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SINGULARITIES OF SOLUTIONS, SPECTRAL SEQUENCES, AND NORMAL FORMS OF LIE ALGEBRAS OF VECTOR FIELDS

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ABSTRACT. A general scheme is presented for constructing solutions of systems of differential equations with a prescribed type of singularities. The scheme is then applied to the homological equation arising in the problem of classifying Lie algebras of vector fields in the neighborhood of a rest (or equilibrium) point. The formal, C^∞ and C^ω variants of the classification problem are discussed. Sufficiency conditions in the contact, symplectic, and general cases are given in terms of spectral sequences.

Bibliography: 18 titles.

There are two approaches to investigating the solubility of systems of differential equations within classes of functions with a given type of singularity: the algebraic—or formal—and the functional. In this paper, we present a general scheme for obtaining conditions for formal solubility. We give the general definition of the type of a singularity below; here we only remark that, in the case of the so-called μ-adic filtration associated to a singularity of the type "order of smallness at a point", our scheme (for regular systems of differential equations) comes in contact with the Spencer-Quillen-Goldschmidt theory of formal solubility [13]. And for singularities of the type "order of smallness on a submanifold of codimension 1" the method includes the classical theory of transfer operators.

Our approach is based on spectral sequences. There are two reasons for this: first, the solubility conditions have a homological nature, and, secondly, the construction of solutions is based on the method of successive approximations. We note that in the spectral sequences that arise, the term (E_0^{pq}, d_0^{pq}) corresponds to the symbolic part, the differential d_1^{pq} in the term (E_1^{pq}, d_1^{pq}) is the transfer operator, and the differentials d_r^{pq}, $r \geq 2$, which were not as a rule considered in the classical approach, are higher transfer operators which are most adequately defined using the machinery of spectral sequences.

From our point of view, the most interesting illustration of our approach (another illustration involving the computation of the stable cohomology of Spencer was presented in [10]) is in obtaining conditions for the formal solubility of the homological equation which arises in classification problems. In this case, spectral sequences not only give solubility conditions, but they also indicate the normal forms to which the classification problem under consideration can be reduced.

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V. I. Arnol′d [2] was the first to use spectral sequences in application to degenerate singular points of functions. Here, we apply spectral sequences to the determination of normal forms of Lie algebras of vector fields (contact, Hamiltonian, or general) in a neighborhood of a rest (or equilibrium) point.

A well-known theorem of Élie Cartan asserts that the (analytic or smooth) action of a compact Lie group in a neighborhood of a fixed point is equivalent to a linear action. In the case of semisimple Lie algebras, Hermann [15] proved that formal linearization is possible. Kushnirenko [4], as well as Guillemin and Sternberg [17], using Weyl's "unitary trick", proved that an analytic action of a semisimple Lie algebra is equivalent to a linear action. In this paper, we consider the case of an arbitrary Lie algebra and the cases when the Lie algebra is realized as an algebra of contact or Hamiltonian fields. The triviality of the first term of the corresponding spectral sequence is analogous to the absence of resonances—the Poincaré conditions—while for a one-dimensional Lie algebra it is analogous to coincidence with these conditions.

For reductive, semisimple and commutative Lie algebras we obtain conditions under which formal equivalence implies C^∞ or C^ω equivalence. Here, a basis role is played by the presence in the algebra of a vector field for which the fixed point is a node ($\operatorname{Re}\lambda > 0$). We note that this coincides with the condition that the Gel′fand-Fuks cohomology be finite-dimensional [14]. This is not by chance. Finite-dimensionality of a Lie algebra $\mathfrak{G}$ represented by vector fields is not essential for consideration of formal questions. In particular, applying the spectral sequences of §2 to representations of filtered algebras on Lie pseudogroups and using the Gel′fand-Fuks cohomology instead of the finite-dimensional Lie algebra cohomology, we obtain analogous results about normal forms of intransitive Lie pseudogroups.

The main results of this paper were announced in the notes [8] and [9].

§1. Singularities

Consider a smooth manifold M with $\dim M = n + m$ and a smooth submanifold $M_0 \subset M$ with $\operatorname{codim} M_0 = m$. Let $\mathscr{F}$ (respectively, $\mathscr{F}_0$) be the **R**-algebra of smooth functions on M (on M_0) and $\mu \subset \mathscr{F}$ the ideal corresponding to M_0:

$$\mu = \{f \in \mathscr{F} \,\big|\, f|_{M_0} = 0\}.$$

In what follows, we shall identify $\mathscr{F}_0$ and $\mathscr{F}/\mu$. More generally, let $\alpha\colon E(\alpha) \to M$ be a smooth vector bundle, and $A = \Gamma(\alpha)$ the module (over $\mathscr{F}$) of smooth sections of α. Then $A_0 = A/\mu A$ can be identified with the $\mathscr{F}_0$-module of smooth sections of the restriction of α to M_0.

If $\beta\colon E(\beta) \to M$ is another vector bundle, then $\operatorname{Diff}_k(A, B)$, where $B = \Gamma(\beta)$, will denote the $\mathscr{F}$-module of differential operators which have order less than or equal to k and which operate from sections of the bundle α to the sections of the bundle β.

1.1. We consider $C^\infty(M \backslash M_0)$ as an $\mathscr{F}$-module and choose a submodule Q together with a decreasing filtration by $\mathscr{F}$-modules:

$$\cdots Q_j \supset Q_{j+1} \supset \cdots, \quad j \in \mathbf{Z}, \qquad Q_{-\infty} = \bigcup_{j\in\mathbf{Z}} Q_j = Q, \qquad Q_\infty = \bigcap_{j\in\mathbf{Z}} Q_j \subset \mathscr{F}.$$

DEFINITION. a) We say that submanifold $M_0 \subset M$ has a *singularity of type* Q if an $\mathscr{F}$-module $Q \subset C^\infty(M\backslash M_0)$ is fixed together with a decreasing filtration $\{Q_j\}$ satisfying the following conditions:

1) *Differential stability*: $\Delta(Q_j) \subset Q_{j-k}$ for all differential operators $\Delta \in \operatorname{Diff}_k(\mathscr{F}, \mathscr{F})$, and $\Delta(Q_j) \subset Q_{j-k+1}$ if $\Delta(\mathscr{F}) \subset \mu$.

2) *Differential completeness*: $Q_{j-k} = \bigcup_{\Delta \in \mathrm{Diff}_k(\mathscr{F},\mathscr{F})} \Delta(Q_j)$ for all $j \in \mathbf{Z}$.

3) *Localness*: if a function $f \in C^\infty(M \backslash M_0)$ is such that for each $x \in M$ there exist a neighborhood $\mathscr{O}_x \ni x$ and a function $q_j \in Q_j$ such that $f \equiv q_j$ in $\mathscr{O}_x \backslash M_0$, then $f \in Q_j$.

b) A function $f \in C^\infty(M \backslash M_0)$ has *a singularity of type Q and order j* if $f \in Q_j$, but $f \notin Q_{j+1}$.

For the vector bundle α we get $Q_j A = Q_j \otimes_{\mathscr{F}} A$. The elements of $Q_j A$ will be considered as sections of α having a singularity of type Q on M and order greater than or equal to j.

EXAMPLE 1. The μ-adic filtration: $Q_j = \mu^j$ for $j \geq 1$ and $Q_j = \mathscr{F}$ for $j \leq 0$.

EXAMPLE 2. Let $M_0 = \{0\} \subset \mathbf{R}^m$, and let $f_\sigma(x_1, \dots, x_m)$, $\sigma = (\sigma_1, \dots, \sigma_m)$, $\sigma_i \in \mathbf{Z}$, be a collection of smooth functions on $\mathbf{R}^m \backslash 0$ such that $\partial f_\sigma / \partial x_i = f_{\sigma - 1_i}$ and $f_\sigma \cdot \varepsilon \in C^\infty(\mathbf{R}^m)$ for all functions ε which are flat at the origin. Let Q be the module generated by functions of the form $x_1^{\tau_1} \cdots x_m^{\tau_m} \cdot f_\sigma(x)$, where $\tau_i \geq 0$, and let Q_j be the submodule generated by those $x^\tau f_\sigma(x)$ for which $|\tau| - |\sigma| \leq j$.

EXAMPLE 3. The preceding example admits the following generalization. Suppose that $M_0 \subset M$ is the submanifold cut out by functions $h_1, \dots, h_m \in C^\infty(M)$, $M_0 = h_1^{-1}(0) \cap \cdots \cap h_m^{-1}(0)$, whose differentials are independent of M_0, and let f_σ be the above collection. Then the module Q generated by $h^\tau f_\sigma(h_1, \dots, h_m)$, $\tau = (\tau_1, \dots, \tau_m)$, with filtration Q_j, $|\tau| - |\sigma| \leq j$, determines a singularity on M_0.

EXAMPLE 4 (a homogeneous filtration). To each variable $x_1, \dots, x_m$ we associate a degree: $k_1, \dots, k_m$. We set $Q = \mathscr{F} = C^\infty(\mathbf{R}^m)$ and $Q_j = Q$ $(j \leq 0)$, and let Q_j $(j \geq 1)$ be the functions of total degree of homogeneity greater than or equal to j. Then Q determines a singularity on $M_0 = \{0\} \subset \mathbf{R}^m$.

REMARK 1. Note that by applying property 1) to the scalar differential operators that are multiplication by functions $f \in \mu$, we obtain $\mu Q_j \subset Q_{j+1}$ for each singularity Q.

REMARK 2. All the preceding examples carry over to a trivial vector bundle α if the elements of $Q_j A$ are understood to be vector-valued functions on $M \backslash M_0$ whose components all lie in Q_j.

1.2. A filtration $\{Q_j\}$ determines an additional filtration on the $\mathscr{F}$-module of differential operators

$$\mathrm{Diff}_*(A, B) = \bigcup_{k \geq 0} \mathrm{Diff}_k(A, B).$$

For this we define the $\mathscr{F}$-module $\mathrm{Diff}_r^Q(A, B)$ of *differential operators of order less than or equal to r with respect to a singularity of type Q* to be the set of differential operators $\Delta \in \mathrm{Diff}_*(A, B)$ such that

1) $\Delta(Q_j) \subset Q_{j-r}$, and

2) $\mathrm{ad}_{f_1} \circ \cdots \circ \mathrm{ad}_{f_k}(\Delta)(Q_j A) \subset Q_{j-r+k}(B)$ for all $f_1, \dots, f_k \in \mathscr{F}$ and $j \in \mathbf{Z}$. Here $\mathrm{ad}_f(\Delta)(a) = \Delta(fa) - f\Delta(a)$ is the commutator of Δ and the operator of multiplication by f.

The inclusions

$$\mathrm{Diff}_r^Q(A, B) \subset \mathrm{Diff}_{r+1}^Q(A, B) \quad \text{and} \quad \mathrm{Diff}_r(A, B) \subset \mathrm{Diff}_r^Q(A, B)$$

are obvious, so that

$$\mathrm{Diff}_*(A, B) = \bigcup_{j \in \mathbf{Z}} \mathrm{Diff}_j^Q(A, B),$$

and $\{\mathrm{Diff}_r^Q(A, B)\}$ defines an increasing filtration in $\mathrm{Diff}_*(A, B)$.

EXAMPLE. Consider the vector field $\nabla = \sum_1^m \nabla_i(x)\partial_i$ on $\mathbf{R}^m$, where $\partial_i = \partial/\partial x_i$ and $\nabla_i(0) = 0$. If Q is a singularity on the submanifold $M_0 = \{0\} \subset \mathbf{R}^m$ and $\nabla_i(x) \notin \mu^2$ for at least one value i, $1 \le i \le m$, then $\nabla \in \operatorname{Diff}_0^Q(\mathcal{F}, \mathcal{F})$. Otherwise, $\nabla \in \operatorname{Diff}_1^Q(\mathcal{F}, \mathcal{F})$.

1.3. The symbolic algebra and the Weyl algebras. Let

$$S_*^Q(A, B) = \sum_{r \in \mathbf{Z}} S_r^Q(A, B)$$

be the graded module associated to the filtration $\{\operatorname{Diff}_r^Q(A, B)\}$; that is,

$$S_r^Q(A, B) = \operatorname{Diff}_r^Q(A, B)/\operatorname{Diff}_{r-1}^Q(A, B).$$

We put $S_*^Q(\mathcal{F}, \mathcal{F}) = S_*(Q)$ and $S_r^Q(\mathcal{F}, \mathcal{F}) = S_r(Q)$. Since $\mu \times \operatorname{Diff}_r^Q(A, B) \subset \operatorname{Diff}_{r-1}^Q(A, B)$, each homogeneous component $S_r^Q(A, B)$ (and, together with it, the entire module $S_*^Q(A, B)$) is an $\mathcal{F}_0$-module.

In the case when $A = B$, composition of differential operators defines the additional structure of a (noncommutative) $\mathcal{F}_0$-algebra on $S_*^Q(A, A)$. We call the $\mathcal{F}_0$-modules $S_*^Q(A, B)$ *symbolic modules* and the $\mathcal{F}_0$-algebra $S_*(Q)$ the *symbolic algebra of the singularity* Q.

EXAMPLE. Let $M_0 = \{0\} \subset \mathbf{R}^m$, and let $Q_j = \mu^j$ ($j \ge 0$) be the μ-adic filtration. Then $S_*(Q)$ can be identified with the algebra of polynomial differential operators $\sum_{\sigma,\tau} a_{\tau,\sigma} x^\sigma \partial^\tau$, where $\sigma = (\sigma_1, \dots, \sigma_m)$ and $\tau = (\tau_1, \dots, \tau_m)$ are multi-indices, and $S_r(Q)$ can be identified with the set of operators of this form for which $|\tau| - |\sigma| = r$.

With this example in mind, we shall call the symbolic algebra of a μ-adic filtration the *infinitesimal Weyl algebra of the ideal* μ (of the submanifold M_0), and denote it by $W(\mu)$.

We give another description of this algebra. Consider the $\mathcal{F}_0$-module $E(\mu) = \nu_0 \oplus \nu_0^*$, where $\nu_0 = \mu/\mu^2$ is the module of sections of the conormal bundle and $\nu_0^* = \operatorname{Hom}_{\mathcal{F}_0}(\nu_0, \mathcal{F}_0)$ is the module of sections of the normal bundle of the submanifold M_0. We define a symplectic structure on $E(\mu)$ using the skew-symmetric two-form $\Omega_\mu \in \Lambda^2(E(\mu)^*)$, where

$$\Omega_\mu(X, Y) = \begin{cases} Y(X), & Y \in \nu_0^*,\ X \in \nu_0, \\ 0 & \text{if either } X, Y \in \nu_0 \text{ or } X, Y \in \nu_0^*. \end{cases}$$

Let $w(\mu)$ be the quotient of the tensor algebra $T(\mu)$ of the module $E(\mu)$ by the ideal generated by

$$X \otimes Y - Y \otimes X - \Omega_\mu(X, Y) \cdot 1, \qquad X, Y \in E(\mu).$$

The following holds.

PROPOSITION. *The infinitesimal Weyl algebra* $W(\mu)$ *is isomorphic to* $w(\mu) \otimes_{\mathcal{F}_0} \operatorname{Diff}_*(\mathcal{F}_0, \mathcal{F}_0)$.

PROOF. Choose a local coordinate system $(x_1, \dots, x_n, y_1, \dots, y_m)$ on a neighborhood $\mathcal{O}$ of the point $a \in M_0$ such that $M_0 \cap \mathcal{O}$ is given by the equations $y_1 = \dots = y_m = 0$. The operator

$$\Delta = \sum a_{\sigma_1,\sigma_2}(x, y)\partial_x^{\sigma_1}\partial_y^{\sigma_2}$$

has order equal to the maximum of the sums $\sigma_2 + \operatorname{ord} a_{\sigma_1,\sigma_2}$, where $\operatorname{ord} a_{\sigma_1,\sigma_2}$ is the order of smallness of the coefficient a_{σ_1,σ_2} on the submanifold M_0. It remains to observe that the elements $y^\tau \partial_y^\sigma$ are elements of $w(\mu)$ and $a_\sigma(x)\partial_x^\sigma \in \operatorname{Diff}_*(\mathcal{F}_0, \mathcal{F}_0)$.

