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CONTACT GEOMETRY AND NON-LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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Introduction

The purpose of this paper is to study non-linear second-order differential equations defined on a smooth manifold M, from the point of view of the contact geometry of the manifold of 1-jets.

A general approach to the geometry of non-linear differential equations, which goes back to Sophus Lie, is based on regarding a non-linear differential equation of the k-th order as a closed subset in the manifold of k-jets ([4], [5], [8], [11], [18]). Then the geometry of second-order equations, as well as that of the manifold $J^2(M)$ turn out to be exceptional compared with the general case $k > 2$ [5].

The following observation is the starting point for the approach proposed in this paper: each differential k-form $\omega \in \Lambda^k(J^1M)$ on the manifold of 1-jets J^1M can be regarded as a non-linear second-order differential operator $\Delta_\omega : C^\infty(M) \to \Lambda^k(M)$ acting according to the rule $\Delta_\omega(h) = \sigma^*_{j_1(h)}(\omega)$, where $\sigma_{j_1(h)}: M \to J^1(M)$ is the section corresponding to the 1-jet of the function $h \in C^\infty(M)$.

In this approach, in contrast to the first, we use the geometrically simpler space J^1M, although, of course, not all non-linear second-order differential equations can then be ~~but only~~ only a certain subclass. Nevertheless, this subclass is broad enough and embraces all the equations that occur in practice: quasi-linear equations and Monge–Ampère equations.

We call operators of the type $\Delta_\omega : C^\infty(M) \to \Lambda^k(\omega)$, $\omega \in \Lambda^k(J^1M)$ Monge–Ampère operators. The motivation behind this definition is the fact that when written in local coordinates the operators Δ_ω lead to non-linearities of the same type as the classical Monge–Ampère operators.

The correspondence $\omega \mapsto \Delta_\infty$ can be used in different ways. First of all, it allows us to transfer directly the contact geometry from $J^1(M)$ to the differential equations and so to determine contact symmetries and to generalize the concept of an automodel solution.

On the other hand, the correspondence $\omega \mapsto \Delta_\omega$ distinguishes additional structures in the algebra of exterior forms on J^1M. In this algebra we consider the ideal $C \subset \Delta^*(J^1M)$ consisting of differential forms that lead to the zero operator,

$$C = \{\omega \in \Lambda^*(J^1M),\ \Delta_\omega = 0\}.$$

Then the Monge–Ampère operators are uniquely determined by elements of the factor-module $\Lambda^*(J^1M)/C$.

The presence of a contact structure on J^2M makes it possible to split C into direct summands and so to describe $\Lambda^*(J^1M)/C$ effectively; the structures used here are similar to those occurring in Kähler geometry ([3], [14]). Just as in Kähler geometry, operators $\top$ and $\perp$ are introduced, corresponding to the effective (= primitive) forms L and Λ, and a theorem on the decomposition into effective forms is proved.

§ § 1 and 2 deal with the details of this algebra.

Differential operators proper appear in §3. In this section, using the correspondence $\omega \mapsto \Delta_\omega$, we explain methods of computing contact symmetries and conservation laws for J^1M. For operators of divergent type we indicate the links between symmetries and conservation laws. A brief discussion of discontinuous solutions and of the Giugonio–Rankin conditions for non-linear differential equations concludes this section.

Appendices I and II deal with the application of the results obtained to concrete non-linear differential equations. In Appendix I we consider the Khoklov–Zabolotskaya equation, which describes the propagation of a bounded sound beam in non-linear media; we calculate the contact symmetries and conservation laws. The symmetries are used to find exact solutions, the conservation laws to study the evolution of the boundary of the sound beam.

In Appendix II V. N. Rubtsov considers equations associated with the non-linear Klein–Gordon equation and calculates the algebras of symmetries and the conservation laws. We mention the dependence of the algebra of symmetries and of the conservation laws for equations of the type $\Box u = \mathcal{F}'(u)$ on the function $\mathcal{F}(u)$, which turn out to be finite-dimensional if $\mathcal{F}''(u)$ and $\mathcal{F}'(u)$ are linearly independent, and infinite-dimensional (on J^1M) otherwise.

In conclusion we dwell on a number of problems closely connected with this paper. There are, first of all, the theorems of Sophus Lie on the contact equivalence of the Monge–Ampère operators [11]. Using the correspondence $\omega \mapsto \Delta_\omega$, we can reduce these theorems to the problem of the contact equivalence of differential forms on J^1M, which we can then study according to the scheme of [15] and [20]. However, we have calculated the conservation laws for the Monge–Ampère operators only for J^1M. The resulting laws can be used to obtain the conservation laws in J^kM if the higher symmetries of the equation are used (see [5] and [19]).

§1. The exterior algebra on a symplectic space

1.0. We fix some notation. As usual, we denote by $\Lambda^k(E)$ (respectively, $\Lambda^k(E^*)$) the space of all k-vectors (k-forms) on E, where E is a vector space over $\mathbf{R}$. If $X \in \Lambda^k(E)$ and $\omega \in \Lambda^s(E^*)$, $s \geqslant k$, we denote by $i_X(\omega)$ or $X \lrcorner\, \omega$ the result of interior multiplication by the k-vector X; if $s < k$, then $i_X(\omega) = 0$.

Now let $\dim E = 2n$ and suppose that a 2-form $\Omega \in \Lambda^2(E^*)$ determines a symplectic structure on E. Then the mapping $\Gamma: E \to E^*$, $\Gamma(X) = i_X(\Omega)$, is an isomorphism, while its exterior powers define isomorphisms $\Gamma_s: \Lambda^s(E) \to \Lambda^s(E^*)$, $\Gamma_s = \Lambda^s(\Gamma)$. For an arbitrary s-form $\omega \in \Lambda^s(E^*)$ we denote by X_ω the s-vector in $\Lambda^s(E)$ corresponding to ω under Γ_s, $\Gamma_s(X_\omega) = \omega$. We denote by i_ω the result of interior multiplication by X_ω.

In particular, for a 1-form $\omega \in E^*$ the vector X_ω is uniquely determined by the relation $i_\omega(\Omega) = \omega$. For decomposable s-forms $\omega = \omega_1 \wedge \ldots \wedge \omega_s$, $\omega_j \in E^*$ we have $X_\omega = X_{\omega_1} \wedge \ldots \wedge X_{\omega_s}$.

To write the operations introduced above in coordinate form we consider a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ in the space E: $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$, $\Omega(e_i, f_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant n$.

Using this basis Ω takes the form

$$\Omega = \sum_1^n e_i^* \wedge f_i^*,$$

where $e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*$ is the dual basis in E^*, while Γ acts as follows:

$$\Gamma\colon e_i \mapsto f_i^*;\quad \Gamma\colon f_i \mapsto -e_i^*.$$

Hence, if for a 1-form $\omega \in E^*$ we have

$$\omega = \sum_1^n (\alpha_i e_i^* + \beta_i f_i^*),\ \alpha_i,\ \beta_i \in \mathbf{R},\ \text{then}\ X_\omega = \sum_1^n (\beta_i e_i - \alpha_i f_i).$$

We note here that the bivector $X_\Omega = \Gamma_2^{-1}(\Omega)$ can be written as

$$X_\Omega = \sum_1^n e_i \wedge f_i.$$

1.1. In the algebra of exterior forms on the symplectic space E

$$\Lambda^*(E^*) = \bigoplus_{s\geqslant 0} \Lambda^s(E^*)$$

we introduce two operators:

$$\top\colon \Lambda^s(E^*) \to \Lambda^{s+2}(E^*)$$

is the operator of exterior multiplication by the 2-form Ω, $\top(\omega) = \Omega \wedge \omega$, and

$$\perp\colon \Lambda^s(E^*) \to \Lambda^{s-2}(E^*)$$

is the operator of interior multiplication by the bivector X_Ω, $\perp(\omega) = X_\Omega \lrcorner\, \omega$. We set

$$\top_k = \frac{1}{k!}\top^k, \quad \perp_k = \frac{1}{k!}\perp^k, \quad \Omega_k = \frac{1}{k!}\Omega^k, \quad \perp_0 = \top_0 = 1.$$

LEMMA 1.1.1. $\perp(\Omega_k) = (n - k + 1)\Omega_{k-1}$.

■ If $e_1, \ldots, e_n, f_1, \ldots, f_n$ is a symplectic basis in E, then

$$i_{e_j}(\Omega_k) = i_{e_j}(\Omega) \wedge \Omega_{k-1} = f_j^* \wedge \Omega_{k-1},$$
$$i_{f_j}(\Omega_k) = i_{f_j}(\Omega) \wedge \Omega_{k-1} = -e_j^* \wedge \Omega_{k-1}.$$

Hence,

$$i_\Omega(\Omega_k) = \sum_{j=1}^n i_{e_j \wedge f_j}(\Omega_k) = \sum_{j=1}^n i_{f_j}(f_j^* \wedge \Omega_{k-1}) =$$
$$= \sum_{j=1}^n (\Omega_{k-1} + f_j^* \wedge e_j^* \wedge \Omega_{k-2}) = (n-k+1)\,\Omega_{k-1}. \ ■$$

LEMMA 1.1.2. *For an arbitrary* 1*-form* $\omega \in E^*$,

$$[\perp, \top](\omega) = (n-1)\omega.$$

■ $[\perp, \top](\omega) = \perp(\top\omega) = \perp(i_\omega \Omega_2) = i_\omega(\perp\Omega_2) = (n-1) i_\omega(\Omega) =$
$= (n-1)\omega.$ ■

LEMMA 1.1.3. *Let* $\theta \in E^*$, $\omega \in \Lambda^s(E^*)$; *then*

$$\perp(\theta \wedge \omega) = \theta \wedge \perp\omega - i_\theta(\omega).$$

■ If, as above, $e_1, \ldots, e_n, f_1, \ldots, f_n$ is a symplectic basis, then

$$\perp(\theta \wedge \omega) = \sum_{j=1}^n i_{e_j \wedge f_j}(\theta \wedge \omega) = \sum_{j=1}^n i_{f_j}(i_{e_j}(\theta)\,\omega - \theta \wedge i_{e_j}\omega) =$$
$$= \sum_{j=1}^n [i_{e_j}(\theta)\, i_{f_j}(\omega) - i_{f_j}(\theta)\, i_{e_j}(\omega)] + \sum_{j=1}^n \theta \wedge i_{f_j}(i_{e_j}\omega) = -i_\theta(\omega) + \theta \wedge \perp\omega. \ ■$$

THEOREM. *Let* $\omega \in \Lambda^k(E^*)$. *Then* $[\perp, \top](\omega) = (n - k)\,\omega$.

■ The proof is by induction on k. The case $k = 1$ is Lemma 1.1.2. Because of the linearity in ω of the relation to be proved we may assume that $\omega = \theta \wedge \bar{\omega}$, where $\theta \in E^*$, $\bar{\omega} \in \Lambda^k(E^*)$. Then

$$[\perp, \top](\theta \wedge \bar{\omega}) = \perp \circ \top(\theta \wedge \bar{\omega}) - \top \circ \perp(\theta \wedge \bar{\omega}) =$$
$$= \perp(\theta \wedge \top \bar{\omega}) - \top(\theta \wedge \perp \bar{\omega} - i_\theta \bar{\omega}) = \theta \wedge \perp \top \bar{\omega} - i_\theta(\top \bar{\omega}) -$$
$$- \theta \wedge \top \perp(\bar{\omega}) + \top i_\theta(\bar{\omega}) = \theta \wedge [\perp, \top]\bar{\omega} - \theta \wedge \bar{\omega} = (n-k-1)\,\theta \wedge \bar{\omega}. \blacksquare$$

COROLLARY 1.1. *If* $\omega \in \Lambda^k(E^*)$, *then*

$$[\perp, \top_s]\omega = (n-k-s+1)\top_{s-1}\omega, \qquad [\perp_s, \top]\omega = (n-k+s-1)\perp_{s-1}\omega.$$

1.2. PROPOSITION. *The mappings*

$$\top\colon \Lambda^k(E^*) \to \Lambda^{k+2}(E^*) \text{ and } \perp\colon \Lambda^s(E^*) \to \Lambda^{s-2}(E^*)$$

are monomorphisms, provided that $k \leqslant n-1$ *and* $s \geqslant n+1$.

■ Suppose that $\omega \in \Lambda^s(E^*)$, where $s \geqslant n+1$, lies in the kernel $\perp$; then, using Corollary 1.1 we obtain

$$0 = \perp^{n+1}\top_{n+1}\omega = \perp^n[\perp, \top_{n+1}]\omega =$$
$$= (-s)\perp^n\top_n\omega = \ldots = (-s)(-s+1)\ldots(-s+n)\omega$$

and so $\omega = 0$. Similarly, if $\omega \in \Lambda^k(E^*)$, $\top\omega = 0$, and $k \leqslant n-1$, then

$$0 = \top^{n+1}\perp_{n+1}\omega = (2n-k)\ldots(n-k)\omega,$$

hence $\omega = 0$.■

1.3. DEFINITION. A form $\omega \in \Lambda^k(E^*)$, $k \leqslant n$, is said to be *effective* if $\perp\omega = 0$.

1.4. THEOREM. *For every k-form* $\omega \in \Lambda^k(E^*)$ *the following Hodge–Lepage expansion holds*:

$$\omega = \omega_0 + \top\omega_1 + \top_2\omega_2 + \ldots,$$

where $\omega_i \in \Lambda^{k-2i}(E^*)$ *are uniquely determined effective forms.*

■ The proof is by induction on k, the case $k = 1$ being trivial. Assuming that the proposition holds for forms of degree less than k, we prove it for forms of degree k.

