetry groupoids are formally integrable. In general, this is not the case and we have to complete them to integrability before equivalence can be checked. Since the natural bundle functors are completely determined by the fibres (see Theorem 3.12), we can translate each bundle of a series of prolongations and projections (see Section 4.1) into natural bundle functors $\mathcal{F}_{(i)}$. The next proposition states that the equivalence of ω and ω' can be tested on $\mathcal{F}_{(i)}$. The complete Vessiot equivalence method will follow in Section 6.1.3.

Proposition 6.6. Two geometric objects ω on $\mathcal{F}(X)$ and ω' on $\mathcal{F}(Y)$ are equivalent if and only if one of the following conditions hold:

(1) $j_r(\omega)$ on $(J_r \circ \mathcal{F})(X)$ and $j_r(\omega')$ on $(J_r \circ \mathcal{F})(Y)$ are equivalent.

(2) $j_r(\omega)K^{q+r}_{q+s}$ on $\mathcal{F}'(X)$ and $j_r(\omega')K^{q+r}_{q+s}$ on $\mathcal{F}'(Y)$ for $0 \leq s < r$ are equivalent. Here $\mathcal{F}'$ is defined by the fibre $F^{(r)}/K^{q+r}_{q+s}$ where F is the fibre of the natural bundle functor $\mathcal{F}$ of order q. ⋄

Proof. We Proceed analogous to the prolongation and projection of symmetry groupoids. Applying J_r to Equation (6.3) and using the canonical embedding, we obtain:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{S}_{q+r}(\omega,\omega') & \longrightarrow & \Pi_{q+r}(X,Y) & \rightrightarrows & (J_r \circ \mathcal{F})(X) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathcal{S}^{(r-s)}_{q+s}(\omega,\omega') & \longrightarrow & \Pi_{q+s}(X,Y) & \rightrightarrows & \mathcal{F}'(X)
\end{array}$$

A diffeomorphism $\varphi : X \to Y$ solves Equation (6.3) if and only if it is a solution of $\mathcal{S}_{q+r}(\omega,\omega')$ and of $\mathcal{S}^{(r-s)}_{q+s}(\omega,\omega')$ for all $0 \leq s < r$. □

6.1.2 The Relative Equivalence Problem

In many situations it is convenient to restrict the possible diffeomorphisms for equivalences and to work with a relative situation. Typical examples deal with the equivalence of

- hypersurfaces on a manifold X under isometries,
- differential equations under contact or point transformations or
- linear partial differential operators under gauge transformations.

The first task is classical and leads to the curvature in the case of surfaces on $\mathbb{R}^3$ (see e.g. [IL03, Ch. 1]). Differential equations were one of the main reasons for Lie and Cartan to consider equivalence problems. We have already seen an example in Section 5.3 and more examples follow in Section 6.3. In Chapter 7 we deal with linear partial differential operators and their factorisation.

The solution to the relative equivalence problem does not differ too much from the functorial approach, so we keep the proofs as short as possible. We start with a functorial definition of the relative situation.

Definition 6.7. Let $\Omega \in \Gamma(\mathcal{G}(X))$ and $\Omega' \in \Gamma(\mathcal{G}(Y))$ be equivalent geometric objects for a natural bundle functor $\mathcal{G}$. Let $\mathcal{F}$ be another natural bundle functor.

Two geometric objects $\omega \in \Gamma(\mathcal{F}(X))$, $\omega' \in \Gamma(\mathcal{F}(Y))$ are *equivalent relative to* Ω *and* Ω' if and only if there exists a diffeomorphism $\varphi : X \to Y$ such that:

$$\mathcal{F}(\varphi^{-1})(\varphi^*(\omega')) = \omega \qquad \text{and} \qquad \mathcal{G}(\varphi^{-1})(\varphi^*(\Omega')) = \Omega. \tag{6.6}$$

Two objects ω and ω' are *locally equivalent relative to* Ω *and* Ω', if there are open neighbourhoods $U \subseteq X$ and $V \subseteq Y$ and a local diffeomorphism $\varphi : U \to V$ satisfying (6.6). ◇

If F and G are the fibres determining $\mathcal{F}$ and $\mathcal{G}$, the relative problem can be solved by constructing the functor $\mathcal{H}$ with fibre $F \times G$. It reduces the relative problem to the equivalence of (ω, Ω) and (ω', Ω').

With the following definition, we can give a more efficient solution for the relative equivalence problem. Additionally, it allows us to treat geometric objects on which only a subgroupoid $\Theta_q \leq \Pi_q$ acts. Examples are found in Section 5.2 and Chapter 7.

Definition 6.8. Let $\mathcal{F}$ be a natural Θ_q-bundle. Then two geometric objects ω, ω' on $\mathcal{F}$ are (locally) *equivalent under (solutions of)* Θ_q if there exists a (local) solution φ of Θ_q such that

$$\varphi^{-1}(\varphi^*(\omega')) = \omega. \tag{6.7}$$

◇

The equivalence of geometric objects on natural Θ_q-bundles is an efficient alternative to relative equivalence problems. The next proposition shows how to proceed.

Proposition 6.9. Using the notation of Definition 6.7, let $\varphi : X \to Y$ be an equivalence between Ω and Ω'. Assume that the symmetry groupoid $\Theta_q = \mathcal{R}_q(\Omega)$ is integrable. Then ω and ω' are equivalent relative to Ω and Ω' if and only if the geometric objects

$$\omega \quad \text{and} \quad \tilde{\omega}' = \varphi^{-1}(\varphi^*(\omega))$$

on the Θ_q-bundle $\mathcal{F}(X)$ are equivalent under Θ_q. ◇

Proof. Let ψ be a solution of Θ_q and an equivalence between ω and $\tilde{\omega}'$. By construction, $\varphi \circ \psi$ is an equivalence between Ω and Ω'. Additionally, we have

$$\begin{aligned} \psi^{-1}(\psi^*(\tilde{\omega}')) &= \psi^{-1}\left(\psi^*\left[\varphi^{-1}(\varphi^*(\omega'))\right]\right) \\ &= \psi^{-1}(\varphi^{-1}\left[\psi^*(\varphi^*(\omega'))\right]) \\ &= (\varphi \circ \psi)^{-1}\left[(\varphi \circ \psi)^*(\omega')\right]. \end{aligned}$$

The converse direction follows with an analogous computation for an equivalence χ between (ω, Ω) and (ω', Ω'). □

In many examples of relative equivalence problems the geometric objects Ω and Ω' are identical, which reduces the equivalence φ to the identity map. These examples can be simplified with Proposition 6.9 right away.

Equivalence Criteria

Again, we can construct a system of PDEs $\mathcal{S}_q(\omega, \omega')$ for the equivalences, which is defined by the exact sequence

$$0 \longrightarrow \mathcal{S}_q(\omega, \omega') \longrightarrow \Theta_q \underset{\omega \circ s}{\overset{\Phi_{\omega'}}{\rightrightarrows}} \mathcal{F}. \tag{6.8}$$

The check of formal equivalence is analogous to Theorem 6.2. The proofs are identical except that we have to trivialise $\mathcal{F}$ around $(x, \omega(x))$, since there might be invariants on the base, if Θ_q is intransitive.

Theorem 6.10. Two geometric objects ω, ω' on the natural Θ_q-bundle $\mathcal{F} \to X$ are formally equivalent if ω, ω' are generic sections and the following conditions hold:

(1) Both symmetry groupoids $\mathcal{R}_q(\omega)$ and $\mathcal{R}_q(\omega')$ have 2-acyclic symbols and are formally integrable with the same equivariant section $c : \mathcal{F} \to \mathcal{F}_{(1)}$. The bundle $\mathcal{F}_{(1)}$ is constructed according to Proposition 3.30.

(2) There exist points $x, y \in X$ such that all invariants on $\mathcal{F}$ coincide

$$\psi^i(\omega(x)) = \psi^i(\omega(y)), \qquad i = 1, \dots, k.$$ ◇

It is also possible to treat geometric objects with non-integrable symmetry groupoids and the proof is a direct translation from Proposition 6.6.

Proposition 6.11. Two geometric objects ω, ω' on the natural Θ_q-bundle $\mathcal{F}$ are equivalent if and only if one of the following conditions hold:

(1) $j_r(\omega)$ and $j_r(\omega')$ on $J_r(\mathcal{F})$ are equivalent.

(2) $j_r(\omega)K^{q+r}_{q+s}$ and $j_r(\omega')K^{q+r}_{q+s}$ on $\mathcal{F}'$ for $0 \leq s < r$ are equivalent. Here $\mathcal{F}'$ is defined by the fibre $F^{(r)}/K^{q+r}_{q+s}$ where F is the fibre of $\mathcal{F}$. ◇

6.1.3 The Vessiot Equivalence Method in Practice

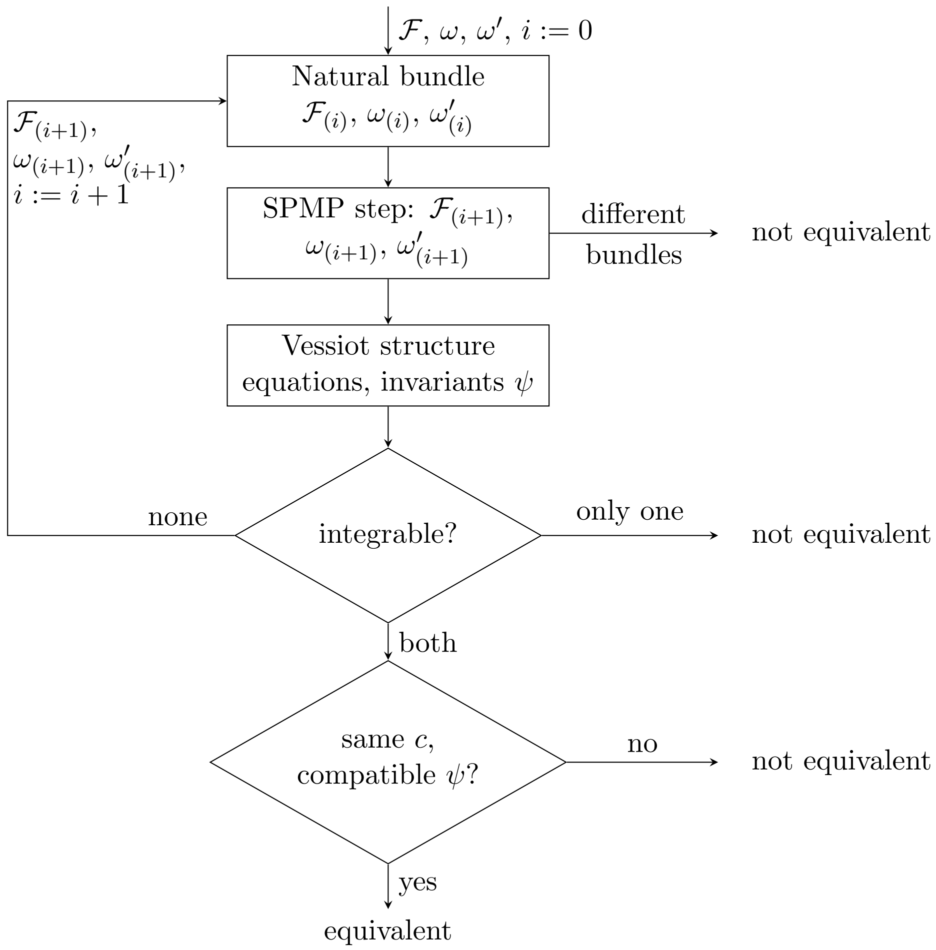
In this section, the equivalence criteria from the previous two sections are combined to develop the Vessiot equivalence method that decides formal equivalence for two geometric objects ω and ω'. The method is identical for the full and the relative equivalence problem and we will formulate it with natural bundles. The translation into the functorial language is immediate.

The key idea is to use Proposition 6.6 (or 6.11) and prolong and project until the symmetry groupoids become integrable. Then the Equivalence Theorem 6.2 (or 6.10) decides whether the objects are equivalent or not. We will again formulate the procedure with flowcharts, as in Chapter 4.

Figure 6.1 shows how to decide equivalence of two geometric objects ω, ω'. It is the direct analogue to Figure 6.3 for Cartan's method (see [Gar89, Fig. 6]). We perform the following steps (for the notation see Section 4.3.1).

- In the SPMP step compute the Spencer cohomology, Prolongation, Minimal bundles and Projection (Details are given in Figure 6.2). If the objects restrict to different subbundles of $\mathcal{F}_{(i+1)} \to \mathcal{F}_{(i)}$, they cannot be equivalent.
- Compute the Vessiot structure equations on $\mathcal{F}_{(i+1)}$ and the invariants on $\mathrm{im}(\varphi_{i,s_{i+1}-1})$, where $\varphi_{i,s_{i+1}}$ is defined in Equation (4.2).
- Check if the symmetry groupoids of $j_{s_{i+1}-1}(\omega_{(i)})$ and $j_{s_{i+1}-1}(\omega'_{(i)})$ are integrable. If only one is integrable, they cannot be equivalent. If both are nonintegrable, proceed with another loop.
- If $j_{s_{i+1}-1}(\omega_{(i)})$ and $j_{s_{i+1}-1}(\omega'_{(i)})$ are integrable and satisfy the conditions of the Equivalence Theorem 6.2 (or 6.10), they are equivalent, otherwise not.

Figure 6.1: The Vessiot equivalence method



Remark 6.12. The Vessiot equivalence method combines the completion of symmetry groupoids to formal integrability from Section 4.1 with the computation of a generating set of invariants from Section 4.2. If the groupoid action on the bundle $\mathcal{F}_{(i)}$ becomes free, but the equivalence could not yet be tested, we recommend to use invariant differential operators to compute the prolongation and projection and Remark 4.2 for the Vessiot structure equations. This is more efficient than computing further prolongations and projections with natural bundles. ◇

The SPMP Step

The process of computing the Spencer cohomology, the minimal subbundle of the prolongation $J_s(\mathcal{F}_{(i)})$ and the projection is nearly identical to the classification of symmetry groupoids in Section 4.1 and Figure 4.2. The only difference is that we restrict the minimal bundle $\operatorname{im}(\varphi_{i,s})$ to the smallest subbundle containing the geometric objects to compare. Due to the similarity of Figures 4.2 and 6.2, we only comment the new steps of Figure 6.2.

- To check if the sections $j_{s_{i+1}}(\omega_{(i)})$ and $j_{s_{i+1}}(\omega'_{(i)})$ are generic on the minimal bundle $\operatorname{im}(\varphi_{i,s_{i+1}})$, we compute the ranks of the algebroid action at a generic point of $\operatorname{im}(\varphi_{i,s_{i+1}})$ and for the prolonged sections at a generic point of X.

  If all three coicide, we take the left turn an proceed with the projection. If only one section is generic, the geometric objects cannot be equivalent, since equivalence implies isomorphic symmetry groupoids. If the ranks for both sections are different, the symmetry groupoids cannot be isomorphic.

- The ranks calculated above give the dimensions of the subbundles to which the sections restrict. Compute them (see Example 6.14) and check if they are identical. If not, the above discussion on symmetry groupoids implies that the symbols cannot be equivalent.

- Proceed with the subbundle of $\operatorname{im}(\varphi_{i,s_{i+1}})$ and project to $\mathcal{F}_{(i+1)}$.

Remark 6.13. The computation of subbundles for nongeneric sections is a hard step, because it depends on a suitable choice of coordinates to identify subbundles.

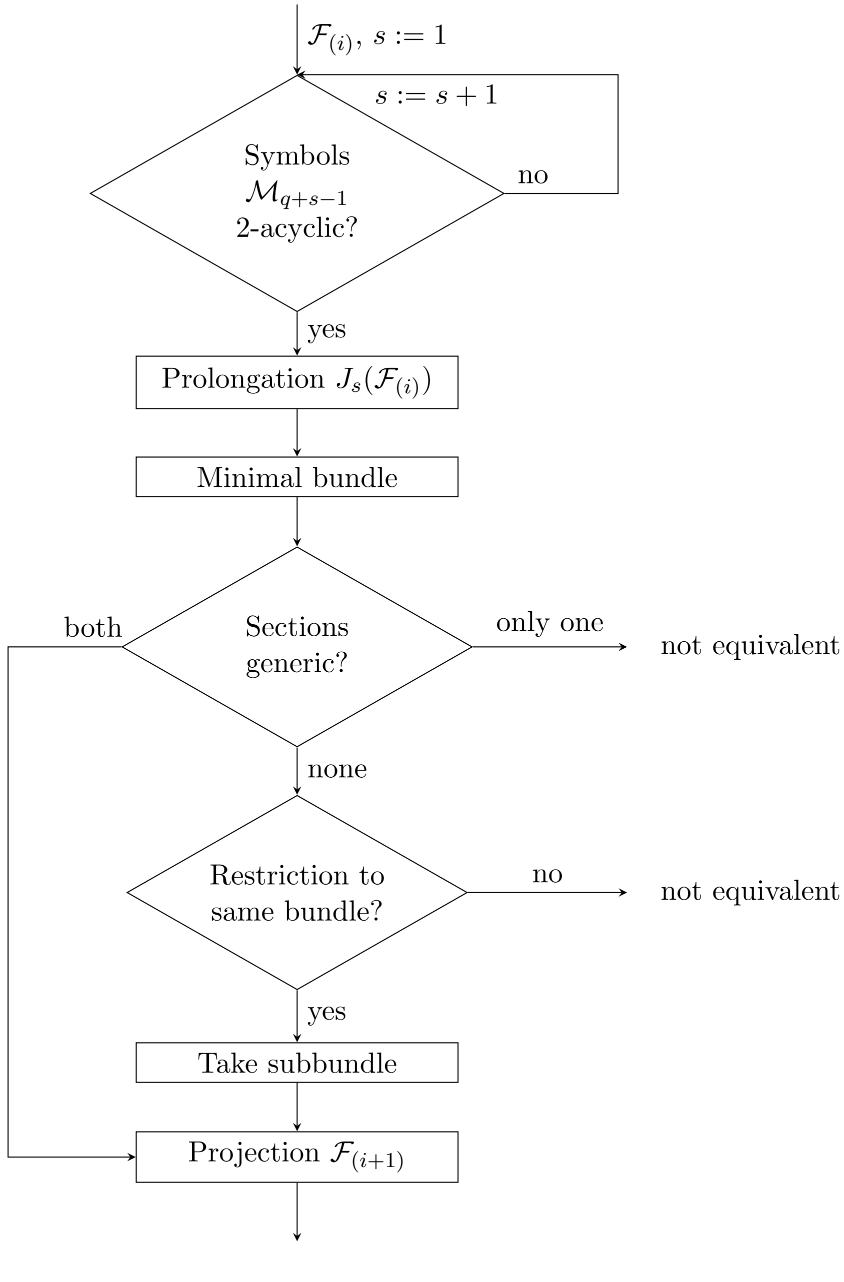
In the example of second order ODE under point transformations in Section 5.3, the prolongation and projection without 2-acyclic symbol reveals possible nongeneric sections. ◇

Here is one example where a subbundle for nongeneric sections can be found.

Example 6.14. The first steps in Example 4.32 on second order ODEs can be computed with Π_1-bundles, but all sections for ODEs become nongeneric. Instead of testing for subbundles on $J_1(\mathcal{F}_{\Pi_1})$, we compute the first bundle of integrability conditions $\mathcal{F}_{(1)} = J_1(\mathcal{F}_{\Pi_1})/K_1^2$ and check here. The fibre coordinates are:

$$v^1 = u^2_x - u^1_y, \quad v^2 = u^3_x - u^1_p, \quad v^3 = u^3_y - u^2_p, \quad v^4 = u^5_p - u^4_x + u^5 u^4_y - u^4 u^5_y.$$

Figure 6.2: SPMP-step for the Vessiot equivalence method



Plugging in the section from Equation (4.11), we obtain

$$v^1 = v^2 = v^3 = 0, \quad v^4 = -1. \tag{6.9}$$

On the ten-dimensional fibre $F_{(1)}$, a generic orbit has dimension nine, but all sections from equation (4.11) restrict to a seven-dimensional subbundle of $\mathcal{F}_{(1)} \to \mathcal{F}$. This can be computed with the `JetGroupoids` command

```
CodimOfAction(F1,0, i->[1,0,0,0,-p,f,  0,0,0,1][i]);
```

by using the section (4.11) and the values for v^i from equation (6.9) to define the generic point. The result is independent from the function $f(x, y, p)$.

On $\mathcal{F}_{(1)}$, all but the last Vessiot structure equations, $v^i = 0$, are satisfied for second order ODEs. This gives a hint for the construction of a suitable subbundle. There exists a projection of natural vector bundles

$$\mathcal{F}_{(1)} \to \mathcal{F}'_{(1)} : (u, v^1, v^2, v^3, v^4) \mapsto (u, v^1, v^2, v^3)$$

omitting the last coordinate. All ODEs restrict to the zero section of $\mathcal{F}'_{(1)} \to \mathcal{F}$, so that Proposition 4.33 yields a smaller Π_1-bundle with fibre coordinate v^4. All sections for second order ODEs restrict to this minimal bundle.

It is possible to continue the calculations from Section 5.3 with the above subbundle of $\mathcal{F}_{(1)} \to \mathcal{F}$, but the computations become lengthy and inefficient. The coordinate expressions are larger by a factor of at least ten and computing the second prolongation takes longer than the whole worksheet in Section 5.3. $\diamond$

6.2 Cartan's Equivalence Method

The Vessiot equivalence method is an alternative to the well-known Cartan equivalence method [Car08, Car10] and this section is intended to compare both approaches. It turns out that they are dual to each other, just as exterior differential systems provide a dual description of PDE systems.

Cartan's method tests the equivalence of G-structures, which are introduced in Section 6.2.1. The following Section 6.2.2 presents the practical algorithm to decide equivalence of G-structures. There exists a large number of excellent introductory texts to the Cartan equivalence method, among them Gardner [Gar89], Ivey and Landsberg [IL03], Olver [Olv95], Sternberg [Ste64] and Stormark [Sto00]. Therefore most proofs are left out and the point of view is heavily biased towards the comparison with Vessiot's approach.

In Section 6.2.3,the comparison between both equivalence methods is completed by interpreting Sternberg's intrinsic structure function of a G-structure in the language of natural bundles and geometric objects. It is worth mentioning the thesis of Neut [Neu03], who implemented Cartan's equivalence method in Maple. Experiments with his packages gave valuable ideas for the comparison.

6.2.1 G-Structures

The primary theoretical object for the Cartan equivalence method are G-structures. This section deals with the theoretical part of Cartan's equivalence method while the practical work is briefly presented in Section 6.2.2. Here, G-structures, equivalence of G-structures and their canonical forms are introduced. In each step, the connection to the Vessiot equivalence method is indicated.

G-structures are very useful to prove that Cartan's and Vessiot's notion of equivalence are 'equivalent'. References for the next definition are [KN63, p. 288] and [Ste64, Def. VII.2.1].

Definition 6.15. Let $\pi : P \to X$ be a principal H-bundle and $G \leq H$ a closed Lie subgroup. A *G-reduction* is a subbundle of $P \to X$ which is a principal G-bundle.

A *G-structure* $\mathcal{G}$ is a G-reduction of the frame bundle $P^1 = P^1(X)$, which is a principal GL_1-bundle (thus $G \leq \mathrm{GL}_1$). If the group $G = \{e\}$ is trivial, then $\mathcal{G}$ is called *e-structure.* ◇

The definition only covers first order G-structures, which are widely treated in the literature. Sample references to textbooks are [Kob72], [Ste64]. It is also possible to define with higher order G-structures, which are then G-reductions of the higher order frame bundles P^q. The only reference found is [Yan92, Ch. VI].

We turn to the correspondence between G-structures and jet groupoids, which is a consequence of the following classical theorem (see e.g. [KN63, p. 57-58]).

Theorem 6.16. Let $\pi : P \to X$ be a principal H-bundle and $G \leq H$ a closed Lie subgroup. There is a pairwise bijective correspondence between:

(1) A G-reduction $Q \hookrightarrow P$,

(2) A global section $\omega : X \to P/G$,

(3) A H-equivariant map $\tilde{\omega} : P \to H/G$ with $\tilde{\omega}(pg) = g^{-1}\tilde{\omega}(p)$. ◇

The first two statements of Theorem 6.16 give rise to an exact sequence. In the case of G-structures, it is

$$0 \longrightarrow \mathcal{G} \longrightarrow P^1 \underset{\omega\circ\pi}{\overset{\mathrm{pr}}{\rightrightarrows}} P^1/G. \tag{6.10}$$

The similarity with an exact sequence (3.5) defining jet groupoids $\mathcal{R}_q(\omega)$ is no coincidence as we will see in the following corollary which establishes a correspondence between G-structures and groupoids.

Corollary 6.17. Every G-structure $\mathcal{G}$ on X defines a natural bundle $\mathcal{F}$ of order 1 and a section $\omega : X \to \mathcal{F}$ such that $\mathcal{R}_1(\omega) \cong \mathrm{Gauge}(\mathcal{G})$. We call ω the *geometric object corresponding to* $\mathcal{G}$. Conversely, each transitive groupoid $\mathcal{R}_1 \leq \Pi_1$ specifies a $G = \mathcal{R}_1(x_0, x_0)$-structure. ◇

Proof. Given a G-structure, the natural bundle $\mathcal{F} = P^1/G$ and the section ω are constructed from the equivalence of (1) and (2) in Theorem 6.16. Since $P^1 \cong \Pi_1(x_0, -)$ for $x_0 \in X$, it follows that $\mathcal{G} \cong \mathcal{R}_1(\omega)(x_0, -)$. As $\mathcal{R}_1(\omega)$ is transitive, Proposition 2.9 implies $\mathcal{R}_1(\omega) \cong \mathrm{Gauge}(\mathcal{G})$.

The converse direction follows from restricting the sequence (3.5) to $\Pi_1(x_0, -)$ and Theorem 6.16. □

Equivalence of G-structures

We now define the equivalence of G-structures (cf. [Neu03, Def. 24], [Ste64, Def. VII.2.2/3]) and show that it coincides with the equivalence of the corresponding geometric objects from Definitions 6.1.

Definition 6.18. Two G-structures $\mathcal{G} \to X$ and $\bar{\mathcal{G}} \to Y$ are *equivalent* if there exists a diffeomorphism $\varphi : X \to Y$ whose lift to the natural bundles $\tilde{\varphi} : P^1(X) \to P^1(Y)$ satisfies $\tilde{\varphi}(\mathcal{G}) = \bar{\mathcal{G}}$. $\mathcal{G}$ and $\bar{\mathcal{G}}$ are called *locally equivalent* if there are neighbourhoods U of $x \in X$ and V of $y \in Y$ such that $\mathcal{G}|_U$ is equivalent to $\bar{\mathcal{G}}|_V$. An *invariant* on $\mathcal{G}$ is a function $\psi : \mathcal{G} \to \mathbb{R}$ with $\tilde{\varphi}^*\psi = \psi$ for all φ with $\tilde{\varphi}(\mathcal{G}) = \mathcal{G}$. ◇

Note that each frame bundle $P^1(X)$ is a natural bundle with fibre GL_1. Denote the corresponding bundle functor again by P^1. Then the lift $\tilde{\varphi}$ is nothing else than $P^1(\varphi)$. This is the key idea for the next theorem.

Theorem 6.19. Let $\mathcal{G}$ and $\bar{\mathcal{G}}$ be G-structures on n-dimensional manifolds X and Y. $\mathcal{G}$ and $\bar{\mathcal{G}}$ are equivalent if and only if their corresponding geometric objects ω and $\bar{\omega}$ are equivalent. ◇

Proof. Consider the exact sequences (6.10) for $\mathcal{G}$ and $\bar{\mathcal{G}}$. By Theorem 3.12, both bundles $P^1(X)/G$ and $P^1(Y)/G$ are the values of a natural bundle functor $\mathcal{F}$ with fibre $F = \mathrm{GL}_1/G$. Each $\varphi : X \to Y$ lifts to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & P^1(X) & \overset{\mathrm{pr}}{\underset{\omega\circ\pi}{\rightrightarrows}} & \mathcal{F}(X) \\ & & & & \downarrow{\scriptstyle P^1(\varphi)} & & \downarrow{\scriptstyle \mathcal{F}(\varphi)} \\ 0 & \longrightarrow & \bar{\mathcal{G}} & \longrightarrow & P^1(Y) & \overset{\mathrm{pr}}{\underset{\bar{\omega}\circ\pi}{\rightrightarrows}} & \mathcal{F}(Y) \end{array} \tag{6.11}$$

built from the exact sequences (6.10). φ is an equivalence if and only if $P^1(\varphi)$ restricts to a morphism $\mathcal{G} \to \bar{\mathcal{G}}$. The condition for the equivalence is on the fibres:

$$\bar{\mathcal{G}}_{\varphi(x)} = P^1(\varphi)(\mathcal{G}_x) \qquad \forall\, x \in X.$$

Because both G-structures are kernels, $\mathcal{G} = \ker_\omega(\mathrm{pr})$ and $\bar{\mathcal{G}} = \ker_{\bar{\omega}}(\mathrm{pr})$, and because of the commutativity of diagram (6.11), this is equivalent to:

$$\begin{aligned} \bar{\omega}(\varphi(y)) = \mathrm{pr}(\bar{\mathcal{G}}_{\varphi(x)}) &= (\mathrm{pr} \circ P^1(\varphi))(\mathcal{G}_x) \\ &= (\mathcal{F}(\varphi) \circ \mathrm{pr})(\mathcal{G}_x) \qquad \forall\, x \in X \\ &= \mathcal{F}(\varphi)(\omega(x)). \end{aligned}$$

Applying $\mathcal{F}(\varphi^{-1})$, we obtain the equivalence condition (6.1) for ω and $\bar{\omega}$. □

At this point, we know theoretically that both equivalence methods can decide the same problems. In Corollary 6.17 we have seen, that only first order natural bundles with a transitive GL_1-action on the fibres corresponds to G-structures. The advantage of Vessiot's method is that both the intransitive case and higher order structures are automatically included.

The Canonical Form of a G-structure

In practice, Cartan's equivalence problems are not given as a G-structure but by a coframe on a manifold X and the structure group G. The link between G-structures $\mathcal{G}$ and the coframe is the canonical form on $\mathcal{G}$. In [Ste64, page 309], Sternberg gives a global and intrinsic definition of the canonical form. For the presentation, we follow Neut [Neu03, §2.2.3].

Some preparations are needed to construct the canonincal form. Fix an n-dimensional manifold X, the frame bundle $\pi : P^1 \to X$ and an n-dimensional vector space V with distinguished basis. Then each $p \in P^1$ defines a basis of T_xP^1 (for $\pi(p) = x$) and induces an isomorphism $p : V \to T_x$.

Definition 6.20. The *canonical form* or *soldering form* θ on P^1 is the map

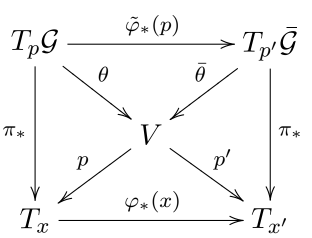
$$\theta : TP^1 \to V : (p, v_p) \mapsto (p^{-1} \circ \pi_*)(v_p), \qquad v_p \in T_pP^1.$$

The canonical form on a G-structure $\mathcal{G}$ is the restriction of θ to $\mathcal{G}$. ◇

If the right action with $g \in \mathrm{GL}_1$ on P^1 is denoted by R_g, then it satisfies $R_g^*\theta = g^{-1}\theta$. With the above definition, θ is a horizontal V-valued 1-form on P^1 (or on $\mathcal{G}$). Using the fixed basis of V, θ splits into n horizontal 1-forms θ^i on P^1. The canonical form also allows to test the equivalence of G-structures.

Proposition 6.21. [Neu03, Prop. 2] Two G-structures $\mathcal{G}$ and $\bar{\mathcal{G}}$ with canonical forms θ and $\bar{\theta}$ are equivalent by a diffeomorphism φ if and only if $\tilde{\varphi}^*(\bar{\theta}) = \theta$. ◇

Proof. The diagram (6.12) commutes if and only if the lower triangle commutes.



(6.12)

The lower triangle commutes if and only if $\varphi(\mathcal{G}) = \bar{\mathcal{G}}$. □

The proposition reduces the equivalence of G-structures to their canonical forms. On an open subset of X, we decompose the canonical form of a G-structure $\mathcal{G}$ into a coframe and a group parameter. This reduces the equivalence of G-structures to the equivalence of coframes modulo a given group.

On an open subset $U \subseteq X$, we can trivialise a G-structure $\mathcal{G}$ by giving a local section $\sigma : X \to \mathcal{G}$. For $x \in U$, $\sigma(x) \in P^1_x$ is a frame and each $p \in \mathcal{G}_x$ differs from $\sigma(x)$ by an element of G. The trivialisation is thus:

$$\mathcal{G}|_U \to U \times G : p \mapsto (x = \pi(p), g), \qquad \text{where } \sigma(x) = pg.$$

To recover the canonical form θ on the trivialisation, we consider the coframe $\omega \in \Gamma(P_1)$, which is dual to σ. Then we set

$$\theta : U \times G \to P_1 : (x, g) \mapsto g \cdot \omega(x). \tag{6.13}$$

In order to get a V-valued 1-form as in Definition 6.20, we have to pull back the coframe bundle $P_1 \subset (T^*)^n$ along the projection $\pi : \mathcal{G} \to X$ to $(T^*\mathcal{G})^n$. On the open subset U, $\mathcal{G}$ is completely determined by the group G and the coframe ω.

Example 6.22. To illustrate the construction of the canonical form from a coframe, we consider the second order ODEs from Example 4.32. They were originally formulated in the language of G-structures (cf. [Neu03, §2.3.3]). The structure group $G \leq \mathrm{GL}_1$ is

$$G = \{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & a_2 & a_3 \end{bmatrix} \mid a_i \in \mathbb{R}, a_1 a_3 \neq 0\}$$

and the coframe defining $\mathcal{G}$ is

$$\omega^1 = dx, \quad \omega^2 = dy - p\,dx, \quad \omega^3 = dp - f(x, y, p)dx. \tag{6.14}$$

Setting $\omega = (\omega^1, \omega^2, \omega^3)^{tr}$ and $\theta = g \cdot \omega$, we obtain the canonical form on $\mathcal{G}$. ◇

Equation (6.13) is very convenient for the comparison between Cartan's and Vessiot's equivalence method. We consider θ as a bundle morphism $\theta : \mathcal{G} \to P_1$ into the coframe bundle. This allows to work with natural bundles that have a right groupoid action:

$$0 \longrightarrow \mathcal{G} \xrightarrow{\ \theta\ } P_1 \underset{\omega \circ \pi}{\overset{\mathrm{pr}}{\rightrightarrows}} \mathcal{F} = P_1 / G. \tag{6.15}$$

The projection of θ to $\mathcal{F}$ is G-independent by construction and thus θ (or even the coframe ω) induces the section of $\mathcal{F} \to X$ defining $\mathcal{G}$.

Remark 6.23. The exact sequence (6.15) provides a transition between Cartan's and Vessiot's equivalence method. The main object is the projection

$$\mathrm{pr} : P_1 \to \mathcal{F} = P_1 / G.$$

Starting with a G-strucure $\mathcal{G}$, we obtain a section of $\mathcal{F} \to X$ by projecting the coframe to $\mathcal{F}$. The map pr is the restriction of Equation (3.7) from Π_1 to P_1.

If $\mathcal{R}_1(\omega)$ defined by the section ω of $\mathcal{F} = P_1/G \to X$ is transitive, we obtain a G-structure by finding a coframe that projects to ω. The coframe is not unique, since the G-dependence has been factored out on $\mathcal{F}$. ◇

Example 6.24. In Example 6.22, the coframe θ for second order ODEs has been constructed. To illustrate Remark 6.23, we explicitly construct a section on a natural bundle from the coframe. Using Maple, this was already done in Section 5.3. For fibre coordinates y^i_j of P_1, the coframe (6.14) is given by:

$$\begin{bmatrix} y^1_1 & y^1_2 & y^1_3 \\ y^2_1 & y^2_2 & y^2_3 \\ y^3_1 & y^3_2 & y^3_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a_1 p & a_1 & 0 \\ -a_2 p - a_3 f & a_2 & a_3 \end{bmatrix}.$$

Using Equation (3.7) to compute the projection $P_1 \to \mathcal{F}$, we obtain

$$u^1 = y^1_1, \;\; u^2 = y^1_2, \;\; u^3 = y^1_3, \;\; u^4 = \frac{y^2_3}{y^2_2}, \; u^5 = \frac{y^2_1}{y^2_2}, \;\; u^6 = \frac{y^3_1 y^2_2 - y^2_1 y^3_2}{y^3_2 y^2_3 - y^2_2 y^3_3}. \qquad (6.16)$$

Plugging in the values for θ, we obtain the section

$$u^1 = 1, \quad u^2 = u^3 = u^4 = 0, \quad u^5 = -p, \quad u^6 = f(x, y, p)$$

of $\mathcal{F} \to X$. For each choice of the function f, it represents a second order ODE. As expected the section is independent from the group parameters a_i. ◇

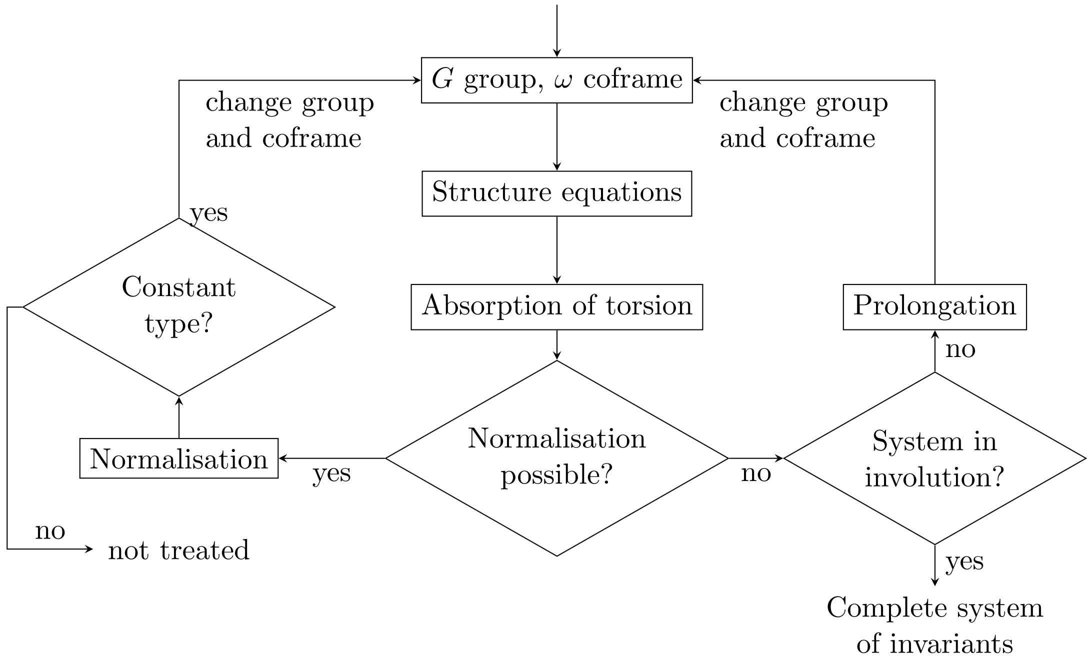
6.2.2 Cartan's Equivalence Method at Work

To decide the local equivalence between two G-structures with Cartan's equivalence method, they are considered as exterior differential systems which are completed to integrability. In the course of this algorithm, the structure group G is subsequently replaced by smaller groups until we end up with e-structures.

Cartan's equivalence method is presented using the flowchart in Figure 6.3, which is adapted from [Gar89]. The most important construction is the structure function developed by Sternberg [Ste64], which is needed for the comparison with Vessiot's equivalence method in Section 6.2.3.

The Cartan equivalence method is formulated with exterior differential forms, which is an alternative description for PDE systems. Note that there is a third way to formulated PDE systems using vector fields. This approach via so-called Vessiot distributions is presented by Fesser [Fes07]. In the work of Vinogradov [Vin01], the distributions are called Cartan distribution.

Figure 6.3: The Cartan equivalence method



Exterior Differential Systems

Cartan's equivalence method is formulated with exterior differential systems. They are an alternative language for PDE systems, as defined in Section 1.3. We follow the treatment of Ivey and Landsberg [IL03] which is very convenient for readers familiar with PDE systems. Several standard references to exterior differential systems are [BCG+91], [Car45], [Gri83] and [Yan92].

On a manifold Y, we consider the space $\Omega^k(Y) = \Gamma(\bigwedge^k T^*)$ of differential k-forms and the algebra $\Omega^*(Y) = \oplus_k \Omega^k(Y)$ of differential forms. A differential ideal is an ideal $\mathcal{I} \leq \Omega^*(Y)$ with $d\mathcal{I} \subseteq \mathcal{I}$.

Definition 6.25. [IL03] An *exterior differential system* (eds) with independence condition on Y is a differential ideal $\mathcal{I} \subset \Omega^*(Y)$ and a nonzero n-form Ω.

