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Finite dimensional dynamics for evolutionary equations

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Abstract We suggest a new method for studying finite dimensional dynamics for evolutionary differential equations. We illustrate this method for the case of the KdV equation. As a side result we give constructive solutions of the boundary problem for the Schrodinger equations whose potentials are solutions of stationary KdV equations and their higher generalizations.

Keywords Evolutionary differential equations · Shuffle symmetry · KdV equation · Finite dimensional dynamic

1 Introduction

We suggest a method for studying finite dimensional dynamics for evolutionary differential equations and we illustrate this method for the KdV equation. We outline the method for scalar evolutionary PDEs in dimension 2 but similar constructions for higher dimensional cases and systems of PDEs can be carried out in the same way by using results in [6] instead of the classical Frobenius theorem.

Let us discuss in more details the case of scalar evolutionary PDEs in dimension $1 + 1$. In this case there are two main points. First of all, if we consider an evolutionary PDE as a "dynamics" on a function space, then a finite-dimensional sub-dynamics can be viewed as a dynamics on the solution space of some ordinary differential equation (ODE). This leads us to the second step, namely description of this finite-dimensional dynamics. We search for such an ODE basing on the requirement that a given evolutionary PDE is the symmetry for the ODE. Putting all this together we will reformulate the problem of finding finite dimensional dynamics for an evolutionary PDE as a problem of finding ODEs for which the given evolutionary PDE is a symmetry. This gives us a differential equation for functions on jet spaces describing ordinary differential equation. In practice it is sufficient to find polynomial solutions of the equation.

The paper is organized as follows. In the first part we present geometrical theory of ODEs in the form suitable for us. Namely, we consider general ordinary differential equations (not necessarily resolved with respect to highest derivative) and recall the theory of shuffling symmetries and their use for integrating ODEs. We will use this type of ordinary differential equations to describe dynamics. We illustrate this approach for the Schrödinger equation. This is done for two reasons: it is instructive to see how shuffling symmetries work in this case, and we will apply these results to the KdV equation. It is worth to note that shuffling symmetries allow us to solve in quadratures the eigenvalue problem for the Schrödinger equations whose potentials satisfy the stationary KdV equation or its higher analogues. In the second part of the paper we describe in the details

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low-dimensional dynamics (up to dimension 4) for the KdV equation.

2 ODEs and dynamics

2.1 Geometry

Denote by $\mathbf{J}^m$ the space of m-jets of scalar functions on $\mathbb{R}$ with canonical coordinates $(p_m, p_{m-1}, \ldots, p_1, p_0, x)$.

In these coordinates the Cartan distribution $\mathcal{C}_m$ on $\mathbf{J}^m$ ([1, 5]) is given by the Cartan differential 1-forms

$$\omega_0 = dp_0 - p_1 dx, \ldots, \omega_{m-1} = dp_{m-1} - p_m\, dx.$$

This is a 2-dimensional distribution generated by two vector fields

$$\mathbf{D}_m = \frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial p_0} + \cdots + p_m \frac{\partial}{\partial p_{m-1}},$$

and

$$\frac{\partial}{\partial p_m}.$$

For any smooth function $f \in C^\infty(\mathbf{J}^m)$ we have

$$df = f_{p_m} dp_m + \mathbf{D}_m(f)\, dx \bmod(\omega_0, ..., \omega_{m-1}).$$

Define a bracket on the algebra $C^\infty(\mathbf{J}^m)$ as follows

$$\{f, g\} = f_{p_m} \mathbf{D}_m(g) - g_{p_m} \mathbf{D}_m(f). \tag{1}$$

This is a skew-symmetric bracket which satisfies the following version of the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \tag{2}$$
$$= f_{p_{m-1}}\{g, h\} - g_{p_{m-1}}\{f, h\} + h_{p_{m-1}}\{f, g\}. \tag{3}$$

The bracket is a bi-derivation and we denote by Z_f the vector field corresponding to the function f:

$$Z_f(g) = \{f, g\}.$$

Then

$$Z_f = \frac{\partial f}{\partial p_m} \mathbf{D}_m - \mathbf{D}_m(f) \frac{\partial}{\partial p_m}.$$

Note that the vector field Z_f belongs to the Cartan distribution, and due to (2)

$$[Z_f, Z_g] = Z_{\{f,g\}} + \frac{\partial f}{\partial p_{m-1}} Z_g - \frac{\partial g}{\partial p_{m-1}} Z_f + \{f, g\} \frac{\partial}{\partial p_{m-1}}.$$