Let $\omega \in \Lambda^k(E^*)$; then $\perp\omega \in \Lambda^{k-2}(E^*)$ and, by the inductive hypothesis,

$$\perp\omega = \alpha_0 + \top\alpha_1 + \top_2\alpha_2 + \ldots,$$

where $\alpha_0, \alpha_1, \alpha_2 \ldots$ are uniquely determined effective forms. Hence, if we set $\omega = x_0 + \top x_1 + \top x_2 + \ldots$, then

$$\perp\omega = (n-k+2)x_1 + (n-k+3)\top x_2 + \ldots$$

if we assume that $x_0, x_1, \ldots$ are effective. Consequently, if we take

$$x_1 = \frac{1}{n-k+2}\alpha_0, \quad x_2 = \frac{1}{n-k+3}\alpha_1, \ldots$$

then

$$\perp(\omega - \top x_1 - \top_2 x_2 - \ldots) = 0,$$

that is,

$$\omega_0 = \omega - \top x_1 - \top_2 x_2 - \ldots$$

is an effective form.■

Let $\omega \in \Lambda^k(E^*)$ and let $\omega = \sum_{s\geq 0} \top_s \omega_s$ be the Hodge-Lepage expansion.
Elementary calculations using Corollary 1.1 lead to the formulae

$$\perp_r\omega = \sum_{s\geq r} \binom{n-k+s+r}{r} \tag{1.4.1}$$

$$\top_r\perp_r\omega = \sum_{s\geq r} \binom{n-k+s+r}{r}\binom{s}{r} \tag{1.4.2}$$

from which it follows, in particular, that the effective part ω_0 of ω can be computed as follows:

$$\omega_0 = \Big[\sum_{s\geq 0} (-1)^s \frac{1}{s+1} \top_s \perp_s\Big](\omega). \tag{1.4.3}$$

1.5. PROPOSITION (1) *The mappings*

$$\perp_k\colon \Lambda^{n+k}(E^*) \to \Lambda^{n-k}(E^*) \quad u \quad \top_k\colon \Lambda^{n-k}(E^*) \to \Lambda^{n+k}(E^*)$$

are isomorphisms, and for effective forms ω: $\perp_k \circ \top_k(\omega) = \omega$.

(2) *The form* $\omega \in \Lambda^{n-k}(E^*)$ *is effective if and only if* $\top_{k+1}(\omega) = 0$.

■ (1) Suppose that $\omega \in \Lambda^{n-k}(E^*)$ and $\top_k\omega = 0$. If $\omega = \omega_0 + \top\omega_1 + \ldots$ is the Hodge-Lepage expansion, then simple calculations using the properties (1.2) show that

$$\perp_k\top_k\omega = \omega_0 + [C^1_{k+1}]^2\top\omega_1 + \ldots = 0,$$

and so, by virtue of the uniqueness ot the expansion $\omega_0 = 0, \omega_1 = 0, \ldots$.

(2) We note first that if $\omega \in \Lambda^{n-k}(E^*)$, then

$$\perp \circ \top_{k+1}(\omega) = \top_{k+1} \circ \perp(\omega)$$

by Corollary 1.1. If $\perp\omega = 0$, then $\perp(\top_{k+1}\omega) = 0$ and $\top_{k+1}\omega = 0$ as follows from Proposition 1.2.

Conversely, if $\top_{k+1}\omega = 0$, then

$$\top_{k+1}(\perp\omega) = \perp(\top_{k+1}\omega) = 0$$

and hence, $\perp\omega = 0$.■

1.6. THEOREM. *Let the effective* $(n-k)$*-forms*

$$\omega_1, \quad \omega_2 \in \Lambda^{n-k}(E^*), \qquad 0 \leq k \leq n,$$

be such that for every isotropic subspace $L \subset E$, $\dim L = n - k$, *on which* $\omega_1|_L = 0$ *the form* ω_2 *also vanishes*, $\omega_2|_L = 0$. *Then* $\omega_2 = \lambda\omega_1$ *for some* $\lambda \in \mathbf{R}$.

The proof is by induction on $\dim E/2$, the case $n = 1$ being trivial. Let E, as before, be a symplectic space of dimension $2n$. We choose an arbitrary pair of covectors θ, $\theta^+ \in E^*$ such that $i_\theta(\theta^+) = 1$. Then the restriction Ω' of the form Ω to

$$E' = \ker \theta \cap \ker \theta^+$$

defines a symplectic structure. We denote by $\top'$ and $\perp'$ the corresponding operators. We note that, since

$$E = E' \oplus \mathbf{R}X_\theta + \mathbf{R}X_{\theta^+},$$

by identifying forms on E' with forms on E that degenerate on $\mathbf{R}X_\theta \oplus \mathbf{R}X_{\theta^+}$, we obtain for an arbitrary form $\alpha \in \Lambda^s(E^*)$ the expansion

$$\alpha = \bar{\alpha}_0 + \theta \wedge \alpha_1 + \theta^+ \wedge \alpha_2 + \theta^+ \wedge \theta \wedge \alpha_3,$$

where $\bar{\alpha}_0, \alpha_1, \alpha_2, \alpha_3$ are uniquely determined forms on E', and

$$i_\theta(\alpha_i) = i_{\theta^+}(\alpha_i) = 0.$$

Let us now assume that α is effective. Then using the expansion

$$X_\Omega = X_{\theta^+} \wedge X_\theta + X_{\Omega'},$$

we obtain

$$0 = \perp\alpha = \perp'\bar{\alpha}_0 + \theta \wedge \perp'\alpha_1 + \theta^+ \wedge \perp'\alpha_2 + \theta^+ \wedge \theta \wedge \perp'\alpha_3 + \alpha_3.$$

Thus, $\alpha_1, \alpha_2, \alpha_3$ are effective, and $\perp' \bar{\alpha}_0 + \alpha_3 = 0$. Let

$$\bar{\alpha}_0 = \alpha_0 + \top' x_1 + \top'_2 x_2 + \ldots$$

be the Hodge-Lepage expansion; then

$$\perp'\bar{\alpha}_0 = (n-s+1)\, x_1 + (n-s+2) \top' x_2 + \ldots$$

hence,

$$x_1 = -\frac{1}{n-s+1}\alpha_3, \quad x_2 = 0, \ldots;$$

finally, we obtain the expansion

$$\alpha = \alpha_0 + \theta \wedge \alpha_1 + \theta^+ \wedge \alpha_2 + \theta^+ \wedge \theta \wedge \alpha_3 - \frac{1}{n-s+1} \top'\alpha_3,$$

in which all the forms α_i are effective.

In particular, for effective forms $\omega_1 \in \Lambda^{n-k}(E^*)$ $(i = 1, 2)$ the following expansion holds:

$$\omega_i = \omega_{0i} + \theta \wedge \omega_{1i} + \theta^+ \wedge \omega_{2i} + \theta^+ \wedge \theta \wedge \omega_{3i} - \frac{1}{k+1} \top'\omega_{3i}.$$

Let us now return to the conditions of the theorem. We construct isotropic subspaces annihilating the forms ω_i in the following way. We choose an arbitrary set of vectors $z_1, \ldots, z_{n-k-2} \in E'$ that are linearly independent and in involution: $\Omega'(z_i, z_j) = 0$, and add to it vectors $u + X_\theta$, $v + X_{\theta^+}$, $u, v \in E'$ in such a way that the vectors $u + X_\theta$, $v + X_{\theta^+}$, $z_1, \ldots, z_{n-k-2}$ are in involution. To do this it is enough to require that

$$\Omega'(z_i, u) = \Omega'(z_i, v) = 0, \text{ and } \Omega'(u, v) = 1.$$

Next, elementary computations show that

$$(u+X_\theta)\wedge(v+X_{\theta^+})\wedge z\,\lrcorner\,\omega_i=u\wedge v\wedge z\,\lrcorner\,\omega_{0i}+\\ +v\wedge z\,\lrcorner\,\omega_{2i}+u\wedge z\,\lrcorner\,\omega_{1i}-\frac{k+2}{k+1}\,z\,\lrcorner\,\omega_{3i},$$

where $z=z_1\wedge\ldots\wedge z_{n-k-2}$.

So we find that for a fixed vector v the u-linear equation

$$u\,\lrcorner\,[v\wedge z\,\lrcorner\,\omega_{02}+z\,\lrcorner\,\omega_{12}]=\frac{k+2}{k+1}z\,\lrcorner\,\omega_{32}-v\wedge z\,\lrcorner\,\omega_{22} \tag{1.6.1}$$

is satisfied whenever

$$\begin{cases} u\,\lrcorner\,[v\wedge z\,\lrcorner\,\omega_{01}+z\,\lrcorner\,\omega_{11}]=\dfrac{k+2}{k+1}z\,\lrcorner\,\omega_{31}-v\wedge z\,\lrcorner\,\omega_{21},\\ u\,\lrcorner\,(v\,\lrcorner\,\Omega')=-1,\quad u\,\lrcorner\,(z_i\,\lrcorner\,\Omega')=0. \end{cases} \tag{1.6.2}$$

Hence, the forms

$$\theta_0=v\wedge z\,\lrcorner\,\omega_{02}+z\,\lrcorner\,\omega_{12},\quad \theta_1=v\wedge z\,\lrcorner\,\omega_{01}+z\,\lrcorner\,\omega_{11},$$
$$\theta_2=\Gamma'(v),\ \ \theta_{i+2}=F'(z_i)\qquad (i=1,\ldots,n-k-2)$$

must be linearly dependent for any choice of vectors $v, z_1, \ldots, z_{n-k-2}$ in involution. Therefore,

$$\theta_0\wedge\theta_1\wedge\theta_2\wedge\Pi=0,\quad\text{where }\Pi=\theta_3\wedge\ldots\wedge\theta_{n-k}. \tag{1.6.3}$$

We consider (1.6.3) as an equation for v:

$$(v\wedge z\,\lrcorner\,\omega_{02})\wedge(v\wedge z\,\lrcorner\,\omega_{01})\wedge\Gamma'(v)\wedge\Pi+\\ +[(v\wedge z\,\lrcorner\,\omega_{02})\wedge(z\,\lrcorner\,\omega_{11})+(z\,\lrcorner\,\omega_{12})\wedge(v\wedge z\,\lrcorner\,\omega_{01})]\wedge\Gamma'(v)\wedge\Pi+\\ +(z\,\lrcorner\,\omega_{12})\wedge(z\,\lrcorner\,\omega_{11})\wedge\Gamma'(v)\wedge\Pi=0, \tag{1.6.4}$$

which for a fixed involutory set $z_1, \ldots, z_{n-k-2}$ must be satisfied for all vectors v for which $\Omega'(z_i, v)=0$ $(i=1,\ldots,n-k-2)$. In what follows we assume that $\omega_{11}\neq 0$ and $z\,\lrcorner\,\omega_{11}\neq 0$; otherwise the proof is simpler.

In (1.6.4) we replace v by tv, differentiate the resulting form with respect to t, and set $t=0$. We find that

$$(z\,\lrcorner\,\omega_{12})\wedge(z\,\lrcorner\,\omega_{11})\wedge\Gamma'(v)\wedge\Pi=0 \tag{1.6.5}$$

for all v, $\Omega'(v, z_i)=0$.

Hence, the restriction of $z\,\lrcorner\,\omega_{12}$ and $z\,\lrcorner\,\omega_{11}$ to $\cap\operatorname{Ker}\Gamma'(z_i)$ for any involutory set $z_1, \ldots, z_{n-k-2}$ leads to linearly dependent forms. This, in particular, means that ω_{12} vanishes on all isotropic subspaces that annihilate ω_{11}, hence, by the inductive hypothesis, $\omega_{12}=\lambda\omega_{11}$.

We now isolate in (1.6.4) the terms of second order in v. To do this we replace v by tv, differentiate the resulting form twice with respect to t, and set $t=0$. We find that

$$[v\wedge z\,\lrcorner\,\omega_{02}-\lambda v\wedge z\,\lrcorner\,\omega_{01}]\wedge(z\,\lrcorner\,\omega_{11})\wedge\Gamma'(v)\wedge\Pi=0, \tag{1.6.6}$$

and thus on $\cap$ Ker $\Gamma'(z_i)$ the forms $v \lrcorner [z]\ (\omega_{02} - \lambda\omega_{01})]$ and $\Gamma'(V)$ are linearly dependent. Therefore, $\omega_{02} - \lambda\omega_{01}$ vanishes on all Lagrangian subspaces, and since $\omega_{02} - \lambda\omega_{01}$ is effective, we see that $\omega_{02} - \lambda\omega_{01} = 0$.

Similar arguments with v replaced by u prove that $\omega_{12} = \lambda'\omega_{21}$. Hence, if $\omega_{01} \neq 0$, $\lambda = \lambda'$, and the left-hand sides of the first two equations in the system (1.6.1)–(1.6.2) are proportional. Consequently,

$$z \lrcorner (\omega_{32} - \lambda\omega_{31}) = 0,$$

for any involutory set $z_1, \ldots, z_{n-k-2}$, and since $\omega_{32} - \lambda\omega_{31}$ is effective, $\omega_{32} = \lambda\omega_{31}$ and $\omega_3 = \lambda\omega_1$. When $\omega_{01} = 0$, then

$$\frac{k+2}{k+1} z \lrcorner (\omega_{31} - \lambda\omega_{31}) - v \wedge z \lrcorner (\lambda - \lambda')\,\omega_{21} = 0,$$

and it follows that

$$(\lambda - \lambda') z \lrcorner \omega_{21} = 0$$

so that either $\omega_{21} = 0$ or $\lambda = \lambda'$.■

The following assertion [3] follows directly from the theorem just proved.