1.4. Consider the graded module $R_*(A) = \sum_{j \in \mathbf{Z}} R_j(A)$ associated to the filtration $Q_j(A)$: that is, $R_j(A) = Q_j A/Q_{j+1} A$. We set $R_j = Q_j/Q_{j+1}$ and $R_* = \sum_{j \in \mathbf{Z}} R_j$.

It follows from the conditions defining the filtration $\{Q_j\}$ that $\mu R_j(A) = 0$. Hence, $R_j(A)$ can be considered as an $\mathscr{F}_0$-module. We also remark that the natural map $R_j \otimes_{\mathscr{F}_0} A_0 \to R_j(A)$ which sends $q_j \bmod Q_{j+1} \otimes a$ and $\bmod \mu A$ to the element $q_j a \bmod Q_{j+1}A$ establishes an isomorphism of $\mathscr{F}_0$-modules.

Let $\Delta \in \mathrm{Diff}_k^Q(A, B)$. We define the jth *symbol of* Δ *with respect to the singularity* Q to be the map

$$\sigma_j^Q(\Delta)\colon R_j(A) \to R_{j-k}(B),$$

where $\sigma_j^Q(\Delta)$ $(a_j \bmod Q_{j+1}A) = \Delta(a_j) \bmod Q_{j-k+1}B$, and we define the *complete symbol* to be the map $\sigma_*^Q(\Delta) = \sum_j \sigma_j^Q(\Delta)\colon R_*(A) \to R_*(B)$. From the conditions on the filtration $\{Q_j\}$ it follows that the symbols $\sigma_j^Q(\Delta)$, $j \in \mathbf{Z}$, are $\mathscr{F}_0$-morphisms.

1.5. Transfer operators. Consider the following problem: find a solution of the equation $\Delta(a) = 0$, $\Delta \in \mathrm{Diff}_*(A, B)$, having a singularity type Q on a submanifold M_0. In other words, find $a_j \in Q_j A$ such that $\Delta(a_j) = 0$. We first consider the cruder problem in which we merely require that $\Delta a_j \in Q_\infty B \subset B$. The natural way to try to solve the latter problem is as follows. Choose an arbitrary element $a_j \in Q_j A$. Then $\Delta a_j \in Q_{j-k}B$ if $\Delta \in \mathrm{Diff}_k^Q(A, B)$. We determine what kind of conditions must be imposed on a_j for $\Delta(a_j)$ to be a "smoother" section; that is, $\Delta(a_j) \in Q_{j-k+1}B$. Using the symbol of the differential operator Δ with respect to the singularity Q, we can describe this condition succinctly as

$$\sigma_j^Q(\Delta)([a_j]) = 0, \quad \text{where } [a_j] = a_j \mod Q_{j+1}A \in R_j(A).$$

Suppose $[a_j] \in \ker \sigma_j^Q(\Delta)$, and consider the conditions under which it is possible to obtain an even "smoother" right-hand side without changing $[a_j]$. For this, it is clearly necessary that $\Delta(a_j + a_{j+1}) \in Q_{j-k+2}B$ for some choice of $a_{j+1} \in Q_{j+1}A$. In other words, it is necessary that $\Delta(a_j) \bmod Q_{j-k+2}B \in \operatorname{im} \sigma_{j+1}^Q(\Delta)$, and so forth.

We formalize this process. To do so, we introduce the $\mathscr{F}_0$-modules

$$\mathscr{K}_j(\Delta) = \ker \sigma_j^Q(\Delta), \qquad \mathscr{C}_j(\Delta) = \operatorname{coker} \sigma_j^Q(\Delta)$$

and consider the operators $\Delta_j^1\colon \mathscr{K}_j(\Delta) \to \mathscr{C}_{j+1}(\Delta)$, where

$$\Delta_j^1([a_j]) = [\Delta(a_j)] \mod \operatorname{im} \sigma_{j+1}^Q(\Delta).$$

THEOREM. *The operators* Δ_j^1, $j \in \mathbf{Z}$, *are differential operators of order less than or equal to* 1 *over* $\mathscr{F}_0$.

PROOF. It is sufficient to verify that $\mathrm{ad}_{f_1} \circ \mathrm{ad}_{f_2}(\Delta_j^1) = 0$ for all $f_1, f_2 \in \mathscr{F}_0$. Let $\bar{f}_1, \bar{f}_2 \in \mathscr{F}$ be extensions of f_1 and f_2. Then

$$\mathrm{ad}_{f_1}(\Delta_j^1)([a_j]) = [\mathrm{ad}_{\bar{f}_1}(\Delta)(a_j)] \mod \operatorname{im} \sigma_{j+1}^Q(\Delta)$$

and

$$\mathrm{ad}_{f_2} \circ \mathrm{ad}_{f_1}(\Delta_j^1)([a_j]) = [\mathrm{ad}_{\bar{f}_2} \circ \mathrm{ad}_{\bar{f}_1}(\Delta)(a_j)] \mod \operatorname{im} \sigma_{j+1}^Q(\Delta),$$

but $\mathrm{ad}_{\bar{f}_2} \circ \mathrm{ad}_{\bar{f}_1}(\Delta)(a_j) \in Q_{j-k+2}B$ and, therefore,

$$[\mathrm{ad}_{\bar{f}_2} \circ \mathrm{ad}_{\bar{f}_1}(\Delta)(a_j)] = 0.$$

1.6. Spectral sequences. In this section, we consider the case when the operators are differentials of degree +1 in a graded complex $\mathscr{A} = \sum A^j$:

$$0 \to A^0 \xrightarrow{\Delta} A^1 \xrightarrow{\Delta} \cdots \xrightarrow{\Delta} A^s \xrightarrow{\Delta} A^{s+1} \to \cdots \xrightarrow{\Delta} A^N \to 0.$$

This case is important, first of all, because considering the Spencer complex associated with the differential operator allows us to obtain more significant (in comparison with §1.5) information about the transfer operator in overdetermined problems. On the other hand, considering complexes of the form $0 \to A^0 = A \xrightarrow{\Delta} A^1 = B \to 0$ allows us to more naturally introduce the higher transfer operators and thereby give a complete solution to the problem formulated in the preceding subsection.

Using the singularity Q, we introduce a filtration on the complex $Q\mathscr{A}$ by setting

$$F_p(A^j) = Q_{p-kj}A^j$$

subject to the condition that $\Delta \in \mathrm{Diff}_k^Q(\mathscr{A}, \mathscr{A})$. The differential Δ is compatible with this filtration. Consider the spectral sequence

$$E_r^{pq} = Z_r^{pq}/Z_{r-1}^{p+1,q-1} + B_{r-1}^{pq},$$

where $Z_r^{pq} = \{a \in F_p(A^{p+q}), \Delta(a) \in F_{p+r}(A^{p+q+1})\}$ and $B_r^{pq} = \Delta(Z_r^{p-r,q+r-1})$. We note that elements in Z_∞^{pq} can be interpreted as formal (that is, considered up to flat ones) cycles in A^{p+q} having a singularity of type Q on M_0. Thus, for a complex of the form $0 \to A \xrightarrow{\Delta} B \to 0$, the elements of $E_\infty^{p,-p}$ are solutions of the problem formulated in §1.5.

We describe the initial terms of the spectral sequence: $E_0^{p,q} = F_p(A^{p+q})/F_{p+1}(A^{p+q})$, and therefore E_0^{pq} is isomorphic to $R_{p(1-k)-qk} \otimes A_0^{p+q}$ and the differential d_0^{pq} coincides with the symbol $\sigma^Q_{p(1-k)-pk}(\Delta)$. Consequently, E_1^{pq} coincides with the cohomology of the symbolic complex

$$\cdots \to R_{p(1-k)-(q-1)k} \otimes A_0^{p+q-1} \xrightarrow{\sigma^Q(\Delta)} R_{p(1-k)-qk} \otimes A_0^{p+q} \xrightarrow{\sigma^Q(\Delta)} R_{p(1-k)+(q+1)k} \otimes A_0^{p+q+1} \to \cdots$$

at the term $R_{p(1-k)-qk} \otimes A_0^{p+q}$.

In particular, since the maps $\sigma_j^Q(\Delta)$ are $\mathscr{F}_0$-homomorphisms, the terms E_1^{pq} are $\mathscr{F}_0$-modules. We remark that for the complex $0 \to A^0 = A \xrightarrow{\Delta} A^1 = B \to 0$ we have $E_1^{pq} = 0$ if $p+q \neq 0, 1$, and $E_1^{p,-p} = \mathscr{K}_p(\Delta)$, while $E_1^{p,1-p} = \mathscr{C}_p(\Delta)$.

THEOREM. *In the spectral sequence (E_r^{pq}, d_r^{pq}), the terms E_1^{pq} are $\mathscr{F}_0$-modules, and the operators $d_1^{pq}\colon E_1^{pq} \to E_1^{p+1,q}$ are differential operators of order less than or equal to 1.*

PROOF. As in the proof of Theorem 1.5, it suffices to verify that $\mathrm{ad}_{f_1} \circ \mathrm{ad}_{f_2}(d_1^{pq})([a_j]) = 0$ for all $f_1, f_2 \in \mathscr{F}_0$ and $j = p(1-k) - qk$, where $[a_j]$ is the image of a_j in E_1^{pq}. But

$$\mathrm{ad}_{f_1} \circ \mathrm{ad}_{f_2}(d_1^{pq})([a_j]) = \mathrm{ad}_{\bar f_1} \circ \mathrm{ad}_{\bar f_2}(\Delta)([a_j]) = 0,$$

because $\mathrm{ad}_{\bar f_1} \circ \mathrm{ad}_{\bar f_2}(\Delta)(a_j) \in F_{p+2}A^{p+q+1}$.

1.7. As a first example of an application of the spectral sequences we have constructed, consider the problem of the existence of first integrals of dynamical systems in a neighborhood of a rest point.

Let $\nabla = \sum_1^m \nabla_i(x)\partial_i$ be a vector field on $\mathbf{R}^m$ for which $0 \in \mathbf{R}^m$ is an equilibrium point; that is, $\nabla_i(0) = 0$ for $1 \le i \le m$. Consider the μ-adic filtration: $Q_j = \mu^j$ $(j \ge 1)$ and $Q_j = \mathscr{F}$ $(j \le 0)$, and the spectral sequence (E_r^{pq}, d_r^{pq}) of the cohomology of the complex $0 \to A_0 = \mathscr{F} \xrightarrow{\nabla} A_1 = \mathscr{F} \to 0$ with respect to this filtration.

We assume that the linear part of ∇ is not identically zero; that is, $\nabla \in \mathrm{Diff}_0^\mu(\mathscr{F}, \mathscr{F})$. In the given case, $R_j = S^jT^*$, where T^* is the cotangent space to $\mathbf{R}^m$ at the point

$0 \in \mathbf{R}^m$ and $\sigma_j^\mu(\nabla)\colon S^jT^* \to S^jT^*$ is the operator of taking the derivative along the linear part of ∇; here $\sigma_1^\mu(\nabla)\colon T^* \to T^*$ is the operator dual to the linear part. It is well known that the spectrum of the operator $\sigma_j^\mu(\nabla)$ coincides with $\sum_1^m n_i\lambda_i$ where $\sum_1^m n_i = j$, $n_1, \dots, n_m$ are natural numbers, and $\lambda_1, \dots, \lambda_m$ is the spectrum of the linear part of ∇. Consequently, $E_1^{p,-p} = 0$ and $E_1^{p,1-p} = 0$ if $\sum_1^m n_i\lambda_i \neq 0$ for all natural numbers n_i such that $\sum_1^m n_i = p$: the terms $E_1^{p,-p} = \ker \sigma_p^\mu(\nabla)$ and $E_1^{p,1-p} = \operatorname{coker} \sigma_p^\mu(\nabla)$ can be nontrivial only if there exist resonances $\sum_1^m n_i\lambda_i = 0$, $\sum n_i = p$. Note also that $k_* = \sum_{p\geq 0} E_1^{p,-p} \subset S^*T^* = \sum_{p\geq 0} S^pT^*$ is a subalgebra of the algebra of homogeneous polynomials on $T = T_0(\mathbf{R}^m)$ and $C_* = \sum_{p\geq 0} E_1^{p,1-p}$ is a module over this subalgebra. Moreover, if the linear part of ∇ is semisimple, then C_* is isomorphic to k_*. The differential $d_1^* = \sum_{p\geq 0} d_1^{p,-p}\colon k_* \to C_*$ is, first, an $\mathbf{R}$-linear operator and, second, a derivation: $d_1^*(ab) = ad_1^*(b) + bd_1^*(a)$. In the semisimple case, d_1^* is a derivation (of degree 1) of the algebra k_*. If $d_1^{p,-p} = \cdots = d_s^{p,-p} = 0$, then

$$E_1^{p,-p} = \cdots = E_{s+1}^{p,p}; \qquad E_1^{p,1-p} = \cdots = E_{s+1}^{p,1-p}$$

and, as above, the differential $d_{s+1}^* = \sum_{p\geq 0} d_{s+1}^{p,-p}\colon k_* \to C_*$ is a derivation.

Thus, the geometrical image which we can associate to the first nontrivial term of the spectral sequence consists in an algebraic manifold P corresponding to algebra k_*, a vector bundle $\mathscr{C}$ over P corresponding to the module C_*, and a derivation d_{s+1}^* with values in $\mathscr{C}$. In the semisimple case, this corresponds to a vector field d_{s+1}^* on P. The next term of the spectral sequence coincides, respectively, with the kernel and cokernel of this vector field (or derivation), and each successive term is obtained by passing to the kernel and cokernel of the appropriate derivation $d_r\colon k_r \to C_r$ determined by the $(r+1)$-jet of the vector field ∇. Here we have the following possibilities: 1) $d_r \equiv 0$, and then $C_{r+1} = C_r$ and $k_{r+1} = k_r$; 2) $d_r \neq 0$, and the action of the generators of k_r on C_r is not nilpotent; or 3) $d_r \neq 0$, but the generators of k_r act on C_r in a nilpotent (or trivial) manner.

Suppose that all the algebras k_r are finitely generated. Then in situation 2) the dimension of k_{r+1} is less than the dimension of k_r, and therefore the spectral sequence stabilizes after a finite number of nontrivial steps (under the condition that each step corresponds to 2)) and $k_\infty = \mathbf{R}$. This corresponds to the situation that all first integrals h of the field ∇, $h(0) = 0$, are functions flat at the origin. Note, too, that here the stabilization conditions for the spectral sequence are determined by a finite jet of the field ∇.

The case in which 3) occurs is the main reason for the lack of finite determinacy in the problem under consideration. In fact, passing to the next term of the spectral sequence kills the finite-dimensional part of k_r and C_r, and leaves us, as before, in situation 3). Therefore, the stabilization cannot be verified in a finite number of steps.

We go into more detail on the low-dimensional cases $m = 2, 3$. Suppose $m = 2$ and let λ_1, λ_2 be the spectrum of the linear part of ∇. Then the term $\sum_{p\geq 0} E_1^{p,-p}$ is nontrivial if either 1) $\lambda_1 = \lambda_2 = 0$ or (after multiplying ∇ by a nonzero number, if necessary) 2) $\lambda_1 = 1$ and $\lambda_2 = -a/b$, where a and b are natural numbers. We first consider case 2) when $\lambda_2 \neq 0$. If $(a,b) = 1$, then $E_1^{p,-p} \simeq E_1^{p,1-p}$ is different from zero if p is a multiple of $(a+b)$ and $k_1 = C_1 = \mathbf{R}[\theta]$, where $\theta = x_1^a x_2^b$ and $x_1, x_2 \in T^*$ is an eigenbasis of $\sigma_1^\mu(\nabla)$. The first nontrivial differential d_r is possible when $r = a+b$ and $d_r(\theta) = \gamma_1\theta^2$, so that $d_r = \gamma_1\theta^2\partial/\partial\theta$. If $\gamma_1 \neq 0$, which is determined by the $(a+b)+1$-jet of ∇, then $E_r^{p,-p} = 0$ for $r > a+b$ and $p > 0$, and $E_r^{p,1-p} = 0$ for $r > a+b$ and $p > a+b$.