An ordinary differential equation (ODE)

$$F\left(x, y, y', \ldots, y^{(m)}\right) = 0 \tag{4}$$

of order m defines in the standard way a subset

$$\mathcal{E} = \{F(x, p_0, \ldots, p_m) = 0\} \subset \mathbf{J}^m.$$

We call point $x_m \in \mathcal{E}$ *singular* if either $\mathcal{E}$ is not a smooth submanifold at x_m, or the tangent space $\mathbf{T}_{x_m}\mathcal{E}$ and the Cartan plane $\mathcal{C}_m(x_m)$ are not transversal, i.e.

$$\mathcal{C}_m(x_m) \subset \mathbf{T}_{x_m}\mathcal{E}.$$

Denote by $\Sigma(\mathcal{E}) \subset \mathcal{E}$ the set of the singular points, and by $\mathcal{E}_0 = \mathcal{E} \backslash \Sigma(\mathcal{E})$ the set of regular points of $\mathcal{E}$.

Note that the subset $\Sigma(\mathcal{E})$ of singular points is defined by the equations

$$F = 0, \quad \frac{\partial F}{\partial p_m} = 0, \quad \mathbf{D}_m(F) = 0 \tag{5}$$

and in general has codimension 2 in $\mathcal{E}$.

The restriction of the Cartan distribution $\mathcal{C}_m$ on the regular part $\mathcal{E}_0$

$$\mathcal{C}_\mathcal{E}: x_m \in \mathcal{E}_0 \mapsto \mathcal{C}_\mathcal{E}(x_m) = \mathcal{C}_m(x_m) \cap \mathbf{T}_{x_m}\mathcal{E}$$

defines a 1-dimensional distribution on $\mathcal{E}_0$.

It is easy to see that this distribution is generated by the vector field Z_F.

By solutions of ODE (4) we shall mean integral curves of the distribution $\mathcal{C}_\mathcal{E}$ or integral curves of Z_F.

Note that the condition $\frac{\partial F}{\partial p_m}(x_m) \neq 0$ at a point $x_m \in \mathcal{E}_0$ implies that the coordinate function x can be used as a local coordinate on the integral curve of Z_F passing through the point, and therefore the curve can be presented at the form

$$p_m = \frac{\partial^m h(x)}{\partial x^m}, \ldots, p_0 = h(x),$$

where the function $h(x)$ is a smooth solution of the ordinary differential Equation (4).

If $\frac{\partial F}{\partial p_m}(x_m) = 0$, then the smooth integral curve of $\mathcal{C}_\mathcal{E}$ shall represent a "multivalued" solution of (4).

Example 1. Consider the hypergeometric ordinary differential equation

$$F = x(1-x)p_2 + (c - (a+b+1)x)p_1 - abp_0,$$

where a, b, c are constants with $ab \neq 0$. Then $\mathcal{E}$ is a 3-dimensional submanifold in $\mathbf{J}^2 = \mathbb{R}^4$ which is diffeomorphic to $\mathbb{R}^3$ and the set $\Sigma\mathcal{E}$ of singular points consists of two straight lines

$$\begin{aligned} &abp_0 + (1+a+b-c)p_1 = 0, \\ &\quad (2+a+b-c)p_2 + (1+a+b+ab) \\ &\quad \times p_1 = 0, \quad x = 1, \end{aligned}$$

and

$$\begin{aligned} &abp_0 - cp_1 = 0, (1+c)p_2 \\ &\quad - (1+a+b+ab)p_1 = 0, x = 0. \end{aligned}$$

Therefore, integral curves which do not coincide with these two lines represent (multivalued) hypergeometric functions.

2.2 Shuffle symmetries

By **Sol**($\mathcal{E}$) we denote the space of solutions of the ordinary differential equation (4), that is, the set of all integral curves of the Cartan distribution $\mathcal{C}_\mathcal{E}$. In general, this set does not possess any "good" topological or smooth structure, so we shall use geometry of jet spaces to induce a geometry on **Sol** ($\mathcal{E}$). In some particular cases, for example when the equation can be resolved with respect to the highest derivative, $F = p_m - F_0(x, p_0, \ldots, p_m - 1)$, $\mathcal{E}$ is diffeomorphic to $\mathbb{R}^{m+1}$, and **Sol** ($\mathcal{E}$) is diffeomorphic to $\mathbb{R}^m$. The last diffeomorphism can be established by taking the initial data.

