GEOMETRIC THEORY OF SINGULARITIES OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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The article is devoted to singularities of integral manifolds which realize solutions of nonlinear partial differential equations and to the algebraic geometric and topological questions related to them.

Singularities, jumps, shock waves, or, in short, catastrophes are one of the most important and intriguing sections of the geometric theory of nonlinear differential equations. Modern theory of singularities explains plausibly enough the mechanism behind the formation of catastrophes (see, for example, [24]). However, despite the simplicity and the vast number of applications (and also speculations), this mechanism is not satisfactory. The point is that, apparently, on the one hand, the majority of processes to which this mechanism is applied are described using differential equations (and, as a rule, partial differential equations) and, on the other hand, its use begins only when the differential equation has been practically forgotten and during further manipulations of the solution a Jacobian has vanished.

In this work we attempt to approach the theory of singularities of solutions of nonlinear differential equations from a unified geometric standpoint and in doing so we indicate a more direct and immediate connection between differential equations and the theory of singularities, applying the latter to projections of integral manifolds.

1. JET BUNDLES

Jet spaces lie at the basis of the geometric approach to the theory of nonlinear differential equations. The concept of the jet was introduced by Ehresmann. However, it should be noted that already the first integrated partial differential equation, the Euler equation of homogeneous functions, was integrated by Euler using the geometry of spaces of first-order jets.

1.1. Let M be a smooth manifold of dimension m + n, where m > 0 is a fixed natural number. For each smooth submanifold $N \subset M$ of codimension m we denote by $[N]_x^k$ a k-jet of this submanifold at point $x \in N$. Suppose that $J_x^k(M, m)$ is the space of all k-jets of submanifolds of codimension m at a fixed point $x \in M$ and

is the submanifold of all k-jets. Reduction of k-jet $[N]_x^k$ to s-jet $[N]_x^s$ for k > s generates projection $\pi_{k,s}: J^k(M, m) \to J^s(M, m)$. Set $\pi_{k,0} = \pi_k$ and $J^0(M, m) = M$.

If manifold M is the total space of smooth bundle $\alpha: M = E(\alpha) \to B$ then we denote by $J^k(\alpha)$ the manifold of all k-jets of the local sections of this bundle. Identifying (local) sections $h: B \to M$ with their graphs, submanifolds in M of form h(B), we obtain imbedding $J^k(\alpha) \subset J^k(E(\alpha), m)$, where $m = \dim \alpha$. Natural projections $\alpha \cdot \pi_k: J^k(\alpha) \to B$ are denoted by α_k. When bundle α is a vector bundle, then so are bundles α_k, $k \geq 0$. Projections $\pi_{k,s}$ define morphisms of these vector bundles, and for s = k − 1 the following exact sequences hold:

$$0 \to \alpha \otimes S^k\tau^* \to \alpha_k \xrightarrow{\pi_{k,k-1}} \alpha_{k-1} \to 0,$$

in which $S^k\tau^*$ denotes the k-th symmetric degree of the cotangent bundle $\tau^*: T^*B \to B$.

1.2. Keeping in mind the description in the general case of the fibers of projections $\pi_{k,k-1}: J^k(M, m) \to J^{k-1}(M, m)$, we consider the action of pseudogroup G(M) of local diffeomorphisms of manifold M in the space $J^k(M, m)$. Denote by $G_{x,y}^k$ the set of all k-jets $[\varphi]_{x,y}^k$ of local diffeomorphisms $\varphi: M \to M$, taking point x onto point y, $\varphi(x) = y$. The composition of the local diffeomorphisms generates the following pairing:

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$$G^k_{y,z} \times G^k_{x,y} \to G^k_{x,z}, \quad [\psi]^k_{y,z} \times [\varphi]^k_{x,y} \mapsto [\psi \circ \varphi]^k_{x,z},$$

which determines the structure of a group in space $G_{x,x}{}^k = G_x{}^k$ provided that x = y = z.

Obviously, $G_x{}^0 = \{e\}$ and $G_x{}^1 = GL(T_xM)$.

Natural projections $\pi_{k,k-1}$: $G_x{}^k \to G_x{}^{k-1}$ are epimorphisms of groups the kernels $H_x{}^k$ of which are Abelian groups isomorphic to tensor products $S^kT^* \otimes T_x$ for $k \geq 2$. Here, by $T_x = T_xM$ and $T_x{}^* = T_x{}^*M$ we denote the tangent and cotangent spaces in manifold M.

Thus, for $k \geq 2$ groups $G_x{}^k$ are obtained by successive extensions of the complete linear group by means of the Abelian groups.

Group $G_x{}^k$ acts in a natural way in the space of k-jets $J_x{}^k(M, m)$: if $[\varphi]_{x,x}{}^k \in G_x{}^k$, $[N]_x{}^k \in J_x{}^k(M, m)$, then $[\varphi]_{x,x}{}^k \cdot ([N]_x{}^k) = [\varphi(N)]_x{}^k$. Under this action, kernel $H_x{}^k$ acts transitively on fibers $F(x_{k-1})$ of projection $\pi_{k,k-1}$: $J^k(M, m) \to J^{k-1}(M, m)$ over elements $x_{k-1} = [N]_x{}^{k-1} \in J^{k-1}(M, m)$. A stationary subgroup (in $H_x{}^k$) of element $[N]_x{}^k \in F(x_{k-1})$ under such action when $k \geq 2$ is the following subspace:

$$(\mathrm{Ann}\, T_xN) \circ S^{k-1}T_x^* \otimes T_x + S^kT_x^* \otimes T_xN,$$

where by $(\mathrm{Ann}\, T_xN) \cdot S^{k-1}T_x{}^*$ we denote the symmetric product of annihilator $\mathrm{Ann}\, T_xN$ and space $S^{k-1}T_x{}^*$.

This remark shows that fibers $F(x_{k-1})$ for $k \geq 2$ are homogeneous spaces relative to the action of a connected Abelian group $H_x{}^k$ and thus carry an affine structure associated with vector space $S^k(T_x{}^*N) \otimes \nu_x$, where by ν_x we denote a subspace normal to submanifold N: $\nu_x = T_xM/T_xN$.

1.3. Every diffeomorphism $\varphi \in G(M)$ is lifted in a natural way to local diffeomorphism $\varphi^{(k)}$: $J^k(M, m) \to J^k(M, m)$, where $\varphi^{(k)}([N]_x{}^k) = [\varphi(N)]_{\varphi(x)}{}^k$. Here, $\varphi^{(0)} = \varphi$, and liftings $\varphi^{(k)}$, $k \geq 1$ are compatible with projections $\varphi_{k,s}$: $\pi_{k,s} \cdot \varphi^{(k)} = \varphi^{(s)} \cdot \varphi_{k,s}$, $k > s$. Obviously, diffeomorphisms $\varphi^{(k)}$, for $k \geq 2$, preserve the affine structure described above. When k = 1, fibers F(x) of bundle $\pi_{1,0}$ are identified with the Grassmannian manifolds of n-dimensional subspaces in tangent space T_xM, $F(x) = G_n(T_x) \simeq G_n(\mathbf{R}^{n+m})$, and automorphisms $\varphi^{(1)}$ are linear collineations of these manifolds generated by differentials $\varphi_{*,x}$.

The following theorem describes the geometric structure of bundles of submanifold jets.

THEOREM. Jet bundles $\pi_{k,k-1}$: $J^k(M, m) \to J^{k-1}(M, m)$ are affine for $k \geq 2$. Here fiber $F(x_{k-1})$ over elements $x_{k-1} = [N]_x{}^{k-1}$ is an affine space with which vector space $S^k(T_x{}^*N) \otimes \nu_x$ is associated. Liftings $\varphi^{(k)}$ of local diffeomorphisms $\varphi \in G(M)$ to spaces $J^k(M, m)$ are affine automorphisms for $k \geq 2$ and are linear collineations of Grassmannian manifolds for k = 1.

Supplement. If manifold M is additionally furnished with a contact (or symplectic) structure and $R^k(M)$ is a manifold of all k-jets of the Legendre (the Lagrange, respectively) submanifolds in M, then, considering instead of G(M) a pseudogroup of contact or symplectic local diffeomorphisms and acting as above, we obtain that for $k \geq 2$ the projections $R^k(M) \to R^{k-1}(M)$ are affine bundles whose fibers over points $[N]_x{}^{k-1}$ are associated with vector spaces $S^{k+1}(T_x{}^*N) \otimes T_xM/C_x$ in the contact case, where C_x is a contact plane at point $x \in M$, and with $S^{k+1}(T_x{}^*N)$ in the symplectic case. Natural liftings of contact symplectic, respectively) local diffeomorphisms in $R^k(M)$ are affine automorphisms for $k \geq 2$.

1.4. Consider some specializations of the given general construction.

1) Let α: $M = E(\alpha) \to B$ be a smooth bundle. Then, identifying local sections of this bundle with submanifolds in M transversal to the fibers, we obtain imbedding κ_k: $J^k(\alpha) \hookrightarrow J^k(M, m)$, where $m = \dim \alpha$. Transversality conditions are the conditions on 1-jet; therefore, the fibers of projections $\pi_{k,k-1}$: $J^k(\alpha) \to J^{k-1}(\alpha)$ coincide for $k \geq 2$ with the fibers of the general projection $\pi_{k,k-1}$: $J^k(M, m) \to J^{k-1}(M, m)$ and thus inherit an affine structure [5, 33]. Liftings of local automorphisms of bundle α are automorphisms of this structure.

2) We shall assume in addition that bundle α is a vector bundle. In that case, bundles $\pi_{k,k-1}$: $J^k(\alpha) \to J^{k-1}(\alpha)$ are also vector bundles and the affine structure defined by the vector one coincides with the one described above. Keeping this in mind, we describe a covering of manifold $J^k(M, m)$ by open sets of form $J^k(\alpha)$ which would preserve affine structures and in doing so would provide a convenient method of realizing arbitrary affine structures. To this end, for each element $x_k = [N]_x{}^k \in J^k(M, m)$ we consider a tubular neighborhood $\mathcal{O}$ of submanifolds $N \subset M$ and the normal bundle α: $\mathcal{O} = E(\alpha) \to N$. Then the image of natural imbedding κ_k: $J^k(\alpha) \hookrightarrow J^k(M, m)$ covers a neighborhood of element x_k and the affine structure on $J^k(M, m)$ in this neighborhood is induced by a linear one.

The described construction is analogous to the construction of affine maps in a projective space. Keeping in mind this analogy we call imbeddings κ_k: $J^k(\alpha) \hookrightarrow J^k(M, m)$ *affine maps* on a manifold of k-jets.

3) For a trivial bundle α: $M \times N \to M$ manifold $J^k(\alpha)$ coincides with manifold $J^k(M, m)$ of k-jets of smooth mappings from M into N. The affine structure described above is, in particular, an invariant of a left—right action.

2. CARTAN DISTRIBUTION

This distribution, which served as the source of the theory of differential forms, enables us to isolate in an inner way among the various submanifolds of a jet space those which correspond to the solutions of differential equations. In other words, the Cartan distribution enables us to reformulate the problem of the integrability of differential equations into a differential geometric problem. The Cartan distribution is a natural generalization of the classical contact structure and the geometry defined by it is the continuation of contact geometry. Many concepts of the theory of differential equations (such as solutions, symmetries, conservation laws, shock waves, and others) are naturally formulated in terms of the arising geometry.

2.1. The Cartan Distribution. Every submanifold $N \subset M$, codim $N = m$ defines a submanifold $J_k(N) \subset J^k(M, m)$, a *k-jet extension* of N, where $j_k(N) = \{[N]_x^k, \forall x \in N\}$. Analogously, each local section h: $B \to E(\alpha)$ of bundle α defines a local section $j_k(h)$: $B \to J^k(\alpha)$ of bundle α_k: $j_k(h)$: $x \to [h]_x^k$, $\forall x \in B$. We call submanifolds $L \subset J^k(M, m)$ [sections θ: $B \to J^k(\alpha)$] that have form $L = j_k(N)$ [$\theta = j_k(h)$] for some manifold M (or section h) *integrable*. If $k \neq 0$, then of course not all the submanifolds are integrable. Keeping in mind the isolation of such manifolds, we consider the following construction.

Each element $x_{k+1} = [N]_x^{k+1} \in J^{k+1}(M, m)$ defines subspace $L(x_{k+1}) \subset T_{x_k}(J^k(M, m))$ tangent to submanifold $j_k(N)$ at point $x_k = [N]_x^k$. Let $C(x_k) \subset T_{x_k}(J^k(M, m))$ be a subspace spanned by a union of various subspaces of form $L(x_{k+1}')$ provided that elements x_{k+1}' pass through the whole fiber $F(x_k)$. In other words, subspace $C(x_k)$ is a linear span of the union of tangent spaces to integrable manifolds passing through point x_k.

Subspaces $C(x_k)$, like distribution $\mathscr{C}$: $x_k \to C(x_k)$ defined by them on a k-jet manifold $J^k(M, m)$, are called *Cartanian*.

The Cartan distribution on manifold $J^k(\alpha)$ is constructed analogously. Furthermore, if κ: $E(\kappa) \to N \subset M$, $\kappa^{(k)}$: $J^k(\kappa) \hookrightarrow J^k(M, m)$ is an affine map, then mapping $\kappa^{(k)}$ carries the Cartan distribution on $J^k(M, m)$ into a Cartan distribution on $J^k(\kappa)$. Using this fact, we can limit ourselves to the consideration of Cartan distributions on manifolds of form $J^k(\alpha)$, where α is a smooth vector bundle.

Example. Manifold $J^1(M, 1)$ is identified in a natural way with the projectivization of skew tangent bundle τ^*: $T^*M \to M$. The Cartan distribution on it is a standard contact structure. Affine maps, in this case, are manifolds of the 1-jets of smooth functions $J^1(1)$ with natural contact structure.

Here and further on by **1** we denote the trivial linear bundle $B \times \mathbb{R} \to B$.

The definition of Cartan subspace $C(x_k)$ directly implies that the differential of projection $\pi_{k,k-1}$ maps this subspace into $L(x_k)$. Furthermore, equality $(\pi_{k,k-1})_{*,x_k}^{-1}(L(x_k)) = C(x_k)$, which gives an alternative definition of the Cartan distribution, is valid. In other words, Cartan subspace $C(x_k)$ can be represented in the form of a direct sum of subspaces $L(x_{k+1})$, $\forall x_{k+1} \in F(x_k)$, and $T_{x_k}(F(x_{k-1}))$, $x_{k-1} = \pi_{k,k-1}(x_k)$.

Submanifolds of form $j_k(N) \subset J^k(M, m)$ are integral manifolds of the Cartan distribution without degeneration projected in manifold M. The converse assertion is also true: each integral manifold L of a Cartan distribution of dimension n for which mapping $\pi_{k,0}$: $L \to M$ is a characteristic imbedding is integrable.

2.2. Cartan Form. Suppose that α: $E(\alpha) \to B$ is a smooth bundle and $\Lambda_0^i(J^k\alpha)$ is a module of differential α_k-horizontal i-forms on a k-jet manifold $J^k(\alpha)$. The operator of exterior differentiation d: $\Lambda^i(B) \to \Lambda^{i+1}(B)$ on manifold B generates the operator of complete differentiation $\hat{d}$: $\Lambda_0^i(J^k\alpha) \to \Lambda_0^{i+1}(J^{k+1}\alpha)$, the value of which on i-form $\omega \in \Lambda_0^i(J^k\alpha)$ is determined by the following universality relation:

$$j_{k+1}^*(h)(\hat{d}\omega) = d\,(j_k^*(h)(\omega)),$$

which is valid for all local sections h: $B \to E(\alpha)$.

Operator $\hat{d}$ is the 1-degree differentiation of graded algebra $\Lambda_0^*(J^k\alpha)$ with values in $\Lambda_0^*(J^{k+1}\alpha)$, while $\hat{d}^2 = 0$.

To each function $f \in C^\infty(J^{k-1}\alpha)$ we can associate two differentials: exterior df and the complete $\hat{d}f$. Their difference

$$U(f) = \pi^*_{k,k-1}(df) - \hat{d}f \in \Lambda^1(J^k\alpha).$$

is called the *Cartan form* corresponding to function f.

The value of this form on vector $X \in T_{x_k}(J^k\alpha)$ is calculated in the following way. Let us represent tangent space $T_{x_{k-1}}(J^{k-1}\alpha)$ in the form of a direct sum of a horizontal space $L(x_k)$ and a vertical space $T_{x_k}{}^v$, a space tangent to a fiber of projection α_k. Corresponding to this expansion, vector $(\pi_{k,k-1})_*(X)$ can be represented in the form $X_0 + X^v$, where $X_0 \in L(x_k)$, $X^v \in T_{x_k}{}^v$. In this notation, equality $U(f)(X) = X^v(f)$ is valid.