A *linear Pfaffian system* (I, J) is an exterior differential system $\mathcal{I}$ which is generated by the set of 1-forms $I = \{\theta^\alpha \,|\, 1 \leq \alpha \leq s\}$ and $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$ for $\omega^i \in \Omega^1(Y)$ such that

$$d\theta^\alpha \equiv 0 \mod J = \{\theta^\alpha, \omega^i\}. \tag{6.17}$$

An *integral manifold* of the eds $\mathcal{I}$ is an immersed submanifold $f : M \to Y$ such that $f^*(\theta) = 0$ for all $\theta \in \mathcal{I}$ and $f^*(\Omega) \neq 0$. ◇

For the remainder of this section we fix a linear Pfaffian system (I, J) on Y.

Remark 6.26. From each system of PDEs $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ it is possible to construct a linear Pfaffian system. Use the coordiates (x, y) for $\mathcal{E}$ and (x, y, y_q) for $J_q(\mathcal{E})$ and define the *standard contact forms*

$$\theta^i_\mu = dy^i_\mu - y^i_{\mu+1_j} dx^j, \qquad 0 \leq |\mu| \leq q-1.$$

By abuse of notation denote the pullback of θ^i_μ to $\mathcal{R}_q$ again by θ^i_μ. Then $I = \{\theta^i_\mu\}$ defines a linear Pfaffian system on $\mathcal{R}_q$ with independence condition $\Omega = dx^1 \wedge \cdots \wedge dx^n$. Integral manifolds are in one-to-one correspondence with solutions of $\mathcal{R}_q$. ◇

Example 6.27. On the bundle $\mathcal{E} = \mathbb{R}^2 \times \mathbb{R}$ with coordinates $(x, y; u)$ consider the system $\mathcal{R}_1 \subset J_1(\mathcal{E})$ defined by

$$u_x = A(x, y, u), \quad u_y = B(x, y, u). \tag{6.18}$$

We have the single contact form $du - u_x\, dx - u_y\, dy$, whose pullback to $\mathcal{R}_1$ is

$$\theta = du - A(x, y, u)dx - B(x, y, u)dy.$$

The independence condition $\Omega = dx \wedge dy \neq 0$ completes the Pfaffian system. We check condition (6.17) on $d\theta$:

$$d\theta = (A_y - B_x)dx \wedge dy + A_u dx \wedge du + B_u dy \wedge du \equiv 0 \mod \{\theta, dx, dy\}.$$

Due to Ω, all integral manifolds are parametrised as

$$f : \mathbb{R}^2 \to \mathcal{R}_1 : (x, y) \mapsto (x, y, u(x, y)).$$

The pullback of θ along f substitutes $du = u_x dx + u_y dy$

$$f^*(\theta) = (u_x - A)dx + (u_y - B)dy \stackrel{!}{=} 0.$$

By the independence condition, the condition for integral manifolds is equivalent to the original PDE system (6.18). ◇

Remark 6.28. The construction of an exterior differential system to check the equivalence of two G-structures $\mathcal{G} \to X$ and $\bar{\mathcal{G}} \to Y$ differs slightly from Remark 6.26. Assume that $\mathcal{G}$ and $\bar{\mathcal{G}}$ are locally given by the group G and their coframes ω and $\bar{\omega}$. This is the starting point of Figure 6.3.

Following Remark 6.26, we consider the PDE system $\mathcal{S}_1(\omega, \bar{\omega})$ for the equivalence constructed in equation (6.3) and pull back the standard contact forms from $\Pi_1(X, Y)$. This is equivalent to the linear Pfaffian system

$$\bar{\omega}^i(y) - g^i_j\, \omega^j(x), \qquad \omega^1 \wedge \cdots \wedge \omega^n \neq 0$$

with the matrix $(g^i_j) \in G$. In shorter notation, we write

$$\bar{\omega}(y) - g \cdot \omega(x).$$

However it is more convenient to work on the manifold $\bar{\mathcal{G}} \times \mathcal{G}$ with the symmetrised Pfaffian system

$$\bar{\theta} - \theta = \bar{g} \cdot \bar{\omega}(y) - g \cdot \omega(x), \qquad \theta^1 \wedge \cdots \wedge \theta^n \neq 0. \tag{6.19}$$

The symmetrised system allows to perform all computations for the canonical forms θ and $\bar{\theta}$ separately. Therefore Figure 6.3 treats only one side. $\diamond$

Structure Equations

So far, we have only considered the algebraic consequences of the linear Pfaffian system (I, J). We will now recall the steps corresponding to the prolongation and projection of PDE systems. The basic observation is that if $f : M \to Y$ is an integral submanifold, then $f^*(\theta^i) = 0$ also implies $f^*(d\theta^i) = 0$. By condition (6.17), there are functions $A^\alpha_{\epsilon j}$ and T^α_{ij} on Y such that $d\theta^\alpha$ has the form

$$d\theta^\alpha \equiv A^\alpha_{\epsilon j} \pi^\epsilon \wedge \omega^j + T^\alpha_{jk} \omega^j \wedge \omega^k \mod I. \tag{6.20}$$

The forms π^ϵ, $1 \leq \epsilon \leq r$, are a complement of J such that $K = \{\theta^\alpha, \omega^i, \pi^\epsilon\}$ is locally a basis of T^*Y. The functions T^α_{jk}, $T^\alpha_{jk} = -T^\alpha_{kj}$, are called *torsion coefficients* and the summand $T^\alpha_{jk}\omega^j \wedge \omega^k$ *apparent torsion*, because it depends on the arbitrary choice of π^ϵ for the complement.

Example 6.29. In Example 6.27, the complement to $\{\theta, dx, dy\}$ is empty and the structure equation depends on a single nonzero torsion coefficient

$$d\theta \equiv (A_y - B_x + A_u B - B_u A)\, dx \wedge dy \mod I.$$

Taking $d\theta$ mod I means replacing $du = Adx + Bdy$. The torsion coefficient is the compatibility condition obtained by taking cross derivatives in (6.18). $\diamond$

Remark 6.30. For the Cartan equivalence method, we take up the symmetrised system (6.19) and compute the exterior derivatives of the coframes θ and $\bar{\theta}$ separately.

$$d\theta = d(g \cdot \omega) = dg\, g^{-1} \wedge \theta + T\, \theta \wedge \theta \tag{6.21}$$

Here $dg\, g^{-1}$ is the right invariant Maurer-Cartan form (see [Gar89, Lecture 2])

$$TG \to \mathfrak{g} = T_{\mathrm{id}}G : (g, v) \mapsto (dg\, g^{-1})(v).$$

Select a basis of forms π^ϵ, $1 \leq \epsilon \leq r = \dim(G)$, among the entries of $dg\, g^{-1}$. Then we find elements $A^i_{\epsilon j} \in \mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ of G such that $(dg\, g^{-1})^i_j = A^i_{\epsilon j}\pi^\epsilon$ and the structure equations have the form

$$d\theta^i = A^i_{\epsilon j}\, \pi^\epsilon \wedge \theta^j + T^i_{jk}\, \theta^j \wedge \theta^k. \tag{6.22}$$

The torsion coefficients depend on the coordinates of $\mathcal{G}$ only. Analogous to θ, we also compute the structure equations for $\bar{\theta}$, where the Lie algebra elements $A^i_{\epsilon j}$ reoccur

$$d\bar{\theta}^i = A^i_{\epsilon j}\,\bar{\pi}^\epsilon \wedge \bar{\theta}^j + \bar{T}^i_{jk}\,\bar{\theta}^j \wedge \bar{\theta}^k. \tag{6.23}$$

Here, the torsion coefficients $\bar{T}$ depend on $\bar{\mathcal{G}}$ only. In the picture of linear Pfaffian systems, the structure equations for $\theta - \bar{\theta}$ are

$$d(\theta^i - \bar{\theta}^i) \equiv A^i_{\epsilon j}(\pi^\epsilon - \bar{\pi}^\epsilon) \wedge \theta^j + (T^i_{jk} - \bar{T}^i_{jk})\theta^j \wedge \theta^k \mod I. \tag{6.24}$$

◇

Example 6.31. Continuing Example 6.22, we compute the Maurer-Cartan form and define the forms π^ϵ by the following equation.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \pi^1 & 0 \\ 0 & \pi^2 & \pi^3 \end{bmatrix} := dg\,g^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & da_1 & 0 \\ 0 & da_2 & da_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & a_2 & a_3 \end{bmatrix}^{-1}$$

With these forms π^ϵ, the structure equations are

$$\begin{array}{lcl} d\theta^1 &=& 0, \\ d\theta^2 &=& \pi^1 \wedge \theta^2 \qquad\qquad\qquad + \; T^2_{12}\,\theta^1 \wedge \theta^2 \; + \; T^2_{13}\,\theta^1 \wedge \theta^3, \\ d\theta^3 &=& \pi^2 \wedge \theta^2 \; + \; \pi^3 \wedge \theta^3 \; + \; T^3_{12}\,\theta^1 \wedge \theta^2 \; + \; T^3_{13}\,\theta^1 \wedge \theta^3. \end{array} \tag{6.25}$$

The values of the torsion coefficients are

$$T^2_{12} = -\frac{a_2}{a_3}, \quad T^2_{13} = \frac{a_1}{a_3}, \quad T^3_{12} = \frac{f_y a_3^2 - f_p a_2 a_3 - a_2^2}{a_1 a_3}, \quad T^3_{23} = \frac{a_2 + f_p a_3}{a_3}.$$

◇

The torsion coefficients in the structure equations still depend on the choice of the forms π^ϵ. The next goal is to eliminate this dependence.

Torsion and the Structure Function

Following Ivey and Landsberg [IL03, §5.5], we construct the *torsion* map, which is a map

$$\tau : Y \to H^2_0(A) : y \mapsto [T^\alpha_{ij}(y)\,\partial_{\theta^\alpha} \otimes \omega^i \wedge \omega^j] \tag{6.26}$$

into a suitable Spencer cohomology group, where the dependence on the forms π^ϵ has been factored out. This allows to formulate the well-known Cartan-Kähler Theorem for the existence of integral manifolds. On a G-structure, τ induces the structure function $c : \mathcal{G} \to H^2_0(A)$ developed by Sternberg [Ste64, p. 316f]. It will be of interest in Section 6.2.3 for the comparison with Vessiot's approach.

Define families of vector spaces over Y by setting for each $y \in Y$

$$V^* = \langle (J/I)_y \rangle = \langle \omega^i|_y \rangle, \qquad W^* = \langle I_y \rangle = \langle \theta^\alpha|_y \rangle.$$

Then evaluate the coefficients $A^\alpha_{\epsilon j}$ of the structure equations (6.20) at y and define the tableau

$$A = A_y = \langle A^\alpha_{\epsilon j}\, \partial_{\theta^\alpha} \otimes \omega^j \,|\, 1 \le \epsilon \le r\rangle \subseteq W \otimes V^*. \tag{6.27}$$

The Spencer δ-map from Appendix A can also be defined for the families of vector spaces V^* and W instead of the bundles T^* and E. The restriction of δ to A yields the skew-symmetrisation map

$$\delta : A \otimes V^* \to W \otimes \bigwedge^2 V^*. \tag{6.28}$$

Now the *torsion* of a linear Pfaffian system is an element of the Spencer cohomology group

$$H^2_0(A) = W \otimes \bigwedge^2 V^* / \delta(A \otimes V^*)$$

by taking the residue classes

$$[T] = [T^\alpha_{ij} \partial_{\theta^\alpha} \otimes \omega^i \wedge \omega^j] \in H^2_0(A).$$

It can be seen as follows. The only way to modify the apparent torsion is to redefine the forms π^ϵ as $\tilde{\pi}^\epsilon = \pi^\epsilon + \lambda^\epsilon_i \omega^i$. This takes the apparent torsion to

$$\tilde{T}^\alpha_{ij} = T^\alpha_{ij} + (A^\alpha_{\epsilon j}\lambda^\epsilon_i - A^\alpha_{\epsilon i}\lambda^\epsilon_j). \tag{6.29}$$

A simple calculation (adding the $\partial_{\theta^\alpha} \otimes \omega^i$ and ω^j) shows that the additional terms for $\tilde{T}^\alpha_{ij}$ are exactly the image of the Spencer δ-map (6.28).

Setting $A_0 = W$, $A_1 = A$ and $A_i = A^{(i)}$ as the i-th prolongation of A (see Definition B.10), we can compute Spencer cohomology groups for the tableau A, since the analogue of Lemma A.4 holds. This allows to formulate the Cartan-Kähler-Theorem for linear Pfaffian system (cf. [IL03, Thm. 5.5.6]).

Theorem 6.32 (Cartan-Kähler). Let (I, J) be a linear Pfaffian system on Y, let $x \in Y$ and let U be a neighbourhood containing x such that for all $y \in U$,

(1) the torsion vanishes $[T]_y = 0$, and

(2) the tableau A_y is involutive.

Then there exist integral manifolds of dimension $n = \dim(J/I)_x$ through x that depend on s_l functions of l variables. ◇

Here s_l is the highest nonvanishing Cartan character (see e.g. [IL03, §4.5], [Pom78, §3.2]), which was not introduced since δ-regular coordinates could be avoided for the computation of all Spencer cohomology groups in Appendix A.

Remark 6.33. The condition $[T]_y = 0$ in the Cartan-Kähler Theorem is the obstruction to finding an appropriate integral manifold. Let $(x^i, y^\alpha, p^\epsilon)$ be coordinates of Y such that $\omega^i = dx^i$, $\theta^\alpha = dy^\alpha$ and $\pi^\epsilon = dp^\epsilon$. By the independence condition, every integral submanifold f can be given by functions $y^\alpha = f^\alpha(x) = 0$ and $p^\epsilon = f^\epsilon(x)$. This implies $f^*\pi^\epsilon = \partial_i f^\epsilon(x)\omega^i$ and

$$\begin{aligned} 0 \stackrel{!}{=} f^*(d\theta^\alpha) &= A^\alpha_{\epsilon i} f^*(\pi^\epsilon) \wedge \omega^i + T^\alpha_{ij}\,\omega^i \wedge \omega^j \\ &= (A^\alpha_{\epsilon j}\,\partial_i f^\epsilon - A^\alpha_{\epsilon i}\,\partial_j f^\epsilon + T^\alpha_{ij})\omega^i \wedge \omega^j \end{aligned}$$

As there are no other conditions on the first order derivatives of $f^\epsilon(x)$, this is solvable if and only if the torsion vanishes. ◇

For the Cartan equivalence method we consider the coframes θ and $\bar\theta$ separately again. In the structure equation 6.22, the torsion coefficients also depend on the choice of π^ϵ and we can construct a structure function (see [Ste64, p. 316f]) analogous to the torsion map.

Definition 6.34. The *structure function* of a G-structure $\mathcal{G}$ is the G-equivariant map

$$c : \mathcal{G} \to H^2_0(A) : (x, g) \mapsto [T^i_{jk}(x, g)\,\partial_{\theta^i} \otimes \theta^j \wedge \theta^k]. \tag{6.30}$$

with $A = \mathfrak{g}_1$ and $H^2_0(A) = V \otimes \bigwedge^2 V^*/\delta(\mathfrak{g}_1 \otimes V^*)$. ◇

Theorem 6.35. [Ste64, Thm. VII.2.1] Let $\mathcal{G}$ and $\bar{\mathcal{G}}$ be G-structures with structure functions c and $\bar c$. If $\mathcal{G}$ and $\bar{\mathcal{G}}$ are equivalent by φ, then $\varphi^*(\bar c) = c$. ◇

The structure function is used for the reduction of $\mathcal{G}$ to a G'-structure $\mathcal{G}' \subset \mathcal{G}$ for a suitable subgroup $G' \leq G$ in the normalisation step. In Section 6.2.3, we give an interpretation of the structure function as a section of a natural bundle.

Absorption of Torsion

To compute the Spencer cohomology group $H^2_0(A)$ of a linear Pfaffian system, we choose a complement C of $\delta(A \otimes V^*)$ such that

$$W \otimes \bigwedge^2 V^* = C \oplus \delta(A \otimes V^*)$$

and identify C with $H^2_0(A)$. Due to the condition $[T] = 0$ in the Cartan-Kähler Theorem, the complement C is chosen such that a maximal number of torsion coefficients is zero. This process, called *absorption of torsion*, is done by redefining the forms π^ϵ such that a maximal number of torsion coefficients vanishes. The remaining torsion coefficients are called *essential* (cf. [Gar89, Lecture 3]).

Example 6.36. In Remark 6.30 we have seen that the torsion coefficients T^i_{jk} depend only on $\mathcal{G}$ while $\bar T^i_{jk}$ depends on $\bar{\mathcal{G}}$. We can thus absorb the torsion of $d\theta$ and $d\bar\theta$ separately. Equation (6.24) shows that the coefficient T^i_{jk} can be absorbed if and only if $\bar T^i_{jk}$ can be absorbed.

For Example 6.31 it is done by the redefinition of π^i as

$$\tilde{\pi}^1 = \pi^1 - T^2_{12}\,\theta^1, \qquad \tilde{\pi}^2 = \pi^2 - T^3_{12}\,\theta^1, \qquad \tilde{\pi}^3 = \pi^3 - T^3_{13}\,\theta^1.$$

Omitting the tilde on the forms π, the structure equations with absorbed torsion contain only a single essential torsion coefficient $T^2_{13} = a_1/a_3$.

$$\begin{array}{rcl} d\theta^1 &=& 0, \\ d\theta^2 &=& \pi^1 \wedge \theta^2 \qquad\qquad\qquad + \; T^2_{13}\,\theta^1 \wedge \theta^3, \\ d\theta^3 &=& \pi^2 \wedge \theta^2 \; + \; \pi^3 \wedge \theta^3. \end{array} \tag{6.31}$$

Here $H^2_0(A)$ is four-dimensional with basis $\partial_{\theta^1} \otimes \theta^i \wedge \theta^j$ and $\partial_{\theta^2} \otimes \theta^1 \wedge \theta^3$ and the structure function is determined by the torsion coefficients $T^1_{ij} = 0$ and T^2_{13}. $\diamond$

Normalisation

After having absorbed the torsion in the structure equations, there might still be essential torsion coefficients T^α_{ij}. In this case, an integral manifold $f : M \to Y$ must also satisfy $f^*(T^\alpha_{ij}) = 0$. We can thus turn to the closed submanifold $Y' \subseteq Y$ defined by $T^\alpha_{ij} = 0$ and pull back the Pfaffian system (I, J). By construction, integral manifolds on Y' are in one-to-one correpondence with integral manifolds of (I, J) on Y. In the picture of PDE systems $Y = \mathcal{R}_q$, the essential torsion coefficients define $Y' = \mathcal{R}^{(1)}_q$ obtained by a single prolongation and projection.

For Cartan's equivalence method, we continue with the symmetrised Pfaffian system from Remark 6.30. Having absorbed the torsion in (6.24), the condition (1) of the Cartan-Kähler Theorem reads

$$T^i_{jk} - \bar{T}^i_{jk} = 0.$$

If the torsion is nonzero, we give a positive answer to the question 'Normalisation possible?' in Figure 6.3 and restrict to a submanifold of $\mathcal{G} \times \bar{\mathcal{G}}$ by setting

$$T^i_{jk} = const^i_{jk} = \bar{T}^i_{jk},$$

since T and $\bar{T}$ depend on disjoint sets of coordinates. This is the normalisation step (cf. [Neu03, §2.6]). On the submanifold of $\mathcal{G} \times \bar{\mathcal{G}}$, we find integral manifolds if and only if there are integral manifolds on $\mathcal{G} \times \bar{\mathcal{G}}$.

Equivalently, we can choose an element $0 \neq w \in H^2_0(A) \cap \operatorname{im}(c)$ and locally define the submanifold and the reduction of the group

$$\begin{array}{rcl} \mathcal{G}' &:=& \{p \in \mathcal{G} \mid c(p) = w\}, \\ G' &:=& \{g \in G \mid \forall p \in G, c(pg) = c(p)\} \end{array} \tag{6.32}$$

and proceed analogously with $\bar{\mathcal{G}}$. In Figure 6.3, this step corresponds to the arrow 'change group and coframe'. Unfortunately, $\mathcal{G}'$ is not necessarily a G'-structure. Gardner [Gar89, Lecture 4] gives a necessary condition for this.

Definition 6.37. A G-structure $\mathcal{G}$ is of *first order constant type* if the image of the structure function c is a single G-orbit on $V \otimes \bigwedge^2 V^*/\delta(\mathfrak{g}_1 \otimes V^*)$. ⋄

Proposition 6.38. Let $\mathcal{G}$ be a G-structure with structure function c and $\mathcal{G}'$ as in Equation (6.32). If $\mathcal{G}$ is of first order constant type then $\mathcal{G}'$ is a G'-structure.⋄

For a proof see [Gar89, p. 38] or [Neu03, Prop 5].

The problem that might occur is that an essential torsion coefficient does not depend on the group parameters and is thus an invariant of the problem. About the treatment of these problems, Gardner writes in [Gar89, p. 37]:

> If an equivalence problem is not of first order constant type, [...]. Cartan actually indicates how this case might be handled, but the added complexity in exposition prohibits the consideration of this difficulty in any generality.

In [Olv95, p. 366f], Olver indicates how to handle equivalence problems of nonconstant type and gives examples (e.g. by Gardner and Shadwick). The Vessiot equivalence method treats equivalence problems of constant and nonconstant type with exactly the same methods. The only difference is that the Vessiot structure equations depend on invariants in the nonconstant case.

Example 6.39. Continue Example 6.36 and normalise the torsion coefficient $T^2_{13} = \frac{a_1}{a_3} = 1$ to eliminate a_1. The original G-structure is of first order constant type and we can continue with another loop through Figure 6.3 (normalisation $T^3_{13} = 0$, eliminating a_2). We obtain the structure equations with absorbed torsion

$$\begin{aligned} d\theta^1 &= 0, \\ d\theta^2 &= \pi^3 \wedge \theta^2 + \theta^1 \wedge \theta^3, \\ d\theta^3 &= \pi^3 \wedge \theta^3 + T^3_{12}\,\theta^1 \wedge \theta^2. \end{aligned}$$

The essential torsion coefficient is the invariant, we have found in Section 4.3.3.

$$T^3_{12} = -\frac{1}{4} f_p^2 + \frac{1}{2} f_y - \frac{1}{2} f_{xp} - \frac{1}{2} p f_{py} - \frac{1}{2} f f_{pp}.$$

The corresponding tableau is not involutive (see [Neu03, §2.7.1]), such that a prolongation becomes necessary. If T^3_{12} is nonconstant, the G-structure is not of first order constant type. Nevertheless, the results in [Neu03, §2.8.1] are correct.⋄

Prolongation

If all torsion is zero, but the tableau A for (I, J) on Y is not involutive, we have to prolong the linear Pfaffian system. Here, we follow [Neu03, §1.6] in a form that is comparable to the prolongation of PDE systems from Section 1.3. For simplicity the torsion $[T]$ is assumed to be zero.

Use the coordinates $(x^i, y^\alpha, p^\epsilon)$ for Y defined in Remark 6.33. The independence condition Ω locally induces a projection $\pi : Y \to X$ to a manifold X with coordinates (x). Define the prolongation $Y' := J_1(Y)$ with additional coordinates $(y^\alpha_i, p^\epsilon_i)$. The standard contact forms on $J_1(Y)$ are

$$\tilde\theta^\alpha = dy^\alpha - y^\alpha_i dx^i, \qquad \pi^\epsilon = dp^\epsilon - p^\epsilon_i dx^i.$$

Set $I' = (\theta^\alpha, \tilde\theta^\alpha, \pi^\epsilon)$ and $J' = (I', \omega^i)$. The prolongation of PDE system $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ was defined as $J_1(\mathcal{R}_q) \cap J_{q+1}(\mathcal{E})$, so we have to find a subbundle of $Y' \to Y$. First set $y^\alpha_i = 0$, since integral manifolds satisfy $f^*(\theta^\alpha) = 0$ and thus $f^*(dy^\alpha - y^\alpha_i dx^i) = y^\alpha_i dx^i \stackrel{!}{=} 0$. Then eliminate the torsion by solving

$$d\theta^\alpha \equiv (A^\alpha_{\epsilon,j} p^\epsilon_i - T^\alpha_{ij})\omega^i \wedge \omega^j \mod I' = 0 \tag{6.33}$$

for a maximal number of p^ϵ_i. An integral manifold must now satisfy $f^*(\pi^\epsilon) = 0$. Writing down the equations for the prolongation of the linear Pfaffian system on the trivial equation $\mathcal{R}_q = J_q(\mathcal{E})$ shows that the restriction of Y' is equivalent to the intersection with $J_{q+1}(\mathcal{E})$ for PDE systems.

For the prolongation of a G-structure $\mathcal{G}$, we compute $J_1(\mathcal{G})$, add the forms $\pi^\epsilon = dp^\epsilon - p^\epsilon_i dx^i$ and obtain a $G^{(1)}$-structure $\mathcal{G}^{(1)} \to \mathcal{G}$ by solving equation (6.33). On $\mathcal{G} \times \bar{\mathcal{G}}$, it is the same as the above prolongation (see [Neu03, §2.5]).

Example 6.40. The prolongation for Example 6.39 is done by adding $\theta^4 = \pi^3$. We obtain an e-structure and structure equations

$$\begin{aligned}
d\theta^1 &= 0, \\
d\theta^2 &= \theta^4 \wedge \theta^2 + \theta^1 \wedge \theta^3, \\
d\theta^3 &= \theta^4 \wedge \theta^3 + T^3_{12}\, \theta^1 \wedge \theta^2, \\
d\theta^4 &= T^4_{12}\, \theta^1 \wedge \theta^2 + T^4_{23}\, \theta^2 \wedge \theta^3.
\end{aligned}$$

with T^3_{12} as in Example 6.39 and

$$T^4_{12} = \frac{f_{yp} + f_{xpp} + p f_{ypp} + 3 f_p f_{pp} + f f_{ppp}}{2a_3}, \qquad T^4_{23} = -\frac{f_{ppp}}{2a_3^2}.$$

Neut shows that a generating set of invariants on the e-structure is given by T^3_{12} and T^4_{23}. This does not coincide with the invariants computed in Section 4.3.3, since we are working over the base $\mathcal{G}$ with coordinates (x, y, p, a_3) instead of X which has the coordinates (x, y, p) as in the case of natural bundles. ◇

Equivalence Conditions

For the equivalence condition, we choose the presentation of Stormark [Sto00], which comes closest to Theorem 6.2 from Vessiot's approach.

Having reduced the structure group of a G-structure $\mathcal{G}$ to the identity group $G' = e$ by several loops in Figure 6.3, all torsion coefficients T^i_{jk} are essential and therefore invariants. Additionally, we have obtained invariant coframes $\theta = \omega$ and $\bar{\theta} = \bar{\omega}$. Compute the dual invariant differential operators $\mathcal{D}_i$ and $\bar{\mathcal{D}}_i$ according to Definition 4.6. There exists an $m \in \mathbb{N}$, such that the derived invariants

$$T^i_{jk,l_1,\ldots,l_m}(x) = \mathcal{D}_{l_1} \ldots \mathcal{D}_{l_m} T^i_{jk}(x), \qquad 1 \leq l_o \leq n$$

contain a maximal number $p \in \mathbb{N}$ of independent invariants $I^1(x), \ldots, I^p(x)$. Use the same choice $\bar{I}^1(\bar{x}), \ldots, \bar{I}^p(\bar{x})$ for the reduction of $\bar{\mathcal{G}}$.

Theorem 6.41. [Sto00, Thm. 15.1.2] Necessary and sufficient conditions for local equivalence are that all the equations

$$T^i_{jk,l_1,\ldots,l_a}(x) = \bar{T}^i_{jk,l_1,\ldots,l_a}(\bar{x}), \quad a = 0, \ldots, m+1$$

are consequences of the equations $I^i(x) = \bar{I}^i(\bar{x})$ for $i = 1, \ldots, p$. Or put somewhat differently: expressing $T^i_{jk,l_1,\ldots,l_a}$ and $\bar{T}^i_{jk,l_1,\ldots,l_a}$ as functions of I and $\bar{I}$ respectively, these functions have to coincide. ◇

If there are no invariants ($p = 0$), we have to compare the constants T^i_{jk} to decide equivalence, since all invariant derivatives are zero.

Example 6.42. In Example 6.40, the invariants T^3_{12} and T^4_{23} are both zero for the equation $y_{xx} = f(x, y, p) = 0$. Any second order ODE is equivalent to $y_{xx} = 0$ if and only if $T^3_{12} = T^4_{23} = 0$ (see [Neu03, Thm. 10]). ◇

6.2.3 Comparison Between Cartan's and Vessiot's Method

In this section, we finish the comparison of the Cartan equivalence method and Vessiot's approach. They stand in duality to each other, since Cartan's method is formulated by linear Pfaffian systems and Vessiot's method with PDE systems. As a result, the following table of translations between both methods is obtained.

Cartan	Vessiot
G-structure $\mathcal{G}$ coframe θ	natural bundle $\mathcal{F}$, geometric object ω
structure equations absorption of torsion normalisation	prolongation and projection
coframe + structure function	geometric object on $\mathcal{F}_{(1)}$
prolongation	prolongation
Equivalence conditions	Vessiot structure equations

The first correspondence is classical and was proved in Corollary 6.17. Although no direct translation was found in the literature, it seems to be known that the structure equations etc. correspond to prolongation and projection on the PDE system side.

In this section we concentrate on the next point and show that the canonical form and the structure function together define a section of the bundle $\mathcal{F}_{(1)}$ of integrability conditions. This result is a new interpretation of the torsion τ and the curvature map κ from Section 3.5.4. As a corollary, we obtain that the normalisation corresponds to the projection of PDE systems. In Vessiot's approach, no choice of constants is necessary.

In contrast to Cartan's method, where a generating set of invariants is computed in order to decide equivalence, the Vessiot structure equations allow to test equivalence before. For given geometric objects, this may avoid prolongation and projection steps. In the general situation, we obtain a finer classification.

Interpreting Coframe and Structure Function

Assume that a G-structure $\mathcal{G}$ with canonical form θ and the corresponding geometric object ω on the natural bundle $\mathcal{F}$ are given. In order to show that coframe and structure function of $\mathcal{G}$ determine the section $I(j_1(\omega))$ on the bundle of integrability conditions $\mathcal{F}_{(1)} \to X$, we use a well-known correspondence between PDE systems and linear Pfaffian systems. Analogous to Section 3.5.4, it is possible to define a curvature map κ for each PDE system $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ (see [Gol67b, Prop. 8.3] and [Pom78, §2.4]). Let (I, J) be the linear Pfaffian system corresponding to $\mathcal{R}_q$. Then under mild assumptions, the curvature κ is identical to the torsion τ from equation (6.26). See [Mal05, §II.3, App. B.1] for a proof.

Equation 3.45 shows that one half of the curvature map κ consists of the section $I(j_1(\omega))$. In the same sense, the structure function of a G-structure $\mathcal{G}$ is one half of the torsion map on $\mathcal{G} \times \bar{\mathcal{G}}$.

Although it may be hidden, this section was strongly inspired by the book of Malgrange [Mal05, §II.3/4], especially by the correspondence between curvature and torsion. Additional sources were the book of Ivey and Landsberg [IL03, §5.7] and the Maple packages of Neut [Neu03].

We need the following preparational lemma.

Lemma 6.43. The bundle $J_1(P_1)/K_1^2$ is isomorphic to $P_1 \times_X (\bigwedge^2 T^*)^n$. ◇

Proof. The GL_1-action on the fibre of P_1 is free (and transitive), such that the GL_2-action on $J_1(P_1)$ is also free. We have $\dim(K_1^2) = \dim(\Pi_2 \to \Pi_1) = n\binom{n+1}{n}$ by Proposition 1.6, such that $\dim(J_1(P_1)/K_1^2 \to P_1) = n^3 - n\binom{n+1}{n} = n\binom{n}{2}$. The dimension of $\bigwedge^2 T^*$ is $\binom{n}{2}$ and for coordinates u^i_j of P_1 ($\omega^i = u^i_j dx^j$), the projection $J_1(P_1) \to P_1 \times_X (\bigwedge^2 T^*)^n$ is given by $(u^i_j, u^i_{j,k}) \mapsto (u^i_j, u^i_{j,k} - u^i_{k,j})$. □

The next theorem gives a direct interpretation of Sternberg's structure function in the context of the Vessiot equivalence method.

Theorem 6.44. Let $\mathcal{G}$ be a G-structure with canonical form θ and let ω be the geometric object on the natural bundle $\mathcal{F} = P_1/G$ corresponding to $\mathcal{G}$. Then the structure function c and the canonical form θ define a morphism

$$\tilde{c} : \mathcal{G} \to \mathcal{F}_{(1)} : (x, g) \mapsto (x, \theta(x, g), c(x, g)). \tag{6.34}$$

which coincides with the section $I(j_1(\omega)) : X \to \mathcal{F}_{(1)}$. Here I is the projection $I : J_1(\mathcal{F}) \to \mathcal{F}_{(1)} = J_1(\mathcal{F})/K_1^2$. $\diamond$

Proof. With A and V^* as in Definition 6.34, Lemma 3.43 implies $\mathcal{M}_1 \cong \mathfrak{k}_0^1 = \mathfrak{g}_1 = A$ and $\mathcal{M}_0 = T^* = V^*$. By Lemma 3.41, $\mathcal{F}_{(1)}$ is a vector bundle that coincides with the Janet bundle $\mathcal{F}_1$ from Equation (3.37). Theorem 3.53 implies that $\mathcal{F}_1 \cong H_0^2(\mathcal{M}_1) = \bigwedge^2 T^* \otimes T/\delta(\mathfrak{g}_1 \otimes T^*)$, which is isomorphic to $H_0^2(A)$ (for appropriate pullbacks). So $\tilde{c}$ is well-defined. To see that $\tilde{c}$ coincides with the section $I(j_1(\omega))$, we apply the exact functor J_1 to the sequence (6.15) and obtain

$$\begin{array}{ccccccc} 0 & \longrightarrow & J_1(\mathcal{G}) & \xrightarrow{p_1(\theta)} & J_1(P_1) & \overset{\longrightarrow}{\underset{j_1(\omega)\circ\pi}{\longrightarrow}} & J_1(\mathcal{F}) \\ & & \| & & \downarrow & & \downarrow I \\ & & J_1(\mathcal{G}) & \xrightarrow{(\theta, d\theta)} & P_1 \times_X (\bigwedge^2 T^*)^n & \overset{\mathrm{pr}}{\rightrightarrows} & \mathcal{F}_{(1)} \end{array} \tag{6.35}$$

The bottom row is constructed by factoring out the K_1^2-action on $J_1(P_1)$ and $J_1(\mathcal{F})$ and using Lemma 6.43. By Proposition B.17 and Theorem 3.15, the diagram commutes. So the map $J_1(\mathcal{G}) \to \mathcal{F}_{(1)}$, which is the composition $\mathrm{pr} \circ (\theta, d\theta)$, coincides with $I(j_1(\omega))$.

An explicit construction shows that it is the map $\tilde{c}$. With coordinates (x, p^ϵ) of $\mathcal{G}$ choose fibre coordinates p_i^ϵ of $J_1(\mathcal{G})$ such that the total derivative on $\mathcal{G}$ is

$$D_{x^j} = \partial_{x^j} + (g\omega)^i_j \, p_i^\epsilon \, \partial_{p^\epsilon}, \qquad \theta = (g\omega)^i_j \, dx^j.$$

Then $d\theta$ is

$$\begin{aligned} d\theta^i &= \pi^\epsilon \wedge D_{p^\epsilon}\theta^i + dx^j \wedge D_{x^j}\theta^i \\ &= A^i_{\epsilon j} \, \pi^\epsilon \wedge \theta^j + A^i_{\epsilon j} \, p_k^\epsilon \, \theta^j \wedge \theta^k + T^i_{jk} \, \theta^j \wedge \theta^k \\ &= A^i_{\epsilon j} \, \pi^\epsilon \wedge \theta^j + (T^i_{jk} + A^i_{\epsilon j} \, p_k^\epsilon - A^i_{\epsilon k} \, p_j^\epsilon) \, \theta^j \wedge \theta^k. \end{aligned}$$

The dependence on $A^i_{\epsilon j} \in \mathfrak{g}_1$ is obtained by comparing

$$\partial_{p^\epsilon}\theta^i = (\partial_{p^\epsilon} g^i_l)\omega^l = ((\partial_{p^\epsilon} g)g^{-1})^i_j \, \theta^j = A^i_{\epsilon j} \, \theta^j.$$

with Equation (6.22). The map $(\theta, d\theta)$ in diagram (6.35) is

$$J_1(\mathcal{G}) \to P_1 \times_X (\bigwedge^2 T^*)^n : (x, p^\epsilon, p_i^\epsilon) \mapsto (x, \, \theta, \, (T^i_{jk} + A^i_{\epsilon j} p_k^\epsilon - A^i_{\epsilon k} p_j^\epsilon)\theta^j \wedge \theta^k).$$

Because the map $\mathcal{G} \to \mathcal{F}$ factors over X, i.e. does not depend on p^ϵ, the composition of $(\theta, d\theta)$ with the projection to $\mathcal{F}_{(1)}$ is independent from p^ϵ and p_i^ϵ and thus coincides with $\tilde{c}$. $\square$

The following corollary connects the prolongation and projection with the normalisation step for G-structures.

Corollary 6.45. Let $\mathcal{G}$, θ, $\mathcal{F}$ and ω be as in Theorem 6.44. Choose $w \in H_0^2(A) \cap \operatorname{im}(c)$ and define $\mathcal{G}'$ by Equation (6.32). Prolong and project the symmetry groupoid $\mathcal{R}_1(\omega)$ to obtain $\mathcal{R}_1^{(1)}(\omega)$. If $\mathcal{R}_1^{(1)}(\omega)$ is transitive there is an $x_0 \in X$ such that $\mathcal{G}' \cong \mathcal{R}_1^{(1)}(\omega)(x_0, -)$. Otherwise, $\mathcal{G}$ is not of first order constant type.$\diamond$

Proof. Since $w \in \operatorname{im}(c)$, there exists an $x_0 \in X$ such that $I(j_1(\omega)(x_0)) = (x_0, \omega(x_0), w)$ by Theorem 6.44. By the proof of Corollary 6.17, we have $\mathcal{G} \cong \mathcal{R}_1(\omega)(x_0, -)$ and $\mathcal{G}' \cong \mathcal{R}_1^{(1)}(\omega)(x_0, -)$ follows. If $\mathcal{R}_1^{(1)}(\omega)$ is intransitive, GL_1 acts intransitively on the fibre F_1 of $\mathcal{F}_{(1)}$, but transitively on the fibre of $\mathcal{F}$. $\square$

For smaller examples, Theorem 6.44 can be checked explicitly.

Example 6.46. We will follow the proof of Theorem 6.44 and compute the section $I(j_1(\omega))$ of $\mathcal{F}_{(1)} \to X$ using the coframe θ and the structure function c for the Example 6.36 of second order ODEs. The starting point is the exact sequence $0 \to \mathcal{G} \to P_1 \to \mathcal{F}$ which was calculated in Example 6.24. We construct the bundles and the maps in diagram (6.35). The bundle coordinates as well as standard maps are summarised in the following table.

P_1	:	$(x,\, y^i_j)$,	
$J_1(P_1)$	:	$(x,\, y^i_j,\, y^i_{j,k} = D_k y^i_j)$,	
$P_1 \times_X (\bigwedge^2 T^*)^n$	:	$(x,\, y^i_j,\, w^i_{jk} = y^i_{j,k} - y^i_{k,j})$,	
$\mathcal{F}$	:	$(x,\, u^\alpha = \Phi^\alpha(y_1))$,	Equation (6.16),
$J_1(\mathcal{F})$	:	$(x,\, u^\alpha, u^\alpha_i = D_i u^\alpha)$,	
$\mathcal{F}_{(1)}$	:	$(x,\, u^\alpha, v^\beta = A^\beta(u) u_x)$,	Example 6.14.