COROLLARY 1.6.1 (LEPAGE'S THEOREM). *The form $\omega \in \Lambda^s(E^*)$ lies in the image of $\top$ if and only if $\omega|_L = 0$ for any isotropic subspace $L \subset E$ with* dim $L = n$.

1.7. THEOREM. *Let $\omega_2 \in \Lambda^s(E^*)$ be such that $\omega_2|_L = 0$ for each isotropic subspace $L \subset E$ with* dim $L = k$ *on which $\omega_1|_L = 0$. If the form $\omega_1 \in \Lambda^k(E^*)$ is effective and $k > s$, then ω_2 lies in the image of $\top$.*

■ We show that $\omega_2|_L = 0$ for any isotropic subspace $L \subset E$. For if this were not the case, we could select an involutive $(k-1)$-vector

$$z = z_1 \wedge \ldots \wedge z_{k-1}$$

in such a way that $(z_1 \wedge \ldots \wedge z_s) \lrcorner \omega_2 \neq 0$. Next we choose as z_k linearly independent solutions of the system of linear equations specified by the 1-forms $\Gamma(z_1), \ldots, \Gamma(z_{k-1})$, $(z_1 \wedge \ldots \wedge z_{k-1}) \lrcorner \omega_1$. Then the subspace L spanned by the vectors $z_1, \ldots, z_k$ is such that $\omega_1|_L = 0$, but $\omega_2|_L \neq 0$.■

§2. Differential forms on $J'M$

2.1. Let $\Phi = J'M$ be the space of 1-jets of smooth functions on M. We recall ([1], [15]) that there is on Φ a universal 1-form $U_1 \in \Lambda'(\Phi)$, which defines the contact structure. Thus, at each point $x \in \Phi$ the restriction $dU_{1,x}$ determines on

$$E_x = \text{Ker } U_{1,x} \subset T_x(\Phi)$$

a symplectic structure, and together with it the operators

$$\top\colon \Lambda^s(E_x^*) \to \Lambda^{s+2}(E_x^*) \text{ and } \perp\colon \Lambda^s(E_x^*) \to \Lambda^{s-2}(E_x^*).$$

The tangent space $T_x(\Phi)$ splits into a direct sum:

$$T_x(\Phi) = E_x \oplus \mathbf{R}X_{1,x},$$

where X_1 is the contact vector field on Φ with generating function 1 (see [4]). Hence, if we denote by $\Lambda^s(E^*)$ the differential s-forms on Φ that vanish along X_1, then, firstly, $\Lambda^s_x(E^*)$ is naturally identified with $\Lambda^s(E^*_x)$, and, secondly,

$$\Lambda^s(\Phi) \rightleftarrows \Lambda^s(E^*) \oplus \Lambda^{s-1}(E^*).$$

Formally, this isomorphism can be expressed as the relation

$$\omega \leftrightharpoons \omega_0 + U_1 \wedge \omega_1,$$

where $\omega \in \Lambda^s(\Phi)$, $\omega_0 = \Lambda^s(E^*)$, $\omega_1 \in \Lambda^{s-1}(E^*)$, and

$$\omega_1 = X_1 \lrcorner\, \omega, \quad \omega_0 = \omega - U_1 \wedge \omega_1.$$

We define a projection $p\colon \Lambda^s(\Phi) \to \Lambda^s(E^*)$ by setting $p(\omega) = \omega_0$ and an operator $d_p : \Lambda^s(E^*) \to \Lambda^{s+1}(E^*)$, $d_p = p \circ d$.

PROPOSITION. *The operator d_p satisfies the following relations:*

(1) $d_p(\lambda_1\omega_1 + \lambda_2\omega_1') = \lambda_1 d_p(\omega_1) + \lambda_2 d_p(\omega_1'),$ $\quad \lambda_1, \lambda_2 \in \mathbb{R}, \quad \omega_1, \omega_1' \in \Lambda^s(E^*),$

(2) $d_p(\omega_1 \wedge \omega_2) = d_p(\omega_1) \wedge \omega_2 + (-1)^s \omega_1 \wedge d_p(\omega_2), \quad \omega_2 \in \Lambda^k(E^*),$

(3) $d_p^2 + \top \circ L_1 = 0,$

(4) $d_p \circ d = 0,$

(5) $\top \circ d_p = d_p \circ \top,$

(6) $i_f(dU_1) = -d_p f,$

where L_f and i_f denote the operators of Lie derivation and of interior multiplication on the contact vector field X_f (see [15]).

Using the properties of d_p it is easy to calculate its action in the special local coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ on Φ:

$$d_p\Big(\sum_{I,J} f_{I,J}(q,u,p)\Big)\, dq_I \wedge dp_J = \sum_{I,J} d_p(f_{I,J}) \wedge dq_I \wedge dp_J,$$

where

$$d_p(f_{I,J}) = \sum_{k=1}^{n} \left[\frac{\partial f_{I,J}}{\partial p_k}\, dp_k + \left(\frac{\partial f_{I,J}}{\partial q_k} + p_k \frac{\partial f_{I,J}}{\partial u} \right) dq_k \right],$$

here I and J are multi-indices and $|I| + |J| = s$ if

$$\omega = \sum_{I,J} f_{I,J}(q,u,p)\, dq_I \wedge dp_J \in \Lambda^s(E^*).$$

2.2. We denote by $C^* \subset \Lambda^*(\Phi)$ the ideal consisting of the differential forms that vanish on all R-manifolds (that is, integral manifolds of U_1 of maximal dimension). From Corollary 1.6.1 it follows that this ideal consists of differential forms of the kind $U_1 \wedge \alpha + dU_1 \wedge \beta$.

Let $C^* = \bigoplus_{s_1+1 \geqslant s \geqslant 0} C^s$ be the decomposition into homogeneous components. Then $C^0 = 0$ and $C^s = \Lambda^s(\Phi)$ if $s \geqslant n+1$, while the elements of the factor module $\Lambda^s_\varepsilon = \Lambda^s(\Phi)/C^s$, by virtue of the Hodge–Lepage expansion, can be identified with effective s-forms on E for $s \leqslant n = \dim M$.

LEMMA. *Any form $\omega \in C^s$ has the decomposition*

$$\omega = U_1 \wedge \omega_1 + dU_1 \wedge \omega_2,$$

where $\omega_1 \in \Lambda^{s-1}(E^*)$ is uniquely determined, and $\omega_2 \in \Lambda^{s-2}(E^*)$ is uniquely determined if $s < n + 2$. Furthermore,

$$d\omega = -U_1 \wedge d_p(\omega_1 + d_p\omega_2) + dU_1 \wedge (\omega_1 + d_p\omega_2).$$

■ For $\omega = U_1 \wedge i_1(\omega) + p(\omega)$, but since $\omega \in C^s$, we see that $p(\omega)$ vanishes on R-manifolds and $p(\omega) = \top\omega_2$ by Corollary 1.6.1, where ω_2 is uniquely determined if $s - 2 < n$ (Proposition 1.2). Moreover, if $\omega = U_1 \wedge \omega_1 + dU_1 \wedge \omega_2$, then $d\omega = -U_1 \wedge d\omega_1 + dU_1 \wedge (\omega_1 + d\omega_2)$; replacing $d\omega_j$ by $d_p(\omega_j) + U_1 \wedge i_1(d\omega_j)$, we obtain $d\omega = -U_1 \wedge (d_p\omega_1 - i_1\, d\omega_2) + dU_1 \wedge (\omega_1 + d_p\omega_2)$ and so the required formula follows from Proposition 2.1(3).

2.3. THEOREM. *The cohomology of the complex*

$$(C)\colon\ 0 \to C^1 \xrightarrow{d} C^2 \to \ldots \to C^s \xrightarrow{d} C^{s+1} \to \ldots \to C^{2n+1} \to 0$$

is trivial for all s except $s = n + 1$, where $n = \dim M$.

■ If $\omega = U_1 \wedge \omega_1 + dU_1 \wedge \omega_2$ is a closed form, then, by the preceding lemma

$$d_p\,(\omega_1 + d_p\omega_2) = 0, \quad \top\,(\omega_1 + d_p\omega_2) = 0.$$

Hence, if $\omega \in C^s$ and $s < n + 1$, then it follows from the second relation that $\omega_1 + d_p\,\omega_2 = 0$, and from Lemma 2.2. that

$$\omega = d(U_1 \wedge \omega_2). \ ■$$

2.4. We consider the factor complex

$$\bar{d}_s\colon\ \Lambda^s(\Phi)/C^s \to \Lambda^{s+1}(\Phi)/C^{s+1},$$

which we shall identify with the complex of effective forms on E;

$$\bar{d}_s\colon\ \Lambda^s_\varepsilon \to \Lambda^{s+1}_\varepsilon.$$

Since (C) is exact in dimensions other than $n + 1$, the cohomology H^s_ε of the complex of effective forms is isomorphic to the de Rham cohomology of M;

$$\lambda_s\colon\ H^s_\varepsilon \rightleftarrows H^s(M) \qquad \text{if} \qquad s \neq n.$$

THEOREM. *A coset $\omega + C^n$ of an effective form $\omega \in \Lambda^n_\varepsilon$ contains a closed form if and only if $\mathscr{E}(\omega) = 0$, where*

$$\mathscr{E} = L_1 + d_p \circ \perp \circ d_p\colon\ \Lambda^n_\varepsilon \to \Lambda^n_\varepsilon.$$

■ Using the fact that $\top\colon \Lambda^{n-1}(E^*) \to \Lambda^{n+1}(E^*)$ is an isomorphism, we can write $d_p\,\omega = \top\tilde{\omega}$.

To find $\tilde{\omega}$ we note that it is effective.

For

$$\top_2\tilde{\omega} = \frac{1}{2}\top d_p\omega = \frac{1}{2} d_p \top \omega = 0.$$

Hence

$$\perp \circ d_p(\omega) = \perp \circ \top(\tilde{\omega}) = [\perp, \top](\tilde{\omega}) = \tilde{\omega}.$$

Next, let $x \in \Lambda^{n-1}(E^*)$ and $y \in \Lambda^{n-2}(E^*)$ be such that

$$d(\omega + U_1 \wedge x + dU_1 \wedge y) = 0.$$

Using Lemma 2.2 we obtain

$$d\omega - U_1 \wedge d_p(x + d_p y) + dU_1 \wedge (x + d_p y) = 0,$$

and since $d\omega = d_p\omega + U_1 \wedge L_1\omega$,

$$U_1 \wedge (L_1\omega - d_p(x + d_p y)) + dU_1 \wedge (\tilde{\omega} + x + d_p y) = 0.$$

Thus, the coset $\omega + C^n$ contains a closed form if and only if the following system is soluble:

$$L_1\omega = d_p(x + d_p y), \qquad \tilde{\omega} = -x - d_p y.$$

A necessary condition for the solubility of this system is that $L_1\omega + d_p(\tilde{\omega}) = 0$, that is $\mathscr{E}(\omega) = 0$. We claim that this condition is also sufficient. For if $\mathscr{E}(\omega) = 0$, then taking $y = 0$ and $x = -\tilde{\omega}$ we obtain $d(\omega - U_1 \wedge \tilde{\omega}) = 0$.■

COROLLARY 2.4.1. *Let ω be an effective form such that $\mathscr{E}(\omega) = 0$; then*

$$d(\omega - U_1 \wedge \perp d_p\omega) = 0.$$

COROLLARY 2.4.2. *The sequence*

$$0 \to \mathrm{Ker}/\mathrm{Im} d \to H^n_\varepsilon \to \Lambda^n_\varepsilon/\mathrm{Ker}\,\mathscr{E} \to 0$$

is exact.

2.5. DEFINITION. The differential operator $\mathscr{E}\colon \Lambda^n_\varepsilon \to \Lambda^n_\varepsilon$ is called the *Euler operator*, and effective forms $\omega \in \mathrm{Ker}\,\mathscr{E}$ are called *divergent forms*.

2.6. THEOREM. *The Euler operator satisfies the following relations*:

(1) $\mathscr{E} \circ L_1 = L_1 \circ \mathscr{E}$,

(2) $d_p \circ \mathscr{E} = 0$,

(3) $\mathscr{E}^2 = L_1 \circ \mathscr{E}$,

(4) $\mathscr{E}(f\omega) = f\mathscr{E}(\omega) + X_1(f)\omega + d_p f \wedge \perp d_p\omega + d_p i_f\omega;\ \omega \in \Lambda^n_\varepsilon$.

■ (2) $d_p \circ \mathscr{E} = d_p \circ L_1 + d_p^2 \circ \perp \circ d_p = d_p \circ L_1 - \top \perp \circ d_p \circ L_1 =$

$$= d_p \circ L_1 + [\perp, \top] \circ d_p \circ L_1 - \perp \circ \top \circ d_p \circ L_1 =$$

$$= d_p \circ L_1 - d_p \circ L_1 - \perp \circ d_p \circ L_1 \circ \top = 0$$

(3) $d_p \circ \perp \circ d_p \circ d_p \circ \perp \circ d_p = -L_1 \circ d_p \circ \perp \circ \top \circ \perp \circ d_p =$

$$= -L_1 \circ d_p \circ [\perp, \top] \circ \perp \circ d_p = -L_1 \circ d_p \circ \perp \circ d_p,$$

therefore $\mathscr{E}^2 = L_1^2 + 2L_1 \circ d_p \circ \perp \circ d_p - L_1 \circ d_p \circ \perp \circ d_p = L_1 \circ \mathscr{E}$.