THEOREM [5]. The distribution on manifold $J^k(\alpha)$ defined by the zeros of Cartan 1-forms $U(f)$, $\forall f \in C^\infty(J^{k-1}\alpha)$, coincides with the Cartan distribution.

2.3. Metasymplectic Structure. Operator $U: C^\infty(J^{k-1}\alpha) \to \Lambda^1(J^k\alpha)$, which associates with function f Cartan form $U(f)$, is a $C^\infty(B)$-linear differentiation over mapping $\alpha_{k,k-1}{}^*$: $U(fg) = fU(g)$ if $f \in C^\infty(B)$, and $U(fg) = fU(g) + gU(f)$, $\forall f, g \in C^\infty(J^{k-1}\alpha)$.

Keeping in mind the description of tangent spaces to integral manifolds of a Cartan distribution, we consider the restrictions of the differentials of Cartan 1-forms to subspaces $C(x_k)$. To this end, we fix element $x_k \in J^k(\alpha)$ and define operator

$$\Omega_{x_k}: C^\infty(J^{k-1}\alpha) \to \Lambda^2(C^*(x_k)),$$

where $\Omega_{x_k}(f)(v_1, v_2) = dU(f)(v_1, v_2)$, $\forall f \in C^\infty(J^{k-1}\alpha)$; $v_1, v_2 \in C(x_k)$.

The properties of operator U imply that $\Omega_{x_k}(fg) = f(x_{k-1})\Omega_{x_k}(g) + g(x_{k-1})\Omega_{x_k}(f)$. Therefore, operator Ω_{x_k} defines operator $\Omega_{x_k}: T_{x_{k-1}}{}^* \to \Lambda^2(C(x_k))$, where $\Omega_{x_k}(\lambda) = \Omega_{x_k}(f)$, $\lambda = d_{x_{k-1}}f$, $f(x_{k-1}) = 0$. Here it is obvious that $\Omega_{x_k}(\lambda) = 0$ if $\lambda \in \operatorname{Im}\alpha_{k-1,k-2}{}^*$. And, since $T_{x_{k-1}}{}^*/\operatorname{Im}\alpha_{k-1,k-2}{}^* = (T_{x_{k-1}}(F(x_{k-2})))^*$, mapping Ω_{x_k} defines operator

$$\Omega: T^*_{x_{k-1}}(F(x_{k-2})) \to \Lambda^2(C^*(x_k)),$$

which we call a *metasymplectic structure* on Cartan space $C(x_k)$.

If α is a vector bundle, then $T_{x_{k-1}}(F(x_{k-2})) = S^{k-1}T_x{}^* \otimes \alpha_x$ and thus the metasymplectic structure here is the operator

$$\Omega: \alpha^*_x \otimes S^{k-1}T_x \to \Lambda^2(C^*(x_k)).$$

Before we calculate the value of this operator, we note that for each covector $\lambda \in T_{x_{k-1}}{}^*(F(x_{k-2}))$ the exterior 2-form $\Omega_\lambda = \Omega(\lambda) \in \Lambda^2(C^*(x_k))$ vanishes on subspaces $L(x_{k+1})$, $\forall x_{k+1} \in F(x_k)$, and $T_{x_k}(F(x_{k-1}))$. Therefore, by virtue of decomposition $C(x_k) = L(x_{k+1}) \oplus T_{x_k}(F(x_{k-1}))$, in order to compute the metasymplectic structure Ω, it suffices to determine the value of the 2-form Ω_λ on the pair of vectors $X \in L(x_{k+1})$, $Y \in T_{x_k}$, $Y \in T_{x_k}(F(x_{k-1}))$. Using the affine structure, we identify tangent space $T_{x_k}(F(x_{k-1}))$ with the vector space $S^kT_x{}'' \otimes \nu_{x_0}$, where $x_0 = \alpha_{k,0}(x_k)$, ν_{x_0} is a tangent space to fiber $\alpha(x)$ at point x_0, and space $L(x_{k+1})$ with tangent space T_x. Under these identifications we have for vectors $X \in T_x$, $Y \in S^kT_x{}^* \otimes \nu$, $\lambda \in S^{k-1}T_x \otimes \nu^*$:

$$\Omega_\lambda(X, Y) = \langle \lambda, X \lrcorner \delta Y \rangle, \tag{1}$$

where by $\delta: S^kT_x{}^* \otimes \nu \to T_x{}^* \otimes S^{k-1}T_x{}^* \otimes \nu$ the Spencer δ-operator is denoted.

Using this formula, it is not hard to describe the degeneration subspace of the exterior 2-form Ω_λ. Namely,

$$\operatorname{Ker}\Omega_\lambda = \{\theta \in S^kT^*_x \otimes \nu \subset C(x_k), \quad \delta\theta \in T^*_x \otimes g_\lambda\}, \tag{2}$$

where $g_\lambda = \operatorname{Ker}\lambda = \{\gamma \in S^{k-1}T_x{}^* \otimes \nu, \langle\lambda, \gamma\rangle = 0\}$.

Thus, the degeneration subspace of Ω_λ coincides with the first prolongation of symbol $g_\lambda \subset S^{k-1}T_x{}^* \otimes \nu$.

2.4. Isotropic Subspaces. Let us say that vectors $X, Y \in C(x_k)$ are found in *involution* under a metasymplectic structure if $\Omega_\lambda(X, Y) = 0$ for all tensors $\lambda \in S^{k-1}T_x \otimes \nu^*$.

Subspace $E \subset C(x_k)$ in which every pair of vectors $X, Y \in E$ is found in involution is called *isotropic*.

Examples. 1) Vectors $X, Y \in L(x_{k+1}) \subset C(x_k)$ are found in involution and $L(x_{k+1})$ is thus an isotropic subspace.

2) Vectors $\theta, \eta \in S^kT_x^* \otimes \nu \subset C(x_k)$ are found in involution; $T_{x_k}(F(x_{k-1})) = S^kT_x^* \otimes \nu$ is an isotropic subspace.

3) Let $\theta \in S^kT_x^* \otimes \nu$, $X \in L(x_{k+1}) \simeq T_x$. Then, by virtue of formula (1), vectors X and θ are found in involution if and only if vector X determines the direction of degeneration of the symmetric tensor θ, i.e., X ? $\delta\theta = 0$.

4) Fix subspace $\mathcal{U} \subset T_x^*$, and using decomposition $C(x_k) = L(x_{k+1}) \oplus S^kT_x^* \otimes \nu$ and identification α_{k*}: $L(x_{k+1}) \rightleftarrows T_x$, we define subspaces $E(x_{k+1}), \mathcal{U}) \subset C(x_k)$, which is spanned by vectors of form $X \oplus \theta$, where $X \in \mathrm{Ann}\,\mathcal{U} \subset T_x$, and $\theta \in S^k\mathcal{U} \otimes \nu \subset S^kT_x^* \otimes \nu$. Then, by virtue of formula (1), subspace $E(x_{k+1}, \mathcal{U})$ is isotropic.

We say that $E \subset C(x_k)$ is a *maximal isotropic subspace* if E is not a proper subset of any other isotropic subspace.

The following proposition gives a complete description of a maximal isotropic subspace.

Proposition. Each maximal isotropic subspace $E \subset C(x_k)$ has the form $E = E(x_{k+1}, \mathcal{U})$ for some subspace $\mathcal{U} \subset T_x^*$. Here subspace $\mathcal{U}$ is uniquely determined by space E; $\mathcal{U} = \mathrm{Ann}\,\alpha_{k*}(E)$.

COROLLARY. Let $E \subset C(x_k)$ be an isotropic space for which mapping α_{k*}: $\mathcal{E} \to T_x$ is an isomorphism. Then E is a maximal isotropic subspace and $E = L(x_{k+1})$ for some element $x_{k+1} \in F(x_k)$.

3. INTEGRAL MANIFOLDS

The introduction of integral manifolds enables us to define a generalized, from a geometric viewpoint, solution of a system of partial differential equations. Such a generalization is analogous in many ways to the approach to integration of first-order equations proposed in the last century by Sophus Lie. The need for such generalizations at present became obvious after the works of V. P. Maslov on asymmetric methods.

3.1. Differential Equations. *A system of differential equations of order at most k* given on a submanifold of codimension m of manifold M is defined to be a smooth submanifold $\mathcal{E} \subset J^k(M, m)$. The *solution* of such a system of differential equations is a smooth submanifold $N \subset M$ the k-jet solution of which lies in manifold $\mathcal{E}$, $j_k(N) \subset \mathcal{E}$.

Let $\mathcal{C}(\mathcal{E})$ be a bounded cartan distribution on submanifold $\mathcal{E}$; $\mathcal{C}(\mathcal{E})$: $\mathcal{E} \ni x_k \to C(x_k) \cap T_{x_k}(\mathcal{E}) = C(\mathcal{E}, x_k)$. We shall say that point $a \in \mathcal{E}$ is the *regular* point of a system of differential equations if in a neighborhood of this point function $x_k \to \dim C(\mathcal{E}, x_k)$ is constant. Otherwise, point $a \in \mathcal{E}$ is called *singular.*

If all the points of the system are regular, then the family of subspaces $\mathcal{C}(\mathcal{E})$ defines a distribution on manifold $\mathcal{E}$, while the solutions of this system are integral manifolds of a distribution of $\mathcal{C}(\mathcal{E})$ having dimension n and being projected without degeneration into manifold M. Waiving the last condition, we arrive at the following geometric generalization of the concept of solution. The *solutions of a system of differential equations* $\mathcal{E}$ are n-dimensional integral manifolds of a distribution of $\mathcal{C}(\mathcal{E})$. n-dimensional integral manifolds of the Cartan distribution are called simply *integral.*

Thus, the solutions of a system of differential equations $\mathcal{E}$ are integral manifolds $L \subset J^k(M, m)$ lying in submanifold $\mathcal{E}$. If $L = j_k(N)$, then such a solution is called *classical* or *smooth*; otherwise, i.e., when mapping $\pi_{k,0}$: $L \to M$ is not a characteristic imbedding, manifold L is called a *generalized* (in the geometric sense) *solution.* The affine variants of the above-given constructions are defined analogously.

Examples. 1) Integral manifolds in contact manifold $J^1(M, 1)$ are Legendre manifolds.

2) Every n-dimensional submanifold $L \subset F(x_{k-1})$ is integral.

3) A natural generalization of the previous example is the following construction. Let N_0 be a submanifold in N and $N \subset M$, codim N = m. Denote by $N_0^{(k-1)} \subset j_{k-1}(\hat{N})$ a submanifold formed by (k − 1)-jets $[N]_x^{k-1}$ provided that point x runs all submanifolds N_0. And suppose that $\hat{N}_0(N)$ is the set of those elements $x_k \in J^k(M, m)$ for which $\pi_{k,k-1}(x_k) \in N_0^{(k-1)}$, and subspace $L(x_k)$ contains a tangent plane to submanifold $N_0^{(k-1)}$ at point x_{k-1}. Then the tangent planes to submanifold $\hat{N}_0(N)$ coincide with the maximal involution subspaces described in the previous section. Therefore, $\hat{N}_0(N)$ is the maximal integral manifold of the Cartan distribution and, consequently, every n-dimensional submanifold is integral.

4) In the case when mapping $\pi_{k,0}$: $L \to M$ has no singularities, i.e., when differential $(\pi_{k,0})_*$ is an imbedding, image $\pi_{k,0}(L)$ is an immersed submanifold in M. For such submanifolds $N \to M$ extension $J_k(N) \subset J^k(M, m)$, which is an integral submanifold for $k \geq 1$, is well-defined.

3.2. Symbols and Prolongations. *A prolongation of order* $l \geq 0$ of a system of differential equations $\mathcal{E} \subset J^k(M, m)$ is defined to be subset $\mathcal{E}^{(l)} \subset J^{K+\ell}(M, m)$ formed by those elements $[N]_x^{k+\ell}$, the k-jet extensions $j_k(N)$ of which are tangent to submanifold $\mathscr{E}$ at point $[N]_x^k \in \mathscr{E}$ with order larger than or equal to l. A system of differential equations $\mathscr{E}$ is called *formally integrable* if for all $l \geq 0$ prolongations $\mathscr{E}^{(l)}$ are smooth submanifolds and projections $\pi_{k+l+1,k+l}$: $\mathscr{E}^{(l+1)} \to \mathscr{E}^{(l)}$ and $\pi_{k,0}$: $\mathscr{E} \to M$ are smooth bundles.

The conditions for formal integrability are naturally formulated in terms of the symbols of differential equations.

Let $\mathscr{E} \subset J^k(M, m)$ be a system of differential equations. The symbol of this system at point $x_k = [N]_x^k \in \mathscr{E}$ is defined to be subspace

$$g(x_k) = T_{x_k}(\mathscr{E}) \cap T_{x_k}(F(x_{k-1})) \subset S^k(T_x^* N) \otimes \nu_x.$$

If all prolongations $\mathscr{E}^{(l)}$ are smooth manifolds, then their symbols $g^{(l)}(x_k) \subset S^{K+l}T_x^*N) \otimes \nu_x$ at points $x_{k+l} = [N]_x^{k+l}$ are l-th prolongations of symbol $g(x_k)$. Therefore, $\delta(g^{(l+1)}(x_k)) \subset T_x^*N \otimes g^{(l)}(x_k)$, and thus at each point $x_k \in \mathscr{E}$ the Spencer δ-complex

$$0 \to g^{(l)}(x_k) \xrightarrow{\delta} g^{(l-1)}(x_k) \otimes T_x^* N \xrightarrow{\delta} g^{(l-2)}(x_k) \otimes \Lambda^2(T_x^* N) \xrightarrow{\delta} \dots$$

is defined.

The cohomologies of this complex in the term $g^{(j)}(x_k) \otimes \Lambda^i(T_x^*N)$ are denoted by $H^{j,i}(\mathscr{E}, x_k)$. They are called Spencer δ-cohomologies of a system of differential equations $\mathscr{E}$ at point $x_k \in \mathscr{E}$.

We shall say that a system of differential equations $\mathscr{E} \subset J^k(M, m)$ is r-*acyclical* if $H^{j,i}(\mathscr{E}, x_k) = 0$ for all values $0 \leq i \leq r$, $j \geq 0$, $\forall x_k \in \mathscr{E}$. n-acyclical systems are called *involutive*. Using affine maps and Goldschmidt's [32] results, we arrive at the following criterion for formal integrability.

THEOREM. Suppose that $\mathscr{E} \subset J^k(M, m)$ is a 2-acylical system of differential equations for which projections $\pi_{k+1,k}$: $\mathscr{E}^{(1)} \to \mathscr{E}$ and $\pi_{k,0}$: $\mathscr{E} \to M$ are smooth bundles. Then $\mathscr{E}$ is a formally integrable system of differential equations.

3.3. Integral Manifolds with Singularities. Suppose that $L \subset J^k(M, m)$ is an integrable manifold and

$$\Sigma L = \{y \in L \,|\, \operatorname{Ker}(\pi_{k,0})_{*,y} \neq 0\}.$$

is a set of singular points of mapping $\pi_{k,0}$: $L \to M$.

Let us decompose $L \backslash \Sigma L = \bigcup_r L_r$ into connected components L_r. For every such component, mapping $\pi_{k,0}$: $L_r \to M$ is an immersion, and thus L_r can be represented as a graph of the k-th prolongation of the immersed submanifold N_r, $L_r = j_k(N_r)$. If, in addition, proper set ΣL has interior points, then at almost every such point (locally) L is a submanifold in a maximal integral manifold of form $\hat{N}_0(N)$ (see Example 3). Image $L' = \pi_{k,s}(L) \subset J^s(M, m)$, $k > s$ of the arbitrary integral manifold $L \subset J^k(M, m)$ is a smooth immersed integral submanifold at points $x_s \notin \pi_{k,s}(\Sigma L)$ and is a smooth manifold with "singularities" at the critical points of this mapping. Consider this process in reverse order. Subset L' defines an integral manifold L at all the nonsingular points. Namely, manifold L is a $(k - s)$-th prolongation of L'. At critical points $x_s \in \pi_{k,s}(\Sigma L)$ a "blowing up" of set L' analogous to the σ-process occurs. Fix the arising type of singularities. An *integral manifold with singularities* is defined to be such subset $L \subset J^k(M, m)$ for every point $x_k \in L$ of which there is a smooth integral manifold $\tilde{L} \subset J^{k+s}(M, m)$, $s = s(x_k)$, the image $\pi_{k+s,k}(\tilde{L})$ of which coincides with L at a neighborhood of point x_k. We shall say that integral manifold $\tilde{L}$ *uniformizes* L at a neighborhood of point x_k.

