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Feedback Differential Invariants

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Abstract The problem of feedback equivalence for control systems is considered. An algebra of differential invariants and criteria for the feedback equivalence for regular control systems are found.

Keywords Autonomous control system · Feedback transformations · Differential invariant

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1 Introduction

In this paper we outline an application of the method of differential invariants to the problem of recognition for control systems.

Namely, we consider the action of feedback transformations on one-dimensional control autonomous systems describing by the second-order ordinary differential equations. We look on such control systems as underdetermined ordinary differential equations. This gives us a representation of feedback transformations as a special type of Lie transformations. We study and find an algebra of differential invariants of this representation.

There are several approaches to study control systems. The more popular ones are based either on EDS methods [2–4], or study affine families of vector fields [1, 5, 11].

It is worth also to note that from the EDS point of view the case of control systems considered here is equivalent to the case of first order control systems considered in [9], but they have different algebras of feedback differential invariants.

It is easy to check that for the control systems (and it is the general case) one has an infinite number of functionally independent invariants, but (and it is also common for such

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problems), they could be organized in algebra of functions on some differential equation (the so-called syzygy). In other words, there is a finite number of differential invariants and invariant derivations such that any differential invariant can be obtained by computing functions of these basic invariants and their derivations [7, 8, 10, 12]. This representation of differential invariants is not unique, in general they satisfy some differential equations (syzygy).

In our case, for the description of the algebra in neighborhoods of regular orbits (Theorem 1), we have three feedback invariant derivations, one first-order differential invariant and two third-order differential invariants and the corresponding syzygy consists of two differential equations. By definition, these differential invariants describe the orbits of jets of the control systems under feedback transformations. Hence, the structure of the differential invariant algebra allows us to establish formal feedback equivalence of the control systems. In Sect. 9 we show that for the case of regular control systems the feedback formal equivalence implies the local smooth feedback equivalence.

To formulate the local feedback equivalence we connect with a regular control system a 3-dimensional submanifold Σ in $\mathbb{R}^8$ and the main result of this paper states that two regular control systems are locally feedback equivalent if and only if the corresponding 3-dimensional submanifolds Σ coincide.

2 Representation of Feedback Pseudogroup

Let

$$y'' = F(y, y', u) \tag{1}$$

be an autonomous one-dimensional control system of the second order.

Here the function $y = y(t)$ describes a dynamic of the state of the system, and $u = u(t)$ is a scalar control parameter.

We shall consider this system as an undetermined second-order ordinary differential equation on sections of two-dimensional bundle $\pi : \mathbb{R}^3 \to \mathbb{R}$, where $\pi : (u, y, t) \longmapsto t$.

Let $\mathcal{E} \subset J^2(\pi)$ be the corresponding submanifold. In the canonical jet coordinates $(t, u, y, u_1, y_1, \ldots)$ this submanifold is given by the equation:

$$y_2 = F(y, y_1, u).$$

It is known (see, for example, [6]) that the Lie transformations in jet bundles $J^k(\pi)$ for two-dimensional bundle π are prolongations of point transformations, that is, prolongations of diffeomorphisms of the total space of the bundle π.

We shall restrict ourselves by point transformations which are automorphisms of the bundle π.

Moreover, if these transformations preserve class (1) of systems, then they should have the form

$$\Phi : (u, y, t) \to (U(u, y), Y(y), t). \tag{2}$$

Diffeomorphisms of form (2) are called *feedback transformations*. The corresponding infinitesimal version of this notion is a *feedback vector field,* i.e., a plane vector field of the form

$$X_{a,b} = a(y)\partial_y + b(u, y)\partial_u.$$

The feedback transformations act on the control systems of type (1) in a natural way:

$$\mathcal{E} \longmapsto \Phi^{(2)}(\mathcal{E}),$$

where $\Phi^{(2)} : J^2(\pi) \to J^2(\pi)$ is the second prolongation of the point transformation Φ.

