LOCAL CLASSIFICATION OF NON-LINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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LOCAL CLASSIFICATION OF NON-LINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Contents

Introduction

0.1. The version of the problem of classifying non-linear partial differential equations formulated here was first stated precisely by A. M. Vinogradov in [6]. There is an obvious connection between this question and the equivalence problem of É. Cartan [35]. A consequence of this connection is the fact that some standard results, such as Darboux's theorem on the local structure of 1-forms, and also those of Sternberg [19], [20] and others, can be interpreted as results on the local classification of first order differential equations.

0.2. The approach to the classification problem we follow here, which allows us to give a meaningful solution of the problem, is based on the geometrical treatment of differential equations going back to Monge, Pfaff, Lie [29], É. Cartan [10], [35] and others. A modern invariant formulation of this approach was given by Vinogradov [6].

0.3. The paper consists of five chapters. The first is of an introductory nature. Its main aim is an exposition of the structures connected with non-linear first order differential equations. A first step in this direction is to give an invariant definition of a (non-linear) differential equation on a smooth manifold M, and of its solution. The natural arena for the action in this case is the manifold $J^1(M)$ of 1-jets of functions as introduced by Ehresmann. In this approach a first order differential equation is most naturally interpreted as being a closed subset $E \subset J^1(M)$. Unless the contrary is specifically stated, we assume throughout that E is a smooth submanifold of $J^1(M)$.

The central feature of Chapter I is an analysis of the concept of solution of a non-linear first order differential equation. Apart from its own independent interest, this question is fundamentally important for us in that its resolution determines the choice of the classifying group; that is, according to Klein, the choice of the corresponding geometry. Indeed, the only and natural requirement imposed on the group is that the transformations it contains must preserve the class of solutions.

For instance, if by a solution of an equation $E \subset J^1(M)$ we understand, as usual, a smooth function $f \in C^\infty(M)$ whose 1-jets $j_1(f)$ lie in E, then the classifying group reduces (as follows easily from Theorem 1.3.3) to the group $T^*(M)$ of canonical diffeomorphisms preserving the fibre of the projection $T^*(M) \to M$. The classification problem for arbitrary non-linear equations relative to this group is void.

A rougher and more interesting classification is achieved at the expense of an extension of the classifying group, induced by an analysis of the many-valued solutions furnished by the R-manifolds. The meaning of the latter is that they are submanifolds of maximal dimension transforming the universal 1-form $U_1 \in \Lambda^1(J^1 M)$ to zero. In a special system of local coordinates $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ in $J^1(M)$, U_1 takes the form $U_1 = -\sum_{i=1}^{n} p_i \, dq_i + du$.

The R-manifolds are closely connected with the Lagrangian submanifolds of $T^*(M)$. In fact, the (local) projection of an R-manifold in $T^*(M)$ is a Langrangian submanifold. However, if we are thinking globally, then the class of R-manifolds is wider than that of Langrangian manifolds. We mention that the necessity for studying R-manifolds was first pointed out by Lie. In recent times the rôle of Langrangian manifolds in linear differential equations has been significantly clarified in papers of Maslov [14] along axiomatic lines and of Hörmander [23] using Fourier integrals. The rôle of R-manifolds for the theory of non-linear differential equations was pointed out by Vinogradov [6], and the term "R-manifold" was proposed by him.

The classifying group obtained on considering many-valued solutions is the group of contact diffeomorphisms of $J^1(M)$ relative to the natural contact structure given by the form U_1. The importance of this group in the theory of differential equations was again first clarified by Lie.

In the Lie algebra of this group (the algebra of contact vector fields) there arises a formalism analogous to the Hamiltonian formalism in symplectic structures (see [2], [18]). We note that this formalism is as vital in the theory of non-linear differential equations as the Hamiltonian formalism is in the theory of linear differential operators. This is traced more carefully in [7] and [6].

0.4. The solution of the classification problem relative to the group of contact diffeomorphisms of $J^1(M)$ is begun in the second chapter. Here we consider the local classification of involutory equations $E^r \subset J^1(M)$, codim $E^r = r$, at a non-singular point, that is, a point $x \in J^1(M)$ where $T_x(E^r)$ and $\Gamma_x = \operatorname{Ker} U_{1,x}$ are transversal. We restrict ourselves to the class of involutory equations, because being involutory is a necessary condition for solubility.

The main result of this chapter is that any two involutory equations $E_1^r \subset J^1(M)$ and $E_2^r \subset J^1(M)$ of the same codimension are locally equivalent at non-singular points. This means in particular, that after a contact diffeomorphism an involutory equation can be written in the neighbourhood of a non-singular point in the form $p_1 = 0, \ldots, p_r = 0$, where $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ is a special system of local coordinates in $J^1(M)$. It follows from this that any two involutory equations of the same codimension in the neighbourhood of a non-singular point have the same stock of many-valued solutions, that is, in particular, they are always locally soluble (in the sense of R-manifolds). As regards the existence of ordinary solutions, it turns out that a necessary and sufficient condition for their existence is that the skew-orthogonal complement to $T_x(E^r) \cap \Gamma_x$ in Γ_x projects non-degenerately onto M.

In particular, if E is the set of zeros of a function $\mathcal{F} \in C^\infty(J^1M)$, $E = \{(q, u, p) \in J^1(M) \mid \mathcal{F}(q, u, p) = 0\}$, then the condition for local solubility in the class of ordinary functions takes the form of a condition of "smooth type", $\left.\frac{\partial \mathcal{F}}{\partial p}\right|_x \neq 0$.

0.5. Beginning in Chapter III, we consider the local equivalence of equations at singular points, that is, points $x \in E^r$ where $T_x(E^r)$ and Γ_x are not transversal, so that $T_x(E^r) \subset \Gamma_x$.

As in the classical Morse theory, we first construct a bilinear form h_ω on $T_x(E^r)$ such that the orbit of h_ω or the orbit of an operator H naturally connected with h_ω under the conformal-symplectic group $CSp(2n)$ is an invariant of the classification problem.

In Chapter III we give a complete list of the invariants characterizing the orbits just mentioned.

Apart from these fundamental questions, we consider necessary conditions for the linearization of equations and we give normal forms of the 2-jet of an equation at a singular point.

0.6. Chapter IV is devoted to the problem of formal equivalence of equations at a singular point. Sufficient conditions are obtained for the formal equivalence of involutory equations E_1^r and E_2^r at a singular point x: if the operators H_k are CSp-equivalent, $k = 1, 2$, and the eigenvalues $\{\lambda_s\}$ are such that $\sum m_i\lambda_i \neq 1$ for all natural numbers m_i whose sum is greater than 2, then the formal Poincaré condition for linearization of the characteristic vector field X_ω at the singular point $x \in E_k^r$ holds when $r = \operatorname{codim} E_k^r = 1$.

For if $\lambda_j = \sum m_i\lambda_i$, with $\sum m_i \geqslant 2$, then since $1 - \lambda_j$ is also an eigenvalue of H_k, we have $1 = (1 - \lambda_j) + \sum m_i\lambda_i$.

Similar conditions for formal equivalence are obtained (see 4.2.11) for equations with a singularity. We consider the most important case, when E is of the form $E = \{\mathcal{F} = 0\}$, where $\mathcal{F}$ is a smooth function on $T^*(M)$, understood as a function on $J^1(M)$ *via* the natural projection $\pi: J^1(M) \to T^*(M)$, and $d\mathcal{F}_x = 0$.

The formal equivalence conditions so obtained allow us to establish the algebraic insolubility of the classification of Hamiltonians in the neighbourhood of a singular point relative to the group of canonical diffeomorphisms.

0.7. In the concluding Chapter V we establish the sufficiency of the formal condition (4.2.9.1) and the CSp-equivalence of the operators H for the local equivalence at a singular point of involutory equations of the same codimension.

We also consider applications of the result so obtained to two classical problems. The first was raised in 1814 by Pfaff, the teacher of Gauss. The problem concerns the reduction of a 1-form $\omega = \sum_{i=1}^{N} a_i(x)dx_i$ to the simplest shape. It was solved by Darboux when the exterior differential $d\omega$ has locally constant rank and ω does not vanish. A full investigation of this case, without the assumption that $d\omega$ has locally constant rank, was carried out by Martinet [13]. The present article contains an investigation of the case when $\omega \in \Lambda'(\mathbf{R}^{2n})$ is involutory (for example, when $d\omega$ has maximal rank) and $\omega_{x_0} = 0$.

The second question is that of the local equivalence of Hamiltonians under the group of canonical diffeomorphisms in the neighbourhood of a singular point. Because of the algebraic insolubility of the question as posed, formal equivalence must be included in the statement of the problem. In this form, and under the condition that the singularity is elementary, the local equivalence problem has already been solved. As above, the classification problem allows us to prove the corresponding results for local solubility.

Finally, I express my deepest thanks to A. M. Vinogradov, who pointed out this circle of problems to me, and who has been a source of constant encouragement and consideration throughout the work. I am grateful also to V. I. Arnol'd for critical remarks and advice, which have helped to improve the article.

CHAPTER I

Geometry of non-linear first order differential equations

The aim of this chapter is to describe the structures connected with non-linear first order equations. It turns out most convenient to represent a differential equation as a closed subset of the manifold of 1-jets of functions. The fundamental structure that arises quite naturally in this approach is the contact structure on the smooth manifold of 1-jets. Here and throughout the sequel, smoothness means belonging to the class C^∞.

§1. Jets

Let M be a smooth manifold, $\dim M = n$, and $C^\infty(M)$ the ring of smooth functions on M.

Consider the maximal ideal $\mu_m \subset C^\infty(M)$, $m \in M$, consisting of the functions that vanish at m:

$$\mu_m = \{f \in C^\infty(M) \mid f(m) = 0\},$$

and the powers of μ_m:

$$\mu_m^k = \{f \in C^\infty(M) \mid f = \sum_{(i_1 \ldots i_k)} g_{i_1} \cdots g_{i_k};\ g_{i_s} \in \mu_m\}, \quad \mu_m^\infty = \bigcap_{k \geq 1} \mu_m^k.$$

1.1.1. DEFINITION. A function $f \in C^\infty(M)$ is said to be *k-flat*, or to be a function of *order of smallness k at m*, if $f \in \mu_m^{k+1}$. The functions $f \in \mu_m^\infty$ are said to be *flat at m*.

This definition is associated with the fact that $f \in \mu_m^{k+1}$ if and only if $\partial^{|\sigma|} f / \partial q_1^{\sigma_1} \ldots \partial q_n^{\sigma_n} |_m = 0$ in every system of local coordinates $q_1, \ldots, q_n$ in the neighbourhood of m, and for every $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $|\sigma| = \sigma_1 + \ldots + \sigma_n \leq k$.

We write $J_m^k(M)$ for the factor ring $C^\infty(M)/\mu_m^{k+1}$.

1.1.2. DEFINITION. The image of $f \in C^\infty(M)$ in $J^k_m(M)$ under the natural projection $C^\infty(M) \to J^k_m(M)$ is called the *k-jet* of f at $m \in M$ and is denoted by $j_k(f)|_m$.

If a system of local coordinates $q_1, \ldots, q_n$ is chosen in the neighbourhood of m, then we can take as representative of $j_k(f)|_m$ the segment of the Taylor series for f up to the terms of degree k, so that the elements of $J^k_m(M)$ can be thought of as the segments up to terms of degree k of the Taylor series of arbitrary functions at m.

1.1.3. Consider $J^k(M) \overset{\text{det}}{=} \bigcup_{m \in M} J^k_m(M)$. We give to $J^k(M)$ the structure of a smooth manifold, $0 \leqslant k < \infty$.

Let $q_1, \ldots, q_n$ be a system of local coordinates in a neighbourhood $\mathcal{U} \subset M$. In $J^k_m(M)$, $m \in \mathcal{U}$, regarded as a vector space, we introduce coordinates p_σ, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $|\sigma| \leqslant k$, relative to the basis $j_k(1)|_m, j_k(q_i - q_i^{(0)})|_m, \ldots, j_k((q_{i_1} - q_{i_1}^{(0)})^{\sigma_1} \ldots (q_{i_k} - q_{i_k}^{(0)})^{\sigma_n})|_m$, where now $q_i^0 = q_i(m)$ are the coordinates of m. Thus, the element $j_k(f)|_m$ can be written in these coordinates as

$$j_k(f)|_m = \left(f(m), \left.\frac{\partial f}{\partial q_1}\right|_m, \ldots, \left.\frac{\partial^{|\sigma|} f}{\partial q_1^{\sigma_1} \ldots \partial q_n^{\sigma_n}}\right|_m, \ldots \right),$$

that is,

$$p_\sigma(j_k(f)|_m) = \left.\frac{\partial^{|\sigma|} f}{\partial q_1^{\sigma_1} \ldots \partial q_n^{\sigma_n}}\right|_m .$$

We now introduce coordinates in the set $\overline{\mathcal{U}} = \bigcup_{m \in \mathcal{U}} J^k_m(M)$, which is by definition open, in the following way. Every point $x \in \overline{\mathcal{U}}$ is uniquely determined by the point $m \in \mathcal{U}$ such that $x \in J^k_m(M)$, and by the coordinates p_σ of x in $J^k_m(M)$. Thus, a point $x \in \overline{\mathcal{U}}$ is uniquely determined by the $(n + 1)$-tuple $(q_1, \ldots, q_n, p_\sigma, |\sigma| \leqslant k)$, where $q_1, \ldots, q_n$ are the coordinates of m and $x \in J^k_m(M)$. It is easily checked that if $x \in \overline{\mathcal{U}}_1 \cap \overline{\mathcal{U}}_2$ and its coordinates $q_1^{(1)}, \ldots, q_n^{(1)}$ in $\mathcal{U}_1$ and $q_1^{(2)}, \ldots, q_n^{(2)}$ in $\mathcal{U}_2$ are smoothly connected, then the transition from the coordinates $(q_1^{(1)}, \ldots, q_n^{(1)}, p_\sigma^{(1)})$ to $(q_1^{(2)}, \ldots, q_n^{(2)}, p_\sigma^{(2)})$ in $\overline{\mathcal{U}}_1 \cap \overline{\mathcal{U}}_2$ is also smooth.

DEFINITION. The set $J^k(M)$ with the smooth structure just introduced is called the *manifold of k-jets* of functions on M.

1.1.4. Let $F\colon M_1 \to M_2$ be a smooth mapping. Then the ring homomorphism $F^*\colon C^\infty(M_2) \to C^\infty(M_1)$, $F^*(f) = f \circ F$, induces a homomorphism $J^k_{m_2}(F)$ of the factor rings: $J^k_{m_2}(F)\colon J^k_{m_2}(M_2) \to J^k_{m_1}(M_1)$, $F(m_1) = m_2$, such that $[J^k_{m_2}(F)]\ (j_k(f)|_{m_2}) = j_k(F^*(f))|_{m_1}$. Taking the union of the $J^k_{m_2}(F)$ for all points $m_2 \in M_2$, we obtain a smooth mapping $J^k(F)\colon J^k(M_2) \to J^k(M_1)$.

PROPOSITION. *Let $F\colon M_1 \to M_2$ and $G\colon M_2 \to M_3$ be two smooth mappings and $1\colon M \to M$ the identity mapping. Then* $J^k(G \circ F) = J^k(F) \circ J^k(G)$, $J^k(1) = 1$.

1.1.5. We consider the mappings π_k: $J^k(M) \to M$, $\pi_{k,l}$: $J^k(M) \to J^l(M)$, $k \geqslant l$, defined by the relations $\pi_k(j_k(f)|_m) = m$, $\pi_{k,l}(j_k(f)|_m) = j_l(f)|_m$. Then π_k turns $J^k(M)$ into a vector bundle over M whose fibre at m is $J^k_m(M)$, with the natural vector space structure.

Let $\mathcal{J}^k(M)$ be the module of smooth sections of this bundle. Then the projection $\pi_{k,l}$: $J^k(M) \to J^l(M)$, which is obviously a morphism of vector bundles, defines a module homomorphism $\nu_{k,l}$: $\mathcal{J}^k(M) \to \mathcal{J}^l(M)$ for $k \geqslant l$.

Let $\tilde{\mathcal{J}}^k(M)$ be the kernel of the homomorphism $\nu_{k,0}$. By definition of $\nu_{k,0}$, the elements of $\tilde{\mathcal{J}}^k(M)$, can be represented as sections of the bundle $\tilde{\pi}_k$: $\tilde{J}^k(M) \to M$, where $\tilde{J}^k(M) = \bigcup_{m \in M} \tilde{J}^k_m(M)$ and $\tilde{J}^k_m(M) = \mu_m / \mu_m^{k+1}$.

EXAMPLE a). Let $k = 0$. Then $J^0_m(M) = \mathbf{R}^1$, so that π_0: $J^0(M) \to M$ is a one-dimensional vector bundle over M. For a section trivializing π_0: $J^0(M) \to M$ we can take $j_0(1)$: $M \to J^0(M)$, $[j_0(1)]\ (m) = j_0(1)|_m$. Thus, $J^0(M) = M \times \mathbf{R}^1$ and $\mathcal{J}^0(M) = C^\infty(M)$.

EXAMPLE b). If $k = 1$, then $\tilde{J}^1(M) = T^*(M)$ is the cotangent manifold to M, and $\tilde{\mathcal{J}}^1(M) = \Lambda^1(M)$ is the module of differentiable 1-forms on M.

In addition to the homomorphisms $\nu_{k,l}$, we introduce the mapping j_k: $C^\infty(M) \to \mathcal{J}^k(M)$, $[j_k(f)](m) = j_k(f)\ |_m$, $m \in M$, $f \in C^\infty(M)$, which (unlike $\nu_{k,l}$) is a differential operator of order k [7].

1.1.6. Let F: $M_1 \to M_2$ be a smooth mapping. Then $J^k(F)$ is a module homomorphism: $\mathcal{J}^k(F)$: $\mathcal{J}^k(M_2) \to \mathcal{J}^k(M_1)$ over the ring homomorphism[1] F^*: $C^\infty(M_2) \to C^\infty(M_1)$: if $s \in \mathcal{J}^k(M_2)$, then $[\mathcal{J}^k(F)(s)](m_1) = [J^k_{m_2}(F)](s(m_2))$, $m_2 = F(m_1)$. The homomorphism $\tilde{\mathcal{J}}^k(F)$: $\tilde{\mathcal{J}}^k(M_2) \to \tilde{\mathcal{J}}^k(M_1)$ is defined similarly; it is also a homomorphism over F^*: $C^\infty(M_2) \to C^\infty(M_1)$.

EXAMPLE a). $k = 0$. Then $\mathcal{J}^0(F) = F^*$: $C^\infty(M_2) \to C^\infty(M_1)$.

EXAMPLE b). $k = 1$. Then $\tilde{\mathcal{J}}^1(F) = F$: $\Lambda^1(M_2) \to \Lambda^1(M_1)$.

As a consequence of Proposition 1.1.4. we find that if G: $M_2 \to M_3$ is a smooth mapping, then

$$\mathcal{J}^k(G \circ F) = \mathcal{J}^k(F) \circ \mathcal{J}^k(G), \quad \mathcal{J}^k(1) = 1.$$

PROPOSITION. *For every smooth manifold M, we have* $\mathcal{J}^k(M) = \tilde{\mathcal{J}}^k(M) \oplus \mathcal{J}^0(M)$; *in addition, if* F: $M_1 \to M_2$ *is smooth, then* $\mathcal{J}^k(F) = \tilde{\mathcal{J}}^k(F) \oplus \mathcal{J}^0(F)$.

PROOF. We define an embedding i_k: $\mathcal{J}^0(M) \to \mathcal{J}^k(M)$, such that $[i_k(f)]\ (m) = j_k(f(m))|_m$, where $f \in \mathcal{J}^0(M) = C^\infty(M)$, $m \in M$.

Then $\nu_{k,0} \circ i_k = 1$, that is, i_k is a right inverse to $\nu_{k,0}$, therefore $\mathcal{J}^k(M) = \operatorname{Ker} \nu_{k,0} \oplus \operatorname{Im} i_k$. The decomposition $\mathcal{J}^k(F) = \tilde{\mathcal{J}}^k(F) \oplus \mathcal{J}^0(F)$ follows from the fact that $\mathcal{J}^k(F)$ and $\nu_{k,0}$, i_k commute:

$$\nu_{k,0} \circ \mathcal{J}^k(F) = \mathcal{J}^0(F) \circ \nu_{k,0}, \quad \mathcal{J}^k(F) \circ i_k = i_k \circ \mathcal{J}^0(F).$$

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[1] We recall that, given a ring homomorphism φ: $K_1 \to K_2$, a mapping f: $E_1 \to E_2$ of a K_1-module E_1 to a K_2-module E_2 is called a *homomorphism over* φ if a) f is additive and b) $f(k_1 \cdot e_1) = \varphi(k_1) f(e_1)$, $k_1 \in K_1$, $e_1 \in E_1$.

COROLLARY 1. $\mathscr{Y}^1(M) = \Lambda^1(M) \oplus C^\infty(M)$, $\mathscr{Y}^1(F) = F^* \oplus F^*$.

COROLLARY 2. $J^1(M) = T^*(M) \times \mathbf{R}^1$.

Thus, the elements of $\mathscr{Y}^1(M)$ can be thought of as pairs (ω, f), with $\omega \in \Lambda^1(M)$, $f \in C^\infty(M)$. If $F\colon M_1 \to M_2$, then $\mathscr{Y}^1(F)(\omega, f) = (F^*(\omega), F^*(f))$. In terms of the direct decomposition of $\mathscr{Y}^1(M)$ the mapping $j_1\colon C^\infty(M) \to \mathscr{Y}^1(M)$ can be written in the form $j_1(f) = (f, df)$.

The representation of $J^1(M)$ as a direct product $T^*(M) \times \mathbf{R}^1$ corresponding to the direct decomposition of $\mathscr{Y}^1(M)$ allows us to define a projection $\pi\colon J^1(M) \to T^*(M)$ and an injection $\alpha\colon T^*(M) \to J^1(M)$, where $\pi(x, t) = x$, $\alpha(x) = (x, 0)$, $x \in T^*(M)$, $t \in \mathbf{R}^1$.

§2. Non-linear differential equations and many-valued solutions

Using the manifold of 1-jets of functions, we can give an invariant definition of a non-linear first order differential equation and its solutions.

1.2.1. DEFINITION 1. A *non-linear* first order partial differential equation on a manifold M is a submanifold $E \subset J^1(M)$.

DEFINITION 2. A *solution* of an equation $E \subset J^1(M)$ is a smooth function $f \in C^\infty(M)$ such that the image of M under $j_1(f)$ lies in E, that is, $[j_1(f)]\ (M) \subset E$.

REMARK. If codim $E > 1$, then E is said to be *overdetermined*.

Let us see how our definitions of equation and solution connect with the classical definitions. To this end we choose local coordinates $q_1, \ldots, q_n$ in some neighbourhood $\mathscr{U} \subset M$. Let $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ be the coordinate system induced in $\overline{\mathscr{U}} = \pi_1^{-1}(\mathscr{U})$, where $u = p_\sigma$, when $\sigma = (0, \ldots, 0)$ and $p_i = p_\sigma$ when $\sigma = (0, \ldots, 1, \ldots, 0)$ with 1 in the i-th position.

In this coordinate system we can find for every point $x \in E$ a neighbourhood $\mathscr{O} \subset \overline{\mathscr{U}}$, $x \in \mathscr{O}$, in which $E \cap \mathscr{O}$ can be given by equations

$$\mathscr{F}_s(q_1, \ldots, q_n, u, p_1, \ldots, p_n) = 0,$$

where $1 \leqslant s \leqslant k$, $k =$ codim E. In this coordinate system the section $j_1(f)$ can be written as

$$[j_1(f)](q_1, \ldots, q_n) = \left(q_1, \ldots, q_n, f(q), \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_n}\right).$$

Therefore, the condition for the image of $[j_1(f)]\ (M)$ in $\mathscr{O}$ to lie in $E \cap \mathscr{O}$, means that

$$\mathscr{F}_s\left(q_1, \ldots, q_n, f(q), \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_n}\right) = 0, \qquad 1 \leqslant s \leqslant k,$$

that is, it corresponds to the usual representation.

Having in mind the above Definition 1.2.1, we endeavour to extract geometrically from the set of all sections of $\pi_1\colon J^1(M) \to M$ those that are "integrable", that is, have the form $j_1(f)$ for some $f \in C^\infty(M)$.

1.2.2. PROPOSITION. *There exists a unique element* $\rho_1 \in \mathscr{Y}^1(J^1M)$ *such* *that for every* $\theta \in \mathcal{J}^1(M)$,

$$[\mathcal{J}^1(\theta)](\rho_1) = \theta. \tag{1.2.2.1}$$

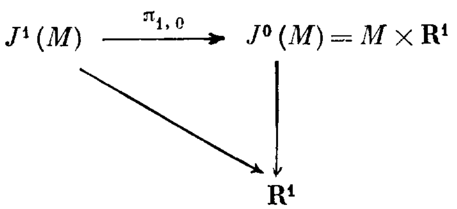
PROOF. Before defining the element ρ_1, we make the following remark. Every point $x \in J^1(M)$ may be interpreted as the 1-jet of some function $f \in C^\infty(M)$ at the point $m = \pi_1(x) \in M$, that is $x = j_1(f)|_m$. We now define ρ_1 as follows: $\rho_1|_x = j_1(\pi_1{}^*(f))|_x$, where $x = j_1(f)|_m$, $m = \pi_1(x)$. Let us check that the element ρ_1 so defined satisfies (1.2.2.1). Let x and f be as above, and $\theta(m) = x$. Then

$$([\mathcal{J}^1(\theta)](\rho_1))|_m = [\mathcal{J}^1(\theta)](j_1(\pi_1^*(f))|_x) = j_1(\theta^*\pi_1^*(f))|_m = j_1(f)|_m = x = \theta(m).$$

Now we give the form of ρ_1 in a special system $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ of local coordinates in $J^1(M)$. Suppose that $x \in J^1(M)$ has the coordinates $(q_1^0, \ldots, q_n^0, u, p_1^0, \ldots, p_n^0)$. Then for an f with $x = j_1(f)|_m$ we can choose the function $u^0 + \sum_{i=1}^{n} p_i^0(q_i - q_i^0)$; hence we obtain, using the definition of ρ_1, that $\rho_1|_x = (u^0, \sum_{i=1}^{n} p_i^0\, dq_i)$ or, counting x as arbitrary, $\rho_1 = (u, \bar{\rho})$, where $\bar{\rho} \in \Lambda^1(J^1 M)$ and in the special local coordinate system $\bar{\rho}$ has the form $\sum_{i=1}^{n} p_i\, dq_i$.

The uniqueness of ρ_1 follows from the simple observation that the only element ρ_1' such that $[\mathcal{J}^1(\theta)](\rho_1') = 0$ for any $\theta \in \mathcal{J}^1(M)$ is $\rho_1' = 0$.

REMARK. The function $u \in C^\infty(J^1M)$, $\rho_1 = (u, \bar{\rho})$, does not depend on the choice of coordinates and can be defined as the composition of two projections:



In addition, if we utilize the direct decomposition of $\mathcal{J}^1$, then u and $\bar{\rho}$ can be defined as the lifting to $J^1(M)$ corresponding to ρ_0 and ρ, where $\rho_0 \in \mathcal{J}^0(J^0M)$ and $\rho \in \Lambda^1(T^*M)$ are determined by the universal properties analogous to (1.2.2.1):

$$[\mathcal{J}^0(\theta)](\rho_0) = \theta \quad \text{if} \quad \theta \in \mathcal{J}^0(M),$$
$$\theta^*(\rho) = \theta \quad \text{if} \quad \theta \in \Lambda^1(M).$$

Thus, $u = \pi_{1,0}^*(\rho_0)$, $\bar{\rho} = \pi^*(\rho)$.

1.2.3. Consider the operator $\mathcal{D}$: $\mathcal{J}^1(M) \to \Lambda^1(M)$, which in terms of the direct decomposition $\mathcal{J}^1(M) = \Lambda^1(M) \oplus C^\infty(M)$, can be written as $\mathcal{D}(f, \omega) = df - \omega$.

PROPOSITION. *The operator* $\mathcal{D}$ *has the following properties*:

a) $\operatorname{Ker} \mathcal{D} = \operatorname{Im} j_1$.

b) $\mathscr{D}$ *is natural, that is, for every smooth mapping* F: $M \to M_1$,

$$\mathscr{D} \circ \mathscr{Y}^1(F) = F^* \circ \mathscr{D}, \text{where}\ F^*: \Lambda^1(M_1) \to \Lambda^1(M).$$

PROOF. Property a) follows from the definition of $\mathscr{D}$, and b) from the direct decomposition $\mathscr{Y}^1(F) = F^* \oplus F^*$ and the fact that the operator of outer differentiation is natural.

REMARK. $\mathscr{D}$: $\mathscr{Y}^1(M) \to \Lambda^1(M)$ is the Spencer operator from the first Spencer resolvent, which in this case has the form

$$0 \to C^\infty(M) \xrightarrow{j_1} \mathscr{Y}^1(M) \xrightarrow{\mathscr{D}} \Lambda^1(M) \to 0.$$

We note that, in contrast to the de Rham sequence, the Spencer sequence is exact on every smooth manifold M.

DEFINITION. The form $U_1 = \mathscr{D}\rho_1 \in \Lambda^1(J^1M)$ is a *classifying element* of U_1 on $J^1(M)$.

We use the proof of Proposition 1.2.2 to establish the form of U in the special coordinate system $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$. As we have seen, $\rho_1 = (u, \bar{\rho})$, where $\bar{\rho} = p\, dq$. Therefore, $U = du - \bar{\rho} = du - p\, dq$.

1.2.4. PROPOSITION. *The section* $\theta \in J^1(M)$ *is "integrable", that is,* $\theta = j_1(f)$ *for some function* $f \in C^\infty(M)$, *if and only if* $\theta^*(U_1) = 0$.

PROOF. By Proposition 1.2.3, $\theta = j_1(f)$ if and only if $\mathscr{D}\theta = 0$. Next we use Proposition 1.2.2 and represent θ in the form $\theta = [\mathscr{Y}^1(\theta)](\rho_1)$. Then $0 = \mathscr{D}\theta = \mathscr{D}[\mathscr{Y}^1(\theta)](\rho_1) = \theta^*(\mathscr{D}\rho_1) = \theta^*(U_1)$.

REMARK. The construction of ρ_1 and of the classifying element U_1 given here was first proposed by A. M. Vinogradov, and is an analogue to one of the possible ways of constructing the universal forms $\rho = p\, dq$ and $d\rho = dp \wedge dq$ on the cotangent manifold $T^*(M)$. Here the module $\Lambda^1(M)$ is naturally replaced by $\mathscr{Y}^1(M)$, and the de Rham sequence by the Spencer resolvent. Note that the assertion corresponding to Proposition 1.2.4 is true for $T^*(M)$ only when M is simply-connected.

1.2.5. DEFINITION. a). A submanifold i: $L \subset J^1(M)$, $\dim L = \dim M$, is said to be an *R-manifold* if $i^*(U_1) = 0$. b) A *many-valued solution* of an equation $E \subset J^1(M)$ is an R-manifold L lying in E.

Thus, Proposition 1.2.4 can be interpreted as a condition for a many-valued solution to correspond to an ordinary (single-valued) solution determined by a smooth function. For a many-valued solution corresponds to an ordinary one if and only if $L = \theta(M)$ for some section θ: $M \to J^1(M)$ or, what is the same thing, if and only if the restriction $\pi_1|_L$: $L \to M$ of π_1 to L is a diffeomorphism.