For every point x_k of an integral manifold with singularities L we choose a neighborhood $\mathcal{O}(x_k) \subset J^k(M, m)$ and a uniformizing integral manifold $\tilde{L}_{x_k} \subset J^{k+s}(M, m)$ such that $\pi_{k+s,k}(\tilde{L}_{x_k}) \cap \mathcal{O}(x_k) = L \cap \mathcal{O}(x_k)$. Suppose that $x_k' \in L$ is another point and the integral manifold $\tilde{L}_{x_k'} \subset J^{k+s'}(M, m)$ uniformizes L in a neighborhood $\mathcal{O}(x_k')$ of point x_k'. If $y_k \in \mathcal{O}(x_k) \cap \mathcal{O}(x_k') \cap L$, and in a neighborhood of this point subset L is a smooth manifold and y_k is not a singular point of mapping $\pi_{k,0}$: $L \to M$, then projections $\pi_{k+s,k+s'}$, providing that $s \geq s'$, are local diffeomorphisms between manifolds $\tilde{L}_x$ and $\tilde{L}_{x_k'}$ everywhere except for the set of singular points. If, however, at point y_k subset L is not a smooth submanifold, then $s = s'$ and submanifold $\tilde{L}_x$ coincides with $\tilde{L}_{x_k'}$ at a neighborhood of the preimage of point y_k. Using this remark and coalescing uniformizing manifolds along the corresponding projections, we obtain a smooth manifold $\hat{L}$, called the "blowing-up" of an integral manifold with singularities L, together with natural mapping $\hat{\pi}$: $\hat{L} \to L$, which is a local diffeomorphism at the nonsingular points of manifold L.

Further generalization of the concept of a solution of a system of differential equations is related to the use of integral manifolds with singularities. And, specifically, an integral manifold with singularities $L \subset J^k(M, m)$ is called a *solution of a system of differential equations* $\mathcal{E} \subset J^k(M, m)$ if for each point $x_k \in L$ the uniformizing integral manifold $\tilde{L} \subset J^{k+s}(M, m)$ lies in the s-th prolongation $\mathcal{E}^{(s)}$.

We say that solution L of a system of differential equations $\mathcal{E}$ has a singularity at point $x_k \in L$ if either (1) L is a smooth submanifold of point x_k but projection $\pi_{k,0}: L \to M$ has a singularity at this point, or (2) L is not a smooth submanifold at this point but this singularity is resolved for some prolongation.

Note that case (2) reduces locally to (1). For this it is enough to pass from a system of differential equations $\mathcal{E}$ to prolongation $\mathcal{E}^{(\ell)}$ and consider here solutions furnished by smooth integral submanifolds.

3.4. Characteristics and Singularities. Consider the system of differential equations $\mathcal{E} \subset J^k(\alpha)$ and let $L \subset \mathcal{E}^{(s)}$ be its solution. Denote by $\Sigma_l L$ the set of those points at which mapping $\alpha_{k+s}: L \to B$ has a Thom–Boardman singularity of type Σ_l. Tangent plane $T_{x_{k+s}}L \subset C(x_{k+s})$ at point $x_{k+s} \in \Sigma_l L$, taking into account the description of maximal isotropic subspaces given above, can be represented as the triad $(x_{k+s+1}, \mathcal{U}, L_0)$, where $\mathcal{U} \subset T_x^*$, $L_0 \subset S^k(\mathcal{U}) \otimes \alpha_x$, $T_{x_{k+s}}L = \mathcal{U}^0(x_{k+s+1}) \oplus L_0$, $\dim \mathcal{U} = \dim L_0 = l$, $\mathcal{U} = \mathrm{Ann}(\alpha_{k+s})_*(T_{x_{k+s}}L)$, $\mathcal{U}^0(x_{k+s+1}) \subset L(x_{k+s+1})$ is the preimage of $\mathrm{Ann}\,\mathcal{U} \subset T_x$ under isomorphism $(\alpha_{k+s})_*: L(x_{k+s+1}) \xrightarrow{\sim} T_x$.

We indicate conditions under which a system of differential equations $\mathcal{E}$ (or its prolongation) has no solutions passing through point $x_{k+s} \in \mathcal{E}^{(s)}$ and having a singularity of type Σ_l at a given point. Everywhere in the sequel, unless particularly specified, we assume that all prolongations $\mathcal{E}^{(s)}$ are smooth submanifolds in $J^{k+s}(M, m)$.

We begin with singularities of type Σ_l. In this case, triad $(x_{k+s+1}, \mathcal{U}, L_0)$, defining tangent plane $T_{x_{k+s}}(L)$, $x_{k+s} \in \Sigma_1 L$, contains two lines: $\mathcal{U} \subset T_x^*$ and $L_0 \subset S^{k+s}\mathcal{U} \otimes \alpha_x$. Suppose that $\lambda \in T_x^*$, $\lambda \neq 0$ is the generating vector of line $\mathcal{U}$; then the generator of line L_0 has the following form: $\theta = \lambda^{k+s} \otimes e$, where $e \in \alpha_x$, $e \neq 0$. Inclusion $T_{x_{k+s}}L \subset T_{x_{k+s}}\mathcal{E}$ implies that $L_0 \subset g^{(s)}(x_k)$, and thus covector λ is characteristic for a system of differential equations at point $x_k \in \mathcal{E}$, $x_k = \pi_{k+s,k}(x_{k+s})$.

THEOREM. For a system of differential equations $\mathcal{E} \subset J^k(\alpha)$ to have solutions L passing through point x_{k+s} and projecting onto subspace $\mathcal{U}^0 \subset T_x$, $x_{k+s} \in \Sigma_1 L$, $(\alpha_{k+s})_*(T_{x_{k+s}}L) = \mathcal{U}^0$, $\mathrm{codim}\,\mathcal{U}^0 = 1$, it is necessary for the generators of line $\mathcal{U} = \mathrm{Ann}\,\mathcal{U}^0$ to be characteristics of system $\mathcal{E}$ at point x_k.

COROLLARY. Elliptic systems of differential equations have no solutions with singularities of type Σ_1.

Note that, generally speaking, elliptic systems can have solutions with singularities of type Σ_l, where $l \geq 2$. Thus, branch points of Riemannian planes give singularities of type Σ_2 for solutions of a system of Cauchy–Riemann differential equations.

3.5. Let us return to the general case. Fix point $x_k \in \mathcal{E} \subset J^k(\alpha)$ and type of singularity Σ_l. We shall say that integral manifold L passes through point x_k and is projected onto $\mathcal{U}^0 = \mathrm{Ann}\,\mathcal{U} \subset T_x$, where $\mathcal{U} \subset T_x^*$ is a subspace, $\dim \mathcal{U} = l$, if $\mathrm{Im}(\alpha_{k+s})_*(T_{x_{k+s}}L) = \mathcal{U}^0$ for some point $x_{k+s} \in L$, $x_k = \pi_{k+s,k}(x_{k+s})$. We give conditions under which a system of differential equations $\mathcal{E}$ has no solutions passing through point x_k and being projected onto $\mathcal{U}^0$. To this end we introduce the following notation. By $V^{\mathbf{C}}$ we shall denote the complexification of vector (over $\mathbf{R}$) field B. Covector $\lambda \in (T_x^*)^{\mathbf{C}}$, $\lambda \neq 0$, is called the *complex characteristic* for a system of differential equations $\mathcal{E}$ at point $x_k \in \mathcal{E}$ if $\lambda^k \otimes e \in (g(x_k))^{\mathbf{C}}$ for a vector $e \in \alpha_x^{\mathbf{C}}$, $e \neq 0$. For each complex characteristic covector λ we denote subspace

$$K(\lambda) = \{e \in \alpha_x^{\mathbf{C}},\ \lambda^k \otimes e \in (g(x_k))^{\mathbf{C}}\}$$

by $K(\lambda) \subset \alpha_x^{\mathbf{C}}$.

Suppose that $\mathbf{P}(T_x^*)^{\mathbf{C}}$ is the projectivization of a complex cotangent space and $\mathbf{P}\,\mathrm{Char}^{\mathbf{C}}(\mathcal{E}, x_k) \subset \mathbf{P}(T_x^*)^{\mathbf{C}}$ is a complex projective characteristic manifold generated by complex characteristic covectors. In this notation the following result is valid.

THEOREM [5, 12]. Let system of differential equations $\mathcal{E} \subset J^k(\alpha)$ and subspace $\mathcal{U} \subset T_x^*$ be such that

1) the linear submanifold $\mathbf{P}(\mathcal{U})^{\mathbf{C}}$ of a complex projective space $\mathbf{P}(T_x^*)^{\mathbf{C}}$ intersects the complex projective characteristic manifold $\mathbf{P}\,\mathrm{Char}^{\mathbf{C}}(\mathcal{E}, x_k)$ in a finite number of points $\lambda_1, \ldots, \lambda_r$, and

2) $$\sum_{j=1}^{r} \dim_{\mathbf{C}} K(\lambda_j) < \dim_{\mathbf{R}} \mathcal{U}.$$

Then there exists a natural number s_0 depending on $\dim\mathcal{U}$, $\dim\alpha$, and on order k of system $\mathcal{E}$ such that all prolongations $\mathcal{E}^{(s)}$ of a system of differential equations $\mathcal{E}$ for $s \geq s_0$ have no solutions given by smooth integral manifolds, which would pass through point $x_k \in \mathcal{E}$ and be projected onto subspace $\mathcal{U}^0$.

Remark. 1) The number s_0 can be calculated exactly like the corresponding number from the Poincaré δ-lemma.

2) Assume that $\mathbf{P}\mathrm{Char}^{\mathbf{C}}(\mathcal{E}, x_k)$ is a complex projective manifold of codimension c and degree d, not lying in any hyperplane. Then in a general position situation $r = d$ if $c = l - 1$ and $r = 0$ if $c > l - 1$. When $c = l - 1$ and $r = d$, we obtain from hypothesis 2) of the theorem that $d \leq l - 1$, that is, $d \leq c$. But (see [18]) for projective manifolds not lying in any hyperplane we always have $d \geq c + 1$. Therefore, in a general position situation, singularities of type Σ_l on integral manifolds that are solutions of system $\mathcal{E}$ for $l \leq c$ are not, as a rule, realized.

4. GEOMETRIZATION OF GENERALIZED FUNCTIONS

In this section we establish a connection between the solutions of linear systems of differential equations that are generalized in a geometric sense and generalized solutions in the sense of distribution theory. The principal constructions of this section (Sec. 4.3) were the result of rethinking the canonical Maslov operator and attempting to carry it over to a more generalized situation. All the bundles, differential operators, and equations in this section are linear.

4.1. Green's Formula. For every smooth vector bundle $\alpha: E(\alpha) \to B$ we denote by $\alpha^t: E(\alpha^t) \to B$ the bundle dual to it: $\alpha^t = \mathrm{Hom}(\alpha, \Lambda^n\tau^*)$, where $\tau^*: T^*B \to B$ is a cotangent bundle and $n = \dim B$. If $a \in \Gamma(\alpha)$ is a smooth section of bundle α, and $a^t \in \Gamma(\alpha^t)$ is a section of the dual bundle, then by $\langle a, a^t\rangle \in \Lambda^n(B)$ we will denote the differential n-form that is the value of section a^t on section a. Each morphism $\sigma: \alpha \to \beta$ of vector bundles over manifold B generates a conjugate-to-it morphism $\sigma^t: \beta^t \to \alpha^t$, where $\langle\sigma(a), b^t\rangle = \langle a, \sigma^t(b^t)\rangle$, $\forall a \in \Gamma(\alpha)$, $b^t \in \Gamma(\beta^t)$. Denote by $\mathrm{Diff}_1(\alpha, \beta)$ the module [over $C^\infty(B)$] of linear first-order differential operators acting from sections of bundle α into sections of bundle β. If $\Delta: \alpha \to \beta$ and $\nabla: \beta^t \to \alpha^t$ are two first-order operators, the symbols $\sigma_\lambda(\Delta)$ and $\sigma_\lambda(\nabla)$ of which on any differential 1-form $\lambda \in \Lambda^1(B)$ are skew adjoint: $\sigma_\lambda{}^t(\Delta) + \sigma_\lambda(\nabla) = 0$, then operator

$$\gamma(\Delta, \nabla) : \Gamma(\alpha\otimes\beta^t)\to\Lambda^n(B),$$

where $\gamma(\Delta, \nabla)(a \otimes b^t) = \langle\Delta(a), b^t\rangle - \langle a, \nabla(b^t)\rangle$, is well defined and is a first-order linear differential operator.

The symbol of this operator determines homomorphism $\omega: \alpha \otimes \beta^t \to \Lambda^{n-1}(B)$ according to the formula

$$\sigma_\lambda(\gamma(\Delta, \nabla))(a\otimes b^t) = \lambda\wedge\omega(a\otimes b^t).$$

So, operators $\gamma(\Delta, \nabla)$ and $d \cdot \omega$ have the same symbol and consequently differ by a zero-order operator. Replacing, if it is necessary, operator ∇ by $\nabla + \nabla'$, where $\nabla' \in \mathrm{Hom}(\beta^t, \alpha^t)$, we obtain for each operator $\Delta \in \mathrm{Diff}_1(\alpha, \beta)$ the single operator $\Delta^t \in \mathrm{Diff}_1(\beta^t, \alpha^t)$ called the *adjoining* to operator Δ for which $\gamma(\Delta, \Delta^t) = d \cdot \omega$.

Note that homomorphism ω is uniquely determined by the symbol of operator Δ since $\sigma_\lambda(\gamma(\Delta, \nabla))(a \otimes b^t) = \langle\sigma_\lambda(\Delta)(a), b^t\rangle$. Thus, $\omega = \omega_\Delta$, and for every operator $\Delta \in \mathrm{Diff}_1(\alpha, \beta)$ there exists a unique adjoint-to-it operator for which Green's formula

$$\langle \Delta(a), b^t \rangle - \langle a, \Delta^t(b^t) \rangle = d\omega_\Delta(a, b^t) \tag{1}$$

is valid.

This formula can be carried over to operators of higher order by decomposing each operator into first-order operators. So, if $\Delta = \Delta_1 \cdot \Delta_2$, where $\Delta \in \mathrm{Diff}_2(\alpha, \gamma)$, $\Delta_1 \in \mathrm{Diff}_1 \in(\beta, \gamma)$, $\Delta_2 \in \mathrm{Diff}_1(\alpha, \beta)$, then $\Delta^t = \Delta_2{}^t \cdot \Delta_1{}^t$, and Green's formula has the form

$$\langle \Delta(a), b^t \rangle - \langle a, \Delta^t(b^t) \rangle = d\omega_\Delta(a, b^t), \tag{2}$$

where $\omega_\Delta(a, b^t) = \omega_{\Delta_1}(\Delta_2 a, b^t) + \omega_{\Delta_2}(a, \Delta_1{}^t b^t)$.

Every k-th order operator $\Delta \in \mathrm{Diff}_k(\alpha, \beta)$ can be realized as a composition of (k − 1)-th and first-order operators. To this end first note that operator Δ defines (and is defined by) a morphism of vector bundles φ_Δ: $\alpha_k \to \beta$, where $\varphi_\Delta([h]_x^k) = \Delta(h)(x)$. And, in addition, bundle α_k is imbedded in a natural way into a bundle of 1-jets of the sections of bundle α_{k-1}, which we will denote by $\alpha_{k-1,1}$. Let F_{k-1} be an additional bundle: $\alpha_{k-1,1} = \alpha_k \oplus F_{k-1}$. Then, extending in a trivial way homomorphism φ_Δ: $\alpha_k \to \beta$ to F_{k-1} and identifying the obtained homomorphism φ: $\alpha_{k-1,1} \to \beta$ with the differential operator $\Delta_F \in \mathrm{Diff}_1(\alpha_{k-1}, \beta)$, we obtain a representation of operator Δ in the following form: $\Delta = \Delta_F \cdot j_{k-1}$, where $j_{k-1} \in \mathrm{Diff}_{k-1}(\alpha, \alpha_{k-1})$, j_{k-1}: $h \to j_{k-1}(h)$. Iterating this process and using formula (2), we obtain Green's formula for operators of any order. Here, standard reasoning shows that operator $\Delta^t \in \mathrm{Diff}_k(\beta^t, \alpha^t)$ does not depend on the choice of additional subbundles F_{k-1}, F_{k-2}, ..., whereas Green's form $\omega_\Delta(a, b^t)$ does. Furthermore, Green's form defines the (k − 1)-th order bidifferential operator ω_Δ: $a \times b^t \to \omega_\Delta(a, b^t)$ with respect to each argument.