(4) $d_p \circ \perp \circ d_p(f\omega) = d_p \circ \perp(d_p f \wedge \omega + f d_p\omega) = d_p i_f\omega +$

$$+ d_p(f \perp d_p\omega) = d_p i_f\omega + d_p f \wedge \perp d_p\omega + f d_p \perp d_p\omega. \ ■$$

2.7. PROPOSITION. *Let* $\omega \in \Lambda^n_\varepsilon$. *Then*

$$p(L_f\omega) = f\mathscr{E}(\omega) + d_p(i_f\omega - f \perp d_p\omega) + \top (i_f \perp d_p\omega).$$

■ Using the infinitesimal Stokes formula $L_f\omega = i_f\, d\omega + di_f\omega$ we obtain

$$\begin{aligned} p(L_f\omega) &= d_p i_f\omega + p(i_f\, d\omega) = d_p i_f\omega + i_f d_p\omega + fL_1\omega = \\ &= f\mathscr{E}(\omega) - fd_p \circ \perp \circ d_p(\omega) + d_p i_f\omega + i_f d_p\omega = \\ &= f\mathscr{E}(\omega) + d_p(i_f\omega - f \perp d_p\omega) + i_f d_p\omega + d_p f \wedge \perp d_p\omega. \end{aligned}$$

But (see 2.4) $d_p\omega \top \perp d_p\omega$, hence

$$i_f d_p\omega = -d_p f \wedge \perp d_p\omega + \top(i_f \perp d_p\omega).\ ■$$

§3. Non-linear differential operators

3.1. For each differential n-form $\omega \in \Lambda^n(\Phi)$ we define a (non-linear) differential operator $\Delta_\omega : C^\infty(M) \to \Lambda^n(M)$ acting according to the following rule:

$$\Delta_\omega(h) = \sigma^*_{j_1(h)}(\omega),$$

where $\sigma_{j_1}(h): M \to \Phi$ is the section determined by the function $h \in C^\infty(M)$

$$\sigma_{j_1(h)}:\ x \to j_1(h)|_x.$$

First of all, we note that two differential forms $\omega_1, \omega_2 \in \Lambda^n(\Phi)$ determine the same operator if and only if $\omega_1 - \omega_2 \in C^n$. Thus, Δ_ω is uniquely determined by the effective part $p(\omega)$. Bearing this in mind we assume in what follows that the operators Δ_ω are specified by effective forms $\omega \in \Lambda^n_\varepsilon$.

EXAMPLE (1). Let Δ: $C^\infty(M) \to C^\infty(M)$ be a linear differential operator of order $\leqslant 2$, and let $\Omega_0 \in \Lambda^n(M)$ be the volume form. We define an operator $\bar{\Delta}$: $C^\infty(M) \to \Lambda^n(M)$ by setting $\bar{\Delta}(h) = \bar{\Delta}(h)\Omega_0$. We claim that there is a unique effective form $\omega \in \Lambda^n_\varepsilon$, such that $\bar{\Delta} = \Delta_\omega$. Uniqueness follows from what has been said above, and existence can be proved by using local coordinates and the relation

$$\frac{\partial^2 h}{\partial q_i\, \partial q_j} dq_1 \wedge \ldots \wedge dq_n = \sigma^*_{j_1(h)}(dq_1 \wedge \ldots \wedge dq_{i-1} \wedge dp_j \wedge dq_{i+1} \wedge \ldots \wedge dq_n).$$

EXAMPLE (2) Let g be a metric on an orientable manifold M, Ω_g the volume form defined by this metric, and $H \in C^\infty(T^*M)$ the Hamiltonian of the metric. The form

$$\omega = di_H(\pi_1^*\Omega_g),$$

where $\pi_1 : \Phi \to M$ is the natural projection, determines the Laplace operator in the sense that $\Delta_\omega(h) = \Delta(h)\Omega_g$, where Δ: $C^\infty(M) \to C^\infty(M)$ is the Laplace operator for the metric g.

3.2. DEFINITION. Operators of the form $\Delta_\omega : C^\infty(M) \to \Lambda^n(M)$, $\omega \in \Lambda^n_\varepsilon$, are called *Monge–Ampère operators*.

3.3. To justify this definition we express the operator Δ_ω in a local coordinate system.

Let $q_1, \ldots, q_n$ be a local coordinate system on M, let $q_1, \ldots, q_n, p_1, \ldots, p_n$ be the corresponding coordinate system in Φ, and suppose that $\omega \in \Lambda^n(E^*)$ is written as

$$\omega = \sum_{I, J} \omega_{I, J}(q, u, p)\, dq_I \wedge dp_J, \qquad |I| + |J| = n.$$

Since

$$\sigma_{j_1(n)}\colon (q_1, \ldots, q_n) \longmapsto \left(q_1, \ldots, q_n, u = h(q), p_1 = \frac{\partial h}{\partial q_1}, \ldots, p_n = \frac{\partial h}{\partial q_n}\right),$$

We see that

$$\Delta_\omega(h) = \left[\sum_{I, J} \omega_{I, J}\left(q, h(q), \frac{\partial h}{\partial q}\right) \det\left|\frac{\partial^2 h}{\partial q_J\, \partial q_{\bar{I}}}\right|\right] dq_1 \wedge \ldots \wedge dq_n,$$

where the multi-index $\bar{I}$ complementary to I is chosen in such a way that

$$dq_I \wedge dq_{\bar{I}} = dq_1 \wedge \ldots \wedge dq_n.$$

3.4. We consider the differential equation determined by Δ_ω.

The (ordinary) solutions of this equation are smooth functions $h \in C^\infty(M)$ such that $\Delta_\omega(h) = 0$. Geometrically, such solutions can be regarded as R-manifolds $L = \sigma_{j_1(h)}(M) \subset \Phi$ of a special form, which are integrals for ω, because

$$\omega|_L = \sigma^*_{j_1(h)}(\omega) = \Delta_\omega(h) = 0.$$

Using this interpretation, we extend the class of solutions by including in it all R-manifolds that are simultaneously integral manifolds of ω.

3.5. Proceeding as in [15] to determine the symmetries of first order equations, we define an action of the group $Ct(\Phi)$ of contact diffeomorphisms of Φ on the Monge–Ampère operators:

$$\alpha(\Delta_\omega) = \Delta_{\alpha^*(\omega)}, \qquad \alpha \in Ct(\Phi).$$

The action of the algebra $ct(\Phi)$ of contact vector fields on Φ is defined similarly:

$$X_f(\Delta_\omega) = \Delta_{L_f(\omega)}, \qquad X_f \in ct(\Phi), \qquad f \in C^\infty(\Phi).$$

We single out the symmetry group of the operator

$$\mathrm{Sym}(\Delta_\omega) = \{\alpha \in Ct(\Phi) \mid \alpha(\Delta_\omega) = \Delta_\omega\}$$

and of the equation

$$\mathrm{Symc}(\Delta_\omega) = \{\alpha \in Ct(\Phi) \mid \alpha(\Delta_\omega) = h_\alpha \Delta_\omega, \quad h_\alpha \in C^\infty(\Phi)\}.$$

We denote the corresponding algebras by

$$\mathrm{symc}(\Delta_\omega) = \{f \in C^\infty(\Phi) \mid X_f(\Delta_\omega) = h_f \cdot \Delta_\omega, \ f_f \in C^\infty(\Phi)\},$$

$$\mathrm{sym}(\Delta_\omega) = \{f \in C^\infty(\Phi) \mid X_f(\Delta_\omega) = 0\}.$$

PROPOSITION. *A function f lies in* $\mathrm{symc}(\Delta_\omega)$, $\omega \in \Lambda^n_\varepsilon$, *if and only if*

$$(3.5.1) \qquad h\omega + d_p i_f \omega + i_f d_p \omega + f L_1 \omega = 0$$

for some function $h \in C^\infty(\Phi)$.

■ We note that if $\omega \in \Lambda^n_\varepsilon$, then $p\,(L_f\omega) \in \Lambda^n_\varepsilon$, therefore, the statement to be proved follows from the fact that

$$p(L_f\omega) = i_f d_p\omega + d_p i_f\omega + fL_1\omega. \blacksquare$$

EXAMPLE. As an illustration of the use of contact transformations in non-linear second-order equations we give a proof of a theorem of Jörgens [7], which states that every function $h_0(q_1, q_2)$ defined on the whole plane and satisfying the Monge–Ampère equation

$$\frac{\partial^2 h_0}{\partial q_1^2}\cdot\frac{\partial^2 h_0}{\partial q_2^2}-\left[\frac{\partial^2 h_0}{\partial q_1\,\partial q_2}\right]^2-1=0 \tag{3.5.2}$$

is a second-order polynomial.

To prove this theorem we note that under the contact transformation

$$\alpha\colon J^1(\mathbf{R}^2)\to J^1(\mathbf{R}^2),\quad \alpha^*(q_1)=p_1,\quad \alpha^*(q_2)=q_2,\quad \alpha^*(p_1)=-q_1,$$
$$\alpha^*(p_2)=p_2,\quad \alpha^*(u)=u-p_1q_1,\quad \alpha^*(q_2)=q$$

the form

$$\omega = dp_1\wedge dp_2 - dq_1\wedge dq_2, \tag{3.5.3}$$

representing the equation (3.5.2) goes over into the form

$$\alpha^*(\omega) = dp_2\wedge dq_1 + dq_2\wedge dp_1, \tag{3.5.4}$$

representing the Laplace operator

$$\alpha(\Delta_\omega)(h) = -\left(\frac{\partial^2 h}{\partial q_1^2}+\frac{\partial^2 h}{\partial q_2^2}\right)dq_1\wedge dq_2.$$

Moreover, let $L=\sigma_{j_1(h_0)}(\mathbf{R}^2)$ be the solution of (3.5.2); then $\alpha(L)$ is a solution of the Laplace equation, and the R-manifold of $\alpha(L)$ is projected without singularities onto the (q_1, q_2)-plane and, therefore, has the form $\sigma_{j_1(h)}(\mathbf{R}^2)$, where $h(q_1, q_2)$ is a harmonic function. For otherwise a vector of the form

$$\lambda_1\frac{\partial}{\partial p_1}+\lambda_2\frac{\partial}{\partial p_2}$$

would touch $\alpha(L)$, hence, the vector

$$-\lambda_1\left(\frac{\partial}{\partial q_1}+p_1\frac{\partial}{\partial u}\right)+\lambda_2\frac{\partial}{\partial p_2}$$

would touch L. But this is impossible, since then at the point of contact we would have

$$\begin{pmatrix}\dfrac{\partial^2 h_0}{\partial q_1} & \dfrac{\partial^2 h_0}{\partial q_1\,\partial q_2}\\[2mm] \dfrac{\partial^2 h_0}{\partial q_1\,\partial q_2} & \dfrac{\partial h_0}{\partial q_2^2}\end{pmatrix}\begin{pmatrix}-\lambda_1\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \lambda_2\end{pmatrix},\qquad \lambda_1^2+\lambda_2^2\neq 0$$

for the solution h_0 of (3.5.2). Next, observe that the second derivatives $\partial^2 h/\partial q_1^2$ and $\partial^2 h/\partial q_2^2$ of h, which determine $\alpha(L)$, are non-zero.

For if at a certain point

$$\frac{\partial^2 h}{\partial q_1^2}=-\frac{\partial^2 h}{\partial q_2^2}=0,\quad\text{and}\quad \frac{\partial^2 h}{\partial q_1\,\partial q_2}=b,$$

then at this point the vector

$$\left(\frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial u}\right) + b \frac{\partial}{\partial p_2}$$

touches $\alpha(L)$, and so

$$-\frac{\partial}{\partial p_1} + b \frac{\partial}{\partial p_2}$$

touches L, which is impossible.

But the functions $\partial^2 h/\partial q_1^2$ and $\partial^2 h/\partial q_2^2$ are harmonic, hence, by Liouville's theorem,

$$\frac{\partial^2 h}{\partial q_1^2} = -\frac{\partial^2 h}{\partial q_2^2} = \text{const} \neq 0.$$

Thus,

$$\frac{\partial^2 h}{\partial q_1 \partial q_2} = \text{const}$$

and $h(q_1, q_2)$, like $h_0(q_1, q_2)$, is a polynomial of the second degree.

3.6. Let $f \in \operatorname{symc}(\Delta_\omega)$ and $A_t \colon \Phi \to \Phi$ be a one-parameter group of shifts along the contact vector field X_f. Then from the definition of the algebra $\operatorname{symc}(\Delta_\omega)$ it follows that $A_t(L)$ is a solution whenever the R-manifold $L \subset \Phi$ is one, where $\omega|_L = 0$.

Therefore, from a known solution L and a symmetry f we can construct a one-parameter system of solutions $L_t = A_t(L)$.

We now consider the special case when $L_t = L$, that is, when L is invariant under the contact vector field X_f. A condition equivalent to the invariance of L, but more easily verifiable, is that L is a solution of the second order equation corresponding to f, $f|_L = 0$ (see [15]).