The references point to places where the maps have been computed. The projection $P_1 \times_X (\bigwedge^2 T^*)^n \to \mathcal{F}_{(1)}$ is given by

$$v^1 = w^1_{12}, \quad v^2 = w^1_{13}, \quad v^3 = w^1_{23}, \quad v^4 = \frac{y^2_3\, w^2_{12} - y^2_2\, w^2_{13} + y^2_1\, w^2_{23}}{(y^2_2)^2}.$$

It is obtained by composing the map $J_1(P_1) \to J_1(\mathcal{F})$ $(u^\alpha_i = D_i \Phi^\alpha(y_1))$ with the projection to $\mathcal{F}_{(1)}$ and then expressing $y^i_{j,k}$ by w^i_{jk}. Instead of the coordinates w^i_{jk}, which correspond to the 2-form $\Omega^i = w^i_{jk} dx^j \wedge dx^k$, it is more convenient to choose the fibre coordinates T^i_{jk} that correspond directly to the torsion coefficients $(\Omega^i = T^i_{jk}\theta^j \wedge \theta^k)$. In these coordinates the map $P_1 \times_X (\bigwedge^2 T^*)^n \to \mathcal{F}_{(1)}$ reads

$$v^1 = T^1_{13}\, a_2 + T^1_{23}\, a_1 a_3 f(x,y,p) + T^1_{12}\, a_1, \qquad v^2 = T^1_{13}\, a_3 - T^1_{23}\, a_1 a_3 p,$$
$$v^3 = T^1_{23}\, a_1 a_3, \qquad v^4 = -T^2_{13}\, \frac{a_3}{a_1}.$$

To obtain the map $\tilde{c}$, we plug in the torsion coefficients from Equation (6.25).

$$T^1_{12} = T^1_{13} = T^1_{23} = 0, \qquad T^2_{13} = \frac{a_1}{a_3}$$

Since $\tilde{c}$ does not depend on the fibre coordinates of $\mathcal{G} \to X$, we can use the simplified torsion coefficients from the absorbed structure equations (6.31). As expected, we recover Equation (6.9):

$$v^1 = v^2 = v^3 = 0, \quad v^4 = -1.$$

There is a caveat, which might lead to confusions. In the absorbed structure equations (6.31), we only see a single essential torsion coefficient T^2_{13}, but the fibre of $\mathcal{F}_{(1)}$ is four-dimensional. Theorem 6.44 implies that the space $H^2_0(A)$ of torsion coefficients is isomorphic to the fibre of $\mathcal{F}_{(1)} \to \mathcal{F}$. In fact, the torsion coefficients T^1_{jk} are also essential, because they cannot be absorbed. They just happen to be zero in this example. It corresponds to the fact that we could find a one-dimensional subbundle of $\mathcal{F}_{(1)} \to \mathcal{F}$ to which all sections for second order ODEs restrict (see Example 6.14). ◇

6.3 Examples

In this section, we present two examples of third and fourth order ODEs of the form

$$\begin{aligned} y_{xxx} &= f(x, y, y_x, y_{xx}), \\ y_{xxxx} &= f(x, y, y_x, y_{xx}, y_{xxx}) \end{aligned}$$

under contact transformations for the direct comparison between Cartans's and Vessiot's equivalence method. With Cartan's equivalence method, the examples have been calculated by Neut [Neu03]. The examples show the limits of both approaches, which can be summarised as follows.

The computations with Cartan's equivalence method cannot be used to decide equivalence of generic third or fourth order ODEs, since the invariants given in [Neu03] still depend on the fibre coordinates of the initial G-structure.

Nevertheless it is possible to give necessary and sufficient conditions for the equivalence to

$$y_{xxx} = 0 \qquad \text{or} \qquad y_{xxx} = y$$

in the case of third order ODEs. For fourth order ODEs, the equivalence to $y_{xxxx} = 0$ can be tested.

With the Vessiot equivalence method only the equivalence of third order ODEs to $y_{xxx} = 0$ can be tested. In all other cases, the computations become too large. On the other hand, it is possible to compute some of the invariants which are needed to decide equivalence in the generic case.

For examples where the Vessiot equivalence method gives a complete solution, we refer to Chapter 7.

6.3.1 Third Order ODEs

Similar to Example 4.32 dealing with second order ODEs under point transformations, we now turn to third order ODEs

$$y_{xxx} = f(x, y, p = y_x, q = y_{xx}) \tag{6.36}$$

under contact transformations. This example has been first considered by Chern [Che40]. In the course of computation, the so-called Wünschmann semi-invariant I [Wün05] occurs and leads to a case splitting. Sato and Yoshikawa [SY98] explicitly give conditions for the equivalence to $y_{xxx} = 0$, which is a special case of the branch with $I = 0$. Neut and Petitot [NP02] (see also [Neu03, Ch. 4]) compute the branch $I \neq 0$.

We take up this example in the Vessiot context and give an interpretation of the Wünschmann semi-invariant. Furthermore, the above computations do not lead to a full classification of third order ODEs. We will discuss this problem and continue the calculations until they become too large to handle with recent computers.

For third order ODEs under contact transformations, we have the coframe

$$\omega^1 = dx, \quad \omega^2 = dy - p dx, \quad \omega^3 = dp - q dx, \quad \omega^4 = dq - f(x, y, p, q)dx$$

on $X = J_3(\mathbb{R} \times \mathbb{R})$ and the structure group

$$G = \begin{bmatrix} X_x & X_y & X_p & 0 \\ 0 & Y_y & 0 & 0 \\ 0 & P_y & P_p & 0 \\ 0 & Q_y & Q_p & Q_q \end{bmatrix}.$$

Here (x, y, p, q) and (X, Y, P, Q) are coordinates of X. In other words, the natural bundle is $\mathcal{F}' = P_1/G$, where G is defined by the equations

$$Y_x = P_x = Q_x = Y_p = X_q = Y_q = P_q = 0.$$

The bundle $\mathcal{F}'$ has a seven-dimensional fibre with coordinates $(u^1, \ldots, u^6, f)$. We are interested in sections ω of the form

$$u^1 = u^2 = u^3 = u^4 = 0, \quad u^5 = -p, \quad u^6 = -q, \quad f = f(x, y, p, q).$$

As in Section 4.3.3, we have a projection to the natural bundle $\mathcal{F}''$ with coordinates $(u^1, \ldots, u^6)$ and all sections of interest project to a single section ω'' of $\mathcal{F}'' \to X$. Its symmetry algebroid $\mathfrak{g} \subset J_1(T)$ is defined by

$$\xi^1_x = -\xi^4_q + \xi^3_p - p\xi^1_y - 2q\xi^1_p, \quad \xi^1_q = \xi^3_q = \xi^3_q = 0, \quad \xi^2_x = \xi^3 - p\xi^3_p - p^2\xi^1_y,$$

$$\xi^2_y = -\xi^4_q + 2\xi^3_p + p\xi^1_y - 2q\xi^1_p, \quad \xi^2_p = \xi^1_p p, \quad \xi^3_x = \xi^4 - q\xi^4_q - \xi^3_y p - q^2\xi^1_p.$$

We use Proposition 4.33 to define $\mathcal{F}$ with single coordinate (f). The infinitesimal $\mathfrak{g}$-action on $\mathcal{F}$ is given by the vector field

$$L = \xi^1\partial_x + \xi^2\partial_y + \xi^3\partial_p + \xi^4\partial_q + (2f\xi^4_q - f\xi^3_p + f\xi^1_p q + \xi^4_x + p\xi^4_y + q\xi^4_p)\partial_f$$

The first three prolongations and projections produce the following bundles $\mathcal{F}_{(i)} \subset J_1(\mathcal{F}_{(i-1)})/K^2_1$ and their corresponding fibre coordinates

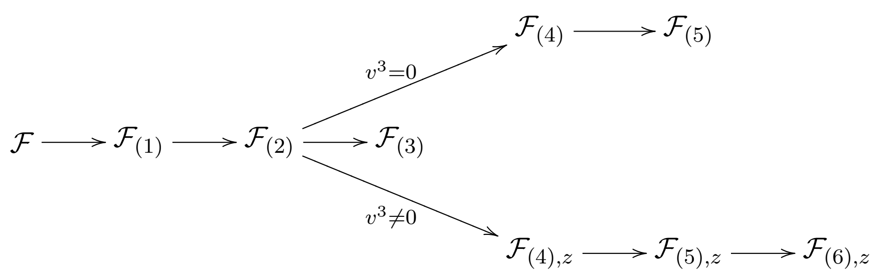
$$\begin{aligned}
\mathcal{F}_{(1)} &: v^1 = f_q,\\
\mathcal{F}_{(2)} &: v^2 = v^1_x + pv^1_y + qv^1_p + fv^1_q - 3f_p,\\
\mathcal{F}_{(3)} &: v^3 = v^2_x + pv^2_y + qv^2_p + fv^2_q + 6f_y - 4v^1 f_p - \frac{4}{9}(v^1)^3 + 2v^1v^2.
\end{aligned}$$

On $\mathcal{F}_{(1)} \to \mathcal{F}$ and $\mathcal{F}_{(2)} \to \mathcal{F}_{(1)}$, there are no equivariant sections, but on $\mathcal{F}_{(3)}$ the Vessiot structure equations are

$$v^3 = v^2_x + pv^2_y + qv^2_p + fv^2_q + 6f_y - 4v^1 f_p - \frac{4}{9}(v^1)^3 + 2v^1v^2 = 0.$$

Plugging in the values of v^1 and v^2, we realise that $v^3 = I$ is exactly the Wünschmann semi-invariant [Neu03, eq. (4.5)]. If the Vessiot structure equations are satisfied, the symmetry groupoids are not necessarily integrable, since the symbols for generic sections of $\mathcal{F}_{(2)} \to X$ are not 2-acyclic. The advantage of computing $\mathcal{F}_{(3)}$ is that we can identify a natural subbundle of $J_1(\mathcal{F}_{(2)}) \to \mathcal{F}_{(2)}$ for third order ODEs which become nongeneric in the next step. Detecting this bundle on $J_2(\mathcal{F}_{(2)})$ is nearly impossible.

In the following, we distinguish the cases $I = v^3 = 0$ and $v^3 \neq 0$ before continuing the computations as displayed in the following diagram. Each arrow stands for a prolongation and projection.



The Case $v^3 = 0$

The equation $v^3 = 0$ defines a subbundle of $\mathcal{F}_{(3)} \to \mathcal{F}_{(2)}$ which is isomorphic to $\mathcal{F}_{(2)}$. Computing a single prolongation and then projecting will not yield any new integrability conditions, so we compute the minimal bundle

$$\mathcal{F}_{(4),z} \subseteq J_2(\mathcal{F}_{(2)})/K^3_2$$

to which all sections from $\mathcal{F} \to X$ with $v^3 = 0$ restrict. For the first prolongation, the coordinates are

$$\begin{array}{llll} w^1 = f_x, & w^2 = f_y, & w^3 = f_p, & \\ & w^4 = v^1_y, & w^5 = v^1_p, & w^6 = v^1_q, \\ & w^7 = v^2_y, & w^8 = v^2_p, & w^9 = v^2_q. \end{array}$$

The fibre coordinates of $\mathcal{F}_{(4),z}$ are

$$b^1 = w^6_q, \qquad b^2 = w^9_q, \qquad b^3 = w^8_q + 2w^4_q - \frac{4}{3}v^1 w^5_q$$

and there are no equivariant sections on $\mathcal{F}_{(4),z} \to J_1(\mathcal{F}_{(2)})$. From here on, the symbols are 2-acyclic such that single prolongations are sufficient. The prolongation in [Neu03] is necessary, because the Cartan equivalence method relies on involutive symbols. The next bundle $\mathcal{F}_{(5),z}$ has a five-dimensional fibre

$$c^1 = b^1_q, \qquad c^2 = b^2_q, \qquad c^4 = b^2_p + 2b^1_y - \frac{4}{3}v^1 b^1_p - \frac{8}{3}w^6 w^5_q$$

$$\begin{aligned} c^5 &= b^3_p - \frac{1}{2}b^2_y + (\frac{2}{9}(v^1)^2 - \frac{1}{3}v^2)b^1_p - \frac{4}{3}w^6 w^5_p + (\frac{8}{9}v^1 w^6 - \frac{4}{3}w^5)w^5_q - \frac{2}{3}w^6 w^8_q \\ c^6 &= b^3_x + pb^3_y + qb^3_p - 4w^6 w^3_p + 2w^4_p - \frac{4}{3}w^5_p v^1 \\ &+ (\frac{4}{9}(v^1)^2 - \frac{2}{3}v^2 + \frac{4}{3}f w^6)w^5_q - w^7_q + w^8_p. \end{aligned}$$

The Vessiot structure equations on $\mathcal{F}_{(5),z}$ are

$$\begin{aligned} c^1 &= 0, \\ c^2 &= 4w^6 b^1, \\ c^3 &= \frac{4}{9}(w^6)^3 - \frac{1}{3}w^6 b^2 + \frac{1}{3}(4w^5 + w^9)b^1, \\ c^4 &= -\frac{2}{27}v^1(w^6)^3 + \frac{2}{3}w^5(w^6)^2 + \frac{1}{18}(v^1 b^2 - 15b^3)w^6 - \frac{1}{3}b^2 w^5 \\ &+ \frac{1}{18}(9w^8 - 6w^4 - 4w^5 v^1 - w^9 v^1)b^1, \\ c^5 &= -\frac{4}{9}f(w^6)^3 + \frac{1}{3}(6w^8 + fb^2 - 3w^4 - 4v^1 w^5)w^6 + 4(w^5)^2 \\ &+ \frac{1}{3}(3w^9 - 4fb^1)w^5 + \frac{1}{3}(-6w^2 - fw^9 + 4v^1 w^3)b^1 - w^3 b^2 - \frac{1}{3}v^1 b^3. \end{aligned}$$

Plugging in the zero section $f = 0$, we are dealing with the case $v^3 = 0$ and the Vessiot equivalence equations are satisfied. Since there are no invariants on $\mathcal{F}_{(4),z}$, Theorem 6.10 and Proposition 6.11 then imply that all geometric objects on $\mathcal{F}$ with $v^3 = 0$ that satisfy the above Vessiot structure equations are eqivalent. In [Neu03], the conditions for the equivalence to $f = 0$ are the following.

Theorem 6.47. [Neu03, Thm 11] A third order ODE (6.36) is equivalent to $f = 0$ under contact transformations if and only if $f_{qqqq} = 0$ and the semi-invariant $v^3 = 0$ vanishes for $f(x, y, p, q)$. ⋄

Expressing the first two Vessiot structure equations by jets of f, we obtain

$$f_{qqqq} = 0, \qquad f_{xqqqq} + pf_{yqqqq} + qf_{pqqqq} + 3f_q f_{qqqq} + ff_{qqqqq} = 0.$$

These two equations are in accordance with Theorem 6.47, but the remaining three equations could not be interpreted this easily. It was not possible to find a simple counterexample and we suppose that these eqations are differential consequences of $f_{qqqq} = 0$ and $v^3 = 0$. Checks by Lange-Hegermann with an experimental version of the differential Thomas algorithm did not terminate.

The Generic Case (with $v^3 = 0$)

Due to the prolongation performed in [Neu03], the expression $I^5 = \frac{f_{qqqq}}{6a_1^3 a_9}$ is called invariant but still depends on the coordinates a_1 and a_9 of the fibre of the original G-structure. With this invariant, it is not possible to decide equivalence between generic third order ODEs with $v^3 = 0$, but the algorithm presented in [Neu03] terminates here.

Remembering that the Cartan equivalence method was defined by linear Pfaffian systems on the cartesian product $\mathcal{G} \times \mathcal{G}'$ of two G-structures, we can apply the Cartan algorithm for linear Pfaffian systems presented in [IL03, Ch. 5]. The invariant I^5 is one of the nonzero essential torsion coefficients which must be set constant in order to shrink $\mathcal{G} \times \mathcal{G}'$. Equivalence may be decided on these subbundles, but the computations become quite large.

Using Vessiot's approach, we are not finished after the construction of $\mathcal{F}_{(5),z}$ and the generic case with $v^3 = 0$ is obtained by further prolongations and projections. At first we turn $\mathcal{F}_{(5)}$ into a vector bundle by changing the coordinates c^i such that all Vessiot structure equations are of the form $c^i = 0$. The single invariant on $\mathcal{F}_{(5)}$ has the form

$$I^{5,z} = \frac{(c^5 - fc^3)(c^1)^{\frac{8}{3}}}{(12v^1c^1c^2 - 9(c^2)^2 - 8(v^1c^1)^2 + 12v^2(c^1)^2 - 36c^1c^3)^{\frac{5}{3}}}.$$

Computing the bundle $\mathcal{F}_{(6),z}$ by another prolongation and projection, the Θ_2-action becomes locally free and there are twelve invariants. Since five of them must be $I^{5,z}$ and its four invariant derivatives, we have seven new invariants. It was only possible to compute a single one

$$I_1^{6,z} = \frac{-9c_q^2c^1 + 7(c^1)^2w^6 - 6c^1c_p^1 + 2c_q^1v^1c^1 + 6c_q^1c^2}{(c^1)^{\frac{11}{5}}(c^5 + fc^3)^{\frac{1}{5}}}.$$

The invariants have been computed with the `JetGroupoids` standard procedure `InvariantsOnNaturalBundle`, which internally uses the MAPLE command `pdsolve`. Since this command improves from version to version, it is possible that a newer version (12 or higher) of MAPLE is capable of finding all invariants on $\mathcal{F}_{(6),z}$. It is likely that the invariants on $\mathcal{F}_{(6),z}$ form a generating set.

The Case $v^3 \neq 0$

For the case $v^3 \neq 0$, we prolong $\mathcal{F}_{(2)}$ twice. The minimal subbundle

$$\mathcal{F}_{(4)} \subseteq J_2(\mathcal{F}_{(2)})/K_2^3 \to \mathcal{F}_{(2)}$$

to which all sections from $\mathcal{F} \to X$ with $v^3 = 0$ restrict has the coordinates

$$\begin{array}{llll} w^1 = f_x, & w^2 = f_y, & w^3 = f_p, & \\ & w^4 = v_y^1, & w^5 = v_p^1, & w^6 = v_q^1, \\ w^7 = v_x^2 & w^8 = v_y^2, & w^9 = v_p^2, & w^{10} = v_q^2. \end{array}$$

for the first prolongation. The fibre of $\mathcal{F}_{(4)}$ is given by

$$\begin{aligned} c^1 &= w_q^6, \\ c^2 &= w_q^{10}, \\ c^3 &= 6w_y^2 - 4v^1 w_p^2 + w_y^7 + pw_y^8 + qw_p^8 + fw_q^8, \\ c^4 &= 6w_y^1 - 4v^1 w_p^1 + w_x^7 + pw_y^7 + qw_p^7 + fw_q^7, \\ c^5 &= 2w_q^4 - 4/3\, v^1 w_q^5 + w_q^9, \\ c^6 &= w_q^7 + pw_q^8 + qw_q^9, \\ c^7 &= 6w_p^2 - 4v^1 w_p^3 + w_p^7 + pw_p^8 + qw_p^9 + fw_q^9. \end{aligned}$$

There are neither invariants nor equivariant sections on $\mathcal{F}_{(4)}$ and the Θ_2-action is not yet free. Another prolongation and projection yields the bundle $\mathcal{F}_{(5)}$ with a 16-dimensional fibre and 15 invariants. The PDE system for equivariant sections on $\mathcal{F}_{(5)} \to \mathcal{F}_{(4)}$ has solutions, but it contains 300 equations and the computations did not terminate. Here, Cartan's equivalence method is more efficient. In [Neu03, Thm. 12], necessary and sufficient conditions for the equivalence to the equation $y_{xxx} = y$ are given.

It was only possible to compute seven of the fifteen invariants on $\mathcal{F}_{(5)}$. Using the coordinates

$$d^1 = v_q^3, \quad d^2 = v_p^3 - \frac{1}{2}v^3 v_q^1, \quad d^3 = v_x^3 + pv_y^3 + qv_q^3,$$

the smallest one has the form

$$I_1^5 = \frac{(3d_q^1\, v^3 - 2(d^2)^2)(v^3)^{\frac{8}{3}}}{(f(d^1)^2 - 3v^3 d^2 - 2v^1 v^3 d^1 + d^1 d^3)^2}$$

and all other invariants can be obtained from [Lor08a]. The coordinates d^i already indicate that they were not computed on $\mathcal{F}_{(5)}$ but on the bundle obtained by prolonging and projection $\mathcal{F}_{(3)}$ twice (ignoring the fact that the symbols do not have 2-acyclic symbols). For brevity, we do not express it in coordinates of $\mathcal{F}_{(5)}$.

6.3.2 Fourth Order ODEs

The example of fourth order ODEs under contact transformations is similar to both Example 4.32 and the third order ODEs in Section 6.3.1. With Cartan's equivalence method it has been calculated in [Neu03, Ch. 5]. We keep the introduction to this example short. The fourth order ODE is defined by

$$y_{xxxx} = f(x, y, p = y_x, q = y_{xx}, r = y_{xxx})$$

and leads to the coframe

$$\omega^1 = dx, \quad \omega^2 = dy - pdx, \quad \omega^3 = dp - qdx, \quad \omega^4 = dq - rdx$$

$$\omega^5 = dr - f(x, y, p, q, r)dx$$

on $X = J_4(\mathbb{R} \times \mathbb{R})$. The structure group is determined by the matrix

$$G = \begin{bmatrix} X_x & X_y & X_p & 0 & 0 \\ 0 & Y_y & 0 & 0 & 0 \\ 0 & P_y & P_p & 0 & 0 \\ 0 & Q_y & Q_p & Q_q & 0 \\ 0 & R_y & R_p & R_q & R_r \end{bmatrix}.$$

Analogous to the previous section, we compute the natural bundle P_1/G and restrict Π_1 to a groupoid Θ_1. The corresponding algebroid action on the subbundle $\mathcal{F}$ of $P_1/G \to X$ with coordinate (f) is given by the vector field

$$L = \xi^1\partial_x + \xi^2\partial_y + \xi^3\partial_p + \xi^4\partial_q + \xi^5\partial_r + (2f\xi^5_r - f\xi^4_q + p\xi^5_y + q\xi^5_p + r\xi^5_q + \xi^5_x)\partial_f.$$

The first prolongation and projection yields the natural bundle $\mathcal{F}_{(1)} = J_1(\mathcal{F})/K^2_1$ with coordinates

$$v^1 = f_r, \qquad v^2 = f_q.$$

There are no equivariant sections on $\mathcal{F}_{(1)} \to \mathcal{F}$ and have to prolong and project further to check equivalence. Computing the Spencer cohomology for generic sections on $\mathcal{F}_{(1)} \to X$ shows that the symbols are not 2-acyclic and we have to prolong twice. The dimensions of the Spencer cohomology groups are

$$\begin{bmatrix} 0 & 0 & 3 & 7 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the bundle $\mathcal{F}_{(2)}$ is the minimal subbundle of $J_2(\mathcal{F}_{(1)})/K^3_2 \to mcF_{(1)}$ to which all sections from $\mathcal{F} \to X$ restrict. It has the coordinates

$$\begin{array}{lllll} w^1 = f_x, & w^2 = f_y, & w^3 = f_p, & & \\ w^4 = v^1_x, & w^5 = v^1_y, & w^6 = v^1_p, & w^7 = v^1_q, & w^8 = v^1_r, \\ w^9 = v^2_x, & w^{10} = v^2_y, & w^{11} = v^2_p, & w^{12} = v^2_q & \end{array}$$

for the $J_1(\mathcal{F}_{(1)})$-part. We have omitted $v^2_r = f_{qr}$, since it coincides with $v^1_q = f_{qr}$ on the minimal subbundle. The coordinates for the highest order fibre are

$$b^1 = w^4_r,\quad b^2 = w^5_r,\quad b^3 = w^6_r,\quad b^4 = w^{12}_r,\quad b^5 = w^7_r,\quad b^6 = w^8_r,$$

$$\begin{aligned}
b^7 &= -2w^{11}_r + w^{12}_q,\\
b^8 &= w^4_y + pw^5_y + qw^5_p + rw^{10}_r,\\
b^9 &= w^4_p + pw^5_p + qw^6_p + \frac{1}{5}rw^{11}_r + \frac{2}{5}rw^{12}_q,\\
b^{10} &= w^9_r + w^{10}_r p + \frac{1}{5}w^{11}_r q + \frac{2}{5}qw^{12}_q,\\
b^{11} &= w^4_x + pw^4_y + qw^4_p + rw^9_r,\\
b^{12} &= w^9_y + pw^{10}_y + qw^{10}_p + rw^{10}_q + fw^{10}_r - 4w^2_p,\\
b^{13} &= w^9_x + pw^9_y + qw^9_p + rw^9_q + fw^9_r - 4w^1_p,\\
b^{14} &= \frac{2}{3}w^{10}_r - 2w^6_p + \frac{2}{15}v^1 w^{11}_r + w^{11}_q + \frac{4}{15}v^1 w^{12}_q,\\
b^{15} &= -4w^3_p + \frac{1}{5}w^{11}_r f + w^9_p + w^{10}_p p + w^{11}_p q + w^{11}_q r + \frac{2}{5}fw^{12}_q,\\
b^{16} &= \frac{2}{5}w^{11}_r r + w^9_q + w^{10}_q p + w^{11}_q q + \frac{4}{5}w^{12}_q r.
\end{aligned}$$

Again, there are no equivariant sections on $\mathcal{F}_{(2)} \to J_1(\mathcal{F}_{(1)})$, but the Θ_2-action is free. It was possible to compute all five invariants on $\mathcal{F}_{(2)}$. The smallest one has the form

$$I^2_1 = \frac{n}{(b^6)^2 d^{\frac{2}{3}}}$$

with numerator

$$\begin{aligned}
n &= 15\left[-84v^2 - 9(v^1)^2 + 36w^7 r + 36qw^6 + 36w^5 p + 36w^4 + 56fw^8\right](b^6)^2\\
&+ 150\left[6b^5 v^1 + 6b^4 - 12b^3 + (w^8 v^1 + 2rb^5 + 2b^1 + 2qb^3 + 2pb^2)w^8\right]b^6\\
&+ 60(w^8)^2 b^5 + 180(b^5)^2 + 5(w^8)^4
\end{aligned}$$

and the factor d of the denominator

$$\begin{aligned}
d &= 5(v^1)^3 - 10(-2v^2 + 3pw^5 + fw^8 + 3rw^7 + 3w^4 + 3qw^6)v^1\\
&+ 20b^{11} + 20w^8 w^1 + 40w^3 - 40w^9 - 40pw^{10} - 16rqb^7 + 40qfb^3\\
&+ 20pw^8 w^2 + 20pb^8 + 40rfb^5 - 20(-2b^1 + w^7 - 2pb^2)f + 20b^4 r^2\\
&+ 20(w^6 - 2w^{12} + w^8 v^2 + b^{10})r + 20(w^5 - 2w^{11} + b^9 + w^8 w^3)q + 20b^6 f^2.
\end{aligned}$$

For obvious reasons, we do not plug in the coordinates v^i, w^i and b^i to pull back the invariant to $J_3(\mathcal{F})$. The remaining four invariants are available at [Lor08a].

To decide the equivalence of fourth order ODEs, the bundle $\mathcal{F}_{(3)}$ must be computed, but even the first prolongation of $\mathcal{F}_{(2)}$ did not finish. Also for fourth order ODEs, Cartan's equivalence method is more efficient.

Chapter 7

Application to Linear Partial Differential Operators

When solving a linear partial differential equation

$$Lu = (a_\mu(x)\partial^\mu)u(x) = 0,$$

a factorisation of the linear partial differential operator (LPDO) $L = L_1 L_2$ will reduce the search for a solution to its factors L_1 and L_2. If the operator is completely factorisable, solutions can be found by quadratures. The factorisation of LPDOs has been of interest recently. Grigoriev and Schwarz [GS04] give an algorithm for separable LPDOs similar to Hensel lifting, Tsarev [Tsa00] considers the problem in connection with Darboux integrability. For bivariate operators, Beals and Kartashova [BK05],[Kar06] show how to separate a first order factor from the operator. Shemyakova and Winkler [SW07b] tackle the problem by introducing the notion of obstacles to a factorisation.

The connection between factorisations of LPDOs and equivalence problems are gauge transformations of the operator L

$$L \mapsto g^{-1} L g,$$

where g is an invertible function. Gauge transformations preserve factorisations $L = L_1 L_2$, so the conditions for factorisation can be formulated by invariants. Shemyakova and Winkler [SW07a] present a generating set of invariants for a third order operator on the plane to express the factorisation conditions [SW08]. More systematically, Mansfield and Shemyakova [MS08] compute the invariants for relevant third order LPDOs on the plane via moving frames.

With a straightforward modification, the Vessiot equivalence method can be applied to compute generating sets of invariants for LPDOs under gauge transformations. Vessiot's method allows to compute examples of order three and four, which are considerably larger than in [MS08]. Instead of using moving frames to compute invariants, we integrate the vector fields of the algebroid action on the natural bundles. In this way, we can avoid the prolongation of finite transformations which become rather large.

The present work was initiated by a talk of Winkler in january 2008 at RWTH Aachen and a visit to RISC at Hagenberg in may 2008. The examples of LPDOs under gauge transformations stimulated the development of Vessiot's methods already presented in the previous chapters. Actually, they provided the first examples of natural Θ_q-bundles and led to the realisation that all proofs in Chapter 3, originally done for full jet groupoids Π_q, are valid for subgroupoids Θ_q.

The chapter is structured as follows. In Section 7.1 we introduce the necessary notation for LPDOs and show how to apply Vessiot's equivalence method. In Section 7.2, we compute generating sets of invariants for LPDOs, especially fourth order ones. For selected LPDOs we give conditions for the existence of a factorisation in terms of invariants.

7.1 LPDOs and Vessiot's Equivalence Method

In this section we describe how to apply Vessiot's equivalence method to LPDOs under gauge transformations. After introducing LPDOs, we construct groupoids of gauge transformations and natural bundles for LPDOs. The only difficulty is that natural bundles are designed to continue diffeomorphisms on the base manifold to the whole bundle, but gauge transformations are not induced by base transformations.

7.1.1 Linear Partial Differential Operators

Let K be a field with n commuting derivations $\partial_1, \dots, \partial_n$ and consider the differential algebra $D = K\langle\partial_1, \dots, \partial_n\rangle$. A standard example is the field of rational functions $K = k(x^1, \dots, x^n)$ with ground field k of characteristic zero and the partial derivatives ∂_{x^i} as derivations. Having fibre bundles in mind, we think of K as smooth functions on a manifold X. Elements of D are linear partial differential operators (LPDOs) of the form

$$L = a_\mu \partial^\mu, \qquad \mu \in (\mathbb{Z}_{\geq 0})^n, |\mu| \leq q,\ a_\mu \in K.$$

The *order* of L is the maximum $\operatorname{ord}(L) = \max\{|\mu| \,|a_\mu \neq 0\}$. The symbol of L is the polynomial $\operatorname{sym}(L) \in K[X^1, \dots, X^n]$ of the form

$$\operatorname{sym}(L) = a_\mu X^\mu, \qquad |\mu| = \operatorname{ord}(L).$$

In this chapter, we consider factorisations $L = L_1 \dots L_k$ for $L_i \in D$, $1 \leq i \leq k \in \mathbb{N}$ where each factor has at least order one ($\operatorname{ord}(L_i) \geq 1$). Then a necessary condition for the existence of a factorisation is that the symbol also factorises:

$$\operatorname{sym}(L) = \operatorname{sym}(L_1) \cdots \operatorname{sym}(L_k).$$

A factorisation $L = L_1 \cdots L_k$ is called of type $(S_1) \dots (S_k)$ if $\operatorname{sym}(L_i) = S_i$.

Definition 7.1. The automorphism $D \to D : L \mapsto g^{-1}Lg$ for $g \in K^*$ is called *gauge transformation.* K^* are the units of K. ◇

Gauge transformations can also be seen as a right K^*-action on D. Obviously, the symbol of a LPDO stays invariant under gauge transformations. The simple calculation

$$(g^{-1}Lg)u = g^{-1}L(gu)$$

shows that solving the PDE for the gauge transformed operator is equivalent to solving the equation for the transformed function gu. Concerning the factorisation of an operator, the associative law of D implies the following important property.

Lemma 7.2. A factorisation $L = L_1 \cdots L_k$ is gauge invariant:

$$g^{-1}Lg = (g^{-1}L_1g) \cdot (g^{-1}L_2g) \cdots (g^{-1}L_kg), \qquad \forall g \in K^*.$$ ◇

Since a factorisation of an LPDO is gauge invariant, also the conditions for the existence of a factorisation must be invariant. This is the reason why one is interested in the invariants for LPDOs under gauge transformations.

Example 7.3. A classical example of LPDOs was considered by Laplace. He studied the second order operator

$$L = \partial_x\partial_y + a\partial_x + b\partial_y + c$$

with parameters $a, b, c \in K$ depending on x and y. It has the symbol $\mathrm{sym}(L) = XY$ and factorisations are of the form $L = L_1L_2$ or $L = L_2L_1$ for

$$L_1 = \partial_x + d, \qquad L_2 = \partial_y + e, \qquad d, e \in K.$$

Both possible factorisations imply $d = b$ and $e = a$ and L has a factorisation of the first type if and only if the first Laplace invariant h vanishes

$$h := L - L_1L_2 = a_x - c + ab.$$

The second factorisation is possible if and only the second Laplace invariant is zero

$$k := L - L_2L_1 = b_y - c + ab.$$

The Laplace invariants $\{h, k\}$ are a generating set of invariants with respect to gauge transformations of L given by

$$g^{-1}Lg = \partial_x\partial_y + \left(a + \frac{g_y}{g}\right)\partial_x + \left(b + \frac{g_x}{g}\right)\partial_y + \left(c + a\,\frac{g_x}{g} + b\,\frac{g_x}{g} + \frac{g_{xy}}{g}\right).$$

Comparing the coefficients of the derivatives of L and $g^{-1}Lg$, a simple calculation shows that h and k are invariant. ◇

7.1.2 Modifications of Vessiot's Approach

In order to deal with LPDOs under gauge transformations, Vessiot's approach must be slightly adapted. Before going into details, we give an outline for the necessary modifications. The requirements are as follows.

- Let $X \subseteq \mathbb{R}^n$ be a manifold with coordinates $(x) = (x^1, \dots x^n)$. Then a LPDO of order q has the form

$$L = a_\mu(x)\partial^\mu, \qquad |\mu| \le q,\ a_\mu \in C^\infty(X) \tag{7.1}$$

  where ∂_{x^i} is the partial derivative with respect to the coordinate x^i and $\partial^\mu = \partial_{x^i}^{\mu_i}$. All coefficients are smooth functions on X.

- Gauge transformations are defined on the fibre bundle $Y = X \times \mathbb{R}$ with coordinates (x, u). Here u plays the role of the solution function. A gauge transformation is diffeomorphism $Y \to Y$ with

$$\hat{x} = x, \qquad \hat{u} = gu, \qquad g \in C^\infty(X). \tag{7.2}$$

The first point implies that LPDOs are sections of the bundle $\tilde{\mathcal{F}} \to X$ with coordinates $(x, a_\mu \,|\, |\mu| \le q)$ and coordinate changes induced by gauge transformations. But the second point requires a natural Θ_q-bundle $\mathcal{F} \to Y$ over the larger base manifold Y in order to apply Vessiot's approach. Here Θ_q is the groupoid of gauge transformations.

The key observation to solve this problem is that gauge transformations respect the bundle structure of $\pi : Y \to X$. We can thus take the pullback

$$\mathcal{F} = \pi^*(\tilde{\mathcal{F}}) \tag{7.3}$$

as natural Θ_q-bundle over Y. By construction of the coordinate changes on $\tilde{\mathcal{F}}$, gauge transformations lift to $\mathcal{F}$. For the prolongation and projection, we will replace $J_r(\mathcal{F}) \to \mathcal{F}$ by the subbundle $\pi^*(J_r(\tilde{\mathcal{F}}))$ containing only x-jets.

Groupoids of Bundle Morphisms

The situation of LPDOs under gauge transformations can be generalised. Let $\pi : Y \to X$ be a fibre bundle and $\mathcal{F} \to Y$ a natural Θ_q-bundle. We ask for conditions under which there are well-defined sections $\omega : Y \to \mathcal{F}$ that depend on X only, namely that $\omega(y) = \omega(y')$ for all $y, y' \in Y$ with $\pi(y) = \pi(y')$. If this is established, we ask for groupoids Θ_q that act on the subbundle $J_{r,X}(\mathcal{F})$ of $J_r(\mathcal{F}) \to \mathcal{F}$ where we have taken only jets with respect to the coordinates of X. They are described in the following definition.

Definition 7.4. Let $\pi : Y \to X$ be a fibre bundle. A groupoid $G \subset \Pi_q(Y \times Y)$ is called *groupoid of bundle morphisms* if all $g \in G$ are q-jets of (local) bundle morphisms $\psi : Y \to Y$ over some morphism $\varphi : X \to X$. ◇

Choose two coordinate systems (x, u) and $(\hat{x}, \hat{u})$ of Y such that (x) and $(\hat{x})$ are coordinates for X. Then $\Pi_{1,\pi}$ defined by the equations $\hat{x}_u = 0$ is the largest groupoid of bundle morphisms, which follows directly from equation (1.1). All groupoids of bundle morphisms Θ_q are subgroupoids of the prolongation $\Pi_{q,\pi}$ of $\Pi_{1,\pi}$ to order q.

Example 7.5. On $Y = X \times \mathbb{R}$, the groupoids of gauge transformations $\Theta_q \subseteq \Pi_{q,\pi}$, defined by the prolongation of

$$\hat{x}^i = x^i, \qquad u\,\hat{u}_u = \hat{u}, \qquad i = 1, \ldots, n, \tag{7.4}$$

are groupoids of bundle morphisms. Solutions of Θ_q coincide with equation (7.2)

$$\hat{x}^i(x, u) = x^i, \qquad \hat{u}(x, u) = g(x)\,u, \qquad g(x) \neq 0.$$

The groupoid of gauge transformations should not be confused with the gauge groupoid Gauge(P) of a principal bundle P. Θ_q is not transitive.

Alternatively, one may add transformations of the base X. The groupoid Θ'_q of gauge and base transformations is defined by the equations

$$\hat{x}_u = 0, \qquad u\,\hat{u}_u = \hat{u}.$$

On linear partial differential operators, it corresponds to transformations of the independent variables combined with gauge transformations.

$$\hat{x}(x, u) = \varphi(x), \qquad \hat{u}(x, u) = g(x)\,u, \qquad g(x) \neq 0.$$

◇

The next lemma deals with X-dependent sections on a natural bundle and shows that the pullback of equation (7.3) is the only possible idea.

Lemma 7.6. Let $\pi : Y \to X$ be a fibre bundle. A fibre bundle $\mathcal{F} \to Y$ has well-defined sections ω that depend only on X if and only if $\mathcal{F}$ is isomorphic to the pullback $\pi^*(\tilde{\mathcal{F}})$ of a bundle $\tilde{\mathcal{F}} \to X$. ◇

Proof. For coordinates (x, u, a) and $(\hat{x}, \hat{u}, \hat{a})$ of $\mathcal{F}$ a coordinate change is of the form

$$\hat{x} = \varphi(x), \quad \hat{u} = \psi(x, u), \quad \hat{a} = \chi(x, u, a).$$

In coordinates $(\hat{x}, \hat{u}, \hat{a})$, an X-dependent section ω is given by $\hat{a}^\alpha = \omega^\alpha(\hat{x})$. Pulling back ω with the coordinate change we obtain

$$\chi^\alpha(x, u, a) = \omega^\alpha(\varphi(x)),$$

which has to be solved for a^β in order to find ω in coordinates (x, u, a). The transformed section depends on u if and only if χ does. Assume $\chi = \chi(x, a)$. Then the projection

$$\mathcal{F} \to \tilde{\mathcal{F}} : (x, u, a) \mapsto (x, a)$$

gives a well-defined bundle $\tilde{\mathcal{F}} \to X$ such that $\mathcal{F} = \pi^*(\tilde{\mathcal{F}})$. The converse is trivial, as the transition functions of $\tilde{\mathcal{F}}$ are of the form $\hat{x} = \varphi(x)$, $\hat{a} = \chi(x, a)$. □

Example 7.7. For the Laplace example 7.3, the bundle of LPDOs $\tilde{\mathcal{F}} \to X = \mathbb{R}^2$ has the coordinates (x, y, a, b, c) and the coordinate changes in Vessiot notation

$$a = \hat{a} + \frac{g_y}{g}, \quad b = \hat{b} + \frac{g_x}{g}, \quad c = \hat{c} + \frac{g_{xy}}{g} + \frac{g_x}{g}\hat{a} + \frac{g_y}{g}\hat{b}. \tag{7.5}$$

Choose two coordinate systems (x, y, u) and $(\hat{x}, \hat{y}, \hat{u})$ for the bundle $Y = X \times \mathbb{R}$. Then the pullback $\mathcal{F} = \pi^*(\tilde{\mathcal{F}})$ is performed by adding the coordinate u and the base transformations

$$\hat{x} = x, \qquad \hat{y} = y, \qquad \hat{u} = g(x, y)u. \tag{7.6}$$

The last step is to recover the action of the groupoid Θ_2 of gauge transformations on $\mathcal{F}$. It is given by

$$a = \hat{a} + \frac{\hat{u}_y}{\hat{u}}, \quad b = \hat{b} + \frac{\hat{u}_x}{\hat{u}}, \quad c = \hat{c} + \frac{\hat{u}_{xy}}{\hat{u}} + \frac{\hat{u}_x}{\hat{u}}\hat{a} + \frac{\hat{u}_y}{\hat{u}}\hat{b}, \tag{7.7}$$

since equation (7.6) implies $\hat{u}_x/\hat{u} = g_x/g$ and so on.