4.2. Green's Formula on Jet Bundles. We carry over Green's formula to bundles α_k. Mapping φ_Δ: $\alpha_k \to \beta$ corresponding to operator $\Delta \in \mathrm{Diff}_k(\alpha, \beta)$ defines a section of the induced bundle $\alpha_k^*(\beta)$, which, as before, we denote by φ_Δ. In particular, the identity operator $\mathrm{id} \in \mathrm{Diff}_k(\alpha, \alpha)$ defines section $\rho_k = \varphi_{\mathrm{id}} \in \Gamma(\alpha_k^*(\alpha))$. For each section $b^t \in \Gamma(\beta^t)$ we denote by $\langle \varphi_\Delta, b^t \rangle$ the α_k-horizontal n-form on manifold $J^k(\alpha)$, the value at point $x_k \in J^k(\alpha)$, $x_k = [h]_x^k$, of which equals $\langle \Delta(h)(x), b^t(x) \rangle$. In an analogous way we associate with Green's form ω_Δ and section b^t an α_{k-1}-horizontal (n − 1)-form $\omega_\Delta(b^t) \in \Lambda_0^{n-1}(J^{k-1}\alpha)$, the value at point x_{k-1} of which equals $\omega_\Delta(h, b^t)(x)$.

In this notation Green's formula

$$\langle \varphi_\Delta, b^t \rangle - \langle \rho_k, \Delta^t b^t \rangle = \hat{d}\omega_\Delta(b^t) \tag{1}$$

is valid on manifold $J^k(\alpha)$. To prove this formula it is enough to restrict the left- and right-hand sides of (1) to a section of form $j_k(h)$, $\forall h \in \Gamma(\alpha)$, and use Green's formula [see (1) from subsection 4.1].

4.3. Generalized Functions. Suppose that $L \subset J^k(\alpha)$ is an oriented integral manifold for which projection α_k: $L \to B$ is a proper mapping. We connect with this integral manifold a generalized function κ_L of class $\mathscr{D}'$ on manifold B the value of which on section $a^t \in \Gamma(\alpha^t)$ with compact support is given by the formula

$$\varkappa_L(a^t) = \int_L \langle \rho_k, a^t \rangle . \tag{1}$$

The use of Green's formula in form (1) from subsection 4.2 leads to the following result.

THEOREM [13]. If $L \subset \mathscr{E}^{(s)} \subset J^{k+s}(\alpha)$ is an integral manifold which is a generalized solution in a geometric sense of a linear system of differential equations $\mathscr{E} \subset J^k(\alpha)$, then κ_L is a solution of this system in the sense of generalized functions. Here the singular support of κ_L is contained in the critical set $\alpha_k(\Sigma L)$.

Supplement. 1) Formula (1) can be carried over in an obvious way to arbitrary integral manifolds with singularities. In this connection, the theorem formulated above remains valid.

2) We present formula (1) in terms of the branches of integral manifold L. Assume that the basis of B is oriented, and let $L \setminus \Sigma L = \bigcup_r L_r$ be the partition into connected components of the complement to the singular set. For each block of L_r, mappings α_{k+s}: $\mathscr{U}_r \to \mathscr{U}_r = \alpha_{k+s}(L_r)$ are local diffeomorphisms, and L_r is the graph of a k-jet of the smooth multivalued section h_r in the neighborhood $\mathscr{U}_r$. We ascribe to each block of L_r the sign $\varepsilon(L_r)$ equal to the sign of the Jacobian mapping α_{k+s}: $L_r \to U_r$. Let $\Omega \in \Lambda^n(B)$ be the volume form on manifold B and $a^t = a^* \cdot \Omega$, where $a^* \in \Gamma(\alpha^*)$, $\alpha^* = \mathrm{Hom}(\alpha, 1)$. Then formula (1) takes the form

$$\varkappa_L(a^*) = \sum_r \varepsilon(L_r) \int_{\mathscr{U}_r} \langle h_r, a^* \rangle \Omega, \tag{1'}$$

where $\int_{\mathscr{U}_r} \langle h_r, a^* \rangle \Omega$ is equal to the sum of the integrals along the branches of h_r.

3) This formula shows how we can connect with any, possibly nonoriented, integral manifold a generalized function on $\Gamma(|\alpha^t|)$, where by $|\alpha^t|$ we denote the bundle geometrically dual to bundle α [29]. Orientation conditions enable us to choose in an unambiguous manner the sign $\varepsilon(L_r)$. In the general case this is possible only if the first

characteristic class w (see below, Sec. 9) dual to the cycle of Σ_1-singularities vanishes on L. When bundle α is functorally connected to manifold B, for example, it is a bundle of half-forms, function $\varepsilon(L_r)$ assumes four values ±1, ±i. Compatibility conditions here are the triviality of class w computed with respect to mod 4 ("quantification conditions"). In this sense classes w are generalizations of the Maslov—Arnol'd classes to arbitrary jet bundles.

4) Solutions of elliptic systems of differential equations do not contain singularities of type Σ_1. Therefore, on solutions of such systems class w_1^k is trivial and the sign $\varepsilon(L_r)$ can be chosen to be equal identically to 1. The previous theorem thus asserts that the sum of the branches of a multivalued solution of an elliptic system of differential equations is a smooth section.

5. SINGULARITIES, TRANSFER OPERATORS, AND SHOCK WAVES

The connection between generalized functions and integral manifolds which was indicated above and also the fact that singularities of generalized solutions of linear differential equations propagate along bicharacteristics permits us to hope that this circumstance has a purely geometric nature. In this section we show that this is indeed so.

5.1. μ-adic Singularities. Let $B_0 \subset B$ be a submanifold of codimension $t \geq 1$ and $\mu = \{f \in C^\infty(B) \mid f_{B_0} = 0\}$ an ideal corresponding to this submanifold. For each vector bundle α: $E(\alpha) \to B$ the module of smooth sections $\Gamma(\alpha^0)$ of the restriction of bundle α to submanifold B_0 can be identified in a natural way with quotient-module $\Gamma(\alpha)/\mu\Gamma(\alpha)$. In particular, $C_\infty(B_0) = C_\infty(B)/\mu$.

Further exposition is a transposition of the general concept of transfer operators as applied to μ-adic filtration [16]. The principal role here is played by the normal ν: $E(\nu) \to B_0$ and conormal ν^*: $E(\nu^*) \to B_0$ bundles to submanifold B_0. The fiber ν_x of the normal bundle ν at point $x \in B_0$ is the quotient space T_xB/T_xB_0 and fiber ν_x^* of the conormal bundle ν^* is, correspondingly, $\operatorname{Ann} T_xB_0 \subset T_x^*B$. Module Λ_ν of smooth sections of the conormal bundle is identified with quotient μ/μ^2, and quotients μ^k/μ^{k+1} are, naturally, identified with k-symmetric powers $S^k\Lambda_\nu$. Correspondingly, for an arbitrary vector bundle α isomorphisms

$$\mu^k\Gamma(\alpha)/\mu^{k+1}\Gamma(\alpha) \simeq \Gamma(\alpha^0) \otimes S^k\Lambda_\nu$$

hold.

5.2. Transfer Operators. Let $\Delta \in \mathrm{Diff}_k(\alpha, \beta)$. Consider the following model problem: find a section $a \in \mu^k\Gamma(\alpha)$, having an ℓ-th order of smallness on submanifold B_0 for which $\Delta(a) = 0 \bmod \mu^\infty\Gamma(\beta)$. The natural course of solving this problem is as follows. For an arbitrary section $a_j \in \mu^{j+k}\Gamma(\alpha)$ inclusion $\Delta(a_j) \in \mu^j\Gamma(\beta)$ is valid. We clarify which conditions should be imposed on section a_j, so that $\Delta(a_j) \in \mu^{j+1}\Gamma(\beta)$. To this end we introduce operator $\sigma_\mu^{(j)}(\Delta)$: $\Gamma(\alpha^0) \otimes S^{j+k}\Lambda_\nu \to \Gamma(\beta^0) \otimes S^j\Lambda_\nu$ acting according to the following rule:

$$\sigma_\mu^{(j)}(\Delta)([a_j]) = \Delta(a_j) \bmod \mu^{j+1}\Gamma(\beta),$$

where $[a_j] = a_j \bmod \mu^{j+1}\Gamma(\alpha)$.

It is easy to verify that operators $\sigma_\mu^{(j)}(\Delta)$ are $C^\infty(B_0)$-homomorphisms. We call them *symbols of operator* Δ *along submanifold* B_0. Using these symbols, conditions $\Delta(a_j) \in \mu^{j+1}\Gamma(\beta)$ can be written more concisely: $\sigma_\mu^{(j)}([a_j]) = 0$.

Suppose that these conditions are fulfilled. Let us see under which conditions, without changing $[a_j]$, we can obtain an even "smaller" right-hand side. For this, obviously, it is necessary that $\Delta(a_j + a_{j+1}) \in \mu^{j+2}\Gamma(\beta)$ for a certain choice of section $a_{j+1} \in \mu^{j+k+1}\Gamma(\alpha)$, i.e., it is necessary that $\Delta(a_j) \bmod \mu^{j+2}\Gamma(\beta) \in \operatorname{Im}\sigma_\mu^{(j+1)}(\Delta)$. Let us formalize this process. To this end, consider $C^\infty(B_0)$-modules $\mathscr{H}_\mu^j(\Delta) = \operatorname{Ker}\sigma_\mu^{(j)}(\Delta) \subset \Gamma(\alpha^0) \otimes S^{j+k}\Lambda_\nu$, $\mathscr{C}_\mu^j(\Delta) = \operatorname{Coker}\sigma_\mu^{(j)}$, and consider operators

$$\Delta_j^1 : \mathscr{H}_\mu^j(\Delta) \to \mathscr{C}_\mu^{j+1}(\Delta),$$
$$\Delta_j^1([\alpha_j]) = (\Delta(a_j) \bmod \mu^{j+2}\Gamma(\beta)) \bmod \operatorname{Im}\sigma_\mu^{(j+1)}(\Delta). \quad (1)$$

The solubility conditions of our problem have the following form on the first two steps: $[a_j] \in \mathscr{H}_\mu^j$ and $\Delta_j^1([a_j]) = 0$. For further progress we need to enlist spectral sequences.

It is easy to verify that operators Δ_j^1 are first-order linear differential operators on submanifold B_0. We call them *transfer operators*. A nontriviality condition for the construction set forth is the requirement that $\mathcal{K}_\mu^j(\Delta) \neq 0$. In this case, submanifold B_0 is said to be *characteristic*.

From a geometric standpoint modules $\mathcal{K}_\mu^j(\Delta)$ [as, too, $\mathcal{C}_\mu^j(\Delta)$] can be considered as the modules of the sections of families of vector spaces $B_0 \ni x \to K^j(x) \subset \alpha_x \otimes S^{j+k}\mathcal{U}_x$, where $\mathcal{U}_x = \operatorname{Ann} T_x B_0$, and mappings $\sigma_{\mu,x}^{(j)}(\Delta)$ as the restrictions of the j-prolongations of the symbol of operator Δ to subspace $\mathcal{U}_x$ for which diagrams

$$\begin{array}{ccc} \alpha_x \otimes S^{j+k}\mathcal{U}_x & \xrightarrow{\sigma_{\mu,x}^{(j)}(\Delta)} & \beta_x \otimes S^j\mathcal{U}_x, \\ \cap & & \cap \\ \alpha_x \otimes S^{j+k}T_x^* & \xrightarrow{\sigma_x^{(j)}(\Delta)} & \beta_x \otimes S^j T_x^* \end{array} \tag{2}$$

are commutative.

Consider the commutative diagram:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}_\mu^{(j)}(\Delta) & \to & \Gamma(\alpha^0)\otimes S^{j+k}\Lambda_\nu & \xrightarrow{\sigma_\mu^{(j)}(\Delta)} & \Gamma(\beta^0)\otimes S^{j}\Lambda_\nu & \to & \mathcal{C}_\mu^{(j)}(\Delta) & \to & 0 \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
0 & \to & \mathcal{K}_\mu^{(j-1)}(\Delta)\otimes\Lambda_\nu & \to & \Gamma(\alpha^0)\otimes S^{(j+k-1)}\Lambda_\nu\otimes\Lambda_\nu & \xrightarrow{\sigma_\mu^{(j-1)}(\Delta)\otimes 1} & \Gamma(\beta^0)\otimes S^{j-1}\Lambda_\nu\otimes\Lambda_\nu & \to & \mathcal{C}_\mu^{(j-1)}(\Delta)\otimes\Lambda_\nu & \to & 0 \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
0 & \to & \mathcal{K}_\mu^{(j-t)}(\Delta)\otimes\Lambda_\nu^t & \to & \Gamma(\alpha^0)\otimes S^{j+k-t}\Lambda_\nu\otimes\Lambda_\nu^t & \xrightarrow{\sigma_\mu^{(j-t)}(\Delta)\otimes 1} & \Gamma(\beta^0)\otimes S^{j-t}\Lambda_\nu\otimes\Lambda_\nu^t & \to & \mathcal{C}_\mu^{(j-t)}(\Delta)\otimes\Lambda_\nu^t & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{3}$$

in which we denote by Λ_ν^i the i-th exterior power of module Λ_ν and by δ the restriction of the Spencer δ-operator.

Suppose that $H_\nu^{j,i}(\Delta)$ are cohomologies of the first column in the term $\mathcal{K}_\mu^{(j)}(\Delta) \otimes \Lambda_\nu^i$, and $H_\nu^{j,k}(\mathcal{C}_\Delta)$ are cohomologies of the second column in the term $\mathcal{C}_\mu^j(\Delta) \otimes \Lambda_\nu^i$. Observe that $H_\nu^{j,i}(\Delta)$ and $H_\nu^{j,i}(\mathcal{C}_\nu)$ are $C^\infty(B_0)$-modules and the values of the former at point $x \in B_0$ equal the δ-cohomologies of the symbol of intersection $g_\Delta(\mathcal{U}_x) = g_x(\Delta) \cap \alpha_x \otimes S^k\mathcal{U}_x$, where $g_x(\Delta)$ is the kernel of the symbol of operator Δ at point $x \in B_0$. For this reason module $H_\nu^{j,i}(\Delta)$ is called the *δ-cohomologies of Spencer along submanifold* B_0.

Note that since the middle two columns in the given diagram are acyclical, isomorphisms

$$H_\nu^{j,i}(\mathcal{C}_\Delta) \simeq H_\nu^{j-2,i+2}(\Delta) \tag{4}$$

hold. In particular, sequence

$$0 \to \mathcal{C}_\nu^{(j)} \xrightarrow{\delta} \mathcal{C}_\nu^{(j-1)} \otimes \Lambda_\nu \xrightarrow{\delta} \mathcal{C}_\nu^{(j-2)} \otimes \Lambda_\nu^2 \tag{5}$$

is exact for all $j \geq 3$ if and only if

$$H_\nu^{j+j,2}(\Delta) = H_\nu^{j,3}(\Delta) = 0$$

for all $j \geq 0$.

The use of diagram (3) and isomorphism (4) enables us to prove, analogously to the proof of formal integration criteria given in [32], the first part of the following assertion [the second part uses the exactness of sequence (5)].

THEOREM. Let submanifold $B_0 \subset B$ and differential operator $\Delta \in \operatorname{Diff}_k(\alpha, \beta)$ be such that

1) families of vector spaces k^0: $x \to K_\nu^{(0)}(x)$ and k^1: $x \to K_\nu^{(1)}(x)$, $\forall x \in B_0$ define smooth vector bundles over manifold B_0 and

2) $H_\nu^{j,2}(\Delta) = 0$, $\forall j \geq 0$. Then for all $j \geq 0$ the families of vector spaces $k^j\colon x \to K_\nu^{(j)}(x)$, $c^j\colon x \to C_\nu^{(j)}(x)$ define smooth vector bundles over manifold B_0.

If, in addition, $H_\nu^{j,3}(\Delta) = 0$, $\forall j \geq 0$, then the symbols of transfer operators Δ_j^1, $j \geq 2$, are defined by the symbol of transfer operator Δ_1^1 from the following commutative diagram:

$$\begin{array}{ccc} \mathscr{K}_\nu^{(j)}(\Delta)\otimes\Lambda^1(B_0) & \xrightarrow{\sigma(\Delta_1^j)} & \mathscr{C}_\nu^{(j+1)}(\Delta) \\ \downarrow\delta & & \downarrow\delta \\ \mathscr{K}_\nu^{(j-1)}(\Delta)\otimes\Lambda_\nu\otimes\Lambda^1(B_0) & \xrightarrow{\sigma(\Delta_1^{j-1})\otimes 1} & \mathscr{C}_\nu^{(j)}(\Delta)\otimes\Lambda_\nu \end{array}$$

and in this sense differential operators Δ_1^j are defined by the first transfer operator.

5.3. Transfer Operators in the Nonlinear Case. Before we carry over the results of the previous subsection to arbitrary differential equations we make a small digression. Suppose that $\Delta \in \mathrm{dif}_k(\alpha, \beta)$ is a nonlinear differential operator acting from a section of bundle α into a section of bundle β and $h \in \Gamma(\alpha)$ is a fixed section. We define linear differential operator $l_h(\Delta) \in \mathrm{Diff}_k(\alpha, \beta)$, the *linearization of operator* Δ *on section* h, by the relation

$$l_h(\Delta)(a) = \frac{d}{dt}\Big|_{t=0} \Delta(h + ta), \tag{1}$$

where $a \in \Gamma(\alpha)$.