DEFINITION. A solution $L \subset \Phi$ is said to be an *f-automodel* solution if L is invariant under the vector field X_f ($f|_L = 0$).

We describe a scheme for construction f-automodel solutions. Let us assume that on the level surface $\{f = 0\} \subset J'(M)$ we can choose a hypersurface $\Gamma \subset \{f = 0\}$ that is transversal to X_f and such that the trajectories of X_f intersect Γ at only one point.

Then every f-automodel solution L, $\omega|_L = 0$, being invariant under X_f ($f|_L = 0$), is uniquely determined by its trace $L_\Gamma = L \cap \Gamma$ on Γ. The manifold L_Γ, $\dim L_\Gamma = n - 1$, is simultaneously integral for the 1-form $U_1|_\Gamma$ and the $(n-1)$-form $i_f(\omega)|_\Gamma$. When there are no singularities of the equation $\{f = 0\}$ on Γ (see [15]), then $U_1|_\Gamma$ defines a contact structure on Γ, therefore, if we choose an $(n-1)$-dimensional manifold M_1 in such a way that $J'(M_1)$ is contact-equivalent to Γ (at least locally, which is always possible), then the construction of L_Γ, and together with it that of the f-automodel solution L, reduces to the solution of the equation defined by the form $i_f(\omega)|_\Gamma$ on M.

3.7 DEFINITION. Operators of the form Δ_ω, where ω is a divergent form, are said to be of *divergent type*.

THEOREM. *An operator Δ_ω, $\omega \in \Lambda^n_\varepsilon$ is of divergent type if and only if* $\mathscr{E}(\omega) = 0$.

3.8. DEFINITION. *A conservation law* for an operator Δ_ω is an $(n-1)$-form $\theta \in \Lambda^{n-1}(\Phi)$ whose restriction to an arbitrary solution L, $\omega_L = 0$, is a closed form.

Let us set the goal of computing conservation laws, say θ. Then $d\theta$ vanishes whenever ω vanishes, hence, by Theorem 1.6, $d\theta - g\omega \in C^n$ for some function $g \in C^\infty(\Phi)$, in other words, the operator $\Delta_{g\omega}$ is of divergent type. Thus, $\mathscr{E}(g\omega) = 0$. Conversely, if $\mathscr{E}(g\omega) = 0$, then from Corollary 2.4.1 it follows that the n-form $g\omega - U_1 \wedge \perp d_p(g\omega)$ is closed. Therefore, if there are no topological obstructions, for example, if $H^n(M) = 0$ then the form $g\omega - U_1 \perp d_p(g\omega)$ is exact and so defines a conservation law θ,

$$d\theta = g\omega - U_1 \wedge \perp d_p(g\omega).$$

We also mention that when M is a compact orientable manifold, then the condition for the n-form $g\omega - U_1 \wedge \perp d_p(g\omega)$ to be exact is equivalent to the fact that $\int_M \sigma^*_{j_1(h)}(g\omega)$ vanishes for some function $h \in C^\infty(M)$. We note that this condition is always satisfied if there is at least one smooth solution, $h_0 \in C^\infty(M)$, $\Delta_\omega(h_0) = 0$. For in this case the integrand is zero, because

$$\sigma^*_{j_1(h_0)}(g\omega) = \sigma^*_{j_1(h_0)}(g)\,\Delta_\omega(h_0) = 0.$$

Thus, we have proved the following theorem.

THEOREM. *The conservation laws, to within closed $(n-1)$-forms on Φ, are in one-to-one correspondence with the solutions of the equation $\mathscr{E}(g\omega) = 0$ if either $H^n(M) = 0$ or if there is at least one smooth solution $h_0 \in C^\infty(M)$, $\Delta_\omega(h_0) = 0$. Here, to each function g with $\mathscr{E}(g\omega) = 0$ there corresponds the conservation law θ satisfying the relation*

$$d\theta = g\omega - U_1 \wedge \perp d_p(g\omega).$$

3.9. COROLLARY. *If Δ_ω is of divergent type, then the functions g defining the conservation laws satisfy the equation*

$$d_p i_g \omega + d_p g \wedge \perp d_p\omega + X_1(g)\omega = 0.$$

3.10. We indicate some explicit formulae for constructing conservation laws from solutions of the equation $\mathscr{E}(g\omega) = 0$. To do this we choose a function $u \in C^\infty(\Phi)$ being determined by the composition of the natural projections $u\colon \Phi \to J^0(M) = M \times \mathbf{R} \to \mathbf{R}$, and a solution $h_0 \in C^\infty(M)$, $\sigma^*_{j_1(h_0)}(g\omega) = 0$. Let $A_t\colon \Phi \to \Phi$ be the one parameter group of shifts along the contact vector field X_{u-h_0}. We note that

$$A_t\colon\ (q, u, p) \to \left(q, (u-h)e^t + h, \left(p - \frac{\partial h}{\partial q}\right)e^t + \frac{\partial h}{\partial q}\right).$$

Next, if θ is a conservation law corresponding to the function g, $d\theta = g\omega - U_1 \wedge \perp d_p(g\omega)$, then using the relations

$$\frac{d}{dt}(A_t^*\theta) = A_t^*(L_{u-h_0}\theta)$$

and

$$L_{u-h_0}(\theta) = i_{u-h_0}(d\theta) + d\,(i_{u-h_0}\theta) = g i_{u-h_0}\omega -$$
$$- (u-h_0) \perp d_p(g\omega) + U_1 \wedge i_{u-h_0}[\perp d_p(g\omega)] + d\,(i_{u-h_0}\theta),$$

we obtain

$$\theta - A^*_{-\infty}(\theta) = \int_{-\infty}^{0} \frac{d}{dt} A^*_t(\theta)\,dt =$$
$$= \int_{-\infty}^{0} A^*_t[g i_{u-h_0}\omega - (u-h_0) \perp d_p(g\omega)]\,dt +$$
$$+ U_1 \wedge \int_{-\infty}^{0} e^t A^*_t [i_{u-h_0} \perp d_p(g\omega)]\,dt + d\left(\int_{-\infty}^{0} A^*_t[i_{u-h_0}\theta]\,dt\right).$$

We set

$$\theta_g = \int_{-\infty}^{0} A^*_t[g i_{u-h_0}\omega - (u-h_0) \perp d_p(g\omega)]\,dt. \qquad (3.10.1)$$

Then the form θ_g determines the same conservation law as θ. In fact, for an arbitrary solution h, $\sigma^*_{j_1(h)}(\omega) = 0$, we have

$$\sigma^*_{j_1(h)}(d\theta - d\theta_g) = \sigma^*_{j_1(h)}(A^{*}_{-\infty}\, d\theta) = (A_{-\infty} \circ \sigma_{j_1(h)})^* (d\theta) = \sigma^*_{j_1(h_0)}(d\theta) = 0.$$

3.11. Now we indicate a connection between contact symmetries and conservation laws.

THEOREM. *For every operator Δ_ω such that $\mathscr{E}(g\omega) = \gamma\omega$ for some function $\gamma \in C^\infty(\Phi)$, in particular, for every operator Δ_ω of divergent type ($\gamma = 0$) and for an arbitrary symmetry $f \in$ symc(Δ_ω), the form $\theta = i_f\,\omega - f \perp d_p\,\omega$ is a conservation law.*

■ Suppose that $f \in$ symc(Δ_ω). Then $p(L_f\,\omega) - h_f\,\omega = 0$ for some function $h_f \in C^\infty(\Phi)$. Hence, by Proposition 2.7,

$$f\ \mathscr{E}(\omega) + d_p(i_f\omega - f \perp d_p\omega) + \top\,(i_f \perp d_p\omega) - h_f\omega =$$
$$= (\gamma f - h_f)\,\omega + d_p(i_f\omega - f \perp d_p\omega) + \top\,(i_f \perp d_p\omega) = 0. \ ■$$

3.12. We note that a conservation law $i_f\,\omega - f \perp d_p\,\omega$ is trivial (see Theorem 2.3) if and only if $\gamma f - h_f = 0$. In particular, for operators of divergent type the trivial conservation laws correspond to the symmetries of the operator $f \in$ sym(Δ_ω). On the other hand, Theorem 1.7 asserts that Δ_ω has non-trivial conservation laws in dimension $n-2$, but not below. However, such laws are quite possible if we limit the class of solutions under consideration.

Let $f \in$ sym(Δ_ω) and let Δ_ω be of divergent type. Then $d(i_f\,\omega - f \perp d_p\,\omega) \in C^n$, and since $i_f\,\omega - f \perp d_p\,\omega \in C^n$, this form determines a cohomology class in

$$H^{n-1}_{\mathscr{E}} \xrightarrow{\ \lambda_{n-1}\ } H^{n-1}(M).$$

Let us assume that this class is trivial:

$$\lambda_{n-1}(i_f\omega - f \perp d_p\omega) = 0.$$

Then there is an $(n-2)$-form θ such that

$$d\theta - (i_f\omega - f \perp d_p\omega) \in C^{n-1}.$$

It turns out that the restriction of θ to any f-automodel solution L, $\omega_L = 0$, is closed.

For in this case $i_f(\omega)_L = 0$, since $\omega|_L = 0$, but L is X_f-invariant and $(f \perp d_p\ \omega)|_L = 0$, because $f_L = 0$. Thus, θ determines an $(n-2)$-dimensional conservation law for f-automodel solutions.

THEOREM. *Let* $f \in \mathrm{sym}(\Delta_\omega)$ *and* $\lambda_{n-1}(i_f\omega - f \perp d_p\ \omega) = 0$. *Then an* $(n-2)$*-form* θ *for which*

$$d\theta - (i_f\omega - f \perp d_p\omega) \in C^{n-1}$$

is a conservation law for f-automodel solutions.

3.13. THEOREM. *When* Δ_ω, $\omega \in \Lambda^n_\varepsilon$, *is of divergent type and such that* $d_p\ \omega = 0$, *then there is a one-to-one correspondence between the Lie algebra* $\mathrm{symc}(\Delta_\omega)/\mathrm{sym}(\Delta_\omega)$ *and the conservation laws (to within trivial laws) in which a symmetry* $f \in \mathrm{symc}(\Delta_\omega)$ *corresponds to the conservation law* $\theta = i_f(\omega)$.

■ Suppose that a function $g \in C^\infty(\Phi)$ determines the conservation law θ. Then $\mathscr{E}(g\omega) = 0$, or by Corollary 3.9, $d_p i_g\ \omega + X_1(g)\ \omega = 0$. On the other hand, $p(L_g\ \omega) = d_p(i_g\ \omega)$. Thus, $g \in \mathrm{symc}(\Delta_\omega)$ and $\theta = i_g\ \omega$ to within a trivial conservation law.

EXAMPLE. The Laplace operator for an arbitrary Riemannian manifold satisfies the conditions of this theorem.

3 14. To conclude this section we return to the definition of solutions of the Monge–Ampère equation and broaden the class by including discontinuous solutions.

Let $M_0 \subset M$ be a submanifold of codimension 1, forming the boundary of a domain $O \subset M$. We consider a discontinuous function

$$v = v^+ \cup v^-, \text{ where } v^+ \in C^\infty(\overline{O}), \quad v^- \in C^\infty(M \setminus O).$$

We call such a function a discontinuous solution corresponding to the conservation law θ, $d\theta - \omega \in C^n$, if

$$\int_M dh \wedge \sigma^*_{j_1(v)}(\theta) = 0$$

for any test function $h \in C_0^\infty(M)$.

THEOREM. *A function v is a discontinuous solution corresponding to a conservation law* θ *if and only if* $\Delta_\omega(v) = 0$ *on* $M \setminus M_0$ *and at points of* M_0 *the Giugonio–Rankin equation holds*:

$$\sigma^*_{j_1(v^+)}(\theta)|_{M_0} = \sigma^*_{j_1(v^-)}(\theta)|_{M_0}.$$

■ Suppose first that $h \in C_0^\infty(M)$ is such that $\mathrm{supp}\ h \cap M_0 = \emptyset$. For example, let supp $h \subset 0$. Then from Stokes' theorem it follows that

$$\int_M dh \wedge \sigma^{*!}_{j_1(v)}(\theta) = \int_M d(h\sigma^{*}_{j_1(v^+)}(\theta)) - \int_M h\Delta_\omega(v^+) = -\int_M h\Delta_\omega(v^+) = 0,$$

hence that $\Delta_\omega(v)$ vanishes on $M \setminus M_0$.

Now if supp $h \cap M_0 \neq \emptyset$, then the above calculation leads to the relation

$$\int_M dh \wedge \sigma^{*}_{j_1(v)}(\theta) = \int_{M_0} h|_{M_0}[\sigma^{*}_{j_1(v^+)}(\theta)|_{M_0} - \sigma^{*}_{j_1(v^-)}(\theta)|_{M_0}] = 0. \quad \blacksquare$$

COROLLARY. *Suppose that M_0 is specified by a function $S \in C^\infty(M)$, $M_0 = \{S = 0\}$, and that $dS \neq 0$ at the points of M_0. Then the Giugonio–Rankin equation is equivalent to the following condition:*

$$[\sigma^{*}_{j_1(v^+)}(\theta) - \sigma^{*}_{j_1(v^-)}(\theta)] \wedge dS = 0$$

at the points of M_0.

§4. The use of contact geometry in the calculus of variations

Below we outline an invariant exposition of the calculus of variations, using the apparatus of effective forms developed in the preceding sections.