Passing to functions F, defining the systems, we get the following action of feedback transformations on these functions:

$$\widehat{\Phi} : F(y, y_1, u) \longmapsto \frac{1}{Y'} F(Y, Y' y_1, U) - \frac{Y''}{Y'} y_1^2. \tag{3}$$

The infinitesimal version of this action leads us to the following presentation of feedback vector fields:

$$\widehat{X_{a,b}} = b(u, y)\partial_u + a(y)\partial_y + a' y_1 \partial_{y_1} + (a'' y_1^2 + a' f)\partial_f. \tag{4}$$

In this formula $\widehat{X_{a,b}}$ is a vector field on the four-dimensional space $\mathbb{R}^4$ with coordinates (y, y_1, u, f). Each control system (1) determines a three-dimensional submanifold $L_F \subset \mathbb{R}^4$, the graph of F:

$$L_F = \{f = F(y, y_1, u)\}.$$

Let A_t be the one-parameter group of shifts along vector field $X_{a,b}$ and let $B_t : \mathbb{R}^4 \to \mathbb{R}^4$ be the corresponding one-parameter group of shifts along $\widehat{X_{a,b}}$. Then these two actions related as follows:

$$L_{\widehat{A_t}(F)} = B_t(L_F).$$

In other words, if we consider an one-dimensional bundle

$$\kappa : \mathbb{R}^4 \to \mathbb{R}^3,$$

where $\kappa((y, y_1, u, f)) = (y, y_1, u)$, then formula (4) defines the representation $X \longmapsto \widehat{X}$ of the Lie algebra of feedback vector fields into the Lie algebra of Lie vector fields on $J^0(\kappa)$, and the action of Lie vector fields $\widehat{X}$ on sections of bundle κ corresponds to the action of feedback vector fields on the right-hand sides of (1) (see [6]).

3 Feedback Differential Invariants

By a *feedback differential invariant* of order $\leq k$ we understand a function $I \in C^\infty(J^k\kappa)$ on the space of k-jets $J^k(\kappa)$, which is invariant under of the prolonged action of feedback transformations.

Namely,

$$\widehat{X_{a,b}}^{(k)}(I) = 0,$$

for all feedback vector fields $X_{a,b}$.

In what follows, we shall omit subscript of order of jet spaces, and say that a function I on the space of infinite jets $I \in C^\infty(J^\infty\kappa)$ is a feedback differential invariant if

$$\widehat{X_{a,b}}^{(\infty)}(I) = 0,$$

where $\widehat{X_{a,b}}^{(\infty)}$ is the prolongation of the vector field $\widehat{X_{a,b}}$ in the space of infinite jets $J^\infty\kappa$.

In a similar way one defines a *feedback invariant derivations* as combinations of total derivatives

$$\nabla = A\frac{d}{dy} + B\frac{d}{dy_1} + C\frac{d}{du}, \quad A, B, C \in C^{\infty}(J^{\infty}\kappa),$$

which are invariant with respect to prolongations of feedback transformations, that is,

$$[\widehat{X_{a,b}}^{(\infty)}, \nabla] = 0$$

for all feedback vector fields $X_{a,b}$.

Remark that for such derivations, the functions $\nabla(I)$ are differential invariants (of order, as a rule, higher then the order of I) for any feedback differential invariant I. This observation allows us to construct new differential invariants from known ones by the differentiations only.

Recall a construction of the Tresse derivations in our case. Let $J_1, J_2, J_3 \in C^{\infty}(J^k\kappa)$ be three feedback differential invariants, and let

$$\widehat{d}J_i = \frac{dJ_i}{dy}dy + \frac{dJ_i}{dy_1}dy_1 + \frac{dJ_i}{du}du$$

be their total differentials.

Assume that we are in a domain $\mathcal{D}$ in $J^k\kappa$, where

$$\widehat{d}J_1 \wedge \widehat{d}J_2 \wedge \widehat{d}J_3 \neq 0.$$

Then, for any function $V \in C^{\infty}(J^l\kappa)$ over domain $\mathcal{D}$, one has decomposition

$$\widehat{d}V = \lambda_1\widehat{d}J_1 + \lambda_2\widehat{d}J_2 + \lambda_3\widehat{d}J_3.$$

Coefficients λ_1, λ_2 and λ_3 of this decomposition are called the *Tresse derivatives* of V and are denoted by

$$\lambda_i = \frac{DV}{DJ_i}.$$

The remarkable property of these derivatives is the fact that they are feedback differential invariants (of higher, as a rule, order then V) each time when V is a feedback differential invariant.