Two notions have the greatest importance for us: functions and vector fields on **Sol** ($\mathcal{E}$).

Namely, by functions on **Sol**($\mathcal{E}$) we mean 1-st integrals of $\mathcal{E}$, or in other words, functions f on $\mathcal{E}$ which are smooth in some domain and are constants on integral curves of $\mathcal{C}_\mathcal{E} : Z_F(f) = 0$, or

$$\{F, f\} = 0$$

on $\mathcal{E}$.

"Vector fields" on **Sol** ($\mathcal{E}$) correspond to (infinitesimal) symmetries of differential Equation (4). One may consider symmetries as vector fields on $\mathcal{E}_0$ that are symmetries of the Cartan distribution $\mathcal{C}_\mathcal{E}$ It is easy to see that all vector fields proportional to Z_F are symmetries, and they are trivial (or characteristic) in the sense that they produce trivial (or identity) transformations on the set **Sol** ($\mathcal{E}$).

Due to triviality we shall consider equivalence classes of symmetries modulo characteristic symmetries. We call them *shuffle symmetries*, (see [1]).

To find them we note that any such class has a representative Y of the form

$$Y = \sum_{i=0}^{m} a_i \frac{\partial}{\partial p_i}.$$

Computing the Lie derivatives of the Cartan forms we get

$$\begin{aligned} \mathbf{L}_Y(\omega_j) &= da_j - a_{j+1}dx = \frac{\partial a_j}{\partial p_m} dp_m + (\mathbf{D}_m(a_j) \\ &\quad - a_{j+1})\, dx \quad \mathrm{mod}(\omega_0, \ldots, \omega_{m-1}) \\ &= \frac{-\frac{\partial a_j}{\partial p_m}\mathbf{D}_m(F) + F_{p_m}(D_m(a_j) - a_{j+1})}{F_{p_m}}\, dx \\ &\quad \mathrm{mod}(\omega_0, \ldots, \omega_{m-1}, dF) \\ &= \left(\frac{[F, a_j]}{F_{p_m}} - a_{j+1}\right) dx \quad \mathrm{mod} \\ &\quad \times (\omega_0, \ldots, \omega_{m-1}, dF). \end{aligned}$$

Since dx does not vanish on $\mathcal{E}_0$, we get

$$a_{j+1} = \delta(a_j) \text{ on } \mathcal{E}_0$$

for all $j = 0, \ldots, m-1$.

Here

$$\delta = \frac{1}{F_{p_m}} Z_F.$$

Summarizing, we see that shuffling symmetries have representatives of the form

$$X_\varphi = \alpha \frac{\partial}{\partial p_0} + \delta(\varphi)\frac{\partial}{\partial p_1} + \cdot + \delta^m(\varphi)\frac{\partial}{\partial p_m}$$

for some functions $\varphi \in C^\infty(\mathcal{E}_0)$.

The requirement that X_φ is tangent to $\mathcal{E}_0$, leads to the Lie equation

$$\sum_{i=0}^{m} \frac{\partial F}{\partial p_i}\delta^i(\varphi) = 0 \quad \text{on} \quad \mathcal{E}_0. \tag{6}$$

We call $\varphi \in C^\infty(\mathcal{E}_0)$ a *generating function* of the shuffling asymmetry.

One can easily check that

$$[X_\varphi, X_\psi] = X_{[\varphi,\psi]},$$

where

$$\begin{aligned}[\varphi, \psi] &= X_\varphi(\psi) - X_\psi(\varphi)\\ &= \sum_{i=0}^{m}\left(\frac{\partial\varphi}{\partial p_i}\delta^i(\psi) - \frac{\partial\psi}{\partial p_i}\Delta^i(\varphi)\right).\end{aligned}$$

The bracket $[\varphi, \psi]$ defines a Lie algebra structure on the space of all shuffling symmetries.

In order to see the analytical meaning of shuffling symmetries let us consider a smooth (low) solution of the ordinary differential equation $h_0(x)$ and let

$$L_0 = \left\{p_0 = h_0, p_1 = h_0', \ldots, p_m = h_0^{(m)}\right\}$$

be the prolongation of $h_0(x)$ to m – jets.