In [MS08], the function $g(x, y)$ is replaced by $\exp(g(x, y))$ to avoid fractions in equation (7.5). This is impossible for the groupoid action, but for practical computations we use the infinitesimal algebroid action. Here the replacement only changes the vector fields by a factor and all computations are identical. $\diamond$

In the remainder of this section, we fix a groupoid of bundle morphisms $\Theta_q \leq \Pi_{q,\pi}$ for a bundle $\pi : Y \to X$ and a natural Θ_q-bundle $\mathcal{F} = \pi^*(\tilde{\mathcal{F}})$. Let (x, u, a) and $(\hat{x}, \hat{u}, \hat{a})$ be coordinate systems of $\mathcal{F}$. If no confusion arises, we suppress all indices of $(x, u, a) = (x^i, u^j, a^\alpha)$. Under the assumption that Θ_q acts on $\mathcal{F}$ via admissible coordinate changes of $\tilde{\mathcal{F}}$, we can replace the prolongation $J_r(\mathcal{F}) \to \mathcal{F}$ by the subbundle $\pi^*(J_r(\tilde{\mathcal{F}})$ where only x-jets appear.

Let $\psi : Y \to Y$ be a bundle morphism over $\varphi : X \to X$ which is a local solution of Θ_q. Then Θ_q acts via admissible coordinate changes if the lift of ψ to the bundle $\mathcal{F}$

$$\hat{x} = \varphi(x), \quad \hat{u} = \psi(x, u), \quad a = \Phi_{\hat{a}}(\varphi(x), \psi(x, u), \varphi_q(x), \psi_q(x, u)).$$

depends on the derivatives of φ and ψ up to order q, but all u-derivatives cancel

$$\partial_u \Phi_{\hat{a}}(\varphi(x), \psi(x, u), \varphi_q(x), \psi_q(x, u)) = 0. \tag{7.8}$$

This has important consequences for the prolongation of the Θ_q-action on $\mathcal{F}$. The Θ_q-action on $\mathcal{F}$ is given by

$$a = \Phi_{\hat{a}}(x, u, \hat{x}, \hat{u}, \hat{x}_q, \hat{u}_q)$$

for $(x, u, \hat{x}, \hat{u}, \hat{x}_q, \hat{u}_q) \in \Theta_q$. We use Remark 1.13 to prolong the action to $J_1(\mathcal{F})$. Equation (7.8) implies

$$a_u = D_u \Phi_{\hat{a}}(\hat{x}, \hat{u}, \hat{x}_q, \hat{u}_q) = (\underbrace{\hat{a}_{\hat{x}}\hat{x}_u}_{=0} + \hat{a}_{\hat{u}}\hat{u}_u)\, \partial_{\hat{a}} \Phi_{\hat{a}}(\hat{x}, \hat{u}, \hat{x}_q, \hat{u}_q). \tag{7.9}$$

Remark 7.8. If $\mathcal{F}$ is a natural $\Pi_{q,\pi}$-bundle, equation (7.8) implies that the dependence on ψ and ψ_q must vanish completely, as they depend arbitrarily on u. For subgroupoids $\Theta_q \subset \Pi_{q,\pi}$, the situation may be different.

The general form of an LPDO (7.1) implies that the action depends only on combinations $\frac{\hat{u}_\mu}{\hat{u}}$ for $\mu \in (\mathbb{Z}_{\geq 0})^{n+1}$ with $\mu_{n+1} = 0$. The calculation

$$D_u\left(\frac{\hat{u}_\mu}{\hat{u}}\right) = \frac{\hat{u}_{\mu+1_u}}{\hat{u}} - \frac{\hat{u}_u \hat{u}_\mu}{\hat{u}^2} = \frac{\hat{u}_\mu}{u\,\hat{u}} - \frac{\hat{u}\,\hat{u}_\mu}{u\,\hat{u}^2} = 0.$$

using the defining equations (7.4) for Θ_q shows that equation (7.8) holds for gauge transformations of LPDOs. ◇

So far, we have constructed the groupoid of gauge transformations and suitable natural bundles for LPDOs. Equation (7.9) is the key observation to eliminate the additional u-jets in the prolongation of $\mathcal{F}$.

Theorem 7.9. Let $\mathcal{F} = \pi^*(\tilde{\mathcal{F}})$ be an affine natural Θ_q-bundle such that the groupoid of bundle morphisms Θ_q acts via admissible coordinate changes. Then there is an exact sequence of natural Θ_{q+r}-bundles

$$0 \longrightarrow \pi^*(J_r(\tilde{\mathcal{F}})) \longrightarrow J_r(\mathcal{F}) \longrightarrow \mathcal{G}_r \longrightarrow 0.$$

The vector bundles $\mathcal{G}_r$ have coordinates $(x, u, a, a_\mu \,|\, \mu = \nu + 1_u, |\nu| < r)$, where each jet contains at least one u-'derivative'. ◇

Proof. Compute the Θ_{q+r}-action on $\mathcal{G}_r$. Each coordinate of $\mathcal{G}_r$ transforms as

$$a_{\nu+1_u} = D_{\mu+1_u}\Phi_{\hat{a}}(\hat{x}, \hat{u}, \hat{x}_q, \hat{u}_q) = D_\mu(\hat{a}_{\hat{u}}\,\hat{u}_u\,\partial_{\hat{a}}\Phi_{\hat{a}}). \tag{7.10}$$

Since $\mathcal{F}$ is an affine natural Θ_q-bundle, $\partial_{\hat{a}}\Phi_{\hat{a}}$ is independent from $\hat{a}$ and thus $\mathcal{G}_r$ is a natural vector bundle. By construction, the map $J_r(\mathcal{F}) \to \mathcal{G}_r$ is a morphism of natural bundles and the preimage of the zero section of $\mathcal{G}_r \to \mathcal{F}$ is $\pi^*(J_r(\tilde{\mathcal{F}}))$. □

For simplicity, we denote the subbundle $\pi^*(J_r(\tilde{\mathcal{F}}))$ of $J_r(\mathcal{F}) \to \mathcal{F}$ by $J_{r,X}(\mathcal{F})$, indicating that only the jets with respect to X are coordinates.

Remark 7.10. If in Theorem 7.9 the assumption that $\mathcal{F}$ is an affine bundle is dropped, there is still an embedding of natural Θ_{q+r}-bundles $J_{r,X}(\mathcal{F}) \hookrightarrow J_r(\mathcal{F})$. ◇

Proof. Equation (7.10) shows that setting $a_{\nu+1_u} = 0$ for $0 \leq |\nu| < r$ defines a subbundle, since after applying D_μ, each summand on the right hand side contains at least one factor $\hat{a}_{\gamma+1_{\hat{u}}}$ with $\hat{u}$-'derivative'. □

We have completed the theoretical preparations to treat LPDOs under gauge transformations. For practical computations with `jets` and `JetGroupoids` in Section 7.2, we define the PDE system $a_u = 0$ on $J_1(\mathcal{F})$ and restrict to the prolongations of this system.

Example 7.11. The bundle $\mathcal{G}_1$ from Theorem 7.9 can be explicitly determined for the Laplace example 7.7. The fibre of $J_1(\mathcal{F})$ has nine coordinates $(a_x, \dots, c_u)$ and the fibre of $\mathcal{G}_1$ has the coordinates (a_u, b_u, c_u) and the Θ_1-action is

$$\begin{aligned} a_u &= D_u(\hat{a} + \frac{\hat{u}_y}{\hat{u}}) = \hat{a}_{\hat{u}}\hat{u}_u = \hat{a}_{\hat{u}}\frac{\hat{u}}{u}, \\ b_u &= \hat{b}_{\hat{u}}\frac{\hat{u}}{u}, \\ b_u &= \hat{c}_{\hat{u}}\frac{\hat{u}}{u} + \hat{a}_{\hat{u}}\frac{\hat{u}_x}{u} + \hat{b}_{\hat{u}}\frac{\hat{u}_y}{u}. \end{aligned}$$

The Θ_2-action on $J_1(\mathcal{F})$ is given by

$$a_x = \hat{a}_X + \hat{a}_{\hat{u}}\hat{u}_x + \frac{\hat{u}_{xy}\,\hat{u} - \hat{u}_x}{\hat{u}^2}, \quad a_y = \hat{a}_Y + \hat{a}_{\hat{u}}\hat{u}_y + \frac{\hat{u}_{yy}\,\hat{u} - \hat{u}_y}{\hat{u}^2}, \quad \dots$$

and for the restriction to $J_{r,X}(\mathcal{F})$, we have to set $\hat{a}_{\hat{u}} = 0$. Alternatively, we can compute $J_1(\tilde{\mathcal{F}})$ and add the base coordinate u. ◇

To compute a generating set of invariants for LPDOs under gauge transformations, we have the following simple, but useful lemma.

Lemma 7.12. Let Θ_q be the groupoid of gauge transformations on the bundle $Y \to X$ with coordinates (x, u). Let $\mathcal{F} = \pi^*(\tilde{\mathcal{F}})$ be a natural Θ_q-bundle and $\psi : J_{r,X}(\mathcal{F}) \to \mathbb{R}$ an invariant. Then $D_x\psi$ is also an invariant on $J_{r+1,X}(\mathcal{F})$. ◇

Proof. Treat ψ as a coordinate. Since it is invariant, the Θ_{q+r}-action on ψ is $\psi = \hat{\psi}$. Its prolongation to $J_{r+1,X}(\mathcal{F})$ is computed by setting $\hat{\psi}_{\hat{u}} = 0$ in

$$\psi_{x^i} = D_{x^i}\psi = \hat{\psi}_{\hat{x}^j}\,\hat{x}^j_{x^i} + \hat{\psi}_{\hat{u}}\,\hat{u}_{x^i}.$$

By equation (7.4), we obtain $\psi_{x^i} = \hat{\psi}_{\hat{x}^i}$. □

7.2 Examples

The last section shows that Vessiot's approach to geometric structures also applies to LPDOs and that it is possible to compute generating sets of invariants. In this section, we present several examples of LPDOs and their invariants. In Section 7.2.1, the trivial Laplace example is used to present the MAPLE worksheet. The relevant examples start in Section 7.2.2 with LPDOs of third order on the plane. They were already computed by Mansfield and Shemyakova [MS08] (see also [SW07a]) using moving frames. A comparison shows that moving frames produces eventually smaller generating sets, while Vessiot's approach detects the minimal order for generating sets of invariants.

The remaining examples are new results, including the full third order LPDO on the plane at the end of Section 7.2.2. To illustrate that Vessiot's approach is not limited to two-dimensional base manifolds we treat an example of a third order operator on a three-dimensional base. It was proposed by Kartashova [Kar06, Ex. 6], who concentrated on LPDOs on the plane.

Furthermore, in Section 7.2.4 we continue the work of Mansfield and Shemyakova [MS08] to fourth order LPDOs on the plane. We compute generating sets of invariants for all operators with completely factorisable symbol. The calculations and invariants are considerably larger than for third order LPDOs.

All results in this chapter are computed as in Section 7.2.1. Due to the size of most results and to avoid the risk of typos, all relevant worksheets and computational results are available in electronical form [Lor08a]. It is recommended to use MAPLE 11 (or newer) to compute the worksheets.

7.2.1 The Laplace Example with MAPLE

The Laplace example is not only useful for the illustration of the theory, but also for the calculations with `jets` and `JetGroupoids`. All higher order LPDOs in the next sections were computed by modifications of the following worksheet. Basically, only the definition of the original LPDO and the number of prolongations and projections has to be adapted.

```
> with(jets): with(JetGroupoids):
```

Define all necessary variables and the groupoid Θ_1 of gauge transformations.

```
> ivar := [x,y,u]: dvar := [X,Y,U]:
> Ivar:=vn(phi,3): Dvar:=vn(xi,3):
> GR := [X[x]=1,X[y]=0,X[u]=0,Y[x]=0,Y[y]=1,Y[u]=0,U[u]=U/u];
```

$$GR := [X_x = 1,\ X_y = 0,\ X_u = 0,\ Y_x = 0,\ Y_y = 1,\ Y_u = 0,\ U_u = \frac{U}{u}]$$

Compute the algebroid R_1 of Θ_1 and its first prolongation R_2. To treat operators of order k, the algebroid R_k is necessary.

```
> R1 := grp2alg(GR,ivar,dvar,Dvar);
> R2 := PrepareAlgebroidRelations(R1,2,ivar,Dvar):
```

$$R1 := [\xi 1_x = 0,\ \xi 1_y = 0,\ \xi 1_u = 0,\ \xi 2_x = 0,\ \xi 2_y = 0,\ \xi 2_u = 0,\ \xi 3_u = \frac{\xi 3}{u}]$$

Gauge Transformation of the Differential Operator

Define the second order hyperbolic operator $L = \partial_x\partial_y + a_{10}\partial_x + a_{01}\partial_y + a_{00}$:

```
> dop := [[1,[X,Y]],[a,[X]],[b,[Y]],[c,[]]];
```

$$dop := [[1,\ [X,\ Y]],\ [a,\ [X]],\ [b,\ [Y]],\ [c,\ []]]$$

The `jets` package contains the command `cchdop` that performs coordinate changes of differential operators. The transformation `tr` shows that the base itself is not changed, but the operator is conjugated with the function $g = \phi^3(x, y)$.

```
> tr := [X=x, Y=y]:
> chdop := cchdop(tr,[1/phi3(x,y),1/phi3(x,y)],dop,ivar[1..2]):
> chdop := eqn2ind(chdop,ivar[1..2],Ivar);
```

$$chdop :=$$
$$[[1, [x, y]], [\frac{\phi3_y + a\,\phi3}{\phi3}, [x]], [\frac{\phi3_x + b\,\phi3}{\phi3}, [y]], [\frac{\phi3_{x,y} + a\,\phi3_x + b\,\phi3_y + c\,\phi3}{\phi3}, []]]$$

Creating the Natural Bundle

The gauge transformed operator defines the natural bundle $\mathcal{F}$ which is necessary for further computations. We determine the fibre coordinates, the gauge groupoid action on $\mathcal{F}$ and the vector fields of the infinitesimal action on $\mathcal{F}$.

```
> uvar := map(a->if type(a[1],symbol) then a[1] fi,dop);
```

$$uvar := [a, b, c]$$

```
> nat := ezip(uvar,map(a->a[1],chdop[2..-1])):
> nat := [x=x, y=y, u=u*phi3, op(nat)];
```

$$nat := [x = x, y = y, u = u\,\phi3, a = \frac{\phi3_y + a\,\phi3}{\phi3}, b = \frac{\phi3_x + b\,\phi3}{\phi3},$$
$$c = \frac{\phi3_{x,y} + a\,\phi3_x + b\,\phi3_y + c\,\phi3}{\phi3}]$$

```
> vec := natfin2inf(nat,ivar,Ivar,Dvar,"")[3..-1];
```

$$vec := [[\xi3, [u]], [-\frac{\xi3_y}{u}, [a]], [-\frac{\xi3_x}{u}, [b]], [-\frac{\xi3_{x,y} + a\,\xi3_x + b\,\xi3_y}{u}, [c]]]$$

We set up the data structure for the natural bundle $\mathcal{F}$. In the second step, we add the information that the coefficients of L do not depend on u.

```
> F:=CreateNaturalBundle(vec,ivar,dvar,uvar,Dvar,"algebroid"=R2):
> F["SUBSvec"] := [[[],[a[u]=0,b[u]=0,c[u]=0],[],uvar]]:
```

Prolongation and Projection

To compute a generating set of invariants, we proceed as in Section 4.2 and perform the usual steps of prolongation and projection. The computation of invariants for higher order LPDOs is *identical* to the procedure below, except that more steps of prolongation and projection are necessary.

```
> J1F := ProlongNaturalBundle(F,1,uu):
> F1  := ProjectNaturalBundle(J1F,v,kernelD):
> F1["inv"];
```

$$[v1 = a_x, v2 = a_y, v3 = b_x, v4 = b_y]$$

The Θ_2-action on $\mathcal{F}_{(1)}$ is free and we compute the first order invariants. They are easily identified as the Laplace invariants h and k.

```
> CodimOfAction(F1);
> Inv1 := InvariantsOnNaturalBundle(F1,"nobase");
```

$$2, 7, 5, 5, 0$$

$$Inv1 := [v1 - c + a\,b, v4 - c + a\,b]$$

For higher order LPDOs it is crucial to change the fibre coordinates of $\mathcal{F}_{(i)}$ such that all invariants are among the coordinates. It simplifies the computations significantly and helps to identify higher order invariants as derivatives of the previous ones. For better readability, we denote the coordinates for the Laplace invariants by h and k in this example.

```
> cF1a := CompleteFibreCoordinates(Inv1,F1["vvar"]):
> cF1a := ezip([h,k,v2,v3],cF1a);
> F1a  := ChangeFibreCoordinates(F1,cF1a):
```

$$cF1a := [h = v1 - c + a\,b,\; k = v4 - c + a\,b,\; v2 = v2,\; v3 = v3]$$

Since gauge transformations do not change the coordinates x and y, it follows that the total derivatives $D_x\psi$ and $D_y\psi$ of an invariant ψ are again invariants. To find small generating sets of invariants, it is convenient to choose a maximal number of coordinates of $\mathcal{F}_{(i+1)}$ as jets of invariants. To do this, we replace the usual internal `jets` routine that picks out the coordinates of $\mathrm{im}(\varphi_{i,s}) \subset J_s(\mathcal{F}_{(i)})$ by a version that prefers the jets of invariants.

```
> `jets/get_vars_to_eliminate_method` := `JetGroupoids/getSolveVarLin`:
> J1F1 := ProlongNaturalBundle(F1a,1,uu):
> unassign('`jets/get_vars_to_eliminate_method`');
```

The projection shows that all new coordinates are jets of h and k and thus invariants. As the action on $\mathcal{F}_{(1)}$ was already free, $\{h, k\}$ is a generating set.

```
> F2   := ProjectNaturalBundle(J1F1,w,kernelD):
> F2["inv"];
```

$$[w1 = h_x,\; w2 = h_y,\; w3 = k_x,\; w4 = k_y]$$

For larger examples, we may have to compute $\mathcal{F}_{(3)}$ and further bundles. If not explicitly indicated, all examples in this chapter are computed by choosing invariants as coordinates and then picking jets of invariants as coordinates for the prolongation. No further modifications are necessary.

7.2.2 Third Order LPDOs on the Plane

In this section, we present generating sets of invariants for third order LPDOs on the plane computed with Vessiot's approach. Except for the full operator

$$\begin{aligned} L_{\text{full}} &= a_{30}\partial_x^3 + a_{21}\partial_x^2\partial_y + a_{12}\partial_x\partial_y^2 + a_{03}\partial_y^3 \\ &+ a_{20}\partial_x^2 + a_{11}\partial_x\partial_y + a_{02}\partial_y^2 + a_{10}\partial_x + a_{01}\partial_y + a_{00}, \end{aligned} \tag{7.11}$$

the results are already computed in [MS08] with moving frames. We use their results for a detailed comparison of both methods, first giving an overview and then explicit results.

Table 7.1 summarises the number of invariants for each order that are contained in a generating set. In two cases (symbols $X\,Y\,(pX + qY)$ and XXY), moving frames produces a smaller generating set. In all but the first example, Vessiot's method produces invariants of smaller order, since the invariants are computed after each step of prolongation and projection. No third order invariants are needed for generating sets. For the operator with symbol X^3, there is a case splitting explained later on.

Table 7.1: Number of invariants for third order LPDOs on the plane

	[MS08]					Vessiot			
Symbol, order	0	1	2	3	total	0	1	2	total
$XY(pX+qY)$	3	3	1	0	7	3	4	1	8
X^3	2	2	1	0	5	2	3	0	5
X^3, (a)	2	0	2	0	4	2	1	1	4
X^3, (b)	1	1	0	1	3	1	1	1	3
X^3, (c)	0	1	1	0	2	0	2	0	2
X^2Y	1	3	1	0	5	1	5	0	6
full	-	-	-	-	-	5	4	1	10

The computation of invariants in Vessiot's approach is done by integrating the involutive distribution that generates the algebroid action on the natural bundle. This involves solving linear PDE systems and so far, we rely on the MAPLE command `pdsolve`, which may produce larger output than necessary. Additionally, the invariants depend heavily on the choice of coordinates in the `CompleteFibreCoordinates` step in Section 7.2.1.

In future, it would be interesting to combine Vessiot's approach of prolongation and projection with moving frames to have both small expressions and invariants of low order. For this, the formulae for the groupoid action on $\mathcal{F}_{(i)}$ is explicitly needed. Since the groupoids of gauge transformations are rather small, it is possible to obtain the action.

We will now explicitly give the invariants for the third order LPDOs from Table 7.1 and express the invariants from [MS08] in terms of the invariants found here. For this, we use the MAPLE package `Janetq` by Robertz, which is a modified version of the `Janet` [BCG+03] package for quasilinear PDEs. If the expressions become too large to be displayed, see [Lor08a] for electronic versions.

Invariants for LPDOs with Symbol $XY(pX+qY)$

The hyperbolic operator

$$L = p\partial_x^2\partial_y + q\partial_x\partial_y^2 + a_{20}\partial_x^2 + a_{11}\partial_x\partial_y + a_{02}\partial_y^2 + a_{10}\partial_x + a_{01}\partial_y + a_{00},$$

has the trivial zero order invariants p and q. Other invariants are

$$\begin{aligned}
I_1^0 &= 2a_{02}p^2 - pqa_{11} + 2q^2a_{20},\\
I_1^1 &= 2(a_{11,y} - 2a_{10})q^3 + (a_{11}^2 - 2a_{11,x}p + 4a_{01}p - 2q_ya_{11} - 4a_{02}p_y)q^2\\
&+ 2(q_xpa_{11} - 2a_{11}pa_{02} + 2q_ypa_{02} + 2p_xa_{02}p)q - 4q_xa_{02}p^2 + 4I_1^0a_{02},\\
I_2^1 &= (a_{01} - a_{11,x})q^2 + (a_{02,x}p - a_{02}a_{11} + q_xa_{11} + 2p_xa_{02})q + a_{02}^2p - 3q_xpa_{02},
\end{aligned}$$

$$\begin{aligned} I_3^1 &= (2a_{02,y} + a_{11,x} - 2a_{01})q^2 + (2a_{02}a_{11} - q_x a_{11} - 2q_y a_{02} - 2p_x a_{02})q \\ &+ 2q_x p a_{02} - 2a_{02}^2 p. \end{aligned}$$

The generating set of invariants consists of two additional large invariants I_4^1 and I_1^2 (see [Lor08a]). Here, the results from [MS08, Thm 5.2] and [SW07a] consists of less invariants, which are also considerably smaller. It turns out that the second order invariant I_1^2 is not needed to express the second order invariant from [MS08].

Invariants for LPDOs with Symbol X^3

The full operator with symbol X^3 is of the form

$$L_{X^3} = \partial_x^3 + a_{20}\partial_x^2 + a_{11}\partial_x\partial_y + a_{02}\partial_y^2 + a_{10}\partial_x + a_{01}\partial_y + a_{00}.$$

Computing a generating set of invariants on the corresponding natural bundle $\mathcal{F}_{X^3}$ yields two zero order invariant coordinates a_{11} and a_{02} as well as

$$\begin{aligned} I_1^1 &= 2(a_{20}^2 + 3a_{20,x} - 3a_{10})a_{02} + (3a_{01} - a_{20}a_{11})a_{11}, \\ I_2^1 &= 6(3a_{02}a_{01,x} - 2a_{02}^2 a_{20,y} - 3a_{01}a_{02,x}) + (3a_{01} - a_{11}a_{20})a_{11}^2 \\ &+ 2(a_{20}^2 a_{02} - 3a_{02}a_{10} + 3a_{02,x}a_{20})a_{11} - 6a_{11,x}a_{02}a_{20}, \\ I_3^1 &= 2(3a_{20,y}a_{11} + 27a_{01,y} + 18a_{10,x} - 54a_{00} + 2[9a_{10} - 2a_{20}^2 - 6a_{20,x}]a_{20})a_{02} \\ &+ 27(a_{01} - 2a_{02,y})a_{01} + 3(a_{11}a_{20} + 2a_{11,x} + 6a_{02,y} - 18a_{01})a_{20}a_{11} \\ &- 18a_{11,y}a_{02}a_{20} - 18a_{11,x}a_{01}. \end{aligned} \tag{7.12}$$

The invariants from [MS08, eq. (13)] (mind the typos) can be expressed as

$$I^{a_{10}} = -\frac{I_1^1}{6a_{02}}, \quad I_x^{a_{01}} = \frac{I_2^1 - I_1^1 a_{11}}{18a_{02}},$$

$$I^{a_{00}} = -6I_{1,x}^1 a_{02} + a_{11}I_2^1 - a_{11}^2 I_1^1 + 6a_{02,x}I_1^1 - I_3^1 a_{02}.$$

Depending on the values of a_{11} and a_{02}, the operators with symbol X^3 split into several cases. Setting one of the invariants to zero yields a natural subbundle $\mathcal{F}' \subset \mathcal{F}_{X^3} \to X$. The above invariants can be restricted to $\mathcal{F}'$, but they may become trivial. In the case of $a_{11} = 0$, the invariants $\{a_{02}, I_1^1, I_2^1, I_3^1\}_{a_{11}=0}$ are still a generating set of invariants. They can be read off equation (7.12) by omitting trailing summands that contain a_{11}.

Invariants for LPDOs with Symbol X^3, Case (a)

For the subbundle $\mathcal{F}_{X^3,a} \subset \mathcal{F}_{X^3} \to X$ defined by $a_{02} = 0$, the dimension of the Θ_3-orbits drops and the invariants have to be computed separately, since we are dealing with a subbundle of $\mathcal{F}_{X^3}$ containing only nongeneric orbits. The coordinate a_{11} and the following invariants are a generating set.

$$\begin{aligned} I_1^0 &= a_{01} - \frac{1}{3}a_{20}a_{11} \\ I_1^1 &= 6a_{11}^2 a_{20,y} + (9a_{10,x} + 3a_{20}a_{20,x} + a_{20}^3 - 27a_{00})a_{11} \\ &+ 3(3a_{20,x} - 3a_{10} + a_{20}^2)a_{11,x} + 9(3a_{10} - 3a_{20,x} - a_{20}^2)a_{01} \\ I_1^2 &= 9a_{11}^2 a_{20,y} + (9a_{20,xx} + a_{20}^3 + 9a_{20}a_{20,x} - 27a_{00})a_{11} \\ &+ 9(3a_{10} - 3a_{20,x} - a_{20}^2)a_{01}. \end{aligned} \tag{7.13}$$

Restricting the first invariant I_1^1 from equation (7.12) to $\mathcal{F}_{X^3,a}$, the order drops by one and we obtain $I_1^0 a_{11}$. All other invariants only reproduce I_1^0 and a_{11}. The comparison with [MS08, eq. (14)] yields

$$I^{a_{01}} = I_1^0, \qquad I_x^{a_{10}} = \frac{I_1^1 - I_1^2}{9a_{11}}, \qquad I^{a_{00}} = -\frac{I_1^2}{27a_{11}}.$$

Invariants for LPDOs with Symbol X^3, Case (b)

The subbundle $\mathcal{F}_{X^3,b} \subset \mathcal{F}_{X^3}$ with $a_{02} = a_{11} = 0$ corresponds to the operator

$$L_{X^3,b} = \partial_x^3 + a_{20}\partial_x^2 + a_{10}\partial_x + a_{01}\partial_y + a_{00}.$$

The restriction of the invariants from equation (7.13) only yields a_{01} and I_1^1, but the complete generating set requires an additional second order invariant.

$$\begin{aligned} I_1^1 &= a_{20,x} - a_{10} + \frac{1}{3}a_{20}^2, \\ I_1^2 &= 9(a_{10,xx} - 3a_{00,x} + a_{01}a_{20,y} + a_{10,x}a_{20})a_{01} \\ &+ (6a_{10}^2 - 3a_{10}a_{20,x} - 4a_{20}^2 a_{10} - 6a_{20}a_{20,xx} - 2a_{20}^2 a_{20,x} + \frac{2}{3}a_{20}^4)a_{01} \\ &- (9a_{10,x} - 27a_{00} + 9a_{20}a_{10} - 6a_{20}a_{20,x} - 2a_{20}^3)a_{01,x}. \end{aligned} \tag{7.14}$$

Compared with [MS08, eq. (15)], the third order part of the invariant $I_x^{a_{00}}$ is only due to the summand containing $I_{1,xx}^1$ in

$$I^{a_{10}} = -I_1^1, \qquad I_x^{a_{00}} = 9I_{1,x}^1 a_{01,x} - I_1^2 + 3(2(I_1^1)^2 - 3I_{1,xx}^1)a_{01}.$$

Invariants for LPDOs with Symbol X^3, Case (c)

Finally, for the operator with $a_{02} = a_{11} = a_{01} = 0$,

$$L_{X^3,c} = \partial_x^3 + a_{20}\partial_x^2 + a_{10}\partial_x + a_{00},$$

the invariant algebra is generated in first order by

$$I_1^1 = a_{20,x} - a_{10} + \frac{1}{3}a_{20}^2, \qquad I_2^1 = a_{10,x} - 3a_{00} + a_{20}a_{10} - \frac{2}{3}a_{20}a_{20,x} - \frac{2}{9}a_{20}^3.$$

The comparison with [MS08, eq. (16)] yields $I^{a_{10}} = -I_1^1$ and $I_x^{a_{00}} = -\frac{1}{3}(I_1^2 + I_{1,x}^1)$. In the last line of equation (7.14), we find the invariant $9I_2^1$ as coefficient of $a_{01,x}$. There may be more connections between the invariants for the operators with symbol X^3.

Invariants for LPDOs with Symbol X^2Y

The operator

$$L_{X^2Y} = \partial_x^2\partial_y + a_{20}\partial_x^2 + a_{11}\partial_x\partial_y + a_{02}\partial_y^2 + a_{10}\partial_x + a_{01}\partial_y + a_{00}$$

has the coordinate a_{02} and five first order invariants as a generating set.

$$\begin{aligned}
I_1^1 &= a_{20,x} - \frac{1}{2}a_{10} + \frac{1}{2}a_{20}a_{11},\\
I_2^1 &= a_{11,x} - 2a_{01} + 4a_{20}a_{02} + \frac{1}{2}a_{11}^2,\\
I_3^1 &= a_{11,y} - a_{10} + a_{20}a_{11},\\
I_4^1 &= a_{10,x} + 2a_{02}a_{20,y} - 2a_{00} + (a_{10} - a_{20,x})a_{11} + (+2a_{01} - 2a_{02}a_{20} - a_{11}^2 - a_{11,x})a_{20},\\
I_5^1 &= a_{01,y} - a_{02}a_{20,y} - a_{00} + (a_{01} - a_{02}a_{20} - 2a_{02,y})a_{20} + \frac{1}{2}(a_{10} - a_{11,y} - a_{20}a_{11})a_{11}
\end{aligned}$$

The comparison with [MS08, eq. (18)] yields

$$I_y^{a_{11}} = I_3^1 - 2I_1^1, \quad I^{a_{10}} = -2I_1^1, \quad I^{a_{01}} = -\frac{1}{2}I_2^1, \quad I^{a_{00}} = -4I_{1,x}^1 - 2I_4^1.$$

Since the invariants from [MS08] are a generating set and I_5^1 does not occur in the above expressions, also $\{a_{20}, I_1^1, I_2^1, I_3^1, I_4^1\}$ represents a generating set of invariants.

Factorisation of Type $(X)(X)(Y)$

The invariants of LPDOs under gauge transformations were computed in order to express conditions for the factorisation in terms of these invariants. For the hyperbolic operator with symbol $XY(pX+qY)$, this was done in [SW08]. We will give a simple example for the factorisation of L_{X^2Y} of type $(X)(X)(Y)$. Basically, we use the first order operators

$$L_1 = \partial_x + b_{00}, \quad L_2 = \partial_x + c_{00}, \quad L_3 = \partial_y + d_{00}$$

with unknown parameters b_{00}, c_{00} and d_{00} and compute

$$\begin{aligned}
L_1L_2L_3 &= \partial_x^2\partial_y + d_{00}\partial_x^2 + (b_{00} + c_{00})\partial_x\partial_y\\
&+ (d_{00,x} + d_{00}(b_{00} + c_{00}))\partial_x + (b_{00}c_{00} + c_{00,x})\partial_y\\
&+ [d_{00,xx} + d_{00,x}(b_{00} + c_{00}) + d_{00}(b_{00}c_{00} + c_{00,x})].
\end{aligned}$$

Comparing the coefficients with L_{X^2Y} in decreasing order, we fix the parameters

$$d_{00} = a_{20}, \quad b_{00} = a_{11} - c_{00}.$$

Plugging this into $L_1L_2L_3$, we obtain the partial Ricatti equation (see [GS04])

$$(a_{11} - c_{00})c_{00} + c_{00,x} = a_{01}$$

for the last coefficient and the two conditions for the existence of a factorisation

$$2I_1^1 = 2a_{20,x} + a_{20}a_{11} - a_{10} = 0, \qquad a_{02} = 0,$$

which are both invariants. For all following factorisation conditions, we apply the strategy of determining parameters from the coefficients of derivatives in decreasing order. For fourth order operators, there are cases, where this strategy breaks down.

Invariants for the Full Third Order LPDO

For the full third order LPDO from equation (7.11), the invariants become rather large, so we display only the zero order ones

$$a_{30}, \quad a_{21}, \quad a_{12}, \quad a_{03},$$

$$I_1^0 = a_{21}a_{12}a_{11} - 2a_{02}a_{21}^2 - 2a_{12}^2a_{20} + 6a_{03}a_{21}a_{20} + 6a_{02}a_{30}a_{12} - 9a_{03}a_{30}a_{11}.$$

For the remaining five invariants see [Lor08a]. Maple was unable to compute the last, second order invariant I_1^2 with `pdsolve`, but using an ad hoc version of moving frames, it was possible to calculate it.

It would be interesting to find relations between the algebra of invariants for the full operator and some more restricted ones. For example, setting $a_{12} = q$ and $a_{30} = a_{03} = 0$, $a_{21} = p$, one obtains the nontrivial zero order invariant for the symbol $XY(pX + qY)$.

The full third order operator served as a test whether generating sets for fourth order operators are computable in reasonable time, since most fourth order operators contain the full third order one as special case.

7.2.3 A Third Order LPDO in Dimension Three

The computation of invariants for LPDOs under gauge transformations using Vessiot's method is not limited to operators on the plane. In this section, we compute the invariants for an LPDO on a three-dimensional manifold. The operator was mentioned in [Kar06, Ex. 6] as an example for problems that may arise in dimension three. It is of the form

$$L = \partial_x^3 + \partial_y^3 + \partial_z^3 - 3\partial_x\,\partial_y\,\partial_z + \sum_{0\le i+j+k\le 2} a_{ijk}\,\partial_x^i\,\partial_y^j\,\partial_z^k$$

and the symbol has the unique factorisation

$$X^3 + Y^3 + Z^3 - 3XYZ = (X + Y + Z)\,(X^2 + Y^2 + Z^2 - XY - XZ - YZ).$$

We give both a generating set of invariants as well as the conditions for factorisation. The zero order invariants are

$$I_1^0 = a_{020} + a_{101}, \quad I_2^0 = a_{011} + a_{200}, \quad I_3^0 = a_{002} + a_{110}.$$

The first order invariants are

$$\begin{aligned}
I_1^1 &= a_{101,x} - a_{110,z} - a_{001} - \frac{1}{3}a_{011}a_{101} + \frac{1}{3}a_{110}^2 - \frac{2}{3}I_3^0 a_{110},\\
I_2^1 &= a_{101,y} + a_{010} - a_{110,x} - \frac{1}{3}a_{101}^2 + \frac{2}{3}a_{101}I_1^0 + \frac{1}{3}a_{110}a_{011},\\
I_3^1 &= a_{101,z} - a_{110,y},\\
I_4^1 &= a_{011,x} + a_{100} - a_{110,y} + \frac{1}{3}a_{110}a_{101} - \frac{1}{3}a_{011}^2 + \frac{2}{3}a_{011}I_2^0,\\
I_5^1 &= a_{011,y} - a_{110,z} - a_{001} - \frac{1}{3}a_{011}a_{101} + \frac{1}{3}a_{110}^2 - \frac{2}{3}I_3^0 a_{110},\\
I_6^1 &= a_{011,z} - a_{110,x},\\
I_7^1 &= a_{001,x} + a_{100,y} + a_{010,z} - \frac{1}{3}a_{101}a_{100} - \frac{1}{3}a_{110}a_{010} + \frac{2}{3}I_3^0 a_{110,x}\\
&+ \frac{2}{3}I_1^0 a_{110,y} + \frac{2}{3}I_2^0 a_{110,z} + \frac{2}{3}I_2^0 a_{001} - \frac{1}{3}a_{011}a_{001} - \frac{2}{3}I_3^1\\
&+ \frac{1}{3}(a_{011,x} + a_{100} - a_{110,y})a_{101} + \frac{2}{3}a_{101}I_{1,z}^0 + \frac{2}{9}I_2^0 a_{101}a_{011} + \frac{2}{3}a_{011}I_{2,y}^0\\
&- \frac{2}{3}a_{011}(a_{011,y} - a_{110,z} - a_{001}) + \frac{1}{3}a_{011}(a_{101,x} - a_{110,z} - a_{001}) - \frac{2}{9}I_2^0 a_{110}^2\\
&+ \frac{1}{3}a_{110}I_6^1 + \frac{2}{3}I_{3,x}^0 a_{110} + \frac{1}{3}a_{110}(a_{101,y} + a_{010} - a_{110,x}) + \frac{4}{9}a_{110}I_3^0 I_2^0,\\
I_8^1 &= a_{001,y} + a_{100,z} + a_{010,x} - \frac{1}{3}a_{110}a_{100} - \frac{1}{3}a_{011}a_{010} + \frac{2}{3}I_2^0 a_{110,x} + \frac{2}{3}I_3^0 a_{110,y}\\
&+ \frac{2}{3}I_1^0 a_{110,z} + \frac{2}{3}a_{001}I_1^0 - \frac{1}{3}a_{001}a_{101} - \frac{2}{3}a_{101}(a_{101,x} - a_{110,z} - a_{001})\\
&+ \frac{1}{3}a_{101}(a_{011,y} - a_{110,z} - a_{001}) + \frac{2}{3}a_{101}I_{1,x}^0 + \frac{1}{3}a_{011}(a_{101,y} + a_{010} - a_{110,x})\\
&+ \frac{2}{9}a_{011}a_{101}I_1^0 + \frac{2}{3}a_{011}I_{2,z}^0 - \frac{2}{3}a_{011}I_6^1 - \frac{2}{9}a_{110}^2 I_1^0\\
&+ \frac{1}{3}a_{110}I_3^1 + \frac{1}{3}a_{110}(a_{011,x} + a_{100} - a_{110,y}) + \frac{4}{9}I_1^0 I_3^0 a_{110} + \frac{2}{3}a_{110}I_{3,y}^0,\\
I_9^1 &= \frac{2}{9}a_{101}(I_1^0)^2 - \frac{1}{3}a_{110}a_{001} - \frac{1}{3}a_{100}a_{011} + \frac{1}{3}a_{100}I_2^0 + \frac{2}{9}a_{011}(I_2^0)^2 - \frac{2}{3}a_{101}a_{011}a_{110}\\
&+ \frac{1}{9}a_{101}a_{110}I_2^0 - 3a_{000} + a_{100,x} + a_{010,y} + a_{001,z} + \frac{1}{9}a_{011}I_1^0 a_{110} + \frac{2}{9}a_{011}^3 + \frac{2}{9}a_{101}^3\\
&+ \frac{1}{3}a_{010}I_1^0 - \frac{4}{9}a_{101}^2 I_1^0 - \frac{1}{3}I_3^0 a_{110}^2 - \frac{4}{9}a_{011}^2 I_2^0 + \frac{1}{3}a_{011}I_3^1 + \frac{2}{3}a_{101}I_{1,y}^0 - \frac{1}{3}I_1^0 a_{110,x}\\
&+ \frac{2}{3}a_{011}I_{2,x}^0 - \frac{1}{3}I_2^0 a_{110,y} - \frac{1}{3}I_3^0 a_{110,z} - \frac{2}{3}a_{101}(a_{101,y} + a_{010} - a_{110,x}) + \frac{1}{3}a_{101}I_6^1\\
&+ \frac{2}{3}a_{110}I_{3,z}^0 + \frac{1}{3}a_{110}(a_{011,y} - a_{110,z} - a_{001}) + \frac{1}{3}a_{110}(a_{101,x} - a_{110,z} - a_{001})\\
&- \frac{2}{3}a_{011}(a_{011,x} + a_{100} - a_{110,y}) - \frac{1}{3}a_{010}a_{101}.
\end{aligned}$$

Factorisation

Since the symbol of L has a unique factorisation, there are two possible factorisations of L with operators

$$\begin{aligned} L_1 &= \partial_x + \partial_y + \partial_z + r, \\ L_2 &= \partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_x \partial_y - \partial_x \partial_z - \partial_y \partial_z + a\,\partial_x + b\,\partial_y + c\,\partial_z + d \end{aligned}$$

depending on the parameters $\{a, b, c, d, r\}$. For both factorisations $L = L_1 L_2$ and $L = L_2 L_1$ they are chosen as

$$\begin{aligned} 3a &= 2a_{200} - a_{020} + a_{110}, \\ 3b &= -a_{200} + 2a_{020} + a_{110}, \\ 3c &= -a_{200} + 2a_{020} + a_{101} - 2a_{110}, \\ 3r &= a_{200} + a_{020} - a_{110}, \\ 3d &= \frac{1}{3}\left(-2a_{200}^2 - a_{200}a_{020} + a_{200}a_{110} + a_{020}^2 - 2a_{020}a_{110} + a_{110}^2\right) \\ &- 2a_{200,y} + a_{020,y} - a_{110,y} - 2a_{200,z} + a_{020,z} - a_{110,z} - 2a_{200,x} \\ &+ a_{020,x} - a_{110,x} + 3a_{100}. \end{aligned}$$

The first factorisation $L = L_1 L_2$ is possible if and only if the following five invariants vanish.

$$I_1^0 - I_2^0 \;=\; I_1^0 - I_3^0 \;=\; I_1^1 + I_2^1 + I_3^1 \;=\; I_4^1 + I_5^1 + I_6^1 = 0,$$

$$\begin{aligned} 0 &= I_7^1 + I_8^1 + I_9^1 - I_{1,xx}^0 - 2I_{1,xy}^0 - 2I_{1,xz}^0 - I_{1,yy}^0 - 2I_{1,yz}^0 - I_{1,zz}^0 \\ &- 2I_1^0(I_{1,x}^0 + I_{1,y}^0 + I_{1,z}^0) - \frac{4}{9}(I_1^0)^3. \end{aligned}$$

The conditions for the second factorisation $L = L_2 L_1$ are slightly larger

$$I_1^0 - I_2^0 \;=\; I_1^0 - I_3^0 \;=\; 0,$$

$$I_1^1 + I_4^1 - I_2^1 - I_5^1 - 2I_{1,x}^0 + 2I_{1,y}^0 \;=\; I_1^1 + I_4^1 - I_3^1 - I_6^1 - 2I_{1,x}^0 + 2I_{1,z}^0 \;=\; 0,$$

$$\begin{aligned} 0 &= I_7^1 + I_8^1 - I_9^1 + I_{1,x}^1 + I_{1,y}^1 - I_{2,x}^1 - I_{3,x}^1 - I_{4,z}^1 + I_{4,x}^1 - 2I_{1,xx}^0 + 2I_{1,xz}^0 \\ &+ \frac{1}{3}(9I_1^1 - 4I_2^1 + 5I_4^1 - 14I_{1,x}^0 + 4I_{1,y}^0 + 4I_{1,z}^0)I_1^0 + \frac{4}{9}(I_1^0)^3. \end{aligned}$$

7.2.4 Fourth Order LPDOs on the Plane

Having successfully computed the invariants for the full third order LPDO, we proceed to fourth order LPDOs, because the complexity of invariants can be managed. In fact, we could compute generating sets of invariants for operators with completely factorisable symbol, as shown in Table 7.2. Up to the knowledge of the author, these results are new. Since they are considerably larger than for order three, we refer to [Lor08a] for most results.