In other words, the Tresse derivatives

$$\frac{D}{DJ_1}, \frac{D}{DJ_2} \text{ and } \frac{D}{DJ_3}$$

are feedback invariant derivations.

4 Dimensions of Orbits

First of all, we remark that the submanifold $y_1^{(-1)}(0)$ is a singular orbit for the feedback action in the space of 0-jets $J^0\kappa$. In what follows, we shall consider orbits of jets at regular points, that is, at such points, where $y_1 \neq 0$.

It is easy to see that the kth prolongation of the feedback vector field $\widehat{X_{a,b}}$ depends on the $(k+2)$-jet of function $a(y)$ and the k-jet of the function $b(u, y)$.

Denote by V_i^k and W_{ij}^k the components of the decomposition

$$\widehat{X_{a,b}}^{(k)} = \sum_{i=0}^{k+2} a^{(i)}(y) V_i^k + \sum_{0 \le i+j \le k} \frac{\partial^{i+j} b}{\partial u^i \partial y^j} W_{ij}^k.$$

Then, by the construction, the vector fields $V_i^k, 0 \le i \le k+2$, and $W_{ij}^k, 0 \le i+j \le k$, generate a completely integrable distribution on the space of k-jets, integral manifolds of which are orbits of the feedback action in $J^k\kappa$.

Let $\mathcal{O}_{k+1}$ be an orbit in $J^{k+1}\kappa$. Then the projection $\mathcal{O}_k = \kappa_{k+1,k}(\mathcal{O}_{k+1}) \subset J^k\kappa$ is an orbit too, and to determine dimensions of the orbits one should find dimensions of the bundles: $\kappa_{k+1,k} : \mathcal{O}_{k+1} \to \mathcal{O}_k$. In order to do this, we should find conditions on the functions a and b under which $\widehat{X_{a,b}}^{(k)} = 0$ at a point $x_k \in J^k\kappa$.

It easy to check that for $k = 1$ at a point x_1, where $f_u \neq 0$, these conditions have the form

$$\begin{aligned} &a(y) = a'(y) = a''(y) = 0, \qquad b(u, y) = 0, \\ &b_u = 0, \qquad a''' y_1^2 - b_y f_u = 0. \end{aligned} \tag{5}$$

Here $(u, y, y_1, f, f_u, f_y, f_{y_1})$ are the canonical coordinates in the 1-jet space $J^1(\kappa)$.

The formula for prolongations of vector fields (see, for example, [6]) shows that the conditions on the functions a and b, such that vector fields $\widehat{X_{a,b}}^{(k)}$ vanish at a point in $J^k\kappa$, are just $(k-1)$-prolongations of (5).

Let

$$\phi = a''(y) y_1^2 + a'(y) f - b(u, y) f_u - a(y) f_y - a'(y) y_1 f_{y_1}$$

be the generating function of the vector field $\widehat{X_{a,b}}$.

Assume that $k > 1$, and that $\widehat{X_{a,b}}^{(k-1)} = 0$ at a point $x_{k-1} \in J^{k-1}\kappa$. Then the vector field $\widehat{X_{a,b}}^{(k)}$ is a $\kappa_{k,k-1}$-vertical over this point. The components

$$\frac{d^k \phi}{du^i dy^j} \frac{\partial}{\partial f_{\sigma_{ij}}}$$

of this vector field, where $\sigma_{ij} = (\underbrace{u, \ldots, u}_{i\text{-times}}, \underbrace{y, \ldots, y}_{j\text{-times}})$, $i + j = k$, and the components

$$\frac{d^k \phi}{dy^{k-1} dy_1} \frac{\partial}{\partial f_\tau},$$

where $\tau = (\underbrace{y, \ldots, y}_{(k-1)\text{-times}}, y_1)$, depend on

$$\frac{\partial^k b}{\partial u^i \partial y^j}$$

and

$$\frac{d^{k+2} a}{dy^{k+2}},$$

respectively.