With each differential n-form $\Omega \in \Lambda^n_\varepsilon$ we associate a functional $\check{\Omega}$ acting according to the following rule:

$$\check{\Omega}(h) = \int_M \sigma^{*}_{j_1(h)}(\Omega), \qquad h \in C_0^\infty(M).$$

The Lagrangian of $\check{\Omega}$, written in local coordinates, contains non-linearities in the second derivatives of the same type as the Monge–Ampère operators.

The following theorem explains why the operator $\mathscr{E}$ of §2 is called the Euler operator.

4.1. THEOREM. *The extremals of the functional $\check{\Omega}$ are solutions of the Euler equation corresponding to the n-form $\omega = \mathscr{E}(\Omega)$.*

■ *Let $h_0 \in C_0^\infty(M)$ be an extremal of $\check{\Omega}$. We fix a function $h_1 \in C_0^\infty(M)$ and consider the function*

$$\varphi(t) = \int_M \sigma^{*}_{j_1(h_0 + th_1)}(\Omega).$$

Then since h_0 is an extremal, $\varphi'(0) = 0$ On the other hand,

$$\varphi'(0) = \int_M \sigma^{*}_{j_1(h_0)}[L_{h_1}\Omega]$$

or, by Proposition 2.7,

$$\varphi'(0) = \int_M \sigma^{*}_{j_1(h_0)}(h_1\mathscr{E}(\Omega)) = \int_M h_1\sigma^{*}_{j_1(h_0)}(\mathscr{E}\Omega).$$

Since h_1 is arbitrary, we obtain

$$\sigma^*_{j_1(h_0)}(\mathscr{E}\Omega) = \Delta_\omega(h_0) = 0. \quad \blacksquare$$

4.2. A contact vector field $X_f \in ct(\Phi)$ is said to be a symmetry of $\check{\Omega}$ if $L_f(\Omega) \in C^n$.

Under the assumption that M is compact this definition can be motivated as follows. Let $A_t\colon \Phi \to \Phi$ be a one-parameter group of shifts along X_f. Then for any function $h \in C^\infty(M)$ and for sufficiently small t, the R-manifolds $L_t = A_t(L)$ and $L = \sigma_{j_1(h)}(M)$ are projected diffeomorphically onto M, hence (see [15]) $L_t = \sigma_{j_1(h_t)}(M)$ for some family $h_t \in C^\infty(M)$.

Moreover, from the fact that $L_f(\Omega) \in C^n$ it follows that $A_t^*(\Omega) - \Omega \in C^n$, consequently,

$$\sigma^*_{j_1(h)}(\Omega) = \sigma^*_{j_1(h)}(A_t^*\Omega) = \sigma^*_{j_1(h_t)}(\Omega),$$

that is, $\check{\Omega}(h_t) = \check{\Omega}(h)$.

4.3. PROPOSITION. *The contact vector field X_f is a symmetry of the variational problem if and only if*

$$d_p i_f \Omega + i_f d_p \Omega + f L_1 \Omega = 0.$$

4 4. THEOREM (Noether). *Suppose that X_f is a symmetry of $\check{\Omega}$. Then the differential $(n-1)$-form $i_f\Omega - f \perp d_p \Omega$ is a conservation law for the Euler equation.*

■ Using Proposition 2.7 we obtain

$$0 = p(L_f\Omega) = f\mathscr{E}(\Omega) + d_p(i_f\Omega - f\perp d_p\Omega) + \top(i_f \perp d_p\Omega),$$

that is,

$$f\omega + d(i_f\Omega - f\perp d_p\Omega) \in C^n. \quad \blacksquare$$

4.5. THEOREM. *Let $L_1(\omega) = 0$ and $\omega = \mathscr{E}(\Omega)$; then every conservation law for the Euler equation is determined by a symmetry of the Euler equation.*

■ We note that the condition for the divergence of Δ_ω is equivalent to $L_1(\omega) = 0$. For $\mathscr{E}(\omega) = \mathscr{E}^2(\Omega) = L_1\mathscr{E}(\Omega) = L_1(\omega)$. Therefore, the assertion follows from Theorem 3.13 and the fact that $d_p \circ \mathscr{E} = 0$. ■

4.6. We consider now the classical case when $\Omega = k\pi_1^*(\Omega_0)$, where $k \in C^\infty(\Phi)$ is the Lagrangian and $\Omega_0 \in \Lambda^n(M)$ is the volume form on M. Then ω has the form

$$\omega = \mathscr{E}(k\pi_1^*\Omega_0) = d_p i_k(\pi_1^*\Omega_0) + X_1(k)\pi_1^*(\Omega_0),$$

and the conservation law corresponding to the symmetry f of $\check{\Omega}$ is

$$\theta = k i_f(\pi_1^*\Omega_0) - f i_k(\pi_1^*\Omega_0).$$

We also write down the equations for finding the symmetries:

$$d_p(k i_f \pi_1^*\Omega_0) + X_f(k)\pi_1^*(\Omega_0) = 0$$

and the conservation laws:

$$d_p i_g \omega + L_1(g\omega) = 0.$$

APPENDIX I

SYMMETRIES, AUTOMODEL SOLUTIONS, AND CONSERVATION LAWS IN NON-LINEAR ACOUSTICS

I.1. The propagation of a bounded three-dimensional sound beam in non-linear media is described in [17] by the Khokhlov–Zabolotskaya equation:

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\rho'}{\partial x}-\frac{\varepsilon}{c_0\rho_0}\rho'\frac{\partial\rho'}{\partial\tau}\right)=\frac{c_0}{2}\left(\frac{\partial^2\rho'}{\partial y^2}+\frac{\partial^2\rho'}{\partial z^2}\right). \tag{I.1.1}$$

Here $\rho' = \rho - \rho_0$ is the deviation from the equilibrium ρ_0 of the density of the medium, c_0 is the velocity of sound in the medium, $\varepsilon = \gamma + 1/2$, γ is the index of the adiabatic curve, $\tau = t - x/c_0$, x is the coordinate in the direction of propagation of the sound beam, and y and z are the transverse coordinates.

If we use the coordinates

$$q_1=\frac{c_0}{\varepsilon}\tau,\quad q_2=x,\quad q_3=(2/\varepsilon)^{1/2}y,\, q_4=(2/\varepsilon)^{1/2}z,$$

then I.1.1. takes the simpler form:

$$\frac{\partial}{\partial q_1}\left(\frac{\partial\rho'}{\partial q_2}-\rho'\frac{\partial\rho'}{\partial q_1}\right)=\frac{\partial^2\rho'}{\partial q_3^2}+\frac{\partial^2\rho'}{\partial q_4^2}. \tag{I.1.2}$$

I.2. We write out the effective form $\omega \in \Lambda^4(\mathbf{R}^9)$ representing the equation (I.1.2):

$$\omega=\frac{1}{2}\,dp_1\wedge dq_1\wedge dq_3\wedge dq_4-\frac{1}{2}\,dp_2\wedge dq_2\wedge dq_3\wedge dq_4+u\,dp_1\wedge dq_2\wedge dq_3\wedge dq_4+$$
$$+p_1^2\,dq_1\wedge dq_2\wedge dq_3\wedge dq_4+$$
$$+dp_3\wedge dq_1\wedge dq_2\wedge dq_4-dp_4\wedge dq_1\wedge dq_2\wedge dq_3.$$

The algebra of symmetries $\operatorname{symc}(\Delta_\omega)$ of the Khokhlov–Zabolotskaya equation can be found from (3.5.1). Solving this equation we see that the generating functions of the symmetries have the form

$$\begin{aligned}(\text{I.2.1})\quad f(q,\,u,\,p) &= [(2\alpha q_2+2c)q_1+\alpha(q_3^2+q_4^2)+\\ &+A'(q_2)q_3+B'(q_2)q_4+K(q_2)]p_1+(3\alpha q_2^2+2\beta q_2+\lambda)p_2+\\ &+[(4\alpha q_2+\beta+c)q_3+\delta q_4+A(q_2)]p_3+[(4\alpha q_2+\beta+c)q_4-\delta q_3+B(q_2)]p_4+\\ &+(4\alpha q_2+2\beta-2c)u+2\alpha q_1+A''(q_2)q_3+B''(q_2)q_4+K'(q_2),\end{aligned}$$

where α, β, δ, c are arbitrary constants and $A(q_2)$, $B(q_2)$, $K(q_2)$ are arbitrary smooth functions.

Thus, as in the case of a bounded two-dimensional beam [6] the algebra of symmetries is infinite-dimensional.

However, if we limit ourselves to physically meaningful solutions $\rho'(q)$ (that is, $\rho'(q) \to 0$ as $r \to \infty$, $r^2 = q_3^2 + q_4^2$) and single out the subalgebra $S \subset \operatorname{symc}(\Delta_\omega)$ of symmetries conserving the class of physical solutions, then the generating functions $f \in S$ have the form (I.2.1), where $\alpha = 0$, $A = A_1 q_2 + A_2$, $B = B_1 q_2 + B_2$, A_1, B_1, A_2, B_2, K are constants, and $\dim S = 9$.

As a basis in S we take the functions

$$f_1 = 2p_2q_2 + p_3q_3 + p_4q_4 + 2u, \quad f_2 = p_2, \quad f_3 = q_4p_3 - q_3p_4,$$
$$f_4 = 2p_1q_1 + p_3q_3 + p_4q_4 - 2u, \quad f_5 = p_1q_3 + p_3q_2, \quad f_6 = p_3,$$
$$f_7 = p_1q_4 + p_4q_2, \quad f_8 = p_4, \quad f_9 = p_1.$$

Here f_2, f_6, f_8, and f_9 correspond to translations in the directions q_2, q_3, q_4, and q_1; f_1 and f_4 are the scaled symmetries; f_3 corresponds to a rotation in the transverse (q_3, q_4)-plane.

The Lie algebra structure in S with respect to the Lagrange bracket is given by the commutation relation:

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
f_1	0	$2f_2$	0	0	$-f_5$	f_6	$-f_7$	f_8	0
f_2	$-2f_2$	0	0	0	$-f_6$	0	$-f_8$	0	0
f_3	0	0	0	0	$-f_7$	$-f_8$	f_5	f_6	0
f_4	0	0	0	0	f_5	f_6	f_7	f_8	$2f_9$
f_5	f_5	f_6	f_7	$-f_5$	0	f_9	0	0	0
f_6	$-f_6$	0	f_8	$-f_6$	$-f_9$	0	0	0	0
f_7	f_7	f_8	$-f_5$	$-f_7$	0	0	0	f_9	0
f_8	$-f_8$	0	$-f_6$	$-f_8$	0	0	$-f_9$	0	0
f_9	0	0	0	$-2f_9$	0	0	0	0	0

From the table it follows that S is soluble. We show how S can be used to construct exact solutions of (I.1.2).

To do this we note that if $\rho(q)$ is a solution of (I.1.2) and $Q_1(t)$, $Q_2(t)$, $Q_3(t)$, $Q_4(t)$ is a solution of the system of ordinary differential equations

$$\text{(I.2.2)} \quad \begin{cases} \dot{Q}_1 = 2cQ_1 + A_1Q_3 + B_1Q_4 + K, & Q_1(0) = q_1, \\ \dot{Q}_2 = 2\beta Q_2 + \lambda, & Q_2(0) = q_3, \\ \dot{Q}_3 = (\beta + c)\, Q_3 + \delta Q_4 + A_1Q_2 + A_2, & Q_3(0) = q_3, \\ \dot{Q}_4 = -\delta Q_3 + (\beta + c)\, Q_4 + B_1Q_2 + B_2, & Q_4(0) = q_4, \end{cases}$$

then for each t, $-\infty < t < \infty$, the function

$$\rho_s(q) = \rho(Q(t)) \cdot \exp[2(\beta - c)t]$$

is displaced by a shift in time t along X_f, where $f \in S$, hence, is a solution of (I.1.2). The initial solution $\rho(q)$ can, for example, be sought in the form

$$\rho(q) = F_1(q_1) \cdot G(r), \quad r = (q_3^2 + q_4^2)^{1/2}.$$

For by substituting the expression for $\rho(q)$ in (I.1.2) and separating the variables we obtain the equations for F and G, respectively, (I.2.3) $$\frac{d}{dq_1}\left(F\frac{dF}{dq_1}\right)=F,$$

(I.2.4) $$\frac{d}{dr}\left(r\frac{dG}{dr}\right)+rG^2=0.$$

(I.2.3) reduces by elementary transformations to an elliptic integral of the second kind

(I.2.5) $$\int F(2F^2+c_1)^{1/2}df=\pm 3^{-1/2}q_1.$$

We note that (I.2.3) has the particular solution $F=(q_1+c)^2/6$. (I.2.4) has a unique solution satisfying the boundary conditions $G(0)=a$, $\lim_{r\to\infty}G(r)=0$, which can be found by using the substitution $G=r^{-2}v(\log r)$, $v'=w(v)$ to reduce the problem to the solution of the second-order equation

(I.2.6) $$w'=4-\frac{v(v+4)}{w}.$$

I.3. Let us describe the conservation laws of (I.2.1) using §§3.8 and 3.10. By solving the equation $\mathscr{E}(g\omega)=0$ we find that a function g defining a conservation law has the form

$$g=a(q_2,q_3,q_4)+q_1b(q_2,q_3,q_4),$$

where $a(q)$ and $b(q)$ satisfy the equations

(I.3.1) $$\frac{\partial^2 b}{\partial q_3^2}+\frac{\partial^2 b}{\partial q_4^2}=0,\qquad \frac{\partial b}{\partial q_2}=\frac{\partial^2 a}{\partial q_3^2}+\frac{\partial^2 a}{\partial q_4^2}.$$