Let A_t be the flow corresponding to a shuffling symmetry X_φ. Then $L_0 \subset \mathcal{E}$ and $L_t = A_t(L_0) \subset \mathcal{E}$. Curves $\{L_t\}$ (at least locally and for small t) are m – jets of the functions $h_t(x)$, that is

$$L_t = \left\{p_0 = h_t, p_1 = h_t', \ldots, p_m = h_t^{(m)}\right\}$$

and $h_t(x)$ satisfies the same ordinary differential equation. Moreover, the function $u(t,x) = h_t(x)$ satisfies the following evolutionary equation

$$\frac{\partial u}{\partial t} = \varphi\left(x, u, u_x, \ldots, \frac{\partial^m u}{\partial x^m}\right) \tag{7}$$

and

$$u|_{t=0} = h_0(x).$$

In other words, if φ is a generating function of a symmetry and $h_0(x)$ is a solution of $\mathcal{E}$, then the function $u(t,x)$ satisfies Equation (7) with the initial data $u(0,x) = h_0(x)$, and $u(t,x)$ is a solution of $\mathcal{E}$ at any fixed moment t.

2.3 Integration by symmetries

We refer to [1] for integration of ordinary differential equations with solvable symmetry group, and we discuss here only of the case of the commutative symmetry algebra.

We begin with the following observation. Let $v_1, \ldots, v_n$ be linearly independent vector fields on domain $D \subset \mathbb{R}^n$, and let

$$[v_i, v_j] = 0$$

for all $i, j = 1, \ldots, n$.

Take independent functions $f_1, \ldots, f_n$ on D and define deferential 1-forms $\theta_1, \ldots, \theta_n$ as follows

$$\theta = W^{-1} df,$$

where

$$\theta = \begin{bmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_n\end{bmatrix}, \quad df = \begin{bmatrix} df_1\\ df_2\\ \vdots\\ df_n\end{bmatrix},$$

and

$$W=\begin{bmatrix} v_1(f_1) & v_2(f_1) & \cdots & v_n(f_1)\\ v_1(f_2) & v_2(f_2) & \cdots & v_n(f_2)\\ \vdots & \vdots & \cdots & \vdots\\ v_1(f_n) & v_2(f_n) & \cdots & v_n(f_n)\end{bmatrix}.$$

Then $\theta_1,\ldots,\theta_n$ constitute the dual basis to v_1, ..., v_n.

Lemma 2. *$d\theta_i=0$ for all $i=1,\ldots,n$.*

Proof: We have $\theta_i(v_j)=\delta_{ij}$, and therefore,

$$d\theta_i(v_a,v_b)=v_a(\theta_i(v_b))-v_b(\theta_i(v_a)) -\theta_i([v_a,v_b])=0.$$

□

We apply this result to the integration of ordinary differential equations.

Assume that in a domain $\mathcal{D}\subset\mathcal{E}_0\subset\mathcal{E}=F^{-1}(0)\subset\mathbf{J}^m$ one has m commuting linear independent shuffling symmetries $\varphi_1,\ldots,\varphi_m$. Then $[Z_F,X_{\varphi_i}]=0$. Indeed, by the definition of symmetry $[Z_F,X_{\varphi_i}]=\lambda Z_F$ for some function λ. Applying both sides to the coordinate function x we get $\lambda=0$.

Therefore, vector fields Z_F, X_{φ_1}, ..., X_{φ_m} commute and linearly independent. To get 1-st integrals we need the following construction. Let us define a *Cartan form w_f corresponding to a function $f\in C^\infty(\mathbf{J}^m)$* as follows

$$\omega_f=\sum_{i=0}^m\frac{\partial f}{\partial p_i}\omega_i,$$

where

$$\omega_i=dp_i-p_{i+1}dx,\quad 0\le i\le m-1,$$

and

$$\omega_m=\frac{F_{pm}d_{p_m}-\mathbf{D}_m(F)dx}{F_{P_m}}.$$

Then

$$\omega_f(X_\varphi)=X_\varphi(f)\quad\text{and}\quad\omega_f(Z_F)=0.$$

Theorem 3. *Let $\varphi_1,\ldots,\varphi_m$ be commuting shuffling symmetries for the ordinary differential equation $\mathcal{E}=F^{-1}(0)\subset\boldsymbol{J}^m$, and let $\mathcal{D}\subset\mathcal{E}_0$ be a domain where vector fields $X_{\varphi_1},\ldots,X_{\varphi_m}$ are linearly independent. Let $f_1,\ldots,f_m$ be functions such that functions $x,f_1,\ldots,f_m$ are independent in $\mathcal{D}$. Then the differential 1-forms $\theta_1,\ldots,\theta_m$ defined by*