In this case, for an arbitrary operator Δ we have

$$\Delta(h+\varepsilon) - \Delta(h) - l_h(\Delta)(\varepsilon) \in \mu^{2j}\Gamma(\beta), \tag{2}$$

if $\varepsilon \in \mu^{j+k}\Gamma(\alpha)$.

And when Δ is a quasilinear operator, we have

$$\Delta(h+\varepsilon) - \Delta(h) - l_h(\Delta)(\varepsilon) \in \mu^{2j+1}\Gamma(\beta). \tag{3}$$

Note also that in this case

$$l_{h+\eta}(\Delta)(\varepsilon) - l_h(\Delta)(\varepsilon) \in \mu^3\Gamma(\beta), \tag{4}$$

if $\eta, \varepsilon \in \mu^{k+1}\Gamma(\alpha)$.

Therefore, the solution of the problem, analogous to the one formulated in subsection 5.2, for a nonlinear operator (i.e., finding solutions different from the given solution h by a "small" solution along submanifold B_0) leads, as above, to transfer operators $\Delta_{1,h}^j$ corresponding to linearization $l_h(\Delta)$. Consider the quasilinear operator $\Delta \in \mathrm{dif}_k(\alpha, \beta)$ and the corresponding to it system of differential equations $\mathscr{E}_\Delta = \varphi_\Delta^{-1}(0) \subset J^k(\alpha)$. Let $L \subset \mathscr{E}_\Delta$ be a solution of this system such that cycle of singularities $\Sigma_t L \subset L$ for some $t > 0$ is diffeomorphically projected onto submanifold $B_0 \subset B$. Define on this cycle the following families of vector spaces: $k_\Sigma(x_k) = \mathrm{Ker}(\alpha_k\,|\,_L)_{*,x_k} \subset \alpha_x \otimes S^k\mathscr{U}_x$, $c_\Sigma(x_k) = \mathrm{Coker}\,\sigma_{x_k}(\Delta)$, where $x_k \in \Sigma_t L$, $\mathscr{U}_x = \mathrm{Ann\,Im}(\alpha_k\,|\,_L)_{*,x_k} \subset T_x^*$, and also $k_\Sigma^{(l)} \subset \alpha_x \otimes S^{k+l}\mathscr{U}_x$ to be the prolongations of spaces k_Σ and $c_\Sigma^{(l)} = \mathrm{coker}\,\sigma_{x_k}^{(l)}(\Delta)$, where $\sigma_{x_k}^{(l)}(\Delta)$ is an l-prolongation of symbol $\sigma_{x_k}(\Delta)\colon \alpha_x \otimes S^k\mathscr{U}_x \to \beta_x$ and $\sigma_{x_k(l)}(\Delta)\colon \alpha_x \otimes S^k\mathscr{U}_x \to \beta_x \otimes S^l\mathscr{U}_x$. Assume (see the Theorem in Sec. 5.2) that these families form smooth vector bundles over manifold $\Sigma_t L$. Choose an arbitrary section $h \in \Gamma(\alpha)$ so that $j_k(h)(B_0) = \Sigma_t L$, and using linearization $l_h(\Delta)$ construct the transfer operator

$$\Delta_\Sigma^1 : \Gamma\left(k_\Sigma^{(1)}\right) \to \Gamma\left(c_\Sigma^{(2)}\right). \tag{5}$$

Relations (4) imply that this operator is independent of the choice of section h which represents $\Sigma_t L$ over manifold B_0 and in so doing unambiguously determines a first-order operator on cycle $\Sigma_t L$. Passing to prolongations $\mathscr{E}^{(l)}$ of the given system of differential equations we have the following result.

THEOREM. Let $L \subset J^{k+l}(\alpha)$ be an integral manifold that defines a solution of a prolonged system of differential equations $\mathscr{E}^{(l)} \subset J^{k+l}(\alpha)$ whose cycle of singularities $\Sigma_t L \subset L$ for some fixed value $t > 0$ is diffeomorphically projected onto submanifold $B_0 \subset B$. Then the restrictions of transfer operators $\Delta_{1,h}^{(l+1)}$ to submanifold $k_\Sigma^{(l+1)}$ define first-order differential operators $\Delta_\Sigma^{(\ell+2)} \in \mathrm{Diff}_1(k_\Sigma^{(k+1)}, c_\Sigma^{(\ell+2)})$ on cycle of singularities $\Sigma_t L$ independently of the choice of representation section $h \in \Gamma(\alpha)$.

Remark. 1) Further on (see Sec. 7) we shall show that subspace $k_\Sigma^{(1)}(x_k)$, $\forall x_k \in \Sigma_t L$ is endowed with a Jordan algebra structure if $k = 1$, and with a commutative algebra structure if $k \geq 2$.

2) For $l = 0$, transfer operators do not depend on the choice of representative section h (or "background") only for the quasilinear operators Δ; for $l \geq 1$, operator Δ can be arbitrary since its extensions $\Delta^{(\ell)}$ are always quasilinear.

5.4. Hamiltonian Formalism. The classical theorem on the propagation of singularities along bicharacteristics is essentially a theorem on the structure of transfer operators for Σ_1-singularities. In order to verify this we shall need the following version of Hamiltonian formalism on jet bundles.

Denote by $\Phi_k(\alpha)$ the total space of bundle $\alpha_k^*(\tau^*)$ induced from cotangent bundle τ^*: $T^*B \to B$ of projections α_k: $J^k(\alpha) \to B$, and by τ_k^*: $\Phi_k(\alpha) \to T^*B$, $\tau_{k,s}$: $\Phi_k(\alpha) \to \varphi_s(\alpha)$ natural projections. Let $\Phi_k(\alpha)$ be an algebra of smooth functions on space $\Phi_k(\alpha)$ and let $\Phi_\infty(\alpha) = \lim_{k\to\infty} \mathrm{proj}\, \Phi_k(\alpha)$.

Each section $h \in \Gamma(\alpha)$ defines section $j_k^\tau(h)$: $T^*B \to \Phi_k(\alpha)$ of bundle τ_k; $j_k^\tau(h)$: $(x, p) \to ([h]_x^k, p)$. Define the *higher Poisson bracket* $\{H, G\}$ on algebra $\Phi_\infty(\alpha)$, $H \in \Phi_k(\alpha)$, $G \in \Phi_l(\alpha)$, $\{H, G\} \in \Phi_s(\alpha)$, where $s = \max(k + 1, l + 1)$, with the help of the following universality property:

$$(j_s^\tau(h))^*\{H, G\} = \{(j_k^\tau(h))^* H, (j_l^\tau(h))^* G\} \tag{1}$$

for all sections $h \in \Gamma(\alpha)$.

On the right-hand side of formula (1) the Poisson bracket on $C^\infty(T^*M)$ is denoted by { , }.

The higher Poisson bracket gives a Hamiltonian structure on algebra $\Phi_\infty(\alpha)$ in the sense that this algebra is a Lie algebra relative to the bracket and, in addition, Leibnitz identity

$$\{H, G_1 \cdot G_2\} = \{H, G_1\} \cdot G_2 + \{H, G_2\} \cdot G_1$$

is fulfilled.

The last property of the Poisson bracket shows that mapping X_H: $\Phi_\infty(\alpha) \to \Phi_\infty(\alpha)$, $X_H(G) = \{G, H\}$ defines Hamiltonian differentiation for every function $H \in \Phi_k(\alpha)$. Here on each finite step X_H: $\Phi_\ell(\alpha) \to \Phi_s(\alpha)$ is a vector field over mapping $\tau_{s,\ell}$.

Suppose now that smooth integral manifold $L \subset J^{k+\ell}(\alpha)$ is a solution of a system of differential equations $\mathscr{E} \subset J^k(\alpha)$, $L \subset \mathscr{E}^{(\ell)}$. Denote by L_0 submanifold $\Sigma_{1,0}L \subset L$, and by $\hat{L}_0 \subset \Phi_{k+\ell}(\alpha)$ its natural lifting: $\hat{L}^0 = \{(x_{k+\ell}, p) \mid x_{k+\ell} \in L_0,\ p \in \mathrm{Ann\,Im}(\alpha_{k+\ell})_*(T_{x_{k+\ell}}L) \subset T_x^*\}$. If a function $S \in C^\infty(B)$ is chosen such that $\alpha_{k+\ell}^*(S) = 0$ on L_0 but $d\alpha_{k+\ell}^*(S) \neq 0$, then by $\hat{L}_{0,s} \subset \hat{L}_0$ we denote the set of points $(x_{k+\ell}, d_x S)$ where $x_{k+\ell}$ runs L_0.

Assume now that $\mathscr{E} = \mathscr{E}_\Delta$ is a determined system of differential equations, $\Delta \in \mathrm{dif}_k(\alpha, \alpha)$, and let $H = H_\Delta$ be the Hamiltonian of operator Δ, $H_\Delta \in \Phi_k(\alpha)$, $H_\Delta(x_k, p) = \det \sigma_{x_k,p}(\Delta)$ $\forall x_k \in J^k(\alpha)$, $p \in T_x^*$, $\alpha_k(x_k) = x$.

Let $\ell(x_{k+\ell})$ be a set of points $x\alpha_{k+\ell+1}$ from $E^{(\ell+1)}$, such that $L(x_{k+\ell+1}) \supset T_{x_{k+\ell}}(L_0)$. Then $\ell(x_{k+\ell})$ is an affine subspace in fiber $F(x_{k+\ell})$ the dimension of which coincides with $\dim \ker \sigma_{x_{k,p}}(\Delta)$, where $p \in T_x^*$ is a covector such that $\mathrm{Ker}\ \alpha_{k+\ell}^*(p) \supset T_{x_{k+\ell}}L_0$. Vectors $X_{H,y} \in T_{x_{k+\ell,p}}(\Phi_{k+\ell}(\alpha))$ do not depend on the choice of elements $y = (x_{k+\ell+1}, p)$, where $x_{k+\ell+1} \in \ell(x_{k+\ell})$, and define at the points of submanifold $\hat{L}_0$ vector field V_H tangent to this submanifold. Here, if submanifold $\hat{L}_{0,S} \subset \hat{L}_0$ is chosen, then the projection of a vector field from points of submanifold $\hat{L}_{0,S}$ onto $J^{k+\ell}(\alpha)$ defines vector field $V_{H,S}$ tangent to submanifold L_0. This field is called a field of bicharacteristics.

THEOREM. For a determined system of differential equations $\mathscr{E}_\Delta \subset J^k(\alpha)$, transfer operators constructed on cycles of Σ_1-singularities cover the field of bicharacteristics on $\Sigma_1 L$ provided that $\dim k_\Sigma = 1$.

5.5. Shock Waves. Projection $\alpha_{k,0}(L) \subset \mathscr{E}(\alpha)$ of an arbitrary integral manifold can be viewed as a multivalued section of bundle α. However, the values occurring in the majority of differential equations are unique. The requirements for solution uniqueness lead to various rules for obtaining a unique but possibly piecewise smooth solution, shock waves, from a given multivalued solution L. Note that in contrast to the usually used approach, when a class is postulated in which a solution is sought, we start from a concrete object: integral manifolds. With this approach many empirical rules (for example, the rule of Maxwell areas) receive a geometric interpretation (for details see [39]). Taking into account the connection with generalized functions it seems to be most natural to conduct a selection of branches from the complement to the singular set $L \setminus \Sigma L$. However, it is not so. And the usually applied here principle of "maximal delay" is valid only for solutions of linear systems of differential equations. In the general case (i.e., taking into account nonlinearity) projection $B_\Sigma = \alpha_k(\Sigma L) \subset B$ describes the "area of metastability" of a solution [39]. A "jumpover," i.e., passage from one branch to another, occurs along a submanifold (possibly with singularities) $B_0 \subset B$ that lies in a "neighborhood" of a critical set.

Consider briefly some methods of branch selection.

a) Gugonio—Rankin Rule. Assume that submanifold $B_0 \subset B$ divides the basis of B into two domains $B_+ \cup B_- = B\backslash B_0$, in each of which one of the branches of solution h^+ or h^- is chosen. And suppose also that the conservation law of θ is given, i.e., we are given an (n — 1)-form $\theta \in \Lambda^{n-1}(J^k\alpha)$, the restriction of whose differential $d\theta$ to an arbitrary solution of a system of differential equations is trivial. Gugonio—Rankin condition corresponding to the conservation of θ consists in the fact the restrictions of differential (n — 1)-forms $\theta_{h^+} = (j_k(h^+))^*(\theta) \in \Lambda^{n-1}(B_+)$ and $\theta_{h^-} = j_k(h^-)^*(\theta) \in \Lambda^{n-1}(B_-)$ to submanifold B_0 coincide (see, for example, [11]). Gugonio—Rankin conditions $\theta_{h^+}|_{B_0} = \theta_{h^-}|_{B_0}$ give a differential equation connecting h^+, h^-, and function S if $B_0 = S^{-1}(0)$, $S \in C^\infty(B)$. For linear systems of equations these conditions coincide with the Hamilton—Jacobi characteristic equation on function S, which enables us to determine the boundary of the discontinuity of B_0. In particular, here $B_0 = B_\Sigma$ always. In the nonlinear case in order to find function S and in so doing the boundary of the shock wave, we need to use additional conservation laws [39].

b) Thermodynamical Principle. Consider the simplest situation when in domains $B_\pm\backslash B_\Sigma$ solution L is a graph of a k-jet of multivalued section $h^\pm$ the branches of which in domain $B_+\backslash B_\Sigma$ are denoted by $h_1^+, \ldots, h_s^+, \ldots$, and in domain $B_-\backslash B_\Sigma$ by $h_1^-, \ldots, h_t^-, \ldots$, respectively. Fix also the conservation law of θ. To each branch $h_i^\pm$ we associate the number $E_i^\pm = \int_{B_i^\pm} \theta_{h_i^\pm}$, where $B_i^\pm$ is the boundary of the domain of branch $h_i^\pm$, which we call *θ-energy of branch* $h_i^\pm$. In addition to this we ascribe to each branch a number $\varepsilon_i^\pm = \varepsilon(j_k(h_i^\pm))$ (an analog to spin) that determines the "decompatibility" of orientation (or a number determined by the index of the intersection with a cycle of Σ_1-singularities, for the nonoriented case see Sec. 9). In both cases $\varepsilon_i^\pm \in \mathbf{Z}_2$. If α is a bundle of half-forms, then $\varepsilon_i^\pm \in \mathbf{Z}_4$, etc. In all the cases, the possibility to associate to a branch the number $\varepsilon_i^\pm$ is equivalent to "quantification conditions": a class of cohomologies w (with respect to mod 2 or mod 4) on integral manifold L is trivial.

State I = (i, j) is defined to be transfer $h_i^+ \to h_j^-$. To each state we ascribe a number, transfer energy $E_I = E_i^+ - E_j^-$ if numbers ε_i^+ and ε_j^- coincide, and $E_I = \infty$ if ε_i^+ and ε_j^- differ. Introducing partition function $Z = \sum_I \exp(-\beta E_I)$, we can compute in the usual manner (see, for example [7]) transfer probabilities, temperature, entropy, etc.

A somewhat different principle is obtained if the "time" axis is isolated on a manifold: $B = \tilde{B} \times \mathbf{R}_t$. Then, considering various sections t = const and performing the situation described above on each section, we obtain a picture of the transfers (which depends on time). This reflects Prigogine's [25, 26] idea that "a description of a system undergoing bifurcation includes both a deterministic and a probabilistic element."

6. INTEGRAL GRASSMANNIANS

Grassmannians of subspaces tangent to integral manifolds or, more concisely, integral Grassmannians are arranged quite curiously. They are projective manifolds, but with the exception of several isolated cases, which also include the Grassmannian of Lagrange subspaces, they are not smooth manifolds. The conditions for the contiguity of natural states to a regular cell lead to Jordan structures on cycles of singularities, which we shall describe in the next section.