For the operator with symbol X^2Y^2, the expressions are small enough and we give both the invariants as well as conditions for the factorisation. They promise new insights for the search of factorisations, as the procedure used for third order operators is not directly applicable. As the focus of this thesis lies in the application of Vessiot's equivalence method to find generating sets of invariants, we only sketch this part.

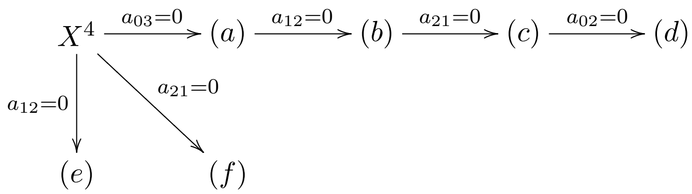
Table 7.2: Number of invariants for fourth order LPDOs on the plane

Symbol, order	0	1	2	3	4
X^4	5	5	1		
X^4 (a)	3	6			
X^4 (b)	3	4			
X^4 (c)	2	4			
X^4 (d)	2	2	1	0	2
X^4 (e)	3	4			
X^4 (f)	3	4			
$X^3\,Y$	4	7	1		
$X^2\,Y^2$	3	10			
$X^3\,(pX+qY)$	5	7	1		
$X^2\,Y\,(pX+qY)$	4	9	1		
$X^2\,(pX+qY)\,(rX+sY)$	5	9	1		
$X\,Y\,(pX+qY)\,(rX+sY)$	5	9			
$X\,Y\,(pX^2+qY^2)$	5	6	1		

In Table 7.2, the number of invariants contained in generating sets for various fourth order LPDOs are displayed. The leading coefficients of the operators have not been normalised to one. This allows to search for connections between the invariants for third and fourth order operators. For example, the operator

$$L_{X^4} = a_{40}\partial_x^4 + \sum_{0\le i+j\le 3} a_{ij}\,\partial_x^i\,\partial_y^j.$$

contains the coefficient a_{40}. Normalising the first order coefficient only decreases the number of zero order invariants by one. The only exception to this convention are the subcases (a)-(f) for the symbol X^4. They correspond to the following subsequent restrictions of L_{X^4} with $a_{40} = 1$.



Restricting the generating set for the full operator with symbol X^4 to case (f) reproduces the generating set of invariants obtained from direct computation. The zero and first order invariants of case X^4 also coincide with case (e), but the second order ones differ.

With one exception, the generating sets contain invariants of maximal order two, similar to third order operators. Without this fact, the computations would have been too large to carry out. Again, the invariants were calculated by integrating the vector fields of the infinitesimal action on the natural bundles $\mathcal{F}_{(i)}$. We expect this to be more efficient than moving frames, since the finite groupoid actions on $J_r(\mathcal{F})$ quickly become too large to compute.

Fourth Order LPDO with Symbol X^2Y^2

The smallest example in Table 7.2 is the operator with symbol X^2Y^2 and we will give the explicit form of a generating set of invariants for this operator. The results of Table 7.2 show that only first order jets of the coefficients of

$$L_{X^2Y^2} = a_{22}\,\partial_x^2\,\partial_y^2 + \sum_{0\le i+j\le 3} a_{ij}\,\partial_x^i\,\partial_y^j$$

are needed. The zero order invariants are simply the coordinates a_{22}, a_{30} and a_{03}. For smaller expressions, we normalise L_{X^2,Y^2} by setting $a_{22} = 1$ in the first order invariants that complete the generating set.

$$\begin{aligned}
I_1^1 &= 2a_{21,x} - a_{11} + a_{12}a_{21},\\
I_2^1 &= 2a_{21,y} - 4a_{20} + 6a_{30}a_{12} + a_{21}^2,\\
I_3^1 &= 2a_{12,x} - 4a_{02} + a_{12}^2 + 6a_{03}a_{21},\\
I_4^1 &= 2a_{12,y} - a_{11} + a_{12}a_{21},\\
I_5^1 &= 4a_{20,x} - 2a_{10} + 2a_{12}a_{20} - 6a_{02}a_{30} - 6a_{12}a_{30,x} - I_1^1a_{21} + 9a_{03}a_{30}a_{21}\\
I_6^1 &= 2a_{11,x} - a_{21}I_3^1 - 4a_{01} + 12a_{03}a_{20} + a_{12}a_{11} - a_{12}I_1^1 - 18a_{12}a_{03}a_{30},\\
I_7^1 &= 2a_{11,y} - a_{21}I_4^1 + a_{11}a_{21} - 18a_{03}a_{30}a_{21} - 4a_{10} + 12a_{02}a_{30} - a_{12}I_2^1,\\
I_8^1 &= 4a_{02,y} - 2a_{01} - 6a_{03}a_{20} + 2a_{02}a_{21} - I_4^1a_{12} - 6a_{21}a_{03,y} + 9a_{12}a_{03}a_{30},\\
I_9^1 &= 4a_{10,x} - 8a_{00} + 4a_{21}a_{01} + 4a_{12}a_{10} + 2a_{12}a_{21}I_1^1 - a_{11}I_1^1 + a_{21}^2I_3^1 - 2a_{20}I_3^1\\
&- 2a_{21}a_{03}I_2^1 - 4a_{30}a_{02,x} + 8a_{03}a_{20,y} + 9a_{12}a_{03}a_{30}a_{21} - 8a_{20}a_{03}a_{21}\\
&+ 4a_{30}a_{12}a_{02} - a_{11}a_{12}a_{21} - 3a_{11}a_{03}a_{30} + 18a_{21}a_{03}a_{30,x} - 12a_{12}a_{30,y}a_{03} + 6a_{21}a_{30}a_{03,x}\\
&+ 4a_{30}a_{12}I_3^1 - 4a_{12}a_{20,x} + 6a_{12}^2a_{30,x} - 12a_{02}a_{30,x} - 2a_{21}a_{11,x} - 2a_{20}a_{12}^2,
\end{aligned}$$

$$\begin{array}{rcl} I^1_{10} & = & 4a_{01,y} - 8a_{00} + 4a_{21}a_{01} + 4a_{12}a_{10} + 2a_{12}a_{21}I^1_4 + 4a_{21}a_{03}I^1_2 + a^2_{12}I^1_2 \\ & - & 2a_{02}I^1_2 - a_{11}I^1_4 + 8a_{30}a_{02,x} - 4a_{03}a_{20,y} + 9a_{12}a_{03}a_{30}a_{21} + 4a_{20}a_{03}a_{21} - 8a_{30}a_{12}a_{02} \\ & - & a_{11}a_{12}a_{21} - 3a_{11}a_{03}a_{30} + 18a_{12}a_{30}a_{03,y} + 6a_{12}a_{30,y}a_{03} - 12a_{21}a_{30}a_{03,x} - 2a_{30}a_{12}I^1_3 \\ & + & 6a_{03,y}a^2_{21} - 12a_{20}a_{03,y} - 2a_{12}a_{11,y} - 4a_{21}a_{02,y} - 2a_{02}a^2_{21} \end{array}$$

Factorisation of $L_{X^2Y^2}$

Now turn to the factorisation of $L_{X^2Y^2}$ under the assumption of $a_{22} = 1$. We use the operators

$$L_1 = \partial_x + b, \quad L_2 = \partial_x + c, \quad L_3 = \partial_y + d, \quad L_4 = \partial_y + e.$$

for the factorisation. Setting the parameters $\{b, c, d, e\}$ such that they obey the following equations

$$b = a_{12} - c, \quad d = a_{21} - e, \quad c_x = a_{02} - ca_{12} + c^2, \quad e_y = a_{20} - ea_{21} + e^2$$

yields the partial factorisation

$$\begin{array}{rcl} 0 & \stackrel{!}{=} & L_{X^2Y^2} - L_1L_2L_3L_4 \\ & = & a_{30}\partial_x^3 + a_{03}\partial_y^3 - I^1_1\partial_x\partial_y \\ & & -\frac{1}{2}(a_{21}I^1_1 - I^1_5)\partial_x - (I^1_6 + 2a_{12}I^1_1 + 2I^1_{1,x})\partial_y \\ & & -2a_{11}I^1_1 - 2a_{21}(I^1_{1,x} + I^1_6) - 2a_{12}I^1_5 - 2I^1_{5,x} - (I^1_1)^2 - I^1_9. \end{array}$$

As in [GS04], the conditions for the parameters include partial Ricatti equations for c and e. A factorisation of type $(X)(X)(Y)(Y)$ with $L = L_1L_2L_3L_4$ is possible if and only if the following invariants are zero

$$a_{30} = a_{03} = I^1_1 = I^1_5 = I^1_6 = I^1_9 = 0.$$

For the factorisation of type $(X)(Y)(X)(Y)$ with $L_{X^2Y^2} = L_1L_3L_2L_4$, the coefficients are

$$\begin{array}{rcl} b & = & \dfrac{a_{00} - a_{02}a_{20} + a_{20,xx} + a_{20}a_{12,x} - a_{10,x}}{a_{10} - a_{12}a_{20} - 2a_{20,x}}, \\ c & = & \dfrac{(a_{20}a_{12} + 2a_{20,x} - a_{10})a_{12} + a_{00} - a_{02}a_{20} + a_{20,xx} + a_{20}a_{12,x} - a_{10,x}}{-a_{10} + a_{12}a_{20} + 2a_{20,x}}, \\ d & = & \dfrac{a_{21}a_{11} - a_{12}a^2_{21} - 2a_{21}a_{21,x} - a_{10} + a_{12}a_{20} + 2a_{20,x}}{a_{11} - a_{21}a_{12} - 2a_{21,x}}, \\ e & = & \dfrac{-a_{10} + a_{12}a_{20} + 2a_{20,x}}{-a_{11} + a_{21}a_{12} + 2a_{21,x}}. \end{array} \quad (7.15)$$

Here the the strategy to compare the coefficients of derivatives of $L_1L_3L_2L_4$ in decreasing order fixes the first two parameters and yields two conditions

$$b = a_{12} - c, \qquad d = a_{21} - e, \qquad a_{30} = 0, \qquad a_{03} = 0.$$

Plugging these equations into $L_{X^2Y^2} - L_1L_3L_2L_4$ yields the operator

$$\begin{aligned}
0 &\stackrel{!}{=} (a_{20} - e_y - a_{21}e + e^2)\partial_x^2 \\
&+ (a_{11} - a_{21}a_{12} - c_y - e_x - a_{21,x})\partial_x\partial_y \\
&+ (a_{02} - c_x - a_{12}c + c^2)\partial_y^2 \\
&+ (a_{10} - ea_{12}a_{21} + a_{12}e^2 - c_ye - 2e_xa_{21} + 3e_xe - ea_{21,x} - e_ya_{12} - 2e_{xy})\partial_x \\
&+ (a_{01} - ca_{12}a_{21} + a_{21}c^2 + ce_x - e_xa_{12} - c_xa_{21} + c_y(c - a_{21}) - ca_{21,x} - e_{xx} - c_{xy})\partial_y \\
&+ a_{00} - e_{xxy}\dots
\end{aligned}$$

Each choice of two coefficients of the second order derivatives is not sufficient to fix c and e completely. For example choosing the equations

$$a_{02} - c_x - a_{12}c + c^2 = 0, \qquad a_{20} - e_y - a_{21}e + e^2 = 0$$

and substituting jets of c_x and e_y in the other coefficients still leaves them c- and e-dependent. Also eliminating the jets c_y and e_x yields conditions for c and e from equations (7.15) and the following invariant conditions for the factorisation.

$$\begin{aligned}
0 &= 2I_5^1I_{1,y}^1 - 2I_1^1I_{5,y}^1 + 2(I_5^1)^2 - (I_1^1)^2I_2^1, \\
0 &= -I_6^1I_5^1 + 2I_{1,x}^1I_5^1 - (I_1^1)^3 + I_1^1I_9^1 - 2I_{5,x}^1I_1^1, \\
0 &= 2I_{5,xy}^1I_5^1(I_1^1)^2 + 2I_{1,y}^1I_5^1(I_1^1)^3 + I_{5,y}^1I_9^1(I_1^1)^2 - (I_1^1)^4I_{5,y}^1 - 2I_{5,x}^1I_{5,y}^1(I_1^1)^2 \\
&\quad -I_{9,y}^1I_5^1(I_1^1)^2 - 2I_6^1(I_5^1)^3 + 4(I_1^1)^3(I_5^1)^2 + 2(I_5^1)^2(I_1^1)^2I_4^1 + 2I_1^1I_9^1(I_5^1)^2, \\
0 &= -8I_3^1(I_5^1)^2 - 4I_6^1I_5^1I_1^1 + 4I_{9,x}^1I_5^1 - 8I_{5,xx}^1I_5^1 - 8I_{5,x}^1I_9^1 + (I_9^1)^2 - 3(I_1^1)^4 \\
&\quad +12(I_{5,x}^1)^2 + 2(I_1^1)^2I_9^1.
\end{aligned}$$

We expect that the factorisation of other fourth order LPDOs leads to similar situations where differential algebra is needed. This may lead to further insight into the factorisation of LPDOs.

The third and last possible factorisation into first order coefficients is $L_{X^2Y^2} = L_3L_1L_2L_4$ of type $(Y)(X)(X)(Y)$. All other possible factorisations into first order operators can be obtained by exchanging x and y.

$$\begin{aligned}
b &= a_{12} - e, \\
c &= a_{21} - d, \\
d_y &= -a_{20} + a_{21,y} + da_{21} - d^2, \\
e &= \frac{-a_{01} + a_{21}a_{02} + 2a_{02,y}}{a_{21}a_{12} - a_{11} + 2a_{12,y}}
\end{aligned}$$

During the calculation of the conditions for factorisation we have assumed that both I_4^1 and I_8^1 are nonzero.

$$\begin{aligned}
0 &= 2I_{4,y}^1 I_8^1 - 2I_{8,y}^1 I_4^1 - I_1^1 (I_4^1)^2 - (I_4^1)^3, \\
0 &= 2I_{3,y}^1 I_8^1 - 2I_4^1 I_{8,x}^1 + 6(I_8^1)^2 - I_6^1 I_8^1 - (I_4^1)^2 I_3^1 \\
0 &= 2I_{1,y}^1 - 4I_5^1 - I_7^1 \\
0 &= I_1^1 (I_4^1)^2 - I_8^1 I_7^1 + I_4^1 I_{10}^1.
\end{aligned}$$

Thus was possible to compute all conditions for the factorisation of $L_{X^2Y^2}$ into first order factors.

Appendix A

Symbols and Spencer Cohomology

In this appendix, the symbol $\mathcal{M}_q$ of a PDE system $\mathcal{R}_q$ is defined, which was already used to test formal integrability in Theorem 1.28. The symbol $\mathcal{M}_q$ contains information about the highest order part of a PDE system $\mathcal{R}_q$. In the linear case, it actually consists of the highest order subsystem. It is possible to check the regularity of the prolongations $\mathcal{R}_{q+r}$ with the symbol.

Based on the symbol, Spencer δ-sequences and Spencer cohomology are introduced. They help to decide at which order $q+r$ new lower order equations may occur because highest order jets cancel.

There are two approaches to the computation of Spencer cohomology. The more widely known version works directly with the Spencer sequences. Over each point of $\mathcal{R}_q$ the Spencer sequences are sequences of vector spaces so the calculation is reduced to linear algebra (see e.g. [Gol67b], [Pom78] or [Spe69]). A second, probably less familiar approach relies on commutative algebra to calculate Spencer cohomology. It is presented in [Qui64] and more recently in [Mal05]. The advantage of the second approach is that all cohomology groups may be determined in a finite computation without using special δ-regular coordinates.

Both ways of computing Spencer cohomology are implemented in a Maple package called `Spencer`. It was developed for this thesis and we give a short introduction to the main functions. A sample calculation is shown in the last part of this appendix.

Before defining the symbol, we construct a power series solution of a linear system $R_q \subseteq J_q(E)$ and show the connection to the symbol. A series solution is computed by induction on the order $q+r$. At each step, inhomogenous linear equations for the jet coordinates y_{q+r} of strict order $q+r$ must be solved. They depend on the lower order jets $y, \ldots, y_{q+r-1}$ already computed. The symbol characterises the solution space for y_{q+r} as it consists of all solutions of the homogenous system with $y = \ldots = y_{q+r-1} = 0$. Geometrically, the homogenous system corresponds to the restriction to the subbundle $S^q T^* \otimes E$ of $J_{q+r}(E) \to X$.

So the symbol is the restriction of R_q to $S^qT^* \otimes E$. For nonlinear systems $\mathcal{R}_q$ we first apply the linearisation $R_q = V(\mathcal{R}_q)$.

Definition A.1. [Gol67b, Def. 7.1] The *symbol* $\mathcal{M}_q$ of a nonlinear system of PDEs $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ is defined as the family of subspaces:

$$\mathcal{M}_q = V(\mathcal{R}_q) \cap S^qT^* \otimes_{\mathcal{R}_q} V(\mathcal{E}).$$

If R_q is a linear system over E, the symbol is defined as

$$\mathcal{M}_q = R_q \cap S^qT^* \otimes E. \qquad \diamond$$

To compute the tensor product $S^qT^* \otimes_{\mathcal{R}_q} V(\mathcal{E})$, first take the pullback bundles $S^qT^* \times_X \mathcal{R}_q$ and $V(\mathcal{E}) \times_{\mathcal{E}} \mathcal{R}_q$ over $\mathcal{R}_q$ and then form the tensor product of their fibres. If no confusion can arise, pullbacks will be omitted.

If $\mathcal{R}_q$ is given by the equations $\Phi^\alpha(x, y, y_q) = \omega^\alpha(x)$, $\alpha = 1, \ldots, k$, then $\mathcal{M}_q$ is defined by restricting the equations for the vertical bundle to the highest order jets $|\mu| = q$:

$$\frac{\partial \Phi^\alpha}{\partial y^i_\mu}(x, y, y_q)\xi^i_\mu = 0, \qquad |\mu| = q. \tag{A.1}$$

Higher order symbols $\mathcal{M}_{q+r}$ obviously depend only on $\mathcal{R}_q$, as their defining equations are:

$$\frac{\partial \Phi^\alpha}{\partial y^i_\mu}(x, y, y_q)\xi^i_{\mu+\nu} = 0, \qquad |\mu| = q, |\nu| = r. \tag{A.2}$$

If $\mathcal{R}_q = R_q$ is a linear system, the vertical derivative simply renames the highest order jets from y^i_μ to ξ^i_μ and up to a pullback both definitions of the symbol coincide.

In Definition 1.24, the prolongation $\mathcal{R}_{q+r}$ was not necessarily regular, i.e. a subbundle of $J_{q+r}(\mathcal{E}) \to X$. With the help of the symbol, regularity in the highest order component can be checked. We have the following proposition.

Proposition A.2. [Gol67b, Prop. 7.1] For a system $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ of PDEs the following statements are equivalent:

(1) $\mathcal{M}_{q+1}$ is a vector bundle over $\mathcal{R}_q$ and $\pi^{q+1}_q : \mathcal{R}_{q+1} \to \mathcal{R}_q$ is surjective.

(2) $\mathcal{R}_{q+1} \to \mathcal{R}_q$ is a subbundle of $J_{q+1}(\mathcal{E})|_{\mathcal{R}_q} \to \mathcal{R}_q$

(3) $\mathcal{R}_{q+1} \to \mathcal{R}_q$ is an affine subbundle of $J_{q+1}(\mathcal{E})|_{\mathcal{R}_q} \to \mathcal{R}_q$ modelled over the vector bundle $\mathcal{M}_{q+1}$.

If one of the assertions is fulfilled, the r-th prolongation $\mathcal{R}_{(q+1)+r}$ of $\mathcal{R}_{q+1}$ is the same as the $r+1$-th prolongation $\mathcal{R}_{q+(r+1)}$ of $\mathcal{R}_q$. $\diamond$

If all symbols $\mathcal{M}_{q+r+1}$ for $0 \leq r \leq s \in \mathbb{N}$ are vector bundles, we can inductively prove that the equivalent conditions hold if q is replaced by $q+r$. In this case, the order of prolongations is irrelevant and we have $\mathcal{R}_{(q+r)+s} = \mathcal{R}_{q+(r+s)}$.

Of course the prolongations $\mathcal{R}_{q+r}$ may be regular even if the projections π^{q+r}_{q+r-1} are not surjective. In this case the new lower order equation must be checked separately (see the example in Section 3.4.3). We give an example for the symbol, where its prolongation is not a vector bundle and the prolongation cannot be regular.

Example A.3. We continue Example 1.25. The symbol $\mathcal{M}_2$ satisfies the equations

$$\begin{aligned} \xi_{yy} - u_{xx}\,\xi_{xx} &= 0, \\ \xi_{xy} - \xi_{xx} &= 0. \end{aligned}$$

The prolongation $\mathcal{M}_3$ is no longer a vector bundle, as the defining equations

$$\begin{aligned} \xi_{xyy} - u_{xx}\,\xi_{xxx} &= 0, \\ \xi_{yyy} - u_{xx}\,\xi_{xxy} &= 0, \\ \xi_{xxy} - \xi_{xxx} &= 0, \\ \xi_{xyy} - \xi_{xxy} &= 0. \end{aligned}$$

also contain

$$(1-u_{xx})\xi_{xxx} = 0.$$

If $u_{xx} \neq 1$, $\mathcal{M}_3$ is zero-dimensional. For $u_{xx} = 1$ there is a rank drop in the equations and R_3 is one-dimensional:

$$u_{xx} = 1 \quad \Rightarrow \quad \xi_{xxx} = \xi_{xxy} = \xi_{xyy} = \xi_{yyy}.$$

So $\mathcal{M}_3$ is only a family of vector spaces. On the open submanifold $u_{xx} \neq 1$ of $J_2(\mathcal{E})$ it is a vector bundle. ◇

Spencer Cohomology

Having defined the symbols $\mathcal{M}_{q+r}$ for a PDE system $\mathcal{R}_q$, we proceed with the Spencer δ-sequences. If all symbols have constant rank, we obtain sequences of vector bundles. The goal of this section is to compute their so-called Spencer cohomology. Therefore we give explicit coordinate representations of the involved maps.

We first define Spencer δ-sequences for the trivial system $R_q = J_q(E)$ with symbols $\mathcal{M}_q = S^qT^* \otimes E$. The Spencer δ-map is a morphism of vector bundles over X:

$$\delta : \bigwedge^s T^* \otimes S^q T^* \to \bigwedge^{s+1} T^* \otimes S^{q-1} T^*, \qquad 0 \leq s \leq n,\ q \in \mathbb{N},$$

defined by the composition

$$\bigwedge^s T^* \otimes S^q T^* \xrightarrow{\iota} \bigwedge^s T^* \otimes T^* \otimes S^{q-1} T^* \xrightarrow{\wedge} \bigwedge^{s+1} T^* \otimes S^{q-1} T^* \tag{A.3}$$

of the canonical embedding $\iota : S^q T^* \hookrightarrow T^* \otimes S^{q-1} T^*$ with the exterior product. Tensoring each $\Lambda^k T^* \otimes S^q T^*$ with a vector bundle E, δ gives rise to the *Spencer δ-sequences*:

$$0 \longrightarrow S^q T^* \otimes E \xrightarrow{\delta} T^* \otimes S^{q-1} T^* \otimes E \xrightarrow{\delta} \dots$$
$$\dots \xrightarrow{\delta} \bigwedge^{n-1} T^* \otimes S^{q-n+1} T^* \otimes E \xrightarrow{\delta} \bigwedge^n T^* \otimes S^{q-n} T^* \otimes E \longrightarrow 0$$

For negative indices $q - r < 0$ set $S^{q-r} T^* = 0$. Interpreting the coordinates of $J_q(E)$ as Taylor coefficients, the δ-map is the formal exterior derivative of homogeneous polynomials of degree q. It follows that the sequences are exact (see also [Spe69, p. 188]).

We derive a coordinate representation for the Spencer δ-map, following [Pom78, p. 86]. At first, we need a local basis $(d\xi^1, \dots, d\xi^n)$ of T^*. The s-fold products

$$d\xi^{i_1} \wedge \dots \wedge d\xi^{i_s}, \qquad i_1 < \dots < i_s$$

are a local basis of $\bigwedge^s T^*$. Let v^k_μ be an element of $S^q T^* \otimes E$ with $1 \le k \le m$ and $|\mu| = q$. Each element of $\bigwedge^s T^* \otimes S^q T^* \otimes E$ can be written as

$$\omega^k_\mu = d\xi^{i_1} \wedge \dots \wedge d\xi^{i_s}\, v^k_{\mu,I}, \qquad I = (i_1, \dots, i_s).$$

Applying δ to $\omega \in \bigwedge^{s-1} T^* \otimes S^{q+1} T^* \otimes E$ yields:

$$(\delta\omega)^k_\mu = d\xi^i \wedge (\delta_i \omega)^k_\omega = d\xi^i \wedge \omega^k_{\mu+1_i}, \tag{A.4}$$

implicitly defining the maps δ_i. In this form it is easy to see $\delta^2 = 0$:

$$\delta^2(\omega)^k_\mu = d\xi^i \wedge d\xi^j \wedge \omega^k_{\mu+1_i+1_j} = 0.$$

The δ-sequences can be restricted to the symbols $\mathcal{M}_{q+r}$ of a PDE system $\mathcal{R}_q$. Due to equation (A.2), the higher order symbols depend on $\mathcal{R}_q$ only, such that we can work with sequences over $\mathcal{R}_q$.

Lemma A.4. [Gol67b, §6] If $\mathcal{M}_q$ is the symbol of a system $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ of PDEs then by setting $\mathcal{M}_k = S^k T^* \otimes V(\mathcal{E})$ for $k < q$ the Spencer δ-sequences can be restricted to sequences

$$0 \longrightarrow \mathcal{M}_{q+r} \xrightarrow{\delta} T^* \otimes \mathcal{M}_{q+r-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigwedge^n T^* \otimes \mathcal{M}_{q-n} \longrightarrow 0. \tag{A.5}$$

◇

Proof. If $\mathcal{M}_q$ is defined by equations $A^\mu_k \eta^k_\mu = 0$ then $\mathcal{M}_{q+1}$ is defined by $A^\mu_k \eta^k_{\mu+1_i} = 0$ for $1 \le i \le n$, being compatible with δ: $A^\mu_k (\delta\eta)^k_\mu = d\xi^i \wedge (A^\mu_k \eta^k_{\mu+1_i}) = 0$. □

It is now possible to define Spencer cohomology and the acyclicity of symbols, which was needed for Theorem 1.28.

Definition A.5. [Gol67b, Def. 6.1] The cohomology of the sequence (A.5) at position $\bigwedge^i T^* \otimes \mathcal{M}_{j-i}$ is denoted by $H^i_{j-i} = H^i_{j-i}(\mathcal{M}_q)$ and called *Spencer cohomology*. $\mathcal{M}_q$ is called *involutive*, if the sequences (A.5) are exact for all $r \geq 0$ and *k-acyclic*, if $H^i_j(\mathcal{M}_q) = 0$ for all $j \leq k$, $i \geq 0$. $\mathcal{M}_q$ is of *finite type*, if $\mathcal{M}_{q+r} = 0$ for some $r \geq 0$. ◇

According to Serre's letter in the appendix of [GS64], involutive symbols correspond to Cartan's [Car04] notion of being in involution.

The most important properties of symbols are 2-acyclicity and involutivity. In addition to the check of formal integrability, we give another consequence of 2-acyclic symbols which is used in Section 4.3.1. It provides a link between Proposition A.2 which deals with the prolongation and Theorem 4.26 which treats the projection of PDE systems. In the linear case, the following proposition is essentially present in [Gol67a, §4].

Proposition A.6. [Pom83, Prop. 1.A.3.30] Let $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ be a system of PDEs with symbol $\mathcal{M}_q$. If $\mathcal{M}_{q+1}$ is a vector bundle over $\mathcal{R}_q$ and $\mathcal{M}_q$ is 2-acyclic then $\mathcal{M}_{q+r}$ is a vector bundle over $\mathcal{R}_q$ for all $r \in \mathbb{N}$. ◇

Theorem 1.28 claims to reduce the test of formal integrability to a finite calculation, but it relies on 2-acyclic symbols. To check a 2-acyclic symbol, there are again infinitely many conditions. This problem can be solved in two ways. It is known that the symbol $\mathcal{M}_{q+r}$ becomes involutive for a sufficiently large $r \in \mathbb{N}$ (see [Spe62] or [EGS65]). There are finite tests for involutive symbols that rely on special δ-regular coordinates (see e.g. [GS64], [Mal05], [Pom78], [Sei02]). We will follow a second approach mentioned in [Qui64] and [Mal05] that uses methods of commutative algebra. It allows to compute the infinite number of cohomology groups in a finite calculation.

Before turning to commutative algebra, we give an example showing the explicit calculation of a single Spencer δ-sequence and its cohomology.

Example A.7. On $\mathcal{E} = \mathbb{R}^3 \times \mathbb{R}$ with coordinates (x^1, x^2, x^3, y) consider the linear system:

$$R_2 : \begin{cases} y_{x^1,x^2} = 0, \\ y_{x^1,x^3} = 0, \\ y_{x^2,x^2} = 0, \\ y_{x^2,x^3} = 0, \\ y_{x^3,x^3} = 0. \end{cases}$$

The symbol $\mathcal{M}_2$ is defined by:

$$\xi_{x^1,x^2} = \xi_{x^1,x^3} = \xi_{x^2,x^2} = \xi_{x^2,x^3} = \xi_{x^3,x^3} = 0.$$

Suitable coordinates for $\mathcal{M}_2$ are (ξ_{x^1,x^1}) and for all higher order symbols $\mathcal{M}_i$ $(\xi_{x^1,\dots,x^1})$ with $i \geq 2$ x^1-derivatives. We compute the δ-sequence starting with $\mathcal{M}_4$:

$$0 \longrightarrow \mathcal{M}_4 \xrightarrow{\delta^{(0)}} T^* \otimes \mathcal{M}_3 \xrightarrow{\delta^{(1)}} \textstyle\bigwedge^2 T^* \otimes \mathcal{M}_2 \xrightarrow{\delta^{(3)}} \textstyle\bigwedge^3 T^* \otimes T^* \longrightarrow 0. \qquad \text{(A.6)}$$

Choosing coordinates for the vector bundles:

$$\begin{array}{rll} \mathcal{M}_4 & (\xi_{x^1,x^1,x^1,x^1}), & \\ T^* \otimes \mathcal{M}_3 & (\xi_{x^1,x^1,x^1} \otimes dx^i) & 1 \leq i \leq 3, \\ \bigwedge^2 T^* \otimes \mathcal{M}_2 & (\xi_{x^1,x^1} \otimes dx^i \wedge dx^j) & 1 \leq i < j \leq 3, \\ \bigwedge^3 T^* & (\xi_{x^i} \otimes dx^1 \wedge dx^2 \wedge dx^3) & 1 \leq i \leq 3, \end{array}$$

we can compute the map $\delta^{(1)}$:

$$\delta(\xi_{x^1,x^1,x^1} dx^i) = \begin{cases} 0, & i = 1 \\ \xi_{x^1,x^1} dx^1 \wedge dx^i, & i \in \{2,3\}. \end{cases}$$

The δ-maps may be written as matrices (in row convention):

$$\delta^{(0)} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \delta^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \delta^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By direct calculation, one verifies that the sequence (A.6) is exact, which means the Spencer cohomology groups vanish:

$$H_4^0(\mathcal{M}_2) = H_3^1(\mathcal{M}_2) = H_2^2(\mathcal{M}_2) = 0. \qquad \diamond$$

Spencer Cohomology and the Koszul Complex

In order to compute all Spencer cohomology groups, we follow an approach of Quillen [Qui64] using a complex of graded modules over a polynomial ring. The link to Spencer δ-sequences is the observation that the dual Spencer δ-sequences are homogenous components of a Koszul complex. We follow the recent presentation of Malgrange [Mal05]. For Koszul complexes (as well as an introduction to commutative algebra) we refer to [Eis95, Ch. 17].

First we shortly introduce the Koszul complex and state properties that are important for the Spencer cohomology. Then we dualise the Spencer δ-sequence and show that the Koszul complex is the direct sum of these dualised sequences. As a corollary, the Spencer cohomology can be computed with the Koszul complex. In the next section, we show a MAPLE package doing these computations.

The Koszul complex is a sequence of modules over a graded polynomial ring $A = k[\xi_1, \dots, \xi_n]$. Let A_i be its homogenous component of degree i and define the vector space $V = A_1 = \langle \xi_1, \ldots, \xi_n \rangle_k$. Let M be a graded A-module. Then the Koszul complex with respect to the sequence $\underline{\xi} = (\xi_1, \ldots, \xi_n) \in A^n$ is denoted by $K_\bullet(\underline{\xi}, M)$. Set the modules as $K_p(\underline{\xi}, M) = \bigwedge^p V \otimes M$ for $0 \leq p \leq n$ and zero otherwise:

$$0 \longrightarrow \bigwedge^n V \otimes M \xrightarrow{d} \bigwedge^{n-1} V \otimes M \xrightarrow{d} \cdots \xrightarrow{d} V \otimes M \xrightarrow{d} M \longrightarrow 0.$$

The differentials d are:

$$d(\xi_{i_1} \wedge \cdots \wedge \xi_{i_p} \otimes m) = \sum_j (-1)^{j+1} \xi_{i_1} \wedge \cdots \wedge \widehat{\xi_{i_j}} \wedge \cdots \wedge \xi_{i_p} \otimes \xi_{i_j} m,$$

where the elements carrying a hat are omitted. By direct computation one shows that $K_\bullet(\underline{\xi}, M)$ is independent from the choice of base $\underline{\xi}$ of V.

Analogous to the Spencer δ-sequences we define acyclicity of the module M, which involves a certain order q. We will see that both definitions coincide.

Definition A.8. Let $H_{p,r}(M) = H_p(\underline{\xi}, M)_r$ be the homogenous part of degree r of the Koszul homology groups. M is called q-involutive if all homology $H_{p,r}(M) = 0$ vanish for $r \geq q$. M_q is called l-acyclic for $l, q \in \mathbb{Z}_{\geq 0}$ if all homology groups $H_{p,r}(M) = 0$ vanish for $r \geq q$ and $p \leq l$. ◇

Basic properties of the Koszul complex are:

Proposition A.9. (1) The Koszul complex for M is the tensor product of $K_\bullet(\underline{\xi}, A)$ with M:

$$K_\bullet(\underline{\xi}, M) \cong K_\bullet(\underline{\xi}, A) \otimes_A M.$$

(2) The homology groups $H_p(\underline{\xi}, M)$ of $K_\bullet(\underline{\xi}, M)$ are all annihilated by ξ^i: $\xi^i H_p(\underline{\xi}, M) = 0$.

(3) If M is finitely generated, then $H_p(\underline{\xi}, M)$ is also finitely generated.

(4) If M is finitely represented, i. e. there is an exact sequence

$$0 \longrightarrow N \longrightarrow A^r \longrightarrow M \longrightarrow 0$$

of finitely generated A-modules and if M, N are generated by elements of degree $\leq l$ then $H_{0,q}(M) = H_{1,q}(M) = 0$ for all $q \geq l$.

(5) Since $H_{p,q}(A^r) = 0$ for $(p, q) \neq (0, 0)$ we have $H_{p+1,q-1}(M) \cong H_{p,q}(N)$.

(6) The Koszul complex decomposes into direct sums of vector spaces:

$$0 \longrightarrow \bigwedge^n V \otimes M_q \longrightarrow \bigwedge^{n-1} V \otimes M_{q+1} \longrightarrow \cdots \longrightarrow M_{q+n} \longrightarrow 0.$$

◇

The first property is trivial and all other properties are proved in [Mal05, Chap. I].

The remaining part of this section establishes the link between Spencer δ-sequences and the Koszul complex. We closely follow Quillen [Qui64, §5]. First we dualise the Spencer δ-sequences. Over each point, they are sequences of vector spaces which are isomorphic to homogenous components of a Koszul complex.

Proposition A.10. The dual sequence of the Spencer δ-sequence (A.5) is:

$$0 \longrightarrow \bigwedge^n T \otimes M_{q-n} \xrightarrow{\delta^*} \cdots \xrightarrow{\delta^*} T \otimes M_{q+r-1} \xrightarrow{\delta^*} M_{q+r} \longrightarrow 0 \tag{A.7}$$

with the family of dual vector spaces $M_q = \mathcal{M}_q^* \subseteq S^q T \otimes E$ for $E = V(\mathcal{E})^*$. The dual map $\delta^* : \bigwedge^s T \otimes M_q \to \bigwedge^{s-1} T \otimes M_{q+1}$ is given by:

$$\delta^*(t_1 \wedge \cdots \wedge t_s \otimes m) = \sum_{j=1}^{q} (-1)^{j+1} t_1 \wedge \cdots \wedge \widehat{t_j} \wedge \cdots \wedge t_q \otimes t_k m,$$

where elements carrying a hat are omitted. The multiplication $t_k m = \delta^*(t_k \otimes m)$ is the restriction of $T \otimes S^q T \otimes E \to S^{q+1} T \otimes E$ to $T \otimes M_q \to M_{q+1}$. It extends to the graded vector bundle $M = \oplus \mathcal{M}_k^*$, where each fibre $M|_{r_q}$ over $r_q \in \mathcal{R}_q$ is a module over the graded algebra

$$S^\bullet T_x = \bigoplus_{k=0}^{\infty} S^k T_x, \qquad S^0 T_x = \mathbb{R}$$

with $x = \pi_0^q(r_q)$. ◇

Proof. It is sufficient to consider the fibres over a point $r_q \in \mathcal{R}_q$. For a basis $(\xi_1, \ldots, \xi_n)$ of T_x and the dual basis $(d\xi^1, \ldots, d\xi^n)$ of T_x^* the Spencer δ-map is $(\delta\omega)_\mu^k = d\xi^i \wedge (\delta_i \omega)_\mu^k$. The interior product $i(d\xi^i)$ is dual to multiplication with $d\xi^i$, which is an antiderivation:

$$\begin{aligned} \delta^*(t_1 \wedge \cdots \wedge t_q \otimes m) &= \sum_k (-1)^{k+1} t_1 \wedge \cdots \wedge \widehat{t_k} \wedge \cdots \wedge t_q \otimes d\xi^i(t_k) \xi^i m \\ &= \sum_k (-1)^{k+1} t_1 \wedge \cdots \wedge \widehat{t_k} \wedge \cdots \wedge t_q \otimes t_k m. \end{aligned}$$

We made use of the fact that δ_i^* is the multiplication with ξ_i.