$$\theta=W^{-1}\omega_f,$$

where

$$\theta=\begin{bmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_m\end{bmatrix},\quad \omega_f=\begin{bmatrix}\omega_{f_1}\\ \omega_{f_2}\\ \vdots\\ \omega_{f_m}\end{bmatrix},$$

and

$$W=\begin{bmatrix} X_{\varphi_1}(f_1) & X_{\varphi_2}(f_1) & \cdots & X_{\varphi_m}(f_1)\\ X_{\varphi_1}(f_2) & X_{\varphi_2}(f_2) & \cdots & X_{\varphi_m}(f_2)\\ \vdots & \vdots & \cdots & \vdots\\ X_{\varphi_1}(f_n) & X_{\varphi_2}(f_n) & \cdots & X_{\varphi_m}(f_m)\end{bmatrix}.$$

are closed in $\mathcal{D}$ and $\theta_i(Z_F)=0$ for all $i=1,\ldots,m$.

Proof: We have

$$dx(Z_f)=1,\omega_{f_i}(Z_f)=0,$$

and

$$dx(X_{\varphi_i})=0,\quad \omega_{f_j}(X_{\varphi_j})=X_{\varphi_j}(f_i)$$

for all $i,j=1,\ldots,m$.

Therefore, the differential 1-forms

$$dx,\theta_1,\ldots,\theta_m$$

constitute the dual basis for

$$Z_F,X_{\varphi_1},\ldots,X_{\varphi_m}$$

and the theorem follows from the above lemma. □

Theorem 4. *Let $\varphi, \varphi_1, \ldots, \varphi_r$ be shuffling symmetries for the ordinary differential equation $\mathcal{E} = F^{-1}(0) \subset \boldsymbol{J}^m$, and let $\mathcal{D} \subset \mathcal{E}_0$ be a domain where vector fields $X_{\varphi_1}, \ldots, X_{\varphi_r}$ are linearly independent. If a vector field X_φ is a linear combination of the fields $X_{\varphi_1}, \ldots, X_{\varphi_r}$ then the shuffling symmetry φ is a linear combination of $\varphi_1, \ldots, \varphi_r$ in the domain $\mathcal{D}$:*

$$\varphi = \lambda_1\varphi_1 + \cdots + \lambda_r\varphi_r,$$

where the coefficients $\lambda_1, \ldots, \lambda_r$ are 1st integrals for $\mathcal{E}$.

Proof: In the domain $\mathcal{D}$ we have

$$X_\varphi = \lambda_1 X_{\varphi_1} + \cdots + \lambda_m X_{\varphi_m}$$

for some functions $\lambda_1, \ldots, \lambda_m$.

Then $\varphi = X_\varphi(p_0) = \lambda_1\varphi_1 + \cdots + \lambda_r\varphi_r$.

As we have seen $[Z_F, X_\psi] = 0$, for all shuffling symmetries, therefore

$$[Z_F, X_\varphi] = Z_F(\lambda_1)X_{\varphi_1} + \cdots + Z_F(\lambda_m)X_{\varphi_m} = 0$$

and $Z_F(\lambda_i) = 0$ for all $i = 1, \ldots, m$. □

2.4 The Shrödinger equation

2.4.1 Linear symmetries

In this section we apply the above approach to the Shrödinger equation

$$y'' + w(x)y = 0. \tag{8}$$

We study the linear symmetries of this equation given by generating functions

$$\varphi = a(x)p_1 + b(x)p_0.$$

Substituting φ into the Lie equation we find that $\varphi = cp_0 + \varphi_z$, where c is a constant, and

$$\varphi_z = z(x)p_1 - \frac{z'(x)}{2}p_0,$$

where the function $z = z(x)$ satisfies the following equation

$$z''' + 4wz' + 2w'z = 0. \tag{9}$$

Note that the symmetries p_0 and φ_z commute and assuming that z is given, one can find first integrals by quadratures. Namely, taking $f_1 = p_0$, $f_2 = p_1$ in theorem (3), one can get two differential 1-forms θ_1 and θ_2 and integrals H_1 and H_2.

The first integral $H = H_1$ can be chosen to be quadratic in p_0, p_1:

$$H = 2zp_1^2 - 2z'p_0p_1 + (z'' + 2wz)p_0^2.$$

We rewrite this integral in the following way. Let us note that Equation (9) is defined by the skew-adjoint operator

$$L = \frac{d^3}{dx^3} + 4w\frac{d}{dx} + 2w'$$

and the Lagrange formula shows that

$$K(z) = 2z(z'' + 2wz) - z'^2$$

is a first integral for Equation (9).