If $d_{a+b} = 0$, then the next nontrivial differential is possible when $r = 2(a+b)$ and $d_r = \gamma_2\theta^3\partial/\partial\theta$, and so forth.

The explicit expression, for example for γ_1, takes a simple form in the coordinates t_1, t_2, $d_0 t_1 = x_1$, $d_0 t_2 = x_2$ in which ∇ has Poincaré-Dulac normal form:

$$\gamma_1 = \frac{a}{(a+1)!b!} \cdot \frac{\partial^{a+b+1}\nabla_1}{\partial t_1^{a+1}\partial t_2^b}(0) + \frac{b}{a!(b+1)!} \cdot \frac{\partial^{a+b+1}\nabla_2}{\partial t_1^a \partial t_2^{b+1}}(0).$$

If $\lambda_1 = 1$ and $\lambda_2 = 0$, then $E_1^{p,-p} = E_1^{p,1-p} = \mathbf{R}x_2^p$ and the first nontrivial differential d_r could occur when $r = 1$, $d_1 = \gamma_1 x_2^2\partial/\partial x_2$; the rest is similar to the case $a \neq 0$.

If $\lambda_1 = \lambda_2 = 0$ and $\sigma_1^\mu(\nabla)$ is nilpotent: $\sigma_1^\mu(\nabla)(x_1) = x_2$ and $\sigma_1^\mu(\nabla)(x_2) = 0$, then $E_1^{p,-p} = \mathbf{R}x_2^p$, $E_1^{p,1-p} = \mathbf{R}x_1^p$, and the action of k_1 on C_1 is trivial. Thus, we are in situation 3).

Finally, if $\sigma_1^\mu(\nabla) = 0$, then $E_1^{p,-p} = E_1^{p,1-p} = S^pT^*$ and $d_1 = P_1(x_1,x_2)\partial/\partial x_1 + P_2(x_1,x_2)\partial/\partial x_2$, where P_1 and P_2 are quadrics on T. If d_1 does not have nontrivial homogeneous first integrals (this can be determined from the 2-jet of ∇), then the spectral sequence stabilizes at the second step.

Let $m = 3$; we restrict ourselves to the case when $\sigma_1^\mu(\nabla)$ is a semisimple operator with spectrum $\lambda_1 = 1$, $\lambda_2 = -a_1/b_1$, $\lambda_3 = a_2/b_2$, and $k_1 = \sum_{p\geq 0} E_1^{p,-p}$ is a free algebra. In this case, the first nontrivial differential is a homogeneous vector field on the plane. Let $(b_1, b_2) = c$, and suppose $(b_i, a_i) = 1$ for $i = 1, 2$. Then it follows from the resonance condition $n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 = 0$ that $n_2 = k_1\beta_1$ and $n_3 = k_2\beta_2$, where $b_1 = c\beta_1$, $b_2 = c\beta_2$, and k_1 and k_2 are natural numbers such that $k_1a_1 - k_2a_2 = 0 \mod c$ and $n_1c = k_1a_1 - k_2a_2$. Choose vectors e_1 and e_2 on the (k_1,k_2)-plane such that $e_1 = (c,0)$ and e_2 is the least vector with positive integral coordinates lying on the line $k_1a_1 - k_2a_2 = 0$. Let $e_i = (k_{1i}, k_{2i})$, $i \geq 3$, be vectors, with positive integral coordinates, lying inside the parallelogram with sides e_1 and e_2 and such that $k_{1i}a_1 - k_{2i}a_2 = 0 \mod c$. To each vector e_i there corresponds a homogeneous polynomial $y_i = x_1^{n_{1i}}x_2^{n_{2i}}x_3^{n_{3i}}$, where $n_{1i}c = k_{1i}a_1 - k_{2i}a_2$, $n_{2i} = k_{1i}\beta_1$, and $n_{3i} = k_{2i}\beta_2$.

The polynomials $y_1, y_2, \ldots$ generate k_1, and the relation $y_i^{s_i} - y_1^{u_i}y_2^{v_i} = 0$ holds for each $i \geq 3$. Consequently, k_1 is a free algebra only in the case when the vectors e_i, $i \geq 3$, are absent. The latter is possible only in the case when $a_1 = 1$. Then $e_1 = (c, 0)$, $e_2 = (a_2, 1)$, $y_1 = x_1x_2^{b_1}$, $y_2 = x_2^{a_2\beta_1}x_3^{\beta_2}$, and $c_1 = k_1 = \mathbf{R}[y_1, y_2]$.

We shall show that it is possible to choose λ_2 and λ_3 so that k_{r+1} acts trivially on C_{r+1}. The first nontrivial differential d_r is a homogeneous vector field on the plane (y_1, y_2), and therefore k_{r+1} will act trivially on C_{r+1} if, for example, $d_r(y_1) = y_2$ and $d_r(y_2) = 0$ or $d_r(y_2) = y_1$ and $d_r(y_1) = 0$. This situation (by reason of the dimension) always occurs if $\deg y_1 < \deg y_2 < \frac{3}{2}\deg y_1$ or $\deg y_2 < \deg y_1 < \frac{3}{2}\deg y_2$.

In view of the form of y_2, the second case is possible only if $\beta_2 = 1$. Finally, we find that if either

$$\left(a_2 - \tfrac{3}{2}c\right)\beta_1 + \beta_2 < \tfrac{3}{2}, \qquad c < a_2 \leq \tfrac{3}{2}c, \tag{1}$$

or

$$\left(\tfrac{3}{2}a_2 - c\right)\beta_1 + \tfrac{1}{2} > 0, \qquad a_2 \leq \tfrac{2}{3}c,\ b_2 = c, \tag{2}$$

holds, then the differential d_r has the form $d_r = \gamma y_2\partial/\partial y_1$ or $d_r = \gamma y_1\partial/\partial y_2$, where $r = (a_2 - c)\beta_1 + \beta_2 - 1$ in case 1) and $r = (1 - \lambda_3)/b_1$ in case 2). In both cases, situation 3) occurs for almost all $(r+1)$-jets of vector fields with a given spectrum.

§2. Spectral sequences and normal forms of Lie algebras of vector fields

In this section we apply the method discussed above for constructing approximate solutions of systems of differential equations to find normal forms of Lie algebras of vector fields. The corresponding system of differential equations is the homological equation. Here, as a rule, the appearance of singularities in the classification problem results in the appearance of singularities in the corresponding homological equation.

In this connection, we only use the μ-adic filtration and we take $M_0 = *$; in fact, all the results carry over word for word to the case of a homogeneous filtration and an arbitrary invariant submanifold M_0.

2.1. Let $D = D(M)$ be the Lie algebra (over $\mathbf{R}$) of vector fields on the manifold M, $M_0 = a \subset M$ a fixed point, and $\mu \subset C^\infty(M)$ the maximal ideal corresponding to this point. Suppose that we are given a representation of a finite-dimensional Lie algebra $\mathfrak{G}$ over $\mathbf{R}$ in the Lie algebra D, $\rho\colon \mathfrak{G} \to D$, for which $a \in M$ is a fixed point, $\rho(\mathfrak{G}) \subset \mu D$. We say that two such representations ρ_1 and ρ_2 are *locally equivalent* if there exists a local diffeomorphism $A\colon M \to M$, $A(a) = a$, such that $A_*(\rho_1(\nabla)) = \rho_2(\nabla)$ for all $\nabla \in \mathfrak{G}$.

2.2. Let $D_a^k = \mu D/\mu^{k+1}D$ be the Lie algebra of k-jets at a of vector fields on M which vanish at a, and let $j_k\colon \mu D \to D_a^k$ denote the natural projection, $1 \leq k \leq \infty$, which gives the k-jet at the point a. The Lie algebra D_a^∞ will also be denoted by D_*.

Let $\rho^{(k)}\colon \mathfrak{G} \to D_a^k$, $\rho^{(k)} = j_k \circ \rho$, be the reduction of the representation ρ to the level of k-jets.

DEFINITION 1. We shall say that the representation ρ_1 and ρ_2 are *k-equivalent* if there exists a local diffeomorphism A for which $(\rho_1 \circ A_*)^{(k)} = \rho_2^{(k)}$. When $k = \infty$, the representations will be said to be *formally equivalent*.

DEFINITION 2. A representation ρ will be called *k-sufficient* if any representation ρ' which is k-equivalent to ρ is equivalent to ρ. If the k-equivalence of the representations ρ and ρ' implies formal equivalence, then the representation ρ will be said to be *k-sufficient in the formal sense.*

DEFINITION 3. A representation ρ is said to be *sufficient* if it is k-sufficient for some k. Otherwise, the representation will be said to be *wild.*

2.3. With a view to obtaining algebraic conditions for formal equivalence of representations of $\mathfrak{G}$ and to finding formal normal forms of such representations, we consider the complex constructed from the representation ρ of the Lie algebra $\mathfrak{G}$:

$$0 \to D_* \xrightarrow{d} \mathfrak{G}^* \underset{\mathbf{R}}{\otimes} D_* \xrightarrow{d} \Lambda^2\mathfrak{G}^* \underset{\mathbf{R}}{\otimes} D_* \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^m\mathfrak{G}^* \underset{\mathbf{R}}{\otimes} D_* \to 0, \tag{1}$$

where $m = \dim_{\mathbf{R}} \mathfrak{G}$ and the differential

$$d\colon \Lambda^s\mathfrak{G}^* \underset{\mathbf{R}}{\otimes} D_* \to \Lambda^{s+1}\mathfrak{G}^* \underset{\mathbf{R}}{\otimes} D_* \tag{2}$$

acts according to the formula

$$\begin{aligned} d\omega(\nabla_1, \dots, \nabla_{s+1}) = &\sum_i (-1)^{i+1}[\rho(\nabla_i), \omega(\nabla_1, \dots, \hat{\nabla}_i, \dots, \nabla_{s+1})] \\ &+ \sum_{i<j} (-1)^{i+j}\omega([\nabla_i, \nabla_j], \nabla_1, \dots, \hat{\nabla}_i, \dots, \hat{\nabla}_j, \dots, \nabla_{s+1}), \end{aligned} \tag{3}$$

in which the elements $\omega \in \Lambda^s\mathfrak{G}^* \otimes_{\mathbf{R}} D_*$ are considered as skew-symmetric forms on the Lie algebra $\mathfrak{G}$ taking values in the Lie algebra of formal vector fields D_*; here, as usual, the caret over an element indicates that the element is omitted, and $\nabla_1, \dots, \nabla_{s+1} \in \mathfrak{G}$.

We postpone the motivation for considering this complex to §2.7. Here, instead, we shall indicate some properties of the operator d. Define the bracket $[\omega_1 \wedge \omega_2] \in \Lambda^{s+1}\mathfrak{G}^* \otimes D_*$ of $\omega_1 \in \Lambda^s\mathfrak{G}^* \otimes D_*$ and $\hat{\omega}_2 \in \Lambda^t\mathfrak{G}^* \otimes D_*$ (see [5]) so that for decomposable elements of the form $\omega_1 = \alpha \otimes X$, $\omega_2 = \beta \otimes Y$, where $\alpha \in \Lambda^s\mathfrak{G}^*$, $\beta \in \Lambda^t\mathfrak{G}^*$, and $X, Y \in D_*$, we have

$$[\omega_1 \wedge \omega_2] = \alpha \wedge \beta \otimes [X, Y]. \tag{4}$$

This bracket defines an $\mathbf{R}$-linear pairing and is such that the following commutation relations hold:

$$[\omega_1 \wedge \omega_2] = (-1)^{st+1}[\omega_2 \wedge \omega_1]. \tag{5}$$

The Jacobi identity in the Lie algebra D_* implies that if $\omega_3 \in \Lambda^k\mathfrak{G}^* \otimes D_*$, then

$$(-1)^{sk}[\omega_1 \wedge [\omega_2 \wedge \omega_3]] + (-1)^{st}[\omega_2 \wedge [\omega_3 \wedge \omega_1]] + (-1)^{tk}[\omega_3 \wedge [\omega_1 \wedge \omega_2]] = 0, \tag{6}$$

and (see [5])

$$d[\omega_1 \wedge \omega_2] = [d\omega_1 \wedge \omega_2] + (-1)^s[\omega_1 \wedge d\omega_2]. \tag{7}$$

2.4. Introduce a filtration on the complex (1) by setting

$$F_{p,q} = \Lambda^{p+q}\mathfrak{G}^* \underset{\mathbf{R}}{\otimes} \mu^{p-1}D_*. \tag{1}$$

In other words, the terms of filtration p are the forms on the algebra $\mathfrak{G}$ taking values in vector fields of order of smallness p. It follows from the condition $\operatorname{im}\rho \subset D_*$ that the differential d is compatible with the filtration $dF_{p,q} \subset F_{p,q+1}$. We set

$$Z_r^{pq} = \{\omega \in F_{p,q};\ d\omega \in F_{p+r,q-r+1}\},$$

$$B_r^{pq} = \{\omega \in F_{p,q};\ \exists\theta \in F_{p-r,q+r-1},\ \omega = d\theta\},$$

where Z_r^{pq} is the set of cycles of order r and B_r^{pq} the set of boundaries of order r, $B_r^{pq} = dZ_r^{p-r,q+r-1}$. The spectral sequence (E_r^{pq}, d_r^{pq}) of the cohomology of the complex (1) constructed with respect to the filtration (1) has the form:

$$E_r^{pq} = Z_r^{pq}/Z_r^{p+1,q-1} + B_{r-1}^{pq},$$

and the differentials $d_r^{pq}\colon E_r^{pq} \to E_r^{p+r,q-r+1}$ are generated by d by passing to quotients.

In the given case $p \geq 0$ and $0 \leq p+q \leq m$; therefore the Dynkin table of this spectral sequence has the form of the band between the lines $p+q=0$ and $p+q=m$, $p \geq 0$.

2.5. PROPOSITION. *The following inclusions hold:*

$$[Z_r^{pq} \wedge Z_r^{p'q'}] \subset Z_r^{p+p'-1,q+q'+1},$$

$$[B_{r-1}^{pq} \wedge Z_r^{p'q'}] \subset B_{r-1}^{p+p'-1,q+q'+1} + Z_{r-1}^{p+p',q+q'}.$$

PROOF. We shall prove the first inclusion. Let $\varepsilon_1 \in Z_r^{pq}$ and $\varepsilon_2 \in Z_r^{p'q'}$. Then, in view of (4) in §2.3,

$$[\varepsilon_1 \wedge \varepsilon_2] \in \Lambda^{p+p'+q+q'}\mathfrak{G}^* \otimes \mu^{p+p'-2}D_*$$

and

$$d[\varepsilon_1 \wedge \varepsilon_2] = [d\varepsilon_1 \wedge \varepsilon_2] + (-1)^{p'+q'}[\varepsilon_1 \wedge d\varepsilon_2].$$

But $d\varepsilon_1 \in \Lambda^{p+q+1}\mathfrak{G}^* \otimes \mu^{p+r-1}D_*$, and so

$$[d\varepsilon_1 \wedge \varepsilon_2] \in \Lambda^{p+q+p'+q'+1}\mathfrak{G}^* \otimes \mu^{p+p'+r-2}D_*.$$

Similarly,

$$[\varepsilon_1 \wedge d\varepsilon_2] \in \Lambda^{p+q+p'+q'+1}\mathfrak{G}^* \otimes \mu^{p+p'+r-2}D_*.$$

Consequently,

$$[\varepsilon_1 \wedge \varepsilon_2] \in F_{p+p'-1,q+q'+1} \quad \text{and} \quad d[\varepsilon_1 \wedge \varepsilon_2] \in F_{p+p'+r-1,q+q'-r+2},$$

whence $[\varepsilon_1 \wedge \varepsilon_2] \in Z_r^{p+p'-1,q+q'+1}$.