According to equation (A.3) in the case of $s = 0$, δ^* is simply the multiplication

$$\delta^* : T \otimes S^q T \to S^{q+1} T,$$

which remains unchanged by tensoring with $\mathcal{M}_k^*$. The extension to $M = \oplus \mathcal{M}_k^*$ is trivial. Since $(\delta^*)^2 = (\delta^2)^* = 0$, we have

$$t_2(t_1 m) - t_1(t_2 m) = (\delta^*)^2 (t_1 \wedge t_2 \otimes m) = 0$$

and thus the multiplication $S^k T \otimes M \to M$ is well-defined for all k. The basis $(\xi_1, \ldots, \xi_n)$ of T_x induces an isomorphism $S^\bullet T_x \cong k[\xi_1, \ldots, \xi_n]$. □

According to [Mal05], M is called *characteristic module*. We can now finish the connection between the Koszul complex and Spencer cohomology. It remains to show that M is a finitely represented module such that the calculation can be implemented on a computer algebra system.

Theorem A.11. Over each $r_q \in \mathcal{R}_q$, the Spencer cohomology can be computed as the homology of the Koszul complex $K_\bullet(\underline{\xi}, M)$:

$$H_r^p(\mathcal{M}_q|_{r_q}) \cong H_{p,r}(M|_{r_q}). \qquad \diamond$$

Proof. By Proposition A.10, the Spencer δ-sequences over r_q are dual to the sequences (A.5). The Koszul complex $K_\bullet(\underline{\xi}, M_{r_q})$ consists of the direct sums of the sequences (A.7), which are then sequences of $S^\bullet T$-modules. We omit the point r_q.

If $\mathcal{M}_q$ is defined by the equations

$$A_k^{\tau,\mu}(r_q)\eta_\mu^k = 0, \qquad 1 \leq \tau \leq s,$$

then the homogenous component M_q is defined by

$$\sum_{\mu,k} A_k^{\tau,\mu}(r_q)\xi^\mu e_k = 0, \qquad 1 \leq \tau \leq s.$$

Here (e_k) is a basis of the fibre of the vertical bundle $V(J_q(\mathcal{E}))$. The higher order symbols and components are defined by:

$$\mathcal{M}_{q+r}: \quad A_k^{\tau,\mu}(r_q)\eta_{\mu+\nu}^k = 0, \quad \Rightarrow \quad M_{q+r}: \quad \xi^\nu(\sum_{\mu,k} A_k^{\tau,\mu}(r_q)\xi^\mu e_k) = 0.$$

It follows that M is finitely represented. □

Because the module M is finitely represented, we are now able to compute all Spencer cohomology groups with Janet or Gröbner basis techniques. We show a Maple implementation in the next section. The 2-acyclicity of a symbol, which was needed in Theorem 1.28, can be read off the dimensions of the cohomology groups.

Remark A.12. To check if a given symbol is acyclic or even involutive, we are interested in the dimensions $h_r^p = \dim(H_r^p(\mathcal{M}_q))$ rather than the cohomology groups themselves. For better readability, it is convenient to display the dimensions $h_r^p = \dim(H_r^p(\mathcal{M}_q))$ in a matrix:

$$\begin{pmatrix} h_1^0 & h_1^1 & \dots & h_1^n \\ h_2^0 & h_2^1 & \dots & h_2^n \\ \vdots & \vdots & & \vdots \end{pmatrix} \tag{A.8}$$

where the colums begin with the zero cohomology groups $H_r^0(\mathcal{M}_q)$. A symbol 2-acyclic if the first three columns of the matrix are filled with zeros. If the whole matrix is zero, the symbol is involutive. $\diamond$

Example A.13. In Example A.7, the cohomology groups $H_4^0 = H_3^1 = H_2^2 = \{0\}$ have been calculated. The above matrix can be filled with the results:

$$\left(\begin{array}{cccc} . & . & . & . \\ \hline . & . & 0 & . \\ . & 0 & . & . \\ 0 & . & . & . \end{array}\right)$$

A dot stands for cohomology groups not yet calculated. As the order of the system is $q = 2$, the groups H_1^p (above the line) are irrelevant. ◇

The Maple package `Spencer`

Both approaches to compute Spencer cohomology groups are implemented in the Maple package `Spencer`, which will be presented in this section. It was developed for the present thesis. A sample calculation follows in the next section.

Independent from this implementation, the `Vessiot` package for Maple V contains a subpackage which is also called `Spencer` (see [ACC+03]). It will be part of the `DifferentialGeometry` package of Maple 14 and it supports the computation of Spencer cohomology groups using sequences of vector bundles and the results from both packages are, up to notation, identical. The commutative algebra version is not implemented in the `Vessiot` subpackage.

Basically, `Spencer` is an application of the `homalg` package for abstract homological algebra by Barakat and Robertz [BR08]. It constructs either Spencer δ-sequences or the module M together with the Koszul complex $K_\bullet(\underline{\xi}, M)$ and uses `homalg` to compute the cohomology groups. While the Spencer δ-sequences involve only linear algebra, the computations via Koszul complex are possible if the following tools are provided:

- The Koszul complex $K_\bullet(\underline{\xi}, A)$.
- The tensor product functor $- \otimes_A M$ to compute $K_\bullet(\underline{\xi}, A) \otimes_A M$.
- A procedure to calculate homology groups $H_p(\underline{\xi}, M)$.
- A procedure to calculate $\dim H_{p,r}(\underline{\xi}, M)$.

To construct the Koszul complex, the package `JanetOre` was extended to the exterior algebra. `JanetOre` was written by Robertz [Rob06], [Rob08]. The tensor product and the homology groups are provided by `homalg`. All computations over $k[\xi_1, \dots, \xi_n]$ are done with `Involutive` [BCG+03], implemented by Cid and Robertz. `Spencer` contains a procedure to calculate $\dim H_{p,r}(\underline{\xi}, M)$ based on `Involutive`'s Hilbert series command. If in future, `homalg` supports graded modules, the last step may be automatic.

For practical work, only three main commands are of interest:

- `SpencerCohomology` – compute the Spencer cohomology.

- `SCohomDim` – display the dimensions $h^{p,r}$ in a matrix.
- `SCZeroSets` – show the restrictions where the computation is valid.

A complete list of commands is found in Appendix D.3. The output of `SpencerCohomology` is usually large and contains a table with sequences and their cohomology groups. The output is best viewed with `SCohomDim`, that prints a matrix of dimensions according to Remark A.12. Theorem A.11 provides the Spencer cohomology computation over a specific point $r_q \in \mathcal{R}_q$. If all coordinates for r_q are left as parameters, `SCZeroSets` prints the domain of validity for the calculation. With a special `homalg` interface, `Involutive` supports this feature.

A Maple Example for Spencer Cohomology

In this section, we show a sample Maple worksheet where `Spencer` is used to compute the Spencer cohomology of a linear system of PDEs. The example is taken from [PQ00, Ex. 3].

On the bundle $\mathcal{E} = \mathbb{R}^3 \times \mathbb{R}$ with coordinates (x^1, x^2, x^3, y) consider the second order linear system with parameters $a, b \in \mathbb{R}$:

$$R_2 : \begin{cases} y_{x^1,x^2} & = & 0, \\ y_{x^1,x^3} & = & 0, \\ y_{x^2,x^3} & = & 0, \\ y_{x^3,x^3} - a\, y_{x^1,x^1} & = & 0, \\ y_{x^2,x^2} - b\, y_{x^1,x^1} & = & 0. \end{cases}$$

Depending on the parameters, we obtain three cases with different Spencer cohomology.

```
> with(jets):
> with(JanetOre):
> with(Involutive):
> with(homalg):
```

The package Spencer to calculate the Spencer cohomology:

```
> with(Spencer);
```

$[KoszulComplexT,\ SCZeroSets,\ SCohomDim,\ SdeltaCosequence,\ SpencerCohomology,\ SymbolModule,\ SymbolOf]$

Declaration of independent and dependent variables:

```
> ivar := [x1, x2, x3]: dvar := [y]: Dvar := [xi1, xi2, xi3]:
```

Define the system R_2:

```
> R2 := [y[x1,x2]=0, y[x1,x3]=0, y[x2,x3]=0, y[x3,x3]-a*y[x1,x1]=0,
> y[x2,x2]-b*y[x1,x1]=0];
```

$$R2 := [y_{x1,x2} = 0,\ y_{x1,x3} = 0,\ y_{x2,x3} = 0,\ y_{x3,x3} - a\, y_{x1,x1} = 0,\ y_{x2,x2} - b\, y_{x1,x1} = 0]$$

The Generic Case:

At first, the Spencer cohomology is calculated for the generic case, where no assumption on the parameters have been made. We compute the Spencer cohomology $H^{p,q}$ for $q \in \{2,3\}$ and $p \in \{0,1,2,3\}$ to see all cohomology groups involving $\mathcal{M}_2$ and $\mathcal{M}_3$:

```
> Sc2_1:=SpencerCohomology(R2,[2,3],[0,1,2,3],ivar,dvar):
```

Print the dimensions of the cohomology groups (dots mean, that nothing has been calculated yet):

```
> SCohomDim(Sc2_1,ivar,dvar);
```

$$\begin{bmatrix} . & . & . & . \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & . \\ 0 & 0 & . & . \\ 0 & . & . & . \end{bmatrix}$$

The symbol $\mathcal{M}_2$ is not involutive, since $H^{3,2}(\mathcal{M}_2)$ is one-dimensional. It is likely that $\mathcal{M}_2$ is 2-acyclic. To prove it, we compute the Spencer cohomology via Koszul complex and display their dimensions.

```
> IZS := `InvolutiveZeroSets/homalg`:
> Sc2_1k:=SpencerCohomology(R2,ivar,dvar,Dvar,IZS):
> SCohomDim(Sc2_1k,Dvar,IZS);
```

$$\begin{bmatrix} . & . & . & . \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The second calculation shows that $\mathcal{M}_2$ is 2-acyclic and $\mathcal{M}_3$ is involutive. If the input of `SCohomDim` has been computed via Koszul complex, a trailing row of zero means that all higher cohomology vanishes. During the calculation, both a and b are assumed to be nonzero (as well as some combinations).

```
> SCZeroSets(Sc2_1k);
```

$$[b,\ a]$$

The previous calculation is not valid for vanishing a or b, so these cases must be treated separately.

The Case $b = 0$:

In this step, we choose a single term from the output of `SCZeroSets` and set it to zero. Our first choice is $b = 0$.

```
> R2_2 := subs(b=0,R2);
```

$$R2\_2 := [y_{x1,\,x2} = 0,\ y_{x1,\,x3} = 0,\ y_{x2,\,x3} = 0,\ y_{x3,\,x3} - a\, y_{x1,\,x1} = 0,\ y_{x2,\,x2} = 0]$$

Compute several cohomology groups to see that $\mathcal{M}_2$ is not 2-acyclic:

```
> Sc2_2:=SpencerCohomology(R2_2,[2,3],[0,1,2,3],ivar,dvar):
> SCohomDim(Sc2_2,ivar,dvar);
```

$$\begin{bmatrix} . & . & . & . \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & . \\ 0 & 0 & . & . \\ 0 & . & . & . \end{bmatrix}$$

Compute the Spencer cohomology via Koszul complex and show the dimensions:

```
> Sc2_2k:=SpencerCohomology(R2_2,ivar,dvar,Dvar,IZS):
> SCohomDim(Sc2_2k,Dvar,IZS);
```

$$\begin{bmatrix} . & . & . & . \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The symbol $\mathcal{M}_3$ is involutive and the calculation is valid for $a \neq 0$:

```
> SCZeroSets(Sc2_2k);
```

$$[a]$$

The Case $a = 0$:

This case is equivalent to $b = 0$ by swapping x^1 and x^3. We expect that this calculation is valid for $b \neq 0$.

```
> R2_3 := subs(a=0,R2);
```

$$R2\_3 := [y_{x1,x2} = 0,\ y_{x1,x3} = 0,\ y_{x2,x3} = 0,\ y_{x3,x3} = 0,\ y_{x2,x2} - b\,y_{x1,x1} = 0]$$

The Spencer cohomology:

```
> Sc2_3k:=SpencerCohomology(R2_3,ivar,dvar,Dvar,IZS):
> SCohomDim(Sc2_3k,Dvar,IZS);
```

$$\begin{bmatrix} . & . & . & . \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The domain of validity is no surprise:

```
> SCZeroSets(Sc2_3k);
```

$$[b]$$

The Case $a = b = 0$:

Both computations for one constant being zero have used the assumption that the second constant is nonzero. So the remaining case where both constants are zero has to be taken into account. It has partly been treated in Example A.7.

```
> R2_4 := subs([a=0,b=0],R2);
```

$$R2\_4 := [y_{x1,x2} = 0,\ y_{x1,x3} = 0,\ y_{x2,x3} = 0,\ y_{x3,x3} = 0,\ y_{x2,x2} = 0]$$

Compute the Spencer cohomology via Spencer δ-sequences:

```
> Sc2_4:=SpencerCohomology(R2_4,[2,3],[0,1,2,3],ivar,dvar):
```

The dimensions:

```
> SCohomDim(Sc2_4,ivar,dvar);
```

$$\begin{bmatrix} . & . & . & . \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & . \\ 0 & 0 & . & . \\ 0 & . & . & . \end{bmatrix}$$

It is very likely that $\mathcal{M}_2$ is involutive. The computation via Koszul complex proves it.

```
> Sc2_4k:=SpencerCohomology(R2_4,ivar,dvar,Dvar,IZS):
> SCohomDim(Sc2_4k,Dvar,IZS);
```

$$\begin{bmatrix} . & . & . & . \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As all parameters were set to zero, this calculation is valid everywhere:

```
> SCZeroSets(Sc2_4k);
```

$$[\,]$$

Further Commands of the Spencer Package

In this part, we show less important commands of the `Spencer` package that might be of use, too. The first procedure computes the symbol of a linear system R_q. It returns two lists. The first contains a local basis of $\mathcal{M}_q$ and the second the defining equations (A.1). Compute the symbol $\mathcal{M}_2$ for $a = b = 0$:

```
> SymbolOf(R2_4,2,ivar,dvar);
```

$$[[y_{x1,\,x1}],\ [y_{x1,\,x2},\ y_{x1,\,x3},\ y_{x2,\,x2},\ y_{x2,\,x3},\ y_{x3,\,x3}]]$$

We observe that both symbols $\mathcal{M}_2$ and $\mathcal{M}_3$ are nonzero:

```
> SymbolOf(R2_4,2,ivar,dvar)[1];
> SymbolOf(R2_4,3,ivar,dvar)[1];
```

$$[y_{x1,\,x1}]$$

$$[y_{x1,\,x1,\,x1}]$$

It is also possible to compute the characteristic module:

```
> SymbolModule(R2_4,ivar,dvar,Dvar,IZS);
```

$$[[1 = 1],\ [\xi 3^2,\ \xi 2\,\xi 3,\ \xi 1\,\xi 3,\ \xi 2^2,\ \xi 1\,\xi 2],\ \text{“Presentation”},\ 1 + 3\,s + \frac{s^2}{1-s},\ [1,\,0,\,0]]$$

The output is a **homalg** presentation of a module (compare [BR08, Fig. 1, App. C.1]). It contains a list of generators $[1 = 1]$, relations on the generators $[\xi_3^2, \ldots]$ and combinatorial data such as the Hilbert series $1 + 3s + \frac{s^2}{1-s}$.

The example is small enough to display a single Spencer δ-sequence. It is actually the sequence (A.6) up to renaming $\xi \to y$. The result is a cosequence in homalg notation containing presentations and matrices of the homomorphisms.

```
> Sp_2_4 := SdeltaCosequence(R2_4,2,ivar,dvar);
```

$$Sp\_2\_4 := \Big[[[1 = 0], [1], \text{"Presentation"}, [1], 0], \begin{bmatrix} 0 \end{bmatrix},$$

$$[[1 = y_{x1,x1,x1,x1}], [0], \text{"Presentation"}, [0], 1], \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, [$$

$$[[1, 0, 0] = y_{x1,x1,x1}\, dx1, [0, 1, 0] = y_{x1,x1,x1}\, dx2, [0, 0, 1] = y_{x1,x1,x1}\, dx3],$$

$$[[0, 0, 0]], \text{"Presentation"}, [0, 0, 0], 3], \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, [$$

$$[[1, 0, 0] = y_{x1,x1}\, dx1\, dx2, [0, 1, 0] = y_{x1,x1}\, dx1\, dx3, [0, 0, 1] = y_{x1,x1}\, dx2\, dx3],$$

$$[[0, 0, 0]], \text{"Presentation"}, [0, 0, 0], 3], \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, [$$

$$[[1, 0, 0] = y_{x1}\, dx1\, dx2\, dx3, [0, 1, 0] = y_{x2}\, dx1\, dx2\, dx3, [0, 0, 1] = y_{x3}\, dx1\, dx2\, dx3]$$

$$, [[0, 0, 0]], \text{"Presentation"}, [0, 0, 0], 3]\Big]$$

The presentations for $\mathcal{M}_4$ and $T^* \otimes \mathcal{M}_3$ are:

```
> Sp_2_4[3];
> Sp_2_4[5];
```

$$[[1 = y_{x1,x1,x1,x1}], [0], \text{"Presentation"}, [0], 1]$$

$[[[1, 0, 0] = y_{x1,x1,x1}\, dx1, [0, 1, 0] = y_{x1,x1,x1}\, dx2, [0, 0, 1] = y_{x1,x1,x1}\, dx3], [[0, 0, 0]],$ "Presentation", $[0, 0, 0], 3]$

The homomorphism $\delta^{(0)}$:

```
> Sp_2_4[4];
```

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

As computed in Example A.7, the Spencer cohomology vanishes. The last entry of the vector space representation shows the dimension, which is zero in all cases:

```
> CSp_2_4 := CohomologyModules(Sp_2_4,[0],PIR);
> map(a->a[-1],CSp_2_4);
```

$$CSp\_2\_4 := [\%1, \%1, \%1]$$
$$\%1 := [[1 = 0], [1], \text{"Presentation"}, [1], 0]$$
$$[0, 0, 0]$$

The Spencer cohomology completes the geometric treatment of PDE systems from Section 1.3.

Appendix B

Jet Groups

Studying the symmetries of geometric objects on a manifold X leads to PDE systems over $\mathcal{E} = X \times X$. Their invertible solutions may be composed, so all germs of solutions, stabilising a single point $x \in X$, form a group. In this appendix the restriction of these groups to q-th order jets, called jet groups, are introduced. Already present in Ehresmann's work [Ehr53], they were thorougly studied by Terng [Ter78] (see also [KMS93]). It turns out that both prolongation and projection of PDE systems can be done with the help of jet groups and their actions on manifolds. This appendix mainly follows Terng [Ter78] and [KMS93, §13].

The most general jet group GL_q consists of all q-jets of diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ that keep the origin fixed. In Chapter 2 on groupoids the GL_q reappears as an isotropy group and in Chapter 3 it is shown that each natural bundle is uniquely defined by a GL_q-action on its fibre.

In many cases, one wants to restrict the symmetries of geometric objects to a subclass of all diffeomorphisms, e.g. to the isometries of a metric. This leads to subgroups $G_q \leq \mathrm{GL}_q$ for all $q \in \mathbb{N}$. But not all choices of subgroups G_q are useful for prolongation and projection, so conditions for suitable groups G_q are developed. The corresponding conditions for their Lie algebras are very much inspired by the results in [GS64] on infinitesimal automorphisms.

B.1 Her Majesty GL_q

At first, we define the jet group GL_q, which is a model for all diffeomorphisms $\varphi : X \to X$ that stabilise a point $x \in X$. As we are dealing with local properties, we may assume $X = \mathbb{R}^n$ and $x = 0$. All germs of diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi(0) = 0$ form a group GL_∞. Taking the q-jet of φ conserves the group properties and defines the Lie group GL_q.

Definition B.1. The *jet group of order q*, $\mathrm{GL}_q = \mathrm{GL}_q(\mathbb{R}^n)$, is the Lie group of q-jets of diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ leaving the origin fixed. The group multiplication is the jet composition. ◇

The truncated Taylor series representation from Remark 1.4 identifies GL_q with polynomial maps $\mathbb{R}^n \to \mathbb{R}^n$ without constant term and nonzero determinant. Under this identification, GL_q is an open subset of $\mathbb{R}[x^1, \ldots, x^n]^n_{0,\leq q}$, which consists of n-tuples of polynomials p of degree $\leq q$ with $p(0) = 0$. The multiplication of $f, g \in \mathrm{GL}_q$ is the truncation of the composition $f \circ g$ of polynomial maps to degree q.

Using the polynomial representation and the jet bundle projections π_q^{q+r}, we obtain the following properties of jet groups (see [Ter78, §2]).

Proposition B.2. For the jet groups $\mathrm{GL}_q(\mathbb{R}^n)$ we have:

(1) $\mathrm{GL}_1(\mathbb{R}^n) \cong \mathrm{GL}(\mathbb{R}^n)$ is the usual general linear group.

(2) For each $q \in \mathbb{N}$ and $r \in \mathbb{Z}_{\geq 0}$ there is an exact sequence of groups:

$$1 \longrightarrow K_q^{q+r} \longrightarrow \mathrm{GL}_{q+r} \xrightarrow{\pi_q^{q+r}} \mathrm{GL}_q \longrightarrow 1 \tag{B.1}$$

defining the normal subgroups K_q^{q+r}. Set $K_{q+1} := K_q^{q+1}$.

(3) The exact sequence splits for π_1^q:

$$1 \longrightarrow K_1^q \longrightarrow \mathrm{GL}_q \underset{\longleftarrow}{\xrightarrow{\pi_1^q}} \mathrm{GL}_1 \longrightarrow 1.$$

(4) K_1^q is nilpotent and $\mathrm{GL}_q \cong \mathrm{GL}_1 \ltimes K_1^q$ is a semidirect product. The normal subgroup $K_{q+1} = K_q^{q+1}$ is abelian. ◇

Proof. The composition of linear polynomials stays linear, so $\mathrm{GL}_1(\mathbb{R}^n) \cong \mathrm{GL}(\mathbb{R}^n)$ follows. The projection of forgetting higher order jets,

$$\pi_q^{q+r} : J_{q+r}(\mathbb{R}^n \times \mathbb{R}^n) \to J_q(\mathbb{R}^n \times \mathbb{R}^n),$$

restricts to the group homomorphism $\pi_q^{q+r} : \mathrm{GL}_{q+r} \to \mathrm{GL}_q$ because $j_q(\psi \circ \varphi) = j_q(\psi) \circ j_q(\varphi)$. The second exact sequence splits, because the composition of linear maps stays linear. It follows from the split that GL_q is a semidirect product. (4) follows from the Lie algebra properties in Proposition B.6. □

The kernels K_q^{q+r} consist of elements whose Taylor expansion differs from the identity map only above order q. K_q^{q+r} is used in Chapter 3 to describe the projection of PDE systems for symmetries efficiently. The projection is prepared in Section B.5.

B.2 The Lie Algebra $\mathfrak{gl}_q$ of GL_q

We also introduce the Lie algebra $\mathfrak{gl}_q$ of GL_q which consists of q-jets of vector fields on $\mathbb{R}^n$ that vanish at the origin. Due to the jet projections, $\mathfrak{gl}_q$ is a graded algebra and we give an explicit basis adapted to the graduation. This basis was implicitly used by Lie [Lie91] and Vessiot [Ves03] and it provides an efficient tool to compute the projections of PDE systems for groupoids in Section 3.4.2.

The algebra $\mathfrak{gl}_q$ reappears in Appendix C.2 as the isotropy algebra of jet algebroids. We have adapted the notation of [Ter78] to match Chapter 2 and Appendix C. Let $J_{q,0}(T\mathbb{R}^n) = \ker(\pi_0^q)$ be the subbundle of $J_q(T\mathbb{R}^n) \to X$ of q-jets $j_q(\xi)(x)$ of vector fields ξ on $\mathbb{R}^n$ which vanish at x ($\xi(x) = 0$). Denote the fibre at the origin $0 \in \mathbb{R}^n$ by $J_{q,0}(T\mathbb{R}^n)_0$. It contains q-jets of vector fields at the origin, which also vanish at the origin.

Proposition B.3. [Ter78, Thm. 2.1] The Lie algebra $\mathfrak{gl}_q$ of GL_q is isomorphic to

$$\mathfrak{gl}_q \cong J_{q,0}(T\mathbb{R}^n)_0.$$

The Lie bracket for $\xi, \eta \in \Gamma(T\mathbb{R}^n)$ with $\xi|_0 = \eta|_0 = 0$ is

$$[j_q(\xi)(0), j_q(\eta)(0)] = -j_q([\xi, \eta](0)).$$

If φ_t is the flow generated by ξ, the exponential map is $\exp(j_q(\xi)(0)) = j_q(\varphi_t)(0)$.$\diamond$

Proof. The tangent space of GL_q at the identity consists of all q-jets $j_q(\xi)(0)$ of vector fields $\xi \in \mathfrak{X}(\mathbb{R}^n) = \Gamma(T\mathbb{R}^n)$ which vanish at the origin: $\xi|_0 = 0$. This is the kernel of $\pi^q_{0,*} : J_q(T\mathbb{R}^n) \to T\mathbb{R}^n$.

Differentiate the flow φ_t generated by ξ to obtain:

$$\left.\frac{d}{dt}\right|_{t=0} j_q(\varphi_t)(0) = j_q(\left.\frac{d}{dt}\right|_{t=0} \varphi_t)(0) = j_k(\xi)(0),$$

since derivatives commute. The Lie bracket is calculated analogously using $j_q(\varphi_t) \circ j_q(\psi_t) = j_q(\varphi_t \circ \psi_t)$. □

Similar to GL_q, elements of $\mathfrak{gl}_q$ can be identified with polynomial vector fields of degree q without constant term. The second property in the next proposition is the infinitesimal analogue to Proposition B.2 (2).

Proposition B.4. For $\mathfrak{gl}_q$ the following properties hold:

(1) The tangent maps $\pi^{q+r}_{q,*}|_{\mathrm{id}} : \mathfrak{gl}_{q+r} \to \mathfrak{gl}_q$ from equation (B.1) are the jet projections

$$\pi^{q+r}_q : J_{q+r,0}(T\mathbb{R}^n) \to J_{q,0}(T\mathbb{R}^n),$$

which are again called π^{q+r}_q.

(2) For each $q \in \mathbb{N}$ and $r \in \mathbb{Z}_{\geq 0}$ there is an exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{k}_q^{q+r} \longrightarrow \mathfrak{gl}_{q+r} \xrightarrow{\pi_q^{q+r}} \mathfrak{gl}_q \longrightarrow 0 \tag{B.2}$$

defining ideals $\mathfrak{k}_q^{q+r} = \ker(\pi_q^{q+r})$ being the Lie algebras of K_q^{q+r}. For $q = 1$ the sequence splits. ◇

For Proposition B.2 (4), we show that $\mathfrak{gl}_q$ is graded. At first, we choose a basis of $\mathfrak{gl}_q$. It plays an important role for infinitesimal diffeomorphisms treated in Appendix C.2 on jet algebroids. The vector fields were introduced by Lie [Lie91, II, §14] and Vessiot [Ves03, eq. 3].

Lemma B.5. The q-jets of the vector fields

$$v_i^\mu = \frac{1}{\mu!} x^\mu \frac{\partial}{\partial x^i}$$

form a basis $(A_i^\mu(q) = j_q(v_i^\mu) | 1 \leq i \leq n,\ 1 \leq |\mu| \leq q)$ of $\mathfrak{gl}_q$ with Lie brackets:

$$[A_i^\mu(q), A_j^\nu(q)] = \begin{cases} \nu_i \frac{(\mu+\nu-1_i)!}{\mu!\,\nu!} A_j^{\mu+\nu-1_i}(q) \quad - \quad \mu_j \frac{(\mu+\nu-1_j)!}{\mu!\,\nu!} A_i^{\mu+\nu-1_j}(q) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } |\mu+\nu| \leq q+1 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases} \tag{B.3}$$

If ξ_μ^i are the fibre coordinates of $J_{q,0}(T\mathbb{R}^n)$, each $A_j^\nu(q)$ corresponds to a single nonzero $\xi_\mu^i = \delta_j^i \delta_\mu^\nu$. ◇

Proof. All v_i^μ form a basis of polynomial vector fields of order $\leq q$. The Lie bracket

$$[x^\mu \partial_{x^i}, x^\nu \partial_{x^j}] = \nu_i\, x^{\mu+\nu-1_i}\, \partial_{x^j} - \mu_i\, x^{\mu+\nu-1_j}\, \partial_{x^i}$$

truncated to order q gives the Lie brackets for $A_i^\mu(q)$ by adjusting coefficients. □

This basis is adapted to the graduation of $\mathfrak{gl}_q$ which will be treated in the next proposition, which summarises several results from [KMS93, §13].

Proposition B.6. For $\mathfrak{gl}_q$ the following properties hold:

(1) The $\mathfrak{k}_i^q$, $1 \leq i \leq q$ are a filtration of $\mathfrak{gl}_q$ which induces a graduation of $\mathfrak{gl}_q$ as a vector space:

$$\mathfrak{gl}_q \cong \mathfrak{gl}_q^1 \oplus \mathfrak{gl}_q^2 \oplus \cdots \oplus \mathfrak{gl}_q^q, \tag{B.4}$$

where the homogenous components $\mathfrak{gl}_q^i := \mathfrak{k}_i^q / \mathfrak{k}_{i+1}^q$ are canonically isomorphic to $S^i \mathbb{R}^{n,*} \otimes \mathbb{R}^n$. So for $i \leq q$, $\mathfrak{gl}_q^i \cong \mathfrak{gl}_{q+1}^i$ and the index q may be omitted if no confusion arises.

(2) Setting $\mathfrak{gl}_q^i = 0$ for $i > q$, the Lie bracket on $\mathfrak{gl}_q$ satisfies:

$$[\mathfrak{gl}_q^i, \mathfrak{gl}_q^j] \subseteq \mathfrak{gl}_q^{i+j-1} . \tag{B.5}$$

(3) Shifting the degrees by one, $\mathfrak{h}_q^i = \mathfrak{gl}_q^{i+1}$, $\mathfrak{gl}_q$ is a graded Lie algebra:

$$[\mathfrak{h}_q^i, \mathfrak{h}_q^j] \subseteq \mathfrak{h}_q^{i+j}. \tag{B.6}$$

(4) All kernels $\mathfrak{k}_i^q$ are nilpotent and therefore solvable. The highest order ideals $\mathfrak{k}_q^{q+1} \cong \mathfrak{gl}_{q+1}^{q+1}$ are abelian. ⋄

Proof. $\mathfrak{k}_i^q \subset \mathfrak{k}_j^q$ for $j \leq i$ implies that

$$\{0\} \subseteq \mathfrak{k}_q^q \subseteq \cdots \subseteq \mathfrak{k}_1^q \subseteq \mathfrak{gl}_q$$

is a filtration of $\mathfrak{gl}_q$. A basis for $\mathfrak{gl}_q^i$ are $(A_j^\mu(q) \,|\, |\mu| = i)$ being q-jets of homogenous polynomial vector fields of degree i. This also proves $\mathfrak{gl}_q^i \cong S^i\mathbb{R}^{n,*} \otimes \mathbb{R}^n$. Equation (B.5) and the graduation (3) then follow directly from Lemma B.5. $\mathfrak{k}_i^q$ is nilpotent, since the derived algebras $[\mathfrak{k}_i^q, \mathfrak{k}_i^q], \ldots$ restrict to higher degrees by equation (B.5). The Lie brackets for the highest order terms $\mathfrak{k}_q^{q+1}$ are thus always zero. □

The graduation of $\mathfrak{gl}_q$ according to equation (B.4) follows the interpretation of $\mathfrak{gl}_q^i$ as jets of strict order i. In this thesis, the jet order is more important than the fact that $\mathfrak{gl}_q$ is a graded Lie algebra with shifted degrees as in equation (B.6).

The isomorphism $\mathfrak{gl}_q^i \cong S^i\mathbb{R}^{n,*} \otimes \mathbb{R}^n$ is connected with the symbol of a PDE system in Appendix A. If $\mathcal{R}_q$ is a PDE system over $\mathcal{E} = \mathbb{R}^n \times \mathbb{R}^n$ of order q, its symbol is isomorphic to a subspace of $\mathfrak{gl}_q^q$ (see Lemma 3.43).

B.3 Subgroups G_q of GL_q

The full jet groups GL_q were introduced as q-jets of all diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi(0) = 0$. But in order to study a smaller class of diffeomorphisms, e.g. rotations in $\mathbb{R}^n$, it is necessary to consider closed subgroups $G_q \leq \mathrm{GL}_q$ for each $q \in \mathbb{N}$. Not all choices of subgroups are induced by diffeomorphisms, so we state conditions on G_q. If they are fulfilled, the next section shows how to prolong the G_q-action on a manifold F to a G_{q+r}-action on the algebraic prolongation $F^{(r)}$ of F, which will be introduced in Section B.5.1. The results may be new, but they are obtained in a straightforward way from [GS64].

An obvious condition to be fulfilled by G_q is that all projections $\pi_q^{q+r} : G_{q+r} \to G_q$ must be well-defined and surjective. Another condition applies to the prolongation, namely that the elements of G_q are defined by PDEs and the groups G_{q+r} with higher jet order should also satisfy the prolonged equations. For GL_q itself, the prolongation property is trivially satisfied.

Proposition B.7. Let $G_q \leq \mathrm{GL}_q$ be closed Lie subgroups for $q \in \mathbb{N}$. The groups G_q are the projections of a closed Lie subgroup $G_\infty \leq \mathrm{GL}_\infty$ to q-th order jets if and only if the following conditions hold for all $q \leq q_0$, $r \in \mathbb{N}$:

(1) The projections π_q^{q+r} restrict to group epimorphisms $\pi_q^{q+r} : G_{q+r} \to G_q$, such that there are exact sequences:

$$1 \longrightarrow K_q^{q+r} \longrightarrow G_{q+r} \xrightarrow{\pi_q^{q+r}} G_q \longrightarrow 1$$

(2) G_{q+r} is a subgroup of $\{j_{q+r}(\varphi) \,|\, \varphi \in \mathrm{GL}_\infty, j_q(\varphi)(0) \in G_q\}$.

In this case, Proposition B.2 (3) and (4) remains valid for the restriction to G_q.$\diamond$

Proof. Let $G_\infty \leq \mathrm{GL}_\infty$. Then properties (1) and (2) follow, because $\pi_q^\infty : G_\infty \to G_q$ is $\pi_q^\infty = \pi_q^{q+r} \circ \pi_{q+r}^\infty$.

If (1) is not valid, there exists a φ such that $j_{q+r}(\varphi)(0) \in G_{q+r}$ but $j_q(\varphi)(0) \notin G_q$. So there cannot be an epimorphism $G_\infty \to G_q$.

If (2) is violated, there exists an element $g_{q+r} \in G_{q+r}$ with taylor series representative φ according to Remark 1.4 and $j_{q+r}(\varphi)(0) \in G_{q+r}$ but $j_q(\varphi)(0) \notin G_q$. So $\pi_q^{q+r} : G_{q+r} \to G_q$ is not well-defined. $\square$

Condition (2) means that for a given G_q, the groups G_{q+r} may not be chosen as the full preimage $(\pi_q^{q+r})^{-1}(G_q)$. Being a closed subgroup of GL_q, there is a system of PDEs defining G_q together with the condition that the origin stays fixed. Then all elements $g_{q+r} \in G_{q+r}$ must be solutions of the same system. It is possible to restrict G_{q+r} further by introducing new equations of order $> q$.

Definition B.8. Let $G_q \leq \mathrm{GL}_q$ be closed Lie subgroups for $q \in \mathbb{N}$ satisfying the conditions of Proposition B.7. The G_q are called *compatible with prolongation.* $\diamond$

The next example shows that condition (2) of Proposition B.7 is nontrivial and may even imply that the prolongation is trivial.

Example B.9. The orthogonal group $O(\mathbb{R}^n) = \{g \in \mathrm{GL}(\mathbb{R}^n) \,|\, gg^{tr} = 1\}$ is a subgroup $G_1 := O(\mathbb{R}^n) \leq \mathrm{GL}_1$ defined by the equations:

$$\sum_j y_j^i y_j^k = \delta^{ik} \qquad \forall\, 1 \leq i, k \leq n.$$

Any solution $y^i = \varphi^i(x)$ also fulfills the second order equations:

$$\sum_j y_{jl}^i \, y_j^k + y_j^i \, y_{jl}^k = 0,$$

which are obtained by prolongation. But these equations are equivalent to $y_{jk}^i = 0$ for all $i, j, k \leq n$. So G_2 and all higher order groups are isomorphic to $O(\mathbb{R}^n)$. The group $O(\mathbb{R}^n)$ is of *finite type.* $\diamond$

B.4 Lie Subalgebras $\mathfrak{g}_q$ of $\mathfrak{gl}_q$

The condition that groups $G_q \leq \mathrm{GL}_q$ for $q \in \mathbb{N}$ are compatible with prolongation can also be stated for their Lie algebras $\mathfrak{g}_q$. This process is very similar to [GS64], where infinitesimal automorphisms of a finite-dimensional vector space are studied. Guillemin and Sternberg derive local conditions on transitive geometries, which have a Lie algebroid structure (see Appendix C). As we are dealing with Lie algebras only, their results are turned into conditions.

At first, we have to define the prolongation of a Lie subalgebra of $\mathfrak{gl}_q$, which is done in [GS64, Def. 2.1]. See also [Kob72] for the case of $q = 1$. The prolongation may be defined using only vector spaces.

Definition B.10. Let P and Q be two vector spaces. A subspace $h \leq \mathrm{Hom}(P, Q)$ is called *tableau*. The *first prolongation* of a tableau h is the subspace $h^{(1)} \leq \mathrm{Hom}(P, h)$ such that $T \in h^{(1)}$ if and only if

$$T(u)v = T(v)u \qquad \forall u, v \in P. \tag{B.7}$$

Higher prolongations are defined by iteration: $h^{(i)} := h^{(1)^i}$. ⋄

Based on the prolongation of Lie algebras, also the prolongation G^1 of a group $G \leq \mathrm{GL}_1$ is defined [Ste64, Def. VII.3.3]. It is may be used for the Cartan equivalence method in Section 6.2. Before we proceed with the prolongation of Lie algebras, we state a useful lemma.

Lemma B.11. [GS64, La. 2.1] For P, Q as in Definition B.10 let $h^0 \leq \mathrm{Hom}(P, Q)$ and h^i $(i \leq 1)$ be a sequence of spaces satisfying $h^{i+1} \subseteq (h^i)^{(1)}$. Then there is an integer q, called the *geometric order*, such that $h^{q+r} = (h^q)^{(r)}$ for all $r \geq 1$. ⋄

The lemma follows directly from Hilbert's basis theorem.

To apply Definition B.10 to the homogenous components of a graded Lie subalgebra $\mathfrak{g}_q \leq \mathfrak{gl}_q$, we use $P = \mathbb{R}^n$ and $Q \leq S^k\mathbb{R}^{n,*} \otimes \mathbb{R}^n$ for suitable $k \in N$. Let

$$\mathfrak{g}_q = \mathfrak{g}^1 \oplus \ldots \oplus \mathfrak{g}^q$$

be the decomposition of $\mathfrak{g}_q$ into homogenous components. Choose the tableau $\mathfrak{g}^k$ for some $k \in \mathbb{N}$. Both the first prolongation $(g^k)^{(1)}$ and $\mathfrak{g}^{k+1}$ are subspaces of $\mathfrak{gl}^{k+1}$. We call those Lie subalgebras $\mathfrak{g}_q$, $q \in \mathbb{N}$ *compatible with prolongation* that satisfy $\mathfrak{g}^{k+1} \subseteq (\mathfrak{g}^k)^{(1)}$. By Lemma B.11, there is a finite geometric order q_0 such that $\mathfrak{g}^{k+1} = (\mathfrak{g}^k)^{(1)}$ for all $k \geq q_0$. The next proposition shows that we have constructed the lie algebras corresponding to subgroups $G_q \leq \mathrm{GL}_q$ which are compatible with prolongation.

Proposition B.12. Let $G_q \leq \mathrm{GL}_q$ be groups which are compatible with prolongation and let $\mathfrak{g}_q \leq \mathfrak{gl}_q$ be their Lie algebras. Then there are subspaces $\mathfrak{g}^k \leq \mathfrak{gl}^k$ with $\mathfrak{g}^{k+1} \subseteq (\mathfrak{g}^k)^{(1)}$ such that

$$\mathfrak{g}_q = \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^q \qquad \forall\, q \in \mathbb{N}$$

can be decomposed as a vector space, inducing a graduation on $\mathfrak{g}_q$.

Conversely, if $\mathfrak{g}_q$, $q \in \mathbb{N}$ are subalgebras which are compatible with prolongation, then formally a subgroups $G_q \leq \mathrm{GL}_q$ compatible with prolongation may be constructed. ◇

Formally means, that it is possible to construct formal power series elements of GL_∞, but no convergence is tested. This is again the difference between formal integrability and the existence of smooth solutions (see Section 1.3.2).