We say that a symmetry φ_z is *elliptic, hyperbolic or parabolic* if $K(z) > 0$, $K(z) < 0$ or $K(z) = 0$ respectively.

Using the symmetry we can rewrite the first integral in the form

$$H = \frac{2\left(\varphi_z^2 + kp_0^2\right)}{z},$$

where $4k = K(z)$.

Taking now $f_1 = H$, $f_2 = p_0$ in theorem () we find two differential 1-forms with

$$\theta_1 = \frac{dH}{2H},$$

and the restriction θ of the second form θ_2 on levels $H = 2c$ equals to

$$\theta = \frac{dp_0 - p_1 dx}{\varphi_z}.$$

Let

$$\alpha = \frac{\varphi_z}{\sqrt{|z|}},$$
$$\beta = \frac{p_0}{\sqrt{|z|}}.$$

Then

$$H = 2(\alpha^2 + k\beta^2) = 2c$$

and the restriction θ takes the following form

$$\theta = \frac{d\beta}{\alpha} - \frac{dx}{z}.$$

Integration of θ gives the following solutions of the Schrödinger equation.

Theorem 5. *Let φ_z be a linear symmetry of (8). Then solutions of the Shrödinger equation have the following form*

- *for elliptic symmetry φ_z,*

$$y(x) = \sqrt{\frac{|cz(x)|}{k}} \sin\left(\sqrt{k}\int \frac{dx}{z(x)}\right),$$

- *for hyperbolic symmetry φ_z,*

$$y(x) = \sqrt{\frac{|cz(x)|}{-k}} \sinh\left(\sqrt{-k}\int \frac{dx}{z(x)}\right),$$

- *for parabolic symmetry φ_z,*

$$y(x) = \sqrt{|cz(x)|}\int \frac{dx}{z(x)}.$$

2.4.2 The spectral problem

In this section we consider such potentials $w(x)$ for which the corresponding eigenvalue problem

$$y'' + w(x)y = \lambda y \tag{10}$$

possesses linear symmetries $z(x, \lambda)$ that are polynomial in λ.

Let

$$z(x,\lambda) = z_0(x)\lambda^n + z_1(x)\lambda^{n-1} + \cdots + z_{n-1}(x)\lambda + z_n(x)$$

be a linear symmetry for Equation (10).

Then the Lie equation gives a polynomial (in λ) of degree $n+1$

$$z'''(x,\lambda) + 4(w(x)-\lambda)z'(x,\lambda) + 2w'(x)z(x,\lambda) = 0$$

and the recursive set of equations on z_k are

$$z'_0 = 0,$$
$$z'_{k+1} = \frac{1}{4}L(z_k),\ k = 0,\ldots,n-1,$$
$$L(z_n) = 0.$$

Setting $z_0 = 1$, we obtain inductively functions $z_k = z_k(w)$ such that

$$z_{k+1}(w) = \frac{1}{4}\int L(z_k(w)), dx,$$

$k = 0, 1, \ldots.$

The first functions are the following:

$$z_1(w) = \frac{w}{2} + \mathbf{c}_1,$$
$$z_2(w) = \frac{w'' + 3w^2}{8} + \frac{\mathbf{c}_1}{2}w + \mathbf{c}_2,$$
$$z_3(w) = \frac{w^{(4)}}{32} + \frac{5}{16}\left(ww'' + \frac{w'^2}{2} + w^3\right) + \frac{c_1}{8}(w'' + 3w^2) + \frac{c_2}{2}w + c_3,$$
$$z_4(w) = \frac{w^{(6)}}{128} + \frac{7w'w^{(3)}}{32} + \frac{7(3w''^2 + 10w^2w'' + 10ww'^2 + 5w^4)}{128} + \frac{\mathbf{c}_1}{32}\left(w^{(4)} + 10(ww'' + \frac{w'^2}{2} + w^3\right) + \frac{\mathbf{c}_2}{8}(w'' + 3w^2) + \frac{c_3}{2}w + c_4.$$

The conditions $L(z_n(w)) = 0$ which can be reformulated also as $z'_{n+1}(w) = 0$ are called the n-th KdV stationary Equations (2).