Now, let $\varepsilon_1 \in B_{r-1}^{pq}$ and $\varepsilon_2 \in Z_r^{p'q'}$, and let $\theta \in \Lambda^{p+q-1}\mathfrak{G}^* \otimes \mu^{p-r}D_*$ be such that $\varepsilon_1 = d\theta$. We have

$$[\varepsilon_1 \wedge \varepsilon_2] = d[\theta \wedge \varepsilon_2] + (-1)^{p+q}[\theta \wedge d\varepsilon_2].$$

But $[\theta \wedge \varepsilon_2] \in \Lambda^{p+q+p'+q'-1}\mathfrak{G}^* \otimes \mu^{p+p'-r-1}D_*$, and therefore

$$d[\theta \wedge \varepsilon_2] \in B_{r-1}^{p+p'-1,q+q'+1}.$$

Furthermore, $[\theta \wedge d\varepsilon_2] \in \Lambda^{p+q+p'+q'}\mathfrak{G}^* \otimes \mu^{p+p'-1}D_*$. In addition,

$$d[\theta \wedge d\varepsilon_2] = [d\theta \wedge d\varepsilon_2] \in \Lambda^{p+q+p'+q'+1}\mathfrak{G}^* \otimes \mu^{p+p'+r-2}D_*,$$

and therefore $[\theta \wedge d\varepsilon_2] \in Z_{r-1}^{p+p',q+q'}$.

COROLLARY. *$[Z_r^{pq} \wedge Z_s^{p'q'}] \subset Z_s^{p+p'-1,q+q'+1}$ for $r > s$.*

In fact, $Z_r^{pq} \subset Z_s^{pq}$ for $r > s$. Using the proof of the proposition, we define a pairing

$$E_r^{pq} \times E_r^{p'q'} \to E_r^{p+p'-1,q+q'+1}, \tag{1}$$

by setting

$$[x_r^{pq} \wedge y_r^{p'q'}] = [x \wedge y] \mod (Z_{r-1}^{p+p',q+q'} + B_{r-1}^{p+p'-1,q+q'+1}) \tag{2}$$

on the cosets

$$x_r^{pq} = x \mod (Z_{r-1}^{p+1,q-1} + B_{r-1}^{pq}), \qquad y_r^{p'q'} = y \mod (Z_{r-1}^{p'+1,q'-1} + B_{r-1}^{p',q'}),$$

where $x \in Z_r^{pq}$ and $y \in Z_r^{p'q'}$.

From §2.3 we find that this bracket satisfies the relations

$$[x_r^{pq} \wedge y_r^{p'q'}] = (-1)^{(p+q)(p'+q'+1)}[y_r^{p'q'} \wedge r_r^{pq}], \tag{3}$$

$$\begin{aligned}(-1)^{sk}[x_r^{pq} \wedge [y_r^{p'q'} \wedge z_r^{p''q''}]] + (-1)^{st}[y_r^{p'q'} \wedge [z_r^{p''q''} \wedge x_r^{pq}]]\\ + (-1)^{tk}[z_r^{p''q''} \wedge [x_r^{pq} \wedge y_r^{p'q'}]] = 0,\end{aligned} \tag{4}$$

where $s = p+q$, $t = p'+q'$, and $k = p''+q''$, and

$$d_r[x_r^{pq} \wedge y_r^{p'q'}] = [d_r x_r^{pq} \wedge y_r^{p'q'}] + (-1)^{p+q}[x_r^{pq} \wedge d_r y_r^{p'q'}]. \tag{5}$$

2.6. We describe the initial terms of the spectral sequence. We have

$$E_0^{pq} = F_{p,q}/F_{p+1,q-1} = \Lambda^{p+q}\mathfrak{G}^* \underset{\mathbf{R}}{\otimes} S^pT^* \underset{\mathbf{R}}{\otimes} T, \tag{1}$$

where T (respectively, T^*) is the tangent (cotangent) space to M at the point $a \in M$; in (1) we used the isomorphism

$$\mu^pD/\mu^{p+1}D \simeq S^pT^* \otimes T.$$

Since $T^* \otimes T = \operatorname{End}_{\mathbf{R}} T$, the reduction $\rho^{(1)}$ of the representation ρ defines a representation of the algebra $\mathfrak{G}$ on the tangent space T; $\rho^{(1)}\colon \mathfrak{G} \to \operatorname{End}_{\mathbf{R}} T$. The representations $S^p(\rho^{(1)})^* \otimes \rho^{(1)}$ thereby define a $\mathfrak{G}$-module structure on all the spaces $S^pT^* \otimes T$, and formula (3) in §2.3 shows that the complexes

$$0 \to S^pT^* \otimes T \xrightarrow{d_0^{p,0}} \mathfrak{G}^* \otimes S^pT^* \otimes T \to \cdots \to \Lambda^m\mathfrak{G}^* \otimes S^pT^* \otimes T \to 0 \tag{2}$$

are Koszul complexes for the cohomology of the Lie algebra $\mathfrak{G}$ with values in the $\mathfrak{G}$-module $S^pT^* \otimes T$. Therefore,

$$E_1^{pq} = H^{p+q}(\mathfrak{G}, S^pT^* \otimes T). \tag{3}$$

In particular, for $p + q = 0$, we have

$$E_1^{p,p} = (S^pT^* \otimes T)^{\#},$$

where if $L^{\#}$ is a $\mathfrak{G}$-module, $L^{\#} = \{x \in L | \nabla(x) = 0, \ \forall\nabla \in \mathfrak{G}\}$ denotes the subspace of $\mathfrak{G}$-invariant elements of L. We note that here $E_1^{1,-1} = \mathrm{End}_{\mathfrak{G}} T$ is the module of $\mathfrak{G}$-homomorphisms of T.

THEOREM. *The filtration $F_{p,q}$ defines a spectral sequence (E_r^{pq}, d_r^{pq}) in which*
1) $E_0^{pq} = \Lambda^{p+q}\mathfrak{G}^* \otimes S^pT^* \otimes T$,
2) $E_1^{pq} = H^{p+q}(\mathfrak{G}, S^pT^* \otimes T)$,
3) *the terms E_r^{pq} carry the structure of a bigraded Lie algebra; that is, there is a bilinear pairing $E_r^{pq} \times E_r^{p'q'} \to E_r^{p+p'-1,q+q'+1}$ on $\bigoplus_{p,q} E_r^{pq}$ satisfying the relations* §2.5 *and the differentials d_r are derivations of this algebra; and*
4) *the spectral sequence stabilizes in the following sense: for each pair of numbers (p, q) there exists a number $r_0 = r_0(p, q)$ such that*

$$E_{r_0}^{pq} = E_{r_0+1}^{pq} = \cdots = E_{\infty}^{pq}, \quad d_r^{pq} = 0, \qquad r \geq r_0,$$

and the stable terms of the spectral sequence are thereby isomorphic to the terms of the graded module associated to the cohomology of the complex (3) *in* §2.3.

It remains to prove the last assertion. To do this, note that there is a finite-dimensional space in each cell (p, q) in which a subspace is then distinguished with successive factorization; in this connection the dimension is not increased.

REMARK. If we put $L_r^0 = \sum_{p+q=0 \mod 2} E_r^{pq}$ and $L_r^1 = \sum_{p+q=1 \mod 2} E_r^{pq}$, then $E_r = L_r^0 \oplus L_r^1$ is a *Lie superalgebra.*

2.7. The use of spectral sequences to obtain normal forms of representations is based on the following remarks. A representation $\rho\colon \mathfrak{G} \to D_*$ determines an element of $\mathfrak{G}^* \otimes D_*$, denoted as above by ρ, which satisfies the Maurer-Cartan equation

$$d\rho - \tfrac{1}{2}[\rho \wedge \rho] = 0. \tag{1}$$

But if the form $\rho + \varepsilon$, where $\varepsilon \in \mathfrak{G}^* \otimes D_*$, also determines a representation of the algebra $\mathfrak{G}$, then

$$[(\rho + \varepsilon)(\nabla_1), (\rho + \varepsilon)(\nabla_2)] = (\rho + \varepsilon)([\nabla_1, \nabla_2])$$

or

$$[\rho(\nabla_1), \varepsilon(\nabla_2)] - [\rho(\nabla_2), \varepsilon(\nabla_1)] - \varepsilon([\nabla_1, \nabla_2]) + [\varepsilon(\nabla_1), \varepsilon(\nabla_2)] = 0,$$

for all $\nabla_1, \nabla_2 \in \mathfrak{G}$. It follows from this that

$$d\varepsilon + \tfrac{1}{2}[\varepsilon \wedge \varepsilon] = 0. \tag{2}$$

Therefore, if $\varepsilon \in F_{p,1-p}$, we have $[\varepsilon \wedge \varepsilon] \in F_{2p-1,3-2p}$ and, consequently, $\varepsilon \in Z_r^{p,1-p}$ for all $r \leq p - 1$. The second remark is connected with the following observation. Let $X \in D_*$ be a formal vector field. Let $A_t = \exp(t \operatorname{ad} X)$ denote the one-parameter group of formal diffeomorphisms corresponding to X. Let $A_{t^*}(\omega) \in \mathfrak{G}^* \otimes D_*$ be the image of the form $\omega \in \mathfrak{G}^* \otimes D_*$, $A_{t^*}(\omega)(\nabla) = A_{t^*}(\omega(\nabla))$, where $A_{t^*}(Y) = (A_t^*)^{-1} \circ Y \circ A_t^*$ is the image of the vector field $Y \in D_*$ under the action of A_t. We have

$$A_{t^*}(\rho)(\nabla) = \rho(\nabla) + t[\rho(\nabla), X] + \tfrac{1}{2}t^2[[\rho(\nabla), X], X] + \cdots,$$

which leads to the following proposition.

PROPOSITION. *If $X \in Z_r^{p,-p}$, then the formal diffeomorphism $A = \exp(\operatorname{ad} X)$ generated by X is such that*

$$A_*(\rho) - \rho - dX \in F_{2p+r-1,2-2p-r}.$$

2.8. THEOREM (the normal form of the pth approximation). *Suppose that $v_1, \dots, v_s$, $v_i \in \mathfrak{G}^* \otimes \mu^p D$, are such that their images generate $E_{p-1}^{p,1-p}$ $(p \geq 2)$. Then for each representation of the Lie algebra $\mathfrak{G}$ of the form $\rho + \varepsilon$, where $\varepsilon \in \mathfrak{G}^* \otimes \mu^p D$ $(p \geq 2)$, there exists a local diffeomorphism A such that*

$$A_*(\rho) - \rho - \varepsilon = \sum_1^s c_i v_i \mod F_{p+1,-p},$$

where $c_i \in \mathbf{R}$, $1 \leq i \leq s$.

PROOF. It suffices to prove that if $\varepsilon \in Z_{p-2}^{p+1,-p} + B_{p-2}^{p,1-p}$, then there is a formal diffeomorphism A such that $A_*(\rho) - \rho - \varepsilon \in F_{p+1,-p}$. Let $\varepsilon = y + dX$, where $y \in Z_{p-2}^{p+1,-p}$ and $X \in F_{2,-2}$. Then it suffices to prove that $A_*(\rho) - \rho - dX \in F_{p+1,p}$ for some diffeomorphism A. It remains to remark that, in view of Proposition 2.7, A can be taken to be $\exp(\operatorname{ad} X)$; since $X \in Z_{p-2}^{2,-2}$, we have $A_*(\rho) - \rho - dX \in F_{3p-3,4-3p} \subset F_{p+1,-p}$ for $p \geq 2$.

REMARK 1. If $\eta \in F_{p,1-p}$ and $X \in F_{2,-2}$, then $A_*(\eta) - \eta \in F_{p+1,-p}$ and A_* thereby induces identity transformations on $E_{p-1}^{p,1-p}$. Therefore, Theorem 2.8 can be reformulated as follows: there exists a local diffeomorphism A for which $A_*(\rho + \varepsilon) - \rho = \sum c_i v_i$ mod $F_{p+1,-p}$.

REMARK 2. The image $[\varepsilon]_{p-1}$ of the element $\sum c_i v_i$ in $E_{p-1}^{p,1-p}$ is not arbitrary; in view of (2) in §2.7, it must satisfy the equation

$$d_{p-1}[\varepsilon]_{p-1} + \tfrac{1}{2}[[\varepsilon]_{p-1} \wedge [\varepsilon]_{p-1}] = 0.$$

2.9. Successive application of Theorem 2.8 leads to the following result.

THEOREM. *If the elements $v_1, \dots, v_s, \dots, v_i \in \mathfrak{G}^* \otimes \mu^{p_i} D$, are such that their images generate all of $E_{p-1}^{p,1-p}$, $p \geq 2$, then for each representation of the Lie algebra $\mathfrak{G}$ of the form $\rho + \varepsilon$, where $\varepsilon \in \mathfrak{G}^* \otimes \mu^p D$, $p \geq 2$, there exists a local diffeomorphism A such that*

$$A_*(\rho) - \rho - \varepsilon = \sum c_i v_i \mod \mathfrak{G}^* \otimes \mu^\infty D.$$

COROLLARY. *In order that the representation ρ be formally sufficient, it is sufficient that $E_{p-1}^{p,1-p} = 0$ for all p starting with some number $p_0 \geq 2$. In this case the representation $\rho^{(\infty)}$ is determined by $\rho^{(p_0)}$.*

2.10. DEFINITION. We call $\varepsilon_{p-1} \in E_{p-1}^{p,1-p}$ a *Maurer-Cartan element* if there exists a representative $\varepsilon \in Z_{p-1}^{p,1-p}$ of the class ε_{p-1} for which $d\varepsilon + \frac{1}{2}[\varepsilon \wedge \varepsilon] = 0$.