All other components

$$\frac{d^k \phi}{dy^r dy_1^s du^t} \frac{\partial}{\partial f_\sigma}$$

are expressed in terms of the $(k-1)$-jet of $b(u, y)$ and the $(k+1)$-jet of function $a(y)$.

This shows that the bundles $\kappa_{k,k-1} : \mathcal{O}_k \to \mathcal{O}_{k-1}$ are $(k+2)$-dimensional if $k > 1$ and $y_1 \neq 0$, $f_u \neq 0$.

We say that the k-jet $[F]_p^k \in J^k\kappa$ of a function F is *weakly regular* if the point p is regular, that is, if $y_1 \neq 0$ at this point, and $F_u \neq 0$.

The orbits of the weakly regular points are called *weakly regular*.

The feedback orbits in the space of 1-jets can be found by direct integration of the six-dimensional completely integrable distribution generated by the vector fields V_i^1, $0 \leq i \leq 3$, and W_{ij}^1, $0 \leq i + j \leq 1$. Summarizing, we get the following result.

Theorem 1

1. *The first non-trivial differential invariants of feedback transformations appear in order one, and they are functions of the basic invariant*

$$J = \frac{y_1 f_{y_1} - 2f}{y_1}.$$

2. *The dimension of a weakly regular orbit of feedback transformations in* $J^k\kappa$, $k > 1$, *is equal to*

$$\frac{(k+2)(k+3)}{2}.$$

3. *There are*

$$\frac{(k+2)(k-1)}{2}$$

independent differential invariants of order k.

5 Invariant Derivations

We'll need the following result which allows us to compute invariant derivations.

Assume that an infinitesimal Lie pseudogroup $\mathfrak{g}$ is represented in the Lie algebra of contact vector fields on the manifold of 1-jets $J^1(\mathbb{R}^n)$.

Moreover, we will identify elements $\mathfrak{g}$ with the corresponding contact vector fields, i.e. we assume that elements of $\mathfrak{g}$ have the form X_f (see [6]), where f is the generating function.

Lemma 1 *Let* $x_1, \dots, x_n$ *be coordinates in* $\mathbb{R}^n$, *and let* $(x_1, \dots, x_n, u, p_1, \dots, p_n)$ *be the corresponding canonical coordinates in the* 1*-jet space* $J^1(\mathbb{R}^n)$.

Then a derivation

$$\nabla = \sum_{i=1}^{n} A_i \frac{d}{dx_i}$$

is $\mathfrak{g}$*-invariant if and only if functions* $A_i \in C^\infty(J^\infty\mathbb{R}^n),\ j=1,\dots,n,$ *are solutions of the following PDE system*:

$$X_f(A_i)+\sum_{j=1}^{n}\frac{d}{dx_j}\left(\frac{\partial f}{\partial p_i}\right)A_j=0, \tag{6}$$

for all $i=1,\dots,n,$ *and* $X_f\in\mathfrak{g}$.

Proof We have [6]:

$$X_f^\infty=\mathbf{E}_f-\sum_{i=1}^{n}\frac{\partial f}{\partial p_i}\frac{d}{dx_i},$$

where

$$\mathbf{E}_f=\sum_\sigma\frac{d^{|\sigma|}f}{dx^\sigma}\frac{\partial}{\partial p_\sigma}$$

is the evolutionary derivation, σ is a multi index and $\{p_\sigma\}$ are the canonical coordinates in $J^\infty\mathbb{R}^n$.

Using the fact that evolutionary derivations commute with the total ones and the relation

$$[\nabla, X_f^\infty]=0,$$

we get

$$\begin{aligned}0&=\left[\sum_{j=1}^{n}A_j\frac{d}{dx_j},\mathbf{E}_f-\sum_{i=1}^{n}\frac{\partial f}{\partial p_i}\frac{d}{dx_i}\right]\\&=-\sum_j\mathbf{E}_f(A_j)\frac{d}{dx_j}+\sum_{i,j}\left(-A_j\frac{d}{dx_j}\left(\frac{\partial f}{\partial p_i}\right)\frac{d}{dx_i}+\frac{\partial f}{\partial p_i}\frac{dA_j}{dx_i}\frac{d}{dx_j}\right)\\&=-\sum_s\left(X_f^\bullet(A_s)+\sum_jA_j\frac{d}{dx_j}\left(\frac{\partial f}{\partial p_s}\right)\right)\frac{d}{dx_s}.\end{aligned}$$