1.2.6. In this subsection we use the direct decomposition $J^1(M) = T^*(M) \times \mathbf{R}^1$ indicated in § 1 to describe a connection between R-manifolds and Lagrangian submanifolds of $T^*(M)$.

First we note that the universal form ρ on $T^*(M)$ can be defined by the equation $\rho = -\alpha^*(U_1)$, because $U_1 = du - \bar{\rho}$, where $\bar{\rho}$ has the form $\bar{\rho} = p\, dq$ in the local coordinate system $q_1, \ldots, q_n, u, p_1, \ldots, p_n$, and

$\alpha(q_1, \ldots, q_n, p_1, \ldots, p_n) = (q_1, \ldots, q_n, 0, p_1, \ldots, p_n)$.

THEOREM a). *Let i: $L \subset J^1(M)$ be an R-manifold. Then for each point $x \in L$ there is a neighbourhood $\mathcal{O} \subset L$, $x \in \mathcal{O}$, whose projection $\pi \circ i$: $\mathcal{O} \to T^*(M)$ in $T^*(M)$ is a Lagrangian submanifold of $T^*(M)$ on which ρ is exact.*

b) *For every connected Lagrangian submanifold i_1: $L \hookrightarrow T^*(M)$ there is an R-manifold i: $L \subset J^1(M)$ such that the mapping $\pi \circ i$: $L \to L_1$ is a covering.*

PROOF. a) As a preliminary we describe the elements of the subspace $V = \operatorname{Ker} \pi_{*,x} \subset T_x(J^1M)$, $x \in J^1(M)$. To do this we note that V is one-dimensional, since π: $J^1(M) \to T^*(M)$ is a projection and $\dim J^1(M) = 1 + \dim T^*(M)$.

Next we show that the restriction of $U_{1,x}$ to V is a non-zero 1-form. For if $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ is a special system of coordinates in the neighbourhood of a point $x \in J^1(M)$ (that is, $q_1, \ldots, q_n$ are local coordinates in the manifold M), then π has the form $\pi(q_1, \ldots, q_n, u, p_1, \ldots, p_n) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$. Consequently,

V is generated by the vector $\left.\frac{\partial}{\partial u}\right|_x$, on which $U_{1,x}\left(\left.\frac{\partial}{\partial u}\right|_x\right) = 1$.

Suppose now that $L \subset J^1(M)$ is an R-manifold and $x \in L$. Then $T_x(L)$ and V intersect in zero alone, because $U_{1,x}$ is zero on $T_x(L)$. Therefore, $\pi_{*,x}$: $T_x(L) \to T_{\pi(x)}(T^*M)$ is a monomorphism, which gives the existence of a neighbourhood $\mathcal{O} \subset L$, as required.

Let us check that the form ρ is exact on $\pi \circ i$: $\mathcal{O} \to T^*(M)$ We have

$$(\pi \circ i)^*(\rho) = (\pi \circ i)^*(\alpha^*(\bar{\rho})) = (\alpha \circ \pi \circ i)^*(du - U_1) = d(i^*(u)).$$

b) We fix a point $l_0 \in L_1$. Then for every point $x \in L_1$ and every path $\gamma = \{x(t)\}$, $x(0) = l_0$, $x(1) = x$, we define a function

$$S(\gamma, x) = \int_\gamma i_1^*(\rho).$$

Further, since the submanifold L_1 is Lagrangian, that is, $d\rho|_{L_1} = 0$, it follows from Stokes's theorem that $S(\gamma, x)$ depends in fact only on the homotopy class $[\gamma]$ of paths joining l_0 and x.

Let

$$L = \bigcup_{([\gamma], x)} (x, S(\gamma, x)), \quad (x, S(\gamma, x)) \in J^1(M).$$

Standard arguments show that L is a smooth submanifold of $J^1(M)$, and by construction we have $U_1|_L = 0$.

It follows from the proof of part a) of the theorem that $\pi|_L$: $L \to L_1$ is a local diffeomorphism. Therefore, to conclude the proof it is enough to compute the pre-image $\pi|_{L_1}^{-1}(x)$ of x.

We consider the homomorphism $\chi\colon \pi^1(L_1) \to \mathbf{R}$ of the fundamental group $\pi^1(L_1)$ of L_1 into the additive group of real numbers:

$$\chi(z) = \int_\gamma i_1^*(\rho), \quad z = [\gamma] \in \pi^1(L_1).$$

By construction of L, the fibre $\pi|_{L_1}^{-1}(x)$, $x \in L_1$, is then isomorphic to the factor group $\pi^1(L_1)/\mathrm{Ker}\ \chi$.

EXAMPLE 1, Assertion a) of the theorem is not true globally. Consider the curve $i\colon R^1 \subset J^1(\mathbf{R}^1)$:

$$i(t) = \left(t^2 - 1,\ \frac{2}{15} t^3 (3t^2 - 5),\ t(t^2 - 1)\right).$$

It is easily checked that $i^*(U_1) = 0$. But the projection of the curve on $T^*(\mathbf{R}^1)$ is a curve with an ordinary double point at $(t^2 - 1, t(t^2 - 1))$.

EXAMPLE 2. Let $M = \mathbf{R}^1$, $L_1 = \{(q, p) \in T^*(\mathbf{R}^1) \mid q^2 + p^2 = 1\}$. Then L is the universal covering over a circle.

§3. Contact diffeomorphisms and symmetries

From the point of view adopted in §2, it is natural to consider not arbitrary diffeomorphisms of $J^1(M)$, but only those that preserve the class of R-manifolds. Such diffeomorphisms must preserve the zero form U_1, and so must multiply U_1 by some function.

1.3.1. DEFINITION. A diffeomorphism $F\colon J^1(M) \to J^1(M)$ is said to be a *contact* (or a U_1-) *diffeomorphism* if $F^*(U_1) = fU_1$, $f \in C^\infty(J^1M)$ (or $F^*(U_1) = U_1$, respectively).

REMARK. Every contact structure on an odd-dimensional manifold N is a "maximally non-integrable" field of hyperplanes $\mathscr{E}$, that is, for each point $x \in N$ there is a subspace $\mathscr{E}_x \subset T_x(N)$, codim $\mathscr{E}_x = 1$, depending smoothly on $x \in N$. "Maximal non-integrability" means that, for any 1-form ω in a neighbourhood $\mathscr{U} \subset N$ of x and such that $\mathrm{Ker}\ \omega = \mathscr{E}$, the restriction of $d\omega$ to $\mathscr{E}$ is non-degenerate. In this situation contact diffeomorphisms $F\colon N \to N$ are those that preserve $\mathscr{E}$, that is, for which $F_{*,x}(\mathscr{E}_x) = \mathscr{E}_{F(x)}$, for every $x \in N$.

In the case of the manifold $J^1(M)$ of 1-jets of functions, a contact structure $\mathscr{E}$ is given by a 1-form U_1: $\mathscr{E}_x = \mathrm{Ker}\ U_{1,x}$, $x \in J^1(M)$, and the contact diffeomorphisms of this structure are just the contact diffeomorphisms in the sense of Definition 1.3.1.

1.3.2. DEFINITION. A *symmetry* of an equation $E \subset J^1(M)$ is a contact diffeomorphism $F\colon J^1(M) \to J^1(M)$ preserving E: $F(E) = E$.

Thus, symmetries are the contact transformations that preserve an equation and the class of its many-valued solutions.

1.3.3. THEOREM. *Let $F\colon J^1(M) \to J^1(M)$ be a U_1-diffeomorphism. Then F preserves the fibre of the projection $\pi\colon J^1(M) \to T^*(M)$ and determines a canonical diffeomorphism $\bar{F}\colon T^*(M) \to T^*(M)$ having a generating function,* *that is, $\bar{F}^*(\rho) - \rho = dS$, where $S \in C^\infty(T^*M)$. Conversely, for every canonical diffeomorphism G having a generating function there is a U_1-diffeomorphism F such that $\bar{F} = G$.*

PROOF. Let us show that F preserves the fibre of π. To do this we look at a vector field X_1 on $J^1(M)$ whose direction at a point $x \in J^1(M)$ is that of the degenerate form $dU_{1,x}$ and which is normed by the condition $U_1(X_1) = 1$. In other words, X_1 can be defined by the equations $X_1 \lrcorner U_1 = 1$, $X_1 \lrcorner dU_1 = 0$, where $\lrcorner$ is the symbol for interior multiplication.

It follows at once from the last definition that a diffeomorphism F preserves the field X_1, $F_*(X_1) = X_1$, whenever it preserves U_1. On the other hand, the trajectories of X_1 are the fibres of π.

We can prove this, for example, by choosing a special system of local coordinates $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ in $J^1(M)$. In this system U_1 has the form $du - p\, dq$, so that $X_1 = \frac{\partial}{\partial u}$. The projection π has in this coordinate system the form $\pi: (q_1, \ldots, q_n, u, p_1, \ldots, p_n) \to (q_1, \ldots, q_n, p_1, \ldots, p_n)$, so that a trajectory of X_1 is a fibre of π.

Let us verify that the diffeomorphism $\bar{F}: T^*(M) \to T^*(M)$ determined by F, $\bar{F}(y) = \pi \circ F \circ \pi^{-1}(y)$, is canonical.

To do this, we represent F in the form $F(x, t) = (\bar{F}(x), t + \pi^*(S)(x))$, where $x \in T^*(M)$, $t \in \mathbf{R}$, $S \in C^\infty(T^*M)$. Note that such a representation is possible, since $F_*(X_1) = X_1$.

Making use now of the equalities $U_1 = du - \bar{\rho}$, $\bar{\rho} = \pi^*(\rho)$, we see that $U_1 = F^*(U_1) = d(F^*(u)) - \pi^*\bar{F}^*(\rho) = \pi^*(dS) + du - \pi^*(\bar{F}^*(\rho))$, that is, $\pi^*(\bar{F}^*(\rho) - \rho) = \pi^*(dS)$.

But, as $F^*(\rho) - \rho \in \Lambda^1(T^*M)$ and $dS \in \Lambda^1(T^*M)$, and as $\pi: J^1(M) \to T^*(M)$ is a projection, it follows that $\bar{F}^*(\rho) - \rho = dS$.

Conversely, suppose that G is a canonical diffeomorphism of $T^*(M)$ and that $S \in C^\infty(T^*M)$ a generating function for it. Then, setting $F(x, t) = (G(x), t + \pi^*(S)(x))$, we obtain the U_1-diffeomorphism F we are looking for.

REMARK. As is clear from the preceding proof, the U_1-diffeomorphisms covering $1: T^*(M) \to T^*(M)$ are translations τ_s along X_1. Therefore, if the U_1-diffeomorphisms F_1 and F_2 cover one and the same canonical diffeomorphism, then $F_1 = \tau_s \circ F_2$ for some $s \in \mathbf{R}^1$.

1.3.4. To conclude this section we describe the special class of U_1-diffeomorphisms corresponding to diffeomorphisms of M.

Let K be a diffeomorphism of M. We construct a diffeomorphism $K': J^1(M) \to J^1(M)$:

$$K'(x) = [J^1_{K(m)}(K)]^{-1}(x),$$

where $x \in J^1(M)$, $m = \pi_1(x)$.

PROPOSITION. *K' is a U-diffeomorphism such that $(K')^*u = u$. Conversely, if F is a U_1-diffeomorphism such that $F^*(u) = u$, then $F = K'$ for some* *diffeomorphism* $K: M \to M$.

PROOF. Let us show that K' preserves ρ_1, that is,

$$[\mathcal{Y}^1(K')](\rho_1) = \rho_1. \tag{1.3.4.1}$$

For this we use the equality

$$K' \circ ([\mathcal{Y}^1(K)](\theta)) = \theta \circ K, \tag{1.3.4.2}$$

which follows at once from the definition of K', and is valid for $\theta \in \mathcal{Y}^1(M)$.

We show next that the element $[\mathcal{Y}^1(K')](\rho_1)$ also satisfies the condition 1.2.2, hence (1.3.4.1) follows, by the uniqueness of ρ_1.

For arbitrary $\theta \in \mathcal{Y}^1(M)$ we have

$$\begin{aligned}[\mathcal{Y}^1(\theta)]([\mathcal{Y}^1(K')](\rho_1)) &= \mathcal{Y}^1(K' \circ \theta)(\rho_1) = \\ &= \mathcal{Y}^1(([\mathcal{Y}^1(K)]^{-1}(\theta)) \circ K)(\rho_1) = \\ &= \mathcal{Y}^1(K) \circ \mathcal{Y}^1([\mathcal{Y}^1(K)]^{-1}(\theta))(\rho_1) = \mathcal{Y}^1(K) \circ [\mathcal{Y}^1(K)]^{-1}(\theta) = \theta.\end{aligned}$$

Further, because of the direct decomposition $\mathcal{Y}^1(K') = (K')^* \oplus (K')^*$, $\rho_1 = (\bar{\rho}, u)$, we see that $(K')^*(\bar{\rho}) = \bar{\rho}$, $(K')^*(u) = u$.

We assume now that the U_1-diffeomorphism F fixes u, $F^*(u) = u$. It follows from the preceding theorem that F has zero generating function, that is, $\bar{F}^*(\rho) = \rho$. Therefore, as is well known, $\bar{F}$ is a lifting of the diffeomorphism $K: M \to M$ to $T^*(M)$, and $K' = F$.

§ 4. The algebra of contact vector fields

1.4.1. We recall some fundamental properties of the Lie derivative.

Let X be a vector field on a manifold N, and T_t a local one-parameter group of translations along X. The Lie derivative $L_X(\omega)$ of a form $\omega \in \Lambda^k(N)$ along X is defined as

$$L_X(\omega) = \lim_{t \to 0} \frac{1}{t}(T_t^*(\omega) - \omega).$$

The basic properties of the Lie derivative that we need in the sequel are summed up in the following proposition. Proofs can be found in [18].

PROPOSITION. *Suppose that* $f \in C^\infty(N)$, $\omega \in \Lambda^k(N)$, $\omega_1 \in \Lambda^r(N)$, *and let* X *and* Y *be vector fields on* N. *Then*

$$L_{X+Y}(\omega) = L_X(\omega) + L_Y(\omega), \tag{1.4.1.1}$$

$$[L_X, L_Y](\omega) = L_{[X,Y]}(\omega), \tag{1.4.1.2}$$

$$L_X(d\omega) = d(L_X\omega), \tag{1.4.1.3}$$

$$L_X(\omega) = X \lrcorner d\omega + d(X \lrcorner \omega), \tag{1.4.1.4}$$

$$L_X(\omega_1 \wedge \omega) = L_X(\omega_1) \wedge \omega + \omega_1 \wedge L_X(\omega). \tag{1.4.1.5}$$

1.4.2. We now consider a local one-parameter group $T_t: J^1(M) \to J^1(M)$ of contact diffeomorphisms, $T^*(U_1) = g_t \cdot U_1$. Let X be the corresponding vector field on $J^1(M)$. Then by the definition of the Lie derivative,

$$L_X(U_1) = g \cdot U_1, \quad \text{where } g = \left.\frac{dg_t}{dt}\right|_{t=0}. \tag{1.4.2.1}$$

On the basis of this last remark, we make the following definition.

DEFINITION. An *infinitesimal contact transformation* or a *contact vector field* is a vector field X on $J^1(M)$ satisfying (1.4.2.1).

REMARK. For any contact structure $\mathscr{E}$ on a manifold N, a vector field X is said to be *contact* if the one-parameter group of translations along X consists of contact diffeomorphisms. Since, in general, a contact structure is not necessarily given by the zeros of a 1-form, the above definition can be reformulated as follows: X is a contact vector field if for every vector field Y such that $Y_x \in \mathscr{E}_x$, $x \in N$ we have $L_{X_1}(Y)\,|_x = [Y, X]_x \in \mathscr{E}_x$.

EXAMPLE 1. The vector field X_1 on $J^1(M)$ defined in the proof of Theorem 1.3.3 is contact. For $L_X(U_1) = X_1 \lrcorner\, dU_1 + d(X_1 \lrcorner\, U_1) = 0$, because $X_1 \lrcorner\, dU_1 = 0$, $X_1 \lrcorner\, U_1 = 1$.

EXAMPLE 2. Let $\bar{T}_t$ be a one-parameter group of translations along a Hamiltonian vector field $\bar{X}$ on $T^*(M)$. Then T_t (see Theorem 1.3.3) is a one-parameter group of translations along a contact vector field X on $J^1(M)$ such that $\pi_*(X) = \bar{X}$.

PROPOSITION. *A vector field X on $J^1(M)$ is contact if and only if the group of translations along X is a one-parameter group of contact diffeomorphisms.*

PROOF. The condition $L_X(U_1) = g \cdot U_1$ is equivalent to the fact that $L_X(U_1) \wedge U_1 = 0$. Therefore, the condition for X to be contact can be written in the form

$$\left.\frac{d}{dt}\right|_{t=0} (T_t^*(U_1)) \wedge U_1 = 0.$$

On the other hand, since $T_{s+t} = T_s \circ T_t$,

$$\left.\frac{d}{dt}\right|_{t=s} (T_t^*(U_1)) \wedge U_1 = 0.$$

Thus, $T_t^*(U_1) \wedge U_1 = T_0^*(U_1) \wedge U_1 = 0$, that is, the T_t are contact diffeomorphisms.

1.4.3. THEOREM. *Every contact vector field X on $J^1(M)$ is uniquely determined by the function $f = U_1(X)$. To every function $f \in C^\infty(J^1M)$ there corresponds a unique contact vector field X_f such that*

$$U_1(X_f) = f, \tag{1.4.3.1}$$

$$L_{X_f}(U_1) = X_1(f) \cdot U_1, \tag{1.4.3.2}$$

$$X_{f+g} = X_f + X_g, \quad g \in C^\infty(J^1M), \tag{1.4.3.3}$$

$$X_{fg} = fX_g + gX_f - fgX_1, \tag{1.4.3.4}$$

$$X_f(f) = X_1(f) \cdot f. \tag{1.4.3.5}$$

PROOF. Let X be a contact vector field on $J^1(M)$, $L_X(U_1) = h \cdot U_1$,

$h \in C^\infty(J^1 M)$. We represent X in the form

(1.4.3.6) $X = f \cdot X_1 + Y$, where $f \in C^\infty(J^1M)$ and $U_1(Y) = 0$.

We now use the property (1.4.1.4) of a Lie derivative:
$L_X(U_1) = X \lrcorner dU_1 + d(X \lrcorner U_1) = h \cdot U_1$, or, substituting (1.4.3.6),

$$Y \lrcorner dU_1 = h \cdot U_1 - df. \tag{1.4.3.7}$$

In particular, by applying the left- and right-hand sides of (1.4.3.7) to X we see that $h = X_1(f)$.

To conclude the proof it is enough to note that (1.4.3.7) determines the field Y uniquely, because the form dU_1 establishes an isomorphism $Y \to Y \lrcorner dU_1$ between the vector fields on which U_1 vanishes and the 1-forms that are zero on X.

If now f is any smooth function on $J^1(M)$, then with Y defined by (1.4.3.7), $h = X_1(f)$, and $X = X_f$ defined by (1.4.3.6), we obtain the required field X_f.

The equalities (1.4.3.1)–(1.4.3.5) for X_f follow from (1.4.3.6), (1.4.3.7), and the properties of the Lie derivative.

DEFINITION. The function $f = U_1(X)$ is called the *Hamiltonian* of the contact vector field X on $J^1(M)$.

EXAMPLE. Every function $H \in C^\infty(T^*M)$ can be regarded as a smooth function on $J^1(M)$ via the projection π: $J^1(M) \to T^*(M)$, $f = \pi^*(H)$. Here the projection of the contact vector field X_f on $T^*(M)$ is the Hamiltonian vector field X_H, and from (1.4.3.2) it follows that $X_H \lrcorner d\rho = dH$, because $dU_1 = -\pi^*(d\rho)$.

We now indicate the form of the contact vector field X_f in local coordinates. Let $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ be a special system of local coordinates in $J^1(M)$. The form of X_f is easy to find from the relations (1.4.3.6) and (1.4.3.7):

$$X_f = -\sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} + \left(f - \sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i}\right)\frac{\partial}{\partial u} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i} + p_i \frac{\partial f}{\partial u}\right)\frac{\partial}{\partial p_i}.$$

REMARK. The assertion analogous to Theorem 1.4.3 is valid for any contact structure $\mathscr{E}$ on N. To formulate it, we consider a 1-dimensional fibration ζ on N whose fibre at a point $x \in N$ is the factor space $T_x(N)/\mathscr{E}_x$. Every vector field Y on N determines a section S_Y of ζ: $S_Y(x)$ is the image of Y_x in the factor space ζ_x under the natural projection $T_x(N) \to T_x(N)/\mathscr{E}_x$.

The analogue to Theorem 1.4.3 can now be formulated as follows. Every contact vector field X on N is uniquely determined by the section S_X of ζ. To every section s there corresponds a contact vector field $X = X_s$ such that $S_X = s$.

If the contact structure is given by a 1-form ω, then ζ is trivialized via a non-zero section S_{X_1}, where X_1 is a vector field such that

$X_1 \rfloor d\rho = 0$, $X_1 \rfloor \omega = 1$. Hence for such structures the contact vector fields are determined by Hamiltonians.

1.6.6. The existence of an isomorphism between contact vector fields on $J^1(M)$ and smooth functions allows us to define various pairings between functions.

(A) THE LAGRANGE BRACKET. Take f and in g $C^\infty(J^1M)$, and let X_f and X_g be contact vector fields with Hamiltonians f and g, respectively. By (1.4.1.2), their commutator $[X_f, X_g]$ is also a contact vector field, so that it has the form X_h for some function h.

DEFINITION. The *Lagrange bracket* $[f, g]$ of two functions f and g is the Hamiltonian of the contact vector field $[X_f, X_g]$, that is, $[f, g] = U_1([X_f, X_g])$.

BASIC PROPERTIES OF THE LAGRANGE BRACKET.

L1. *Bilinearity*:

$$[af_1 + bf_2, g] = a[f_1, g] + b[f_2, g], \quad a, b \in \mathbf{R}$$
$$[f, ag_1 + bg_2] = a[f, g_1] + b[f, g_2].$$

L2. *Antisymmetry*: $[f, f] = 0$.

L3. *Jacobi identity*:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0.$$

(B) THE JACOBI BRACKET.

DEFINITION. The *Jacobi bracket* $\{f, g\}$ of two functions f and g in $C^\infty(J^1M)$ is the function $\{f, g\} = dU_1(X_f, X_g)$.

BASIC PROPERTIES OF THE JACOBI BRACKET.

J1. *Bilinearity*:

$$\{af_1 + bf_2, g\} = a\,\{f_1, g\} + b\,\{f_2, g\},$$
$$\{f, ag_1 + bg_2\} = a\,\{f, g_1\} + b\,\{f, g_2\}, \quad a, b \in \mathbf{R}.$$

J2. *Antisymmetry*: $\{f, f\} = 0$.

J3. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = X_1(f)\,\{g, h\} + X_1(g)\,\{h, f\} + X_1(h)\,\{f, g\}$.

J4. $\{f, g\} = X_f(g) - fX_1(g)$.

(C) THE POISSON BRACKET.

DEFINITION. The *Poisson bracket* (f, g) of two functions f and g in $C^\infty(J^1M)$ is the function $(f, g) = X_f(g)$.

BASIC PROPERTIES OF THE POISSON BRACKET.

P1. *Bilinearity*:

$$(af_1 + bf_2, g) = a(f_1, g) + b(f_2, g),$$
$$(f, ag_1 + bg_2) = a(f, g_1) + b(f, g_2), \quad a, b \in \mathbf{R}.$$

P2. $(f, g) + (g, f) = X_1(f)g + X_1(g)f$.

P3. $(f, g) = [f, g] + X_1(f)g$.

P4. $(f, g) = \{f, g\} + X_1(g)f$.

REMARK. If we consider only functions $f \in C^\infty(J^1M)$ that are liftings of functions in $T^*(M)$, $f = \pi^*(H)$, $H \in C^\infty(T^*M)$, then for these functions the three brackets $[\,,\,]$, $\{\,,\,\}$, $(\,,\,)$ coincide with the Poisson bracket in $C^\infty(T^*M)$, because $X_1(\pi^*(H)) = 0$.

Let us compute these brackets in special local coordinates $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ in $J^1(M)$. Using the expression for X_f in these coordinates, together with properties L4 and J4, we find

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) + \sum_{i=1}^{n} p_i\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial p_i}\right) + f\frac{\partial g}{\partial u} - g\frac{\partial f}{\partial u},$$

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) + \sum_{i=1}^{n} p_i\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial p_i}\right),$$

$$(f, g) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) + \sum_{i=1}^{n} p_i\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial p_i}\right) + f\frac{\partial g}{\partial u}.$$

§5. Cauchy's problem

In this section we consider differential equations of the form

$$E_f = \{x \in J^1(M) \mid f(x) = 0\}, \quad f \in C^\infty(J^1M).$$

We describe the one-parameter group T_t of symmetries of the equation E_f. It follows from the results of the preceding section that T_t is a group of translations along the contact vector field X_g with Hamiltonian g. The requirement that $T_t(E_f) = E_f$ means that X_g is tangent to E_f, or that $df(X_g)|E_f = (g, f)|_{E_f} = 0$.

1.5.1. DEFINITION. An *infinitesimal symmetry* of E_f is any contact vector field X_g whose Hamiltonian is such that $(g, f)|_{E_f} = 0$.

We note that E_f always has at least one symmetry, namely the field X_f. For $(f, f)|_{E_f} = X_1(f)\cdot f|_{E_f} = 0$.

REMARK. The vector field $Y = X_f - fX_1$ is called the characteristic vector field for E. But for us the only important thing is the action of the field on E_f; therefore, Y can be replaced by X_f, which, unlike Y, is contact.

1.5.2. PROPOSITION. *Every R-manifold $L \subset E_f$ is invariant under X_f.*

PROOF. Let $x \in L$, and let $T_x(L)$ be the tangent space to L at x. We must prove that $X_{f,x} \in T_x(L)$. To do this we consider the linear span V of the subspaces $T_x(L)$ and $X_{f,x}$ in $T_x(J^1M)$. We claim that V has the following two properties: 1) $U_{1,x}(Y) = 0$ for $Y \in V$;
2) $dU_{1,x}(Y_1, Y_2) = 0$ for $Y_1, Y_2 \in V$.

The first property is clear, because L is an R-manifold, that is, $U_1|_L = 0$, and also $U_{1,x}(X_{f,x}) = f(x) = 0$. Let us establish the second property. It is obviously true if $Y_1, Y_2 \in T_x(L)$. Thus, it is enough to consider the pair $Y_1 = X_{f,x}$, $Y_2 \in T_x(L)$. Using (1.4.3.7) we find that

$dU_{1,x}(X_{f,x}, Y_2) = X_1(f)|_x \cdot U_{1,x}(Y) - df_x(Y_2) = 0.$

Thus, $V \subset \operatorname{Ker} U_{1,x}$ and it is Langrangian with respect to $dU_{1,x}$. Therefore, $\dim V \leqslant n$, or, since $\dim T_x(L) = n$, $X_{f,x} \in T_x(L)$.

COROLLARY. *If the R-manifold L is tangent to E_f at $x \in E_f$, then $X_{f,x} \in T_x(L)$.*

1.5.3. DEFINITION. We say that a generalized Cauchy problem is posed for an equation $E_f \subset J^1(M)$ if there is given an $(n-1)$-dimensional submanifold i: $L' \subset E_f$ such that $i^*(U_1) = 0$ and $X_{f,x} \in T_x(L')$ for any $x \in L'$. The submanifold L' is called the *Cauchy data.*

PROPOSITION. *A generalized Cauchy problem $L' \subset E_f$ has a unique solution, that is, there is an R-manifold $L \subset E_f$, $L \supset L'$, and any two such R-manifolds coincide in a neighbourhood of L'.*

PROOF. Consider the embedding i_1: $\mathcal{O} \subset E_f$, where $\mathcal{O} \subset L' \times (-\varepsilon, \varepsilon)$ — is some neighbourhood of $L' \times 0$, $\varepsilon > 0$; here $i_1(x, t) = T_t(x)$, $x \in L'$, $(x, t) \in \mathcal{O}$, and T_t is a one-parameter group of translations along X_f.

Since the T_t are contact diffeomorphisms and $U_1(X_f)|_{E_f} = 0$, we have $i^*(U_1) = 0$. Thus, i_1: $\mathcal{O} \subset E_f$ is a solution of the Cauchy problem $L' \subset E_f$.

On the other hand, any many-valued solution of the Cauchy problem $L' \subset E_f$ must be invariant under X_f, so that it must coincide with $\mathcal{O}$ in some neighbourhood of L'.

COROLLARY. *We assume that the translation group T_t is defined for all values of t, $-\infty < t < \infty$, and that the manifold $L = \bigcup_t T_t(L')$ projects diffeomorphically onto M. Then the generalized Cauchy problem $L' \subset E_f$ has a solution $f \in C^\infty(M)$ given by the formula*

$$f(m) = u(l') + \int_\gamma \bar{\rho}, \tag{1.5.3.1}$$

where $\gamma(t)$ is a trajectory of X_f such that $\pi_1(\gamma(1)) = m$, $\gamma(0) = l' \in L'$.

PROOF. We note that L is an R-manifold, and, as L projects diffeomorphically onto M, we see that $L = [j_1(h)]\ (M)$ for some function $h \in C^\infty(M)$.

It follows from the definition of $u \in C^\infty(J^1M)$ that $u(l) = h(m)$ if $l \in L$, $m = \pi_1(l) \in M$. Next, $X_f(u)|_L = du(X_f)|_L = U_1(X_f)|_L + \bar{\rho}(X_f)|_L = \bar{\rho}(X_f)|_L$, that is, if $\gamma(t)$ is a trajectory of X_f, then $du(\gamma(t))/dt = \bar{\rho}(X_f)|_{\gamma(t)}$ and (1.5.3.1) follows.

1.5.4. Suppose that we are given a submanifold ν: $N \subset M$, codim $N = 1$, and an R-manifold i: $L' \subset J^1(N)$. The embedding ν: $N \subset M$ induces a mapping $\tilde{\nu}$: $J^1_\nu(M) \cap E_f \to J^1(N)$, where $J^1_\nu(M)$ is the restriction of π_1: $J^1(M) \to M$ to N, and $\tilde{\nu}$ that of $J^1(\nu)$ to $J^1_\nu(M) \cap E_f$.

DEFINITION. a). A point $y \in J^1(N)$ is said to be *free* relative to E_f if $\tilde{\nu}$ is a diffeomorphism at the point. If not, then y is said to be *characteristic.*

b). An R-manifold $L \subset J^1(N)$ is said to be *free* (*characteristic*) relative to E_f if all its points are free (characteristic) relative to E_f.

PROPOSITION. *Let $L \subset J^1(N)$ be an R-manifold that is free relative to E_f. Then $L' = \tilde{\nu}^{-1}(L)$ gives us Cauchy data for E.*

PROOF. As a preliminary, we establish the following equation:

$$\tilde{\nu}^* (U_1^N) = U_1^M) |_{J^1_\nu(M) \cap E_f}, \tag{1.5.4.1}$$

where $U_1^N (U_1^M)$ is a classifying element of the manifold N(or M, respectively).