We shall obtain a condition which singles out a Maurer-Cartan element. For $\varepsilon_{p-1} \in E_{p-1}^{p,1-p}$ to be a Maurer-Cartan element, it is necessary that

$$\varphi_0(\varepsilon_{p-1}) = d_{p-1}\varepsilon_{p-1} + \tfrac{1}{2}[\varepsilon_{p-1} \wedge \varepsilon_{p-1}] = 0 \in E_{p-1}^{2p-1,3-2p}. \tag{1}$$

Furthermore, suppose that condition (1) holds. Then for an arbitrary representative $\varepsilon \in Z_{p-1}^{p,1-p}$ we have

$$d\varepsilon + \tfrac{1}{2}[\varepsilon \wedge \varepsilon] = x + dy,$$

where $x \in Z^{2p,2-2p}_{p-2}$ and $y \in Z^{p+1,-p}_{p-2}$. Replacing ε by an $\varepsilon - y$ that is not an element of the class ε_{p-1} and taking advantage of §2.5, we find that the representative $\varepsilon \in Z^{p,1-p}_{p-1}$ can be chosen so that

$$d\varepsilon + \tfrac{1}{2}[\varepsilon \wedge \varepsilon] = x \in Z^{2p,2-2p}_{p-2}.$$

In fact, $x \in Z^{2p,2-2p}_{p-1}$, because

$$dx = [d\varepsilon \wedge \varepsilon] = [x \wedge \varepsilon] - \tfrac{1}{2}[\varepsilon \wedge [\varepsilon \wedge \varepsilon]] = [x \wedge \varepsilon].$$

Here we have used the fact that $[\varepsilon \wedge [\varepsilon \wedge \varepsilon]] = 0$ (which follows from the Jacobi identity). Therefore, $dx = [x \wedge \varepsilon] \in F_{3p-1,4-3p}$ and, consequently, $x \in Z^{2p,2-2p}_{p-1}$. Let $\bar{x} \in E^{2p,2-2p}_{p-1}$ be the image of the element x. We find out to what extent the element ε_{p-1} is uniquely determined. Let $\varepsilon + \eta$ where $\eta = a + db$ for $a \in Z^{p+1,-p}_{p-2}$, and let $b \in Z^{2,-2}_{p-2}$ be another representative of ε_{p-1} such that

$$d(\varepsilon + \eta) + \tfrac{1}{2}[(\varepsilon + \eta) \wedge (\varepsilon + \eta)] \in Z^{2p,2-2p}_{p-2}. \tag{2}$$

Expanding the brackets and using the fact that $d\varepsilon + \frac{1}{2}[\varepsilon \wedge \varepsilon] \in Z^{2p,2-2p}_{p-2}$, we find that the element

$$d(a + \tfrac{1}{2}[b \wedge db] - [\varepsilon \wedge b]) + [\varepsilon \wedge a] + [d\varepsilon \wedge b] + \tfrac{1}{2}[a \wedge a] + [a \wedge db]$$

lies in $Z^{2p,2-2p}_{p-2}$. But, in view of §2.5,

$$\begin{aligned}
[\varepsilon \wedge a] &\in [Z^{p,1-p}_{p-1} \wedge Z^{p+1,-p}_{p-2}] \subset Z^{2p,2-2p}_{p-2},\\
[d\varepsilon \wedge b] &\in [Z^{2p-1,3-2p}_{p-1} \wedge Z^{2,-2}_{p-2}] \subset Z^{2p,2-2p}_{p-2},\\
[a \wedge a] &\in [Z^{p+1,-p}_{p-2} \wedge Z^{p+1,-p}_{p-2}] \subset Z^{2p+1,1-2p}_{p-2},\\
[a \wedge db] &\in [Z^{p+1,-p}_{p-2} \wedge Z^{p,1-p}_{p-2}] \subset Z^{2p,2-2p}_{p-2}.
\end{aligned}$$

On the other hand, by the same considerations

$$\theta = a + \tfrac{1}{2}[b \wedge db] - [\varepsilon \wedge b] \in Z^{p+1,-p}_{p-2}.$$

Therefore, condition (2) means that $d\theta \in Z^{2p,2-2p}_{p-2}$ or, equivalently,

$$\theta \in Z^{p+1,-p}_{p-1}. \tag{3}$$

Representing $a = \theta + [\varepsilon \wedge b] - \frac{1}{2}[b \wedge db]$, we find that the element $x \in Z^{2p,2-2p}_{p-2}$ translates to $x + \lambda$, where

$$\begin{aligned}
\lambda = {} & d\theta + [\varepsilon \wedge \theta] + [d\varepsilon \wedge b] + [\varepsilon \wedge [\varepsilon \wedge b]] + [\theta \wedge db]\\
& - \tfrac{1}{2}[\varepsilon \wedge [b \wedge db]] - \tfrac{1}{2}[[b \wedge db] \wedge db]\\
& + [[\varepsilon \wedge b] \wedge db] + \tfrac{1}{2}[a \wedge a].
\end{aligned}$$

It follows from the Jacobi identity (6) in §2.3 that $[\varepsilon \wedge [\varepsilon \wedge b]] = \frac{1}{2}[[\varepsilon \wedge \varepsilon] \wedge b]$ and, therefore,

$$[d\varepsilon \wedge b] + [\varepsilon \wedge [\varepsilon \wedge b]] = [x \wedge b] \in Z^{2p+1,1-2p}_{p-2}.$$

Furthermore,

$$[\theta \wedge db] \in [Z^{p+1,-p}_{p-1} \wedge B^{p,1-p}_{p-2}] \in Z^{2p+1,1-2p}_{p-2} + B^{2p,2-2p}_{p-2}.$$

In addition,

$$d[[b \wedge db] \wedge b] = [[db \wedge db] \wedge b] - [[b \wedge db] \wedge db]. \tag{4}$$

Using the Jacobi identity, we obtain

$$-[db \wedge [b \wedge db]] + [b \wedge [db \wedge db]] + [db \wedge [db \wedge b]] = 0,$$

and so $2[db \wedge [b \wedge db]] = [b \wedge [db \wedge db]]$. Putting the latter equality in (4), we finally obtain

$$[[b \wedge db] \wedge db] = -\tfrac{1}{3}\, d\gamma,$$

where $\gamma = [[b \wedge db] \wedge b]$. Therefore, $[[b \wedge db] \wedge db] \in B_{p-2}^{2p+1,1-2p}$. Furthermore, suppose that $\omega = [[\varepsilon \wedge b] \wedge b]$. Then

$$d\omega = [[d\varepsilon \wedge b] \wedge b] - [[\varepsilon \wedge db] \wedge b] - [[\varepsilon \wedge b] \wedge db].$$

Exploiting the Jacobi identity, we get

$$-[\varepsilon \wedge [b \wedge db]] + [b \wedge [db \wedge \varepsilon]] + [db \wedge [\varepsilon \wedge b]] = 0.$$

From this we have

$$\begin{aligned}\tfrac{1}{2}[\varepsilon \wedge [db \wedge b]] - [[b \wedge \varepsilon] \wedge db] &= -\tfrac{1}{2}[b \wedge [\varepsilon \wedge db]] - \tfrac{1}{2}[db \wedge [b \wedge \varepsilon]] \\ &= -\tfrac{1}{2}\, d\omega + \tfrac{1}{2}[[d\varepsilon \wedge b] \wedge b].\end{aligned}$$

But $\omega \in Z_{p-2}^{p+2,-1-p}$ and, therefore, $d\omega \in B_{p-2}^{2p,2-2p}$; and, in addition, $[[d\varepsilon \wedge b] \wedge b] \in Z_{p-2}^{2p+1,1-2p}$. So, finally, the class $\bar{x}$ gets carried to

$$\bar{x} + d_{p-1}\bar{\theta} + [\varepsilon_{p-1} \wedge \bar{\theta}], \tag{5}$$

where $\bar{\theta} \in E_{p-1}^{p+1,-p}$ is the equivalence class of the element θ. We define the operator

$$L(\varepsilon_{p-1})\colon E_{p-1}^{p+r,-p-r+q} \to E_{p-1}^{2p+r-1,-2p-r+q+2}$$

by the equality

$$L(\varepsilon_{p-1})(\theta) = d_{p-1}\theta + [\varepsilon_{p-1} \wedge \theta]. \tag{6}$$

It follows from the Jacobi identity that the 1-form ε_{p-1} satisfies the relation

$$[\varepsilon_{p-1} \wedge [\varepsilon_{p-1} \wedge \gamma]] = \tfrac{1}{2}[[\varepsilon_{p-1} \wedge \varepsilon_{p-1}] \wedge \gamma].$$

Therefore,

$$\begin{aligned}(L(\varepsilon_{p-1}))^2(\theta) &= d_{p-1}[\varepsilon_{p-1} \wedge \theta] + [\varepsilon_{p-1} \wedge d_{p-1}\theta] + [\varepsilon_{p-1} \wedge [\varepsilon_{p-1} \wedge \theta]] \\ &= [d_{p-1}\varepsilon_{p-1} \wedge \theta] + \tfrac{1}{2}[[\varepsilon_{p-1} \wedge \varepsilon_{p-1}] \wedge \theta] = 0,\end{aligned}$$

so that $L(\varepsilon_{p-1})$ under the condition $d_{p-1}\varepsilon_{p-1} + \frac{1}{2}[\varepsilon_{p-1} \wedge \varepsilon_{p-1}] = 0$ determines the complex at the term E_{p-1}. We let $H^{s,t}(\varepsilon_{p-1})$ denote the cohomology of this complex at the term $E_{p-1}^{s,t}$.

We now return to relation (5), which shows that the class $\bar{x}$ modulo $\operatorname{im} L(\varepsilon_{p-1})$ is well-defined by the element ε_{p-1}. On the other hand,

$$L(\varepsilon_{p-1})(\bar{x}) = (dx + [\varepsilon \wedge x]) \mod (Z_{p-2}^{3p,3-3p} + B_{p-2}^{3p-1,4-3p}),$$

but, since $x = d\varepsilon + \frac{1}{2}[\varepsilon \wedge \varepsilon]$,

$$dx + [\varepsilon \wedge x] = [d\varepsilon \wedge \varepsilon] + [\varepsilon \wedge d\varepsilon] + \tfrac{1}{2}[\varepsilon \wedge [\varepsilon \wedge \varepsilon]] = 0.$$

Consequently, $\bar{x}$ is a cycle with respect to the differential $L(\varepsilon_{p-1})$, and the cohomology class

$$\varphi_1(\varepsilon_{-p-1}) = \bar{x} \mod \operatorname{im} L(\varepsilon_{p-1}) \in H^{2p,2-2p}(\varepsilon_{p-1}) \tag{7}$$

is the first obstruction to ε_{p-1} being a Maurer-Cartan element.

Furthermore, if this obstruction is trivial, then we can find a representative $\varepsilon \in Z_{p-1}^{p,1-p}$ of the class ε_{p-1} for which

$$d\varepsilon + \tfrac{1}{2}[\varepsilon \wedge \varepsilon] \in Z_{p-2}^{2p+1,1-2p} + B_{p-2}^{2p,2-2p}.$$

Let $d\varepsilon + \frac{1}{2}[\varepsilon \wedge \varepsilon] = x + dy$, where $x \in Z_{p-2}^{2p+1,1-2p}$ and $y \in Z_{p-2}^{p+2,-1-p}$. Upon replacing ε by $\varepsilon - y$, we can assume that

$$d\varepsilon + \tfrac{1}{2}[\varepsilon \wedge \varepsilon] = x \in Z_{p-2}^{2p+1,1-2p}. \tag{8}$$

We now use the following assertion, whose proof completely repeats the construction of the first obstruction $\varphi_1(\varepsilon_{p-1})$.

LEMMA. *Let $\varepsilon \in Z_{p-1}^{p,1-p}$ be a representative of the class $\varepsilon_{p-1} \in E_{p-1}^{p,1-p}$ such that*

$$d\varepsilon + \tfrac{1}{2}[\varepsilon \wedge \varepsilon] = x \in Z_{p-1}^{2p+r,2-2p-r}. \tag{9}$$

Then $x \in Z_{p-1}^{2p+r,2-2p-r}$, and the class $\bar{x} \in E_{p-1}^{2p+r,2-2p-r}$ is a cocycle with respect to the differential $L(\varepsilon_{p-1})$, while the cohomology class $\varphi_{r+1}(\varepsilon_{p-1}) \in H^{2p+r,2-2p-r}(\varepsilon_{p-1})$ of the element $\bar{x}$ is equal to zero if and only if there exists a representative ε' of the class ε_{p-1} differing from ε by an element in $Z_{p-2}^{p+r+1,-p-r}$ and for which $d\varepsilon' + \frac{1}{2}[\varepsilon' \wedge \varepsilon'] \in Z_{p-2}^{2p+r+1,1-2p-r}$.

PROPOSITION. *For the element $\varepsilon_{p-1} \in E_{p-1}^{p,1-p}$ to be a Maurer-Cartan element, it is necessary that $d_{p-1}\varepsilon_{p-1} + \frac{1}{2}[\varepsilon_{p-1} \wedge \varepsilon_{p-1}] = 0$ and $\varphi_1(\varepsilon_{p-1}) = 0$, and sufficient that all obstructions $\varphi_i(\varepsilon_{p-1})$, $i = 0, 1, \dots,$ be trivial.*

COROLLARY 1. *Let p_0 be a number such that $E_{p-1}^{s,2-s} = 0$ for $s \geq p_0$ and $p_0 \leq 2p$. Then each element $\varepsilon_{p-1} \in E_{p-1}^{p,1-p}$ for which $d_{p-1}\varepsilon_{p-1} + \frac{1}{2}[\varepsilon_{p-1} \wedge \varepsilon_{p-1}] = 0$ is a Maurer-Cartan element.*

COROLLARY 2. *Let $\varepsilon_{p-1} \in E_{p-1}^{p,1-p}$ be an element such that*

$$d_{p-1}\varepsilon_{p-1} + \tfrac{1}{2}[\varepsilon_{p-1} \wedge \varepsilon_{p-1}] = 0$$

and the second cohomology $H^{s,2-s}(\varepsilon_{p-1})$ is trivial for $s \geq 2p$. Then ε_{p-1} is a Maurer-Cartan element.

SUPPLEMENT TO THEOREM 2.8. *The elements $\sum c_i v_i$ occurring in the normal form of a p-approximation are Maurer-Cartan elements.*

2.11. Let (M, Ω) be a symplectic manifold, where $\Omega \in \Lambda^2(M)$ is the 2-form defining the symplectic structure, and let $\rho\colon \mathfrak{G} \to \operatorname{Ham}(M)$ be a representation of the Lie algebra $\mathfrak{G}$ by Hamiltonian fields which have a fixed point $a \in M$, $\operatorname{im}\rho \subset \mu D(M)$. The use of the correspondence $f \in C^\infty(M) \mapsto X_f \in \operatorname{Ham}(M)$ between $\operatorname{Ham}(M)$ and $C^\infty(M)$, where the Hamiltonian field X_f is determined from the equality $X_f \lrcorner \Omega = df$, allows us to assume that a representation $\rho\colon \mathfrak{G} \to C^\infty(M)$ is given, where $C^\infty(M)$ is considered as a Lie algebra with respect to the Poisson bracket: $(f, g) = X_f(g)$; $f, g \in C^\infty(M)$. In this connection, $\operatorname{im}\rho \subset \mu^2$.

Suppose that $\mu_* \subset C^\infty(M)/\mu^\infty = \mathcal{J}_a^\infty$ is the image of the ideal μ in the space $\mathcal{J}_a^\infty$ of formal power series at the point $a \in M$. As in §2.3, we consider a complex constructed from the representation ρ:

$$0 \to \mathcal{J}_a^\infty \xrightarrow{d} \mathfrak{G}^* \underset{\mathbf{R}}{\otimes} \mathcal{J}_a^\infty \xrightarrow{d} \Lambda^2\mathfrak{G}^* \underset{\mathbf{R}}{\otimes} \mathcal{J}_a^\infty \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^m\mathfrak{G}^* \underset{\mathbf{R}}{\otimes} \mathcal{J}_a^\infty \to 0, \tag{1}$$

where

$$d\omega(\nabla_1,\dots,\nabla_{s+1}) = \sum_i (-1)^{i+1}(\rho(\nabla_i),\omega(\nabla_1,\dots,\hat{\nabla}_i,\dots,\nabla_{s+1})) + \sum_{i<j}(-1)^{i+j}\omega([\nabla_i,\nabla_j],\nabla_1,\dots,\hat{\nabla}_i,\dots,\hat{\nabla}_j,\dots,\nabla_{s+1}). \quad (2)$$

The elements $\omega \in \Lambda^s\mathfrak{G}^* \otimes_{\mathbf{R}} \mathscr{J}_a^\infty$ can be considered to be skew-symmetric s-forms on the Lie algebra $\mathfrak{G}$ taking values in the Lie algebra of formal power series on M.

As above, we define the bracket $(\omega_1 \wedge \omega_2) \in \Lambda^{s+t}\mathfrak{G}^* \otimes_{\mathbf{R}} \mathscr{J}_a^\infty$ of the elements $\omega_1 \in \Lambda^s\mathfrak{G}^* \otimes_{\mathbf{R}} \mathscr{J}_a^\infty$ and $\omega_2 \in \Lambda^t\mathfrak{G}^* \otimes_{\mathbf{R}} \mathscr{J}_a^\infty$ so that on decomposable elements $\omega_1 = \alpha \otimes f$ and $\omega_2 = \beta \otimes g$ we have

$$(\omega_1 \wedge \omega_2) = \alpha \wedge \beta \otimes (f,g). \quad (3)$$

This bracket determines a bilinear pairing for which the commutation relations (5)–(7) in §2.3 hold.