□

In our case we expect to have three linear independent feedback invariant derivations. The straightforward computations in order ≤ 2 show that they are of the form

$$\begin{aligned}\nabla_u&=\frac{y_1}{f_u}\frac{d}{du},\\\nabla_y&=-\frac{y_1^3f_{y_1u}-2z^2f_y+y_1^2f\ f_{y_1y_1}-2y_1\,f\ f_{y_1}+2f^2}{y_1(-2f_u+y_1f_{y_1u})}\frac{d}{du}+y_1\frac{d}{dy}+f\frac{d}{dy_1},\\\nabla_{y_1}&=y_1\frac{d}{dy_1}.\end{aligned}$$

It is easy to check that these derivations obey the following commutation relations:

$$\begin{aligned}[\nabla_u,\nabla_y]&=\frac{L-J_{y_1y_1}}{J_u}\nabla_u+\nabla_{y_1},\\[\nabla_u,\nabla_{y_1}]&=(1+J_u)\nabla_u,\end{aligned}$$

$$[\nabla_y, \nabla_{y_1}] = \frac{J_u K + J_{y_1}(J_{y_1} - J_u + J J_u) - J_{y_1 y_1}(J_{y_1} - J_u)}{J_u^2} \nabla_u \\ - \nabla_y - J\,\nabla_{y_1}, \tag{7}$$

where K and L are third-order differential invariants (see below).

6 Second-Order Differential Invariants

Theorem 1 shows that there are two independent differential invariants whose order is precisely two. We can get them by applying invariant derivations to the first-order invariant J:

$$J_u \overset{\text{def}}{=} \nabla_u(J) = \frac{y_1 f_{y_1 u} - 2 f_u}{f},$$

$$J_{y_1} \overset{\text{def}}{=} \nabla_{y_1}(J) = \frac{y_1^2 f_{y_1 y_1} - 2 y_1 f_{y_1} + 2 f}{y_1^2}.$$

However,

$$\nabla_y(J) = 0.$$

7 Third-Order Differential Invariants

Theorem 1 shows that there are five independent third-order differential invariants. We get three of them by the invariant differentiation:

$$J_{u\,u} = \nabla_u \nabla_u(J), \qquad J_{u\,y_1} = \nabla_u \nabla_{y_1}(J), \qquad J_{y_1 y_1} = \nabla_{y_1} \nabla_{y_1}(J).$$

In order to find the last two differential invariants, we remark that the 3-prolongations of feedback vector fields are affine along the fibres

$$J^3(\kappa) \overset{\kappa_{3,2}}{\to} J^2(\kappa)$$

(see, for example, [6]).

Therefore, one can try to find the differential invariants as functions which are affine along the fibres $\kappa_{3,2}$.

Finally, we get

$$K = y_1^2 f_{u y_1 y_1} - \frac{3 y_1\ J_u\ - y_1 J_{y_1 u}}{J_u} f_{y y_1} - y_1 J_u f_{y y_1} + \frac{2(J_{y_1 u} + 2 J_u)}{J_u} f_u \\ + 2 J_u f_y - \frac{(J_u J_{y_1} + J_{y_1} - J_{y_1 y_1}) J_u + J_{y_1} J_{y_1 u}}{y_1\ J_u} f + \frac{(J_u - J_{y_1})(J_{y_1 y_1} + J_{y_1})}{J_u},$$

and

$$L = \frac{y_1^2}{f_u} f_{u y y_1} - \frac{(2 + J_u)\ y_1}{f_u} f_{u y} - \frac{y_1 J_{uu}}{J_u} f_{y y_1} + 2 \frac{J_{uu}}{J_u} f_y + \left(\frac{J_{y_1 u}}{y_1} - \frac{J_{y_1} J_{uu}}{y_1 J_u} \right) f \\ + J_{y_1} + J_{y_1 y_1}.$$

8 The Algebra of Feedback Differential Invariants

In order to use the above computations, one should reinforce the notion of regularity. We give the following definition.