To prove (1.5.4.1) we choose a system of local coordinates $q_1, \ldots, q_n$ in M such that N is given by the equation $q_n = 0$. Then

$$\tilde{\nu}^* (U_1^N) = \left(du - \sum_{i=1}^{n-1} p_i \, dq_i\right) |_{E_f} = U_1^M |_{J^1_\nu(M) \cap E_f}.$$

Further, taking (1.5.4.1) into account, we see that $U_1^M |_{L'} = \tilde{\nu}(U_1^N)|_{L'} = U_1^N |_L = 0$. In a similar way it can be checked that L is free if $X_{f,x} \notin T_x(L')$ whenever $x \in L'$.

§6. Involutory equations

1.6.1. Let Γ be a vector space, dim $= 2n$, carrying a symplectic structure, that is, a non-degenerate skew-symmetric 2-form Ω.

IMPORTANT EXAMPLE. Let $\Gamma = \Gamma_x = \operatorname{Ker} U_{1,x} \subset T_x(J^1M)$, $x \in J^1(M)$. Then the restriction of $dU_{1,x}$ to Γ gives a symplectic structure.

For every subspace $\Gamma_1 \subset \Gamma$ we define the skew-orthogonal complement $\Gamma_1^\perp$ to be the set of all vectors $X \in \Gamma$ such that $\Omega(X, Y) = 0$ for every $Y \in \Gamma_1$. Since Ω is non-degenerate, dim $\Gamma_1^\perp$ = codim Γ_1.

DEFINITION. A subspace $\Gamma_1 \subset \Gamma$ is said to be *involutory* if $\Gamma_1^\perp \subset \Gamma_1$.

EXAMPLE 1. Every subspace $\Gamma_1 \subset \Gamma$ with codim $\Gamma_1 = 1$ is involutory. This is because a generator $X \in \Gamma_1^\perp$, dim $\Gamma_1^\perp = 1$, is determined by the condition $\Omega(X, Y) = \alpha(Y)$ for arbitrary $Y \in \Gamma$, where $\alpha \neq 0$ is a 1-form such that $\alpha|_{\Gamma_1} = 0$. Since Ω is non-degenerate, such an X always exists and $\alpha(X) = \Omega(X, X) = 0$, that is, $X \in \Gamma_1$.

EXAMPLE 2. Every Lagrangian (=maximal isotropic) subspace $\Gamma_1 \subset \Gamma$ is involutory, in fact $\Gamma_1^\perp = \Gamma_1$. For if $X \in \Gamma_1^\perp$ but $X_{\Gamma_1} \notin \Gamma_1$, then the linear span of Γ_1 and X would be an $(n + 1)$-dimensional Lagrangian subspace.

LEMMA. *If $\Gamma_1 \subset \Gamma$ is an involutory subspace and $\Gamma_2 \supset \Gamma_1$, then Γ_2 is also involutory.*

PROOF. If $X \in \Gamma_2^\perp$ then $X \in \Gamma_1^\perp$, so that $X \in \Gamma_1$; that is, $\Gamma_2^\perp \subset \Gamma_1^\perp \subset \Gamma_1 \subset \Gamma_2$.

PROPOSITION (CRITERION FOR A SUBSPACE TO BE INVOLUTORY). *A subspace $\Gamma_1 \subset \Gamma$ is involutory if and only if there exists a Lagrangian subspace $\Gamma_2 \subset \Gamma_1$.*

PROOF. Suppose that Γ_1 is involutory. Then $\Gamma_1 = \Gamma_1^\perp \oplus \Gamma_3$, and the restriction of Ω to Γ_3 is a non-degenerate 2-form. Let $\Gamma_4 \subset \Gamma_3$ be a Lagrangian subspace. Then $\Gamma_2 = \Gamma_1^\perp \oplus \Gamma_4 \subset \Gamma_1$ is the required Lagrangian subspace of Γ_1.

The converse assertion follows immediately from Lemma 1.6.1 and Example

COROLLARY 1. *If* Γ_1 *is involutory, then* $\Gamma_1^{\perp}$ *lies in all Lagrangian subspaces* $\Gamma_2 \subset \Gamma_1$.

COROLLARY 2. *If* $\Gamma_1 \subset \Gamma$ *is involutory, then* $\dim \Gamma_1 \geqslant \frac{1}{2} \dim \Gamma$.

REMARK. Suppose that Γ_1 is an involutory subspace with $\operatorname{codim} \Gamma_1 = k$. Then the form $\Omega|_{\Gamma_1}$ is of rank $2(n-k)$, and $\Gamma_1^{\perp}$ is the degeneracy subspace of the 2-form $\Omega|_{\Gamma_1}$.

1.6.2. DEFINITION. An equation $E \subset J^1(M)$ is said to be *involutory at a point* $x \in E$ if the subspace $T_x(E) \cap \Gamma_x$ is involutory in $\Gamma_x = \operatorname{Ker} U_{1,x}$.

EXAMPLE. Every equation $E \subset J^1(M)$ such that $\operatorname{codim} E = 1$ is involutory at every point $x \in E$.

Thus, the concept of being involutory has real force only for overdetermined equations E, $\operatorname{codim} E > 1$.

PROPOSITION (CRITERION FOR AN EQUATION TO BE INVOLUTORY). *An equation* $E \subset J^1(M)$ *is involutory at a point* x *if and only if there exists an R-manifold* $L \subset J^1(M)$ *tangent to* E *at* x.

PROOF. The assertion follows immediately from Proposition 1.6.1 and the fact that a Lagrangian subspace of Γ_x is the tangent space $T_x(L)$ of some R-manifold $L \subset J^1(M)$.

1.6.3. We consider an equation of the following form:

$$E = E_{f_1, \dots, f_k} = \{x \in J^1(M) \mid f_i(x) = 0, \quad 1 \leqslant i \leqslant k\},$$

where the $f_i \in C^\infty(J^1M)$ are independent functions on E.

DEFINITION. A system of independent functions $f_1, \dots, f_k$ is said to be *involutory at a point* x if the equation $E_{f_1,\dots,f_k}$ is involutory at $x \in E_{f_1,\dots,f_k}$.

PROPOSITION. *A system of independent functions* $f_1, \dots, f_k$ *is involutory at* $x \in E_{f_1,\dots,f_k}$ *if and only if*
$[f_i, f_j]_x = (f_i, f_j)|_x = \{f_i, f_j\}|_x = 0, \ 1 \leqslant i \leqslant k.$

PROOF. We consider the subspace
$\Gamma_1 = T_x(E_{f_1,\dots,f_k}) \cap \Gamma_x = \operatorname{Ker} df_{1,x} \cap \dots \cap \operatorname{Ker} df_{k,x} \cap \Gamma_x$. Since the $f_1, \dots, f_k$ are independent, two cases are possible:

1) $df_{1,x}, \dots, df_{k,x}, U_{1,x}$ are linearly independent;

2) $U_{1,x} = \sum_{i=1}^{k} \lambda_i df_{i,x}, \quad \lambda_i \in \mathbf{R}$.

These two cases are determined by the conditions:

$$\operatorname{codim} \Gamma_1 = k, \tag{1.6.3.1}$$

$$\operatorname{codim} \Gamma_1 = k - 1. \tag{1.6.3.2}$$

Further, it follows from the definition of the vector field X_{f_i} that

$$U_{1,x}(X_{f_i,x}) = f_i(x) = 0, \quad X_{f_i,x} \lrcorner\, \Omega_x = -df_i|_{\Gamma_x},$$

where $\Omega_x = dU_{1,x}|_{\Gamma_x}$. Thus, if (1.6.3.1) (or (1.6.3.2)) is satisfied, then the vectors $X_{f_i,x} \in \Gamma_x$ are linearly independent (or linearly dependent). Let Γ_2 denote the linear span of the vectors $X_{f_1,x}, \dots, X_{f_k,x}$. We show that

$\Gamma_2 = \Gamma_1^{\perp}$. By what we have said above, it is enough to establish that $\Gamma_2 \subset \Gamma_1^{\perp}$. Suppose that $Y \in \Gamma_1$; then, using (1.4.3.7), we obtain

$$\Omega(X_{f_i,x}, Y) = dU_{1,x}(X_{f_i,x}, Y) = X_1(f_i)|_x \cdot U_{1,x}(Y) - df_{i,x}(Y) = 0.$$

Thus, the equation $E_{f_1,\ldots,f_k}$ is involutory at x if and only if $X_{f_i,x} \in T_x(E)$ or $X_{f_i,x}(f_j) = (f_i, f_j)|_x = 0$, $1 \leqslant i, j \leqslant k$.

COROLLARY. *Let $L \subset J^1(M)$ be a many-valued solution of the involutory equation $E_{f_1,\ldots,f_k}$. Then L is invariant under the vector fields X_{f_i}, $1 \leqslant i \leqslant k$.*

1.6.4. DEFINITION. a) An equation $E \subset J^1(M)$ is said to be *involutory* if it is involutory at each of its points.

b) A system of functions $f_1, \ldots, f_k$ is *involutory* if $[f_i, f_j] = 0$, $1 \leqslant i, j \leqslant k$.

EXAMPLE. The maximum codimension of an involutory equation E is $n + 1$. Let $E \subset J^1(M)$ be such an equation. Then $T_x(E) \subset \Gamma_x$, $x \in E$, so that E is an R-manifold. Thus, the R-manifolds are the involutory equations of maximal codimension.

1.6.5. Let $E \subset J^1(M)$ be an involutory equation, codim $E = k$. Consider the field of subspaces $C_x = \Gamma_{1,x}^{\perp}$. In general, the dimension of C_x varies as x varies. In fact, dim $C_x = k$ if $T_x(E) \subset \Gamma_x$.

DEFINITION. An involutory equation $E \subset J^1(M)$ is said to be *regular* (at a point $x_0 \in E$) if $T_x(E)$ and Γ_x are transversal at every point $x \in E$ (at $x_0 \in E$).

REMARK 1. An involutory equation of the form $E_{f_1,\ldots,f_k}$ is regular only if the vector fields $X_{f_1}, \ldots, X_{f_k}$ are linearly independent at every point of $E_{f_1,\ldots,f_k}$.

REMARK 2. If E is involutory and regular, then the field of subspaces C_x determines a differentiable distribution C_E on E.

PROPOSITION. *Let E be an involutory regular equation with codim $E = k$. Then C_E is a completely integrable distribution on E, and dim $C_E = k$.*

PROOF. Since the assertion is of local character, we assume that E has the form $E_{f_1,\ldots,f_k}$ and that the X_{f_i} are linearly independent on E.

Let X, Y be vector fields on E such that $X_x \in C_x$, $Y_x \in C_x$, $\forall x \in E$. We claim that $[X, Y]_x \in C_x$, $\forall x \in E$. Let X_i denote the restriction of X_{f_i} to E, $1 \leqslant i \leqslant k$. Then the $X_{i,x}$ form a basis for C_x, and therefore,

$$X = \sum_{i=1}^{k} a_i(x) X_i, \quad Y = \sum_{i=1}^{k} b_i(x) X_i.$$

Let $\bar{a}_i$ and $\bar{b}_i$ be any extensions of a_i and b_i to smooth functions on $J^1(M)$. We consider contact vector fields X_h and X_g, where

$h=\sum_{i=1}^{k}\bar{a}_i f_i$, $g=\sum_{i=1}^{k}\bar{b}_i f_i$. The restrictions of X_h and X_g to E coincide on X and Y respectively, since for $x \in E$ we have, for instance,

$$X_{h,x}=X_{\sum_{i=1}^{k}\bar{a}_i f_i,\,x}=\sum_{i=1}^{k} X_{\bar{a}_i,\,f_i,\,x}=$$

$$=\sum_{i=1}^{k}\left(\bar{a}_i(x)X_{f_i,\,x}+f_i(x)X_{\bar{a}_i,\,x}-\bar{a}_i(x)f_i(x)X_{1,\,x}\right)=X_x.$$

Therefore, $[X, Y]_x = [X_h, X_g]_x = X_{[h,g],x}$, and by a straightforward calculation we see that $[X, Y]_x \in C_x$.

COROLLARY. *Let X be a vector field on a regular involutory equation $E_{f_1,\dots,f_k}$, and suppose that $X_x \in C_x$, $\forall x \in E_{f_1,\dots,f_k}$. Then there is a smooth function $f \in C^\infty(J^1M)$ such that $f|_{E_{f_1,\dots,f_k}} = 0$ and the restriction of X_f to $E_{f_1,\dots,f_k}$ is X.*

REMARK. The generalized Cauchy problem for an involutory equation $E_{f_1,\dots,f_k}$ consists in specifying an $(n-k)$-dimensional submanifold $i\colon L \subset E_{f_1,\dots,f_k}$ for which $i^*(U_1) = 0$ and $C_x \not\subset T_x(L)$, $\forall x \in L$. In this case the solution of the Cauchy problem is (locally) unique and is $\bigcup_{x\in L} N_x$, when N_x is an integral manifold for the distribution $C_{E_{f_1,\dots,f_k}}$, passing through $x \in L$, with $\dim N_x = k$.

§7. The theorems of Darboux and Chern

1.7.1. PROPOSITION. Let $E \subset J^1(M)$ be an involutory equation with $\operatorname{codim} E = k$, and $\omega \in \Lambda^1(E)$ the restriction of U_1 to E, $\omega = U_1|_E$. Then

$$\omega \wedge \underbrace{d\omega \wedge \dots \wedge d\omega}_{n-k+1 \text{ times}} \equiv 0. \tag{1.7.1.1}$$

PROOF. If E is not regular at $x \in E$, then $T_x(E) \subset \Gamma_x$, so that $\omega_x = 0$ and (1.7.1.1) is satisfied.

Now let $x \in E$ be a regular point, $\omega_x \neq 0$. Let $\Gamma_{2,x} \subset T_x(E)$ denote the degeneracy subspace of $d\omega_x$. We consider first the case when $X_{t,x} \in T_x(E)$ (where X_1 is a contact field, and $U_1(X_1) = 1$); then $\Gamma_{2,x}$ is the linear span of $X_{1,x}$ and $\Gamma_{1,x}^{\perp}$, $\Gamma_{1,x} = T_x(E) \cap \Gamma_x$, and the rank of $d\omega_x$ is $2(n-k)$ in this case, so that (1.7.1.1) is satisfied.

Suppose that $X_{1,x} \notin T_x(E)$. Then $\Gamma_{2,x} \subset \Gamma_{1,x}^{\perp}$ is a subspace of codimension 1, because a vector $Y \in \Gamma_{1,x}^{\perp}$ lies in $\Gamma_{2,x}$ if and only if $d\omega_x(X, Y) = 0$ for some $X \in T_x)$, $X \notin \Gamma_{1,x}$.

Thus, the rank of $d\omega_k$ is $2(n-k+1)$ in this case. Next, it is clear that the form (1.7.1.1) is non-zero only on vectors $Y_0, Y_1, \dots, Y_{2(n-k+1)}$ for which (after renumbering if necessary) $Y_0 \notin \Gamma_{1,x}$ and $Y_1, \dots, Y_{2(n-k+1)}$

can be regarded as lying in $\Gamma_{1,x}$. But in this case (1.7.1.1) is also satisfied, because the restriction of $d\omega_x$ to $\Gamma_{1,x}$ is a form of rank $2(n - k + 1)$.

COROLLARY. *Let $E \subset J^1(M)$, codim $E = k$, be an involutory equation such that the restriction of $\pi_{*,x}$ to $T_x(E)$, $x \in E$, is a monomorphism. Then the rank of $d\omega_x$ is $2(n - k + 1)$.*

1.7.2. DEFINITION. A form $\omega \in \Lambda^1(N)$ is said to be *involutory* if the rank of $d\omega$ is $2k$ and

$$\omega \wedge \underbrace{d\omega \wedge \ldots \wedge d\omega}_{k \text{ times}} \equiv 0.$$

EXAMPLE. Every 1-form ω on an even-dimensional manifold such that $d\omega$ is a form of maximal rank is involutory.

1.7.3. We recall that a local diffeomorphism carrying $x \in M_1$ into $y \in M_2$ is a diffeomorphism $F\colon \mathcal{O}_1 \to \mathcal{O}_2$ of some neighbourhoods $\mathcal{O}_1 \subset M_1$ and $\mathcal{O}_2 \subset M_2$, such that $x \in \mathcal{O}_1$ and $y \in \mathcal{O}_2$.

If F is a local diffeomorphism and $\theta \in \Lambda^k(M_2)$, we understand by $F^*(\theta)$ the k-form $F^*(\theta|_{\mathcal{O}_2})$ on $\mathcal{O}_1$.

EXAMPLE. If $M_1 = J^1(M)$ and $M_2 = J^1(M)$, then F is called a *local contact diffeomorphism* if $F^*(U_1) = f \cdot U_1$, $f \in C^\infty(\mathcal{O}_1)$.

1.7.4. DEFINITION. Two forms $\theta_1 \in \Lambda^r(M_1)$ and $\theta_2 \in \Lambda^r(M_2)$ are *locally equivalent* at $m_1 \in M_1$ and $m_2 \in M_2$ if there exists a local diffeomorphism F with $F(m_1) = m_2$, such that $F^*(\theta_2) = \theta_1$.

THEOREM. *Let $\omega_1 \in \Lambda^1(M)$ and $\omega_2 \in \Lambda^1(M_2)$ be two involutory forms in neighbourhoods of $m_1 \in M_1$ and $m_2 \in M_2$, $\omega_{1,m_1} \neq 0$, $\omega_{2,m_2} \neq 0$, such that the ranks of $d\omega_{1,m_1}$ and $d\omega_{2,m_2}$ are the same. Then ω_1 and ω_2 are locally equivalent at $m_1 \in M_1$ and $m_2 \in M_2$.*

THEOREM (Darboux). *Let $\omega \in \Lambda^1(M)$ be an involutory 1-form in a neighbourhood of $m \in M$ such that $\omega_m \neq 0$ and $d\omega_m$ is of rank $2k$. Then there exist local coordinates $x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_l$ in a neighbourhood of $m \in M$ such that* $\omega = (1 + y_1)dx_1 + \sum_{i=2}^{k} y_i\, dx_i$.

A proof of Darboux's theorem can be found in [10], [18]; Theorem 1.7.4 follows in an obvious fashion from it.

1.7.5. Let N_1 and N_2 be two odd-dimensional manifolds, dim N_1 = dim N_2, and suppose that the 1-form $\omega_i \in \Lambda^1(N_i)$ gives a contact structure in some neighbourhood of $x_i \in N_i$, that is,

$$\omega_i \wedge \underbrace{d\omega_i \wedge \ldots \wedge d\omega_i}_{k \text{ times}} \neq 0, \text{ where } k = \tfrac{1}{2}(\dim N_i - 1).$$

THEOREM. *Suppose that the forms $\omega_i \in \Lambda^1(N_i)$ satisfy the conditions stated above in neighbourhoods of two points $x_1 \in N_1$ and $x_2 \in N_2$. Then the forms ω_1 and ω_2 are locally equivalent at $x_1 \in N_1$, $x_2 \in N_2$.*

COROLLARY 1. *If the form $\omega \in \Lambda^1(N)$ is as above, then we can choose* *a system of local coordinates* $x_1, \ldots, x_k, y_1, \ldots, y_k, z$, $\dim N = 2k+1$, *in a neighbourhood of* x *in which*

$$\omega = dz - \sum_{i=1}^{k} y_i \, dx_i.$$

A proof of this lemma can also be found in [18].

COROLLARY 2. *Let* $f \in C^\infty(J^1M)$ *be a function such that* $f(x) \neq 0$. *Then there exists a local contact diffeomorphism* F: $F(x) = x$, $F^*(U_1) = g \cdot U_1$, *such that* f *and* g *coincide in some neighbourhood of* $x \in J^1(M)$.

PROOF. The assertion follows from Theorem 1.7.5 and the fact that

$$(fU_1) \wedge \underbrace{d(fU_1) \wedge \ldots \wedge d(fU_1)}_{n \text{ times}} = f^{n+1}(U_1 \wedge \underbrace{dU_1 \wedge \ldots \wedge dU_1}_{n \text{ times}}) \neq 0,$$

where $n = \dim M$.

1.7.6. The following theorem concerns the question of the local equivalence of two vector fields X and Y on M at a point $m \in M$ such that $X_m = Y_m = 0$, in other words, the question of the existence of a local diffeomorphism F such that $F(m) = m$, $F_*(X) = Y$.

Before formulating the theorem, we give the following definition.

DEFINITION. A vector field Z on M is said to be *flat* at $m \in M$ if Z: $C^\infty(M) \to \mu_m^\infty$.

In other words, Z is flat at $m \in M$ if, in any local coordinate system $x_1, \ldots, x_n$ in the neighbourhood of m, Z can be written in the form $Z = \sum_{i=1}^{n} Z_i(x) \frac{\partial}{\partial x_i}$, where the $Z_i(x)$ are flat functions at m, $1 \leqslant i \leqslant n$.

Suppose now that the eigenvalues $\{\lambda_k\}$ of the linear part of the vector field X at m, $X_m = 0$, are such that $\operatorname{Re} \lambda_k \neq 0$.

Then, as is well known, there exists a Lyapunov function for X, that is, a function G having $m \in M$ as a non-degenerate critical point and such that $X(G) > 0$ in some neighbourhood of m (except at m itself).

THEOREM (Chern). *Suppose that* X *is as above and* $Y = X + Z$, *where* Z *is a field that is flat at* $m \in M$. *Then there exists a local diffeomorphism* F, $F(m) = m$, *such that*

$$F_*(X) = Y \text{ and } F \,|_{\{G=0\}} = 1.$$

A proof of this theorem can be found in [24].

CHAPTER 2

Local classification of regular differential equations

The main problem in any classification question is the choice of the classifying group. For second order differential equations two natural classification problems arise, depending on the classes of solutions that are considered. The first – and rougher – of these is defined by the group of contract diffeomorphisms of $J^1(M)$, that is, those that preserve the class of many-valued solutions. The second is defined by the group of contact diffeomorphisms of $J^1(M)$ that preserve the projection $\pi_1: J^1(M) \to M$, that is, those defined by the class or ordinary solutions.

This paper is devoted to the classification with respect to the first of these two groups.

§ 1. Statement of the problem

2.1.1. DEFINITION. Two equations $E_1 \subset J^1(M)$ and $E_2 \subset J^1(M)$ are locally equivalent (respectively, U_1-equivalent) at points $x_1 \in E_1$ and $x_2 \in E_2$ if there exist neighbourhoods $\mathcal{O}_1 \subset J^1(M)$ and $\mathcal{O}_2 \subset J^1(M)$, $x_1 \in \mathcal{O}_1$, $x_2 \in \mathcal{O}_2$, and a local contact (respectively, U_1 –) diffeomorphism $F: \mathcal{O}_1 \to \mathcal{O}_2$, $F(x_1) = x_2$, such that $F(E_1 \cap \mathcal{O}_1) = E_2 \cap \mathcal{O}_2$.

In what follows, we are interested in problems of local equivalence. We may, therefore, assume that M is connected. In that case any two points $x_1 \in J^1(M)$ and $x_2 \in J^1(M)$ can be carried on to the other by a U_1-diffeomorphism, so that henceforth we assume that $x_1 = x_2 = x$.

2.1.2. THEOREM. *Suppose that the equations $i_k: E_k \subset J^1(M)$ $(k = 1, 2)$ are such that the differential $(\pi \circ i_k)_{*,x}: T_x(E_k) \to T_{\pi(x)}(T^*M)$ is an isomorphism. Then the equations E_k are locally U_1-equivalent at a point $x \in E_k$ if and only if the* 1*-forms $\omega_k = U_1|_{E_k}$ are locally equivalent at x.*

PROOF. The necessity is clear. Therefore we just prove the sufficiency. Suppose that $\mathcal{O}_k \subset J^1(M)$, $x \in \mathcal{O}_k$ $(k = 1, 2)$ are neighbourhoods such that the $\pi(\mathcal{O}_k)$ are simply-connected, that the inverse mapping $(\pi \circ i_k)^{-1} \to E_k \cap \mathcal{O}_k$ exists in $\pi(\mathcal{O}_k)$ and that $F: E_1 \cap \mathcal{O}_1 \to E_2 \cap \mathcal{O}_2$ is a diffeomorphism with $F^*(\omega_2) = \omega_1$.

Let $\bar{G}: \pi(\mathcal{O}_1) \to \pi(\mathcal{O}_2)$ be the diffeomorphism defined by the formula $\bar{G} = (\pi \circ i_2) \circ F \circ (\pi \circ i_1)^{-1}$. We claim that $\bar{G}$ is a canonical diffeomorphism. To see this, we use the equations $\pi^*(d\rho) = -dU_1$, $\alpha \circ \pi = 1$, and obtain

$$\begin{aligned}\bar{G}^*(d\rho) &= (\pi \circ i_1)^{-1*} \circ F^* \circ (\pi \circ i_2)^*(d\rho) = -(\pi \circ i_1)^{-1*} \circ F^*(d\omega_2) = \\ &= -(\pi \circ i_1)^{-1*}(d\omega_1) = -(\pi \circ i_1)^{-1*} \circ i_1^*(dU_1) = \\ &= -(\alpha \circ \pi \circ i_1 \circ (\pi \circ i_1)^{-1})^*(dU_1) = -\alpha^*(dU_1) = d\rho.\end{aligned}$$

Next, the neighbourhoods $\mathcal{O}_k \subset J^1(M)$ have been chosen so that the $\pi(\mathcal{O}_k)$ are simply-connected; thus, the diffeomorphism $\bar{G}$ has a generating function in $S \in C^\infty(\pi(\mathcal{O}_1))$.

Let $G: \mathcal{O}_1 \to \mathcal{O}_2$ be a U_1-diffeomorphism covering $\bar{G}$, with $G(x) = x$. We consider the diagram

$$\begin{array}{ccccc} E_1 \cap \mathcal{O}_1 & \underset{i_1}{\hookrightarrow} & \mathcal{O}_1 & \xrightarrow{\pi} & \pi(\mathcal{O}_1) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle \bar{G}} \\ E_2 \cap \mathcal{O}_2 & \underset{i_2}{\hookrightarrow} & \mathcal{O}_2 & \xrightarrow{\pi} & \pi(\mathcal{O}_2) \end{array}$$

The outer and the right-hand squares in this diagram are commutative, so that there exists a translation τ along the fibres of π that makes the diagram commute, that is,

(2.1.2.1) $$i_2 \circ F = \tau \circ G \circ i_1.$$

Next we represent τ in the form

(2.1.2.2) $\tau(x, t) = (x, t + \mathcal{F}(x)),\ x \in T^*(M),\ t \in \mathbf{R},\ \mathcal{F} \in C^\infty(\pi(\mathcal{O}_2))$.

Applying the right- and left-hand sides of (2.1.2.1) to the form U_1 and using (2.1.2.2), we get

$$\omega_1 = (i_1 \circ F)^*(U_1) = (\tau \circ G \circ i_1)^*(U_1) = i_1^* \circ G^* \circ \tau^*(du - \bar{\rho}) =$$
$$= i_1^* \circ G^*(du - \bar{\rho} + d\mathcal{F}) = i_1^* \circ G^*(U_1 + d\mathcal{F}) = \omega_1 + i_1^* \circ G^*(d\mathcal{F}).$$

Thus, $i_1^* \circ G^*(d\mathcal{F}) = 0$, so that $\mathcal{F} = \text{const}$. But $G(x) = x$, so that $\mathcal{F} = 0$ and $\tau = 1$.

§2. Local classification of regular equations

2.2.1. THEOREM. *Let $i_k: E_k \subset J^1(M)$ be regular equations with* codim $E_k = 1$, *such that the differentials $(\pi \circ i_k)_{*,x}$ are isomorphisms. Then the E_k are locally U_1-equivalent at $x \in E_k$ $(k = 1, 2)$.*

PROOF. We show that the forms ω_k are locally equivalent at x. To this end we note that the form $d\omega_{k,x}$ has the maximal rank $2n$, $n = \dim M$, because $(\pi \circ i_k)_{*,x}$ is an isomorphism, and $d\omega_{k,x} = (\pi \circ i_k)_x^*(d\rho|_{\pi(x)})$. The regularity condition means that $\omega_{k,x} \neq 0$. The local equivalence of the forms ω_k at x now follows from Darboux's theorem.

2.2.2. LEMMA. *Let X_f be a contact vector field on $J^1(M)$, $F: J^1(M) \to J^1(M)$ a contact diffeomorphism, and $F^*(U_1) = \lambda U_1$, $\lambda \in C^\infty(J^1M)$. Then $F_*(X_f) = X_1$ if and only if $\lambda f = 1$.*

PROOF. $F_*(X_f)$ is a contact vector field whose Hamiltonian g is $g|_{F(x)} = U_{1,f(x)}(F_*(X_f)|_{F(x)}) = F^*(U_1)|_x(X_{f,x}) = (\lambda f)|_x$, that is, $F^*(g) = \lambda f$.

2.2.3. LEMMA. *For every equation $i: E \subset J^1(M)$ there exists a local contact diffeomorphism F, $F(x) = x$, $x \in F$, such that the differential $(\pi \circ i_1)_{*,x}$ of the equation $i: E_1 = F(E) \subset J^1(M)$ is a monomorphism.*

PROOF. Note that if E is not regular at x, then $T_x(E) \subset \Gamma_x$, so that $(\pi \circ i)_{*,x}$ is a monomorphism. Suppose now that E is regular at $x \in E$, but that $(\pi \circ i)_{*,x}$ is not a monomorphism, that is, $X_{1,x} \in T_x(E)$. We choose a function f such that $f(x) \neq 0$, $X_{f,x} \notin T_x(E)$. Let F be a local contact diffeomorphism such that $F^*(U_1) = \frac{1}{f} \cdot U_1$, $F(x) = x$ (see 1.7.5). We show that F is the required diffeomorphism. Indeed, by the preceding lemma we have $F_*(X_f) = X_1$, and $X_{1,x} \notin T_x(F(E))$ since $X_{f,x} \notin T_x(E)$; that is, $(\pi \circ i_1)_{*,x}$ is a monomorphism.

2.2.4. THEOREM. *Any two regular equations $E_k \subset J^1(M)$, codim $E_k = 1$, are locally equivalent at $x \in E_k$ $(k = 1, 2)$.*

2.2.5. In this subsection we consider one of the ways of constructing involutory equations.

PROPOSITION. *Let $E^r \subset J^1(M)$ be an involutory equation,* codim $E = r > 1$, *and T_t the group of translations along the contact vector field X_f, $X_{f,x} \notin T_x(E)$. Then the equation $E^{r-1} = \bigcup_t T_t(E^r)$, in some neighbourhood of x, is involutory; and $E^{r-1} \supset E^r$,* codim $E^{r-1} = r - 1$.

PROOF. We shall show that for each point $y \in E^{r-1}$ there is an R-manifold touching E^{r-1} at y. Let $L \subset J^1(M)$ be an R-manifold touching E^r at $x = T_t(y)$; then $T_t(L)$ is obviously an R-manifold of the sort required.

2.2.6. PROPOSITION. *Let i_k^r: $E_k^r \subset J^1(M)$,* codim $E_k^r = r > 1$ $(k = 1, 2)$ *be involutory equations, where the $(\pi \circ i_k^r)_{*,x}$ are monomorphisms. Then for every local diffeomorphism F_r: $E_1^r \to E_2^r$, $F_r(x) = x$, that establishes the equivalence of the* 1-forms $\omega_k^2 = U_1|_{E_k^r}$, *there is an involutory equation* i_k^{r-1}: $E_k^{r-1} \subset J^1(M)$, codim $E_k^{r-1} = r - 1$, $E_k^{r-1} \supset E_k^r$ $(k = 1, 2)$ *and a local diffeomorphism F_{r-1}, $F_{r-1}(x) = x$, establishing the equivalence of the forms $\omega_k^{r-1} = U_1|_{E_1^{r-1}}$. In addition the differential $(\pi \circ i_k^{r-1})_{*,x}$ is a monomorphism and the diagram*

$$\begin{array}{ccc} E_1^r & \subset & E_1^{r-1} \\ \downarrow{\scriptstyle F_r} & & \downarrow{\scriptstyle F_{r-1}} \\ E_2^r & \subset & E_2^{r-1} \end{array}$$

commutes.