We introduce a filtration into the complex (1) by setting

$$F_{p,q} = \Lambda^{p+q}\mathscr{J}^* \otimes \mu_*^{p+1}. \quad (4)$$

The differential d preserves this filtration. We consider the spectral sequence (E_r^{pq}, d_r^{pq}) constructed from (4). The proof of the next theorem is similar to that of Theorem 2.6.

THEOREM. *The filtration $F_{p,q}$ determines a spectral sequence (E_r^{pq}, d_r^{pq}) in which*

1) $E_0^{pq} = \Lambda^{p+q}\mathfrak{G}^* \otimes_{\mathbf{R}} S^{p+1}T^*$,

and the differential $d_0^{pq}\colon E_0^{pq} \to E_0^{p,q+1}$ is the differential in the Koszul complex of the Lie algebra $\mathfrak{G}$ constructed from the symmetric power of the reduction $\rho^{(1)}\colon \mathfrak{G} \to S^2T^* \simeq$ $\mathrm{sp}(T)$ of the representation of the Lie algebra $\mathfrak{G}$ in the symplectic algebra $\mathrm{sp}(T)$. Therefore,

2) $E_1^{pq} = H^{p+q}(\mathfrak{G}, S^{p+1}T^*)$,

3) *there is a bilinear pairing of terms of the spectral sequence $E_r^{pq} \times E_r^{p'q'} \to E_r^{p+p'-1,q+q'+1}$ that satisfies relations* (3)–(5) *in* §2.5,

4) *the spectral sequence (E_r^{pq}, d_r^{pq}) stabilizes in the following sense: for each pair (p,q) there exists a number $r_0 = r_0(p,q)$ such that*

$$E_{r_0}^{pq} = E_{r_0+1}^{pq} = \cdots = E_\infty^{pq}, \quad d_r^{pq} = 0, \qquad r \geq r_0,$$

and the terms of the spectral sequence are isomorphic to the terms of the graded module associated to the cohomology of the complex (1).

2.12. We say that two representations ρ_1 and ρ_2 of a Lie algebra $\mathfrak{G}$ into a Lie algebra of Hamiltonian vector fields are (*formally*) *equivalent* if there exists a local (formal) symplectic diffeomorphism A carrying ρ_1 into ρ_2 (respectively, $\rho_1^{(\infty)}$ into $\rho_2^{(\infty)}$). Similarly, when we speak of *k-sufficiency* (respectively, *k-sufficiency in the formal sense*) in this context, we shall understand k-sufficiency with respect to the group of symplectic diffeomorphisms. The proof of the following theorem is word for word the same as the proof of Theorems 2.8 and 2.9.

THEOREM. 1) *Let $v_1,\dots,v_s$, $v_i \in \mathfrak{G}^* \otimes \mu_*^{p+1}$, be such that their images generate $E_{p-1}^{p,1-p}$, $p \geq 2$. Then for each representation by Hamiltonian fields of the Lie algebra $\mathfrak{G}$ of the form $\rho+\varepsilon$, where $\varepsilon \in \mathfrak{G}^* \otimes \mu^{p+1}$, $p \geq 2$, there exists a local diffeomorphism A such that*

$$A_*(\rho) - \rho - \varepsilon = \sum_{i=1}^{s} c_i v_i \quad \mod F_{p+1,-p}.$$

where $c_i \in \mathbf{R}$ for $1 \le i \le s$ and the image of the element $\sum c_i v_i$ in $E_{p-1}^{p,1-p}$ is a Maurer-Cartan element.

2) *Let $v_1, \dots, v_s, \dots, v_i \in \mathfrak{G}^* \otimes \mu^{p_i+1}$, be such that their images generate $E_{p-1}^{p,1-p}$ for $p \ge 2$. Then for each representation by Hamiltonian fields of the Lie algebra $\mathfrak{G}$ of the form $\rho + \varepsilon$, where $\varepsilon \in \mathfrak{G}^* \otimes \mu^{p+1}$, $p \ge 2$, there exists a local diffeomorphism A such that*

$$A_*(\rho) - \rho - \varepsilon = \sum c_i v_i \mod \mathfrak{G}^* \otimes \mu^\infty.$$

COROLLARY. *If $E_{p-1}^{p,1-p} = 0$ in the spectral sequence for all p, starting with some number p_0, then the representation ρ is p_0-sufficient (in the formal sense).*

REMARK. The description of the Maurer-Cartan element presented in §2.10 is also valid in the symplectic case.

2.13. Let (M, ω) be a contact manifold with contact structure given by the 1-form $\omega \in \Lambda^1(M)$, and let $\operatorname{ct}(M)$ be the Lie algebra of contact vector fields. In order to describe the elements of this algebra [6], [7], we consider the one-dimensional bundle l over M whose fiber at the point $x \in M$ is the quotient space $T_x/E_x = l_x$, where $E_x = \ker \omega_x$. Each contact vector field X on M is uniquely determined by a section S_X of this bundle, where $S_X(x) = X_x \mod E_x$ (see [6]) and each section uniquely determines a contact vector field on M. Choosing the structural form ω determining the contact structure is equivalent to choosing a basis in the dual bundle l^* whose fiber at a point x can be identified with $\operatorname{Ann} E_x \subset T_x^*$. Let X_1^ω be a contact field whose generating section determines a basis dual to ω. This field is uniquely determined by the conditions

$$X_1^\omega \lrcorner d\omega = 0, \qquad X_1^\omega \lrcorner \omega = 1. \tag{1}$$

Then each section S_X can be represented in the form $S_X = f \cdot S_{X_1^\omega}$, where f is the generating function of the field X with respect to ω: $f = X \lrcorner \omega$. Using this isomorphism with the Lie algebra of contact vector fields, we can also introduce a Lie algebra structure on the module $\Gamma(l)$ of smooth sections of the bundle l by setting

$$[S_X, S_Y] = S_{[X,Y]} \tag{2}$$

for arbitrary contact vector fields X and Y. In terms of the generating functions (with respect to a fixed structure from ω) the bracket (2), called the *Lagrange bracket*, has the form

$$[f, g] = X_f^\omega(g) - X_1^\omega(f) \cdot g, \tag{3}$$

where X_f^ω is the contact vector field corresponding to the function f.

More generally, for an arbitrary section $S \in \Gamma(l)$, we let X_S denote the contact vector field corresponding to S. We remark that the correspondence $S \to X_S$ is a first order linear differential operator (see [6]).

Let $\mathscr{J}_a^\infty = C^\infty(M)/\mu_a^\infty$ be the algebra of formal series at the point a, and let $\mathscr{J}_a^\infty(l) = \Gamma(l)/\mu_a^\infty \Gamma(l)$ be the module over the $\mathscr{J}_a^\infty$-jets of infinite order of sections of the bundle l at the point a; $\mathscr{J}_a^\infty(l) = \mathscr{J}_a^\infty \otimes_{\mathbf{R}} l_a$.

For a given representation $\rho\colon \mathfrak{G} \to \operatorname{ct}(M)$ of a Lie algebra into the Lie algebra of contact vector fields, we consider the complex

$$\begin{aligned} 0 \to \mathscr{J}_a^\infty(l) &\xrightarrow{d} \mathfrak{G}^* \underset{\mathbf{R}}{\otimes} \mathscr{J}_a^\infty(l) \xrightarrow{d} \Lambda^2 \mathfrak{G}^* \underset{\mathbf{R}}{\otimes} \mathscr{J}_a^\infty(l) \\ &\to \cdots \xrightarrow{d} \Lambda^m \mathfrak{G}^* \otimes \mathscr{J}_a^\infty(l) \to 0, \end{aligned} \tag{4}$$

where

$$d\omega(\nabla_1,\dots,\nabla_{k+1}) = \sum_i(-1)^{i+1}[S_{\rho(\nabla_i)},\omega(\nabla_1,\dots,\hat{\nabla}_i,\dots,\nabla_{k+1})] + \sum_{i<j}(-1)^{i+j}\omega([\nabla_i,\nabla_j],\nabla_1,\dots,\hat{\nabla}_i,\dots,\hat{\nabla}_j,\dots,\nabla_{k+1})$$

for all $\omega \in \Lambda^k\mathfrak{G}^*\otimes\mathscr{J}_a^\infty(l)$ and $\nabla_1,\dots,\nabla_{k+1}\in\mathfrak{G}$. As above, we use the Lagrange bracket (2) to introduce a bilinear pairing $[\omega_1\wedge\omega_2]$ for which relations (5)–(7) in §2.5 hold.

Let $C_p\subset\mathscr{J}_a^\infty(l)$ be the set of ∞-jets of sections of l for which the contact field X_S has a zero at $a\in M$ of order greater than or equal to p. The quotient algebra C_p/C_{p+1} is a commutative Lie algebra and admits the following description.

A generating function f of a contact field which has a zero at $a\in M$ of order k can be represented in the form $f=\lambda h^k+\varepsilon$, where $\lambda\in\mathbf{R}$, h is a function such that $d_ah=\omega_a$ and $h(a)=0$, and ε is a function of order of smallness $k+1$. Here the coefficient λ is uniquely determined by the function f (and the structure form ω); $k!\lambda=(X_1^\omega)^k(f)(a)$, where $(X_1^\omega)^k=X_1^\omega\circ\dots\circ X_1^\omega$ is the kth iterate of the differentiation operator X_1^ω. The map $C_p/C_{p+1}\xrightarrow{\lambda}S^{p-1}l_a^*$ which takes the contact field X_f^ω to the tensor $(1/p!)(X_1^\omega)^p(f)(a)\cdot\omega_a^{p-1}$, does not depend on the choice of the structure form ω, and is an epimorphism. In view of the above description of the elements of C_p, its kernel is $S^{p+1}T_a^*/S^{p+1}l_a^*\otimes l_a$.

Thus, the sequence

$$0\to S^{p+1}T_a^*/S^{p+1}l_a^*\otimes l_a\to C_p/C_{p+1}\to S^{p-1}l_a^*\to 0 \tag{5}$$

of commutative (for $p\geq 2$) Lie algebras is exact.

Let $\rho\colon\mathfrak{G}\to\mathrm{ct}(M)$ be a representation such that $\operatorname{im}\rho\subset\mu D(M)$; we introduce a filtration on the complex (4) by setting

$$F_{p,q}=\Lambda^{p+q}\mathfrak{G}^*\otimes C_p. \tag{6}$$

The differential d preserves this filtration; we consider the spectral sequence (E_r^{pq},d_r^{pq}) constructed from this filtration.

The term E_0^{pq} in this spectral has the form $E_0^{pq}=\Lambda^{p+q}\mathfrak{G}^*\otimes C_p/C_{p+1}$ and, therefore, is contained in an exact sequence

$$0\to\Lambda^{p+q}\mathfrak{G}^*\otimes S^{p+1}T_a^*/S^{p+1}l_a^*\otimes l_a\to E_0^{pq}\to\Lambda^{p+q}\mathfrak{G}^*\otimes S^{p-1}l_a^*\to 0. \tag{7}$$

The spaces $S^{p+1}T_a^*$, $S^{p+1}l_a^*$, and C_p/C_{p+1} are $\mathfrak{G}$-modules with respect to the linear part of the representation ρ; $\rho^{(1)}:\mathfrak{G}\to C_1/C_2\hookrightarrow\operatorname{End}_{\mathbf{R}}T_a$, and the differential d_0^{pq} is the differential of the Koszul complex of the cohomology of the algebra $\mathfrak{G}$ with coefficients in C_p/C_{p+1}. From the short exact sequence (5) we obtain a long exact cohomology sequence

$$0\to H^0(\mathfrak{G},S^{p+1}T_a^*/S^{p+1}l_a^*\otimes l_a)\to E_1^{p,-p}\to H^0(\mathfrak{G},S^{p-1}l_a^*)\to\cdots\to H^{p+q}(\mathfrak{G},S^{p+1}T_a^*/S^{p+1}l_a^*\otimes l_a)\to E_1^{p,q}\to H^{p+q}(\mathfrak{G},S^{p-1}l_a^*)\to\cdots.$$

Finally, we obtain the following description of the spectral sequence.

THEOREM. *The filtration* (6) *determines a spectral sequence* (E_r^{pq},d_r^{pq}) *in which the following conditions are satisfied:*

1) *The term* $E_0^{pq}=\Lambda^{p+q}\mathfrak{G}^*\otimes C_p/C_{p+1}$ *is included in an exact sequence*

$$0\to\Lambda^{p+q}\mathfrak{G}^*\otimes S^{p+1}T_a^*/S^{p+1}l_a^*\otimes l_a\to E_0^{pq}\to\Lambda^{p+q}\mathfrak{G}^*\otimes S^{p-1}l_a^*\to 0,$$

and the differential d_0^{pq} is a differential in the Koszul complex of the Lie algebra $\mathfrak{G}$ constructed from the representation in C_p/C_{p+1} induced by $\rho^{(1)}$. Therefore,

2) *The terms $E_1^{pq} = H^{p+q}(\mathfrak{G}, C_p/C_{p+1})$ are contained in an exact sequence*

$$\cdots \to H^{p+q}(\mathfrak{G}, S^{p+1}T_a^*/S^{p+1}l_a^* \otimes l_a) \to E_1^{pq} \to H^{p+q}(\mathfrak{G}, S^{p-1}l_a^*)$$
$$\to H^{p+q+1}(\mathfrak{G}, S^{p+1}T_a^*/S^{p+1}l_a^* \otimes l_a) \to E_1^{p,q+1} \to H^{p+q+1}(\mathfrak{G}, S^{p-1}l_a^*) \to \cdots .$$

3) *There is a bilinear pairing $E_r^{pq} \times E_r^{p'q'} \to E_r^{p+p'-1,q+q'+1}$ induced by the bracket*

$$[\Lambda^s\mathfrak{G}^* \times \mathcal{J}_a^\infty(l) \wedge \Delta^t\mathfrak{G}^* \otimes \mathcal{J}_a^\infty(l)] \to \Lambda^{s+t}\mathfrak{G}^* \otimes \mathcal{J}_a^\infty(l)$$

which converts E_r into a bigraded Lie algebra; d_r is a derivation of this algebra.

4) *The spectral sequence stabilizes in the following sense: for each pair (p, q) there exists a number $r_0 = r_0(p, q)$ such that*

$$E_{r_0}^{pq} = E_{r_0+1}^{pq} = \cdots = E_\infty^{pq}, \quad d_r^{pq} = 0, \qquad r \geq r_0.$$

2.14. The concepts of (formal) equivalence and sufficiency of representations of an algebra $\mathfrak{G}$ into a Lie algebra of contact vector fields carries over in the obvious way to the contact case: all diffeomorphisms in the definitions are contact diffeomorphisms.

THEOREM. 1) *Let $v_1, \ldots, v_s$, $v_i \in \mathfrak{G}^* \otimes C_p$, be such that their images generate $E_{p-1}^{p,1-p}$, $p \geq 2$. Then for each representation by contact vector fields of the Lie algebra $\mathfrak{G}$, of the form $\rho + \varepsilon$, where $\varepsilon \in \mathfrak{G}^* \otimes C_p$, $p \geq 2$, there exists a local contact diffeomorphism A such that*

$$A_*(\rho) - \rho - \varepsilon = \sum c_i v_i \quad \operatorname{mod} F_{p+1,-p},$$

where $c_i \in \mathbf{R}$ for $1 \leq i \leq s$ and the image of the element $\sum c_i v_i$ in $E_{p-1}^{p,1-p}$ is a Maurer-Cartan element.