6.1. Integral Planes. An *integral plane* in Cartan subspace $C(x_k)$, $x_k \in J^k(\alpha)$ is defined to be an arbitrary n-dimensional isotropic subspace. Recall that every such plane L is defined by triad $(x_{k+1}, \mathcal{U}, L_0)$, where $x_{k+1} \in F(x_k)$, $L_0 = \mathrm{Ker}(\alpha_{k*}: L \to T_x)$, $\mathcal{U} = \mathrm{Ann\,Im}(\alpha_{k*}|_L)$, $\dim \mathcal{U} = \dim L_0$; moreover, $L_0 \subset \alpha_x \otimes S^k\mathcal{U}$ and $L = \mathcal{U}(x_{k+1}) \oplus L_0$, where $\mathcal{U}(x_{k+1}) \subset L(x_{k+1})$ is the preimage of $\mathcal{U}^0 = \mathrm{Im}(\alpha_{k*}|_L)$ under isomorphism $\alpha_{k*}: L(x_{k+1}) \to T_x$. Subspaces $\mathcal{U}$ and L_0, as it follows from their description, are uniquely defined by integral plane L. Two triads $(x_{k+1}, \mathcal{U}, L_0)$ and $(x_{k+1}', \mathcal{U}, L_0)$, where $x_{k+1}' = x_{k+1} + \theta$, $\theta \in \alpha_x \otimes S^{k+1}T_x^*$, define the same integral plane if $\theta \in T_x^* \cdot L_0 + \alpha_x \otimes S^k\mathcal{U}$, where by $T_x^* \cdot L_0 \subset \alpha_x \otimes S^{k+1}T_x^*$ we denote a subspace spanned by symmetric products $a \cdot b$, $a \in T_x^*$, $b \in L_0$.

6.2. Integral Grassmannians. Denote by $I(x_k)$ the Grassmannian of all integral planes lying in $C(x_k)$ and let $I_l = \{L \in I(x_k) \mid \dim \alpha_{k*}(L) = n - l\}$. The set of integral planes isomorphically projected onto a tangent space T_x is identified with $F(x_k)$ and is therefore isomorphic to $\alpha_x \otimes S^{k+1}T_x^*$. To describe subspaces $I_l \subset I(x_k)$, $l \geq 1$, we need additional constructions. Let $G_l(V)$ denote the Grassmannian of l-dimensional subspaces of vector space V. Consider manifold H_l, the elements of which are pairs $(\mathcal{U}, L_0)$, where $\mathcal{U} \in G_l(T_x^*)$, $L_0 \in G_l(\alpha_x \otimes S^k\mathcal{U})$, and bundle $h_l: H_l \to G_l(T_x^*)$, $h_l(\mathcal{U}, L_0) = \mathcal{U}$. Natural mapping $k_l: I_l \to H_l$, associating with integral plane $L = (x_{k+1}, \mathcal{U}, L_0)$ pair $(\mathcal{U}, L_0) \in H_l$, converts I_ℓ into a total space of a smooth affine bundle over H_ℓ the fiber of which at point (U, L_0) is associated with vector space

$$\alpha_x \otimes S^{k+1}T_x^* / T_x^* \circ L_0 + \alpha_x \otimes S^{k+1}\mathcal{U}.$$

Therefore,

$$\dim I_l = m\binom{n+k}{k+1} + mk\binom{l+k-1}{k+1} - l^2. \tag{1}$$

We derive a number of consequences from the obtained formula. First, observe that number $m\binom{n+k}{k+1}$ coincides with the dimension of regular cell I_0. Therefore, number $\nu(m, k, l) = \dim I_0 - \dim I_\ell$ can be interpreted as "codimension" relative to the regular cell. Next, function $\nu(m, k, l)$ decreases in variables m and k and for $l = 1$ we have $\nu(m, k, 1) = 1$; for $k = 1$ we have $\nu(m, 1, l) = (m/2)l - (m/2 - 1)l^2$. Therefore, $\nu(m, 1, l) < 0$ if $l \geq 2$, $m > 4$. For bundles α and the dimension which does not exceed 4, all the cases when $\nu \geq 0$ are indicated in the tables given below.

k \ ℓ	1	2	3	4
1	1	3	6	10
2	1	2	1	-4
3	1	1	-6	-29
4	1	0	-15	-48
5	1	-1	-26	-124

m=1

k \ ℓ	1	2	3	4
1	1	2	3	4
2	1	0	-7	-24
3	1	-2	-21	-74

m=2

k \ ℓ	1	2	3	4
1	1	1	0	-2
2	1	-2	-15	-44

m=3

k \ ℓ	1	2	3
1	1	0	-3
2	1	-4	-23

m=4

Note that for k = 1 and m = 1, 2, we have $\nu(1, 1, l) = \binom{l}{2}$, $\nu(2, 1, l) = l$.

These cases, together with $l = 1$ and k = 1, m = 3, $l = 2$, are called exceptional.

6.3. R-Planes and R-Grassmannians. In nonexceptional cases $\dim I_\ell \geq \dim I_0$ if $l \geq 2$. This fact can be interpreted in the following way. Let $L \subset J^k(\alpha)$ be an integral manifold. Then at each point $x_k \in L$ tangent space $T_{x_k}L$ is an integral plane and, consequently, $T_{x_k}L \in I(x_k)$. The condition that $T_{x_k}L \in I_l(x_k)$ is equivalent to the fact that at this point mapping $\alpha_k\colon L \to B$ has a Thom—Boardman singularity of type Σ_ℓ. The calculation of dimensions conducted above shows that if integral plane $T_{x_k}L$ does not lie on the boundary of regular cell $I_0(x_k)$ and a nonexceptional case holds, then some neighborhood of point x_k consists wholly of the singular points of mapping $\alpha_k\colon L \to B$. Keeping in mind this circumstance we separately isolate a class of R-*planes*, i.e., a class of integral planes lying in the closure of the regular cell $I_0(x_k)$. The Grassmannian of all R-planes at point x_k is denoted by $RI(x_k)$.

Integral manifold L is called an R-*manifold* if singular set ΣL is nowhere compact in L. The tangent plane at each point of R-manifold is an R-plane. The converse is, of course, not true.

To describe R-planes we introduce the following stratification in the R-Grassmannian. Set $RI_\ell = I_\ell(x_k) \cap \overline{I}_0(x_k)$. Next, inductively define subsets $RI_{\ell_1,\ldots,\ell_r} = I_{\ell_r}(x_k) \cap RI_{\ell_1,\ldots,\ell_{r-1}}$, corresponding to sequence $0 < l_1 < \ldots < l_r \leq n$. R-planes occurring in $RI_{\ell_1,\ldots,\ell_r}$, can be characterized as R-planes $L \in RI_{\ell_r}$ adjoining (i.e., lying in the closure) to subsets RI_{ℓ_i}, $0 \leq i \leq r - 1$.

THEOREM. Integral plane L defined by triad $(x_{k+1}, \mathcal{U}, L_0)$ is an R-plane if and only if there exists a sequence of imbedded subspaces $T_x^* \supset \mathcal{U} = \mathcal{U}_0 \supset \mathcal{U}_1\ \mathcal{U} \ldots \supset \mathcal{U}_r \supset 0$ and tensors $\theta_j \in \alpha_x \otimes S^{k+1}\mathcal{U}_{j-1}$, $1 \leq j \leq r + 1$, such that subspace $L_0 \subset \alpha_x \otimes S^k\mathcal{U}$ is generated by tensors of form $v \lrcorner \delta\theta_j$ for all $1 \leq j \leq r + 1$ and $v \in \mathrm{Ann}\,\mathcal{U}_j$.

Supplement. 1) In the hypotheses of the theorem the R-plane L lies in subset $RI_{\ell_1,\ldots,\ell_r}$, where $l_i = \dim \mathcal{U}_{r-i+1}$.

2) Tensors θ_j involved in the formulation of the theorem are not arbitrary. Dimensionality considerations imply that subspaces $L_j = \{v \lrcorner \delta\theta_j, \forall v \in \mathrm{Ann}\,\mathcal{U}_{j+1}\}$ have a dimension equal to $\dim\mathcal{U}_{j+1} - \dim\mathcal{U}_j$.

7. JORDAN STRUCTURES ASSOCIATED TO REGULAR INTEGRAL PLANES

The description of R-planes given in the previous section enables us to connect in a natural way to each "regular" R-plane a Jordan algebra or an associative algebra with involution. For convenience of notation fix point $x_k \in J^k(\alpha)$ and set $\alpha_x = W$, $T_x/\operatorname{Ann} U = V$.

7.1. Tensors. Every symmetric tensor $\theta \in W \otimes S^{k+1}V^*$ defines a symmetric mapping $M_\theta\colon V \to W \otimes S^kV^*$, where $M_\theta(v) = v \rfloor \delta\theta$. Set $L(\theta) = \operatorname{Im} M_\theta$. Number $\dim L(\theta)$ is called the *rank* ($\operatorname{rk}\theta$) *of symmetric tensor* θ. Tensor θ is called *nondegenerate* if $\operatorname{rk}\theta = \dim V = n$.

Let $Cd_r^{k+1}(W, V) \subset W \otimes S^{k+1}V^*$ be a set of tensors of rank r and $Cd^{k+1}(W, V)$ a cone of degenerate tensors. If $\operatorname{Ker}\theta = \operatorname{Ker} M_\theta \subset V$ is a degeneration subspace of tensor θ, then restriction $\bar\theta \in W \otimes S^{k+1}(V/\operatorname{Ker}\theta)^*$ is well-defined; moreover, tensor $\bar\theta$ is already nondegenerate. To describe space $Cd_r^{k+1}(W, V)$ we consider three bundles over Grassmannian $G_r(V)$ of r-dimensional subspaces: (1) η is a tautological bundle, (2) η_W and η_V are trivial bundles with fibers W and V, respectively. Elements $a(V_1)$, $V_1 \in G_r(V)$ of the total space of tensor product $\eta_W \otimes S^{k+1}(\eta_V/\eta)^*$ can be viewed as tensors $\theta \in W \otimes S^{k+1}V^*$, such that $\operatorname{Ker}\theta \supset V_1$. Therefore, set $Cd_r^{k+1}(W, V)$ is a smooth open submanifold in $W \otimes S^{k+1}V^*$ of codimension $\nu_r(m, n, k+1) = m\binom{n+k}{k+1} - m\binom{n+k-r}{k+1} - r(n-r)$, where $m = \dim W$. And the codimension of the cone of degenerate tensors $Cd^{k+1}(W, V)$ is equal to $m\binom{n+k-1}{k} - n + 1$.

So, for example, the codimension of degenerate quadrics in $W \otimes S^2V^*$ is equal to $(m-1)n + 1$, and the codimension of degenerate homogeneous polynomials in $S^{k+1}V^*$, $\dim V = 2$, is equal to k.

7.2. Regular Subspaces. Subspace $L \subset W \otimes S^kV^*$ is called *regular* if its first prolongation $L^{(1)} \subset W \otimes S^{k+1}V^*$ is nontrivial and is not contained in the cone of degenerate tensors. Note that if the R-plane is defined by triad $(x_{k+1}, \mathcal{U}, L_0)$, then $L_0 = L(\theta)$ for some tensor $\theta \in d_x \otimes S^{k+1}\mathcal{U}$ i.e., in the representation of the theorem of subsection 6.3 we can choose a chain of length 1 if and only if L_0 is a regular subspace.

Examples. 1) if $\dim\mathcal{U} = 1$, then $L_0 = L(\theta)$ for some tensor θ.

2) If $\dim\mathcal{U} = 2$, then triad $(x_{k+1}, \mathcal{U}, L_0)$ defines an R-plane if and only if either subspace L_0 is regular or $L_0 \cap Cd_1^k(\alpha_x, T_x(\operatorname{Ann}\mathcal{U})) \neq \varnothing$.

7.3. Jordan Algebras. Assume that nondegenerate tensors θ and θ' define the same space $L = L(\theta) = L(\theta')$. In this case, $M_{\theta'} = M_\theta \cdot a$ for some isomorphism $a\colon V \to V$, i.e., $v \rfloor \delta\theta' = av \rfloor \delta\theta$, $\forall v \in V$, or $\delta\theta' = (1 \otimes a^*)\delta\theta$. Thus, there exists tensor $\theta' = \theta_a$ corresponding to operator $a \in \operatorname{End}V$, if and only if $\delta(1 \otimes a^*)\delta\theta = 0$. Let $\operatorname{Jor}\theta \subset \operatorname{End}V$ be the set of operators satisfying this equality.

Nondegenerate operators $a \in \operatorname{Jor}\theta$ are put in correspondence to various representations of space $L(\theta)$ in the form of $L(\theta_a)$. For arbitrary operators $a \in \operatorname{Jor}\theta$ correspondence $a \to \theta_a$ defines an isomorphism between $\operatorname{Jor}\theta$ and the first extension $L^{(1)}(\theta)$.

An equivalent description of the elements of $\operatorname{Jor}\theta$ can be given in terms of the second differential θ. Namely, define the second differential $\delta_2\theta$ of tensor θ as mapping $\delta_2\theta\colon S^2V \to W \otimes S^{k+1}V^*$, where $\delta^2\theta(v^1, v_2) = v_1 \rfloor \delta(v_2 \rfloor \delta\theta)$. It is not hard to verify that operator $a \in \operatorname{End}V$ lies in set $\operatorname{Jor}\theta$ if and only if

$$\delta_2\theta(av_1, v_2) = \delta_2\theta(v_1, av_2) \tag{1}$$

for all vectors $v_1, v_2 \in V$.

This description implies that set $\operatorname{Jor}\theta$ forms a Jordan algebra, i.e., $\operatorname{Jor}\theta \subset \operatorname{End}V$ is a vector subspace closed with respect to anticommutator $\{a_1, a_2\} = 1/2(a_1a_2 + a_2a_1)$.

Examples. Let $\dim V = 2$, $k = 2$, $\dim W = 1$; then each nondegenerate tensor $\theta \in S^3V^*$ in the suitable coordinates $x, y \in V^*$ can be written in one of the following standard forms:

1) elliptic case: $\theta = x^3 - 3xy^2$,

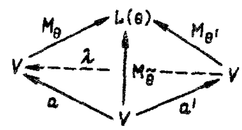
2) hyperbolic case: $\theta = x^3 + y^3$,

3) parabolic case: $\theta = xy^2$.

Each of these cases is determined by the type of intersection of plane $L(\theta)$ and cone $Cd^2(\mathbf{R}, V)$ of degenerate quadrics: in case 1) we have $L(\theta) \cap Cd^2(\mathbf{R}, V) = 0$; in case 2) $L(\theta) \cap Cd^2(\mathbf{R}, V)$ is a pair of distinct lines; and in case 3) $L(\theta) \cap Cd^2(\mathbf{R}, V)$ is one straight line. The corresponding Jordan algebras are as follows: 1) $\operatorname{Jor}\theta = \mathbf{R}[X]/X^2 + 1$; 2) $\operatorname{Jor}\theta = \mathbf{R}[X]/X^2 - 1$; 3) $\operatorname{Jor}\theta = \mathbf{R}[X]/X^2$.

7.4. Associative Algebras. Denote by $\mathrm{As}(\theta) \subset \mathrm{End}\,V$ the associative algebra generated by Jordan algebra Jor θ. Relation $\delta_2\theta(av_1, v_2) = \delta_2\theta(v_1, a^{\#}v_2)$ for all $v_1, v_2 \in V$ unambiguously determines for each operator $a \in \mathrm{As}(\theta)$ operator $a^{\#} \in \mathrm{As}(\theta)$, and in so doing also involution $\#$: $\mathrm{As}(\theta) \to \mathrm{As}(\theta)$. This involution enables us to represent algebra $\mathrm{As}(\theta)$ as direct sum $\mathrm{As}(\theta) = A_0(\theta) \oplus A_1(\theta)$, where $A_0(\theta) = \mathrm{Jor}\,\theta = \{a \in \mathrm{As}(\theta) \mid a^{\#} = a\}$ is a space of Hermitian elements and $A_1(\theta) = \{a \in \mathrm{As}(\theta) \mid a^{\#} = -a\}$ is a space of skew symmetric Hermitian elements.

Choose nondegenerate tensors θ', $\tilde\theta \in L^{(1)}(\theta)$ and consider the following commutative diagram:



in which $a \in \mathrm{Jor}\,\theta$ is the representative of tensor $\tilde\theta$ and $a' \in \mathrm{Jor}\,\theta'$ of tensor $\tilde\theta'$. Then $\mathrm{Jor}\,\theta = \lambda(\mathrm{Jor}\,\theta')$, where $\lambda = M_\theta^{-1}M_{\theta'}$. Relation (1) of subsection 7.3 implies that $\lambda, \lambda^{-1} \in \mathrm{Jor}\,\theta$ and, therefore, $\mathrm{As}(\theta) \subset \mathrm{As}(\theta')$. Analogously, $\mathrm{As}(\theta') \subset \mathrm{As}(\theta)$ and, therefore, algebras $\mathrm{As}(\theta)$ and $\mathrm{As}(\theta')$ coincide. Involution $\#'$ in algebra $\mathrm{As}\,\theta'$, defined by tensor θ', is conjugate to involution $\#$, $\#' = \lambda^{-1} \cdot \# \cdot \lambda$. Thus, the change of a generator in $L^{(1)}(\theta)$ leads to a "turn" of the Jordan algebra inside algebra $\mathrm{As}(\theta)$ and to the conjugation of the involution.