PROOF. We construct E_2^{r-1} by means of translations along the Hamiltonian vector field X_f (see 2.2.5): $E_2^{r-1} = \bigcup_t T_t^2(E_2^r)$. We also construct E_1^{r-1} by means of translations along the Hamiltonian vector field X_g: $E_1^{r-1} = \bigcup_t T_t^1(E_1^r)$, where now we choose the Hamiltonian g so that $F_r^*(f|_{E_2^r}) = g|_{E_1^r}$. The choice of g is always possible in some neighbourhood of x, because $(\pi \circ i_k^r)_{*,x}$ is a monomorphism.

We define a local diffeomorphism F_{r-1} as follows: $F_{r-1}(y) = T_t^2 \circ F_r \circ T_{-t}^1(y)$, where $t \in (-\varepsilon, \varepsilon)$ is such that $T_{-t}^1(y) \in E_1^r$.

Let us check that $F_{r-1}^*(\omega_2^{r-1}) = \omega_1^{r-1}$. To do this we note that T_t^1 preserves U_1, so that the restriction of T_t^1 to E_1^{r-1} preserves ω_1^{r-1}. Similarly, the restriction of T_t^2 to E_2^{r-1} preserves ω_2^{r-1}. Thus, the equality of the forms $F_{r-1}^*(\omega_2^{r-1})$ and ω_1^{r-1} has been verified at points $y \in E_1^r$. But, by construction, these forms coincide on vectors tangent to E_1^r. On $X_{g,y}$ the form $\omega_{1,y}$ is equal to $g(y)$, and

$$F^*_{r-1}(\omega_2^{r-1})(X_{g,y}) = F_*^{r-1}(\omega_2^{r-1}(F_{r-1\,*}\cdot(X_{g\,y}))) = F_r^*(f)(y) = g(y).$$

2.2.7. THEOREM. *Suppose that* $i_k^r\colon E_k^r \subset J^1(M)$, codim $E_k^r = r$ $(k = 1, 2)$ *are regular equations,* $x \in E_k^r$. *Then* E_1^r *and* E_2^r *are locally equivalent at* x.

PROOF. Using Lemma 2.2.3, if necessary, we may assume that the $(\pi \circ i_k^r)_{*,x}$ are monomorphisms. Further, we use the preceding proposition and construct a chain of involutory regular equations E_k^{r-i} and local diffeomorphisms F_{r-i} $(i = 0, \dots, r-1)$ establishing the local equivalence of the forms ω_k^{r-i} at x. (The existence of F_r follows from Darboux's theorem).

In this situation the diagram

$$\begin{array}{ccccccc} E_1^r & \subset & E_1^{r-1} & \subset & \dots & \subset & E_1 \\ \downarrow{\scriptstyle F_r} & & \downarrow{\scriptstyle F_{r-1}} & & & & \downarrow{\scriptstyle F_1} \\ E_2^r & \subset & E_2^{r-1} & \subset & \dots & \subset & E_2^1 \end{array}$$

commutes, and $(\pi \circ i_k^{r-j})_{*,x}$ is a monomorphism.

According to Theorem 2.2.1, there exists a local U_1-diffeomorphism F, $F(x) = x$, such that $F|_{E_1^1} = F_1$. Therefore $F|_{E_1^{r-i}} = F_{r-i}$, that is, F establishes the local U_1-equivalence of E_1^{r-i} and E_2^{r-i} at x, $i = 0, 1, \dots, r-1$.

COROLLARY 1. *If the regular involutory equations* $i_k^r\colon E_k^r \subset J^1(M)$ *are such that the* $(\pi \circ i_k^r)_{*,x}$ *are monomorphisms, then* E_1^r *and* E_2^r *are* U_1*-equivalent at* x.

COROLLARY 2. *Let* $i^r\colon E^r \subset J^1(M)$ *be a regular involutory equation. Then there exists a local contact diffeomorphism* F, $F(x) = x$, *such that* $F(E^r)$ *is given in a neighbourhood of* x *by a system of equations* $p_1 = 0, \dots, p_r = 0$ *in a special local coordinate system* $q_1, \dots, q_n, u, p_1, \dots, p_n$.

2.2.8. REMARK. The U_1-classification of regular involutory equations $i^r\colon E^r \subset J^1(M)$ can be considered at all points $x \in E^r$ where $(\pi \circ i^r)_{*,x}$ is degenerate. By Thom's transversality theorem ([1], [15]), we can arrange by small displacements of the embedding i^r that the degenerate singularities are in general position, so that the set of degenerate points is a submanifold of codimension 1 in E^r. In this case the condition of general position is the same as that for the 1-forms ω^r at the points where the rank of $d\omega^r$ is not maximal. Therefore, by Martinet's theorem (see [9]), the forms ω_k^r are locally equivalent, whereas the problem of local U_1-equivalence remains open, because, in general, Proposition 2.2.6 is not true when $(\pi \circ i^r)_*$ is degenerate.

§3. Local solubility of regular involutory equations

In this section we investigate the local solubility of involutory equations, using the classification theorems of §2.

2.3.1. THEOREM. *An involutory equation* $E^r \subset J^1(M)$ *is locally soluble in the neighbourhood of every regular point* $x \in E^r$.

2.3.2. As an illustration we consider the following example. Suppose that the equation $E \subset J^1(M)$ has the form $u = 0$ in the special local coordinate system $q_1, \ldots, q_n, u, p_1, \ldots, p_n$, and let $x \in E$ be the point with the coordinates $(q_1^0, \ldots, q_n^0, 0, p_1^0, \ldots, p_n^0$, where $\sum_{i=1}^{n} (p_i^0)^2 \neq 0$, so that x is a regular point of E. In invariant terms the equation $u = 0$ specifies a submanifold $T^*(M) \subset J^1(M)$. The solutions of this equation are the Lagrangian submanifolds $L \subset T^*(M)$ passing through x and transforming the form ρ to zero, $\rho|_L = 0$. It is not hard to see that these solutions are essentially many-valued, that is, even locally they do not have the form of the graph of a section corresponding to the 1-jet of some function.

We remark incidentally that the store of many-valued solutions of this equation is greater than that of generalized solutions (in the usual sense), since the latter are altogether absent.

2.3.3. THEOREM. *There exists a (locally) ordinary solution of a regular involutory equation $E^r \subset J^1(M)$ passing through the point $x \in E^r$,* codim $E^r = r$, *if and only if $\pi_{1*,x}: C_x \to T_{\pi_1(x)}(M)$ is a monomorphism.*

PROOF. Let $L \subset E^r$ be an ordinary solution at $x \in L$, that is, locally $L = [j_1(f)]\,(M)$ for some function $f \in C^\infty(M)$. Then $C_x \subset T_x(L)$, so that $\pi_{1*,x}$ is a monomorphism on C_x.

Assume now that there exists a Lagrangian subspace $L_x \subset \Gamma_x$ such that $C_x \subset L_x$ and L_x projects onto $T_{\pi_1(x)}(M)$ without degeneracy. We choose a local diffeomorphism F, $F(x) = x$, such that $F(E^r)$ can be written in the form $p_1 = 0, \ldots, p_r = 0$ in some special local coordinate system $q_1, \ldots, q_n, u, p_1, \ldots, p_n$. Let $L \subset F(E^r)$ be an R-manifold whose projection on $T^*(M)$ is invariant relative to the fields $\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_r}$ and which is tangent to $F_{*,x}(L_x)$. Then $F^{-1}(L) \subset E^r$ defines the required R-manifold.

Thus, it is enough to show the existence of a subspace $L_x \subset \Gamma_x$. To do this we note that $dU_{1,x}|_{C_x} = 0$, since $C_x = \Gamma_{1,x}^{\perp}$. Therefore C_x is an isotropic subspace. If dim $C_x = n$ (= dim M), then with $C_x = L_x$ we get the required subspace.

Otherwise we choose any Lagrangian subspace $L_x \supset C_x$ for which $L_x \cap R_x = \{0\}$, where $R_x = \operatorname{Ker} \pi_{1,x}$ is also Lagrangian.

2.3.4. If an involutory equation $E^r \subset J^1(M)$ is given by a system

$$f_1(q_1, \ldots, q_n, u, p_1, \ldots, p_n) = 0, \ldots, f_r(q_1, \ldots, q_n, u, p_1, \ldots, p_n) = 0$$

in a special local coordinate system $q_1, \ldots, q_n, u, p_1, \ldots, p_n$, then the condition of local solubility in Theorem 2.3.3 means that $\frac{\partial f_1}{\partial p}(x) \ldots, \frac{\partial f_r}{\partial p}(x)$ are linearly independent. In particular, if $r = 1$, then this condition is equivalent to the condition "of smooth type" $\frac{\partial f}{\partial p}(x) \neq 0$.

CHAPTER III

Singular points of first order differential equations

In the preceding chapter we have studied the structure of an involutory differential equation in the neighbourhood of a regular point, that is, a point $x \in E^r$ such that the subspaces $T_x(E^r)$ and Γ_x are transversal. In this chapter we proceed to a study of the structure of an involutory differential equation in the neighbourhood of a singular point $x \in E^r$, $T_x(E^r) \subset \Gamma_x$. In contrast to the regular case, there is now a continuum of equivalence classes. The fundamental invariant distinguishing these classes is the Hessian of the equation.

§ 1. The Hessian of an involutory equation at a singular point

3.1.1. Let E^r be an involutory equation, $E^r \subset J^1(M)$, codim $E^r = r$, and let $x \in E^r$ be a singular point, that is, $T_x(E^r) \subset \Gamma_x$. If ω denotes the restriction of the form U_1 to E^r, $\omega = U_1|_{E^r}$, then the condition for $x \in E^r$ to be singular is equivalent to $\omega_x = 0$.

We consider any two vectors $X, Y \in T_x(E^r)$ and extensions $\bar{X}, \bar{Y}$ of them to E^r, so that $\bar{X}_2 = X, \bar{Y}_x = Y$.

We set $h_\omega(X, Y) = \bar{X}(\omega(Y))|_x$.

LEMMA–DEFINITION. *h_ω is a bilinear form on $T_x(E^r)$ and is called the Hessian of E^r at the singular point x.*

PROOF. We show that h_ω is well-defined. Note first that the expression $\bar{X}(\omega(\bar{Y}))|_x$ does not depend on the extension $\bar{X}$ and is determined by the value $\bar{X}_x = X$. Next, we make use of the standard formula $d\omega(\bar{X}, \bar{Y}) = \bar{X}(\omega(\bar{Y})) - \bar{Y}(\omega(\bar{X})) - \omega([\bar{X}, \bar{Y}])$, which at a singular point $x \in E^r$ takes the form

$$d\omega_x(X, Y) = \bar{X}(\omega(\bar{Y}))|_x - \bar{Y}(\omega(\bar{X}))|_x. \tag{3.1.1.1}$$

The left-hand side of (3.1.1.1) does not depend on the extensions $\bar{X}, \bar{Y}$, therefore $\bar{Y}(\omega(\bar{X}))|_x$ does not depend on $\bar{X}$. Thus, h_ω is well-defined on $T_x(E^r)$.

COROLLARY. *For all $X, Y \in T_x(E^r)$,*

$$h_\omega(X, Y) - h_\omega(Y, X) = d\omega_x(X, Y). \tag{3.1.1.2}$$

3.1.2. We indicate the form of h_ω in local coordinates. For this purpose we note that $d\omega_y$ is of rank $2(n - r + 1)$ if $y \in E^r$ is sufficiently near the singular point $x \in E^r$ (see 1.7.1). Therefore, we can choose a system of local coordinates $x_1, \ldots, x_{n-r+1}, y_1, \ldots, y_{n-r+1}, z_1, \ldots, z_{r-1}$ in the neighbourhood of x in which $d\omega = \sum_{i=1}^{n-r+1} dy_i \wedge dx_i$. Further, since ω is involutory, in this system of local coordinates we have

$$\omega = d\mathcal{F}(x, y) - \sum_{i=1}^{n-r+1} y_i\, dx_i.$$

It follows at once from the definition of h_ω that h_ω is given in these coordinates by the matrix:

(3.1.2.1)
$$\left\| \begin{matrix} \left.\frac{\partial^2 \mathcal{F}}{\partial x_i \partial x_j}\right|_x & \left.\frac{\partial^2 \mathcal{F}}{\partial x_i \partial y_j}\right|_x & 0 \\ \left.\frac{\partial^2 \mathcal{F}}{\partial y_i \partial y_j}\right|_x - \delta_{ij} & \left.\frac{\partial^2 \mathcal{F}}{\partial y_i \partial y_j}\right|_x & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

For example, $h_\omega \left(\left.\frac{\partial}{\partial x_i}\right|_x, \left.\frac{\partial}{\partial y_j}\right|_x \right) = \left.\frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{F}}{\partial y_j}\right)\right|_x$

3.1.3. PROPOSITION. *Let* $X \in T_x(E^r)$, $Y \in C_x = T_x(E^r)^\perp$; *then* $h_\omega(X, Y) = h_\omega(Y, X) = 0$.

PROOF. Since the assertion is of local character, we may assume that $E^r = E_{f_1, \dots, f_r}$. Then C_x is the linear span of the vectors $X_{f_1,x}, \dots, X_{f_r,x}$. Therefore $Y = \sum_{i=1}^{r} a_i X_{f_i,x}$.

For the extension $\bar{Y}$ we take the restriction of the vector field X_f to E^r, $f = \sum_{i=1}^{r} a_i f_i$. Then

$$h_\omega(X, Y) = X(\omega(\bar{Y})) = X(U_1(X_f)) = \sum_{i=1}^{r} a_i\, df_{i,x}(X) = 0,$$

since $X \in T_x(E^r)$.

That $h_\omega(Y, X) = 0$ now follows from (3.1.1.2) and the fact that C_x is the degeneracy subspace of $d\omega_x$.

3.1.4. We consider the factor-space $V = T_x(E^r)/C_x$. Since C_x is the degeneracy subspace of $d\omega_x$, this form defines a non-degenerate 2-form Ω on V and, thus, provides a symplectic structure on V. Here $\dim V = 2(n - r + 1)$. Proposition 3.1.3 shows that, in its turn h_ω defines a bilinear form $\bar{h}_\omega$ on V, and by (3.1.1.2),

(3.1.4.1) $$\bar{h}_\omega(X, Y) - \bar{h}_\omega(Y, X) = \Omega(X, Y),\quad X, Y \in V.$$

The existence of a symplectic structure and a bilinear form on V allows us to define an operator H: $V \to V$ in the standard way:

$$\bar{h}_\omega(X, Y) = \Omega(HX, Y).$$

Also, (3.1.4.1) takes the form

(3.1.4.2) $$\Omega(HX, Y) - \Omega(X, HY) = \Omega(X, Y).$$

3.1.5. Before considering how the operator H and the form h_ω are connected with the problem of the local equivalence of equations, we make the following remark.

PROPOSITION. *Let $E_k^r \subset J^1(M)$ ($k = 1, 2$), codim $E_k^r = r$, be involutory equations, and let $x \in E_k^r$ be a singular point. Then there exists a local contact diffeomorphism F, $F(x) = x$, such that $F_{*,x}(T_x(E_1^r)) = T_x(E_2^r)$.*

PROOF. It is enough if we take for F any contact diffeomorphism extending a symplectic map A: $\Gamma_x \to \Gamma_2$ such that $A(T_x(E_1^r)) = T_x(E_2^r)$. The fact that such an A exists can be proved, for example, as follows. Let A_1: $C_{x,1} \to C_{x,2}$, $C_{x,1} = T_x(E_1^r)^{\perp}$, $C_{x,2} = T_x(E_2^r)^{\perp}$ be any non-degenerate linear mappings. Then by Witt's theorem (see [25]), there is an extension of A_1 to a symplectic map A and the mapping so obtained is easily seen to be what we are looking for.

3.1.6. With Proposition 3.1.5 in mind, we consider in what follows equations E_k^r such that $T_x(E_1^r) = T_x(E_2^r)$. Let F be a local contact diffeomorphism, $F(x) = x$, establishing the equivalence of E_1^r and E_2^r at x. Then its differential $F_{*,x}$ induces a conformally-symplectic map A: $\Gamma_x \to \Gamma_x$, where $dU_{1,x}(AX, AY) = \lambda\, dU_{1,x}(X, Y)$ for all $X, Y \in \Gamma_x$, $\lambda \neq 0$.

For, since $F^*(U_1) = g \cdot U_1$, we have $F^*(dU_{1,x}) = d(F^*(U_1))|_x = dg_x \wedge U_{1,x} + g(x)dU_{1,x}$, hence on restricting to Γ_x we find that $\lambda = g(x)$.

Further, since $A = F_{*,x}|_{\Gamma_x}$ is a conformally-symplectic transformation, it follows that A preserves skew-orthogonal complements; that is, in particular, A: $C_x \to C_x$, $C_x = T_x(E_k^r)^{\perp}$. Consequently, A defines a mapping $\bar{A}$: $V \to V$, which is also conformally-symplectic with respect to the form Ω, $\bar{A}^*(\Omega) = \lambda\Omega$. Here, as usual, $\bar{A}^*(\Omega)(X, Y) = \Omega(\bar{A}X, \bar{A}Y)$.

Thus the differential $F_{*,x}$ of a local contact diffeomorphism determines a conformally-symplectic map on Γ_x; if $F_{*,x}$ preserves $T_x(E_k^r)$, then it determines a conformally-symplectic transformation on V. Therefore, it is enough to study the behaviour of h_ω and H under such transformations.

3.1.7. PROPOSITION. *Let F be a local contact diffeomorphism, $F(x) = x$, such that $F(E_1^r) = E_2^r$. Then $A^*(h_{\omega_2}) = \lambda\, h_{\omega_1}$, where $A = F_{*,x}|_{Tx(E_1^r)}$ $\lambda = g(x)$, $F^*(U_1) = g \cdot U_1$.*

PROOF. By definition we have $A^*(h_{\omega_2})(XY) = h_{\omega_2}(AX, AY) = \overline{AX}(\omega_2(AY))|_x$. We choose extensions $\bar{X}$ and $\bar{Y}$ such that $\overline{AX} = F_*(\bar{X})$, $\overline{AY} = F_*(\bar{Y})$. Then

$$h_{\omega_2}(AX, AY) = \overline{AX}(\omega_2(A\bar{Y}))\,|_x = F_*(\bar{X})(\omega_2(F_*(\bar{Y})))\,|_x =$$

$$= (F^*)^{-1}(\bar{X}(F^*(\omega_2)(\bar{Y})))\,|_x = \lambda h_{\omega_1}(X, Y),$$

since $F^*(\omega_2) = g \cdot \omega_1$.

COROLLARY. *Let $\bar{A}$ be the transformation on V defined by A; then $\bar{A}H_1 = H_2\bar{A}$, where the H_k are the operators generated by h_{ω_k} ($k = 1, 2$).*

PROOF. Since $\bar{A}^*(\Omega) = \lambda\Omega$, it follows that $\Omega(H_1X, Y) = \lambda^{-i}\Omega(\bar{A}H_1X, \bar{A}Y)$. On the other hand $\bar{A}^*(h_{\omega_2}) = \lambda\bar{h}_{\omega_1}$ or

$\Omega(H_2\bar{A}X, \bar{A}Y) = \lambda\Omega(H_1X, Y)$. Comparing the two equations so obtained, we find that $\Omega(H_2\bar{A}X, \bar{A}Y) = \Omega(\bar{A}H_1X, \bar{A}Y)$, that is, $H_2\bar{A} = \bar{A}H_1$.

3.1.8. Let $CSp(n - r + 1)$ denote the group of all conformally-symplectic transformations on a $2(n - r + 1)$-dimensional symplectic space (V, Ω).

We define an action of $A \in CSp(n - r + 1)$ on bilinear forms as follows:

$$A[b] = \lambda^{-1}A^*(b), \tag{3.1.8.1}$$

where b is a bilinear form and $A^*(\Omega) = \lambda\Omega$.

Combining Proposition 3.1.7 and Corollary 3.1.7, we now get the following assertion.

THEOREM. *A necessary condition for involutory differential equations $E_k^r \subset J^1(M)$ to be locally equivalent at a singular point $x \in E_k^r$ $(k = 1, 2)$ is that the Hessians or the operators H_k are equivalent under $CSp(n - r + 1)$.*

3.1.9. Together with the form $\bar{h}_\omega$ we can consider its symmetric part $\bar{h}_\omega^s$, $\bar{h}_\omega^s(X, Y) = \frac{1}{2}(\bar{h}_\omega(X, Y) + \bar{h}_\omega(Y, X))$, $X, Y \in V$, and its skew-symmetric part $\bar{h}_\omega^a$, $\bar{h}_\omega^a(X, Y) = \frac{1}{2}(\bar{h}_\omega(X, Y) - \bar{h}_\omega(Y, X))$. It follows from (3.1.4.2) that $\bar{h}_\omega^a = \frac{1}{2}\Omega$. The symmetric form $\bar{h}_\omega^s$ can be arbitrary (see 3.1.2). Therefore, Theorem 3.1.8 can be reformulated as follows.

THEOREM. *A necessary condition for involutory differential equations $E_k^r \subset J^1(M)$ to be equivalent at a singular point $x \in E_k^r$ $(k = 1, 2)$ is that the symmetric forms $\bar{h}_{\omega_k}^s$ are $CSp(n - r + 1)$-equivalent with respect to the action* (3.1.8.1).

3.1.10. REMARK 1. Results analogous to Theorem 3.1.8 can be obtained for the U_1-equivalence of equations at a singular point, the only difference being that we have to use the symplectic group $Sp(n - r + 1)$.

REMARK 2. As is clear from the necessary conditions just mentioned, the local classification of equations at a singular point is not discrete if $r \neq n + 1$.

§2. *Csp*-classification

In this section we investigate the question of the $CSp(n - r + 1)$-equivalence of operators $H: V \to V$ satisfying the conditions in 3.1.4.

3.2.1. We denote by ${}^{\mathbf{C}}V$, ${}^{\mathbf{C}}H$, ${}^{\mathbf{C}}\bar{h}_\omega$, ${}^{\mathbf{C}}\Omega$ the complexifications of V, H, $\bar{h}_\omega$, Ω, respectively. The following analogue to (3.1.4.2) holds:

$$ {}^{\mathbf{C}}\Omega({}^{\mathbf{C}}HX, Y) + {}^{\mathbf{C}}\Omega({}^{\mathbf{C}}HY, X) = {}^{\mathbf{C}}\Omega(X, Y),$$

$$ {}^{\mathbf{C}}\Omega({}^{\mathbf{C}}HX, Y) = {}^{\mathbf{C}}\bar{h}_\omega(X, Y),$$

where $X, Y \in {}^{\mathbf{C}}V$, and ${}^{\mathbf{C}}\Omega$ is a form of maximal rank.

Let $K_\lambda \subset {}^{\mathbf{C}}V$ be the subspace consisting of the vectors annihilated by some power of ${}^{\mathbf{C}}H - \lambda$, $\lambda \in \mathbf{C}$, and $C_\lambda \subset K_\lambda \subset {}^{\mathbf{C}}V$ the eigenspace of ${}^{\mathbf{C}}H$

corresponding to the eigenvalue λ.

The following lemma (see 3.2.2 b) is an analogue to the Poincaré-Lyapunov theorem on eigenvalues of Hamiltonian systems (see [2]).

3.2.2. LEMMA. a) *Suppose that* $X \in K_{\lambda_1}$, $Y \in K_{\lambda_2}$, *and* $\lambda_1 + \lambda_2 \neq 1$. *Then* ${}^{C}\Omega(X, Y) = 0$, *that is, the subspaces* K_{λ_1} *and* K_{λ_2} *are skew-orthogonal.*

b) *Let* $\lambda \in \mathbf{C}$ *be an eigenvalue of the operator* H; *then* $\bar{\lambda}$, $1 - \lambda$ *and* $1 - \bar{\lambda}$ *are also eigenvalues of* H.

PROOF. a) We use 3.2.1. If ${}^{C}HX = \lambda_1 X$, ${}^{C}HY = \lambda_2 Y$, we obtain $(\lambda_1 + \lambda_2)\,{}^{C}\Omega(X, Y) = {}^{C}\Omega(X, Y)$. But $\lambda_1 + \lambda_2 \neq 1$, so that ${}^{C}\Omega(X, Y) = 0$. Suppose now that $X \in K_{\lambda_1}$, $Y \in K_{\lambda_2}$ are arbitrary. Then there exist n_1 and n_2 such that $({}^{C}H - \lambda_1)^{n_1} X \in C_{\lambda_1}$, $({}^{C}H - \lambda_2)^{n_2} Y \in C_{\lambda_2}$. We choose n_1 and n_2 minimal, and set $X_k = ({}^{C}H - \lambda_1)^{n_1 - k} X$, $Y_k = ({}^{C}H - \lambda_2)^{n_2 - k} Y$. Then $X_0 \in C_{\lambda_1}$, $Y_0 \in C_{\lambda_2}$, and therefore ${}^{C}\Omega(X_0, Y_0) = 0$. We now prove by induction that ${}^{C}\Omega(X_k, X_0) = 0$. We assume that this last equation has been proved for X_{k-1}; then, replacing X_k and Y_0 in 3.2.1 by X and Y, we get

$$(-1 + \lambda_1 + \lambda_2)\,{}^{C}\Omega(X_k, Y_0) = {}^{C}\Omega(X_{k-1}, Y_0) = 0.$$

Similarly we can prove by induction on s that ${}^{C}\Omega(X_k, Y_s) = 0$ for arbitrary k and s. In particular, on setting $k = n_1$ and $s = n_2$ we find that ${}^{C}\Omega(X, Y) = 0$.

b) We assume the contrary, that is, that there exists an eigenvalue $\lambda_1 \in \mathbf{C}$ such that $1 - \lambda_1$ is not an eigenvalue of H. It follows from a) that if $0 \neq \eta \in K_{\lambda_1}$, then ${}^{C}\Omega(\eta, X) = 0$ for arbitrary $X \in {}^{C}V$, since ${}^{C}V = \oplus K_{\lambda}$. Thus, the vector η is skew-orthogonal to the whole subspace ${}^{C}V$, which is impossible because the form ${}^{C}\Omega$ is non-degenerate.

3.2.3. Lemma 3.2.2, therefore, shows that the whole space ${}^{C}V$ splits into the direct sum ${}^{C}V = \bigoplus_{(\lambda, 1-\lambda)} (K_{\lambda} \oplus K_{1-\lambda})$, whose summands are invariant under ${}^{C}H$ and skew-orthogonal relative to ${}^{C}\Omega$. Here the restriction of ${}^{C}\Omega$ to each direct summand $K_{\lambda} \oplus K_{1-\lambda}$ is non-degenerate.

Thus, the investigation of H relative to the (symplectic) conformally-symplectic group reduces to an investigation of the restriction of ${}^{C}H$ to $K_{\lambda} \oplus K_{1-\lambda}$.

We introduce the following notation: $H_1 = {}^{C}H - \lambda$, $H_2 = {}^{C}H - (1 - \lambda)$. Then (3.1.4.2) can be generalized as follows:

$$\text{(3.2.3.1)} \qquad {}^{C}\Omega(H_1^k X, Y) = (-1)^k\,{}^{C}\Omega(X, H_2^k Y),$$

where $k \geqslant 0$ is a natural number, and $X, Y \in {}^{C}V$.

Let us prove these relations. They are obviously true for $k = 0, 1$. Assuming that they are true for $k \leqslant N$, we find

$${}^{C}\Omega(H_1^{N+1} X, Y) = (-1)^N\,{}^{C}\Omega(H_1 X, H_2^N Y) = (-1)^{N+1}\,{}^{C}\Omega(X, H_2^{N+1} Y).$$

Next, using the relations so obtained, we describe the process of constructing the canonical basis (with respect to ${}^{C}\Omega$) of $K_\lambda \oplus K_{1-\lambda}$, relative to which the matrix of ${}^{C}H$ takes the Jordan form. We divide the construction into several lemmas.

3.2.4. LEMMA. *A basis* $\ldots, f_i^1, \ldots, f_i^{n_i}, g_i^1, \ldots, g_i^{n_i}, \ldots$ $(i = 1, \ldots, \dim C_\lambda)$ *can be chosen in* $K_\lambda \oplus K_{1-\lambda}$, *with* $f_i^1 \in C_\lambda$, $g_i^1 \in C_{1-\lambda}$, $f_i^k = H_1^{n_i-k} f_i^{n_i}$, $g_i^k = H_2^{n_i-k} g_i^{n_i}$, *such that* $H_1^{n_i} f_i^{n_i} = H_2^{n_i} g_i^{n_i} = 0$. *Moreover*

$$(3.2.4.1) \qquad {}^{C}\Omega(g_j^k, f_i^m) = (-1)^m \delta_{ij}\delta_{k, n_i - m+1}.$$

PROOF. We carry out the construction of a basis of this kind by induction on the dimension of K_λ. We assume that $f^{n_1} \in K_\lambda$ has maximal height with respect to H_1 among the vectors of K_λ, that is, $H_1^{n_1} X = 0$, $X \in K_\lambda$, but $H_1^{n_1-1} f^{n_1} \neq 0$. The vector $\tilde{g}^{n_1} \in K_{1-\lambda}$ is chosen so that ${}^{C}\Omega(f^1, \tilde{g}^{n_1}) = 1$, where $f^k = H_1^{n_1-k} f^{n_1}$. As is easily seen, there always exists a vector $\tilde{g}^{n_1}$ as indicated, since the restriction of the form ${}^{C}\Omega$ to $K_\lambda \oplus K_{1-\lambda}$ is non-degenerate. The vector $\tilde{g}^{n_1}$ is of height n_1, because ${}^{C}\Omega(f^{n_1}, H_2^{n_1-1}\tilde{g}^{n_1}) = (-1)^{n_1-1}\,{}^{C}\Omega(f^1, \tilde{g}^{n_1}) \neq 0$, that is, $H_2^{n_1-1}\tilde{g}^{n_1} \neq 0$, while on the other hand, ${}^{C}\Omega(X, H_2^{n_1}\tilde{g}^{n_1}) = (-1)^n\, {}^{C}\Omega(H_1^{n_1} X, \tilde{g}^{n_1}) = 0$ for all $X \in K_\lambda$, so that $H_2^{n_1}\tilde{g}^{n_1} = 0$.