2) *Let $v_1, \ldots, v_s, \ldots, v_i \in \mathfrak{G}^* \otimes C_{p_i}$, be such that their images generate $E_{p-1}^{p,1-p}$ for $p \geq 2$. Then for each representation by contact vector fields of the Lie algebra of the form $\rho + \varepsilon$, where $\varepsilon \in \mathfrak{G}^* \otimes C_p$, $p \geq 2$, there exists a local contact diffeomorphism A such that*

$$A_*(\rho) - \rho - \varepsilon = \sum c_i v_i \quad \operatorname{mod} \mathfrak{G}^* \otimes \mu^\infty D.$$

COROLLARY. *If $E_{p-1}^{p,1-p} = 0$ in the spectral sequence for all $p \geq p_0$, then the representation ρ is p_0-sufficient in the formal sense.*

§3. Applications of spectral sequences

3.1. Let $\mathfrak{G}$ be a semisimple Lie algebra and $\rho\colon \mathfrak{G} \to \mu D(M)$ a representation into the Lie algebra of vector fields with fixed point at $a \in M$. In this case the terms $E_{p-1}^{p,1-p}$ are trivial (see [11]) and we obtain the following result from Theorem 2.9.

THEOREM [15]. *The representation $\rho\colon \mathfrak{G} \to \mu D(M)$ of the Lie algebra $\mathfrak{G}$ is 1-sufficient in the formal sense.*

3.2. By considerations similar to those is the contact and symplectic cases, we have $E_{p-1}^{p,1-p} = 0$ for representations of a semisimple Lie algebra. Therefore, we obtain the following result.

THEOREM. *A representation of a semisimple Lie algebra $\mathfrak{G}$ by Hamiltonian (contact) vector fields is 1-sufficient in the formal sense with respect to the group of symplectic (contact) diffeomorphisms.*

3.3. Let $\rho\colon \mathfrak{G} \to \mathrm{ct}(M)$ be a representation of a semisimple Lie algebra $\mathfrak{G}$ by analytic contact vector fields. Recall that in special local coordinates, the contact vector field corresponding to the generating function f has the form

$$X_f = X_f^\omega = -\sum_{i=1}^n \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} + \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} + p_i \frac{\partial f}{\partial u}\right)\frac{\partial}{\partial p_i} + \left(f - \sum_{i=1}^n p_i \frac{\partial f}{\partial p_i}\right)\frac{\partial}{\partial u}, \tag{1}$$

where $\omega = du - \sum_1^n p_i\, dq_i$.

We note that if the function f has the form $f = u - \frac{1}{2}\sum_1^n p_i q_i + \varepsilon$, where ε is a function of third order of smallness, then the linear part of X_f has the spectrum $\lambda_1 = \cdots = \lambda_{2n} = \frac{1}{2}$, $\lambda_{2n+1} = 1$.

LEMMA 1 [19]. *An analytic contact vector field with generating function $f = u - \frac{1}{2}\sum_1^n p_i q_i + \varepsilon$ can be reduced to a linear field by a local analytic contact diffeomorphism.*

LEMMA 2. *The generating function of an analytic contact vector field X_f, $X_{f,0} = 0$, commutes with X_{f_0}, where $f_0 = u - \frac{1}{2}\sum_1^n p_i q_i$ has the form $f = ku + H(p,q)$ in which $H(p,q)$ is a quadric.*

THEOREM 1. *A representation of a semisimple Lie algebra by analytic contact vector fields in a neighborhood of a common fixed point is contact analytically 1-sufficient.*

PROOF. The proof is analogous to that of Theorem 1 in [4]. The crucial point is the construction of a contact vector field which commutes with $\rho(\mathfrak{G})$ and has the form cited in Lemma 1. To construct such a field, we consider the representation $\rho^{(1)}\colon \mathfrak{G} \to \mathrm{End}_{\mathbf{R}} T$. The subspace $E = \ker \omega_a$ is invariant with respect to this representation, and since $\mathfrak{G}$ is semisimple, there exists a complementary invariant subspace F such that $F \oplus E = T$. We now choose a structure from ω so that $X_{1,a}^\omega$ will be a generator in F. Furthermore, since one-dimensional representations of semisimple algebras are trivial, the representation $\rho^{(1)}$ is a direct sum of a trivial representation on F and some representation by symplectic transformations in E. The linear part of the vector field X_f, where $f = u - \frac{1}{2}\sum_1^n p_i q_i$, is a homothety of the space E with coefficient $\frac{1}{2}$ and is the identity on F. Therefore, passing to the compact real form $\mathfrak{G} \otimes \mathbf{C}$ and averaging with respect to the action of the corresponding compact group (see [4]), we obtain a contact vector field X_f with generating function $f = u - \frac{1}{2}\sum p_i q_i + \varepsilon$, where ε is a function of third order of smallness, which commutes with all vector fields $\rho(\nabla)$, $\nabla \in \mathfrak{G}$. According to Lemma 1, X_f can be reduced by an analytic contact diffeomorphism to X_{f_0}; it remains to use Lemma 2.

Similarly, by using an averaging procedure with respect to the Haar measure, together with the fact that Lemmas 1 and 2 are also true in the C^∞ situation, we can prove the following generalization of a theorem due to Cartan.

THEOREM 2. *A smooth (C^∞) action of a compact Lie group by contact diffeomorphisms in a neighborhood of a fixed point is contact 1-sufficient.*

Using contactification of symplectic manifolds, together with the theorems proved above, we obtain the following result.

THEOREM 3. 1) *An analytic representation of a semisimple Lie algebra by Hamiltonian vector fields in a neighborhood of a fixed point is contact equivalent to a linear one.*

2) *A smooth (C^∞) action of a compact Lie group by symplectic transformations in a neighborhood of a fixed point is contact equivalent to a linear action.*

3.4. In the general case, the coincidence of conditions for formal and C^∞ equivalence of representations is based on the following lemma.

LEMMA. *Let X be a smooth vector field on M, $\dim M = r$, with an equilibrium point $a \in M$, $X_a = 0$, at which the spectrum $\nu_1, \dots, \nu_r$ of the linear part satisfies the conditions* $\operatorname{Re}\nu_j > 0$, $j = 1, \dots, r$. *Then each flat vector field $Y \in \mu_a^\infty D(M)$ which satisfies an equation $(L_X - \lambda)^k Y = 0$, where L_X is the Lie derivative along X, λ is an arbitrary constant, and k is a natural number, is zero in a neighborhood of a.*

PROOF. Let $Y_1, \dots, Y_r$ be the components of a vector field Y with respect to a system of local coordinates $(q_1, \dots, q_r)$ centered at the point $a \in M$. If $q(t)$ is a trajectory of the field X and $A(t)$ the matrix obtained by restricting the matrix $||\partial_i X_j||$ to $q(t)$, then on this trajectory the equation $(L_X - \lambda)^k Y = 0$ has the form

$$[d/dt - A(t) - \lambda]^k Y(t) = 0, \tag{1}$$

where $Y(t)$ denotes the vector-valued function with components $Y_j(t) = Y_j(q(t))$.

It follows from (1) that the norm of $Y(t)$ is bounded in a sufficiently small neighborhood of a as follows:

$$C_2 \exp(\beta t) \le ||Y(t)|| \le C_1 \exp(\alpha t) \tag{2}$$

for some constants C_1, C_2, α, and β. In addition, $C_1 \ge C_2 > 0$ if $Y(t) \ne 0$. On the other hand, in a sufficiently small neighborhood

$$K_2 \exp(\delta t) \le ||q(t)|| \le K_1 \exp(\gamma t) \tag{3}$$

for positive constants $K_1 \ge K_2$, δ and γ. Hence, $||Y(t)|| \ge C_2 K_2^{-\beta/\delta} ||q(t)||^{\beta/\delta}$ on those trajectories $q(t)$ for which $Y(t) \ne 0$. On the other hand, from the fact that $Y \in \mu_a^\infty D(M)$, it follows that $||Y(t)|| \le C_N ||q(t)||^N$ for all natural numbers N.

COROLLARY. *If, under the conditions of the lemma, Y is a flat vector field which commutes with X, then $Y = 0$.*

THEOREM 1. *Let $\rho\colon \mathfrak{G} \to \mu D(M)$ be a representation of the Lie algebra $\mathfrak{G}$ for which there exists an element $\nabla_0 \in \mathfrak{G}$ with the property that the spectrum of the linear part $\rho^{(1)}(\nabla_0)$ lies in the right half-plane* ($\operatorname{Re}\lambda > 0$). *Then each representation $\rho'\colon \mathfrak{G} \to \mu D(M)$ which is formally equivalent to ρ is C^∞ equivalent to ρ.*

PROOF. Using a lemma of Borel, we may assume that $\varepsilon(\nabla) = \rho(\nabla) - \rho'(\nabla) \in \mu^\infty D(M)$ for all $\nabla \in \mathfrak{G}$. Moreover, the conditions on the spectrum of $\rho^{(1)}(\nabla_0)$ and a theorem due to Chen [16] show that we may assume that $\varepsilon(\nabla_0) = 0$. We shall prove that $\varepsilon(\nabla) = 0$ for all $\nabla \in \mathfrak{G}$.

For this we represent the Lie algebra $\mathfrak{G}$ as a direct sum $\mathfrak{G} = \bigoplus_{\{\lambda\}} \mathfrak{G}_\lambda$, where (λ) is the spectrum of $\operatorname{ad}_{\nabla_0}\colon \mathfrak{G} \to \mathfrak{G}$ and $\mathfrak{G}_\lambda$ is the invariant subspace corresponding to the eigenvalue λ. It evidently suffices to prove that $\varepsilon(\nabla) = 0$ for any element $\nabla \in \mathfrak{G}_\lambda$. But then $(\operatorname{ad}_{\nabla_0} - \lambda)^k(\nabla) = 0$ for some natural number k. Therefore, if we set $X = \rho(\nabla_0)$ and $Y = \varepsilon(\nabla)$, we have

$$\begin{aligned}(L_X - \lambda)^k(Y) &= \rho((\operatorname{ad}_{\nabla_0} - \lambda)^k)(\rho(\nabla)) - \rho'((\operatorname{ad}_{\nabla_0} - \lambda)^k)(\rho'(\nabla)) \\ &= \rho((\operatorname{ad}_{\nabla_0} - \lambda)^k(\nabla)) - \rho'((\operatorname{ad}_{\nabla_0} - \lambda)^k(\nabla)) = 0.\end{aligned}$$

It remains to use the previous lemma.

The contact analogue of Theorem 1 is proved similarly.

THEOREM 2. *If ρ is a representation of a Lie algebra $\mathfrak{G}$ by contact vector fields which satisfies the conditions of the preceding theorem, then each representation ρ' formally contact equivalent to ρ is C^∞ contact equivalent to ρ.*

3.5. We consider representations of commutative Lie algebras.

LEMMA 1. *Let $\mathfrak{G}$ be a commutative Lie algebra and $\alpha\colon \mathfrak{G} \to \operatorname{End}_{\mathbf{R}} V$ a completely reducible, finite-dimensional representation. Then $H^k(\mathfrak{G}, V) = \Lambda^k\mathfrak{G}^* \otimes V^{\#}$, where $V^{\#} = \{v \in V | \alpha(V)(v) = 0$ for all $\nabla \in \mathfrak{G}\}$, is the subspace of fixed elements.*

PROOF. Let $V = \bigoplus_i V_i$ be a decomposition of V into irreducible components. Then $H^k(\mathfrak{G}, V) = \bigoplus_i H^k(\mathfrak{G}, V_i)$, and, consequently, it suffices to calculate $H^k(\mathfrak{G}, V)$ for simple $\mathfrak{G}$-modules V. If the action of $\mathfrak{G}$ on V is trivial, then $H^k(\mathfrak{G}, V) = \Lambda^k\mathfrak{G}^* \otimes V$. We show that if $\mathfrak{G}$ acts in a nontrivial manner on V, then $H^k(\mathfrak{G}, V) = 0$. Without loss of generality, we may assume that V is one-dimensional (by passing, if necessary, to the complexification). In this case, the action of $\mathfrak{G}$ on V is determined by the 1-form $\lambda \in \mathfrak{G}^*$ (the weight of the representation): $\alpha(\nabla)v = \lambda(\nabla)v$, $v \in V$. Let $v_0 \in V$ be a generator of V and $f \in \Lambda^k\mathfrak{G}^* \otimes V$ a cocycle. Representing f in the form $f = f_0 \otimes v_0$, where $f_0 \in \Lambda^k\mathfrak{G}^*$, we obtain

$$df(\nabla_1, \dots, \nabla_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \lambda(\nabla_i) f_0(\nabla_1, \dots, \nabla_i, \dots, \nabla_{k+1}) v_0 = 0$$

for all $\nabla_1, \dots, \nabla_{k+1} \in \mathfrak{G}$. Thus, $\lambda \wedge f_0 = 0$. Consequently, there exists a $(k-1)$-form $h \in \Lambda^{k-1}\mathfrak{G}^*$ such that $f_0 = \lambda \wedge h$. But then $d(h \otimes v_0) = f$. The lemma is proved.

LEMMA 2. *Let $\alpha\colon \mathfrak{G} \to \operatorname{End}_{\mathbf{R}} V$ be a finite-dimensional representation of a commutative algebra, and $V_0^{\#} = \{v \in V | \forall \nabla \in \mathfrak{G},\ \exists k,\ (\alpha(\nabla))^k v = 0\}$ the subspace associated to the zero weight. Then $H^k(\mathfrak{G}, V) = H^k(\mathfrak{G}, V_0^{\#})$.*

PROOF. By Engel's theorem, all operators $\alpha(\nabla)$ (over the field $\mathbf{C}$) can be reduced to triangular form. Suppose that the weights of the representation α are nontrivial: then a basis $\nabla_1, \dots, \nabla_m$ of the algebra $\mathfrak{G}$, subject to the condition that all weights are not equal to zero on ∇_1, forms a regular sequence (see [12]); in this case $H^k(\mathfrak{G}^*, V) = 0$.

Consider the short exact sequence of $\mathfrak{G}$-modules $0 \to V_0^{\#} \to V \to V/V_0^{\#} \to 0$, to which corresponds the long exact cohomology sequence

$$\cdots \to H^{i-1}(\mathfrak{G}, V/V_0^{\#}) \to H^i(\mathfrak{G}, V_0^{\#}) \to H^i(\mathfrak{G}, V) \to H^i(\mathfrak{G}, V/V_0^{\#}) \to \cdots.$$

The representation $\bar{\alpha}\colon \mathfrak{G} \to \operatorname{End}_{\mathbf{R}} V/V_0^{\#}$, being a quotient representation of α, does not have zero weights. Therefore $H^i(\mathfrak{G}, V/V_0^{\#}) = 0$, and the inclusion $H^i(\mathfrak{G}, V_0^{\#}) \to H^i(\mathfrak{G}, V)$ is an isomorphism.

COROLLARY. *Let $\alpha\colon \mathfrak{G} \to \operatorname{End}_{\mathbf{R}} V$ be a finite-dimensional representation of a commutative Lie algebra. Then $H^i(\mathfrak{G}, V) = 0$ for $i \geq 0$ if $V^{\#} = 0$ or, equivalently, the representation α does not have zero weight.*

3.6. Let $\rho\colon \mathfrak{G} \to \mu D(M)$ be a representation of a commutative Lie algebra $\mathfrak{G}$ whose linear part $\rho^{(1)}\colon \mathfrak{G} \to \operatorname{End}_{\mathbf{R}} T$ is completely reducible. Using the lemma proved above, we find that the first term of the spectral sequence of §2.6 has the form

$$E_1^{pq} = \Lambda^{p+q}\mathfrak{G}^* \otimes (S^pT^* \otimes T)^{\#}.$$

Let $\lambda_1, \dots, \lambda_r \in \mathfrak{G}^*$ be the weights of the representation $\rho^{(1)}\colon \mathfrak{G} \to \operatorname{End}_{\mathbf{R}} T$ (generally speaking, complex). Then the representation $S^p(\rho^{(1)})^* \otimes \rho^{(1)}$ is also completely reducible, and its weights are equal to

$$-\sum_{i=1}^{r} m_i\lambda_i + \lambda_j,$$

respectively, where $m_1, \dots, m_r$ are natural numbers and $\sum_1^r m_i = p$. Therefore, $E_1^{pq} \neq 0$ if and only if there exists a resonance on the level p: $\sum_1^r m_i\lambda_i = \lambda_j$ for some weight λ_j and natural numbers $m_1, \dots, m_r$ such that $\sum_1^r m_i = p$.