7.5. To describe the structure of algebra $\mathrm{As}\,\theta$, we define symmetric form $F(a, b) = \mathrm{tr}\{a, b^{\#}\}$, $a, b \in \mathrm{As}(\theta)$. Degeneration subspace Ker F coincides with radical $\mathrm{Rad}(\theta)$ of algebra $\mathrm{As}(\theta)$. Therefore, algebra $\mathrm{As}(\theta)$ is semisimple if and only if F is a nondegenerate symmetric form.

Relation $\delta_2\theta(av_1, v_2) = \delta_2\theta(v_1, a^{\#}v_2)$ implies that $\theta(av_1, \ldots, v_{k+1}) = \theta(v_1, a^{\#}v_2, \ldots, v_{k+1}) = \theta(v_1, v_2, av_3, \ldots, v_{(k+1)}) = \theta(a^{\#}v_1, v_2, \ldots, v_{k+1})$, and since tensor θ is nondegenerate, then for $k \geq 2$ we have $\# = \mathrm{id}$ and, consequently, $\mathrm{As}\,\theta$ is a commutative algebra.

Radical $\mathrm{Rad}(\theta)$ of algebra $\mathrm{As}(\theta)$ is a nilpotent ideal. Denote by $l = l(\theta)$ the degree of nilpotency of this ideal: $(\mathrm{Rad}(\theta))^{l+1} = 0$, but $(\mathrm{Rad}(\theta))^{l} \neq 0$. To indicate a more direct connection between the degree of nilpotency of $l(\theta)$ and tensor θ we introduce the following concept.

Let $(A, \#)$ be an arbitrary associative subalgebra in algebra End V. Consider subspace

$$\eta(A, \#) = \{\lambda \in S^2V^* \mid \lambda(av_1, v_2) = \lambda(v_1, a^{\#}v_2),\ \forall v_1, v_2 \in V,\ \forall a \in A\}$$

and define the *index of tensor* θ in the following way:

$$\mathrm{ind}(\theta) = \min_{\lambda \in \eta(\mathrm{As}\theta, \#)} (\min(n_+(\lambda), n_-(\lambda) + n_0(\lambda))), \tag{2}$$

where by $n_\pm(\lambda)$ we denote the positive (negative, respectively) inertia index, and by $n_0(\lambda)$ the defect of quadric $\lambda \in S^2V^*$.

In other words, the index of tensor θ is equal to the smallest dimension of the maximal isotropic subspaces of quadrics $\lambda \in \eta(\mathrm{As}(\theta), \#)$. Inertia index and the degree of nilpotency are related by the following inequality:

$$\left[\frac{1}{2} l(\theta)\right] \leqslant \frac{2}{\varkappa(\theta)} \mathrm{ind}(\theta), \tag{3}$$

where $\kappa(\theta)$ is the dimension of the minimal irreducible representation of semisimple algebra $\mathrm{As}(\theta)/\mathrm{Rad}(\theta)$.

As a consequence of this inequality, we get the following assertion: if subspace $\eta(\mathrm{As}(\theta), \#)$ contains a nondegenerate definite quadric, then algebra $\mathrm{As}(\theta)$ is semisimple.

7.6. Realization. We indicate first the conditions for the realization of an algebra with involution $(A, \#) \subset \mathrm{End}\,V$ as algebra $(\mathrm{As}\,\theta, \#)$ for some nondegenerate symmetric tensor θ under the assumption that subspace $\eta(A, \#)$ contains a nondegenerate quadric h. In this case, involution $\#_h$: End V $\to$ End V generated by this quadric with restriction to algebra A coincides with involution $\#$. Denote by $\theta_A \in C_0^*(A) \otimes S^2V^*$, where $C_0(A)$ is the set of Hermitian elements (relative to $\#_h$) of the centralizer $C(A)$ of algebra A in End V, the quadric defined by relation

$$\langle b, \theta_A \rangle (v_1, v_2) = h(bv_1, v_2), \tag{4}$$

where $b \in C_0(A)$, $v_1, v_2 \in V$.

This quadric is nondegenerate since $\langle 1, \theta_A \rangle = h$.

THEOREM 1. For an arbitrary algebra with involution $(A, \#) \subset (\mathrm{End}\, V, \#_h)$, generated by quadric $h \in S^2V^*$, inclusion $(A, \#) \subset (\mathrm{As}(\theta_A), \#)$ holds. If centralizer C(A), as an algebra over **R**, is generated by the Hermitian part $C_0(A)$ and double centralizer C(C(A)) coincides with algebra A, then this inclusion is isomorphic.

COROLLARY. Suppose that $A \subset \mathrm{End}\, V$ is a semisimple algebra with involution generated by quadric h, the centralizer's Hermitian part $C_0(A)$ of which generates C(A); then $A = \mathrm{As}(\theta_A)$.

If we discard the conditions that algebra A be represented in vector space V and tensor θ lie in $W \otimes S^2V^*$, then the previous theorem can be augmented.

THEOREM 2. Let (A, #) be a unitary semisimple finite-dimensional algebra over **R**, the Jordan subalgebra Jor $(A, \#) = \{a \in A, a^{\#} = a\}$ of which generates A. Then algebra (A, #) can be realized as algebra $(\mathrm{As}\,\theta_A, \#)$ for nondegenerate symmetric tensor $\theta_A \in \mathrm{Jor}(A, \#)^* \otimes S^2A^*$ on algebra A given by formula $\langle b, \theta_A \rangle (a_1, a_2) = F_A(R_b a_1, a_2)$.

Here, by $F_A(x, y)$ we denote a quadric on algebra A; $F_A(x, y) = \mathrm{tr}(L_x \cdot L_{y\#})$, and by $L: A \to \mathrm{End} A$ and $R: A^0 \to \mathrm{End}\, A$ we denote the left and right regular representations $L_x(a) = xa$, $R_x(a) = ax$, $\forall x, a \in A$.

And, finally, an arbitrary unitary finite-dimensional commutative algebra over **R** can be realized as algebra As (θ_A) for symmetric tensor $\theta_A \in A \otimes S^2A^*$, defined by the multiplication $\theta_A(a_1, a_2) = a_1a_2$, $\forall a_1, a_2 \in A$.

8. PRIMITIVE SINGULARITIES AND CLIFFORD STRUCTURES

8.1. Primitive Tensors. Nondegenerate symmetric tensor $\theta \in W \otimes S^{k+1}V^*$ is called *primitive* if we cannot indicate a chain of subspaces $V^* = \mathcal{U}_0 \supset \mathcal{U}_1 \supset \ldots \supset \mathcal{U}_2 = 0$, $r > 1$ and tensors $\theta_j \in W \otimes S^{k+1}\mathcal{U}_j$, $1 \leq j \leq r$, such that subspace $L(\theta)$ is generated by tensors of form $v \lrcorner\, \delta\theta_j$, where $v \in \mathrm{Ann}\, U_j$. Tensor θ is called *decomposable* if $\theta = \theta_1 + \ldots + \theta_r$, where $\theta_i \in W \otimes S^{k+1}\mathcal{U}_i$ and $\mathcal{U}_i \subset V^*$ are subspaces such that $V^* = \mathcal{U}_1 \oplus \ldots \oplus \mathcal{U}_r$.

Obviously primitive tensors are nondecomposable.

Geometrically the condition that tensor $\theta \in \alpha_x \otimes S^{k+1}\mathcal{U}_x$ be primitive means that R-planes L of form $(x_{k+1}, \mathcal{U}_x, L(\theta))$ lie in $\mathrm{RI}_\ell(x_k)$, where $l = \dim \mathcal{U}_x$, but they do not lie in closures $\mathrm{RI}_{\ell'}(x_k)$, $0 < l' < l$. Such R-planes are also called *primitive*.

We indicate some connections between Jordan algebra Jor θ and the conditions of decomposability and primitiveness.

First, if tensor θ is primitive, then Jor θ is a simple Jordan algebra. In addition, decompositions of a Jordan algebra into a direct sum of ideals are put in correspondence to the corresponding decomposition of tensor θ into a direct sum. Moreover, if $k \geq 2$, then in the decomposition of tensor θ into a sum of indecomposable terms the latter are determined to within a permutation. This decomposition is put in correspondence to the decomposition of algebra As θ = Jor θ into a sum of local Artinian algebras over **R**.

A complete description of the structure of algebras As(θ) for primitive tensors is given in the following theorem.

THEOREM. (1) Tensor $\theta \in W \otimes S^2V^*$ is primitive if and only if algebra As(θ) is a Clifford algebra with $r \leq 3$ generators and involution # coincides with the standard involutory automorphism of a Clifford algebra. (2) Tensor $\theta \in W \otimes S^{k+1}V^*$, $k \geq 2$, is primitive if and only if algebra As θ = Jor θ is isomorphic to **R** or to **C**.

We give a sketch of the proof of this theorem. First, note that if tensor θ is primitive, then algebra Jor θ contains no nontrivial nilpotents. Therefore, each operator $a \in \mathrm{Jor}\theta$ is semisimple. Next, the factorization of the characteristic polynomial of operator $a \in \mathrm{Jor}\theta$ into relatively prime factors is put in correspondence to a direct decomposition of tensor θ. Therefore, all the elements of $a \in \mathrm{Jor}\theta$ that are not a homothety satisfy the irreducible quadratic equation $a^2 + pa + q = 0$, $p, q \in \mathbf{R}$.

Consider now the restriction of form F, associated with the representation of algebra As(θ) in End V (see Sec. 7.5), to subspace $\mathrm{Jor}^0\theta = \{a \in \mathrm{Jor}\,\theta, \mathrm{tr}\, a = 0\}$. This form is negative definite and the Jordan product can be computed in its terms: $\{a, b\} = 1/nF(a, b)$, where $a, b \in \mathrm{Jor}^0\theta$, and $n = \dim V$. This implies that As(θ) is a Clifford algebra over $\mathrm{Jor}^0\theta$ associated with quadric F. It remains to show that $\dim \mathrm{Jor}\,\theta \leq 4$ for $k = 1$. To this end, consider in Clifford algebra As(θ) element $b = a_1a_2a_3a_4$, where $a_i \in \mathrm{Jor}^0\theta$. Then $b^{\#} = b$; consequently, $b \in \mathrm{Jor}\,\theta$. But $b^2 = b \cdot b^{\#} = 1$ if a_1, a_2, a_3, a_4 are orthogonal (or of length 1) relative to quadric F. Consequently, by virtue of the aforementioned, $b = \pm 1$, whence $a_1 = \pm a_2a_3a_4$ and thus $a_1{}^2 = 1$, which contradicts the negative definiteness of form F.

8.2. A-Types. We call R-plane $L \in RI_l(x_k)$ *regular* if in its representation $L = (x_{k+1}, \mathcal{U}, L_0)$ subspace L_0 is regular, i.e., $L_0 = L(\theta)$ for some nondegenerate tensor $\theta \in \alpha_x \oplus S^{k+1}\mathcal{U}$. We say that regular R-plane $L = (x_{k+1}, \mathcal{U}, L_0)$ has A-*type*, where A is a finite-dimensional associative algebra with involution, if algebras A and $As(\theta)$ are isomorphic as algebras with involution. The theorem of subsection 8.1 leads to the following result.

THEOREM. Primitive R-planes are regular and have A-type, where $A = Cl_r$ is a Clifford algebra with $r \le 3$ generators for $k = 1$ and with $r \le 1$ generator for $k \ge 2$. Algebra $As(\theta)$ corresponding to primitive R-plane $L = (x_{k+1}, \mathcal{U}, L_0)$ defines in $T_x/\mathcal{U}^0 = \operatorname{coker}(\alpha_{k*} \mid_L)$ a module structure over a Clifford algebra.

8.3. Codimensions. We indicate conditions for the realization of Cl_r-types and their codimensions in the Grassmannian of R-planes.

THEOREM. (1) Primitive R-planes $L \in RI_l(x_k)$ are possible for the following values of l, k, $m = \dim \alpha$:

R-types for all values of k, l, and m except $m = 2$, $l \ge 2$ and $m = 1$, $l = 2$, $k = 2$.

C-types, #-id for all values of k, $l = 0, \bmod 2$, m, except $m = 2$, $l \ge 2$, $k = 1$ and $m = 1$, $k = 2$, $l = 4$.

$Cl_2 = \mathbf{H}$-type for all values $m \ge 3$, $l = 0 \bmod 4$, $k = 1$.

Cl_3-type for all values $k = 1$, $l = 0 \bmod 4$, $l \ge 8$, $m \ge 2 + \max\begin{pmatrix}\frac{l_+}{l_-}, & \frac{l_-}{l_+}\end{pmatrix}$ where $4l_+ = \dim\operatorname{Ker}(C - 1)$, $4l_- = \dim\operatorname{Ker}(C + 1)$, C is the generator of the center of algebra Cl_3 and $l_+ \ge 1$, $l_- \ge 1$.

(2) Codimensions of primitive R-planes of a given type in the Grassmannian of all R-planes are given in the following table:

type / k	R	$C = Cl_1$	$Cl_2 = H$	$Cl_3 = H \oplus H$, (l_+, l_-)
$k = 1$	1	$\frac{1}{4}(m-2)t^2 + 2$	$\frac{3}{8}(m-2)l^2 + \frac{1}{4}ml + 3$	$\frac{1}{4}(2m-3)l^2 - 4(m+2)l_+l_- + \frac{1}{2}ml + 4$
$k \geqslant 2$	1	$m\binom{k+l}{k+1} - 2m\binom{\frac{1}{2}l+k}{k+1} - \frac{1}{2}l^2 + 2$	—	—

Remark. a) Cl_3-types are divided into subtypes corresponding to pairs of natural numbers (l_+, l_-), where $(1/4)l = l_+ + l_-$, $l_+ \ge 1$, $l_- \ge 1$. The table gives the codimensions of these types.

2) The table implies that **R**-types can be realized for all values of k and m in codimension 1, and the smallest codimension of C-types equals mk, for **H**-types this codimension equals $7m - 9$, while for Cl_3-types it is $32m - 52$, where $m \ge 3$.

9. THE TOPOLOGY OF INTEGRAL GRASSMANNIANS

9.1. Fix element $x_k \in J^k(M, m)$. The Grassmannian of integral planes in Cartan subspace $C(x_k)$ will be denoted by $I_k(n, m) (= I(x_k))$. Set $I_k(n, m)$ is obtained by intersecting a certain number of quadrics in Grassmann manifold $G_n(C(x_k))$ and is therefore an algebraic manifold. In the following two cases, the topology of these Grassmannians is well known. First, if $n = 1$, then the integrability conditions are missing and all 1-dimensional subspaces in $C(x_k)$ are integrable. Therefore, $I_k(1, m) \simeq \mathbf{RP}^m$. In addition, if $m = k = 1$, then the Cartan distribution on $J^1(M, 1)$ coincides with the standard contact structure and, consequently, Grassmannian $I_1(n, 1)$ is the Grassmannian of n-dimensional Lagrange subspaces in the 2n-dimensional symplectic space $C(x_1)$. Its cohomologies with coefficients in $\mathbf{Z}_2$ were computed by Borel:

$$H^*(I_1(n, 1); \mathbf{Z}_2) = \mathbf{Z}_2[w_1^{(1)}, \ldots, w_n^{(1)}]/((w_1^{(1)})^2, \ldots, (w_n^{(1)})^2),$$

where $w_i^{(1)}$ are Stiefel—Whitney classes of a tautological bundle over the Grassmannian.

Both of these cases are special from the standpoint of the geometry of differential equations. In this section we describe cohomology rings of integral Grassmannian manifolds $I_k(n, m)$ with coefficients in $\mathbf{Z}_2$ provided that $km \ge 2$. We will also limit ourselves to the Grassmannian of nonoriented planes; in the oriented case the results are analogous [14].

THEOREM [14, 40]. Cohomology ring $H^*(I_k(n, m); \mathbf{Z}_2)$ up to dimension n is isomorphic to ring of polynomials $\mathbf{Z}_2[w_1^{(k)}, ..., w_n^{(k)}]$ in Stiefel—Whitney classes $w_1^{(k)}, ..., w_n^{(k)}$ of a tautological bundle over $I_k(n, m)$.

Supplement. Up to dimension n — 1, Stiefel—Whitney classes $w_i^{(k)}$ are algebraically independent. In dimension n algebraic relations are possible only provided that k = 1, m = 2.

9.2. Grassmannians Associated with Differential Equations. Let $\mathcal{E} \subset J^k(M, m)$ be a system of differential equations. For each element $x_k \in \mathcal{E}$ we denote by $I\mathcal{E}(x_k)$ the Grassmannian of the integral planes lying in tangent spaces $Tx_k(\mathcal{E})$. Space $I\mathcal{E}(x_k)$ is called the *integral Grassmannian associated with a system of differential equations* $\mathcal{E}$.

We begin the description of the cohomologies of these spaces with determined spaces of first-order differential equations.