Definition 1 We say that a weakly regular orbit is *regular* if $J_u \neq 0$, on the orbit.

Remark that for *irregular* or *singular* control systems one has $J_u \equiv 0$, and therefore, they have the form

$$y'' = A(y, y') + B(u, y)y'^2.$$

Counting dimensions shows that the differential invariants J, K and L are generators in the algebra of feedback differential invariants, and considering symbols of differential invariants shows that they satisfy two syzygy relations.

Theorem 2

1. *In a neighborhood of regular orbits, the algebra of feedback differential invariants is generated by the first-order differential invariant J, the third-order differential invariants K and L and all their invariant derivatives.*
2. *Syzygies for this algebra have two generators of the form*

$$J_y = 0,$$

$$K_u - L_{y_1} + \frac{J_{y_1 u} + J_u - J_x^2}{J_u} L - \frac{J_{u\,u}}{J_u} K = \Phi(J, J_u, J_{y_1}).$$

Remark 1 In a similar way, for irregular systems, we get the following description of the differential invariants algebra.

The algebra of differential invariants for systems with $J_u \equiv 0$, but $y_1 \neq 0$, is generated by the first-order differential invariant J, the third-order differential invariant M,

$$M = y_1 f_{y_1 y_1 y_1} + y_1^2 f_{y\, y_1 y_1} - f f_{y_1 y_1} - 2 y_1 f_{y_1 y} + \frac{2 f f_{y_1}}{y_1} + 2 f_y - \frac{2 f^2}{y_1^2},$$

and all invariant derivatives

$$\nabla_{y_1}^a J,\ \nabla_{y_1}^a \nabla_u^b M.$$

9 The Feedback Equivalence Problem

Consider two control systems given by functions F and G. Then, to establish feedback equivalence, one should solve the differential equation

$$\frac{1}{Y'} F(Y, Y' y_1, U) - \frac{Y''}{Y'} y_1^2 - G(y, y_1, u) = 0 \tag{8}$$

with respect to the functions $Y(y)$ and $U(u, y)$.

Let us denote the left-hand side of (8) by H. Then, assuming the general position, we can find the functions U, Y, Y' and Y'' from the equations

$$H = H_{y_1} = H_{y_1 y_1} = H_{y_1 y_1 y_1} = 0.$$

Remark, that the above general conditions are feedback invariant, depend on finite jet of the system and hold in a dense open domain of the jet space. Therefore, they hold in regular points.

Suppose that these functions are

$$U = A(y, y_1, u), \qquad Y = B(y, y_1, u),$$
$$Y' = C(y, y_1, u), \qquad Y'' = D(y, y_1, u).$$

Then the conditions

$$A_{y_1} = B_{y_1} = C_{y_1} = D_{y_1} = 0,$$
$$B_u = C_u = D_u = 0$$

and

$$C = B_y, \qquad D = C_y$$

show that if (8) has a formal solution at each point (y, y_1, u) in some domain, then this equation has a smooth solution.

On the other hand, if a system F at a point $p = (y^0, y_1^0, u^0)$ and a system G at a point $\widetilde{p} = (\widetilde{y}^0, \widetilde{y}_1^0, \widetilde{u}^0)$ have the same differential invariants, then, by the definition, there is a formal feedback transformation which sends the infinite jet of F at the point p to the infinite jet of G at the point $\widetilde{p}$.

Keeping in mind these observations and results of Theorem 2, we consider the space $\mathbb{R}^3$ with coordinates (u, y, y_1) and the space $\mathbb{R}^8$ with coordinates $(j, j_1, j_3, j_{11}, j_{13}, j_{33}, k, l)$.

Then any control system, given by the function $F(u, y, y_1)$, defines a map

$$\sigma_F : \mathbb{R}^3 \to \mathbb{R}^8$$

by

$$j = J^F, \qquad j_1 = J_u^F, \qquad j_3 = J_{y_1}^F,$$
$$j_{11} = J_{u\,u}^F, \qquad j_{13} = J_{u\,y_1}^F, \qquad j_{33} = J_{y_1 y_1}^F,$$
$$k = K^F, \qquad l = L^F,$$

where the subscript F means that the differential invariants are evaluated due to the system.