Below we list the first KdV equations:
The 0-th KdV equation

$$w' = 0,$$

The 1st KdV equation

$$w''' + 6ww' + 4\mathbf{c}_1 w' = 0,$$

The 2nd KdV equation

$$\begin{aligned} &w^{(5)} + 10(ww''' + 2w'w'' + 3w^2 w') \\ &\quad +4c_1(w'' + 6ww') + 16\mathbf{c}_2 w' = 0. \end{aligned}$$

We conclude that potentials w which satisfy the n-th stationary KdV equation possess linear symmetry φ_{S_n} with

$$S_n = \lambda^n + \sum_{k=1}^{n} z_k \lambda^{n-k}.$$

As we have seen the function $K = 2z(z'' + 2(w - \lambda)z) - z'^2$ is the first integral of the Lie equation and therefore, coefficients of the polynomials

$$Q_n = 2S_n(S''_n + 2(w - \lambda)S_n) - S'^2_n$$

are first integrals for the n-th KdV equation.

For example, for the classical (first) KdV equation, $w''' + 6ww' + 4c_1 w' = 0$, one has

$$S_1 = \lambda + \frac{w}{2} + c_1$$

and

$$Q_1 = -4\lambda^3 = 8c_1\lambda^2 + q_{11}\lambda + q_{10},$$

where

$$\begin{aligned} q_{11} &= w'' + 3w^2 + 4\mathbf{c}_1 w - 4c_1^2, \\ q_{10} &= \frac{2ww'' - w'^2 + 4w^3}{4} + c_1(w'' + 4w^2 + 4\mathbf{c}_1 w) \end{aligned}$$

are first integrals.

Solving the KdV equation together with the equations $q_{11} = \text{const}$, $q_{10} = \text{const}$ we find the 1st order ODE for w:

$$\begin{aligned} w'^2 = &-2w^3 - 4c_1 w^2 + 2\left(q_{11} + 4c_1^2\right)w \\ &+4\left(c_1 q_{11} - q_{10} + 4c_1^3\right) \end{aligned}$$

and solutions

$$w = -2\wp(x + \mathbf{c}, g_2, g_3) - \frac{2c_1}{3},$$

where $\wp(x, g_2, g_3)$ is the Weierstrass elliptic function with invariants

$$g_2 = \frac{4c_1^2 - 6c_1}{3}, \quad g_3 = -\frac{152c_1^3}{27} - \frac{5q_{11}c_1}{3} + 2q_{10}.$$

For the second KdV equation,

$$\begin{aligned} w^{(5)} \; &+10(ww''' + 2w'w'' + 3w^2 w') \\ &+ 4c_1(w'' + 6ww') + 16c_2 w' = 0, \end{aligned}$$

one has

$$S_2 = \lambda^2 + \left(\frac{w}{2} + c_1\right)\lambda + \frac{w'' + 3w^2}{8} + \frac{c_1}{2}w + c_2,$$

and

$$\begin{aligned} Q_2 = &-4\lambda^5 - 8c_1\lambda^4 - 4\left(c_1^2 + 2c_2\right)\lambda^3 \\ &+q_{22}\lambda^2 + q_{21}\lambda + q_{20}, \end{aligned}$$

where

$$\begin{aligned} q_{22} &= \frac{10w^3 + 5w'^2 + 10ww'' + w^{(4)}}{10} \\ &\quad -8c_1c_2 + (w'' + 3w^2)c_1 + 4wc_2, \\ q_{21} &= \frac{2ww^{(4)} - 2w'w''' + w''^2 + 20w^2w'' + 15w^4)}{16} \\ &\quad +(w'' + 3w^2)c_1^2 + 4wc_1c_2 - 4c_2^2 + \frac{4w^{(4)} + 12ww'' + 4w'^2 + 14w^3}{4}c_1 + w^2c_2, \\ q_{20} &= \frac{2w''w^{(4)} + 6w^2w^{(4)} + 4w(4w''^2 - 3w'w''') - w'''^2 + 12w'^2w'' + 60w^3w'' + 36w^5}{64} \\ &\quad + \frac{4w^3 - w'^2 + 2ww''}{4}\mathbf{c}_1^2 + (w'' + 4w^2)\mathbf{c}_1\mathbf{c}_2 + 4w\mathbf{c}_2^2 \\ &\quad + \frac{12w^4 + 13w^2w'' + w''^2 - w'w''' + ww^{(4)}}{8}\mathbf{c}_1 + \frac{12w^3 + 6w'^2 + 10ww'' + w^{(4)}}{4}\mathbf{c}_2 \end{aligned}$$

are the first integrals for the second KdV.