Proof. Propositions B.3 and B.7 imply that there are exact sequences of Lie algebras

$$0 \longrightarrow \mathfrak{k}_q^{q+r} \longrightarrow \mathfrak{g}_{q+r} \longrightarrow \mathfrak{g}_q \longrightarrow 0$$

defining ideals $\mathfrak{k}_q^{q+r} \leq \mathfrak{g}_{q+r}$ analogous to equation (B.2). Again, $\mathfrak{k}_q^{q+r}$ is the Lie algebra of K_q^{q+r} and all $\mathfrak{k}_i^q$ induce a filtration of $\mathfrak{g}_q$. Setting $\mathfrak{g}^k = \mathfrak{k}_k^q / \mathfrak{g}_{k+1}^q$ to be the homogenous components, there is a decomposition of $\mathfrak{g}_q$ as a vector space:

$$\mathfrak{g}_q = \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^q$$

As $\mathfrak{k}_{q-1}^q \leq S^q\mathbb{R}^{n,*} \otimes \mathbb{R}^n$, each $\mathfrak{g}^k$ is a subspace of $\mathfrak{gl}^k$. The graduation of the Lie algebra $\mathfrak{g}_q$ follows from $\mathfrak{gl}_q$.

If $G_q \leq \mathrm{GL}_q$ is defined by the equations $\Phi^\tau(y_q) = 0$, $1 \leq \tau \leq l$, then condition (2) of Proposition B.7 implies that all total derivatives $D_j \Phi^\tau(y_{q+1}) = 0$ of the equations are valid on G_{q+1}.

According to Proposition B.3, $\mathfrak{g}_q$ is then defined by the equations $\mathrm{id}^*\,\delta(\Phi^\tau) = 0$, where $\delta = \xi_\mu^i \partial_{y_\mu^i}$ is the vertical derivative and ξ^i are the coordinates of $J_q(T\mathbb{R}^n)$. By the above graduation, one obtains homogenous equations $A_i^{\tau,\mu}\xi_\mu^i = 0$. On $\mathfrak{g}_{q+1}$, the equations $A_i^{\tau,\mu}\xi_{\mu+1_j}^i = 0$ must be valid, because vertical and total derivatives commute: $\delta D_j = D_j \delta$ by Proposition 1.10 (5).

These are exactly the conditions from Definition B.10. Equation (B.7) implies that $(\mathfrak{g}^k)^{(1)} \leq \mathfrak{gl}^{k+1}$. Let $\mathfrak{g}^k$ be defined by equations $A_i^{\tau,\mu}\xi_\mu^i$ for $|\mu| = k$. Then an element

$$\xi_{\mu+1_j}^i x^{\mu+1_j} \partial_{x^i} \in (\mathfrak{g}^k)^{(1)}$$

must be a map $\mathbb{R}^n \to S^k\mathbb{R}^{n,*} \otimes \mathbb{R}^n$, so we decompose it to

$$x^j \otimes \xi_{\mu+1_j}^i x^\mu \partial_{x^i}.$$

But then $\xi_{\mu+1_j}^i x^\mu \partial_{x^i}$ must be an element of $\mathfrak{g}^k$, thus $A_i^{\tau,\mu}\xi_{\mu+1_j}^i = 0$ must be valid.

Conversely, Proposition B.3 allows to construct a chain of closed Lie subgroups G_q that are compatible with prolongation. However the power series vector field constructed for increasing q is only a formal one. Take G_∞ as the inverse limit of all G_q. □

Example B.13. We continue Example B.9 on the orthogonal group. Its Lie algebra $\mathfrak{o} = \mathfrak{o}(\mathbb{R}^n)$ is defined by:

$$\mathrm{id}^*(\delta(\sum_j y^i_j y^k_j)) = id^*(\xi^i_j y^k_j + y^i_j \xi^k_j) = \xi^i_k + \xi^k_i = 0.$$

The second order equations,

$$\xi^i_{kl} + \xi^k_{il} = 0,$$

can easily be solved for all second order derivatives ξ^i_{kl} by means of linear algebra. The first prolongation $\mathfrak{o}^{(1)}$ is zero, which means that all higher prolongations must also vanish. See [GS64, eq. (2.2)] for a proof for metrics with arbitrary signature. ⋄

If in the following, a group $G_q \leq \mathrm{GL}_q$ is mentioned, we silently assume that G_q stands for subgroups G_q for each $q \in \mathbb{N}$ which are compatible with prolongation.

B.5 G_q-actions on Manifolds

Having treated the full jet groups GL_q and subgroups $G_q \leq \mathrm{GL}_q$ compatible with prolongation, we now turn to their actions on a manifold F. The prolongation properties of G_q are chosen such that the jet bundle functor J_r from Section 1.2.1 induces a G_{q+r}-action on the algebraic prolongation $F^{(r)}$ of F. This includes the special case of $G_q = \mathrm{GL}_q$.

As the subgroups $K^{q+r}_q \leq G_{q+r}$ are all normal subgroups, a projection to order q is locally well-defined by taking the space of K^{q+r}_q-orbits on $F^{(r)}$. Both prolongation and projection will be used on the natural bundles in Chapter 3. Essentially, a natural bundle $\mathcal{F}$ is defined by its fibre F and the prolongation $J_r(\mathcal{F})$ by $F^{(r)}$ respectively. So we treat the fibres separately.

B.5.1 Prolongation

Definition B.14. Let F be a manifold with a G_q-action. The r-th *algebraic prolongation* $F^{(r)}$ of F is defined as $F^{(r)} := J_r(\mathbb{R}^n \times F)|_0$, being all r-jets of maps from the origin of $\mathbb{R}^n$ to F. ⋄

In [KMS93, §12.8], $F^{(r)}$ is called the space of n-velocities of order r on F. In Section 3.2, F will be the fibre of a natural bundle $\mathcal{F}$. The algebraic prolongation is the most simple way of obtaining the fibre $F^{(k)}$ of $J_r(\mathcal{F})$.

Lemma B.15. Let F be a manifold with a G_q-action. The jet functor J_r induces a G_{q+r}-action on the algebraic prolongation $F^{(r)}$ such that the following diagram commutes:

$$\begin{array}{ccccc} G_{q+r} & \times & F^{(r)} & \longrightarrow & F^{(r)} \\ \downarrow{\scriptstyle \pi^{q+r}_q} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G_q & \times & F & \longrightarrow & F \end{array}$$

⋄

Proof. $J_r(\mathbb{R}^n \times F) \cong \mathbb{R}^n \times F^{(r)}$ is a bundle over $\mathbb{R}^n \times F$, so $\pi : F^{(r)} \to F$ is a bundle. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism with $\varphi(0) = 0$ and $j_q(\varphi)(0) \in G_q$. Then φ acts on F and induces a map

$$J_q(\mathbb{R}^n \times \mathbb{R}^n) \times F \to \mathbb{R}^n \times F,$$

which is continued by Proposition 1.14 to

$$J_{q+r}(\mathbb{R}^n \times \mathbb{R}^n) \times F^{(r)} \to \mathbb{R}^n \times F^{(r)}.$$

The induced map $p_r(\varphi)$ still leaves the origin invariant and $p_r(\varphi)$ restricts to a map $F^{(r)} \to F^{(r)}$. So G_{q+r} acts on $F^{(r)}$. □

B.5.2 Projection

If F is a manifold with an GL_{q+r}-action, we project down to a manifold with G_q-action by taking the orbit space F/K_q^{q+r}. It is irrelevant whether F is the algebraic prolongation of another manifold F' or not.

Proposition B.16. The orbit space of K_q^{q+r} on a manifold F with G_{q+r}-action is locally a manifold F/K_q^{q+r} with G_q-action. ◇

Proof. As $K_q^{q+r} \trianglelefteq G_{q+r}$ is a normal subgroup, $G_q \cong G_{q+r}/K_q^{q+r}$ acts on the orbits of K_q^{q+r} on F. Take the Lie algebra $\mathfrak{k}_q^{q+r} \trianglelefteq \mathfrak{g}_{q+r}$ and the infinitesimal action

$$\mathfrak{k}_q^{q+r} \to \Gamma(TF^{(r)}),$$

where the local flows correspond to the K_q^{q+r}-action. Nilpotency of $\mathfrak{k}_q^{q+r}$ implies that its image is an involutive distribution. By smoothness, it is locally of constant rank, so Frobenius' theorem is applicable. □

Especially for vector spaces and linear G_q-actions, the origin is a single orbit where the rank of the distribution drops to zero. As we are only interested in local problems, we can choose an open subset of $F^{(k)}$.

If the G_{q+r}-action on F is intransitive on an open subset of F, the local invariants $F \to \mathbb{R}$ can by obtained by factoring out the whole G_q-action. It is recommendable to project down order by order.

B.5.3 Equivariant Maps

As a preparation for Chapter 3 on natural bundles, we shortly introduce equivariant maps. They will be useful for morphisms of natural bundles. A smooth map $\varphi : F \to F'$ is called G_q-equivariant if $\varphi(gf) = g\varphi(f)$ for all $g \in G_q$ and $f \in F$.

Proposition B.17. Let $\varphi : F \to F'$ be a G_q-equivariant map.

(1) If φ is injective, surjective or bijective, then the map induced by J_r, $p_r(\varphi)$: $F^{(r)} \to F'^{(r)}$ is also injective, surjective or bijective.

(2) The map $F \to F/K_q^{q+r}$ from Proposition B.16 is equivariant.

(3) If φ is injective, surjective or bijective, then the map on the orbit spaces $\bar{\varphi} : F/K_{q-s}^q \to F'/K_{q-s}^q$ is also injective, surjective or bijective. ◇

Proof. (1) is a consequence of J_r being an exact functor. (2) follows from $G_q \cong G_{q+r}/K_q^{q+r}$. In (3), φ maps G_q-orbits into G_q-orbits. Factoring out K_{q-s}^q changes the orbits but not the orbit space. □

Appendix C

Lie Algebroids

In complete analogy to a Lie group, a Lie groupoid has an infinitesimal structure called Lie algebroid, which is introduced in this section. Lie algebroids were originally defined by Pradines [Pra66]. Starting with the definition of Lie algebroids and their actions, the Lie algebroid of a Lie groupoid is constructed in Section C.1.1. If not indicated otherwise, the section is based on [MM03, Ch. 6].

In Section C.2, jet algebroids corresponding to jet groupoids are treated in detail, since they are necessary in Chapter 3.

C.1 Lie Algebroids

We directly start with the definition of a Lie algebroid.

Definition C.1. A *Lie algebroid* $\mathfrak{g}$ over the manifold X is a vector bundle $\mathfrak{g} \to X$ with a Lie bracket, that is defined on the sections of $\mathfrak{g} \to X$:

$$[-,-] : \Gamma(\mathfrak{g}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$$

and an anchor map $\mathrm{an} : \mathfrak{g} \to TX$ over X, which is a Lie algebra homomorphism and satisfies the Leibniz identity

$$[\xi, f\eta] = f[\xi, \eta] + \mathrm{an}(\xi)(f)\eta \tag{C.1}$$

for all $\xi, \eta \in \Gamma(\mathfrak{g})$ and $f \in C^\infty(X)$. ⋄

As there is no need for distinctions, we will usually speak of algebroids. A *morphism of algebroids* is a morphism of vector bundles over the same base X commuting with the anchor map and the Lie bracket. Due to the Lie bracket being defined on the sections of $\mathfrak{g} \to X$, general morphisms of algebroids are more complicated, see [Mac05, §4.3]. An algebroid is called *transitive* if the anchor map is fibrewise surjective. A subalgebroid is a subbundle $\mathfrak{h} \subseteq \mathfrak{g} \to X$ such that anchor and bracket restrict to $\mathfrak{h}$.

The next example takes up previous examples for Lie groupoids and exhibits the corresponding Lie algebroids (see Examples 2.2, 2.3).

Example C.2. (1) A Lie algebra is a Lie algebroid over a point. It is the Lie algebroid of a Lie group.

(2) The trivial bundle $M \times \{0\}$ is an algebroid with zero anchor and bracket.

(3) The tangent bundle $\mathfrak{g} = TX$ of a smooth manifold X together with the bracket of vector fields and $\mathrm{an} = \mathrm{id}_{TX}$ is an algebroid corresponding to the pair groupoid.

(4) Let $\gamma : \mathfrak{h} \to \mathfrak{X}(M)$ be the infinitesimal action of a Lie algebra $\mathfrak{h}$ on a manifold M. $\mathfrak{X}(M)$ denotes the vector fields on M. Then γ is a homomorphism of Lie algebras. The *action algebroid* is the trivial bundle $\mathfrak{g} = M \times \mathfrak{h}$ and the anchor map is $\mathrm{an}(\xi, x) = \gamma(\xi)_x$. Therefore the Lie bracket is:

$$[\xi, \eta](x) = [\xi(x), \eta(x)] + [\gamma(\xi), \gamma(\eta)](x) \qquad \forall \xi, \eta \in \Gamma(\mathfrak{g}).$$

The first bracket is the Lie bracket on $\mathfrak{h}$ and the second one the bracket of vector fields on M. It is well-defined since γ is a Lie algebra homomorphism.

(5) If the action γ is trivial, then the anchor map is zero and $\mathfrak{g} = M \times \mathfrak{h}$ is only a bundle of Lie algebras with fibrewise bracket $[\xi, \eta](x) = [\xi(x), \eta(x)]$. ◇

In most applications, a Lie groupoid action on a bundle $\mathcal{F}$ has the tendency to have large coordinate expressions. It is more convenient to work in the infinitesimal context, where the algebroid acts on $\mathcal{F}$.

Definition C.3. [Mac05, Def. 4.1.1] Let $\mathfrak{g} \to X$ be a Lie algebroid and $\pi : \mathcal{F} \to X$ a bundle. A *Lie algebroid action* is a map of vector bundles $\Gamma(\mathfrak{g}) \to \mathfrak{X}\mathcal{F} : \xi \mapsto \xi^\dagger$ satisfying the conditions:

$$(\xi + \eta)^\dagger = \xi^\dagger + \eta^\dagger, \qquad \text{(C.2)}$$
$$(f\xi)^\dagger = (f \circ \pi)\xi^\dagger, \qquad \text{(C.3)}$$
$$[\xi, \eta]^\dagger = [\xi^\dagger, \eta^\dagger], \qquad \text{(C.4)}$$
$$\pi_* \xi^\dagger = \mathrm{an}(\xi). \qquad \text{(C.5)}$$

◇

For examples of algebroid actions we refer to Section C.2. We now turn to the algebroid of a given Lie groupoid.

C.1.1 The Lie Algebroid of a Lie Groupoid

The Lie algebroid of a Lie groupoid G consists of all infinitesimal transformations of a Lie groupoid on itself. We construct the Lie algebroid completely analogous to the Lie algebra of a Lie group, except that we have to deal with bundles. For a Lie group H, the following steps are taken:

- Extend the right multiplication R_h with $h \in H$ to a H-action on the tangent bundle TH.
- Consider right invariant vector fields $\xi \in \mathfrak{X}H = \Gamma(TH)$ that satisfy
$$R_{h,*}(\xi_g) = \xi_{gh} \qquad \forall\, g, h \in H.$$
- Each right invariant vector field ξ is uniquely defined by its value ξ_{1_H} on the Lie algebra $\mathfrak{h} = T_1 H$.

Analogously, the Lie algebroid of a Lie groupoid is constructed. We follow the approach in [MM03, §6.1]. Note that it is possible to construct Lie algebroids with left invariant vector fields as well. We shall shortly mention the relevant steps, which are needed in Section 3.3.

Construction via Right Invariant Vector Fields

We first extend the right multiplication action on Lie groupoid G. Due to the restricted multiplication, it will not lift to the full tangent space $TG^{(1)}$ but only to the vertical bundle $VG^{(1)}$. Interpreting right invariant vector fields as the infinitesimal left multiplication, it corresponds to the fact that left multiplication leaves the source invariant.

The right multiplication with an element $h \in G(x, y)$, defined on the source fibres:
$$R_h : G(y, -) \to G(x, -) : g \mapsto gh,$$
lifts to the tangent map is $R_{h,*} : TG(y, -) \to TG(x, -)$. We show that the bundle $TG(x, -)$ is the restriction of the vertical bundle $VG^{(1)}$ to all elements $g \in G^{(1)}$ with source $s(g) = x$. $G(x, -)$ is specified by the exact sequence of bundles over the target copy of $G^{(0)}$:

$$0 \longrightarrow G(x,-) \longrightarrow G^{(1)} \underset{c_x}{\overset{(s,t)}{\rightrightarrows}} G^{(0)} \times G^{(0)}.$$

In other words, $G(x, -)$ is the kernel $\ker_{c_x}(s, t)$ with respect to the constant map
$$c_x : G^{(0)} \to G^{(0)} \times G^{(0)} : y \mapsto (x, y).$$

Apply the exact tangent functor yields:

$$0 \longrightarrow TG(x,-) \longrightarrow TG^{(1)} \underset{c_{x,*}}{\overset{(s_*,t_*)}{\rightrightarrows}} TG^{(0)} \times TG^{(0)}$$

with
$$c_{x,*} : TG^{(0)} \to TG^{(0)} \times TG^{(0)} : v_y \mapsto (0_x, v_y).$$

The sequence restricts to the vertical bundle

$$0 \longrightarrow TG(x,-) \longrightarrow VG^{(1)} \overset{(s,t_*)}{\underset{c_x}{\rightrightarrows}} G^{(0)} \times TG^{(0)},$$

such that $TG(x,-)$ consists of all vertical vectors over groupoid elements with source x.

Denote the vertical bundle by $T^sG^{(1)} = VG^{(1)}$. Then the construction via left invariant vector fields involves the normal bundle $T^tG^{(1)} = \ker(t_*)$. The right multiplication action lifts to a G-action on $T^sG^{(1)}$:

$$T^s(G^{(1)}) \underset{G^{(0)}}{{}_{s_*}\times_t} G^{(1)} \to T^s(G^{(1)}) : (\xi, h) \mapsto R_{h*}(\xi). \tag{C.6}$$

By abuse of notation, s_* is considered as a map to $G^{(0)}$ because its image coincides with the zero embedding $G^{(0)} \hookrightarrow TG^{(0)} : x \mapsto 0_x$.

We now turn to right invariant vector fields. The G-action of equation (C.6) continues to vertical vector fields, which are denoted by:

$$\mathfrak{X}^s = \mathfrak{X}^s(G^{(1)}) = \Gamma(T^sG^{(1)}).$$

In the next proposition it is shown that the right invariant vector fields,

$$\mathfrak{X}^s_{\text{inv}} = \mathfrak{X}^s_{\text{inv}}(G) := \{\xi \in \mathfrak{X}^s \,|\, \xi_{gh} = \xi_g h \ \ \forall (g,h) \in G^{(2)}\},$$

form a subalgebra of all vector fields on $G^{(1)}$. Additionally the target map is a good candidate for the anchor.

Proposition C.4. [MM03, Prop. 6.1] Let G be a Lie groupoid. Then

(1) $\mathfrak{X}^s_{\text{inv}}$ is a Lie subalgebra of $\mathfrak{X}(G^{(1)})$.

(2) any G-invariant vector field on $G^{(1)}$ is projectable along the target map t_* to $TG^{(0)}$.

(3) the derivative of the target map induces a Lie algebra homomorphism

$$t_* : \mathfrak{X}^s_{\text{inv}}(G) \to \mathfrak{X}(G^{(0)}).$$

◇

Proof. (1) Take $\xi, \eta \in \mathfrak{X}^s_{\text{inv}}(G)$. We have $[\xi,\eta] \in \mathfrak{X}^s(G^{(1)})$, while for any $g, h \in G^{(1)}$ with $s(g) = t(h)$ there is:

$$[\xi,\eta]_g h = R_{h*}([\xi,\eta]_g) = [R_{h*}(\xi), R_{h*}(\eta)]_{gh} = [\xi,\eta]_{gh}.$$

(2) For any arrow $g : x \to y \in G^{(1)}$ we have:

$$t_*(\xi_g) = t_*(\xi_{1_y} g) = t_*(R_{g_*}(\xi_{1_y})) = (t \circ R_g)_*(\xi_{1_y}) = t_* \xi_{1_y}.$$

(3) This is clear from (1) and (2). □

In the next step we construct the bundle $\mathfrak{g}$ of the algebroid. By the unit embedding ι, we identify the set of units $\{1_x | x \in G^{(0)}\}$ with $G^{(0)}$. Due to the right invariance, each vector field $\eta \in \mathfrak{X}^s_{\text{inv}}(G)$ is uniquely determined by its restriction to $\iota(G^{(0)})$. Therefore define the algebroid $\mathfrak{g}$ as the pullback of the vector bundle $T^s(G^{(1)})$ along ι:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & T^s(G^{(1)}) \\ \big\downarrow \pi & & \big\downarrow \pi \\ G^{(0)} & \xrightarrow{\ \iota\ } & G^{(1)} \end{array}$$

So far, we have constructed isomorphic vector spaces $\mathfrak{X}^s_{\text{inv}}(G)$ and $\Gamma(\mathfrak{g})$, where the isomorphism is given by the right multiplication:

$$\sharp : \mathfrak{g} \; {}^{\pi}\!\underset{G^{(0)}}{\curlywedge}{}^{t} \; G^{(1)} \to T^s(G^{(1)}) : (\eta_{1_{t(g)}}, g) \mapsto \eta_g = \eta_{1_{t(g)}} g. \tag{C.7}$$

To show that it is also an isomorphism of Lie algebras, we need a Lie bracket on $\mathfrak{g}$. Extend $\mathfrak{g}$, $\sharp$ to sections of $\mathfrak{g} \to X$:

$$\sharp : \Gamma(\mathfrak{g}) \to \mathfrak{X}^s_{\text{inv}}(G^{(1)}) : \eta \mapsto (g \mapsto \eta g). \tag{C.8}$$

Then $\sharp$ is $C^\infty(G^{(0)})$-linear by evaluating the function $f \in C^\infty(G^{(0)})$ at the target:

$$\sharp(f\eta) = (f \circ t)\sharp(\eta)$$

The unique Lie bracket on $\mathfrak{g}$, such that $\Gamma(\mathfrak{g})$ and $\mathfrak{X}^s_{\text{inv}}(G)$ are isomorphic as Lie algebras, is defined by pulling back the Lie bracket on G:

$$[\xi, \eta] = \text{id}^*([\sharp(\xi), \sharp(\eta)]) \qquad \xi, \eta \in \Gamma(\mathfrak{g}). \tag{C.9}$$

To complete the Lie algebroid, the anchor map is given by $\text{an} := t_*$. Using Proposition (C.4 (3)), it is possible to verify equation (C.1).

Although the algebroid of a Lie groupoid may be complicated to construct, it is very helpful to have the following example in mind.

Example C.5. The algebroid $T = TX$ of the pair groupoid $G^{(1)} = X \times X$ from Example C.2 (3) can be explicitly constructed. Although quite simple, it plays an important role for jet algebroids in Section C.2.1.

The tangent bundle of $G^{(1)}$ is $TX \times TX$ and therefore $T^s G^{(1)} = X \times TX$. The groupoid action on $T^s G^{(1)}$ is still the translation action of the pair groupoid:

$$X \times TX \; {}^{\text{pr}_1}\!\underset{X}{\curlywedge}{}^{\text{pr}_2} \; X \times X \to X \times TX : ((y, v_z), (x, y)) \mapsto (x, v_z).$$

A vector field on $X \times TX$ is given by a map $(x, y) \mapsto (x, y, v(x, y))$. It is right invariant if and only if $v = v(y)$ depends on the target only. The algebroid $\mathfrak{g} = TX$ is constructed by the pullback diagram:

$$\begin{array}{ccc} TX & \xrightarrow{(\pi,\mathrm{id})} & X \times TX \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times\pi} \\ X & \xrightarrow{\iota=(\mathrm{id},\mathrm{id})} & X \times X \end{array}$$

By writing out the definition for $\sharp$, we see that it is the identity map:

$$\sharp : TX \;{}_{\pi}\!\curlywedge_{X}^{\mathrm{pr}_2}\; X \times X = X \times TX \to X \times TX$$

and it follows that the Lie bracket on TX is the usual bracket of vector fields. The anchor map is $t_* = \mathrm{id}_{TX}$.

In Section C.2 on jet algebroids, $T = TX$ is the algebroid of the zero order jet groupoid $\Pi_0 = X \times X$ (the pair groupoid) and the algebroids $J_q(T)$ of higher order jet groupoids Π_q are constructed from T by using the jet functor J_q. ◇

Construction via Left Invariant Vector Fields

Analogous to the right invariant case, the left multiplication extends to an action on the bundle $T^tG^{(1)} = \ker(t_*)$ as:

$$G^{(1)} \;{}_{s}\!\curlywedge_{G^{(0)}}^{t_*}\; T^t(G^{(1)}) \to T^t(G^{(1)}) : (g, \xi) \mapsto L_{g_*}(\xi).$$

All left invariant vector fields $\mathfrak{X}^t_{\mathrm{inv}}(G)$ are uniquely defined on the pullback $\mathfrak{g}_t$ of $T^tG^{(1)}$ along ι. The isomorphism

$$\flat : G^{(1)} \times_{G^{(0)}} \mathfrak{g}_t \to T^tG^{(1)}. \tag{C.10}$$

also extends to $\flat : \mathfrak{g}_t \to \mathfrak{X}^t_{\mathrm{inv}}$. The anchor map in this case is $\mathrm{an}_t = s_*$.

C.2 Jet Algebroids

In this section the jet algebroids of Π_q for $q \in \mathbb{N}$ are treated in detail. Just as elements of Π_q are q-jets of local diffeomorphisms $X \to X$, the algebroid consists of q-jets of infinitesimal transformations. An infinitesimal diffeomorphism is a vector field on X or equivalently a section of its tangent bundle $T = TX$. Thus the algebroid of Π_q is $J_q(T)$.

We follow Section C.1.1 and start with the algebroid T of the zero order jet groupoid $\Pi_0 = X \times X$, which was already presented in Example C.5. The following extension to higher order algebroids $J_q(T)$ involves only prolongation and the jet bundle functor from Section 1.2.1. This recovers the infinitesimal results of Pommaret [Pom78, §7.1], [Pom83, §2.A.1] in the language of algebroids introduced here.

Since both left and right invariant vector fields are needed in Chapter 3, the isomorphisms $\sharp$ and $\flat$ between sections of the algebroid and invariant vector fields are explicitly constructed. They consist of a translational part and an infinitesimal action of $\mathfrak{gl}_q$ on Π_q. As a last result of this section, an explicit formula for the Lie bracket on $J_q(T)$ is given, which was originally derived by Pommaret [Pom83, Def. 2.A.1.19].

C.2.1 The Algebroid of Π_0

The results for the algebroid T of Π_0 from Example C.5 are rather simple. They serve mainly as a preparation for the higher order algebroids. Using right invariant vector fields on the vertical bundle $T^s\Pi_0 = V(\Pi_0)$, each section $\eta \in \Gamma(T)$ of the algebroid is a vector field on X. A representation in local coordinates (y) of X is given by:

$$\eta = \eta^j(y)\frac{\partial}{\partial y^j}.$$

Because T is the pullback of $V(\Pi_0)$ along the (diagonal) unit embedding

$$\iota : X \to X \times X : x \mapsto 1_x = (x, x),$$

we can embed T in $V(\Pi_0)$ as:

$$T \to V(\Pi_0) : (y, v_y) \mapsto (y, y, 0_y, v_y).$$

Via right translation, we construct $\sharp$:

$$\sharp : T\,{}^{\pi}\!\times^{t}_{X} \Pi_0 \to V(\Pi_0) : ((y, v_y), (x, y)) \mapsto (x, y, 0_x, v_y)$$

For Π_0, the bijection between between vector fields on X and right invariant vector fields on $T^s\Pi_0$ is trivial:

$$\sharp : \Gamma(T) = \mathfrak{X}X \to \mathfrak{X}^s_{\text{inv}}(\Pi_0) : \eta^j(y)\frac{\partial}{\partial y^j} \mapsto \eta^j(y)\frac{\partial}{\partial y^j}. \tag{C.11}$$

We observe that for $\text{an} = t_*$, the composition $\text{an} \circ \sharp = \text{id}_{\Gamma(T)}$ is the identity map for vector fields on X. The Lie bracket on T is the ordinary bracket of vector fields, so the explicit coordinate representation is:

$$[\eta, \xi] = \left(\eta^i(y)\frac{\partial \xi^j}{\partial y^j}(y) - \xi^i(y)\frac{\partial \eta^j}{\partial y^j}(y)\right)\frac{\partial}{\partial y^j}.$$

The analogous left invariant construction using the normal bundle $T^t(\Pi_0) = \ker(t_*)$ yields the map

$$\flat : \Gamma(T) = \mathfrak{X}X \to \mathfrak{X}^t_{\text{inv}}(\Pi_0) : \xi^j(x)\frac{\partial}{\partial x^j} \mapsto \xi^j(x)\frac{\partial}{\partial x^j}. \tag{C.12}$$

These are the basic formulae which are necessary to construct $\sharp$ and $\flat$ for Π_q.

C.2.2 The Algebroid of Π_q

To construct the algebroid $J_q(T)$ of Π_q we prolong the results of the previous section. The bundle T for the algebroid is defined by the pullback diagram:

$$\begin{array}{ccc} T & \longrightarrow & V(\Pi_0) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota} & \Pi_0 \end{array}$$

Applying the jet functor J_q, we obtain:

$$\begin{array}{ccc} J_q(T) & \longrightarrow & J_q(V(\Pi_0)) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota} & J_q(\Pi_0) \end{array}$$

Then the canonical isomorphism $V(J_q(\mathcal{E})) \cong J_q(V(\mathcal{E}))$ from Proposition 1.10 (5) implies $J_q(V(\Pi_0)) = V(J_q(X \times X))$. We restrict the diagram to the open subset $\Pi_q \subset J_q(X \times X)$ of invertible jets:

$$\begin{array}{ccccc} J_q(T) & \longrightarrow & V(\Pi_q) & \subset & J_q(V(X \times X)) \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\iota} & \Pi_q & \subset & J_q(X \times X) \end{array}$$

From now on, we will restrict our discussion to invertible jets without further notice. We have constructed the algebroid $J_q(T) = \iota^*(V(\Pi_q))$ only by applying the jet functor. This is also possible for left invariant vector fields, since Π_0 is a trivial bundle. In this case, the action of taking the horizontal bundle commutes with the jet bundle functor J_q: $J_q(T^t\Pi_0) = T^t J_q(\Pi_0)$.

Both for left and right invariant vector fields, the algebroid of Π_q is $J_q(T)$, but the Lie brackets are different. The brackets are defined using the isomorphisms $\sharp$ and $\flat$ between sections of the algebroid and invariant vector fields on the groupoid.

C.2.3 Prolongation Formulae for $\sharp$ and $\flat$

We construct $\sharp$ and $\flat$ by prolonging equations (C.11) and (C.12). To distinguish between algebroids of different jet orders, $\sharp_q$ denotes the map $\sharp_q : J_q(T) \to V(\Pi_q)$. In the following sections, the index will be omitted again.

The first step towards $\sharp_q$ is the prolongation of $\sharp_0$ from equation (C.11):

$$\rho_q \circ \sharp_0 : \mathfrak{X}X \to \mathfrak{X}^s_{\text{inv}}\Pi_q : \eta^j(y)\frac{\partial}{\partial y^j} \mapsto \rho_q(\eta^j(y)\frac{\partial}{\partial y^j}) = D_\mu \eta^j(y)\frac{\partial}{\partial y^j_\mu}.$$

Here, Definition 1.18 with the characteristic $Q^j = \eta^j(y)$ has been used. $\rho_q \circ \sharp_0$ is not yet $\sharp_q$, since it is defined on $\Gamma(T)$ rather than on $\Gamma(J_q(T))$. However it is easily possible to recover the pointwise version of $\sharp_q$ from $\rho_q \circ \sharp_0$ (see equation (C.7)). Due to the total derivative in the prolongation formula, derivatives of η up to order q appear in the vector field $(\rho_q \circ \sharp_0)(\eta)$ on Π_q. For each groupoid element $(x, y, y_q) \in \Pi_q$, the vector field depends on the q-jet $j_q(\eta)(y)$ at the target. So there is a unique pointwise continuation:

$$\sharp_q : J_q(T) \,{}^{\pi}\!\underset{X}{\times}{}^{t}\, \Pi_q \to V(\Pi_q) : ((y, \eta, \eta_q), (x, y, y_q)) \mapsto D_\mu \eta^j \frac{\partial}{\partial y^j_\mu}. \tag{C.13}$$

The continuation $\sharp_q$ uses the generalised total derivative

$$D_i = \partial_{x^i} + y^j_{\mu+1_i} \frac{\partial}{\partial y^j_\mu} + y^k_i \eta^j_{\mu+1_k} \frac{\partial}{\partial \eta^j_\mu}$$

which considers η as y-dependent. Similar to equation (C.8), $\sharp_q$ can be considered as a map $\sharp_q : \Gamma(J_q(T)) \to \mathfrak{X}^s_{\text{inv}} \Pi_q$.

For further computations, it is very convenient to split $\sharp_q$ into an infinitesimal translation and the infinitesimal action of the isotropy algebra $\mathfrak{gl}_q$. The translational part $\eta^j \partial_{y^j}$ depends on zero order jets only. In [Ves03, §3], Vessiot introduced basis vector fields $B^\mu_i(q)$ for the isotropy part, which are a representation of the Lie algebra $\mathfrak{gl}_q$ on Π_q. Integrating their flow results in the left $\mathrm{GL}_q \cong \Pi_q(y, y)$-action on $\Pi_q(-, y)$. See Appendix B for the jet groups GL_q and their Lie algebras $\mathfrak{gl}_q$.

To obtain $B^\mu_i(q)$, we first apply $\sharp_q$ to a section (η, η_q) of $J_q(T) \to X$ and collect for the jet coordinates η^i_μ:

$$\sharp_q(\eta, \eta_q) = \eta^i \frac{\partial}{\partial y^i} + \eta^i_\mu B^\mu_i(q), \qquad 1 \leq |\mu| \leq q. \tag{C.14}$$

The vector fields $B^\mu_i(q)$ on Π_q are both tangent to the source and target fibre. This means, $B^\mu_i(q)$ contains neither x-derivatives nor y-derivatives. By the total derivative, $B^\mu_i(q)$ depends on the jet coordinates $(y_1, \dots, y_q)$ starting with first order jets, but not on (x, y). We can now prove the $\mathfrak{gl}_q$-representation:

Proposition C.6. The vector fields $B^\mu_i(q)$ on Π_q are a representation of the Lie algebra $\mathfrak{gl}_q$. ◇

Proof. To show that the $B^\mu_i(q)$ are a representation of $\mathfrak{gl}_q$, we have to check the Lie brackets from Lemma B.5. The vector fields

$$b^\mu_i = \frac{1}{\mu!} y^\mu \frac{\partial}{\partial y^i} \;\in\; \mathfrak{X}X$$

are a basis of $\mathfrak{gl}_q$. Their q-jets $j_q(b^\mu_i)(0)$ correspond to the point $(0, \eta, \eta_q) \in J_q(T)$ with $\eta^i_\mu = 1$ and all other $\eta^j_\nu = 0$. As $\sharp_q$ is the unique pointwise continuation of $\rho_q \circ \sharp_0$, the Lie brackets for b^μ_i and the section $(\eta^i = 0, \eta^j_\nu = \delta^{i,j}\delta_{\mu,\nu})$ of $J_q(T) \to X$ must coincide:

$$\sharp_q(0, \delta^{i,j}\delta_{\mu,\nu}) = B^\mu_i(q).$$

So the Lie brackets for $B^\mu_i(q)$ are given by equation (B.3). □

The analogue for left invariant vector fields is the map $\flat_q$ obtained by prolongation from $\flat_0$:

$$\rho_q \circ \flat_0 : \mathfrak{X}X \to \mathfrak{X}^t_{\mathrm{inv}}\Pi_q : \xi^i(x)\frac{\partial}{\partial x^i} \mapsto \xi^i(x)D_i - D_\mu(y^j_i\xi^i(x))\frac{\partial}{\partial y^j_\mu}.$$

Again the pointwise continuation can be collected for the jets of ξ:

$$\begin{aligned}\flat_q : \Pi_q \, {}^s\!\underset{X}{\times}{}^\pi \, J_q(T) : ((x,y,y_q),(x,\xi,\xi_q)) &\mapsto \xi^i D_i - \tilde{D}_\mu(y^j_i\xi^i)\frac{\partial}{\partial y^j_\mu} \\ &= \xi^i\frac{\partial}{\partial x^i} + \xi^i_\mu A^\mu_i(q). \qquad \text{(C.15)}\end{aligned}$$

$\flat_q$ involves both the usual total derivative D_i from Definition 1.11 as well as a generalisation

$$\tilde{D}_i = \partial_{x^i} + y^j_{\mu+1_i}\frac{\partial}{\partial y^j_\mu} + \xi^j_{\mu+1_i}\frac{\partial}{\partial \xi^j_\mu}$$

which considers ξ as x-dependent. The summands containing y_{q+1} in $\flat_q$ cancel.

Integrating the flows of $A^\mu_i(q)$, results in the right $\mathrm{GL}_q \cong \Pi_q(x,x)$-action on $\Pi_q(x,-)$, so $A^\mu_i(q)$ differs from $B^\mu_i(q)$, but their properties are very similar. All $A^\mu_i(q)$ are tangent to the source and target fibres and they are also independent of x and y.

Proposition C.7. The vector fields $A^\mu_i(q)$ on Π_q are a representation of the Lie algebra $\mathfrak{gl}_q$. ◇

The proof is the same as for Proposition C.6. The $A^\mu_i(q)$ can be given by an explicit formula (see [Ves03, eq. (5)]).

Proposition C.8. The vector fields $A^\mu_i(q)$ have the coordinate expression

$$A^\mu_i(q) = -\sum_{\substack{\nu\geq\mu \\ |\nu|\leq q}} \binom{\nu}{\mu} y^j_{\nu-\mu+1_i}\frac{\partial}{\partial y^j_\nu}, \qquad |\mu| \leq q,$$

with $A^\mu_i(q) = 0$ for $|\mu| > q$. ◇

Proof. In $D_\nu(y^j_i\xi^i(x))$, the term $y^j_{\nu-\mu+1_i}\frac{\partial^{|\mu|}\xi^i(x)}{\partial x^\mu}$ occurs $\binom{\nu}{\mu} = \prod_k \binom{\nu_k}{\mu_k}$ times. □

The next proposition is a consequence of $[x^\mu\frac{\partial}{\partial x^i}, y^\nu\frac{\partial}{\partial y^j}] = 0$. It is the infinitesimal version of the associative law saying that right multiplication on Π_q commutes with left multiplication.

Proposition C.9. The Lie bracket of $A^\mu_i(q)$ and $B^\nu_j(q)$ vanishes:

$$[A^\mu_i(q), B^\nu_j(q)] = 0 \qquad \forall\, \mu,\nu \in \mathbb{Z}^n_{\geq 0},\ 1 \leq i,j \leq n.$$ ◇

It will be very useful in Section 3.3 for treating differential invariants. There, it is used in the following sense: If $R_q \leq J_q(T)$ is a subalgebroid and $\{\Phi^\tau : \Pi_q \to \mathbb{R}\}$ are a complete set of invariants under the infinitesimal left action of R_q, then $J_q(T)$ still acts on the invariants from the right.

The Lie Bracket on $J_q(T)$[1]

In this section, we derive an explicit formula for the Lie bracket on $J_q(T)$ using right invariant vector fields. The bracket must satisfy equation (C.9):

$$[\xi, \eta] = \mathrm{id}^*([\sharp(\xi), \sharp(\eta)]) \qquad \xi, \eta \in \Gamma(J_q(T)).$$

Instead of using the basis vector fields $B^i_\mu(q)$ directly, we follow an approach used by Pommaret [Pom78, La. 2.29 *bis*, p. 295]. The Lie bracket is constructed in two steps. First, Lemma 1.20 defines the Lie bracket for prolongations $j_q(\xi)$, $j_q(\eta)$ of sections ξ, η of $T \to X$. The result is called algebraic bracket. However Remark 1.9 shows that there are more sections on $J_q(T) \to X$ than those prolonged from T, which are covered by the complete differential bracket.

We begin with the algebraic bracket. The Lie brackets for $\xi, \eta \in \mathfrak{X}X$,

$$[\xi^i(y)\frac{\partial}{\partial y^i}, \eta^j(y)\frac{\partial}{\partial y^j}] = (\xi^i \frac{\partial \eta^k}{\partial y^i} - \eta^j \frac{\partial \xi^k}{\partial y^j})\frac{\partial}{\partial y^k},$$

already depends on the first order derivatives of ξ and η, so there is a pointwise version

$$[\ ,\ ] : J_1(T) \times J_1(T) \to T$$

by substituting jets for derivatives. According to Proposition 1.14, there is a unique prolongation compatible with j_q to a map

$$J_{q+1}(T) \times J_{q+1}(T) \to J_q(T).$$

It is the first building block for the Lie bracket on $J_q(T)$. Note that it is C^∞-linear.