System of differential equations $\mathcal{E} \subset J^1(M, m)$ is called *determined* if $\operatorname{codim}\mathcal{E} = m$, and at each point $x_1 \in \mathcal{E}$, $x_1 = [N]_x^1$, not all covectors $\lambda \in T_x^*N\backslash 0$ are characteristic.

THEOREM. (A) Let $\mathcal{E} \subset J^1(M, m)$ be a determined system of differential equations; then imbedding $I\mathcal{E}(x_1) \hookrightarrow I(x_1)$ induces an isomorphism of the algebras of cohomologies with coefficients in $\mathbf{Z}_2$ up to dimension n in all cases, with the exception of the following.

1) m = 2, n ≥ 3. Here cohomology ring $H^*(IE(x_1; \mathbf{Z}_2)$ up to dimension n is isomorphic to the algebra of polynomials $\mathbf{Z}_2[w_1^{(1)}, ..., w_n^{(1)}, U_{n-1}, \operatorname{Sq} U_{n-1}]$, where we denote by Sq Steenrod's square and degree U_{n-1} equals n — 1.

2) m = 3, n = 2. In this case, cohomology ring $H^*(IE(x_1); \mathbf{Z}_2)$ up to dimension 2 is isomorphic to the algebra of polynomials $\mathbf{Z}_2[w_1^{(1)}, w_2^{(1)}, \alpha_1, ..., \alpha_r]$, where degrees α_i are equal to 2, and r is the number of connected components of characteristic manifold $\operatorname{Char}(\mathcal{E}, x_1)$ over which kernel bundle k is a line bundle.

3) m = n = 2. In this case, integral Grassmannian $I\mathcal{E}(x_1)$ is diffeomorphic either to torus $S^1 \times S^2$ if system $\mathcal{E}$ is hyperbolic at point $x_1 \in \mathcal{E}$ or to complex projective straight line CP^1 if system $\mathcal{E}$ is elliptic at point x_1.

(B) For determined system $\mathcal{E} \subset J^2(M, 1)$ imbedding $I\mathcal{E}(x_2) \subset I(x_2)$ induces an isomorphism of cohomology algebras up to dimension n in all cases except when n = 2. In the latter case, $I\mathcal{E}(x_2)$ is diffeomorphic to torus $S^1 \times S^1$ if $\mathcal{E}$ is hyperbolic at point x_2 and is diffeomorphic to CP^1 if **E** is elliptic at this point.

9.3. Second-order differential equations $\mathcal{E} \subset J^2(1)$ of Monge—Ampere type can be determined with the help of effective n-forms on contact manifold $J^1(1)$ [11]. This enables us, compared to the general case, to go down one order and determine Grassmannians $I(\omega) \subset I_1(n, 1)$ associated with Monge—Ampere equations. Here $\omega \in \Lambda^n(J^1(1))$ is a differential n-form determining the given Monge—Ampere equation and $I(\omega)$ is the Grassmannian of n-dimensional Lagrange subspaces annihilating this form.

In dimension n = 2 Grassmannians $I(\omega)$ are arranged just as in the general case: if the equation is hyperbolic, then $I(\omega) = S^1 \times S^1$, and if it is elliptic then $I(\omega) = CP^1$.

In dimension n = 3 we can connect with each effective form $\omega \in \Lambda^3(C^*(x_1)$ a quadratic form q_ω [17], which differentiates the orbits of symplectic group Sp(6): $q_\omega(X, Y) = -1/4\perp^2(X \rfloor \omega \wedge Y \rfloor \omega)$. If form q_ω is nondegenerate, then its signature $\operatorname{sign} q_\omega$ is equal to either 0 or 2. The value of the signature differentiates two types of orbits. Note that for n = 3, orbits not containing quasilinear representatives exist. Specifically, they are put in correspondence to nondegenerate forms q_ω.

THEOREM [6]. (1) If form q_ω is nondegenerate and $\operatorname{sign} q_\omega = 0$, cohomology ring $H^*(I(\omega); \mathbf{Z}_2)$ up to dimension 3 is isomorphic to the quotient of algebra $\mathbf{Z}_2[w_1^{(1)}, w_2^{(1)}, U_2]$ by the ideal spanned by polynomials $(w_1^{(1)})^2$ and $w_1^{(1)}U_2$: the degree of cohomology class U_2 is equal to 2. If, however, $\operatorname{sign} q_\omega = 2$, then cohomology ring $H^*(I(\omega); \mathbf{Z}_2)$ up to dimension 3 is isomorphic to the quotient of algebra $\mathbf{Z}_2[w_1^{(1)}, w_2^{(1)}, U_2]$ by the ideal generated by $(w_1^{(1)})^2$.

(2) If form q_ω is degenerate, then for elliptic differential equations cohomology ring $H^*(I(\omega); \mathbf{Z}_2)$ up to dimension 3 is isomorphic to the quotient of algebra $H^*(I_1(1, n); \mathbf{Z}_2)$ by the ideal generated by $w_1^{(1)}w_2^{(1)}$. If, however, the equation is hyperbolic at a given point, then imbedding $I(\omega) \hookrightarrow I_1(n, 1)$ induces an isomorphism of cohomologies to dimension 3.

To formulate a general (n ≥ 3) result, we introduce the important concept of the degree of nonlinearity of a Monge—Ampere differential equation at a given point $x_1 \in J^1(1)$. The *degree of nonlinearity* k(L) of effective form $\omega \in \Lambda^n(C^*(x_1))$ relative to Lagrange subspace $L \in I(\omega)$ is defined to be the minimal number among numbers k, such that k-vector $X = X_1 \wedge ... \wedge X_k$, $X_i \in L$ annihilates form ω: $X \rfloor \omega = 0$. The *degree of nonlinearity* $k(\omega)$ (at point x_1) *of a Monge—Ampere differential equation* that corresponds to effective n-form ω is defined to be the smallest among numbers k(L) provided that L runs the whole Grassmannian $I(\omega)$.

THEOREM [6]. If a Monge—Ampere equation is quasilinear at point x_1 [i.e., $k(\omega) = 1$], and n ≥ 4, then imbedding $I(\omega \subset I_1(n, 1)$ induces an isomorphism of cohomologies up to dimension n.

Supplement. In [6], Zil'bergleit also proved that the assertion of the theorem under certain additional conditions remains valid if the degree of nonlinearity k of this equation and the dimension of the basis n are connected by the inequality $n > k + 1/2 + \sqrt{2k + 1/4}$.

9.4. Let us return to the general case. To describe cohomology ring $H^*(IE(x_k); \mathbf{Z}_2)$ in the general case is an insoluble problem. However, we can hope that this ring "stabilizes" for prolongations of a system of differential equations $\mathscr{E}$. This assertion, which is the *topological variant of the Cartan—Kuranishi theorem on prolongation*, constitutes the substance of the following theorem.

Before we formulate this theorem, we introduce the concept of characteristic regularity for systems of first-order differential equations $\mathscr{E} \subset J^1(M, m)$. We note in this connection that integral plane L representable by triad $(x_{l+2}, \mathscr{U}, L_0)$ lies in Grassmannian $I\mathscr{E}^{(l)}(x_{l+1})$ if $L_0 \subset g^{(l)}(\mathscr{U}, x_1)$, where $g^{(l)}(\mathscr{U}, x_1)$ is the *l*th prolongation of symbol $g(\mathscr{U}, x_1) = g(x_1) \cap \nu_x \otimes \mathscr{U}$, $\mathscr{U} \subset T_x{}^*N$, $x_1 = [N]_x{}^1$. Set $\mathrm{Char}_s(\mathscr{E}, x_1) = \{\mathscr{U} \subset T_x{}^*N, \dim \mathscr{U} = s \mid g(\mathscr{U}, x_1) \neq 0\}$. The elements of this set will be called *s-dimensional characteristics*. For s = 1, 1-dimensional characteristics coincide with the usual ones, so that $\mathrm{Char}_1(\mathscr{E}, x_1) = \mathrm{Char}(\mathscr{E}, x_1)$. The condition that subspace $\mathscr{U} \subset T_x{}^*N$ be characteristic can be reformulated differently if we use Guillemin's theorem [35]. Namely, subspace $\mathscr{U}$ is characteristic if and only if its complexification $\mathscr{U}^{\mathbf{C}}$, viewed as a linear submanifold in $\mathbf{P}(T_x{}^*N)^{\mathbf{C}}$ intersects complex characteristic manifold $\mathrm{Char}^{\mathbf{C}}(\mathscr{E}, x_1)$. Denote by $\mathscr{E}_{l+1}(\mathscr{U}) \subset S^{l+2}T_x{}^*N \otimes \nu_x$ a subspace formed by tensors θ such that $\delta\theta \in T_x{}^*N \otimes g^{(l)}(x_1) + \mathscr{U} \otimes S^{l+1}T_x{}^* \otimes \nu_x$, and by $\xi_s{}^{(l+1)}$ the family of vector spaces over $\mathrm{Char}_{n-s}(\mathscr{E}, x_1)$, the fiber at point $\mathscr{U}$ of which is subspace $\mathscr{E}_{l+1}(\mathscr{U})$.

We say a formal integrable system of differential equations $\mathscr{E} \subset J^1(M, m)$ is *characteristically regular at point* x_1 if families of vector spaces $\xi_s{}^{(l+1)}$ are vector bundles over $\mathrm{Char}_{n-s}(E, x_1)$ for all values $1 \leq s \leq n - 1$ and *l* beginning with some number l_0.

THEOREM [14, 40]. Let $\mathscr{E} \subset J^1(M, m)$ be a formal integrable system of differential equations, the complex characteristic manifold $\mathrm{Char}^{\mathbf{C}}(\mathscr{E}, x_1)$ of which does not lie in any hyperplane and $\dim \mathrm{Char}^{\mathbf{C}}(\mathscr{U}, x_1) > 0$. If $\mathscr{E}$ is characteristically regular at point x_1, then imbedding $I\mathscr{E}^{(l)}(x_{l+1}) \hookrightarrow I(x_{l+1})$, $\pi_{l+1,1}(x_{l+1}) = x_1$ for sufficiently large values of *l* induces an isomorphism up to dimension n, inclusive.

Supplement. Since $\delta(\mathscr{E}_{l+1}(\mathscr{U})) \subset E_l(\mathscr{U}) \otimes T_x{}^*N$, it makes sense to speak of Spencer δ-cohomologies of bundles $\xi_s{}^{(l)}$ for every value of s. Analogously to Goldschmidt's formal integrabiity criteria we can obtain the following criterion for characteristic regularity.

Proposition. Suppose that $\xi_s{}^{(l_0)}$ and $\xi_s{}^{(l_0+1)}$ are vector bundles, the fibers of which are 2-acyclical. Then $\xi_s{}^{(l)}$ are vector bundles for all values of $l > l_0$.

10. CHARACTERISTIC CLASSES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

10.1. We unite integral Grassmannians $I(x_k)$, $x_k \in J^k(M, m)$ into total space

$$I^k(M, m) = \bigcup_{x_k \in J^k(M, m)} I(x_k)$$

of bundle $i_k: I^k(M, m) \to J^k(M, m)$, where $i_k(x_k, L) = x_k$, if $(x_k, L) \in I^k(M, m)$, $L \in I(x_k)$.

Each integral manifold $L \subset J^k(M, m)$ defines a *tangential mapping* $t_L: L \to I^k(M, m)$, $t_L(x_k) = (x_k, T_{x_k}(L))$, and each cohomology class $\omega \in H^j(I^k(M, m); \mathbf{Z}_2)$ is a characteristic class on integral manifolds: $L \to \omega(L) = t_L{}^*(\omega) \in H^j(L; \mathbf{Z}_2)$, $0 \leq j \leq n$. Hirsch's theorem and the theorem from subsection 9.1 imply that the cohomology ring of space $I^k(M, m)$ up to dimension n, as algebra over $H^*(J^k(M, m); \mathbf{Z}_2)$, is generated by Stiefel—Whitney classes of a tautological bundle over $I^k(M, m)$. Taking into account that bundles $\pi_{k,k-1}: J^k(M, m) \to J^{k-1}(M, m)$ are affine for $k \geq 2$ and bundle $\pi_{1,0}: J^1(M, m) \to M$ is the fibering of Grassmannians, we obtain the following result.

THEOREM. Cohomology ring $H^*(I^k(M, m); \mathbf{Z}_2)$ to dimension $n = \dim M - m$, as algebra over $H(J^k(M, m); \mathbf{Z}_2)$, is generated by Stiefel—Whitney classes $w_1{}^{(k)}, \ldots, w_n{}^{(k)}$ of a tautological bundle over $I^k(M, m)$. Here algebra $H^*(J^k(M, m); \mathbf{Z}_2)$, as algebra over $H^*(M; \mathbf{Z}_2)$, is generated by Stiefel—Whitney classes $w_1, \ldots, w_n$ of a tautological bundle over $J^1(M, m)$.

10.2. Let $\mathscr{E} \subset J^k(M, m)$ be a formally integrable system of differential equations and $I\mathscr{E}^{(l)} \subset I^{k+l}(M, m)$ is a subset formed by pairs $(x_{k+l}, L) \in I^{k+l}(M, m)$, such that $L \in I\mathscr{E}^{(l)}(x_{k+l})$ and $x_{k+l} \in \mathscr{E}^{(l)}$. The restriction of mapping i_{k+l}

to $I\mathscr{E}^{(l)}$ defines mapping i_{k+l}: $I\mathscr{E}^{(l)} \to \mathscr{E}^{(l)}$, the fibers of which are Grassmannians of integral planes associated with a given system of differential equations. And, since imbeddings $I\mathscr{E}_{(x_{k+l})} \hookrightarrow I(x_{k+l})$ for sufficiently large l induce isomorphisms of cohomologies in dimensions much larger than n, according to Whitehead's theorem, an n-skeleton in the cell partition of $I\mathscr{E}^{(l)}(x_{k+l})$,can be chosen, for sufficiently large values of l, homotopic to an n-skeleton in absolute
the cell partition of $I\mathscr{E}^{(l)}(x_{k+l})$ can be chosen, for sufficiently large values of l, homotopic to an n-skeleton in absolute
Grassmannian $I(x_{k+l})$, Therefore, for large l for formally integrable systems satisfying the hypothesis of the theorem from subsection 9.4, cohomology ring, as module over $H^*(E^{(l)}; Z_2)$, is generated by Stiefel—Whitney classes $w_1^{(k+l}$, ..., $w_n^{(k+l)}$ of a tautological bundle over $I\mathscr{E}^{(l)}$. Note also that since bundles $\mathscr{E}^{(l)} \to \mathscr{E}^{(l-1)}$ are affine for $l \geq 1$, $H^*(\mathscr{E}^{(l)}; Z_2) = H^*(\mathscr{E}, Z_2)$.

10.3. The *frame of the integral manifold* of a Cartan distribution $\Gamma \subset \mathscr{E}^{(l)}$, of dimension less than n is defined to be section h: $\mathscr{E}^{(l)} \to I\mathscr{E}^{(l)}$ of bundle i_{k+l} such that $h(x_{k+l}) \supset T_{x_{k+l}}(\Gamma)$ for all elements $x_{k+l} \in \Gamma$. Framed (n — 1)-dimensional integral manifold $\Gamma \subset \mathscr{E}^{(l)}$ is called *Cauchy data* (Γ, h). The *solution of Cauchy problem* with initial data (Γ, h) is defined to be integral manifold L with boundary ∂L, such that $L \subset \mathscr{E}^{(l)}$, $\partial L = \Gamma$ and $T_{x_{k+l}}(L) = h(x_{k+l})$, $\forall x_{k+l} \in \partial L$.

With each cohomology class $\omega \in H^{n-1}(I\mathscr{E}^{(l)}; Z_2)$ and Cauchy data (Γ, h) we connect characteristic number $\chi_\omega(\Gamma, h) = \langle h^*\omega, Z_\Gamma\rangle$, where $Z_\Gamma \in H_{n-1}(\Gamma; Z_2)$ is the fundamental cycle of manifold Γ. In particular, for an arbitrary partition of number (n — 1) into sum $n - 1 = i_1 + ... + i_r + j_1 + ... + j_s$, where $1 \leq i_1 \leq ... \leq i_r$, $1 \leq j_1 \leq ... \leq j_s$, we can determine characteristic number $\chi_{i_1,...,i_r,j_1,...,j_s}(\Gamma, h) = \chi_\omega(\Gamma, h)$, where $\omega = w_{i_1}\cdot...\cdot w_{i_r}\cdot w_{j_1}^{(k+l)}\cdot...\cdot w_{j_s}^{(k+l)}$.

For formally integrable systems of differential equations satisfying the hypothesis of the theorem from subsection 9.4, for sufficiently large values of the number l, all the characteristic numbers are exhausted by numbers of form $\chi_{i_1,...,i_r,j_1,...,j_s}$.

THEOREM. To solve Cauchy problem with initial data (Γ, h) it is necessary that all characteristic numbers $\chi_\omega(\Gamma, h)$ be equal to zero.

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