We set $\tilde{g}^k = H_2^{n_1-k}\tilde{g}^{n_1}$ and claim that $\tilde{g}^k$ can be replaced by g^k in such a way that the following relations are satisfied: ${}^{C}\Omega(g^k, f^{n_1}) = (-1)^{n_1}\delta_{1k}$, $1 \leqslant k \leqslant n_1$, and $H_2^{n_1-k} g^{n_1} = g^k$. To see this we choose $g^1 = \tilde{g}^1$, $g^2 = \tilde{g}^2 - a_2\tilde{g}^1$, where $a_2 = (-1)^{n_1}\,{}^{C}\Omega(\tilde{g}^2, f^{n_1})$; $g^3 = \tilde{g}^3 - a_2\tilde{g}^2 - a_3\tilde{g}^1$, where $a_3 = (-1)^{n_1}\,{}^{C}\Omega(\tilde{g}^3, f^{n_1}) - a_2^2, \ldots,$ $g^k = \tilde{g}^k - a_2\tilde{g}^{k-1} - a_3\tilde{g}^{k-2} - \ldots - a_k\tilde{g}^1$, where the coefficient a_k, under the assumption that $a_2, \ldots, a_{k-1}$ are known, is obtained from the formula

$$a_k = (-1)^{n_1}\,{}^{C}\Omega(\tilde{g}^k, f^{n_1}) - a_2a_{k-1} - a_3a_{k-2} - \ldots - a_{k-1}a_2.$$

Replacing X by g^k and Y by f^s in (3.1.4.2), $1 \leqslant k, s \leqslant n_1$, we find

$$(3.2.4.2) \qquad {}^{C}\Omega(g^k, f^{s-1}) + {}^{C}\Omega(g^{k-1}, f^s) = 0.$$

But since ${}^{C}\Omega(g^1, f^{n_1}) = (-1)^{n_1}$, ${}^{C}\Omega(g^k, f^{n_1}) = 0$, $k > 1$, it follows at once from (3.2.4.2) that ${}^{C}\Omega(g^k, f^s) = (-1)^s\delta_{k,n_1-s+1}$. We consider now the subspaces $M_\lambda \subset K_\lambda$, $M_{1-\lambda} \subset K_{1-\lambda}$, where $M_\lambda = \{X \in K_\lambda \mid {}^{C}\Omega(X, g^k) = 0, \forall k\}$, $M_{1-\lambda} = \{X \in K_{1-\lambda} \mid {}^{C}\Omega(X, f^k) = 0, \forall k\}$. Let $[f]$ (respectively, $[g]$) be the linear span of the vectors $f^1, \ldots, f^{n_1}$ (respectively, $g^1, \ldots, g^{n_1}$). Then $K_\lambda = M_\lambda \oplus [f]$, $K_{1-\lambda} = M_{1-\lambda} \oplus [g]$. For suppose that $X \in K_\lambda$ and ${}^{C}\Omega(X, g^k) = a_k$, for example; then clearly $\tilde{X} = X - \sum_{k=1}^{n_1} (-1)^{n_1-k} a_k f^{n_1-k+1}$ lies in M_λ and $X = \tilde{X} + \sum_{k=1}^{n_1} (-1)^{n_1-k} a_k f^{n_1-k+1}$. Moreover, it is clear that

$M_\lambda \cap [f] = 0$. The decompositions $K_\lambda = M_\lambda \oplus [f]$ and $K_{1-\lambda} = M_{1-\lambda} \oplus [g]$ are such that the subspaces $[f]$ and $M_{1-\lambda}$ (respectively, $[g]$ and M_λ) are skew-orthogonal by construction, and invariant relative to ${}^{C}H$. For instance, if $X \in M_\lambda$, then $H_1 X \in M_\lambda$, since ${}^{C}\Omega(H_1 X, g^k) = (-1)\,{}^{C}\Omega(X, H_2 g^k) = (-1)\,{}^{C}\Omega(K, g^{k-1}) = 0$. Thus, the restrictions of ${}^{C}H$ and ${}^{C}\Omega$ to $M_\lambda \oplus M_{1-\lambda}$ are such that the restriction of ${}^{C}\Omega$ is a form of maximal rank, and $M_{1-\lambda}$ and M_λ are invariant relative to ${}^{C}H$; therefore, the inductive hypothesis now yields a basis of the kind required.

The lemma just proved allows us to obtain the following intermediate result.

3.2.5. THEOREM. *Suppose that the eigenvalues $\{\lambda_s\}$ of the operators $H^{(k)}$ $(k = 1, 2)$ are such that* $\operatorname{Re} \lambda_s \neq \frac{1}{2}$. *Then the operators $H^{(k)}$ are symplectically (conformally-symplectically) equivalent if and only if they are equivalent under the full linear group.*

PROOF. The set of all eigenvalues of the $H^{(k)}$ can be represented as a union of disjoint quadruples $(\lambda_s, \bar\lambda_s, 1-\lambda_s, 1-\bar\lambda_s)$. By Lemma 3.2.4, for each operator $H^{(k)}$ there is a basis $\ldots f^r_m, \ldots, g^r_m, \ldots$ of $K^{(k)}_\lambda \oplus K^{(k)}_{1-\lambda}$ satisfying (3.2.4.1), and for the similar basis in $K^{(k)}_{\bar\lambda} \oplus K^{(k)}_{1-\bar\lambda}$ we choose $\ldots \bar f^r_m, \ldots, \bar g^r_m, \ldots$. Then in $V \cap (K^{(k)}_\lambda \oplus K^{(k)}_{1-\lambda} \oplus K^{(k)}_{\bar\lambda} \oplus K^{(k)}_{1-\bar\lambda})$ we can choose a basis consisting of vectors

$$\frac{1}{\sqrt{2}}(f^r_m + \bar f^r_m),\quad \frac{1}{i\sqrt{2}}(f^r_m - \bar f^r_m),\quad \frac{1}{\sqrt{2}}(g^r_m + \bar g^r_m),\quad \frac{1}{i\sqrt{2}}(g^r_m - \bar g^r_m),$$

relative to which the matrix of $H^{(k)}$ has generalized Jordan form. If corresponding bases are chosen for $H^{(1)}$ and $H^{(2)}$, then transition from one to the other is a symplectic transformation.

3.2.6. We consider now the subspaces $K_\lambda \oplus K_{1-\lambda}$ with $\lambda = \frac{1}{2} + i\mu$, $\mu \neq 0$. The basis whose existence is asserted in Lemma 3.2.4 is unsatisfactory in this case, because, in general, $f^r_m \neq \bar g^r_m$, so that we cannot find a canonical basis in $V \cap (K_\lambda \oplus K_{1-\lambda})$ by this method. To investigate this case as well we give a proof of Lemma 3.2.4 when $\lambda = \frac{1}{2} + i\mu$, that is, when $f^r_m = \bar g^r_m$.

3.2.7. LEMMA. *Suppose that $\lambda = \frac{1}{2} + i\mu$, $\mu \neq 0$; then a basis $\ldots, f^1_r, \ldots, f^{n_r}_r, \bar f^{n_r}_r, \ldots, \bar f^{n_r}_r, \ldots,$ can be chosen in $K_\lambda \oplus K_{1-\lambda}$ in which $f^k_r \in K_\lambda, \bar f^k_r \in K_{1-\lambda} = K_{\bar\lambda}$ are vectors of height k, and such that $f^k_r = H^{n_r-k}_r f^{n_r}_r, \bar f^k_r = H^{n_r-k}_2 \bar f^{n_r}_r, {}^{C}\Omega(f^k_s, \bar f^m_r) = (-1)^{m-1}\cdot \varepsilon_{n_r}\cdot\delta_{sr}\cdot\delta_{n_r-k,\, m-1}$,*

where $\varepsilon_{n_r} = \begin{cases} i \text{ if } n_r \text{ is odd,} \\ 1 \text{ if } n_r \text{ is even.}\end{cases}$

PROOF. As in the proof of Lemma 3.2.4, the required basis is constructed by induction on the dimension of K_λ.

Let $f^1, \ldots, f^{n_1} \in K_\lambda$, $g^1, \ldots, g^{n_1} \in K_{1-\lambda}$ be as in the proof of Lemma 3.2.4. We choose $a, b \in \mathbf{C}$ so that

$$(3.2.7.1)\qquad {}^{C}\Omega\,(af^{n_r}+\overline{bg^{n_1}},\ \overline{af^1}+bg^1)\neq 0,$$

hence $H_1^{n_1}(af^{n_1}+\overline{bg^{n_1}})=0$, since $af^{n_1}+\overline{bg^{n_1}}\in K_\lambda$. On the other hand, $H_1^{n_1-1}(af^{n_1}+\overline{bg^{n_1}})=af^1+\overline{bg^1}\neq 0$. Rewriting (3.2.7.1) and using the fact that ${}^C\Omega(\overline{X},\overline{Y})={}^C\overline{\Omega(X,Y)}$, we find that

$$(3.2.7.2)\qquad |a|^2\,{}^C\Omega\,(f^{n_1},\ \overline{f}^1)+|b|^2\,{}^C\Omega\,(\overline{g}^{n_1},\ g^1)+((-1)^{n_1-1}ab-\overline{ab})\neq 0.$$

Further, ${}^C\Omega(f^{n_1},\overline{f}^1)=\overline{{}^C\Omega(\overline{f}^{n_1},f^1)}=(-1)^{n_1}\,\overline{{}^C\Omega(f^{n_1},\overline{f}^1)}$, that is, ${}^C\Omega(f^{n_1},\overline{f}^1)$ and ${}^C\Omega(g^{n_1},\overline{g}^1)$ are purely imaginary if n_1 is odd and real if n_1 is even. Therefore, we can choose $a,\ b\in\mathbf{C}$ so that ${}^C\Omega(af^{n_1}+\overline{bg^{n_1}},\ \overline{af^1}+bg^1)=\varepsilon_{n_1}$.

The rest of the proof is completely analogous to that of Lemma 3.2.4.

3.2.8. We consider now the case $\lambda=\frac{1}{2}$. The following lemma is an analogue to Lemmas 3.2.4 and 3.2.7.

LEMMA. *In $K_{1/2}\subset V$ there is a basis*

$$\ldots,\ f_i^1,\ \ldots,\ f_i^{n_i},\ g_i^1,\ \ldots,\ g_i^{n_i},\ \ldots,\ h_j^1,\ \ldots,\ h_j^{m_j},\ \ldots,$$

where m_j is even and the g_i^s (respectively, f_i^s, h_j^s) are vectors of height s with respect to $H=H_1=H_2$, and

$$\Omega\,(f_i^s,\ f_j^k)=\Omega\,(g_i^s,\ g_j^k)=\Omega\,(f_i^s,\ h_j^k)=\Omega\,(g_i^s,\ h_j^k)=0,$$

$$\Omega\,(f_i^s,\ g_j^k)=(-1)^{s+1}\,\delta_{ij}\delta_{s,\ n_i-k+1},$$

$$\Omega\,(h_j^s,\ h_i^k)=(\mp 1)^{s+1}\,\delta_{ij}\delta_{s,\ n_i-k+1}.$$

PROOF. As above, the proof is by induction on the dimension of $K_{1/2}$. Let f^{n_1} be a vector of maximal height n_1, $f^k=H^{n_1-k}f^{n_1}$. Since the restriction of Ω to $K_{1/2}$ has maximal rank, there exists a vector $g^{n_1}\in K_{1/2}$ such that $\Omega(f^1,g^{n_1})=1$. There are two possibilities:

a) Representatives of g^{n_1} and f^{n_1} in $K_{1/2}^{(n_1)}/K_{1/2}^{(n_1-1)}$ are linearly independent, where now $K_{1/2}^{(s)}$ is the subspace of $K_{1/2}$ consisting of the vectors of height $\leqslant s$.

b) The representatives of g^{n_1} and f^{n_1} for any g^{n_1} with $\Omega(f^1,g^{n_1})=1$ in $K_{1/2}^{(n_1)}/K_{1/2}^{(n_1-1)}$ are linearly dependent.

We consider a) and claim that we can change g^k and f^s so that they satisfy the relations

$$(3.2.8.1)\qquad \begin{cases}\Omega\,(f^s,\ g^k)=(-1)^{s+1}\,\delta_{s,\ n_i-k+1},\\ \Omega\,(g^k,\ g^s)=\Omega\,(f^k,\ f^s)=0.\end{cases}$$

Let k be the smallest number such that $\Omega(f^{n_1},f^k)=a_k\neq 0$. Then, replacing f^{n_1} by $f^{n_1}-(-1)^{k+1}a_kg^{n_1-k}$, we get $\Omega(f^{n_1},f^s)=0$ for

$s \leqslant k + 1$. Because $\Omega(f^1, g^s) = 0$ for $s < n_1$, we have $\Omega(f^k, g^s) = 0$ if $k + s \leqslant n_1$. Proceeding in this manner, we obtain vectors f^k, g^s satisfying (3.2.8.1).

Now we consider case b) and note that $\Omega(f^1, f^{n_1}) \neq 0$, since $\Omega(f^1, X) = 0$ for any $X \in K_{1/2}^{n_1-1}$. We may assume here that $\Omega(f^1, f^{n_1}) = \pm 1$. It follows from (3.2.3.1) that $\Omega(f^1, f^{n_1}) = (-1)^{n_1}\Omega(f^1, f^{n_1})$; therefore, n_1 must be even. Next, as in a), we can replace the f^i by h^i for which $\Omega(h^s, h^k) = (\pm 1)^k\, \delta_{n_1-s,k-1}$, where the ± 1 is chosen in accordance with the sign of $\Omega(h^1, h^{n_1}) = \Omega(f^1, f^{n_1})$.

The rest of the proof is analogous to that of Lemma 3.2.4.

3.2.9. PROPOSITION. *There exists a decomposition of V as a skew-orthogonal direct sum (relative to Ω) of subspaces that are invariant under H, $V = \bigoplus_{(\lambda, 1-\lambda, j)} E_{\lambda,j}$, $\bigoplus_{(j)} E_{\lambda,j} = V \cap (K_\lambda \oplus K_{1-\lambda} \oplus K_{\bar\lambda} \oplus K_{1-\bar\lambda})$, such that the restriction of H to the direct summand $E_{\lambda,j}$ is of one of the following types:*

1) *If $\lambda \in \mathbf{R}$, $\lambda \neq \frac{1}{2}$ and* $\dim E_{\lambda,j} = 2n_j$, *then there is a basis $a_{j,1}, \ldots, a_{j,n_j}, b_{j,1}, \ldots, b_{j,n_j}$ in which $\Omega(a_{j,k}, a_{j,s}) = \Omega(b_{j,k}, b_{j,s}) = 0$, $\Omega(a_{j,k}, b_{j,s}) = (-1)^k \delta_{n_j-s,k-1}$ for $1 \leqslant k, s \leqslant n_j$, and the matrix of the restriction of H to $E_{\lambda,j}$ has the form*

$$\left\|\begin{array}{c|c} \begin{matrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 1-\lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1-\lambda \end{matrix} \end{array}\right\|.$$

2) *If $\lambda = \sigma + i\tau$, where $\sigma \neq \frac{1}{2}$ and $\tau \neq 0$,* $\dim E_{\lambda,j} = 4n_j$, *then there is a basis $a_{j,1}, a'_{j,1}, \ldots, a_{j,n_j}, a'_{j,n_j}, b_{j,1}, b'_{j,1}, \ldots, b_{j,n_j}, b'_{j,n_j}$ of $E_{\lambda,j}$ in which $\Omega(a_{j,k}, a'_{j,s}) = \Omega(b_{j,k}, b'_{j,s}) = \Omega(a_{j,k}, b'_{j,s}) = \Omega(a'_{j,k}, b_{j,s}) = 0$, $\Omega(a_{j,k}, b_{j,s}) = \Omega(b'_{i,s}, a'_{j,k}) = (-1)^k\, \delta_{n_j-s,k-1}$ for $1 \leqslant k, s \leqslant n_j$, and the matrix of the restriction of H to $E_{\lambda,j}$ is of the form*

$$\left\|\begin{array}{c|c} \begin{matrix} M_\lambda & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & M_\lambda \end{matrix} & 0 \\ \hline 0 & \begin{matrix} M_\lambda & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & M_\lambda \end{matrix} \end{array}\right\|,$$

where $M_\lambda = \begin{Vmatrix} \sigma & \tau \\ -\tau & \sigma \end{Vmatrix}$, $I_2 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$.

3) *If* $\lambda = \frac{1}{2} + i\mu$, $\mu \neq 0$, *and* $\dim E_{\lambda,j} = 2n_j$ *with* n_j *odd, then there is a basis* $a_{j,1}, b_{j,1}, \dots, a_{j,n_j}, b_{j,n_j}$ *of* $E_{\lambda,j}$ *in which* $\Omega(a_{j,k}, a_{j,s}) = \Omega(b_{j,k}, b_{j,s}) = 0$, $\Omega(a_{j,k}, b_{j,s}) = (-1)^s \delta_{s-1,n_j-k}$ *for* $1 \leqslant s, k \leqslant n_j$, *and the matrix of the restriction of* H *to* $E_{\lambda,j}$ *is of the form*

$$\begin{Vmatrix} M_\lambda & I_2 & 0 \\ & \ddots & \ddots & I_2 \\ 0 & & \ddots & M_\lambda \end{Vmatrix},$$

where

$$M_\lambda = \begin{Vmatrix} \frac{1}{2} & \pm\mu \\ \mp\mu & \frac{1}{2} \end{Vmatrix}.$$

4) *If* $\lambda = \frac{1}{2} + i\mu$, $\mu \neq 0$, *and* $\dim E_{\lambda,j} = 2n_j$ *with* n_j *even, then there is a basis* $a_{j,1}, b_{j,1}, \dots, a_{j,n_j}, b_{j,n_j}$ *of* $E_{\lambda,j}$ *in which* $\Omega(a_{j,k}, b_{j,s}) = 0$, $\Omega(a_{j,k}, a_{j,s}) = \Omega(b_{j,k}, b_{j,s}) = (-1)^s \delta_{k-1,n_j-s}$ *for* $1 \leqslant k, s \leqslant n_j$, *and the matrix of the restriction of* H *to* $E_{\lambda,j}$ *is of the form*

$$\begin{Vmatrix} M_\lambda & I_2 & 0 \\ & \ddots & \ddots & I_2 \\ 0 & & \ddots & M_\lambda \end{Vmatrix},$$

where $M_\lambda = \begin{Vmatrix} \frac{1}{2} & \mu \\ -\mu & \frac{1}{2} \end{Vmatrix}$.

5) *If* $\lambda = \frac{1}{2}$ *and* $\dim E_{\lambda,j} = 2n_j$, *then one of the following holds for* $E_{\lambda,j}$:

a) *there is a basis* $a_{j,1}, \dots, a_{j,n_j}, b_{j,1}, \dots, b_{j,n_j}$ *in which* $\Omega(a_{j,s}, a_{j,k}) = \Omega(b_{j,s}, b_{j,k}) = 0$, $\Omega(a_{j,s}, b_{j,k}) = (-1)^{s+1} \delta_{s,n_j-k+1}$ *for* $1 \leqslant k, s \leqslant n_j$ *and the matrix of the restriction of* H *to* $E_{\lambda,j}$ *is of the form*

$$\left\|\begin{array}{ccc|ccc} \frac{1}{2} & 1 & 0 & & & \\ & \ddots & \ddots\ 1 & & 0 & \\ 0 & & \frac{1}{2} & & & \\ \hline & & & \frac{1}{2} & 1 & 0 \\ & 0 & & & \ddots & \ddots\ 1 \\ & & & 0 & & \frac{1}{2} \end{array}\right\|,$$

b) *there is a basis* $a_{j,1}, \ldots, a_{j,2n_j}$ *in which*

$$\Omega(a_{j,s}, a_{j,k}) = (\mp 1)^{s+1}\delta_{s,2n_j-k+1},$$

and the matrix of the restriction of H *to* $E_{\lambda,j}$ *has the Jordan form*

$$\left\|\begin{array}{ccc} \frac{1}{2} & 1 & 0 \\ & \ddots & \ddots\ 1 \\ 0 & & \frac{1}{2} \end{array}\right\|.$$

PROOF. The five types listed follow immediately from the lemmas proved above on transition to the basis

$$\frac{1}{\sqrt{2}}(f_j^k+\bar{f}_j^k),\quad \frac{1}{i\sqrt{2}}(f_j^k-\bar{f}_j^k),\quad \frac{1}{\sqrt{2}}(g_j^k+\bar{g}_j^k),\quad \frac{1}{i\sqrt{2}}(g_j^k-\bar{g}_j^k)$$

for complex λ, and to the basis consisting of vectors $f_j^k, \ldots, g_j^k, \ldots, h_j^s, \ldots$ in the case of real λ.

We make a remark concerning the sign of Im λ in the four cases that arise.

As is well-known, the Jordan form relative to the full linear group is the same if the Jordan blocks are chosen corresponding to $\bar{\lambda}$ instead of λ. This is true also for the classification of the operators H relative to the symplectic group when Re $\lambda \neq \frac{1}{2}$, since in that case the linear transformation of $V \cap (K_\lambda \oplus K_{1-\lambda})$ generated by the mapping $f_m^r \to \bar{f}_m^r$, $g_m^r \to \bar{g}_m^r$ of the basis is clearly a symplectic transformation. However, the latter is not always true when $\lambda = \frac{1}{2} + i\mu$. Indeed, if n_j is even, then the conjugate mapping $f_s^k \to \bar{f}_s^k$ induces a symplectic transformation in $V \cap (K_\lambda \oplus K_{\bar{\lambda}})$, and if n_j is odd, this is a conformally-symplectic transformation with the factor (-1). Therefore, the Jordan form in this case is determined to within $\pm\mu$.

3.2.10. For a given $\lambda = \frac{1}{2} + i\mu$, where for definiteness we take μ positive, let $\nu_k^+(\mu)$ (respectively, $\nu_k^-(\mu)$) stand for the number of Jordan blocks of size $2k \times 2k$, k odd, that occur in the above decomposition of H (type 3)) with

$+\mu$ (respectively, with $-\mu$). We claim that $\nu_k^\pm(\mu)$ are invariants of H relative to the symplectic group.

We note that to prove the invariance of $\nu_k^\pm(\mu)$ relative to the symplectic group it is enough to prove that of $\nu_k^+(\mu) - \nu_k^-(\mu)$, because $\nu_k^+(\mu) + \nu_k^-(\mu)$ is the number of Jordan blocks of size $2k \times 2k$ corresponding to $\lambda = \frac{1}{2} + i\mu$. Let $V_k \subset V \cap (K_\lambda \oplus K_{\bar\lambda})$ denote the subspace consisting of the vectors of height $\leqslant k$ relative to the operator $H^2 - H + (\mu^2 + \frac{1}{4})$, and $\chi_k^\pm(\mu)$ the positive (negative) index of the restriction of $\bar h_\omega^s$ to V_k. We evaluate $\nu_k^+(\mu) - \nu_k^-(\mu)$ in terms of $\chi_k^\pm(\mu)$.

As a preliminary we make the following remark. If $L \subset V$ is a subspace invariant under H and if $\Omega|_L = 0$, then $\bar h_\omega^s|_L = 0$. This follows at once from the formula

$$\bar h_\omega^s(X, Y) = \Omega(HX, Y) - \frac{1}{2}\Omega(X, Y). \tag{3.2.10.1}$$

Next we note that if an operator H in some space $\mathbf{R}^{2k}$, together with a skew-symmetric form, is defined by the relations in Proposition 3.2.9 (types 3) and 4)), then for the symmetric form $\bar h_\omega^s$ defined by (3.2.10.1) the difference between the positive and negative indices of inertia is zero if k is even and 2 if k is odd.

We now compute $\nu_1^+(\mu) - \nu_1^-(\mu)$. We consider the decomposition of V into the $E_{\lambda,j}$. Then $V_1 = \oplus V_{1,j}$, where $V_{1,j} = V_1 \cap E_{\lambda,j}$, $\lambda = \frac{1}{2} + i\mu$ and $\bar h_\omega^s|_{V_{1,j}} = 0$ if $\dim E_{\lambda,j} > 2$. Therefore $2(\nu_1^+(\mu) - \nu_1^-(\mu)) = \chi_1^+(\mu) - \chi_1^-(\mu)$. We now consider the restriction of $\bar h_\omega^s$ to $V_k \cap E_{\lambda,j}$. It follows from Proposition 3.2.9 that if $k < n_j$, where $2n_j = \dim E_{\lambda,j}$ (with n_j odd), or if $k \leqslant n_j$ (n_j even), then $\Omega|_{V_k \cap E_{\lambda,j}} = 0$, so that $\bar h_\omega^s|_{V_k \cap E_{\lambda,j}} = 0$. Further, if $k \geqslant n_j$ (n_j odd) or $k > n_j$ (n_j even), then the difference between the positive and negative indices of the forms $\bar h_\omega^s|_{V_k \cap E_{\lambda,j}}$ is zero if n_j is even and ± 2 if n_j is odd (the $\pm$ sign is chosen in accordance with the sign with which μ occurs in the Jordan block). For example, suppose that n_j is odd. Then, as $V_{2n_j-k+1} \cap E_{\lambda,j}$ is the degeneracy subspace of $\bar h_\omega^s|_{V_k \cap E_{\lambda,j}}$, it follows that $\bar h_\omega^s$ can be extended to a form $\hat h_\omega^s$ on the factor space $V_k \cap E_{\lambda,j}/V_{2n_j-k-1} \cap E_{\lambda,j}$. By what was said above, the difference between the positive and negative indices of inertia of the form $\hat h_\omega^s$ so obtained is ± 2. Thus, for arbitrary s and μ,

$$\chi_s^+(\mu) - \chi_s^-(\mu) = 2\sum_{j=0}^{s} (\nu_{2j+1}^+(\mu) - \nu_{2j+1}^-(\mu)). \tag{3.2.10.2}$$

Using this formula, we can compute $\nu_{2s+1}^+ - \nu_{2s+1}^-$ on the assumption that the preceding differences $\nu_{2j+1}^+ - \nu_{2j+1}^-$, $j < s$, are known.

In a similar way it can be proved that the number of blocks of type 5 b) such that $\Omega(a_{j,1}, a_{j,2n_j}) = +1$ (respectively, -1) is an invariant of H relative to the symplectic group. As above, this invariant is evaluated in terms of

$\chi_s^{\mp}(0)$, the positive (negative) index of the restriction of $\bar{h}_\omega^s$ to $K_{1/2}^{(s)}$.

3.2.11. Combining Proposition 3.2.9 and Theorem 3.2.5, we come to the following assertion.

THEOREM a) *Operators $H^{(k)}$: $V \to V$ $(k = 1, 2)$ are symplectically equivalent if and only if they are equivalent under the full linear group and if the invariants $\chi_s^{\pm}(\mu)$, $\chi_{s,1}^{\pm}(\mu) = \chi_{s,2}^{\pm}(\mu)$ coincide for all s and all $\lambda = \frac{1}{2} + i\mu$.*

b) *Operators $H^{(k)}$: $V \to V$ $(k = 1, 2)$ are conformally-symplectically equivalent if and only if they are equivalent under the full linear group and either $\chi_{s,1}^{\pm}(\mu) = \chi_{s,2}^{\pm}(\mu)$ or $\chi_{s,1}^{\pm}(\mu) = \chi_{s,2}^{\mp}(\mu)$ for all s and all $\lambda = \frac{1}{2} + i\mu$.*

PROOF. Since we have already proved a), we turn to a proof of b). It follows from Proposition 3.2.9 that b) is sufficient. So we prove the necessity. Let R be a conformally-symplectic transformation of V, $R^*(\Omega) = a\Omega$, $a \neq 0$, establishing the equivalence of the $H^{(k)}$; as $R^*(\bar{h}_{\omega_2}) = ah_{\omega_1}$, we then have $R^*(\bar{h}_{\omega_2}^s) = a\bar{h}_{\omega_1}^s$. It follows that $\chi_{s,1}^{\pm}(\mu) = \chi_{s,2}^{\pm}(\mu)$ if $a > 0$ and $\chi_{s,1}^{\pm}(\mu) = \chi_{s,2}^{\mp}(\mu)$ otherwise.

COROLLARY. *For the existence of a Lagrangian subspace of V, invariant under H, it is sufficient that $\chi_s^+(\mu) = \chi_s^-(\mu)$ for all s and $\lambda = \frac{1}{2} + i\mu$, $\mu \neq 0$.*

PROOF. We observe that a subspace $E_{\lambda,j}$ with $\lambda \neq \frac{1}{2} + i\mu$, $\mu \neq 0$, contains an invariant Lagrangian subspace, which we denote by $L_{\lambda,j}$. Next, for every E_{λ,j_1} with $\lambda = \frac{1}{2} + i\mu$, $\mu \neq 0$, when the matrix of H in the decomposition 3.2.9 is a Jordan block taken with $+\mu$, there exists a subspace E_{λ,j_2} such that the matrix of H is a Jordan block with $-\mu$. Thus, there is an invariant Lagrangian subspace L_{λ,j_1,j_2} in $E_{\lambda,j_1} \oplus E_{\lambda,j_2}$. The subspace $L = \bigoplus_{(\lambda,j)} L_{\lambda,j} \bigoplus_{j(\lambda,j_1,j_2)} L_{\lambda,j_1,j_2}$ is also Lagrangian and invariant under H, since the $E_{\lambda,j}$ are skew-orthogonal.

3.2.12. We consider an equation $E \subset J^1(M)$, codim $E = 1$, with a singular point $x \in E$. Let us find out when it is that E is locally equivalent to a linear equation in the neighbourhood of x, that is, to an equation of the form $u = \mathcal{F}(q, p)$, where $\frac{\partial^2 \mathcal{F}}{\partial p^2} = 0$ in a special system of local coordinates $q_1, \ldots, q_n, u, p_1, \ldots, p_n$. For this purpose we observe that a linear equation $E_1 = \{u - \mathcal{F}(q, p) = 0\}$ at a singular point x, that is, $\left.\frac{\partial \mathcal{F}}{\partial p}\right|_x = 0$, $\left.\left(\frac{\partial \mathcal{F}}{\partial q_i} - p_i\right)\right|_x = 0$, $i \leqslant n$, always has a Lagrangian subspace L invariant under the operator H: $T_x(E_1) \to T_x(E_1)$.

For L we can take, for example, the subspace spanned by the vectors $\left.\frac{\partial}{\partial p_1}\right|_x, \ldots, \left.\frac{\partial}{\partial p_n}\right|_x$, which is clearly Lagrangian and lies in $T_x(E_1)$, because $\left.\frac{\partial \mathcal{F}}{\partial p_i}\right|_x = 0$. In addition, $\Omega\left(H\left.\frac{\partial}{\partial p_i}\right|_x, \left.\frac{\partial}{\partial p_j}\right|_x\right) = h_\omega\left(\left.\frac{\partial}{\partial p_i}\right|_x, \left.\frac{\partial}{\partial p_j}\right|_x\right) = \left.\frac{\partial^2 \mathcal{F}}{\partial p_i \partial p_j}\right|_x = 0$, that is, $H\left(\left.\frac{\partial}{\partial p_i}\right|_x\right) \in L$.

We therefore have the following assertion:

PROPOSITION. *For an equation* $E \subset J^1(M)$, codim $E = 1$, *to be equivalent to a linear equation in the neighbourhood of a singular point* $x \in E$, *it is necessary that there exists a Lagrangian subspace of* $T_x(E)$ *that is invariant under* H: $T_x(E) \to T_x(E)$.

3.2.13. REMARK 1. In what follows, when we obtain conditions under which the *CSp*-equivalence of operators H is sufficient for local equivalence at a singular point, Proposition 3.2.12 and Corollary 3.2.11 determine conditions that are sufficient for the linearization of the equation at the singular point.

REMARK 2. By (3.2.10.1), the *CSp*-classification of operators H in this section is also a *CSp*-classification of quadratic forms under the action indicated above (see 3.1.8).

REMARK 3. The *Sp*-classification of symplectic transformations was carried out by Williamson [4], [5]. Theorem 3.2.11 differs from Williamson's main theorem (see [4], Theorem 4) in that the invariants χ are more constructive.