Definition C.10. The *algebraic bracket* $\{\ ,\ \}$ is defined as the q-th prolongation

$$\{\ ,\ \} := p_q([\ ,\ ]) : J_{q+1}(T) \times J_{q+1}(T) \to J_q(T)$$

of the usual Lie bracket $[\ ,\ ]$ on T. ◇

By construction the algebraic bracket works fine for prolonged sections $j_q(\xi)$, where the prolongation to order $q+1$ is done by differentiating the highest order jet components. We give an example of the algebraic bracket.

Example C.11. For $X = \mathbb{R}$ and $q = 1$ the algebraic bracket for ξ_2, η_2 is

$$\begin{aligned} \{j_2(\xi), j_2(\eta)\} &= j_1(\xi\, \partial_x \eta - \eta\, \partial_x \xi) \\ &= (\xi\, \partial_x \eta - \eta\, \partial_x \xi,\ \xi\, \partial_x^2 \eta - \eta\, \partial_x^2 \xi). \end{aligned}$$

The formula for the algebraic bracket is therefore

$$\{\xi_2,\ \eta_2\} = (\xi\, \eta_x - \eta\, \xi_x,\ \xi\, \eta_{xx} - \eta\, \xi_{xx}).$$

By coincidence, all $\xi_x \eta_x$-terms cancel. It is a pointwise map, since no derivatives occur and one could choose explicit values for all jets of ξ and η. ◇

[1] The results of this section will not be used in this thesis, but they could be used to check if a PDE system on $J_q(T)$ defines a subalgebroid without using vector fields on Π_q.

In order to complete the Lie bracket on $J_q(T)$, we need a differential operator of order one which vanishes on prolonged sections $j_q(\eta)$.

Proposition C.12. [Qui64, Prop. 4.1] Let $E \to X$ be a vector bundle. There is a unique map $D : \Gamma(J_{q+1}(E)) \to \Gamma(T^* \otimes J_q(E))$ such that

(1) $D(fs) = fDs + df \otimes \pi_q^{q+1}(s)$ for $f \in C^\infty(X)$, $s \in \Gamma(J_{q+1}(E))$,

(2) $D \circ j_{q+1} = 0$.

D is called *canonical operator* or *Spencer operator* (cf. [Pom78, Def/ 2.1.20]) and a coordinate expression of D is:

$$D : J_{q+1}(E) \to T^* J_q(E) : (x, \xi^i_\mu(x)) \mapsto (x, (\frac{\partial \xi^i_\mu(x)}{\partial x^k} - \xi^i_{\mu+1_k}(x))dx^k). \qquad \diamond$$

Proof. Existence is assured by the coordinate expressions, for a direct proof see [Qui64]. Obviously (2) is fulfilled, since for $j_{q+1}(\xi)$ we have $\xi^i_{\mu+1_k} = \partial_{x^{\mu+1_k}} \xi^i$. Check condition (1) by looking at the components:

$$\begin{aligned} D(f(x)\xi_q(x))^i_\mu &= (\frac{\partial(f(x)\xi^i_\mu(x))}{\partial x^k} - f(x)\xi^i_{\mu+1_k}(x))dx^k \\ &= f(x)(\frac{\partial \xi^i_\mu(x)}{\partial x^k} - \xi^i_{\mu+1_k}(x))dx^k + \frac{\partial f(x)}{\partial x^k}\xi^i_\mu(x)dx^k \\ &= f(x)D(\xi_q(x))^i_\mu + \xi^i_\mu(x)df. \end{aligned}$$

(1) implies that the difference $D - D'$ between two possible Spencer operators D and D' is $C^\infty(X)$-linear. Uniqueness then follows from the fact that each local section s of $J_{q+1}(E) \to X$ is a $C^\infty(X)$-linear combinations of sections $j_{q+1}(s')$. □

On vector bundles, we have seen the Spencer operator being defined by an uniqueness condition. On arbitrary bundles, condition (1) does not make sense anymore, but there is nevertheless a nonlinear Spencer operator $J_{q+1}(\mathcal{E}) \to V(J_q(\mathcal{E})) \otimes T^*$. See [Pom83, Def. I.A.3.32] for details.

The complete Lie bracket on $J_q(T)$ should depend on the algebraic bracket, which already gave the correct result for prolonged vector fields and an additional part containing the Spencer operator. As D maps to $T^* \otimes J_q(E)$, the interior product i provides a suitable map to $J_q(E)$. The most simple candidate for the Lie bracket is:

Definition C.13. Let ξ_q, η_q be two sections of $J_q(T) \to X$. The differential bracket is defined by

$$\begin{aligned} [\ ,\ ] \ &: \ \Gamma(J_q(T)) \times \Gamma(J_q(T)) \to \Gamma(J_q(T)) \\ &: \ (\xi_q, \eta_q) \mapsto \{\xi_{q+1}, \eta_{q+1}\} + i(\xi)D\eta_{q+1} - i(\eta)D\xi_{q+1} \end{aligned}$$

for arbitrary lifts ξ_{q+1}, η_{q+1} projecting onto ξ_q, η_q. $\diamond$

With the definition of the algebraic bracket and the Spencer operator, one can easily prove that the differential bracket does not depend on the lifts ξ_{q+1}, η_{q+1}, because the highest jets cancel.

Theorem C.14. The differential bracket on $\Gamma(J_q(TX))$ coincides with the one defined in Equation (C.9):

$$[\xi_q, \eta_q] = \iota^*[\sharp(\xi_q), \sharp(\eta_q)] \qquad \forall\, \xi_q, \eta_q \in \Gamma(J_q(TX)). \qquad \diamond$$

Proof. The formula above has to be verified. Let ξ_q and η_q be two sections of $J_q(T) \to X$. The corresponding right invariant vector fields are:

$$\begin{aligned} \sharp(\xi_q) &= \xi^i(y)\frac{\partial}{\partial y^i} + \xi^i_\mu(y) B^\mu_i(q), \\ \sharp(\eta_q) &= \eta^j(y)\frac{\partial}{\partial y^j} + \eta^j_\nu(y) B^\nu_j(q), \end{aligned}$$

and their commutator is:

$$[\sharp(\xi_q), \sharp(\eta_q)] = [\xi, \eta]^i \frac{\partial}{\partial y^i} + (\xi^i \partial_k \eta^j_\mu - \eta^j \partial_j \xi^k_\mu)\, B^\mu_k(q) + \xi^i_\mu \eta^j_\nu\, [B^\mu_i(q), B^\nu_j(q)].$$

By expressing the last summand in terms of $B^\sigma_k(q)$ the pullback along ι is easily performed. The last summand is a pointwise expression, so the prolongation from sections of $T \to X$ can be applied. With the help of Lemma 1.20 and the construction of $\sharp$ we use

$$[\sharp(j_q(\xi)), \sharp(j_q(\eta))] = [\rho_q(\xi), \rho_q(\eta)] = \sharp(j_q([\xi, \eta]))$$

to obtain:

$$\begin{aligned} [\sharp(j_q(\xi)), \sharp(j_q(\eta))] &= [\xi, \eta]^i \frac{\partial}{\partial y^i} + (\xi^i \partial_{\mu+1_k} \eta^j - \eta^j \partial_{\mu+1_j} \xi^k)\, B^\mu_k(q) \\ &\quad + \partial_\mu \xi^i \partial_\nu \eta^j\, [B^\mu_i(q), B^\nu_j(q)] \\ &= [\xi, \eta]^i \frac{\partial}{\partial y^i} + \partial_\mu [\xi, \eta]^i B^\mu_i(q). \end{aligned}$$

The unique pointwise continuation to $J_q(TX)$ of this equation (solved for the $[B, B]$-term) is:

$$\xi^i_\mu \eta^j_\nu\, [B^\mu_i(q), B^\nu_j(q)] = \Big(\{\xi_{q+1}, \eta_{q+1}\}^k_\mu - \xi^i \eta^k_{\mu+1_i} + \eta^j \xi^k_{\mu+1_j}\Big)\, B^\mu_k(q).$$

It is now easy to identify the components of the Spencer operator. □

We complete Example C.11 to the differential bracket.

Example C.15. For $X = \mathbb{R}$ and $q = 1$ the differential bracket for sections ξ_1, η_1 of $J_1(T) \to X$ is defined by

$$\begin{aligned}
[\xi_1, \eta_1] &= (\xi\,\eta_x - \eta\,\xi_x,\, \xi\,\eta_{xx} - \eta\,\xi_{xx}) \\
&+ (\partial_x\eta - \eta_x,\, \partial_x\eta_x - \eta_{xx})dx(\xi\frac{\partial}{\partial x}) - (\partial_x\xi - \xi_x,\, \partial_x\xi_x - \xi_{xx})dx(\eta\frac{\partial}{\partial x}) \\
&= (\xi\partial_x\eta - \eta\partial_x\xi,\, \xi\partial_x\eta_x - \eta\partial_x\xi_x).
\end{aligned}$$

Clearly, the second order jets cancel, but the derivatives of the first order jets remain present. ◇

Appendix D

Implementation

All examples in this thesis are computed with the Maple packages `jets`, `JetGroupoids` and `Spencer` and in this appendix, we give an overview over the relevant commands.

The package `jets`, written by Barakat and Hartjen [Bar01], implements the jets calculus for Maple. It allows to compute generalised symmetries of differential equations. Barakat extended this with basic commands for jet groupoids, algebroids and natural Π_q-bundles. This development was joined by the author of this thesis with further extensions and various efficiency improvements.

Due to the size of `jets`, which contains approximately 400 procedures, the package `JetGroupoids` was created as an addon for `jets`. It extends the `jets` procedures to natural Θ_q-bundles. The package also contains data structures and procedures for efficient prolongation and projection of natural bundles.

In the first section, we give a short summary of `JetGroupoids` commands. For detailed Maple help pages, see [Lor08a]. The relevant `jets` commands are presented in Section D.2. We continue with a short introduction to the Spencer package in Section D.3 and finish the appendix with a sample Maple session which was used (in various alterations) for all examples of this thesis.

The following expressions in the arguments of the procedures have a fixed meaning throughout the appendix.

`ivar`	independent variables,
`dvar`	dependent variables,
`Dvar`	fibre coordinates the tangent bundle T (for algebroids),
`uvar`	fibre coordinates of $\mathcal{F}$,
`F`	natural bundle $\mathcal{F}$,
`Fi`	natural bundle $\mathcal{F}_{(i)}$ obtained by prolongation and projection,
`JrF`	prolonged natural bundle $J_r(\mathcal{F})$,
`vec`	vector field, usually for an algebroid action.
`gP`	generic point procedure, e.g. `i->10+i+i*i`.

D.1 The MAPLE Package `JetGroupoids`

D.1.1 Natural Bundle Commands

`ChangeFibreCoordinates(F,lst)`

Change the fibre coordinates of `F` according to the new coordinates in `lst`.

`CodimOfAction(F,[i,gP])`

Computes the following dimensions for the `i`-th prolongation of natural Θ_q-bundle `F` at a given generic point `gP`. Let G_q be the isotropy groupoid of Θ_q and F the fibre of `F`.

- dimension of orbit space of G_{q+i} on $F^{(i)}$ at `gP`.
- $\dim(F^{(i)})$,
- dimension of the G_{q+i}-orbits on $F^{(i)}$ through `gP`.
- $\dim(G_{q+i})$,
- dimension of the stabiliser of `gP`.

The default is `i=0` and if `gP` is omitted, then the computation is valid for generic points. This is a wrapper to the `jets` procedure `codim_of_action`.

`CompleteFibreCoordinates(lst,var)`

Complete the new coordinates from the list `lst` with coordinates from `var` to a coordinate system of the complete fibre.

`CreateNaturalBundle({inv,vec},ivar,dvar,uvar,Dvar,["algebroid"=R])`

Sets up a natural bundle data structure, either from defining equations `inv` of a jet groupoid in Lie form or directly for the algebroid action `vec` on the natural bundle.

`EquivariantSections(Fi,[lsti])`

Computes the equivariant sections $c : J_{r-1}(\mathcal{F}_{(i-1)}) \to \mathcal{F}_{(i)}$. The additional argument `lsti` may contain lists of lists of integers indicating in which order the vector fields are integrated.

`IsNaturalBundle(F,vec,ivar,dvar)`

Checks if the vector fields generating the algebroid action on `F` have commutation relations compatible with the distribution `vec` defining a reference algebroid.

`ProjectNaturalBundle(JrF,v)`

Project a prolonged natural bundle `JrF` a single step down and use the symbol `v` for coordinates.

`ProlongNaturalBundle(Fi,num,u)`

Prolongs the natural bundle `Fi` `num` times and uses the symbol `u` for new coordinates. Internally uses `prolnatinf` which calculates the minimal bundle to which all sections from $\mathcal{F} \to X$ restrict and the `jets` command `natinfG`.

`PullbackToF(expr,Fi)`

Pull back a map `Fi`$\to \mathbb{R}$, i.e. an expression in the coordinates of `Fi` to a suitable prolongation of the original natural bundle `F`.

`PushToNB(expr,Fi)`

Tries to express `expr` on $J_r(\mathcal{F})$ by coordinates of `Fi`. Uses `Janetq` internally.

`RestrictNaturalBundle(F,SUBS,[inv,SUBSinv])`

Restrict the natural bundle `F` according to the substitutions `SUBS`. If the optional `inv` and `SUBSinv` are given, they replace the entries `F["inv"]` and `F["SUBSinv"]`.

`VessiotStructureEquations(Fi,[lsti],[""])`

Uses `EquivariantSections` to calculate the Vessiot structure equations on `Fi`. If the additional string is given, the coordinates of $\mathcal{F}_i \to J_{r-1}(\mathcal{F}_{(i-1)})$ are used as left hand sides rather than the projection $J_r(\mathcal{F}_{(i-1)}) \to \mathcal{F}_{(i)}$.

D.1.2 Invariants and Related Commands

`DualDerivatives(coframe,ivar,dxvar)`

Computes the differential operators $\mathcal{D}_i$ dual to a `coframe` with differentials `dxvar`.

`InvariantDifferentialOperators(Fi,gP,nat,GR,dvar)`

Computes the invariant differential operators dual to an invariant coframe on the natural bundle $\mathcal{F}_{(i)}$. Also given are a generic point `gP` and the finite groupoid action `nat` of `GR` on the natural bundle $\mathcal{F}$ one started the prolongation and projection.

`InvariantsOnNaturalBundle(F,[lsti],["nobase"])`

Returns the invariants $\mathcal{F} \to \mathbb{R}$ of with `lsti` as for `EquivariantSections`. If a coordinate of the base manifold is also invariant, it can be excluded from the output by adding the optional parameter `"nobase"`.

D.1.3 Exterior Differential Forms

`ExtDeriv(expr,ivar,dvar,evar)`

Computes the exterior derivative of the expression `expr`, such that

```
ExtDeriv(ivar[i],ivar,dvar,evar) = evar[i].
```

`Fsubs(SUBS,form,evar,nevar)`

Substitutes `SUBS` into the differential `form`, taking care of the antisymmetric wedge product (using the package `JanetOre`).

D.1.4 Tool Procedures

`CommutatorTable(lvec,ivar,[dvar,""])`

Computes the commutator table for an algebroid generated by the list of vector fields `lvec` with independent and dependent variables `ivar` and `dvar`. If an optional string is given, also the relations between the vector fields are returned.

`FF1coor(vec,ivar,uvar,udvar,Dvar,[kernelproc])`

For a given algebroid action `vec` on $J_r(\mathcal{F})$, the coordinates of $\mathcal{F}_{(1)} = J_r(\mathcal{F})/K_q^{q+1}$ are computed. `ndvar` denotes the fibre coordinates of $J_r(\mathcal{F}) \to J_{r-1}(\mathcal{F})$. Typically, the optional kernel procedure is `kernelD`. Internally used in `ProjectNaturalBundle`.

`kernelByDegree(M,deg,var)`

Computes the kernel of the polynomial matrix `M` with an ansatz of polynomials up to degree `deg`. It is meant as a replacement of `linalg[kernel]`, which depends on random numbers and usually returns far too large results.

`kernelByDegreeEliminationStep(ker,old,var)`

Eliminates all generators from `old` (the output of `kernelByDegree`) that are linear combinations of some given generators `ker`.

`kernelD(M,var)`

Uses `kernelByDegree` to calculate a polynomial basis of the kernel of `M` by searching for generators of increasing degree.

`kernelN(M)`

A wrapper to `linalg[kernel]` that computes the kernel of a polynomial matrix and then multiplies the generators with the least common multiple of all denominators, such that all entries are polynomial.

`PrepareAlgebroidRelations(R,i,ivar,dvar)`

From a subalgebroid `R` of $J_q(T)$, a substitution list for the `i`-th prolongation of `R` is computed using the `Janet` package. This avoids problems with the `jets` command `jsubs`.

`PrepareGroupoidRelations(GLF,omega0,ord,ivar,dvar,wvar)`

Prepares a substitution list for the *quasilinear* groupoid equations `GLF` in general Lie form (`omega0` defines the groupoid).

`prolnatinf(vec,num,ivar,dvar,Dvar,u,[SUBSvec],[uvar])`

Given an infinitesimal action on a natural bundle $\mathcal{F}$, `prolnatinf` calculates $J_r(\mathcal{F})$ and its minimal subbundle where all sections from $\mathcal{F} \to X$ restrict to. It is internally used in `ProlongNaturalBundle`.

`UseRamdisk(strPath)`

Copies all files found under the paths in `libname` to the path `strPath` and sets `libname` to it. Very useful if `strPath` is the mountpoint of a ram disk. Then it speeds up `kernelByDegree` substantially by avoiding disk or net traffic.

D.2 The MAPLE Package `jets`

In this section, several commands from the MAPLE package `jets` [Bar01] are documented. We restrict to those commands which are relevant for jet groupoids and natural bundles. Most of the procedures are originally written by Barakat.

D.2.1 Groupoid and Algebroid Commands

`grp2alg(GLF,ivar,dvar,Dvar)`

Convert the defining equations `GLF` of a jet groupoid into the corresponding algebroid equations.

`invtarget(T,ivar,dvar,Tvar,[""])`

For the algebroid `T` of a groupoid G, compute the defining equations for the differential invariants that determine G (for target transformations). If the optional string as last argument is given, the equations are solved.

`inv2LF(Phi,ivar,dvar)`

Given differential invariants `Phi` for a groupoid G compute the defining equations in Lie Form.

`isoalg(T,ivar,dvar,Tvar)`

Compute a basis of vector fields for the algebroid action of `T` on the groupoid Π_q (infinitesimal target transformations). Integrating these vector fields is equivalent to solving the equations from `invtarget`.

`LieForm(T,ivar,dvar,Tvar)`

Compute the defining equations of a jet groupoid in Lie form using `invtarget`.

`LieFormG(nat,ivar,dvar,Ivar,wvar)`

For a given groupoid action `nat` and a general section `wvar` of a natural bundle compute the equations for the symmetry groupoid in Lie form.

D.2.2 Natural Bundle Commands

In their original form, the following commands deal with natural Π_q-bundles only. In some cases (e.g. `F1coor`) they are superseded by `JetGroupoids` commands, in other cases (e. g. `natinf(G)`) they are internally used by `JetGroupoids`, as they generalise easily to Θ_q-bundles.

`codim_of_action(vec,num,ivar,uvar,Dvar,[gP],[""])`

Calculates the dimension of orbit space of the G_{q+i}-action given by `vec` on the fibre $F^{(\mathrm{num})}$ at `gP`. If the optional string as last argument is given, the output is as described for `CodimOfAction` which internally uses this procedure.

`F1coor(vec,ivar,Dvar,uvar)`
Computes the fibre coordinates of the bundle $\mathcal{F}_{(1)} = J_1(\mathcal{F})/K_q^{q+1}$ for a natural Π_q-bundle $\mathcal{F}$.

`F1dim(vec,ivar,Dvar,uvar,[gP])`
Computes the fibre dimension of $\mathcal{F}_{(1)} = J_1(\mathcal{F})/K_q^{q+1}$. If a generic point `gP` is given, it computed at this point.

`inf2MF(vec,ivar,uvar,wvar)`
Convert the algebroid action `vec` on $\mathcal{F}$ into the Medolaghi form of the symmetry algebroid of a section of $\mathcal{F} \to X$ given by `wvar`.

`JacobiCond(VSE,ivar,uvar)`
Computes the Jacobi conditions for given Vessiot structure equations on $\mathcal{F}_{(i)}$.

`natfin(inv,ivar,dvar,uvar,Ivar)`
Determine the Π_q-action on a natural bundle $\mathcal{F}$ defined by the Lie form `inv` of a groupoid.

`natfinG(nat,inv1,ivar,vvar)`
For a natural bundle $\mathcal{F}$, given the groupoid action `nat`, compute the finite action on a projection of $J_r(\mathcal{F}) \to \mathcal{F}_{(i)}$ given by `inv1` with coordinates `vvar`. Usually, `inv1` is the output of `F1coor`.

`natinf(inv,ivar,dvar,uvar)`
Similar to `natfin`, the vector field `vec` for the algebroid action is computed from groupoid equations `inv` in Lie form.

`natinfG(vec,inv1,ivar,uvar,vvar,Dvar)`
Similar to `natfinG`, the vector field for the algebroid action on a projection $J_r(\mathcal{F}) \to \mathcal{F}_{(i)}$ given by `inv1` with coordinates `vvar`. Usually, `inv1` is the output of `F1coor` or `FF1coor`.

`natfin2inf(nat,ivar,Ivar,Dvar,"")`
Converts the groupoid action `nat` into a vector field for the algebroid action.

D.3 The Maple Package `Spencer`

The Maple package `Spencer` mainly calculates the Spencer cohomology for a given linear system of PDEs. It depends on `jets`, `homalg` [BR08] and suitable ring packages (such as `Involutive` [BCG+03] or `JanetOre`).

There exists another package called `Spencer`, which is part of the `Vessiot` package for Maple V (see [ACC+03]). In Maple 14, it will be a part of the `DifferentialGeometry` package. It contains an independent implementation to compute Spencer cohomology groups using sequences of vector bundles (see Appendix A). The commutative algebra approach based on the Koszul complex is not supported.

`SCohomDim(sC)`

Displays the dimensions of the Spencer cohomology groups from the output `sC` of `SpencerCohomology` as in Remark A.12. See below for details.

`SCZeroSets(sC)`

If `SpencerCohomology` is invoked using the module approach, `SCZeroSets` displays all expressions that were assumed to be nonzero during calculation.

`SpencerCohomology(R,ivar,dvar,Tvar,RP)`

`SpencerCohomology(F,ivar,Dvar,Tvar,RP)`

`SpencerCohomology(R,lstp,lstq,ivar,dvar)`

Computes the Spencer cohomology for a given linear system of PDEs `R`. Here, `Tvar` are the variables for the polynomial ring $S^\bullet T_x$. Alternatively, we can directly give a natural bundle `F`. With the third calling sequence, the Spencer cohomology is computed via Spencer δ-sequences of vector bundles for all combinations (p, q) given in `lstp` and `lstq`.

`SymbolOf(R,ord,ivar,dvar)`

Obtains the symbol $\mathcal{M}_{\texttt{ord}}$ of a given linear system of PDEs `R`.

`SymbolModule(R,ivar,dvar,Dvar,RP)`

The characteristic module M of a given linear system of PDEs `R` using the `homalg` ring package `RP`.

D.3.1 `JanetOre` Extension for the Exterior Algebra

The computation of Koszul complexes relies on the exerior algebra $\bigwedge V$ of a vector space V. To compute in the exterior algebra, we extended the package `JanetOre` written by Robertz [Rob06], [Rob08]. As the ideas are quite simple, we outline them briefly.

- The exterior algebra may be seen as the quotient of an iterated skew polynomial ring $k[dx^1, \sigma_1, \delta_1] \dots [dx^n, \sigma_n, \delta_n]$ with

  $$dx^i dx^j = -dx^j dx^i \qquad i \neq j$$

  modulo the ideal $\langle (dx^1)^2, \dots (dx^n)^2 \rangle$.

- Like for Ore algebras, we use normal forms $dx^2 \wedge dx^1 = -dx^1 \wedge dx^2$ for monomials and $(dx^i)^2 = 0$ during multiplication.
- $(dx^i)^2 = 0 \Rightarrow$ the multiplicative variables for dx^i may not contain dx^i. With this trivial change it is possible to compute Janet bases over the exterior algebra. It is not important how the leading term is eliminated, as long as it is eliminated.

D.3.2 ZeroSets Extension for `Involutive`

The main procedure `InvolutiveBasis` of `Involutive` which computes a Janet basis and stores all expressions by which the procedure has divided during calculations in a global variable. The contents can be displayed using `PolZeroSets`, which has to be done right after the call of `InvolutiveBasis`. Each call of `InvolutiveBasis` erases the previous results.

To obtain all the expressions `Involutive` divided by during the Spencer cohomology computation, the output of `PolZeroSets` has to be stored in a new global variable where all new denominators are added rather than erasing old results.

The `homalg` translation table `` `InvolutiveZeroSets/homalg` `` provides the global variable `_InvZeroSets` together with the procedure `ResetInvZeroSets()` that sets `_InvZeroSets := []`.

D.4 Sample Worksheet

In this section we present commands that are typically used to compute a jet groupoid, the corresponding natural bundle and then start to prolong and project. Most examples in this thesis are variations of this worksheet and the reader may take it as a starting point for own computations.

```
## Loading the package and declaration of variables:
################

with(jets): with(JetGroupoids): with(Spencer):

n := 3;
ivar := vn(x,n);   dvar := vn(y,n);
Ivar := vn(phi,n); Dvar := vn(xi,n); Tvar := vn(eta,n);

## A jet groupoid in Lie form
################

## Groupoid and algebroid:
```

```
GR := [...groupoid definition...];
R  := grp2alg(GR,ivar,dvar,Dvar);

## target algebroid and involutive distribution:

T    := grp2alg(GR, ivar, dvar, Tvar, "");
Tred := nrsolve(T, getSolveVar(T, dvar, Tvar));
iso  := isoalg(Tred, ivar, dvar, Tvar);

## Lie form of GR:

Phi := invtarget(Tred,ivar,dvar,Tvar,"");
LF  := inv2LF(Phi,ivar,dvar);

## Compute the natural bundle:
################

uvar := [...fibre coordinates of F...];
inv  := ezip(uvar,Phi);

## groupoid and algebroid action:

nat := natfin(inv,ivar,dvar,uvar);
vec := natfin(inv,ivar,dvar,uvar,Dvar,"");
## or:
vec := natfin2inf(nat,ivar,dvar,Dvar,"");

## data structure:

F := CreateNaturalBundle(vec,ivar,dvar,uvar,Dvar);
## or
F := CreateNaturalBundle(inv,ivar,dvar,uvar,Dvar);

## Prolongation, Projection and Vessiot structure equations:
################

J1F := ProlongNaturalBundle(F,1,uu):
F1 := ProjectNaturalBundle(J1F,v1):

VSE1 := VessiotStructureEquations(F1);

CodimOfAction(F1);
Inv1 := InvariantsOnNaturalBundle(F1);
```

Bibliography

[ACC+03] I. Anderson, F. Catrina, B. Croft, C. Hillyard, J. Humphries, and C. Miller. *The* MAPLE *package* `Vessiot`*, version 1.3*, 2003. (http://www.math.usu.edu/~fg_mp/).

[Bar01] M. Barakat. Jets. A MAPLE-package for formal differential geometry. In *Computer algebra in scientific computing (Konstanz, 2001)*, pages 1–12. Springer, Berlin, 2001.

[BCG+91] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1991.

[BCG+03] Y. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, and D. Robertz. The MAPLE Package "Janet": I. Polynomial Systems, II. Linear Partial Differential Equations. In *Proceedings of the 6th International Workshop on Computer Algebra in Scientific Computing, Sep. 20-26, 2003, Passau*, pages 31–54, 2003.

[BK05] R. Beals and E. A. Kartashova. Constructive factorization of linear partial differential operators. *Teoret. Mat. Fiz.*, 145(2):165–180, 2005.

[BR08] M. Barakat and D. Robertz. `homalg`: a meta-package for homological algebra. *J. Algebra Appl.*, 7(3):299–317, 2008.

[Bra27] H. Brandt. Über eine Verallgemeinerung des Gruppenbegriffes. *Math. Ann.*, 96(1):360–366, 1927.

[Car04] É. Cartan. Sur la structure des groupes infinis de transformation. *Ann. Sci. École Norm. Sup. (3)*, 21:153–206, 1904.

[Car08] É. Cartan. Les sous-groupes des groupes continus de transformations. *Ann. Sci. École Norm. Sup. (3)*, 25:57–194, 1908.

[Car10] É. Cartan. Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Sci. École Norm. Sup. (3)*, 27:109–192, 1910.

[Car45] É. Cartan. *Les systèmes différentiels extérieurs et leurs applications géométriques.* Actualités Sci. Ind., no. 994. Hermann et Cie., Paris, 1945.

[Che40] S.-S. Chern. The geometry of the differential equation $y''' = F(x, y, y', y'')$. *Sci. Rep. Nat. Tsing Hua Univ. (A)*, 4:97–111, 1940.

[Dar82] G. Darboux. Sur le problème de Pfaff. *Bull. Sci. Math. (2)*, 6:14–36, 49–68, 1882.

[dC92] M. P. do Carmo. *Riemannian geometry.* Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[EGS65] L. Ehrenpreis, V. W. Guillemin, and S. Sternberg. On Spencer's estimate for δ-Poincaré. *Ann. of Math. (2)*, 82:128–138, 1965.

[Ehr53] C. Ehresmann. Introduction à la théorie des structures infinitésimales et des pseudogroupes de Lie. In *Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 11*, page 16. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

[Ehr54] C. Ehresmann. Extension du calcul des jets aux jets non holonomes. *C. R. Acad. Sci. Paris*, 239:1762–1764, 1954.

[Ehr55] C. Ehresmann. Les prolongements d'un espace fibré différentiable. *C. R. Acad. Sci. Paris*, 240:1755–1757, 1955.

[Eis95] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[ET79] D. B. A. Epstein and W. P. Thurston. Transformation groups and natural bundles. *Proc. London Math. Soc. (3)*, 38(2):219–236, 1979.

[Fes07] D. Fesser. *On Vessiot's Theory of Partial Differential Equations.* PhD thesis, Universität Kassel, 2007.

[Gar89] R. B. Gardner. *The method of equivalence and its applications*, volume 58 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.

[Gol67a] H. Goldschmidt. Existence theorems for analytic linear partial differential equations. *Ann. of Math. (2)*, 86:246–270, 1967.

[Gol67b] H. Goldschmidt. Integrability criteria for systems of nonlinear partial differential equations. *J. Differential Geometry*, 1:269–307, 1967.

[Gol68] H. Goldschmidt. Prolongations of linear partial differential equations. I. A conjecture of Élie Cartan. *Ann. Sci. École Norm. Sup. (4)*, 1:417–444, 1968.

[Gri83] P. A. Griffiths. *Exterior differential systems and the calculus of variations*, volume 25 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1983.

[GS64] V. W. Guillemin and S. Sternberg. An algebraic model of transitive differential geometry. *Bull. Amer. Math. Soc.*, 70:16–47, 1964.

[GS67] V. W. Guillemin and S. Sternberg. The Lewy counterexample and the local equivalence problem for G-structures. *J. Differential Geometry*, 1:127–131, 1967.

[GS04] D. Grigoriev and F. Schwarz. Factoring and solving linear partial differential equations. *Computing*, 73(2):179–197, 2004.

[IL03] T. A. Ivey and J. M. Landsberg. *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, volume 61 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[Kar06] E. A. Kartashova. A hierarchy of generalized invariants for linear partial differential operators. *Teoret. Mat. Fiz.*, 147(3):470–478, 2006.

[KL06] B. Kruglikov and V. Lychagin. Invariants of pseudogroup actions: homological methods and finiteness theorem. *Int. J. Geom. Methods Mod. Phys.*, 3(5-6):1131–1165, 2006.

[KMS93] I. Kolář, P. W. Michor, and J. Slovák. *Natural operations in differential geometry*. Springer-Verlag, Berlin, 1993.

[KN63] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963.

[Kob72] S. Kobayashi. *Transformation groups in differential geometry*. Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70.

[KS72] A. Kumpera and D. Spencer. *Lie equations. Vol. I: General theory*. Princeton University Press, Princeton, N.J., 1972. Annals of Mathematics Studies, No. 73.

[Kum75] A. Kumpera. Invariants différentiels d'un pseudogroupe de Lie. I/II. *J. Differential Geometry*, 10(2):289–416, 1975.

[Lew57] H. Lewy. An example of a smooth linear partial differential equation without solution. *Ann. of Math. (2)*, 66:155–158, 1957.

[Lie91] S. Lie. Die Grundlagen für die Theorie der unendlichen kontinuierlichen Transformationsgruppen I/II. *Leipz. Ber.*, III:316–393, 1891.

[Lor08a] A. Lorenz. *Groupoids with* `jets` *and* `JetGroupoids` *project*, 2008. (http://wwwb.math.rwth-aachen.de/~arne/Groupoids).

[Lor08b] A. Lorenz. On local integrability conditions of jet groupoids. *Acta Appl. Math.*, 101:205–213, 2008.

[LSE93] S. Lie, E. Study, and F. Engel. *Beiträge zur Theorie der Differentialinvarianten*. Teubner-Archiv zur Mathematik [Teubner Archive on Mathematics], 17. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1993. Edited and with a foreword and an appendix by G. Czichowski and B. Fritzsche.

[Mac05] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[Mal05] B. Malgrange. *Systèmes différentiels involutifs*, volume 19 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2005.

[Man08] E. L. Mansfield. *The Invariant Calculus for smooth and discrete problems*. Cambridge University Press, Cambridge, 2008. To appear.

[MM03] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.

[MS08] E. L. Mansfield and E. Shemyakova. Moving frames for Laplace invariants. *ISSAC'08*, 2008.

[Neu03] S. Neut. *Implantation et nouvelles applications de la méthode d'équivalence*. PhD thesis, L'Université des Sciences et Technologies de Lille I, 2003.

[Nij72] A. Nijenhuis. Natural bundles and their general properties. Geometric objects revisited. In *Differential geometry (in honor of Kentaro Yano)*, pages 317–334. Kinokuniya, Tokyo, 1972.

[NP02] S. Neut and M. Petitot. La géométrie de l'équation $y''' = f(x, y, y', y'')$. *C. R. Math. Acad. Sci. Paris*, 335(6):515–518, 2002.

[Olv93] P. J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1993.

[Olv95] P. J. Olver. *Equivalence, invariants, and symmetry.* Cambridge University Press, Cambridge, 1995.

[OP] P. J. Olver and J. Pohjanpelto. Moving frames for Lie pseudo-groups. *Canad. J. Math.* to appear.

[OP07a] P. J. Olver and J. Pohjanpelto. Differential invariant algebras of Lie pseudo-groups. preprint, University of Minnesota, 2007.

[OP07b] P. J. Olver and J. Pohjanpelto. Differential invariants for Lie pseudo-groups. In *Gröbner bases in symbolic analysis*, volume 2 of *Radon Ser. Comput. Appl. Math.*, pages 217–243. Walter de Gruyter, Berlin, 2007.

[OP08] P. J. Olver and J. Pohjanpelto. Pseudo-groups, moving frames, and differential invariants. In *Symmetries and overdetermined systems of partial differential equations*, volume 144 of *IMA Vol. Math. Appl.*, pages 127–149. Springer, New York, 2008.

[Pom78] J.-F. Pommaret. *Systems of partial differential equations and Lie pseudogroups*, volume 14 of *Mathematics and its Applications.* Gordon & Breach Science Publishers, New York, 1978. With a preface by André Lichnerowicz.

[Pom83] J.-F. Pommaret. *Differential Galois theory*, volume 15 of *Mathematics and its Applications.* Gordon & Breach Science Publishers, New York, 1983.

[PQ00] J. F. Pommaret and A. Quadrat. Formal elimination for multidimensional systems and applications to control theory. *Math. Control Signals Systems*, 13(3):193–215, 2000.

[Pra66] J. Pradines. Théorie de Lie pour les groupoïdes différentiables. Relations entre propriétés locales et globales. *C. R. Acad. Sci. Paris Sér. A-B*, 263:A907–A910, 1966.

[PT77] R. S. Palais and C. L. Terng. Natural bundles have finite order. *Topology*, 19(3):271–277, 1977.

[Qui64] D. G. Quillen. *Formal Properties of Over-Determined Systems of Linear Partial Differential Equations.* PhD thesis, Harvard University, 1964.

[Rob06] D. Robertz. *Formal Computational Methods for Control Theory*. PhD thesis, RWTH Aachen, Germany, June 2006. This thesis is available at http://darwin.bth.rwth-aachen.de/opus/volltexte/2006/1586.

[Rob08] D. Robertz. JanetOre*: A* Maple *package to compute a* JANET *basis for modules over* ORE *algebras*, 2003-2008. (http://wwwb.math.rwth-aachen.de:8040).

[Sei02] W. M. Seiler. *Involution*. Habilitation thesis, Universität Mannheim, 2002.

[Sha97] R. W. Sharpe. *Differential geometry*, volume 166 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Cartan's generalization of Klein's Erlangen program, With a foreword by S. S. Chern.

[Spe62] D. C. Spencer. Deformation of structures on manifolds defined by transitive, continuous pseudogroups. I. Infinitesimal deformations of structure. *Ann. of Math. (2)*, 76:306–398, 1962.

[Spe69] D. C. Spencer. Overdetermined systems of linear partial differential equations. *Bull. Amer. Math. Soc.*, 75:179–239, 1969.

[Ste64] S. Sternberg. *Lectures on differential geometry*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[Sto00] O. Stormark. *Lie's structural approach to PDE systems*, volume 80 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000.

[SW07a] E. S. Shemyakova and F. Winkler. A Full System of Invariants for Third-Order Linear Partial Differential Operators in General Form. *Lecture Notes in Comput. Sci.*, 4770:360–369, 2007.

[SW07b] E. S. Shemyakova and F. Winkler. Obstacles to the factorization of partial differential operators into several factors. *Programmirovanie*, (2):17–25, 2007.

[SW08] E. S. Shemyakova and F. Winkler. On the Invariant Properties of Hyperbolic Bivariate Third-Order Linear Partial Differential Operators. In D. Kapur, editor, *Computer Mathematics - Proceeding 8th Asian Symp. on Computer Mathematics (ASCM, 2007)*, pages 199–212. Singapore, Dec. 2007; Springer LNAI 5081, 2008, 2008.

[SY98] H. Sato and A. Y. Yoshikawa. Third order ordinary differential equations and Legendre connections. *J. Math. Soc. Japan*, 50(4):993–1013, 1998.

[Ter78] C. L. Terng. Natural vector bundles and natural differential operators. *Amer. J. Math.*, 100(4):775–828, 1978.

[Tre94] A. Tresse. Sur les invariants différentiels des groupes continus de transformations. *Acta Math.*, 18(1):1–3, 1894.

[Tsa00] S. P. Tsarev. Factorization of linear partial differential operators and the Darboux method for integrating nonlinear partial differential equations. *Teoret. Mat. Fiz.*, 122(1):144–160, 2000.

[Ves03] E. Vessiot. Sur la théorie des groupes continus. *Ann. Sci. École Norm. Sup. (3)*, 20:411–451, 1903.

[Vin01] A. M. Vinogradov. *Cohomological analysis of partial differential equations and secondary calculus*, volume 204 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the Russian manuscript by Joseph Krasil′shchik.

[Wün05] K. Wünschmann. *Über Berührungsbedingungen bei Integralkurven von Differentialgleichungen*. PhD thesis, Greifswald. Leipzig: B. G. Teubner, 35 S, 1905.

[Yan92] K. Yang. *Exterior differential systems and equivalence problems*, volume 73 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1992.

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Curriculum Vitae

Persönliche Daten:

Name:	Arne Lorenz
Geburtsdatum:	8. Januar 1980
Geburtsort:	Elmshorn
Familienstand:	verheiratet, keine Kinder

Schulbildung:

1986-1990	Besuch der Grund- und Hauptschule St. Georgsberg, Ratzeburg
1990-1999	Besuch der Lauenburgische Gelehrtenschule Ratzeburg (Gymnasium)
Juni 1999	Abitur, Leistungskurse Mathematik und Physik

Zivildienst:

1999-2000	Zivildienst am Bauhof der Stadt Ratzeburg

Studium:

Oktober 2000	Beginn des Physikstudiums an der RWTH Aachen
September 2002	Vordiplom in Physik
August 2004	Beginn der Diplomarbeit am Institut für Theoretische Physik (RWTH) bei Prof. Dr. W. Bernreuther
Juli 2005	Diplom in Physik mit der Gesamtnote "sehr gut"
Juli 2005:	Beginn der Promotion am Lehrstuhl B für Mathematik (RWTH) bei Prof. Dr. W. Plesken

Bisherige Tätigkeiten:

2003-2005	Studentische Hilfskraft am Lehrstuhl B für Mathematik
Juli 2005- Juli 2007	Stipendiat des DFG-Graduiertenkollegs "Hierarchie und Symmetrie in mathematischen Modellen"
August 2007 - Juli 2008	Wissenschaftlicher Angestellter am Lehrstuhl B für Mathematik
August 2008 - Dezember 2008	Wissenschaftliche Hilfskraft am Lehrstuhl B für Mathematik