Let $M_p = E_1^{p,-p}$. Then $M_* = \sum_{p\geq 0} M_p$ is a Lie algebra and the differential $d_1^{p,-p}\colon M_p \to \mathfrak{G}^* \otimes M_{p+1}$ determines a graded $\mathfrak{G}$-module structure on M_*. Let $H^{p+q,p}(\mathfrak{G}, M_*)$ be the Koszul cohomology of this module at the term $\Lambda^{p+q}\mathfrak{G}^* \otimes M_p$. Theorem 2.6 can be supplemented as follows.

THEOREM. *Suppose that $\rho\colon \mathfrak{G} \to \mu D(M)$ is a representation of a commutative Lie algebra for which the linear part $\rho^{(1)}$ is completely reducible. Then, in the spectral sequence of §2.6, the following is true:*

1) $E_1^{pq} = \Lambda^{p+q}\mathfrak{G}^* \otimes M_p$, *where $M_p = (S^pT^* \otimes T)^{\#}$ is the space of invariants of $\mathfrak{G}$ corresponding to resonances of the terms* (1). *The differential $d_1^{p,-p}$ determine a graded $\mathfrak{G}$-module structure on $M_* = \sum_{p\geq 0} M_p$.*

2) $E_2^{pq} = H^{p+q,p}(\mathfrak{G}, M_*)$ *coincides with Koszul cohomology of the Lie algebra $\mathfrak{G}$ with values in M_*.*

Supplement. If the representation $\rho^{(1)}$ is not completely reducible, then

$$E_1^{pq} = H^{p+q}(\mathfrak{G}, (S^pT^* \otimes T)_0^{\#})$$

in view of Lemma 3.5(2). Therefore $E_1^{pq} = 0$ if $\sum m_i\lambda_i \neq \lambda_j$ for all j and natural numbers m_i such that $\sum m_i = p$.

3.7. THEOREM. *Let $\rho\colon \mathfrak{G} \to \mu D(M)$ be a representation of a commutative Lie algebra and $\lambda_1, \dots, \lambda_r$ the weights (over $\mathbf{C}$) of the representation $\rho^{(1)}$.*

1) *If $\sum_1^r m_i\lambda_i \neq \lambda_j$ for all natural numbers $m_1, \dots, m_r$ such that $\sum_1^r m_i \geq p_0$, then the representation ρ is p_0-sufficient in the formal sense.*

2) *If, in addition, $\Lambda_+ \neq \varnothing$, where*

$$\Lambda_+ = \{\nabla \in \mathfrak{G} \mid \operatorname{Re}\lambda_j(\nabla) > 0,\ j = 1, \dots, r\},$$

then the representation ρ is p_0-sufficient.

The first part of the assertion is a corollary of Theorems 3.6 and 2.9. The second part follows from Theorem 3.4(1).

REMARK 1. Assertion 1) of Theorem 3.7 admits the following generalization for a completely reducible representation $\rho^{(1)}$: if $H^{1,p}(\mathfrak{G}, M_*) = 0$ for all $p \geq p_0$, then the representation ρ is p_0-sufficient in the formal sense.

REMARK 2. Theorem 2.9, applied in the given situation, gives a resonant normal form for vector fields in $\rho(\mathfrak{G})$.

REMARK 3. In the case of a one-dimensional Lie algebra, Theorem 2.9 under the conditions $\operatorname{Re}\lambda_j \neq 0$ (in view of Chen's theorem [16]), gives a description of p_0-sufficient orbits in the C^∞-case.

3.8. Let $\rho\colon \mathfrak{G} \to \operatorname{ct}(M)$, $\operatorname{im}\rho \subset \mu D$, be a representation of a commutative Lie algebra by contact vector fields. We suppose that the linear part $\rho^{(1)}$ is completely reducible. We denote the weights of $\rho^{(1)}$ by $\lambda_0, \lambda_1, \dots, \lambda_{2n}$, where λ_0 is the weight of the representation $\rho^{(1)}$ on l_a. There is the following relation among the weights [7]: for each λ_i there exists a λ_j $(i, j \neq 0)$ such that $\lambda_i + \lambda_j = \lambda_0$.

If we suppose that $\lambda_0 \neq 0$, then $H^k(\mathfrak{G}, S^{p-1}l_a^*) = 0$ for all $k \geq 0$ and $p \geq 2$, and, consequently, in the spectral sequence of §2.13, the term E_1^{pq} appears in the following form:

$$E_1^{pq} = H^{p+q}(\mathfrak{G}, S^{p+1}T^*/S^{p+1}l_a^* \otimes l_a), \tag{1}$$

where $p \geq 2$. Using the exact cohomology sequence of a pair

$$0 \to S^{p+1}l_a^* \to S^{p+1}T^* \to S^{p+1}T^*/S^{p+1}l_a^* \to 0,$$

we obtain isomorphisms

$$E_1^{pq} = H^{p+q}(\mathfrak{G}, S^{p+1}T^* \otimes l_a) = \Lambda^{p+q}\mathfrak{G}^* \otimes (S^{p+1}T^* \otimes l_a)^{\#}.$$

In terms of the generating functions, the action of $\mathfrak{G}$ on $S^{p+1}T^*$ has the form

$$\nabla([f]_a^{p+1}) = [\rho(\nabla)(f) - X_1(f_\nabla)f]_a^{p+1}, \tag{2}$$

where f_∇ is a generating function of $\rho(\nabla)$, $f \in \mu^{p+1}$, and

$$[f]_a^{p+1} = f \mod \mu^{p+2} \in S^{p+1}T^*.$$

Since $\lambda_0(\nabla) = X_1(f_\nabla)(a)$, it follows from (2) that the weights of the representation of $\mathfrak{G}$ in $S^{p+1}T^* \otimes l_a$ have the form $-\sum_0^{2n} m_i\lambda_i + \lambda_0$ under the condition that $\sum_0^{2n} m_i = p+1$. We set $M_p = E_1^{p,-p} = (S^{p+1}T^* \otimes l_a)^{\#}$. Then $M_* = \sum_{p\geq 0} M_p$ is a Lie algebra and the differential $d_1^{p,-p}\colon M_p \to \mathfrak{G}^* \otimes M_{p+1}$ determines the structure of a graded $\mathfrak{G}$-module on M_*. As in §3.7, let $H^{p+q,p}(\mathfrak{G}, M_*)$ be the Koszul cohomology of this module at the term $\Lambda^{p+q}\mathfrak{G}^* \otimes M_p$; Theorem 2.13 can be supplemented as follows.

THEOREM. *Let $\rho\colon \mathfrak{G} \to \mathrm{ct}(M)$, $\operatorname{im}\rho \subset \mu D$, be a representation of a commutative Lie algebra by contact vector fields for which the linear part of $\rho^{(1)}$ is completely reducible. In addition, let $\lambda_0, \lambda_1, \dots, \lambda_{2n}$, $2n+1 = \dim M$, be the weights (over $\mathbf{C}$) of the representation $\rho^{(1)}$, where λ_0 is the weight of the representation $\rho^{(1)}$ on l_a. Then, if $\lambda_0 \neq 0$, the spectral sequence of* §2.13 *is such that the following conditions hold:*

1) *The term $E_1^{pq} = \Lambda^{p+q}\mathfrak{G}^* \otimes M_p$, where $M_p = (S^{p+1}T^* \otimes l_a)^{\#}$, is the space of invariants of $\mathfrak{G}$ corresponding to resonant terms $M_p = 0$, if $\lambda_0 \neq \sum_0^{2n} m_i\lambda_i$ for all natural numbers $m_0, \dots, m_{2n}$ such that $\sum_0^{2n} m_i = p+1$, $p \geq 2$. The differential $d_1^{p,-p}\colon M_p \to \mathfrak{G}^* \otimes M_{p+1}$ determines a $\mathfrak{G}$-module structure on $M_* = \sum_{p\geq 0} M_p$.*

2) *The term $E_2^{p,q} = H^{p+q,p}(\mathfrak{G}, M_*)$ coincides with the Koszul cohomology of the Lie algebra $\mathfrak{G}$ with values in M_*.*

3.9. The proof of the following theorem is similar to that of 3.5.

THEOREM. *With the notation of the preceding theorem:*

1) *The representation ρ is p_0-sufficient in the formal sense if $H^{1,p}(\mathfrak{G}, M_*) = 0$ for all $p \geq p_0$. In particular, this is always the case if conditions* 3.6 *are satisfied for all natural numbers $m_0, \dots, m_{2n}$ such that $\sum_0^{2n} m_i \geq p_0 + 2$.*

2) *If, in addition, $\Lambda_+ \neq \varnothing$, then the representation ρ is p_0-sufficient in the C^∞ sense.*

REMARK 1. In the case of a one-dimensional Lie algebra, when $\operatorname{Re}\lambda_j \neq 0$, Theorem 2.14, in view of the results of [7], gives a description of the p_0-sufficient orbits in the C^∞ case.

REMARK 2. The assumption that $\rho^{(1)}$ be completely reducible can be omitted when the conditions of 3.6 hold.

3.10. In conclusion, we consider representations of reductive algebras.

LEMMA. *Let $\alpha\colon \mathfrak{G} \to \operatorname{End}_{\mathbf{R}} V$ be a finite-dimensional representation of a reductive Lie algebra whose restriction to the center $\mathfrak{Z} \subset \mathfrak{G}$ is completely reducible. Then*

$$H^k(\mathfrak{G}, V) = \sum_{i+j=k} H^i(\mathfrak{G}_0, \mathbf{R}) \underset{\mathbf{R}}{\otimes} \Lambda^j\mathfrak{Z}^* \otimes V^{\#},$$

where $\mathfrak{G}_0 = [\mathfrak{G}, \mathfrak{G}]$ is the semisimple part of $\mathfrak{G}$.

PROOF. From the Serre-Hochschild spectral sequence for the pair $(\mathfrak{G}, \mathfrak{Z})$ it follows that

$$H^k(\mathfrak{G}, V) = \sum_{i+j=k} H^i(\mathfrak{G}_0, \mathbf{R}) \underset{\mathbf{R}}{\otimes} (H^j(\mathfrak{Z}, V))^{\#},$$

and it remains to use Lemma 3.5. The lemma is proved.

3.11. Let $\rho\colon \mathfrak{G} \to \mu D(M)$ be a representation of a reductive Lie algebra $\mathfrak{G}$ for which the restriction of the linear part $\rho^{(1)}$ to the center $\mathfrak{Z}$ is completely reducible. Then the term E_1^{pq} of the spectral sequence in §2.6 has the form

$$E_1^{pq} = \sum_{i+j=p+q} H^i(\mathfrak{G}_0, \mathbf{R}) \underset{\mathbf{R}}{\otimes} \Lambda^j \mathfrak{Z}^* \otimes (S^pT^* \otimes T)^{\#}. \tag{1}$$

Let $T = V_1 \oplus \cdots \otimes V_r$ be the decomposition of T into a direct sum of irreducible $\mathfrak{G}_0 = [\mathfrak{G}, \mathfrak{G}]$-modules. Then the restriction of the representation $\rho^{(1)}$ to $\mathfrak{Z}$ is scalar on each subspace V_1:

$$\rho^{(1)}(z)(v_i) = \lambda_i(z)v_i, \qquad v_i \in V_i \underset{\mathbf{R}}{\otimes} \mathbf{C},$$

for some weight $\lambda_i \in (\mathfrak{Z} \otimes_{\mathbf{R}} \mathbf{C})^*$.

Let $(S^pT^* \otimes T)^{\mathfrak{G}_0}$ be the module of invariants of the algebra $\mathfrak{G}_0$. Then $\sum_{p\geq 0}(S^pT^* \otimes T)^{\mathfrak{G}_0}$ is a module over the algebra of invariants $\sum_{p\geq 0}(S^pT^*)^{\mathfrak{G}_0}$, and therefore $(S^pT^* \otimes T)^{\mathfrak{G}_0}$ is nontrivial for arbitrarily large values of p. The weights (over $\mathbf{C}$) of the representation of $\mathfrak{Z}$ in $S^pT^* \otimes T$ are equal to $\lambda_j - \sum m_i\lambda_i$, where the m_i are natural numbers and $\sum m_i = p$. Consequently, $E_1^{pq} = 0$ for $p \geq p_0$ if resonances are absent in the representation $\rho^{(1)}$: $\sum m_i\lambda_i \neq \lambda_j$ for all natural numbers $m_1, \ldots, m_r$ such that $\sum m_i \geq p_0$.

THEOREM. *Let $\rho\colon \mathfrak{G} \to \mu D(M)$ be a representation of a reductive Lie algebra $\mathfrak{G}$ whose linear part restricts to a completely reducible representation on the center $\mathfrak{Z} \subset \mathfrak{G}$. Let $\lambda_1, \ldots, \lambda_r \in (\mathfrak{Z} \otimes_{\mathbf{R}} \mathbf{C})^*$ be the weights of the restriction of the representation $\rho^{(1)}$ to $\mathfrak{Z}$.*

1) *If $\sum m_i\lambda_i \neq \lambda_j$ for all natural numbers $m_1, \ldots, m_r$ for which $\sum m_i \geq p_0$, the representation ρ is p_0-sufficient in the formal sense.*

2) *If $\Lambda_+ \neq \varnothing$ where $\Lambda_+ = \{z \in \mathfrak{Z} \mid \operatorname{Re}\lambda_j(z) > 0,\ 1 \leq j \leq r\}$ and condition 1) is satisfied, the representation ρ is p_0-sufficient in the class C^∞.*

3) *If $\Lambda_+ \neq \varnothing$, then formal and C^∞ equivalence of representations of the Lie algebra $\mathfrak{G}$ are equivalent.*

COROLLARY. *If $\Lambda_+ \neq \varnothing$, then Theorem 2.9 gives normal forms of representations in the class C^∞.*

The first two assertions of the theorem were proved above; the equivalence of formal and C^∞ sufficiency follows from Theorem 3.4.

3.12. Comparing the results of §§3.10 and 3.8, we obtain the following result.

THEOREM. *Let $\rho\colon \mathfrak{G} \to \operatorname{ct}(M)$, $\operatorname{im}\rho \subset \mu D$, be a representation of a Lie algebra $\mathfrak{G}$ by contact vector fields, whose linear part $\rho^{(1)}$ restricts to a completely reducible representation on the center $\mathfrak{Z} \subset \mathfrak{G}$. Let $\lambda_0, \lambda_1, \ldots, \lambda_r$ be the weights (over $\mathbf{C}$) of the restriction of $\rho^{(1)}$ to $\mathfrak{Z}$ where $\lambda_0 \in \mathfrak{Z}^*$ for all representations of $\mathfrak{Z}$ in l_a. If $\lambda_0 \neq 0$ and*

1) *$\lambda_0 \neq \sum m_i\lambda_i$ for all natural numbers $m_0, \ldots, m_r$, for which $\sum m_i \geq p_0 + 2$, then the representation ρ is p_0-sufficient in the formal sense;*

2) *$\Lambda_+ \neq \varnothing$ and condition 1) is satisfied, then the representation ρ is p_0-sufficient in the C^∞-sense;*

3) *$\Lambda_+ \neq \varnothing$, then formal and C^∞-equivalence are equivalent and Theorem 2.14 gives normal forms of representations in the class C^∞.*

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