§3. Normal forms

3.3.1. We consider first an equation $E \subset J^1(M)$, codim $E = 1$. Let $q_1, \dots, q_n, u, p_1, \dots, p_n$ be a special system of local coordinates in the neighbourhood of a singular point $x \in E$. We assume that the coordinates of x are all zero.

In this system, E takes the form

$$u = \mathcal{F}(q_1, \dots, q_n, p_1, \dots, p_n),$$

where $\mathcal{F}(0) = 0, \frac{\partial \mathcal{F}}{\partial p}(0) = \frac{\partial \mathcal{F}}{\partial q}(0) = 0, \ \omega = d\mathcal{F} - p\, dq.$

The operator H determines the 2-jet of $\mathcal{F}$ at x according to the formulae

$$(3.3.1.1) \quad \begin{cases} \left.\dfrac{\partial^2 \mathcal{F}}{\partial q_i \partial q_j}\right|_x = h_\omega\left(\left.\dfrac{\partial}{\partial q_i}\right|_x, \left.\dfrac{\partial}{\partial q_j}\right|_x\right) = \Omega\left(H\left.\dfrac{\partial}{\partial q_i}\right|_x, \left.\dfrac{\partial}{\partial q_j}\right|_x\right), \\ \left.\dfrac{\partial^2 \mathcal{F}}{\partial q_i \partial p_j}\right|_x = h_\omega\left(\left.\dfrac{\partial}{\partial q_i}\right|_x, \left.\dfrac{\partial}{\partial p_j}\right|_x\right) = \Omega\left(H\left.\dfrac{\partial}{\partial q_i}\right|_x, \left.\dfrac{\partial}{\partial p_j}\right|_x\right), \\ \left.\dfrac{\partial^2 \mathcal{F}}{\partial p_i \partial p_j}\right|_x = h_\omega\left(\left.\dfrac{\partial}{\partial p_i}\right|_x, \left.\dfrac{\partial}{\partial p_j}\right|_x\right) = \Omega\left(H\left.\dfrac{\partial}{\partial p_i}\right|_x, \left.\dfrac{\partial}{\partial p_j}\right|_x\right), \end{cases}$$

3.3.2. We now use Proposition 3.2.9 to establish normal forms for the 2-jet of $\mathcal{F}$ at x. To do this we assume that when

1) $\lambda \in \mathbf{R}$, dim $E_{\lambda,j} = 2n_j$, the coordinates $q_1, \dots, q_n, u, p_1, \dots, p_n$ are chosen so that $a_{i,j} = \left.\frac{\partial}{\partial q_i}\right|_x, \left.\frac{\partial}{\partial p_i}\right|_x = (-1)^i b_{n_j - i+1, j}$,

2) $\lambda = \sigma + i\tau$, $\sigma \neq \frac{1}{2}$, dim $E_{\lambda,j} = 4n_j$ the coordinates are chosen in two ways, $q_1, \dots, q_{n_j}, q'_1, \dots, q'_{n_j}, p_1, \dots, p_{n_j}, p'_1, \dots, p'_{n_j}$, so that

$$\frac{\partial}{\partial q_i}\bigg|_x = a_{i,j},\ \frac{\partial}{\partial q'_i}\bigg|_x = a'_{i,j},\ \frac{\partial}{\partial p_i}\bigg|_x = (-1)^i b_{n_j-i+1,j},\ \frac{\partial}{\partial p'_i}\bigg|_x = (-1)^i b'_{n_j-i+1,j},$$

3) $\lambda = \frac{1}{2} + i\mu$, $\mu \neq 0$, $\dim E_{\lambda,j} = 2n_j$, n_j odd, the coordinates $q_1, \ldots, q_{n_j}, p_1, \ldots, p_{n_j}$ are chosen so that

$$\frac{\partial}{\partial q_i}\bigg|_x = a_{i,j},\ \frac{\partial}{\partial p_i}\bigg|_x = (-1)^i b_{n_j-i+1,j},$$

4) $\lambda = \frac{1}{2} + i\mu$, $\mu \neq 0$, $\dim E_{\lambda,j} = 2n_j$, n_j even, the coordinates $q_1, \ldots, q_{n_j}, p_1, \ldots, p_{n_j}$ are chosen so that

$$\frac{\partial}{\partial q_{2i-1}}\bigg|_x = a_{i,j} \qquad \frac{\partial}{\partial q_{2i}}\bigg|_x = b_{i,j},$$

$$\frac{\partial}{\partial p_{2i-1}}\bigg|_x = (-1)^{i+1} a_{n_j-i+1,j},\ \frac{\partial}{\partial p_{2i}}\bigg|_x = (-1)^{i+1} b_{n_j-i+1,j},$$

5) $\lambda = \frac{1}{2}$, $\dim E_{\lambda,j} = 2n_j$ the coordinates $q_1, \ldots, q_{n_j}, p_1, \ldots, p_{n_j}$ are chosen so that

a) $\displaystyle \frac{\partial}{\partial q_i}\bigg|_x = a_{i,j},\ \frac{\partial}{\partial p_i}\bigg|_x = (-1)^{i+1} b_{n_j-i+1,j};$

b) $\displaystyle \frac{\partial}{\partial q_i}\bigg|_x = a_{i,j},\ \frac{\partial}{\partial p_i}\bigg|_x = (\mp 1)^{i+1} a_{2n_j-i+1,j}.$

When we now use (3.3.1.1) and the form of H in the basis $\ldots, a_{i,j}, b_{i,j}, \ldots,$ we obtain the following normal forms for the 2-jet of the equation at the singular point:

(I) $\displaystyle \lambda \sum_{i=1}^{n_j} p_i q_i + \sum_{i=1}^{n_j-1} q_i p_{i+1},$

(II) $\displaystyle \sigma \sum_{i=1}^{n_j} (p_i q_i + p'_i q'_i) + \tau \sum_{i=1}^{n_j} (p'_i q_i - p_i q'_i) + \sum_{i=1}^{n_j-1} (q_i p_{i+1} + q'_i p'_{i+1}),$

(III) $\displaystyle \frac{1}{2} \sum_{i=1}^{n_j} p_i q_i \pm \mu \sum_{i=1}^{n_j} (-1)^{i+1} (q_i q_{n_j-i+1} + p_i p_{n_j-i+1}) + \sum_{i=1}^{n_j-1} q_i p_{i+1},$

(IV) $\displaystyle \frac{1}{2} \sum_{i=1}^{n_j} p_i q_i + \mu \sum_{i=1}^{n_j} (-1)^{i+1} q_i p_{i-(-1)^i} + \sum_{i=1}^{n_j-2} q_i p_{i+2},$

(V) $\begin{cases} \text{a) } \displaystyle \frac{1}{2} \sum_{i=1}^{n_j} p_i q_i + \sum_{i=1}^{n_j-1} q_i p_{i+1}, \\ \text{b) } \displaystyle \frac{1}{2} \sum_{i=1}^{n_j} p_i q_i + \sum_{i=1}^{n_j-1} q_i p_{i+1} \mp q_{n_j}^2. \end{cases}$

3.3.3. PROPOSITION. *For every equation* $E \subset J^1(M)$, codim $E = 1$, *and every singular point* $x \in E$ *there exists a local contact diffeomorphism* F, $F(x) = x$, *such that the equation* $F(E)$ *can be represented in some special* *system of local coordinates* $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ *at* x *in the form* $u = \mathcal{F}(q_1, \ldots, q_n, p_1, \ldots, p_n)$, *where the 2-jet of* $\mathcal{F}$ *at* x *is a direct sum of forms of types* (I) – (V).

3.3.4. EXAMPLE 1. Suppose that all the eigenvalues λ of the operator H: $T_x(E) \to T_x(E)$ are real and distinct, and $\lambda \neq \frac{1}{2}$. Then the 2-jet of E at x takes the form $u - \sum_{i=1}^{n} \lambda_i p_i q_i$.

EXAMPLE 2. If all the eigenvalues λ are distinct and $\operatorname{Re} \lambda = \frac{1}{2}$, $\mu_s = \operatorname{Im} \lambda_s \neq 0$, $1 \leqslant s \leqslant n$, then the 2-jet of the equation is:

$$u - \sum_{s=1}^{n} [\pm \mu_s (q_s^2 + p_s^2)] - \frac{1}{2} \sum_{s=1}^{n} p_s q_s.$$

3.3.5. Now let $E^r \subset J^1(M)$ be any involutory equation, codim $E^r = r > 1$, and $x \in E^r$ a singular point. We choose an arbitrary contact field X_f such that $f(x) \neq 0$. Then $X_{fx} \notin T_x(E^r)$, since $T_x(E^r) \subset \Gamma_x$ and $X_{f,x} \notin \Gamma_x$. The equation $E^{r-1} = \bigcup_t T_t(E^r)$, where T_t is a local one-parameter group of translations along X_f and t is sufficiently small, is also involutory; but x is not singular for E^{r-1}, because $T_x(E^{r-1})$ and Γ_x are transversal. Thus, using the results of Chapter II, we can choose a local contact diffeomorphism F, $F(x) = x$, such that the equation $F(E^{r-1})$ takes the form $p_1 = 0, \ldots, p_{r-1} = 0$ in some system of special local coordinates $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ (we assume here that $x = (0, 0, \ldots, 0)$). In this same coordinate system the equation of $F(E^r)$ can then be written as $p_1 = 0, \ldots, p_{r-1} = 0$, $u = F(q_r, \ldots, q_n, p_r, \ldots, p_n)$, where now $F(0) = 0$, $df(0) = 0$.

Further, as in 3.3.4, we can choose a coordinate system $q_r, \ldots, q_n, p_r, \ldots, p_n$ such that the 2-jet of F at x is represented as a direct sum of forms of types (I) – (V).

Thus, we have established the following proposition.

PROPOSITION. *Every involutory equation* $E^r \subset J^1(M)$, codim $E^r = r$, *is equivalent in the neighbourhood of a singular point to an equation of the form* $p_1 = 0, \ldots, p_{r-1} = 0$, $u = \mathcal{F}(q_r, \ldots, q_n, p_r, \ldots, p_n)$, *where the 2-jet of* $\mathcal{F}$ *at* x *is a direct sum of forms of types* (I) – (V).

CHAPTER IV

Formal classification of equations at singular points

Let $E^r \subset J^1(M)$, codim $E^r = r$, be an involutory equation with singular point x. We say that two such equations E_1^r and E_2^r are formally equivalent at $x \in E_k^r$ ($k = 1, 2$) if there is a local contact diffeomorphism $F, F(x) = x$, such that $F(E_1^r)$ and E_2^r have contact of infinite order at x. It is clear that formal equivalence of equations is necessary for local equivalence. In this chapter we explain when the condition of formal equivalence follows from

CSp-equivalence of the operators H_k.

§ 1. The connection between local equivalence and local solubility

4.1.1. We replace the construction of a local contact diffeomorphism by the construction of its graph. To do this we denote by $pr_k: E_1^r \times E_2^r \to E_k^r$ the projection onto the k-th component ($k = 1, 2$), and $\hat{F}: E_1^r \to E_1^r \times E_2^r$ the graph of F, $\hat{F}(y) = (y, F(y))$.

Similarly, if $A: T_x(E_1^r) \to T_x(E_2^r)$ is a linear transformation, then we define the graph $\hat{A}: T_x(E_1^r) \to T_{(x,x)}(E_1^r \times E_2^r)$ by identifying $T_{(x,x)}(E_1^r \times E_2^r)$ with $T_x(E_1^r) \oplus T_x(E_2^r)$; here $\hat{F}_{*,x} = (\hat{F})_{*,x}$.

Let $\theta = pr_1^*(\omega_1) - pr_2^*(\omega_2)$ be a 1-form on $E_1^r \times E_2^r$, where $\omega_k = U_1|_{E_k^r}$ ($k = 1, 2$), as above.

PROPOSITION. *There exists an embedding $E_1^r \times E_2^r \subset J^1(M \times M)$ of some neighbourhood of (x, x) such that $\theta = U_1|_{E_1^r \times E_2^r}$.*

PROOF. Note that the rank of $d\omega_k$ is $4(n - r + 1)$ in some neighbourhood of (x, x), since the rank of $d\omega_k$ in some neighbourhood of x is $2(n - r + 1)$, $k = 1, 2$.

Further, as well as ω_1, ω_2, the form θ is involutory, because

$$\theta \wedge \underbrace{d\theta \wedge \dots \wedge d\theta}_{2(n-r+1)} =$$

$$= (pr_1^*(\omega_1) - pr_2^*(\omega_2)) \wedge pr_1^*(\underbrace{d\omega_1 \wedge \dots \wedge d\omega_1}_{n-r+1}) \wedge pr_2^*(\underbrace{d\omega_2 \wedge \dots \wedge d\omega_2}_{n-r+1}) \equiv 0;$$

therefore, since the rank of $d\theta$ is constant in some neighbourhood of (x, x), we can choose coordinates $x_1, \dots, x_{2(n-r+1)}, y_1, \dots, y_{2(n-r+1)}, z_1, \dots, z_{2r-2}$ in some neighbourhood of $(x, x) \in E_1^r \times E_2^r$ in which $d\theta = \sum_{i=1}^{2(n-r+1)} dx_i \wedge dy_i$.

In these coordinates θ has the form $\theta = d\mathcal{F} - y\,dx$, where $\mathcal{F}$ is a function of the variables $x_1, \dots, x_{2(n-r+1)}, y_1, \dots, y_{2(n-r+1)}$, since θ is involutory. We now specify the embedding of $E_1^r \times E_2^r$ in $J^1(M \times M)$ in the following way:

$$u = \mathcal{F}, \quad y_i = p_i, \quad x_i = q_i, \quad 1 \leqslant i \leqslant 2(n - r + 1),$$
$$z_j = q_{j+2(n-r+1)}, \quad j \leqslant 2r - 2, \quad p_j = 0, \quad j > 2(n - r + 1),$$

where $q_1, \dots, q_{2n}, u, p_1, \dots, p_{2n}$ is some special coordinate system in $J^1(M \times M)$.

4.1.2. PROPOSITION. *A local U_1-diffeomorphism F, $F(x) = x$, carries E_1^r to E_2^r if and only if $\hat{F}(E_1^r)$ is a many-valued solution of the equation $E_1^r \times E_2^r \subset J^1(M \times M)$ passing through (x, x).*

PROOF. Since F establishes the local equivalence of E_1^r and E_2^r at x if and only if its restriction to E_1^r establishes the local equivalence of the forms ω_k (see Chapter II), it is enough to check that $\hat{F}^*(\theta) = 0$ if and only if $(F|_{E_1^r})^*(\omega_2) = \omega_1$. Let us compute $\hat{F}^*(\theta)$. We have

$$\hat{F}^*(\theta) = \hat{F}^* \circ pr_1^*(\omega_1) - \hat{F}^* \circ pr(\omega_2) = (F|_{E_1^r})^*(\omega_1) - \omega_2, \quad \text{q.e.d.}$$

4.1.3. Now let $\bar{A}$: $V \to V$ be a transformation establishing the Sp-equivalence of the operators H_k ($k = 1, 2$). As in Chapter III, we assume here that $T_x(E_1^r) = T_x(E_2^r)$, and that $V = T_x(E_1^r)/C_x$ is symplectic with the form Ω.

PROPOSITION. *For every extension A: $T_x(E_1^r) \to T_x(E_2^r)$ of a symplectic transformation $\bar{A}$ the subspace $\hat{A}(T_x(E_1^r))$ of $T_x(E_1^r \times E_2^r) \subset T_{(x,x)}(J^1(M \times M))$ is a Lagrangian subspace of $\Gamma_{(x,x)} = \text{Ker } U_{1,(x,x)} \subset T_{(x,x)}(J^1(M \times M))$ on which h_θ vanishes.*

PROOF. The fact that $\hat{A}(T_x(E_1^r))$ is Lagrangian is checked as in 4.1.2. The fact that $\bar{A}$ establishes the equivalence of H_1 and H_2 means that the image of $\hat{A}$ in $V_{1,2} = T_{(x,x)}(E_1^r \times E_2^r)/C_{(x,x)}$ is invariant under the operator $H_{1,2} = H_1 \oplus H_2$. Indeed, $V_{1,2} = V \oplus V$, since $\theta = pr_1^*(\omega_1) - pr_2^*(\omega_2)$. Thus, $H_{1,2} = H_1 \oplus H_2$, and the elements lying in the image of $\hat{A}$ in $V_{1,2}$ have the form $(X, \bar{A}X)$ with $X \in V$. But $H_{1,2}(X, \bar{A}X) = (H_1X, H_2\bar{A}X)$, so that $H_{1,2}(X, \bar{A}X)$ lies in the image only when $\bar{A}H_1X = H_2\bar{A}X$, that is, when $\bar{A}$ establishes the equivalence of H_1 and H_2. The fact that h_θ vanishes on $\hat{A}(T_x(E_1^r))$ follows from the following remark.

4.1.4. LEMMA. *Let $E^r \subset J^1(M)$ be an involutory equation and $x \in E^r$ a singular point. Then the form h_ω vanishes on a subspace $L \subset V$, $\dim L = \frac{1}{2} \dim V$, if and only if L is a Lagrangian subspace and invariant under the operator H: $V \to V$.*

PROOF. If L is Lagrangian and invariant under H, then $\bar{h}_\omega(X, Y) = \Omega(HX, Y) = 0$ for all $X, Y \in L$, since $HX \in L$.

Suppose now that $\bar{h}_\omega(X, Y) = 0$ for all $X, Y \in L$. If $X \in L$, then HX is skew-orthogonal to L, since $\Omega(HX, Y) = \bar{h}_\omega(X, Y) = 0$, and therefore $HX \in L^\perp = L$.

4.1.5. Summarizing what was said above, we see that the question of the sufficiency of Sp-equivalence of the operators H_k for the local U_1-equivalence of the equations reduces to the following problems on the local solubility of an equation at a singular point: (i) how to choose a Lagrangian subspace $L \subset T_x(E^r)$ on which h_ω vanishes (that is, such that the image of L in V is invariant under H); (*f*) is there a many-valued solution of E^r passing through x and tangent to L?

4.1.6. REMARK. If the operators H_k are CSp-equivalent, then the question of the existence of a local contact diffeomorphism establishing the equivalence of the equations and extending a conformally-symplectic transformation $\bar{A}$, $\bar{A}H_1 = H_2\bar{A}$, $\bar{A}^*(\Omega) = \lambda\Omega$, also reduces to the question about local solubility just mentioned, with ω_1 replaced by $\lambda\omega_1$, $\lambda \in \mathbf{R}$.

§2. Formal solubility of equations at a singular point

With the results of the preceding section in mind, we are concerned in what follows with the local solubility question as indicated there.

4.2.1. Let L^n be a many-valued solution of an involutory equation $E^r \subset J^1(M)$ passing through the singular point x.

We consider the tangent space $L = T_x(L^n)$; then h_ω vanishes on L. For if $X, Y \in L$, we can choose extensions $\bar{X}, \bar{Y}$ so that they are tangent to L^n. In this case $h_\omega(X, Y) = X(\omega(\bar{Y})) = 0$, because $\omega(\bar{Y})$ is a function that is identically zero on L^n.

Further, since $L \subset T_x(E^r)$ is Lagrangian, so that $L \supset C_x$, because E^r is involutory, the projection $\bar{L}$ of L in V is defined, and $\bar{L}$ is also a Lagrangian subspace. In addition, $\bar{h}_\omega = \bar{h}^s_\omega + \frac{1}{2}\Omega$, so that $\bar{h}^s_\omega$ also vanishes on L. Thus, we get the following proposition.

PROPOSITION. *If there is a many-valued solution of an involutory equation $E^r \subset J^1(M)$ passing through a singular point x, then there is a Lagrangian subspace $L \subset T_x(E^r)$ for which the following equivalent conditions hold*:

a) $h_\omega|_L = 0$;

b) $\bar{h}^s_\omega|_{\bar{L}} = 0$;

c) $\bar{L} \subset V$ *is invariant under H.*

4.2.2. REMARK. This proposition shows that the conditions for CSp-equivalence of the operators H_k, which are necessary for local equivalence, go over after the reduction (i, f) into necessary conditions for the existence of a solution passing through a singular point for the corresponding differential equation.

4.2.3. EXAMPLE. Suppose that in the special coordinate system $q_1, \dots, q_n, u, p_1, \dots, p_n$ the equation E takes the form

$$u - \frac{1}{2}\sum_{i=1}^{n} p_i q_i - \sum_{i=1}^{n} (\lambda_i q_i^2 + \mu_i p_i^2) + \varepsilon(q_1, \dots, q_n, u, p_1, \dots, p_n) = 0,$$

where $\lambda_i > 0$, $\mu_i > 0$, $1 \leqslant i \leqslant n$, and $\varepsilon(q_1, \dots, q_n, u, p_1, \dots, p_n)$ is a function of third order of smallness at $(0, \dots, 0)$. Then in these coordinates the form h^s_ω is given by the matrix

$$\begin{Vmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_n & & & \\ & & & \mu_1 & & \\ & 0 & & & \ddots & \\ & & & & & \mu_n \end{Vmatrix}$$

Therefore, no many-valued solution passing through $(0, \dots, 0)$ exists (and in particular, no ordinary solution exists).

4.2.4. The necessary conditions in Proposition 4.2.1 become particularly transparent in the case $r = \operatorname{codim} E^r = 1$.

We consider the vector field X_ω in some neighbourhood of the singular point $x \in E^r = E$ that is defined in the following way:

$$(4.2.4.1) \qquad X_\omega \lrcorner\, d\omega = \omega.$$

This field X_ω exists and is unique, since the form $d\omega$ is of maximal rank (= dim E) in the neighbourhood of a singular point. Also, $X_\omega = 0$ if and only if $\omega = 0$. In particular, $X_{\omega, x} = 0$.

PROPOSITION. *Let* $T = \lim_{t\to 0} \frac{T^t_{*,x} - T^0_{*,x}}{t}$ *be the linear part of the field* X_ω *at* x. *Here* T^t *is a local one-parameter group of translations along* X_ω. *Then* $H = T$.

PROOF. Let f be a smooth function and Y a vector field given in some neighbourhood of the singular point $x \in E$. Then, by definition,

$$(TY_x)(f) = \lim_{t\to 0} \frac{1}{t}(T^t_{*,x}(Y) - Y_x)(f) = [Y, X_\omega]|_x(f) = Y_x(X_\omega f),$$

since $X_{\omega,x} = 0$.

Denoting by $\widetilde{df}$ the vector field for which $\widetilde{df} \lrcorner\, d\omega = df$, we obtain

$$(TY_x)(f) = Y_x(X_\omega(f)) = Y_x(df(X_\omega)) = Y_x(d\omega(\widetilde{df}, X_\omega)) =$$
$$= -Y_x(\omega(\widetilde{df})) = -h\omega(Y_x, \widetilde{df}_x) = d\omega_x(\widetilde{df}_x, HY_x) = df(HY_x) = HY_x(f).$$

Thus, $HY_x = TY_x$ for every $Y_x \in T_x(E)$, so that $H = T$.

We consider now a function f such that $f|_E = 0$, $X_1(f) = 1$. Then it is easy to see that the restriction of X_f to E is X_ω, since $X_f(U_1) = U_1$. On the other hand, every solution L^n of E passing through x must be invariant under X_f, in particular, $T_x(L^n)$ is invariant under the linear part of X_f at x, that is, under H. So we have obtained another proof of Proposition 4.2.1 in the case codim $E^r = 1$.

4.2.5. Next we describe the situation we have achieved in local coordinates. Applying a contact diffeomorphism if necessary, we assume that the equation $E^r \subset J^1(M)$ is given in the neighbourhood of the singular point x as follows: $p_1 = 0, \ldots, p_{r-1} = 0, u = \mathcal{F}(q_r, \ldots, q_n, p_r, \ldots, p_n)$ and $L = T_x(L^n)$ is the linear span of the vectors $\frac{\partial}{\partial q_1}\Big|_x, \ldots, \frac{\partial}{\partial q_n}\Big|_x$. Then $\omega = d\mathcal{F} - \sum_{i=r}^{n} p_i\, dq_i$. The condition that $h\omega|_L = 0$ means that

$$(4.2.5.1) \qquad h_\omega\left(\frac{\partial}{\partial q_i}\Big|_x, \frac{\partial}{\partial q_j}\Big|_x\right) = \frac{\partial^2 \mathcal{F}}{\partial q_i\, \partial q_j}\Big|_x = 0.$$

Further, the space V is spanned by $\frac{\partial}{\partial q_r}, \ldots, \frac{\partial}{\partial q_n}, \frac{\partial}{\partial p_r}, \ldots, \frac{\partial}{\partial p_n}$, and $\Omega = \sum_{i=r}^{n} dq_i|_x \wedge dp_i|_x$. The matrix of the restriction of H to $\bar{L}$ is $\left\| \frac{\partial^2 \mathcal{F}}{\partial q_i\, \partial p_j}\Big|_x \right\|$, $r \leqslant i, j \leqslant n$, in this basis.

Note that incidentally we get another interpretation of the condition $H: \bar{L} \to \bar{L}$, or $h_\omega|_L = 0$. Namely, if we consider any R-manifold L^n touching L, then this manifold had contact of order $\geqslant 3$ with E^r at x. This can be proved in the following way. Locally, L^n is given by the system $u = f(q_1, \ldots, q_n)$, $p_i = \frac{\partial f}{\partial q_i}$. The condition of tangency with L means that $f(0) = 0$, $\frac{\partial f}{\partial q_i}(0) = 0$, $\frac{\partial^2 f}{\partial q_i \partial q_j}(0) = 0$, $x = (0, \ldots, 0, \ldots, 0)$, hence, using (4.2.5.1), we get

$$f(q) - \mathscr{F}\left(q_r, \ldots, q_n, \frac{\partial f}{\partial q_r}, \ldots, \frac{\partial f}{\partial q_n}\right) = o(q_1^2 + \ldots + q_n^2).$$

This remark points to the need of studying R-manifolds that are not exact solutions of E^r, but that have contact of sufficiently high order with E^r at x. Moreover, the existence of R-manifolds having contact of infinite order with E^r is a necessary condition for the existence of a solution passing through x.

4.2.6. We say that two R-manifolds L_1 and L_2 are *s-equivalent at x* if they have contact of order $\geqslant s$ there.

DEFINITION. *The jet L_x^s of order s* of an R-manifold L at x is the class of R-manifolds that are $(s + 1)$-equivalent to L at $x \in J^1(M)$.

EXAMPLE. If L projects diffeomorphically into M at x, $L \subset J^1(M)$, then locally $L = [j_1(f)]\ (M)$. Therefore $L_x^s = j_{s+1}(f)|_{\pi_1(x)} \in J^{s+1}_{\pi_1(x)}(M), f \in C^\infty(M)$.

4.2.7. DEFINITION. The *extension $E_x^{r,s}$ of order s* of the equation $E^r \subset J^1(M)$ at $x \in E^r$ is the set of jets of R-manifolds of order s at x whose representatives have contact of order s with E^r at x.

EXAMPLE. $E_x^{r,0} = x$, $E_x^{r,1} \neq \varnothing$ if and only if E^r is involutory at x.

REMARK. Usually (see [8], [7]), the definition of extensions of an equation uses R-manifolds without the restriction of projecting onto M. In this case, $\bigcup_{x \in E^r} E_x^{r,s} \subset J^{s+1}(M)$. We consider arbitrary R-manifolds; therefore, $\bigcup_{x\in E^r} E_x^{r,s} \subset \widetilde{J^{s+1}(M)}$, where $\widetilde{J^{s+1}(M)}$ is the augmented manifold of $(s + 1)$-jets, whose points are s-jets of R-manifolds. Clearly, $J^{s+1}(M) \subset \widetilde{J^{s+1}(M)}$.

4.2.8. Let $\Pi_x^s\colon E_x^{r,s} \to E_x^{r,s-1}$ denote the natural projection. Then every solution $L \subset E^r$ passing through x defines a sequence $L_x^s \in E_x^{r,s}$, $0 \leqslant s < \infty$, with $L_x^0 = x$, $\Pi_x^s(L_x^s) = L_x^{s-1}$.

DEFINITION. An equation $E^r \subset J^1(M)$ is said to be *formally integrable at $x \in E^r$* if there exists a sequence L_x^s of s-jets of R-manifolds at $x \in E^r$ such that $L_x^s \in E_x^s$ and $\Pi_x^s(L_x^s) = L_x^{s-1}$. The sequence L_x^s, $0 \leqslant s < \infty$, is called a *formal solution of the equation E^r* at x.

EXAMPLE. Let $x \in E^r$ be a singular point, codim $E^r = r = 1$. Then $E_x^{r,0} = x$, and $E_x^{r,1}$ is the set of Lagrangian subspaces of Γ_x. Let $L \subset E_x^{r,1}$ be such a subspace. Then the requirement that $(\Pi_x^2)^{-1}(L) \neq \varnothing$ means that $h_\omega|_L = 0$ (see 4.2.1).

4.2.9. THEOREM. *Let $E \subset J^1(M)$ be a first order differential equation and $x \in E$ a singular point,* codim $E = 1$, $\omega_x = 0$. *If a formal solution $\{L_x^s\}$ exists, then the Lagrangian subspace $L = L_x^1$ is invariant under H. If $L \subset T_x(E)$ is a Lagrangian subspace invariant under H, if the eigenvalues $\{\lambda_k\}$ of the restriction of H to L satisfies the condition*

$$(4.2.9.1) \qquad \sum m_i\lambda_i \neq 1,$$

and if $\sum m_i \geqslant 3$, where the $m_i \geqslant 0$ are natural numbers, then there exists a unique formal solution $\{L_x^s\}$ of E such that $L_x^1 = L$.

PROOF. The necessity was proved above. For the sufficiency proof, we choose a system of local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ on E such that the embedding $E \underset{i}{\hookrightarrow} J^1(M)$ in some neighbourhood of $x \in E$ is given by: $i^*(q_k) = x_k$, $i^*(p_k) = y_k$, $i^*(u) = \mathcal{F}(x, y)$, $1 \leqslant k \leqslant n$, where $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ is a special system of local coordinates in $J^1(M)$. Replacing E by $\alpha(E)$, if necessary, where α is a local diffeomorphism, we may assume that L is the linear span of the vectors $\left.\frac{\partial}{\partial x_1}\right|_x, \ldots, \left.\frac{\partial}{\partial x_n}\right|_x$. As we have seen, the condition $H\colon L \to L$ then means that $\left.\frac{\partial^2 \mathcal{F}}{\partial x_i \partial x_j}\right|_x = 0$, $1 \leqslant i, j \leqslant n$, while the restriction of H to L is given by the matrix $\left\| \left.\frac{\partial^2\mathcal{F}}{\partial x_i \partial y_j}\right|_x \right\|$. Since L projects onto M without degeneracy, every R-manifold tangent to L is given locally as follows: $y_k = \frac{\partial f}{\partial x_k}$, where $f(x_1, \ldots, x_n) \in C^\infty$ is of at least the third order of smallness at zero ($x_k(x) = 0$, $1 \leqslant k \leqslant n$). Therefore, the existence of the sequence L_x^s is equivalent to that of a formal series $\hat{f} \in \mathbf{R}[[x_1, \ldots, x_n]]$; $\hat{f}$ has order of smallness $\geqslant 3$, and the series is such that

$$\hat{f} - \hat{\mathcal{F}}\left(x_1, \ldots, x_n \frac{\partial \hat{f}}{\partial x_1}, \ldots, \frac{\partial \hat{f}}{\partial x_n}\right) = 0,$$

where $\hat{\mathcal{F}}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is the Maclaurin series for $\mathcal{F}$.