Thus, each conservation law is uniquely determined by a pair of functions $a(q_2,q_3,q_4)$, $b_0(q_3,q_4)$, where $a(q_2,q_3,q_4)$ is biharmonic in q_3 and q_4, and $b_0(q_3,q_4)$ is harmonic. Here

(I.3.2) $$b(q_2,q_3,q_4)=b_0(q_3,q_4)+\int_0^{q_2}\left(\frac{\partial^2 a}{\partial q_3^2}+\frac{\partial^2 a}{\partial q_4^2}\right)dq_2.$$

Evaluating the integral in (3.10.1) we now find the conservation laws

(I.3.3) $$\begin{aligned}\theta_g=&\left(-g\cdot p_4+\frac{\partial p}{\partial q_4}u\right)dq_1\wedge dq_2\wedge dq_3+\\&+\left(g\cdot p_3-\frac{\partial g}{\partial q_3}u\right)dq_1\wedge dq_2\wedge dq_4+gp_1\,dq_1\wedge dq_3\wedge dq_4+\\&+\left(gup_1+\frac{\partial g}{\partial q_2}u-\frac{1}{2}\frac{\partial g}{\partial q_1}u^2\right)dq_2\wedge dq_3\wedge dq_4.\end{aligned}$$

Let us point out certain consequences of (I.3.3). To do this we integrate $\sigma^*_{j_1(p)}(\theta_g)$ around the boundary of the domain D: $0\leqslant q_1\leqslant\tau$, $0\leqslant q_2\leqslant c$ for the following functions g:

(i) if $g=f_\varepsilon(q_2)\lambda_0(q_3,q_4)$, where f_ε is shaped like the delta function and λ_0 is harmonic, then

$$\text{(I.3.4)}\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\rho\frac{\partial\rho}{\partial q_1}-\frac{\partial\rho}{\partial q_2}\right)\Big|_{q_1=0}^{q_1=\tau}\cdot\lambda_0(q_3,\ q_4)\,dq_3\wedge dq_4=0,$$

(ii) if $g=h'_\varepsilon(q_2)\,\lambda_1(q_3,\ q_4)+h_\varepsilon(q_2)\cdot q_1$, where $\partial^2\lambda_1/\partial q_3^2+\partial^2\lambda_1/\partial q_4^2=1$ and $h_\varepsilon(q_2)$ converges (in the sense of generalized functions) to the Heaviside function $\theta(q_2-c)$ then

$$\text{(I.3.5)}\qquad \iint_{-\infty}^{\infty}\left(\rho\frac{\partial\rho}{\partial q_1}-\frac{\partial\rho}{\partial q_2}\right)\Big|_{q_1=0}^{q_1=\tau}\lambda_1(q_3,\ q_4)\,dq_3\wedge dq_4+$$

$$+\iint_{-\infty}^{\infty}\int_0^{\tau}\rho\,dq_1\wedge dq_3\wedge dq_4=0,$$

or, bearing in mind (I.3.4),

$$\text{(I.3.6)}\qquad \iint_{-\infty}^{\infty}\left(\rho\frac{\partial\rho}{\partial q_1}-\frac{\partial\rho}{\partial q_2}\right)\Big|_{q_1=0}^{q_1=\tau}\frac{r^2}{2}\,dq_3\wedge dq_4+\iint_{-\infty}^{\infty}\int_0^{\tau}\rho\,dq_1\wedge dq_3\wedge dq_4=0.$$

I.4. Let us use the conservation laws we have found to describe the propagation of perturbations in non-linear acoustics. We assume that the boundary of the second beam is given by the function $S(q_1,\ q_2,\ q_3,\ q_4)$, that is, $\rho(q)=0$ if $S(q)>0$, $\rho(q)\neq 0$ if $S(q)<0$, and $S(q)=0$ is the propagation law of the boundary of the sound beam.

We choose an arbitrary conservation law θ_g and write out the corresponding Giugonio–Rankin conditions. In our case, independently of the choice of θ_g, they lead to the Hamilton–Jacobi equation for S:

$$\text{(I.4.1)}\qquad \frac{\partial S}{\partial q_1}\cdot\frac{\partial S}{\partial q_2}=\left(\frac{\partial S}{\partial q_3}\right)^2+\left(\frac{\partial S}{\partial q_4}\right)^2.$$

Suppose now, for example, that the disturbance for $q_2=0$ is localized in a circle of radius $r(q_1)$. Then, solving the Cauchy problem for (I.4.1) with the initial conditions

$$S(q_1,\ 0,\ q_3,\ q_4)=q_3^2+q_4^2-r^2(q_1),$$

we find that in the section $q_2=\text{const}>0$ the disturbance is localized in a circle of radius

$$\text{(I.4.2)}\qquad r(q_1,\ q_2)=\left|r(q_1^*)+\frac{2q_2}{r'(q_1^*)}\right|.$$

where q_1^*, $0\leqslant q_1^*\leqslant q_1$, is a solution of

$$\text{(I.4.3)}\qquad \frac{q_1-q_1^*}{q_2}=\frac{1}{[r'(q_1^*)]^2}.$$

We consider some particular cases connected with singularities of the solutions of (I.4.3).

Firstly, for disturbances whose boundary grows when $q_2=0$ at a constant rate ε, $r(q_1)=r_0+\varepsilon q_1$, we find

$$r(q_1, q_2) = r_0 + \varepsilon q_1 + \frac{1}{\varepsilon} q_2.$$

Thus, to obtain a stable bounded beam, what is necessary is an auto-oscillatory regime of change of the boundary of the initial disturbance. Let us consider, for example, the following case:

$$r(q_1) = r_0 + A \sin \omega q_1.$$

Then the equation (I.4.3) for determining q_1^* takes the form

$$\text{(I.4.4)} \qquad \frac{A^2\omega^2}{q_2}(q_1 - q_1^*) = \frac{1}{\cos^2 \omega q_1^*},$$

which, depending on q_1 and q_2 may have various sets of solutions, $0 < q_1^* < q_1$.

Physically, this means that in a given section q_2 = const, as q_1 grows, there must arise a stratification of the sound beam; this is the phenomenon of "self-diffraction". The stage at which the stratification occurs and the radii of the resulting rings can be determined from (I.4.4) and (I.4.2).

APPENDIX II

ON CONSERVATION LAWS AND SYMMETRIES OF NON-LINEAR EQUATIONS OF KLEIN–GORDON TYPE

V. N. Rubtsov

This appendix deals with an application of the theory developed above, to find the conservation laws and the symmetry algebras (of infinitesimal symmetries) of non-linear partial differential equations. We wish to emphasize the fact that the presence of a large number of conservation laws is not just peculiar to the "selected" equations (for example, the Korteweg-de Vries or the "Sine–Gordon" equation) but quite generally, is a common phenomenon for the majority of natural equations of mathematical physics. The family of equations considered in this Appendix is grouped around the non-linear Klein–Gordon equation:

$$\Box u = f(u), \text{ where } \quad \Box = \frac{\partial^2}{\partial t^2} - \Delta, \ \Delta \text{ is the Laplace operator.}$$

In particular, the standard model "Sine–Gordon" equation is obviously included in the class under consideration. With it we begin our study of the conservation laws on J^1M.

II.1. The "Sine–Gordon" equation. The initial manifold is $M = \mathbf{R}^2$, $J^1\mathbf{R}^2 \cong \mathbf{R}^5$, and the equation is

$$\text{(II.1.1)} \qquad u_{q_1, q_2} = \sin u,$$

where q_1 and q_2 are coordinates on $\mathbf{R}^2$ and (q_1, q_2, p_1, p_2, u) are the corresponding coordinates on $J^1\mathbf{R}^2 \cong \mathbf{R}^5$. The effective form corresponding to

(II.1.1) is

$$\text{(II.1.2)} \qquad \omega = -\frac{1}{2}\,dp_1 \wedge dq_1 + \frac{1}{2}\,dp_2 \wedge dq_2 - \sin u\, dq_1 \wedge dq_2,$$
$$\omega \in \Lambda^2 (J^1 \mathbf{R}^2).$$

To find the conservation law $\theta \in \Lambda^1 (J^1 \mathbf{R}^2)$ it is necessary to find $g \in C^\infty(J^1 \mathbf{R}^2)$ from

$$\text{(II.1.3)} \qquad \mathscr{E}\,(g\omega) = 0,$$

where $\mathscr{E}$ is an Euler operator.

In our case, since

$$d\omega = -\cos u\, du \wedge dq_1 \wedge dq_2 \quad \text{и} \quad d_p\omega = p \circ d\omega = 0,$$

the action of the Euler operator on $g\omega$ can be written as follows:

$$\text{(II.1.4)} \qquad \mathscr{E}(g\omega) = gL_1\omega + d_p(X_g \lrcorner\, \omega) + L_1(g)\omega.$$

Computing the three terms in (II.1.4), we obtain

$$gL_1\omega = -g\cos u\, dq_1 \wedge dq_2,$$
$$d_p (X_g \lrcorner\, \omega) = d_p \Big\{ -\Big[\frac{1}{2}(g_{q_1} + p_1 g_u) + \sin u\, g_{p_2}\Big] dq_1 +$$
$$+ \Big[\frac{1}{2}(g_{q_2} + p_2 g_u) + \sin u\, g_{p_1}\Big] dq_2 - \frac{1}{2}\, g_{p_1}\, dp_1 + \frac{1}{2}\, g_{p_2}\, dp_2 \Big\},$$
$$L_1 (g)\, \omega = \frac{\partial g}{\partial u}\, \omega.$$

Equating to zero the coefficients for the corresponding forms in (II.1.3), we find g after simple manipulations:

$$\text{(II.1.5)} \qquad g = (aq_1 + b)p_1 + (-aq_2 + c)p_2,$$

where a, b, and c are arbitrary constants.

After substituting it in the formula $d\theta = g\omega - U_1 \wedge X_g \lrcorner\, \omega$, we obtain

$$d\theta = \Big(\frac{1}{2}\, g_{q_1} + g_{p_2} \sin u\Big)\, du \wedge dq_1 -$$
$$- \Big(\frac{1}{2}\, g_{q_2} + g_{p_1} \sin u\Big)\, du \wedge dq_2 +$$
$$+ \frac{1}{2}\, g_{p_1}\, du \wedge dp_1 - \frac{1}{2}\, g_{p_2}\, du \wedge dp_2 +$$
$$+ \frac{1}{2}\, p_2 g_{p_1}\, dp_1 \wedge dq_2 - \frac{1}{2}\, p_1 g_{p_2}\, dp_2 \wedge dq_1 -$$
$$- \frac{1}{2}\, p_2 g_{p_2}\, dp_1 \wedge dq_1 + \frac{1}{2}\, p_1 g_{p_1}\, dp_2 \wedge dq_2.$$

To find θ we evaluate the integral (3.10.1) and obtain the conservation law:

$$\text{(II.1.6)}\quad \theta_g=\left(\frac{1}{4}ug_{q_1}-g_{p_2}\cos u-\frac{1}{2}p_1p_2g_{p_2}\right)dq_1+$$
$$+\left(-\frac{1}{4}ug_{q_2}+g_{p_1}\cos u+\frac{1}{2}p_1p_2g_{p_1}\right)dq_2+$$
$$+\frac{1}{4}(p_2g_{p_2}-p_1g_{p_1})\,du+\frac{1}{4}ug_{p_1}\,dp_1-\frac{1}{4}ug_{p_2}\,dp_2.$$

Thus, as follows from (II.1.5) and (II.1.6), the space of conservation laws for (II.1.1) generated by $J^1(\mathbf{R}^2)$ is three-dimensional. We write out the basic conservation laws:

$$\theta_1=\left[-\frac{1}{4}\frac{\partial u}{\partial q_1}\frac{\partial u}{\partial q_2}-\cos u-\frac{1}{4}\sin u\right]dq_1+\left[-\frac{1}{4}u\frac{\partial^2 u}{\partial q_2^2}+\frac{1}{4}\left(\frac{\partial u}{\partial q_2}\right)^2\right]dq_2,$$
$$\theta_2=\left[\frac{1}{4}\frac{\partial u}{\partial q_1}+\frac{\partial u}{\partial q_2}+\cos u+\frac{1}{4}\sin u\right]dq_2+\left[\frac{1}{4}u\frac{\partial^2 u}{\partial q_1^2}-\frac{1}{4}\left(\frac{\partial u}{\partial q_1}\right)^2\right]dq_1,$$
$$\theta_3=\left[\frac{1}{4}u\frac{\partial u}{\partial q_1}+q_2\cos u+\frac{1}{4}q_2\frac{\partial u}{\partial q_1}\frac{\partial u}{\partial q_2}+\right.$$
$$\left.+\frac{1}{4}uq_1\frac{\partial^2 u}{\partial q_1^2}+\frac{1}{4}uq_2\sin u-\frac{1}{4}q_1\left(\frac{\partial u}{\partial q_1}\right)^2\right]dq_1+$$
$$+\left[\frac{1}{4}u\frac{\partial u}{\partial q_2}+q_1\cos u+\frac{1}{4}q_1\frac{\partial u}{\partial q_1}\frac{\partial u}{\partial q_2}+\right.$$
$$\left.+\frac{1}{4}uq_2\frac{\partial^2 u}{\partial q_2^2}+\frac{1}{4}uq_1\sin u-\frac{1}{4}q_2\left(\frac{\partial u}{\partial q_2}\right)^2\right]dq_2.$$

NOTE 1. In the form

$$u_{xx}-u_{tt}=\sin u,$$

to which it is transformed by the substitution

$$\text{(II.1.7)}\qquad q_1=\frac{x-t}{\sqrt{2}},\qquad q_2=\frac{x+t}{\sqrt{2}},$$

(II.1.1) can be regarded as the Euler–Lagrange equation for the Lagrangian

$$\text{(II.1.8)}\qquad \Omega=\left(\frac{p_1^2-p_2^2}{2}+\cos u-1\right)dq_1\wedge dq_2.$$

In the case of (II.1.1) the algebras sym(Δ_ω) and symc(Δ_ω) are the same, namely, the semidirect product of the algebra of translations and that of hyperbolic rotations of the plane. All the conservation laws θ_1, θ_2, and θ_3 are Noetherian. Using (II.1.7) and (II.1.8), we see that the laws θ_1 and θ_2 correspond to two directions of translation, that is, they represent the laws of conservation of energy and of momentum while the law θ_3 corresponds to the conservation of the relativistic moment of momentum for a hyperbolic rotation. In conclusion we give the general form of the vector field X_f, $f\in$ sym(Δ_ω), for (II.1.1):

$$X_f=-(aq_1+b)\frac{\partial}{\partial q_1}-(-aq_2+c)\frac{\partial}{\partial q_2}+ap_1\frac{\partial}{\partial p_1}-ap_2\frac{\partial}{\partial p_2}.$$

NOTE 2. Using $f\in$ sym(Δ_ω) and a method similar to that used in Appendix I for the Khokhlov–Zabolotskaya equation, we can obtain f-automodel solutions of (II.1.1), which are familiar to physicists [2] and can be expressed in terms of elliptic functions.