Let

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^3$$

be a feedback transformation.

Then from the definition of the feedback differential invariants it follows that

$$\sigma_F \circ \Phi = \sigma_{\widehat{\Phi}(F)}.$$

Therefore, the geometrical image

$$\Sigma_F = \mathrm{Im}(\sigma_F) \subset \mathbb{R}^8$$

does depend on the feedback equivalence class of F only.

We say that a system F is *regular* in a domain $D \subset \mathbb{R}^3$ if

1. 3-jets of F belong to regular orbits,
2. $\sigma_F(D)$ is a smooth three-dimensional submanifold in $\mathbb{R}^8$, and
3. the functions j, j_1 and j_3 are coordinates on Σ_F.

The following lemma gives a relation between the Tresse derivatives and invariant differentiations ∇_u, ∇_y and ∇_{y_1}.

Lemma 2 *Let*

$$\frac{D}{DJ}, \frac{D}{DJ_u}, \frac{D}{DJ_{y_1}}$$

be the Tresse derivatives with respect to the differential invariants J, J_u and J_{y_1}.

Then the following decompositions hold:

$$\nabla_u = J_u \frac{D}{DJ} + J_{u\,u} \frac{D}{DJ_u} + J_{u\,y_1} \frac{D}{DJ_{y_1}},$$

$$\nabla_y = (J_{y_1 y_1} - J_{y_1} - L) \frac{D}{DJ_u} + \frac{J_u K + J_{y_1}(J_{y_1} - J_u) - J_{y_1 y_1}(J_{y_1} - J_u)}{J_u} \frac{D}{DJ_{y_1}},$$

$$\nabla_{y_1} = J_{y_1} \frac{D}{DJ} + (J_{u\,y_1} - J_u - J_u^2) \frac{D}{DJ_u} + J_{y_1 y_1} \frac{D}{DJ_{y_1}}.$$

Proof The proof follows directly from the definition of the Tresse derivatives and commutation relations (7). □

Theorem 3 *Two regular systems F and G are locally feedback equivalent if and only if*

$$\Sigma_F = \Sigma_G. \tag{9}$$

Proof Let us show that condition (9) implies a local feedback equivalence.

Assume that

$$J_{uu}^F = j_{11}^F(J^F, J_u^F, J_{y_1}^F), \qquad J_{uy_1}^F = j_{12}^F(J^F, J_u^F, J_{y_1}^F), \qquad J_{y_1 y_1}^F = j_{22}^F(J^F, J_u^F, J_{y_1}^F),$$
$$K^F = k^F(J^F, J_u^F, J_{y_1}^F), \qquad L^F = l^F(J^F, J_u^F, J_{y_1}^F)$$

on Σ_F, and

$$J_{uu}^G = j_{11}^G(J^G, J_u^G, J_{y_1}^G), \qquad J_{uy_1}^G = j_{12}^G(J^G, J_u^G, J_{y_1}^G), \qquad J_{y_1 y_1}^G = j_{22}^G(J^G, J_u^G, J_{y_1}^G),$$
$$K^G = k^G(J^G, J_u^G, J_{y_1}^G), \qquad L^G = l^G(J^G, J_u^G, J_{y_1}^G)$$

on Σ_G.

Then condition (9) shows that $j_{11}^F = j_{11}^G$, $j_{12}^F = j_{12}^G$, $j_{22}^F = j_{22}^G$ and $k^F = k^G$, $l^F = l^G$. Moreover, as we have seen, the invariant derivations ∇_u, ∇_y and ∇_{y_1} are linear combinations of the Tresse derivatives.

In other words, the functions

$$j_{11}^F, j_{12}^F, j_{22}^F, k^F, l^F$$

and their partial derivatives in j, j_1 and j_3 determine the restrictions of all differential invariants.

Therefore, condition (9) equalize restrictions of differential invariants not only to order ≤ 4 but in all orders, and provides formal and therefore local feedback equivalence between F and G. □

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