So we have to solve this formal equation. To do this we choose a basis $k_1, \ldots, k_n$ in L such that the matrix of $H\colon L \to L$ has the generalized Jordan form. Let $y_1, \ldots, y_n$ be functions such that $\widetilde{dy_i}|_0 = k$, where the vector field $\widetilde{dy_i}$ on E is defined by the equation $dy_i = \widetilde{dy_i} \lrcorner\, d\omega$, and $d\omega(\widetilde{dy_i}, \widetilde{dy_j}) = 0$. We now supplement (see [45], [34]) $y_1, \ldots, y_n$ to a canonical system of coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ ($d\omega = \sum_{i=1}^n dx_i \wedge dy_i$).

In this system we have $\omega = d\mathcal{F} - \sum_{i=1}^n y_i\, dx_i$, $k_i = \left.\frac{\partial}{\partial x_i}\right|_0 = \widetilde{dy_i}|_0$ and the matrix of $H\colon L \to L$ takes the Jordan form, that is,

$$H = \begin{Vmatrix} \left.\frac{\partial^2\mathcal{F}}{\partial x_1 \partial y_1}\right|_x & \cdots & \left.\frac{\partial^2\mathcal{F}}{\partial x_n \partial y_1}\right|_x \\ \cdots & \cdots & \cdots \\ \left.\frac{\partial^2\mathcal{F}}{\partial x_1 \partial y_n}\right|_x & \cdots & \left.\frac{\partial^2\mathcal{F}}{\partial x_n \partial y_n}\right|_x \end{Vmatrix} = \begin{Vmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_k \end{Vmatrix},$$

where

$$H_s = \left\| \begin{matrix} \sigma_s & \tau_s & 1 & 0 & & 0 \\ -\tau_s & \sigma_s & 0 & 1 & & \\ & & \ddots & & \ddots & \\ & & & \cdot 0 & & \cdot 1 \\ 0 & & & \cdot \sigma_s & & \tau_s \\ & & & -\tau_s & & \sigma_s \end{matrix} \right\|,$$

if $\lambda_s = \sigma_s + i\tau_s$, or else

$$H_s = \left\| \begin{matrix} \lambda_s & 1 & & \\ & \ddots & \ddots & 0 \\ & & \ddots & 1 \\ 0 & & & \lambda_s \end{matrix} \right\|,$$

if λ_s is real.

We solve the equation $\hat{f}(x) - \hat{\mathscr{F}}\left(x, \frac{\partial \hat{f}}{\partial x}\right) = 0$ formally in this system of coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. We solve this equation by induction on the degrees of $\hat{f}$, that is, we look for polynomials $\hat{f}_N(x_1, \ldots, x_n)$ of degree N such that

$$\hat{f}_N(x_1, \ldots, x_n) - \hat{\mathscr{F}}\left(x_1, \ldots, x_n, \frac{\partial \hat{f}_N}{\partial x_1}, \ldots, \frac{\partial \hat{f}_N}{\partial x_n}\right) = 0 \bmod \mu^{N+1},$$

where μ is a maximal ideal in the ring $\mathbf{R}[[x_1, \ldots, x_n]]$ of formal power series.

Suppose, then, that we have found polynomials $\hat{f}_N$, $N \geqslant 2$, satisfying the above equation. To find the series $\hat{f}_{N+1}$, we first apply a canonical transformation α: $(x, y) \to \left(x, y - \frac{\partial f_N}{\partial x}\right)$, where f_N is a smooth function whose Maclaurin expansion is precisely $\hat{f}_N$. The equation for the polynomial $\hat{f}_{N+1}$, which must now be homogeneous of degree $N + 1$, takes the following form:

$$(4.2.9.2) \qquad \hat{f}_{N+1}(x_1, \ldots, x_n) - \\ - \hat{\mathscr{F}}'\left(x_1, \ldots, x_n, \frac{\partial \hat{f}_{N+1}}{\partial x_1}, \ldots, \frac{\partial \hat{f}_{N+1}}{\partial x_n}\right) = 0 \bmod \mu^{N+2},$$

where $\mathscr{F}' = \alpha^*(\mathscr{F}) - f_N$ obviously satisfies the following relations:

$$\frac{\partial}{\partial x_i}\bigg|_0 (\mathscr{F}') = \frac{\partial}{\partial y_i}(\mathscr{F}')\bigg|_0 = 0, \qquad 1 \leqslant i \leqslant n,$$

$$\frac{\partial^{|k|}(\mathscr{F}')}{\partial x^k}\bigg|_0 = 0, \qquad |k| \leqslant N,$$

$k = (k_1, \ldots, k_n)$, $|k| = \sum k_i$,

$$\frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} = \frac{\partial^{|k|}}{\partial k^k}, \quad \frac{\partial^2 \mathscr{F}'}{\partial x_i \, \partial y_i}\bigg|_0 = \frac{\partial^2 \mathscr{F}}{\partial x_i \, \partial y_j}\bigg|_0, \qquad 1 \leqslant i, \; j \leqslant n.$$

It is more convenient to solve (4.2.9.2) in the ring $\mathbf{C}[[z_1, \ldots, z_n]]$ of complex formal power series.

First we make a few remarks. With every series $\hat{f} \in \mathbf{R}[[x_1, \ldots, x_n]]$ we can associate its complexification ${}^C\hat{f}$, where now ${}^C\left(\frac{\partial \hat{f}}{\partial x}\right)=\frac{\partial\, {}^C\hat{f}}{\partial z}$, z being the complexification of x, $z = {}^C x$. Further, as in the real case, every change $z' = \varphi(z)$ of coordinates can be thought of as a change $\eta' = \tilde{\varphi}(\eta)$, where η is the complexification of y and $\tilde{\varphi}(\eta) = [J'(\varphi)]^{-1}(\eta)$, and $J(\varphi)$ is the Jacobian matrix of φ. The lifting $\eta' = \tilde{\varphi}(\eta)$ is such that if $\eta=\frac{\partial f}{\partial z}$ for some $f \in C[[z_1, \ldots, z_n]]$, then after the change $\eta' = \frac{\partial f'}{\partial z'}$, where $f'(z') = f(\varphi^{-1}(z'))$ (a complex canonical transformation).

We choose the change $z' = \varphi(z)$ of coordinates so that the operator ${}^C H$ (the complexification of H: $L \to L$) has the Jordan form in the basis $\frac{\partial}{\partial z_1'}, \ldots, \frac{\partial}{\partial z_n'}$.

This can be done as follows. Suppose that H_s has size $2n_s \times 2n_s$ and acts on the vectors

$$\frac{\partial}{\partial x_{m_1}}, \ldots, \frac{\partial}{\partial x_{m_1+2n_s-1}}, \quad \lambda_s=\sigma_s+i\tau_s, \qquad \tau_s\neq 0.$$

Then we put

$$z'_{m_1}=z_{m_1}+iz_{m_1+1}, \quad z'_{m_1+1}=z_{m_1}-iz_{m_1+1}, \ldots, z'_{m_1+2n_s-1}=z_{m_1+2n_s-1}+iz_{m_1+2n_s},$$
$$z'_{m_1+2n_s}=z_{m_1+2n_s-1}-iz_{m_1+2n_s}.$$

In this case

$$\eta_{m_1}=\frac{1}{2}\eta_{m_1}+\frac{1}{2i}\eta_{m_1+1}, \quad \eta'_{m_1+1}=\frac{1}{2}\eta_{m_1}-\frac{1}{2}\eta_{m_1+1}, \ldots$$

It is not hard to see that, after this change of coordinates, the equation

$${}^C\hat{f}'_{N+1}-\hat{\mathscr{F}}'\left(z, \frac{\partial \hat{f}_{N+1}}{\partial z}\right)=0 \mod \mu^{N+2}$$

becomes

$$\hat{f}'_{N+1}-\hat{\mathscr{F}}'\left(z, \frac{\partial \hat{f}_{N+1}}{\partial z}\right)=0 \mod \mu^{N+2}, \tag{4.2.9.3}$$

where the matrix $\left\|\frac{\partial^2\mathscr{F}}{\partial z_k\,\partial\eta_j}\Big|_0\right\|$ has the Jordan form

$$\left\|\begin{matrix} \frac{\partial^2\mathscr{F}}{\partial z_1\,\partial\eta_1}\Big|_0 & \cdots & \frac{\partial^2\mathscr{F}}{\partial z_n\,\partial\eta_1}\Big|_0 \\ \cdots & \cdots & \cdots \\ \frac{\partial^2\mathscr{F}}{\partial z_1\,\partial\eta_n}\Big|_0 & \cdots & \frac{\partial^2\mathscr{F}}{\partial z_n\,\partial\eta_n}\Big|_0 \end{matrix}\right\| = \left\|\begin{matrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_k \end{matrix}\right\|, \tag{4.2.9.4}$$

$$H_s=\left\|\begin{matrix} \lambda_s & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_s \end{matrix}\right\|, \quad \lambda_s=\sigma_s+i\tau_s.$$

We look for $\hat{f}'_{N+1}$ in the form

$$\hat{f}'_{N+1}=\sum_{k_1,\ldots,k_n} c_{k_1,\ldots,k_n} z_1^{k_1}\cdots z_n^{k_n},$$

where $\sum k_i = N + 1$, since $\hat{f}'_{N+1}$ must be a homogeneous polynomial. Substituting the expression in (4.2.9.3), we find the coefficients of the terms of degree $N + 1$. Note that the terms of degree $N + 1$ on the left-hand side of (4.2.9.3) consist of monomials of degree $N + 1$ of $\hat{f}'_{N+1}$, monomials of degree $N + 1$ of $\hat{\mathcal{F}}'$ in $z_1, \ldots, z_n$, and finally, after the substitution $\eta_j = \frac{\partial f_{N+1}}{\partial z_j}$ of monomials of the form $z_k \eta_j$ of $\hat{F}'$, where the coefficients of these last monomials are elements of the Jordan matrix (4.2.9.4). As a result, we get a system of linear equations for the coefficients $c_{k_1, \ldots, k_n}$. We solve this system by induction on $(k_1, \ldots, k_n)$. To do this, we introduce the following ordering on the set of n-tuples $(k_1, \ldots, k_n)$ such that $\sum k_i = N + 1$. We take $(k_1, \ldots, k_n) < (k'_1, \ldots, k'_n)$ if $k'_n > k_n$, or if $k_n = k'_n, \ldots, k_{n-s} = k'_{n-s}$, but $k'_{n-s-1} > k_{n-s-1}$ for some $s < n - 1$.

FIRST STEP OF THE INDUCTION. We find the coefficient $c_{N+1,0,\ldots,0}$. To this end we write out the coefficient of z_1^{N+1} in (4.2.9.3) and find

$$(1-(N+1)\lambda_1)\, c_{N+1,0,\ldots,0} = \frac{1}{(N+1)!} \frac{\partial^{N+1} \hat{\mathcal{F}}'}{\partial z_1^{N+1}}\bigg|_0 ,$$

which always has a solution, because $1 - (N + 1)\lambda_1 \neq 0$.

GENERAL STEP OF THE INDUCTION. Suppose that we have found the coefficients $c_{k_1,\ldots,k_n}$ for all $(k_1, \ldots, k_n) < (k_1^0, \ldots, k_n^0)$. Let us find $c_{k_1^0,\ldots,k_n^0}$. On writing out the coefficient of $z_1^{k_1^0} \ldots z_n^{k_n^0}$ in (4.2.9.3), we have

$$(4.2.9.5) \qquad c_{k_1^0,\ldots,k_n^0} - \sum_{i=1}^{n} r_i = \frac{1}{k_1^0! \ldots k_n^0!} \frac{\partial^{N+1} \hat{\mathcal{F}}'}{\partial z_1^{k_1^0} \ldots \partial z_n^{k_n^0}}\bigg|_0 ,$$

where r_s is the term corresponding to the s-th column of $\left\| \frac{\partial^2 \mathcal{F}}{\partial z_k \, \partial \eta_j}\bigg|_0 \right\|$. If the s-th column contains only λ_m ($m = m(s)$ depends on s), then $r_s = k_s^0 \lambda_m c_{k_1^0,\ldots,k_n^0}$. If the s-th column also contains 1, then

$$r_s = k_s^0 \lambda_m c_{k_1^0,\ldots,k_n^0} + (k_{s-1}^0 + 1)\, c_{k_1^0,\ldots,k_{s-1}^0+1,k_s^0-1,k_{s+1}^0,\ldots,k_n^0}.$$

By the inductive hypothesis, the coefficient $c_{k_1^0,\ldots,k_{s-1}^0+1,k_s^0-1,k_{s+1}^0,\ldots,k_n^0}$ is known, because $(k_1^0, \ldots, k_{s-1}^0 + 1, k_s^0 - 1, k_{s+1}^0, \ldots, k_n^0) < (k_1^0, \ldots, k_n^0)$. Thus, (4.2.9.5) has the form $(1 - \sum k_s^0 \lambda_{m(s)}) c_{k_1^0, \ldots, k_n^0} = \ldots$, where the right-hand side consists of terms already known. Since $1 - \sum k_s^0 \lambda_{m(s)} \neq 0$, (4.2.9.5) has one, and in fact only one, solution.

Thus, if the conditions of Theorem 4.2.9 are satisfied, then there exists a unique element $\hat{f}_{N+1} \in \mathbf{C}\,[[z_1, \ldots, z_n]]$ satisfying the complexified equation (4.2.9.3), so that the real part of its restriction to $x_1, \ldots, x_n$ gives us (also uniquely) an element $\hat{f}_{N+1} \in \mathbf{R}\,[[x_1, \ldots, x_n]]$ satisfying

(4.2.9.2). Hence, there exists a formal power series $\hat{f}$ satisfying the formal equation $\hat{f}-\hat{\mathcal{F}}\left(x, \frac{\partial f}{\partial x}\right)=0$.

REMARK 1. a) As we have seen, a solution of (4.2.9.2) is equivalent to one of a certain system $A_{|k|}c_k = v_{|k|}$, $k = (k_1, \ldots, k_n)$ of linear equations, for each $|k|$. About this system we remark, firstly, that $v_{|k|}$ can be homogeneous polynomial of degree k, and, secondly, that the above arguments show that the eigenvalues of $A_{|k|}$, more accurately of $A_{|k|}(L)$, are $1 - \sum m_i\lambda_i$, where $\sum m_i = |k|$ and the λ_i are the eigenvalues of the restriction of $H: L \to L$. Therefore, the formal conditions 4.2.9 mean that the operators $A_{|k|}(L)$ are non-degenerate for $|k| \geqslant 3$.

b) Let us investigate how restrictive the conditions of Theorem 4.2.9 are. For this purpose we consider the set of points $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ whose coordinates do not satisfy (4.2.9.1). For every $|k|$, the set of $(\lambda_1, \ldots, \lambda_n)$ such that $\sum m_i\lambda_i = 1$ and $\sum m_i = |k|$ is obviously closed and nowhere dense in $\mathbf{C}^n$, and therefore, the set of $(\lambda_1, \ldots, \lambda_n)$ such that $\sum m_i\lambda_i=1$ and $\sum m_i \geqslant |k|$ is thin in $\mathbf{C}^n$. Baire's theorem now shows at once that the set of all $(\lambda_1, \ldots, \lambda_n)$ satisfying (4.2.9.1) is everywhere dense in $\mathbf{C}^n$.

REMARK 2. The formal conditions (4.2.9.1) are a generalization of Poincaré's δ-lemma (see [9], [17]) to the non-linear case, since they ensure the absence of any obstacle to the construction of the sequence L_x^s.

Moreover, Poincaré's lemma applies to formally integrable equations. The following definition is the natural generalization of the concept of formal integrability to the non-linear case (see [8]).

Let $E^s = \bigcup_{x\in E} E_x^s$. Then the equation $E \subset J^1(M)$ is said to be *formally integrable* if E^s is a smooth manifold for each $s \geqslant 0$, and $\Pi^s: E^s \to E^{s-1}$ is a fibred manifold.

In our case, that is, when E contains a singular point $x \in E$, the whole equation E^s is not a smooth manifold; however, if there exists a sequence $L_x^s \in E_x^s$ analogous to that described in Theorem 4.2.9, then some neighbourhood of $L_x^s \in E_x^s$ is a smooth submanifold of $J^{s+1}(M)$.

EXAMPLE. The conditions (4.2.9.1) are always satisfied when $\mathrm{Re}\,\lambda_k = \frac{1}{2}$ for all λ_k.

4.2.10. Suppose now that $E^r \subset J^1(M)$ is an involutory equation with $\mathrm{codim}\, E^r = r > 1$. As before, we assume that coordinates $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ have been chosen so that the Lagrangian subspace $L \subset T_x(E)$, $h_{\omega}|_L = 0$, is the linear span of the vectors $\left.\frac{\partial}{\partial q_1}\right|_x, \ldots, \left.\frac{\partial}{\partial q_n}\right|_x$, and E^r is given by the system $p_1 = 0, \ldots, p_{r-1} = 0$, $u = \mathcal{F}(q_1, \ldots, q_n, p_r, \ldots, p_n)$. In this basis, the matrix of $H: V \to V$ is $\left\|\frac{\partial^2\mathcal{F}}{\partial q_i \partial p_j}\right\|$, $r > i, j \leqslant n$; therefore, similarly to 4.2.9, we get the following assertion:

THEOREM. *Let $E^r \subset J^1(M)$ be an involutory differential equation and x a singular point. If a formal solution $\{L_x^s\}$ exists, then the image $\bar{L}$ of the Lagrangian subspace L_x^1 in V is a Lagrangian subspace invariant under H. If $L \subset T_x(E^r)$ is a Lagrangian subspace such that its image $\bar{L} \subset V$ is invariant under H and the eigenvalues $\{\lambda_k\}$ of the restriction of H to $\bar{L}$ satisfy* (4.2.9.1), *then there exists a unique formal solution $\{L_x^s\}$ of E^r such that $L_x^1 = L$.*

4.2.11. To conclude this section we consider the condition of formal integrability for an equation with a singularity. Suppose that $\mathcal{F}_1 \in C^\infty(T^*M)$. Because of the natural projection π: $J^1(M) \to T^*(M)$ we can then regard $\mathcal{F}_1$ as a function on $J^1(M)$, $\mathcal{F} = \pi^*(\mathcal{F}_1)$, and the functions $\mathcal{F} \in C^\infty(T^*M)$ of the form $\pi^*(\mathcal{F}_1)$ are characterized by the fact that $X_1(\mathcal{F}) = 0$.

We assume now that $E = \{\mathcal{F} = 0\}$, where $X_1(\mathcal{F}) = 0$, and that $x \in E$ is a point for which $d\mathcal{F}_x = 0$. In this case E is not a smooth submanifold of $J^1(M)$ at all points of the trajectory of the field X_1 passing through x.

Let H: $T_{\pi(x)}(T^*M) \to T_{\pi(x)}(T^*M)$ be the operator corresponding to the Hessian $h_{\mathcal{F}_1}$ of $\mathcal{F}_1$ at $\pi(x)$ relative to the canonical form $d\rho \in \Lambda^2(T^*M)$, that is, $h_{\mathcal{F}_1}(X, Y) = d\rho_{(\pi)x}(HX, Y)$ for all $X, Y \in T_{\pi(x)}(T^*M)$.

Just as in 4.2.1, it can be shown that a necessary condition for the existence of a solution L^n of E passing through x is the existence of a subspace $L \subset T_{\pi(x)}(T^*M)$ that is Lagrangian (relative to the form $d\rho\pi(x)$) and invariant under H. The following is an analogue to Theorem 4.2.9.

THEOREM. *If there exists a formal solution $\{L_x^s\}$ of the equation E at the point x, then $L_x^1 = L$ is a Lagrangian subspace of $T_{\pi(x)}(T^*M)$ invariant under H. If $L \subset T_{\pi(x)}(T^*M)$ is a Lagrangian subspace invariant under H and such that the eigenvalues $\{\lambda_k\}$ of the restriction of H to L satisfy the condition*

$$\text{(4.2.11.1)} \qquad \sum m_i\lambda_i \neq 0$$

where the m_i are natural numbers and $\sum m_i \geqslant 3$, then there exists a unique formal solution $\{L_x^s\}$ such that $\pi_{*,x}(L_x^1) = L$.

PROOF. The proof of Theorem 4.2.11 is analogous to that of Theorem 4.2.9, with the difference that it is necessary to solve the formal equation $\tilde{\mathcal{F}}(q_1, \ldots, q_n, p_1, \ldots, p_n) = 0$, where $\left\| \frac{\partial^2 \mathcal{F}}{\partial p_i \, \partial q_j} \right\|$ is the matrix of H: $L \to L$ in the basis $\left.\frac{\partial}{\partial q_1}\right|_x, \ldots, \left.\frac{\partial}{\partial q_n}\right|_x$. (Here, as before, it is assumed that local coordinates $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ are chosen such that L is the linear span of $\left.\frac{\partial}{\partial q_1}\right|_x, \ldots, \left.\frac{\partial}{\partial q_n}\right|_x$.)

REMARK. Theorem 4.2.11 carries over in an obvious fashion to involutory equations of codimension > 1 having a singularity of the type indicated.

4.2.12. We draw some consequences from Theorem 4.2.10 concerning the question of local equivalence.

PROPOSITION. *Let* $\{L_x^s\}$ *be a formal solution of the equation* $E^r \subset J^1(M)$ *at* $x \in E^r$. *Then there exists an R-manifold* $L \subset J^1(M)$ *having contact of infinite order with* E^r *at* x *and such that* $[L]_x^s = L_x^s$, *where* $[L]_x^s$ *is the s-jet of* L *at* x.

PROOF. We choose a local U_1-diffeomorphism α such that the representatives $\alpha_*(L_x^1)$ at x project without singularity onto M, $\alpha(x) = x$. By a theorem of Borel (see [15]), there then exists a smooth function f defined in some neighbourhood of $\pi_1(x) \in M$ such that $j_s(f)|_{\pi_1(x)} = \alpha_*(L_x^{s-1})$ for all $s \geqslant 1$. If we now let L_1 denote the R-manifold corresponding to $j_1(f)$, then the R-manifold $L = \alpha^{-1}(L_1)$ is what we want.

4.2.13. Using Proposition 4.1.2, we now obtain the following result.

THEOREM. *A necessary condition for the formal equivalence of involutory equations* $E_k^r \subset J^1(M)$ *at a singular point* $x \in E_k^r$ *is that the operators* H_k $(k = 1, 2)$ *are CSp-equivalent. If the eigenvalues of the* H_k *satisfy* (4.2.9.1), *then the Csp-equivalence of the* H_k *is sufficient for the formal equivalence of the equations.*

§3. The algebraic insolubility of the local classification of Hamiltonians

4.3.1. In this section we consider the question of the local equivalence of two Hamiltonians $G_1, G_2 \in C^\infty(T^*M)$ in the neighbourhood of a point $x \in T^*(M)$, where $G_1(x) = G_2(x) = 0$, $dG_{1,x} = dG_{2,x} = 0$, that is, of the existence of a local canonical diffeomorphism $\alpha\colon T^*(M) \to T^*(M)$, $\alpha(x) = x$, such that $\alpha^*(G_2) = G_1$. As in 4.1.1, the construction of α can be replaced by the construction of a solution of the equation $E \subset J^1(M \times M)$, where $E = \{G = 0\}$, $G = pr_1^*(G_1) - pr_2^*(G_2)$, $G \in C^\infty(T^*(M \times M))$, and $pr_k^*\colon T^*(M \times M) \to T^*(M)$ is the projection onto the k-th component $(k = 1, 2)$.

4.3.2. THEOREM. *The problem of the local classification of Hamiltonians in the neighbourhood of a singular point is not algebraically soluble, that is, there is no natural number k such that the local equivalence of Hamiltonians relative to the group of canonical diffeomorphisms follows from the equivalence of the k-jets of Hamiltonians relative to the same group.*

PROOF. We assume that such a k exists. Then for any Hamiltonians G_1 and G_2 having equal k-jets at the singular point $x \in T^*(M)$, there is a canonical diffeomorphism α such that $\alpha^*(G_2) = G_1$.

We now use the fact that the construction of α is equivalent to the solution of the equation $E \subset J^1(M \times M)$. In the language of formal equations, this last equation means: if $f_k \in \mathbf{R}[q_1, \ldots, q_n]$ satisfies the equation

$$\hat{G}\left(q_1, \ldots, q_n, \frac{\partial f_k}{\partial q_1}, \ldots, \frac{\partial f_k}{\partial q_n}\right) = 0 \mod \mu^{k+1},$$

then there exists a series $g \in \mathbf{R}[[q_1, \ldots, q_n]]$ extending f_k and such that

$$\hat{G}\left(q_1, \ldots, q_n, \frac{\partial g}{\partial q_1}, \ldots, \frac{\partial g}{\partial q_n}\right) = 0. \tag{4.3.2.1}$$

Suppose now that $L_0 \subset T_{(x,x)}(T^*(M \times M))$ is the graph of the differential of α at (x, x) or, what is the same thing, the plane tangent to dg at (x, x). By the Poincaré-Lyapunov theorem (see [27]), the eigenvalues $\{\lambda_s\}$ of the restriction of $H_{\tilde{G}}$ to L_0 then split into the union of quadruples of type $(\lambda_s, \bar{\lambda}_s, -\lambda_s, -\bar{\lambda}_s)$. Therefore, the condition (4.2.11.1) of formal integrability is violated for arbitrarily large k.

Furthermore, the solution of the formal equation (4.3.2.1), as was the case for the solution of the corresponding equation in the proof of Theorem 4.2.9, is equivalent (see Remark 1 a)) to the solution of the system of linear equations $A_{|k|}(L_0)c_{k_1, \ldots, k_n} = v_{|k|}$ for each $|k|$; clearly, Remark 1 a) is valid here, except that the eigenvalues of the operator $A_{|k|}(L_0)$ are of the form $\sum m_i\lambda_i$, where $\sum m_i = |k|$, and the λ_i are the eigenvalues of $H: L_0 \to L_0$. But since the eigenvalues are distributed in pairs $(\lambda_s, -\lambda_s)$, the corank $r(|k|)$ of $A_{|k|}(L_0)$ (the codimension of the image) is at any rate not less than the number of solutions in natural numbers of the equation $2\sum n_i = |k|$ (summation is over the pairs $(\lambda_s, -\lambda_s)$).

We now choose $|k|$ large enough so that $r(|k|) > n(2n + 1)$ (note that $n(2n + 1) = \dim Sp(2n)$). As is not difficult to see, we can then choose $v_{|k|} \in \mathbf{R}[q_1, \ldots, q_n]$ so that the equation $A_{|k|}(L_0)C_k = v_{|k|}$ is insoluble for any $L_0 \in T_{(x,x)}(T^*(M \times M))$, which is the graph of the differential of a canonical diffeomorphism and invariant under H.

Thus, in particular, there exist Hamiltonians G_1 and G_2 whose k-jets are canonically equivalent (with k any fixed number), whereas (4.3.2.1) is insoluble.

CHAPTER V

Local classification at a singular point

In the two preceding chapters we have obtained necessary and sufficient conditions for the formal equivalence of equations at a singular point. In this chapter we investigate the sufficiency of formal equivalence of equations for local equivalence at a singular point, and we also give some applications of our results.

§ 1. Sufficient conditions

5.1.1. With Theorem 4.2.13 in mind, we consider in this section a pair of involutory equations E_1^r and E_2^r which have infinite contact at a singular point $x \in E_k^r$ $(k = 1, 2)$, and we take E_2^r, say, as a model. We require that E_2^r satisfy the following two conditions:

(A) In some special local coordinate system $q_1^0, \dots, q_n^0, u, p_1^0, \dots, p_n^0$ in the neighbourhood of x, E_2^r has the form
$p_1^0 = 0, \dots, p_{r-1}^0 = 0$, $u^0 = \mathscr{F}_2(q_r^0, \dots, q_n^0, p_r^0, \dots, p_n^0)$, where $\mathscr{F}_2(q_r^0, \dots, q_n^0, p_r^0, \dots, p_n^0)$ is a quadratic function of its arguments.

(B) The eigenvalues $\{\lambda_s\}$ of the operators $H_k : V \to V$ are such that $\operatorname{Re} \lambda_s \neq 0$.

Note, firstly, that every involutory equation E^r such that H satisfies (4.2.9.1) can be brought by a U_1-diffeomorphism to an equation having contact of infinite order with E_2^r, that is, E_1^r, can be any involutory equation E^r with a singular point.

Moreover, condition (B) is automatically satisfied if (4.2.9.1) is. Otherwise, for every λ_s with $\operatorname{Re} \lambda_s = 0$, we would have $N(\lambda_s + \bar{\lambda}_s) + \lambda_s + (1 - \lambda_s) = 1$ for every natural number N, therefore violating (4.2.9.1).

5.1.2. Thus, we consider a pair of involutory equations $E_k^r \subset J^1(M)$ ($k = 1, 2$) for which conditions (A) and (B) are satisfied.

Let $L_0 \subset T_x(E_1^2)$ be a Lagrangian subspace invariant under H_1. We may suppose that local coordinates $q_1^0, \dots, q_n^0, u^0, p_1^0, \dots, p_n^0$ (see (A)) have been chosen so that L_0 is the linear span of the vectors $\left.\frac{\partial}{\partial q_1^0}\right|_0, \dots, \left.\frac{\partial}{\partial q_n^0}\right|_0$, $x = (0, \dots, 0)$. For this can always be achieved by applying to E_1^r and E_2^r a U_1-diffeomorphism that is linear in the coordinate system $q_1^0, \dots, q_n^0, p_1^0, \dots, p_n^0$ in (A) and takes L_0 to the linear span of the vectors $\left.\frac{\partial}{\partial q_1^0}\right|_0, \dots, \left.\frac{\partial}{\partial q_n^0}\right|_0$. The existence of such a transformation is a consequence of Witt's theorem [25]. Suppose now that $L \subset E_2^r$ is a solution of E_2^r. For L we may take the R-manifold corresponding to $j_1(0)$, for example. We show how to construct a solution of E_1^r tangent to L_0, using conditions (A), (B) and the solution L.

We consider first the case when $\operatorname{codim} E_k^r = r = 1$.

5.1.3. Let G be a smooth function on $J^1(M)$, $G(x) = 0$, having a non-degenerate singularity at x. The set $\{y \mid G(y) > 0\}$ is called a *conical neighbourhood* of x.

5.1.4. PROPOSITION. *There exists a smooth function $\mathscr{F}$ defined in some neighbourhood of $\pi_1(x)$ and quadratic in the coordinates $q_1^0, \dots, q_n^0$* (see (A)) *for which*

(5.1.4.1) $$\pi_{1*}(X_{\omega_{k,y}})(\mathscr{F}) > 0$$

for all $y \neq x$ in the complement of some conical neighbourhood K of x. Also, K can be chosen so that $L \setminus \{x\}$ lies in the interior of the complement of K.

PROOF. Let $\mathscr{F}$ be a Lyapunov function for the restriction of X_{ω_2} to L. As X_{ω_2} is a linear vector field in the coordinates $q_1^0, \dots, q_n^0, u^0, p_1^0, \dots, p_1^0$, we can choose $\mathscr{F}$ to be a quadratic form. The equations E_1 and E_2 have contact of infinite order at x; therefore, the set of points y for which

$\pi_{1*}(X_{\omega_{1,\,\eta}})(\mathcal{F}) \leqslant 0$ can be included in a conical neighbourhood of K, $L \setminus \{x\} \not\subset \bar{K}$.