For the following equations we give only the final results.

II.2. The equation $u_{q_1 q_2} = \mathcal{F}'(u)$. If $\mathcal{F}'(u)$ and $\mathcal{F}''(u)$ are linearly independent, then there is a three-dimensional space of conservation laws on $J^1(\mathbf{R}^2)$ for which the laws $\tilde{\theta}_1$, $\tilde{\theta}_2$, and $\tilde{\theta}_3$ serve as a basis. They are obtained from the corresponding θ_1, θ_2, and θ_3 for (II.1.1) after replacing $\sin u$ by $\mathcal{F}'(u)$, $\cos u$ by $\mathcal{F}''(u)$, and making the obvious sign change. The generating function g is the same as in (II.1).

If $\mathcal{F}'(u)$ and $\mathcal{F}''(u)$ are linearly dependent, then the equation is essentially of the form

$$u_{q_1 q_2} = e^{\alpha u}.$$

In this case the space of conservation laws on $J^1(\mathbf{R}^2)$ becomes infinite-dimensional. Each conservation law is determined by a pair of functions of a single variable, and the generating function is

$$g = f(q_1)\,p + \varphi(q_2)\,p_2 + \frac{1}{\alpha}\left(f'(q_1) + \varphi'(q_2)\right).$$

II.3. The Kruskal transformation ([13]). The Kruskal transformation

$$u = v \pm \arcsin \varepsilon v_{q_1} \tag{II.3.1}$$

takes solutions of the equation

$$v_{q_1 q_2} = \sqrt{1 \pm \varepsilon^2 v_{q_1}^2}\,\sin v, \qquad 0 < \varepsilon < 1, \tag{II.3.2}$$

into solutions of (II.1.1)

$$u_{q_1 q_2} = \sin u.$$

The form corresponding to (II.3.2) is

$$\omega = -\frac{1}{2}\,dp_1 \wedge dq_1 + \frac{1}{2}\,dp_2 \wedge dq_2 - \sqrt{1 + \varepsilon^2 p_1^2}\,\sin u\, dq_1 \wedge dq_2. \tag{II.3.3}$$

For definiteness we take the negative sign under the radical.

In the case (II.3.3) $d_p \omega \neq 0$

$$d_p \omega = d\omega - U_1 \wedge (X_1 \lrcorner\, d\omega) = \frac{\varepsilon^2 p_1}{\sqrt{1 - \varepsilon^2 p_1^2}}\,\sin u\, dp_1 \wedge dq_1 \wedge dq_2.$$

After some manipulations we obtain the generating function $g \in C^\infty(J'(\mathbf{R}^2))$

$$g = \frac{g_1 - \varepsilon^2 g_2 \cos u}{\sqrt{1 - \varepsilon^2 p_1^2}}\, p_1 + g\, p_2,$$

where $g_1 = aq_1 + b$, $g_2 = -aq_2 + c$, and a, b and c are arbitrary constants.

We note that, as $\varepsilon \to +0$, the function g for (II.3.2) goes over into the corresponding function g for (II.1.1). The conservation laws on $J^1(\mathbf{R}^2)$ form a three-dimensional space. The basic forms for (II.3.2) go over, as $\varepsilon \to +0$, into the basic forms for the Sine–Gordon equation.

II.4. The non-linear Klein–Gordon equation in $\mathbf{R}^4$. This equation has the form

$$\text{(II.4.1)} \qquad -\frac{\partial^2 u}{\partial q_1}+\frac{\partial^2 u}{\partial q_2}+\frac{\partial^2 u}{\partial q_3}+\frac{\partial^2 u}{\partial q_4}-f'(u)=0,$$
$$M=\mathbf{R}^4, \quad J^1(\mathbf{R}^4)\cong \mathbf{R}^9.$$

An effective form corresponding to (II.4.1) is

$$\text{(II.4.2)} \quad \omega = -dp_1 \wedge dq_2 \wedge dq_3 \wedge dq_4 - dp_2 \wedge dq_1 \wedge dq_3 \wedge dq_4 + \\ + dp_3 \wedge dq_1 \wedge dq_2 \wedge dq_4 - dp_4 \wedge dq_1 \wedge dq_2 \wedge dq_3 - \\ - f(u) dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4;$$

The generating function $g \in C^\infty(J^1\mathbf{R}^4)$ for (II.4.1) is

$$g = (aq_2 + bq_3 + cq_4 + d)p_1 + (aq_1 + eq_3 + fq_4 + h)p_2 + \\ + (bq_1 - eq_2 + pq_4 + r)p_3 + (cq_1 - fq_2 - pq_3 + l)p_4,$$

where $a, b, c, d, e, f, h, p, r$, and l, are arbitrary constants. The space of conservation laws on $J^1(\mathbf{R}^4)$ for (II.4.1) is ten-dimensional if $f'(u)$ and $f''(u)$ are linearly independent.

The algebra $\mathrm{sym}(\Delta_\omega)$ in this case is the Poincaré algebra. The basic conservation laws, as in II.1, are Noetherian.

Four conservation laws are generated by translations and six by rotations conserving the Minkowski metric. As an example we write out one of the basic conservation laws

$$\theta_g = ({}^1/_2 p_1^2 q_4 - {}^1/_2 p_3^2 q_4 - {}^1/_2 p_4^2 q_4 - {}^1/_2 p_2^2 q_4 - {}^1/_2 u p_4 + q_4 f) dq_1 \wedge dq_3 \wedge dq_4 + \\ + (-{}^1/_2 p_1^2 q_2 + {}^1/_2 p_2^2 q_2 - {}^1/_2 p_3^2 q_2 + {}^1/_2 p_4^2 q_2 + {}^1/_2 u p_2 - q_2 f) dq_1 \wedge dq_2 \wedge dq_3 - \\ - {}^1/_2 p_1 p_2 q_4 dq_2 \wedge dq_3 \wedge dq_4 + {}^1/_2 u q_4\, dp_1 \wedge dq_3 \wedge dq_4 - {}^1/_2 p_1 q_4\, du \wedge dq_3 \wedge dq_4 - \\ - {}^1/_2 u q_2\, dp_1 \wedge dq_2 \wedge dq_3 + {}^1/_2 p_1 q_2\, du \wedge dq_2 \wedge dq_3 - \\ - {}^1/_2 u q_2\, dp_2 \wedge dq_1 \wedge dq_3 + ({}^1/_2 p_4 q_4 + {}^1/_2 q_2 p_2) du \wedge dq_1 \wedge dq_3 + \\ + {}^1/_2 u q_4\, dp_3 \wedge dq_1 \wedge dq_4 - {}^1/_2 p_3 q_4\, du \wedge dq_1 \wedge dq_4 + {}^1/_2 u q_2\, dp_3 \wedge dq_1 \wedge dq_2 - \\ - {}^1/_2 p_3 q_2\, du \wedge dq_1 \wedge dq_2 - {}^1/_2 u q_4\, dp_4 \wedge dq_1 \wedge dq_3.$$

This corresponds to the conservation of the relativistic moment of momentum for a hyperbolic rotation in the (q_1, q_2)-plane.

II.5. The "stationary" Klein–Gordon equation in $\mathbf{R}^2$. This equation has the form

$$\text{(II.5.1)} \qquad \Delta u = f'(u), \quad f'(0) = 0.$$

A corresponding effective form is

$$\text{(II.5.2)} \qquad \omega = dp_1 \wedge dq_2 - dp_2 \wedge dq_1 - f'(u) dq_1 \wedge dq_2.$$

Since $d_p\omega = 0$, the Euler operator acts according to (II.1.4). Solving the corresponding equation (II.1.3), we find the generating function for the conservation laws of (II.5.1).

As in II.2, if $\alpha f'' + \beta f' \neq 0$, then

$$g = (-aq_2 + c)p_1 + (aq_1 + b)p_2,$$

where a, b, and c are arbitrary constants.

The space of conservation laws on $J^1(\mathbf{R}^2)$ is in this case three-dimensional. A basis is formed by the laws

$$\theta_1 = \left[\frac{1}{2} u \frac{\partial^2 u}{\partial q_1 \partial q_2} - \frac{1}{2}\frac{\partial u}{\partial q_1}\frac{\partial u}{\partial q_2}\right] dq_1 + \left[\frac{1}{2}\left(\frac{\partial u}{\partial q_1}\right)^2 - f(u) + \frac{1}{2} u \frac{\partial^2 u}{\partial q_2^2}\right] dq_2,$$

$$\theta_2 = \left[-\frac{1}{2}\left(\frac{\partial u}{\partial q_2}\right)^2 + \mathcal{F}(u) - \frac{1}{2} u \frac{\partial^2 u}{\partial q_1^2}\right] dq_1 + \left[\frac{1}{2}\frac{\partial u}{\partial q_1}\frac{\partial u}{\partial q_2} - \frac{1}{2} u \frac{\partial^2 u}{\partial q_1 \partial q_2}\right] dq_2,$$

$$\theta_3 = \left\{q_1\left[-\frac{1}{2}\left(\frac{\partial u}{\partial q_2}\right)^2 - \mathcal{F}(u) - \frac{1}{2} u \frac{\partial^2 u}{\partial q_1^2}\right] - \frac{1}{2} u \frac{\partial u}{\partial q_1} - \right.$$
$$\left. - q_2\left[\frac{1}{2} u \frac{\partial^2 u}{\partial q_1 \partial q_2} - \frac{1}{2}\frac{\partial u}{\partial q_1}\frac{\partial u}{\partial q_2}\right]\right\} dq_1 + \left\{-q_2\left[\frac{1}{2}\left(\frac{\partial u}{\partial q_1}\right)^2 - \mathcal{F}(u) + \frac{1}{2} u \frac{\partial^2 u}{\partial q_2^2}\right] - \right.$$
$$\left. - \frac{1}{2} u \frac{\partial u}{\partial q_2} + q_1\left[\frac{1}{2}\frac{\partial u}{\partial q_1}\frac{\partial u}{\partial q_2} - \frac{1}{2}\frac{\partial u}{\partial q_1}\frac{\partial u}{\partial q_2} - \frac{1}{2} u \frac{\partial^2 u}{\partial q_1 \partial q_2}\right]\right\} dq_2.$$

The symmetry algebra for (II.5.1) is the same as the algebra of the group of motions of the Euclidean plane. The vector field X_f, $f \in \operatorname{sym} \Delta_\omega$, for (II.5.1) has the form

$$X_f = -(aq_2 + c)\frac{\partial}{\partial q_1} - (aq_1 + b)\frac{\partial}{\partial q_2} + ap_2\frac{\partial}{\partial p_1} - ap_1\frac{\partial}{\partial p_2}.$$

All the conservation laws $\theta_1, \theta_2, \theta_3$ are Noetherian, θ_1 and θ_2 correspond to conservation of momentum for parallel displacements, while θ_3 corresponds to conservation of moment of momentum under a rotation.

Because of the importance of (II.5.1) in hydrodynamics [10] the f-automodel solutions, which can be easily constructed by the method of §3, are of particular interest.

If $\alpha f'' + \beta f' = 0$, then the equation is in effect

(II.5.3) $$\Delta u = e^{\alpha u},$$

which was studied and solved already by Poincaré and Picard. The conservation laws on $J^1(\mathbf{R}^2)$ in this case, as with II.2, form an infinite-dimensional space, and each conservation law is determined by a pair of functions of a single variable, but of different arguments.

NOTE. The results about the construction of the symmetry algebra in II.4 and II.5 are also valid only when

$$\alpha f'' + \beta f' \neq 0.$$

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