5.1.5. We assume now that $m > 0$ of the eigenvalues $\{\lambda_s\}$ of the operator H: $L_0 \to L_0$ have Re $\lambda_s > 0$, and the remaining $n - m > 0$ have Re $\lambda_s < 0$.

We write $\mathcal{K}_t = \{\mathcal{F} = t\}$. Then for sufficiently small t the $\mathcal{K}_t$ are submanifolds of some neighbourhood of $\pi_1(x)$, $t \neq 0$. It is known that $\mathcal{K}_t$ is diffeomorphic to $\mathbf{R}^m \times \mathbf{S}^{n-m-1}$ for $t > 0$, and to $\mathbf{R}^{n-m} \times \mathbf{S}^{m-1}$ for $t > 0$; but $\mathcal{K}_0$ is a cone, so that $\mathcal{K}_0 - \{\pi_1(x)\}$ is a submanifold.

PROPOSITION. *The Cauchy initial data $\varphi_t = 0$ on the submanifold $\mathcal{K}_t$ for $t \neq 0$ and on $\mathcal{K}_0 \setminus \{\pi_1(x)\}$ for $t = 0$ are free both for E_1 and for E_2.*

PROOF. For sufficiently small t, since $\varphi_t = 0$, the check on the characteristic property can be carried out in the complement to a conical neighbourhood K (see 5.1.4). Suppose that a local coordinate system $q_1, \ldots, q_n$ on M in some neighbourhood of the point $y_t \in \mathcal{K}_t$, that does not intersect K, is chosen so that $\mathcal{F}(q_1, \ldots, q_n) = q_n$ in this neighbourhood. If E_1 can be written in the local coordinate system $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ induced by $q_1, \ldots, q_n$ as

$$u - \mathcal{F}_3(q_1, \ldots, q_n, p_1, \ldots, p_n) = 0,$$

then the condition for the initial data $\varphi_t = 0$ to be characteristic is equivalent to $\frac{\partial \mathcal{F}_3}{\partial p_n} \neq 0$ at the pre-images of the points $j_1(\varphi_t)$. But the latter lie in the complement of K, where $\frac{\partial \mathcal{F}_3}{\partial p_n} = \pi_{1*}(X_\omega)(\mathcal{F}) > 0$.

5.1.6. The family of functions φ_t (see 1.5.4) determines Cauchy initial data $L_t^1 \subset E_1$ for sufficiently small t; we take L_1^0 to be supplemented by x.

PROPOSITION. *Suppose that $L^1 = \bigcup_t L_t^1$, with t sufficiently small. Then L^1 is a smooth submanifold of E_1 having contact of infinite order with L at x.*

PROOF. Let $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ be local coordinates in a neighbourhood of x as in conditions (A) and (B). Then E_2 is given by the equation $u - \mathcal{F}_2(q_1, \ldots, q_n, p_1, \ldots, p_n) = 0$, where $\mathcal{F}_2(q, p)$ is a quadratic form in $q_1, \ldots, q_n, p_1, \ldots, p_n$. Since E_1 has contact of infinite order with E_2 at x, E_1 is given by the equation $u - \mathcal{F}_1(q_1, \ldots, q_n, p_1, \ldots, p_n) = 0$, where $\varepsilon(q, p) = \mathcal{F}_1(q, p) - \mathcal{F}_2(q, p)$ is a flat function at $x = (0, \ldots, 0)$. Moreover, we have chosen $\mathcal{F}(q_1, \ldots, q_n)$ to be quadratic in $q_1, \ldots, q_n$. In this case L^1 is the set of points of the form $(q_1, \ldots, q_n, 0, p_1(q), \ldots, p_n(q))$, since $\varphi_t = 0$. The functions $p_i(q)$ satisfy the equations

$$(5.1.6.1) \qquad \begin{cases} -\sum\limits_{i=1}^{n} p_i(q)\, dq_i = \lambda(q)\, d\mathcal{F}, \\ \mathcal{F}_1(q_1, \ldots, q_n, p_1(q), \ldots, p_n(q)) = 0. \end{cases}$$

The first of the equations (5.1.6.1) follows from the fact that the restriction of the form $U_1 = du - \sum_{i=1}^{n} p_i\, dq_i$ to the submanifold L_t^1 must vanish; the second is a consequence of the fact that $L^1 = \bigcup_t L_t^1 \subset E_1$.

Thus, the equations (5.1.6.1) show that L^1 is determined by some function $\lambda(q)$, which must satisfy the equation

$$(5.1.6.2) \qquad \mathcal{F}_1\left(q_1, \ldots, q_n - \lambda\frac{\partial \mathcal{F}}{\partial q_1}, \ldots, -\lambda\frac{\partial \mathcal{F}}{\partial q_n}\right) = 0,$$

since $-p_i(q) = \lambda(q)\frac{\partial \mathcal{F}}{\partial q_i}$. Now we use the fact that $\mathcal{F}_1 = \mathcal{F}_2 + \varepsilon$, where $\mathcal{F}_2(q, p)$ is a quadratic form and ε is a flat function at zero. Then (5.1.6.2) takes the form

$$(5.1.6.3) \quad -\lambda \sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{F}_2}{\partial p_i\, \partial q_j} q_j \frac{\partial \mathcal{F}}{\partial q_i} + \lambda^2 \sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{F}_2}{\partial p_i\, \partial p_j} \frac{\partial \mathcal{F}}{\partial q_i} \frac{\partial \mathcal{F}}{\partial q_j} = -\varepsilon\left(q, -\lambda\frac{\partial \mathcal{F}}{\partial q}\right).$$

It follows from 5.1.3 that $X_{\omega_2}(\mathcal{F}) > 0$ at points of L other than x. But at $(q_1, \ldots, q_n, 0, \ldots, 0)$, X_{ω_2} takes the value $\sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{F}_2}{\partial p_i\, \partial q_j} q_j \frac{\partial}{\partial q_i}$, so that

$$\sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{F}_2}{\partial p_i\, \partial q_i} q_j \frac{\partial \mathcal{F}}{\partial q_i}$$

is a positive definite quadratic form. We rewrite (5.1.6.3) as follows:

$$(5.1.6.4) \quad \lambda = -\varepsilon\left(q, -\lambda\frac{\partial \mathcal{F}}{\partial q}\right)\left(\sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{F}_2}{\partial q_j\, \partial p_i} q_j \frac{\partial \mathcal{F}}{\partial q_i} - \lambda \sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{F}_2}{\partial p_i\, \partial p_j} \frac{\partial \mathcal{F}}{\partial q_i} \frac{\partial \mathcal{F}}{\partial q_j}\right)^{-1}.$$

We claim that the function on the right of (5.1.6.4), as a function of the variables $(q_1, \ldots, q_n, \lambda)$, is smooth and flat at zero. We consider a neighbourhood of $(0, \ldots, 0, 0)$ that is sufficiently small so that in it

$$g(q, \lambda) = \sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{F}_2}{\partial q_i\, \partial p_j} q_i \frac{\partial \mathcal{F}}{\partial q_j} - \lambda \sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{F}_2}{\partial p_i\, \partial p_j} \frac{\partial \mathcal{F}}{\partial q_i} \frac{\partial \mathcal{F}}{\partial q_j}$$

is a positive definite quadratic form in $(q_1, \ldots, q_n)$. We note next that the Taylor series of $g(q, \lambda)$ in the neighbourhood of $(0, \ldots, 0,0)$ under discussion divides the Taylor series of $\varepsilon\left(q, -\lambda\frac{\partial \mathcal{F}}{\partial q}\right)$.

For it is easy to see that the zeros of $g(q, \lambda)$ in this neighbourhood are the points $(0, \ldots, 0, \lambda)$. But $\varepsilon\left(q, -\lambda\frac{\partial \mathcal{F}}{\partial q}\right)$ is flat at these points, and $g(q, \lambda)$ is a quadratic. Thus, the Taylor series of $g(q, \lambda)$ in the neighbourhood divides that of $\varepsilon\left(q, -\lambda\frac{\partial \mathcal{F}}{\partial q}\right)$. Further, by a theorem of Lojasiewicz and Hörmander (see [11], [22]), there exists a smooth function $h(q, \lambda)$ such that $\varepsilon\left(q, -\lambda\frac{\partial\mathcal{F}}{\partial q}\right) = h(q, \lambda)g(q, \lambda)$; as follows from a discussion of the formal power series corresponding to h, g, ε, we also see that h is flat at the points $(0, \lambda)$. Thus, (5.1.6.4) is equivalent to:

$$\lambda = -h(q, \lambda), \tag{5.1.6.5}$$

where $h(q, \lambda)$ is a flat function at $(0, \lambda)$. It is not hard to see that (5.1.6.5) has a unique smooth solution. This solution $\lambda(q)$ is flat at zero, as follows at once from the properties of $h(q, \lambda)$ if the solution of (5.1.6.5) is considered in formal power series.

This last remark concludes the proof of the proposition.

5.1.7. Using the proposition just proved, we can establish the existence of a solution as an equation $E \subset J^1(M)$ that is tangent to a Lagrangian subspace $L_0 \subset T_x(E)$ invariant under H, and $\omega_x = 0$.

PROPOSITION. *Suppose that we are given an equation* $E \subset J^1(M)$, $\omega_x = 0$, *that the eigenvalues of* H *satisfy* (4.2.9.1), *that among the eigenvalues of* H: $L_0 \to L_0$ *there are some with real parts of opposite signs, where* L_0 *is a Lagrangian subspace of* $T_x(E)$. *Then there is a solution of* $E \subset J^1(M)$ *tangent to* L_0.

5.1.8. We prove a more general assertion from which the one just formulated follows.

PROPOSITION. *Suppose that we are given equations* $E_k \subset J^1(M)$, $\omega_{k,x} = 0$, codim $E_k = 1$, *having contact of infinite order at* x *and satisfying the conditions* (A) *and* (B). *Suppose also that* $L_0 \subset T_x(E_1)$ *is a Lagrangian subspace such that some of the eigenvalues of* H: $L_0 \to L_0$ *have real parts of opposite signs. Then there exists a solution of* $E_1 \subset J^1(M)$ *tangent to* L_0.

PROOF OF PROPOSITION 5.1.8. We consider the manifolds L_t^1 (with the notation of Proposition 5.1.6), while L_0^1 is to be a manifold with a singularity at $x \in J^1(M)$.

The vector field X_{ω_1} was defined by the equation $X_{\omega_1} \lrcorner\, d_{\omega_1} = \omega_1$, from which it follows that $L_{X_{\omega_1}}(\omega_1) = \omega_1$. Therefore, if T_s, $-\infty < s < \infty$, is a one-parameter group of translations along X_{ω_1}, then $T_s^*(\omega_1) = e^s\omega_1$. Moreover, it follows from 5.1.3 that X_{ω_1} is transversal to L_0^1 everywhere except at the singular point $x \in J^1(M)$. Thus, the set $L_1 = \bigcup_s T_s(L_0^1 \setminus x)$ in the neighbourhood of x is a smooth submanifold of E_1. Further, since $U_1|_{L_0^1 \setminus x} = 0$ and $T_s^*(\omega_1) = e^s\omega_1$, $\omega_1(X_{\omega_1}) = 0$, it follows that $U_1|_{L_1} = 0$.

Let $\bar{L}_1$ be the closure of L_1; we show that $\bar{L}_1$ is the solution of $E_1 \subset J^1(M)$ that we are looking for. Clearly, it is enough to prove that $\bar{L}_1$ is a smooth submanifold of E_1.

We choose a U_1-diffeomorphism σ_1 such that $\sigma_1(x) = x$, $\sigma_1(L_0^1) \subset L_0^2$, where $L_0^2 = \{\pi_1^*(\mathcal{F}) = 0\} \cap L$, and σ_1 differs from the identity diffeomorphism by a mapping that is flat at x.

Let us show that such a U_1-diffeomorphism σ_1 exists. By Whitney's extension theorem (see [11]), there is an R-manifold L_3 containing L_0^1 and having contact of infinite order with L at x. Then, in some neighbourhood of $\pi_1(x)$, L_3 is the graph of the 1-jet of a function $f(q)$ that is flat at $\pi_1(x)$. If we now define

$$\sigma_1(q_1, \ldots, q_n, u, p_1, \ldots, p_n) = \left(q_1, \ldots, q_n, u - f(q), p_1 - \frac{\partial f}{\partial q_1}, \ldots, p_n - \frac{\partial f}{\partial q_n}\right),$$

where the coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ are as in conditions (A) and (B), then clearly $\sigma_1(L_0^1) \subset L_0^2$.

Let G be a Lyapunov function for the vector field X_{ω_2} such that $G|_L = \pi_1^*(F)|_L$ on L. We consider a local diffeomorphism σ_2: $J^1(M) \to J^1(M)$, $\sigma_2(x) = x$, differing from the identity by a flat function at $x \in J^1(M)$ and such that σ_2: $\sigma_1(E_1) \to E_2$, $\sigma_2|_{L_0^2} = 1$. The vector fields X_{ω_2} and $\sigma_{2*} \circ \sigma_{1*}(X_{\omega_1})$ on E_2 differ by a vector field that is flat at x; therefore, by Chern's theorem (see Ch. I, §7), there exists a local diffeomorphism σ_3: $E_2 \to E_2$ that fixes the cone $G = 0$ pointwise and is such that $\sigma_{3*} \circ \sigma_{2*} \circ \sigma_{1*}(X_{\omega_1}) = X_{\omega_2}$. We note that the choice of G ensures that σ_3 fixes L_0^2 pointwise, therefore, $\bar{L}_1$ is taken by $\sigma_3 \circ \sigma_2 \circ \sigma_1$ to $\sigma_3 \circ \sigma_2 \circ \sigma_1(\bar{L}_1) = \sigma_3 \circ \sigma_2 \circ \sigma_1(\bigcup_s T_s(L_0^1 \setminus x)) = L$, or $\bar{L}_1 = \sigma_1^{-1} \circ \sigma_2^{-1} \circ \sigma_3^{-1}(L)$, that is, $\bar{L}_1$, as the image of a smooth submanifold under a diffeomorphism, is also a smooth submanifold of E_1, touches L_0, and is therefore the required solution of E_1.

PROOF OF PROPOSITION 5.1.7. We choose an equation $E_2 \subset J^1(M)$, $\omega_{2,x} = 0$, touching E_1 and satisfying conditions (A) and (B). By Theorem 4.2.13, there exists a local U_1-diffeomorphism α: $J^1(M) \to J^1(M)$, $\alpha(x) = x$, such that E_2 and $\alpha(E_1)$ have contact of infinite order at x, and consequently satisfy 5.1.8.

5.1.9. We now consider the case when $\operatorname{Re} \lambda_s > 0$ for the eigenvalues $\{\lambda_s\}$ of H: $T_x(E_1) \to T_x(E_1)$. Replacing the cone $F = 0$ by the sphere $F = 1$, we can then prove the following propositions in complete analogy to Propositions 5.1.7 and 5.1.8.

PROPOSITION. a) *Suppose that we are given equations* $E_k \subset J^1(M)$, $\omega_{k,x} = 0$ $(k = 1, 2)$ *having contact of infinite order at x and satisfying conditions* (A) *and* (B); *suppose that the eigenvalues of* H: $T_x(E_1) \to T_x(E_1)$ *have real parts of the same sign only. Then, for every Lagrangian subspace* $L_0 \subset T_x(E)$ *invariant under H, the equation* $E_1 \subset J^1(M)$ *has a solution tangent to* L_0.

b) *Let* $E \subset J^1(M)$, $\omega_x = 0$, *be an equation such that the eigenvalues of* H: $T_x(E) \to T_x(E)$ *satisfy* (4.2.9.1) *and have real parts of the same sign. Then, for every Lagrangian subspace* $L_0 \subset T_x(E)$ *invariant under H, the equation has a solution tangent to* L_0.

5.1.10. THEOREM. *Suppose that the equations* $E_k \subset J^1(M)$, codim $E_k = 1$, $\omega_{k,x} = 0$ *are such that the operators* $H_k: T_x(E_k) \to T_x(E_k)$ $(k = 1,2)$ *are conformally-symplectically (symplectically) equivalent, and their eigenvalues satisfy* (4.2.9.1). *Then the equations* $E_k \subset J^1(M)$ $(k = 1, 2)$ *are locally equivalent* (U_1*-equivalent*) *at* $x \in J^1(M)$.

PROOF. We use Proposition 4.2.12. Since the operators H_k $(k = 1, 2)$ are symplectic (conformally-symplectic) and the conditions (4.2.9.1) are satisfied, there exists a (contact) U_1-diffeomorphism α_1 such that $\alpha_1(E_1)$ and E_2 have contact of infinite order at x.

Lemma 3.2.2 shows that the eigenvalues of the H_k $(k = 1, 2)$ satisfy the conditions either of Proposition 5.1.7 or of Proposition 5.1.9. It was proved in Ch. IV that the local classification problem is equivalent to that of the existence of a solution of some differential equation $E \subset J^1(M \times M)$; therefore, the theorem follows from Propositions 5.1.7 and 5.1.9.

5.1.11. THEOREM. *Suppose that we are given an equation* $E \subset J^1(M)$, codim $E = 1$, $\omega_x = 0$, *and that the eigenvalues of* H: $T_x(E) \to T_x(E)$ *satisfy* (4.2.9.1). *Then, for every Lagrangian subspace* $L_0 \subset T_x(E)$ *invariant under* H, *the equation has a solution tangent to* L_0.

5.1.12. REMARK. Theorem 5.1.11 merely establishes the existence of a solution. As for the question of uniqueness, it does not hold in the usual sense of the word. However, there is uniqueness of the following type. Let L_1 and L_2 be solutions of the equation $E \subset J^1(M)$, $\omega_x = 0$, satisfying (4.2.9.1). If L_1 and L_2 are tangent at the singular point $x \in E$, then they have contact of infinite order at that point.

5.1.13. We consider now singular points of involutory equations $E^r \subset J^1(M)$, where codim $E^r = r > 1$.

In this case the classification problem reduces, as it did above, to that of local solubility of an equation at a singular point. The equation obtained here is also involutory, and so in suitable coordinates it has the form

$$p_1 = 0, \ldots, p_{r-1} = 0, \qquad u = \mathscr{F}(q_r, \ldots, q_n, p_r, \ldots, p_n),$$

that is, it reduces, in fact, to the solubility of the equation $u = \mathscr{F}(q_r, \ldots, q_n, p_r, \ldots, p_n)$ at the singular point. Therefore, using Theorems 5.1.10 and 5.1.11, we obtain the following assertion.

THEOREM. a) *If the involutory equations* $E_k^r \subset J^1(M)$ *are such that the operators* H_k: $V \to V$ *are conformally-symplectically (symplectically) equivalent at a singular point, and their eigenvalues satisfy* (4.2.9.1), *then the equations* $E_k (k = 1, 2)$ *are locally equivalent* (U_1*-equivalent*) *at that point.*

b) *Let* E^r *be an involutory equation satisfying* (4.2.9.1) *at the singular point* x; *then for every Lagrangian subspace* $L_0 \subset T_x(E^r)$ *such that* $h_{\omega|L_0} = 0$, *there exists a many-valued solution of* E^r *passing through* x *and tangent to* L_0.

5.1.14. REMARK. If E^r is an involutory equation as above, then there exists a local contact diffeomorphism α, $\alpha(x) = x$, such that $\alpha(E^r)$ has the form $p_1 = 0, \ldots, p_{r-1} = 0$, $u = \mathscr{F}(q_r, \ldots, q_n, p_r, \ldots, p_n)$, where $\mathscr{F}(q_r, \ldots, q_n, p_r, \ldots, p_n) = \sum_{\lambda,j} \mathscr{F}_{\lambda,j}$ is a quadratic form, and the types of $\mathscr{F}_{\lambda,j}$ are as listed in 3.3.2.

§2. Local classification of even-dimensional Pfaffian forms in the *neighbourhood of a singular point*

5.2.1. Suppose that we are given 1-forms $\omega_k \in \Lambda^1(\mathbf{R}^{2n})$ $(k = 1, 2)$. We want to find out when these forms are equivalent at $0 \in \mathbf{R}^{2n}$. If they have constant rank r, $0 \leqslant r \leqslant n$, in some neighbourhood of $0 \in \mathbf{R}^{2n}$, if they are involutory, and if $\omega_{k,0} \neq 0$, then Darboux's theorem asserts that they are equivalent.

In particular, if $d\omega_k$ has maximal rank $2n$ at 0, then the conditions $\omega_k \neq 0$ turn out to be sufficient for the local equivalence of the ω_k. Furthermore, we can discard either the condition that the rank of $d\omega_k$ is constant in the neighbourhood of $0 \in \mathbf{R}^{2n}$, or the condition that $\omega_{k,0} \neq 0$. The first possibility was investigated by Martinet [13]. We consider the case when the $d\omega_k$ are forms of maximal rank and $\omega_{k,0} = 0$.

5.2.2. In this subsection we consider an arbitrary 1-form $\omega \in \Lambda^1(\mathbf{R}^{2n})$ that vanishes at $0 \in \mathbf{R}^{2n}$, $\omega|_0 = 0$.

Acting as in Ch. III, we define the Hessian h_ω of ω at 0 as follows: $h_\omega(X, Y) = \bar{X}(\omega(\bar{Y}))|_0$, where $X, Y \in T_0(\mathbf{R}^{2n})$, and $\bar{X}$, $\bar{Y}$ are any vector fields extending X and Y, $\bar{Y}_0 = Y$, $\bar{X}_0 = X$.

It follows from the formula $d\omega_0(X, Y) = X\omega(\bar{Y}) - Y\omega(\bar{X})$ that the bilinear form h_ω is well-defined and that $h_\omega(X, Y) - h_\omega(Y, X) = d\omega_0(X, Y)$. Thus, $h_\omega = \frac{1}{2}d\omega_0 + h^s_\omega$, where h^s_ω is a bilinear symmetric form.

If now $\omega_k \in \Lambda^1(\mathbf{R}^{2n})$ $(k = 1, 2)$ and $\omega_{k,0} = 0$, then for local equivalence of these forms at $0 \in \mathbf{R}^{2n}$ the equivalence of the Hessians h_{ω_k} under the full linear group $GL(2n, \mathbf{R})$ is necessary. Every linear transformation $A\colon T_0(\mathbf{R}^{2n}) \to T_0(\mathbf{R}^{2n})$ establishing the equivalence of h_{ω_1} and h_{ω_2} must take the forms $d\omega_{1,0}$ and $h^s_{\omega_1}$ to $d\omega_{2,0}$ and $h^s_{\omega_2}$, respectively; that is, for all $X, Y \in T_0(\mathbf{R}^{2n})$,

$$d\omega_{1,0}(AX, AY) = d\omega_{2,0}(X, Y), \; h^s_{\omega_1}(AX, AY) = h^s_{\omega_2}(X, Y).$$

5.2.3. We turn next to the case when $\omega_k \in \Lambda^1(\mathbf{R}^{2n})$, $\omega_{k,0} = 0$, and the $d\omega_k$ have the maximal rank $2n$ at $0 \in \mathbf{R}^{2n}$. Then we may assume that $d\omega_{1,0} = d\omega_{2,0}$. The necessary condition for the local equivalence of the forms ω_k at $0 \in \mathbf{R}^{2n}$ in the preceding subsection reduces to the following: the quadratic forms $h^s_{\omega_1}$ and $h^s_{\omega_2}$ must lie in the same orbit of $Sp(n)$. Thus, the orbits of $Sp(n)$ in the space of all quadratic forms on $T_0(\mathbf{R}^{2n}) \approx \mathbf{R}^{2n}$ are invariants of the local classification problem.

A description of these orbits based on the operators H_k, $d\omega_{k,0}(H_k, X, Y) = h_{\omega,k}(X, Y)$ was given in Ch. III.

5.2.4. Let ω_k be forms as above. We claim that they can be realized as $\omega_k = U_1|_{E_k}$ for certain equations $E_k \subset J^1(M)$, $M = \mathbf{R}^n$. We use Darboux's theorem and choose coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ in some neighbourhood of $0 \in \mathbf{R}^{2n}$ in which $d\omega_k = \sum_{i=1}^{n} dx_i \wedge dy_i$. In these coordinates ω_k becomes $\omega_k = d\mathcal{F}_k(x, y) - \sum_{i=1}^{n} y_i\, dx_i$. Therefore, E_k can be realized by the embedding $i_k\colon \mathbf{R}^{2n} \hookrightarrow J^1(\mathbf{R}^n)$, where

$$i_k^*(q_j) = x_j, \quad i_k^*(p_j) = y_j, \quad 1 \leqslant j \leqslant n, \quad i_k^*(u) = \mathcal{F}_k(x, y)$$

for some special coordinate system $q_1, \ldots, q_n, u, p_1, \ldots, p_n$ in $J^1(\mathbf{R}^n)$.

When we now use the results of the preceding section, we obtain the following result.

5.2.5. THEOREM. *A necessary condition for the local equivalence of forms $\omega_k \in \Lambda^1(\mathbf{R}^{2n})$, where $\omega_{k,0} = 0$ and $d\omega_{k,0}$ is of maximal rank, $k = 1, 2$, is that the quadratic forms $h^s_{\omega_k}$ (= the operators H_k) are $Sp(n)$-equivalent. This condition is sufficient if the eigenvalues $\{\lambda_j\}$ of the H_k satisfy the condition*

$$\sum m_j\lambda_j \neq 1 \qquad (5.2.5.1)$$

for all natural numbers m_j such that $\sum m_j \geqslant 3$.

COROLLARY. *Suppose that $\omega \in \Lambda^1(\mathbf{R}^{2n})$ reduces to the zero form at the point $0 \in \mathbf{R}^{2n}$, where $d\omega$ is a form of maximal rank and the eigenvalues $\{\lambda_j\}$ of H satisfy* (5.2.5.1); *then there is a local coordinate system $x_1, \ldots, x_n, y_1, \ldots, y_n$ in the neighbourhood of $0 \in \mathbf{R}^{2n}$ in which ω takes the form $\omega = d\mathcal{F} - \sum_{i=1}^{n} y_i\, dx_i$, where $\mathcal{F}$ is a quadratic form in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ u $\mathcal{F} = \sum_{\lambda, j} \mathcal{F}_{\lambda, j}$, and the types of the $\mathcal{F}_{\lambda, j}$ are as listed in Ch.* IV, §3.

EXAMPLE. Suppose that the eigenvalues of $H - \frac{1}{2}$ are purely imaginary and distinct. Then the conditions (5.2.5.1) are satisfied, so that, in some coordinate system $x_1, \ldots, x_n, y_1, \ldots, y_n$, ω takes the form

$$\omega = \sum_{j=1}^{n} \left(\mu_j\, d\,(x_j^2 + y_j^2) + \frac{1}{2}\,(x_j\, dy_j - y_j\, dx_j)\right),$$

where the μ_j are the imaginary parts of the eigenvalues of $H - \frac{1}{2}$.

5.2.6. Theorem 5.2.5 allows of a generalization analogous to Chern's theorem for vector fields. Namely, suppose that the form ω_1 can be written as $\omega_1 = d\mathcal{F}_1 - \sum_{i=1}^{n} y_i\, dx_i$ in some coordinate system $x_1, \ldots, x_n, y_1, \ldots, y_n$, where $\mathcal{F}_1$ is a quadratic form. If $\omega_2 - \omega_1 = d\varepsilon$, where ε is a flat function at the origin, we derive the following result from Propositions 5.1.8 and 5.1.9.

THEOREM. *Suppose that ω_1 and ω_2 are forms as above and that* Re $\lambda_j \neq 0$ *for the eigenvalues* $\{\lambda_j\}$ *of the operators* $H_k (k = 1, 2)$. *Then ω_1 and ω_2 are locally equivalent at* $0 \in \mathbf{R}^{2n}$.

5.2.7. REMARK. The results of this section go over in an obvious way to involutory 1-forms for which $\underbrace{d\omega \wedge \dots \wedge d\omega}_{k} \neq 0$ and $\omega \wedge \underbrace{d\omega \wedge \dots \wedge d\omega}_{k} \equiv 0$. In this case the operator H is induced by the Hessian h_ω on the factor space $T_0(\mathbf{R}^{2n})/C_0$, where C_0 is the degeneracy subspace of the 2-form $d\omega_0$, so that locally ω can be written as

$$\omega = d\mathcal{F}(x_1, \dots, x_k, y_1, \dots, y_k) - \sum_{i=1}^{k} y_i \, dx_i,$$

where $\mathcal{F}(x_1, \dots, x_k, y_1, \dots, y_k)$ is a quadratic form.

§3. Local classification of Hamiltonians in the neighbourhood of a singular point

So far we have mainly considered first order equations on M corresponding to smooth submanifolds of $J^1(M)$. In this section we consider the case when the equations are given by submanifolds with a singularity. Suppose, then, that $\mathcal{F} \in C^\infty(T^*M)$, and that $x \in T^*(M)$ is a non-degenerate singular point, $\mathcal{F}(x) = 0$, $d\mathcal{F}|_x = 0$. In this case the equation $E_{\mathcal{F}}$ has a singular line $(x, u) \subset J^1(M)$, at the points of which $E_{\mathcal{F}}$ fails to be smooth.

5.3.1. We consider two questions, which are obviously connected with one another.

1) Let H be the operator generated by the Hessian $h_{\mathcal{F}}$ of $\mathcal{F}$ relative to the canonical 2-form $d\rho$:

$$d\rho_x(HX, Y) = h_{\mathcal{F}}(X, Y) \quad X, Y \in T_x(T^*M),$$

and let $L_0 \subset T_x(T^*M)$ be a Lagrangian subspace invariant under H. Is there a Lagrangian submanifold $L \subset \{\mathcal{F} = 0\}$ tangent to L_0?

2) Let $\mathcal{F}_1, \mathcal{F}_2 \in C^\infty(T^*M)$ be functions for which $x \in T^*(M)$ is a non-degenerate singular point, $d\mathcal{F}_{k,x} = 0$, $\mathcal{F}_k(x) = 0$ $(k = 1, 2)$. Is there a local canonical diffeomorphism σ, $\sigma(x) = x$, such that $\sigma^*(\mathcal{F}_2) = \mathcal{F}_1$?

We solve these problems simultaneously, as we did for equations without singularities. It was proved above (Theorem 4.3.2) that to solve 2) it is not enough to give some finite jet of $\mathcal{F}_k$ at x. Therefore, we have to go into a study of "elementary" singular points.

5.3.2. DEFINITION. A point $x \in T^*(M)$ at which $\mathcal{F}(x) = 0$, $d\mathcal{F}_x = 0$, is said to be an *elementary singular point* for a smooth function $\mathcal{F}$ if Re $\lambda_s \neq 0$ for the eigenvalues $\{\lambda_s\}$ of H and $j_\infty(\mathcal{F})|_x$ is a polynomial in some system of local canonical coordinates.

5.3.3. THEOREM. *Let $x \in T^*(M)$ be an elementary singular point for $\mathcal{F}_1$ and suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are formally equivalent at x relative to the group of canonical diffeomorphisms; then $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent at x relative to the group of canonical diffeomorphisms.*

The proof of this theorem is analogous to that of Proposition 5.1.8.

5.3.4. COROLLARY. *Let $x \in T^*(M)$ be an elementary singular point for $\mathcal{F} \in C^\infty(T^*M)$ and suppose that $j_\infty(\mathcal{F})|_x$ is a quadratic form in some system of local canonical coordinates in a neighbourhood of x. Then, for every Lagrangian subspace $L_0 \subset T_x(T^*M)$ invariant under H, there is a Lagrangian submanifold $L \subset T^*(M)$ tangent to L_0 and such that $\mathcal{F}|_L = 0$.*

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