Using these integrals one can reduce the 2-nd KdV equation to the following 2-nd order ODE $\mathcal{E}_{JT}$:

$$
\begin{aligned}
&25w^8 + 80w^7c_1 + 32w^6\big(2c_1^2 + 5c_2\big)\\
&\quad -16w^5(24c_1c_2 + 5Q_{22}) + 32w^4\big(-72c_1^2c_2 + 28c_2^2\\
&\quad + 5Q_{21} - 9c_1Q_{22}\big) + 256\big(-8c_1^2c_2 + 4c_2^2\\
&\quad + Q_{21} - c_1Q_{22}\big)^2 + 256W(8c_1c_2 + Q_{22})\\
&\quad \times \big(8c_1^2c_2 - 4c_2^2 - Q_{21} + c_1Q_{22}\big)\\
&\quad -256w^3\big(8c_1^3c_2 - c_1\big(-4c_2^2 + Q_{21}\big)\big) - 256w^3\\
&\quad \times \big(c_1^2Q_{22} + c_2Q_{22}\big) + 64w^2\big(32c_2^3 + Q_{22}^2\\
&\quad + 8c_2(Q_{21} + c_1Q_{22})\big) + 76w^5(w')^2 + 152w^4c_1\\
&\quad \times (w')^2 + 64w^3\big(c_1^2 + 3c_2\big)(w')^2 - 64w\\
&\quad \times \big(4c_2^2 + Q_{21}\big)(w')^2 - 64w^2\big(6c_1c_2 + Q_{22}(w')^2\\
&\quad + 128(8c_1^3c_2 - c_1\big(20c_2^2 + Q_{21}\big) + c_1^2Q_{22}\\
&\quad + 2(2Q_{20} - c_2Q_{22}))(w')^2 - 20w^2(w')^4\\
&\quad -16wc_1(w')^4 + 16\big(c_1^2 - 4c_2\big)(w')^4\\
&\quad + 80w^3(w')^2w'' + 96w^2c_1(w')^2w''\\
&\quad + 128wc_2(w')^2w'' - 32(8c_1c_2 + Q_{22})(w')^2\\
&\quad \times w'' - 8(w')^4w'' - 10w^4(w'')^2 - 16w^3c_1(w'')^2\\
&\quad -32w^2c_2(w'')^2 + 16w(8c_1c_2 + Q_{22})(w'')^2\\
&\quad -32\big(-8c_1^2c_2 + 4c_2^2 + Q_{21} - c_1Q_{22}\big)(w'')^2\\
&\quad + (20w + 8c_1)(w')^2(w'')^2 + (w'')^4 = 0.
\end{aligned}
$$

Two cases when $\mathbf{c} = 0$, and $\mathbf{c} = 0, q = 0$ give us shorter ODEs:

$$
\begin{aligned}
c = 0&25w^8 + 160w^4Q_{21} + 256Q_{21}^2 - 80w^5Q_{22}\\
&- 256wQ_{21}Q_{22} + 64w^2Q_{22}^2 + 76w^5(w')^2\\
&+ 512Q_{20}(w')^2 - 64wQ_{21}(w')^2\\
&- 64w^2Q_{22}(w') - 20w^2(w')^4\\
&+ 80w^3(w')^2w'' - 32Q_{22}(w')^2w''\\
&- 8(w')^4w'' - 10w^4(w'')^2 - 32Q_{21}(w'')^2\\
&+ 16wQ_{22}(w'')^2 + 20w(w')^2(w'')^2 + (w'')^4 = 0
\end{aligned}
$$

$c = 0, q = 0$

$$
\begin{aligned}
&25w^8 + 76w^5(w')^2 - 20w^2(w')^4\\
&+ 80w^3(w')^2w'' - 8(w')^4w'' - 10w^4(w'')^2\\
&+ 20w(w')^2(w'')^2 + (w'')^4 = 0
\end{aligned}
$$

We shall see later on that these equations has a 2-dimensional commutative symmetry Lie algebra generated by translations and the 1st KdV, and therefore, they can be solved in quadratures.

Now we apply theorem (5) to the spectral problems for the Shrödinger ordinary differential equations in which potentials are solutions of the n-th KdV equations. This gives us complete and explicit solutions of the spectral problems.

We illustrate this method for potentials which satisfy the first KdV equation (this is the case of a special Lame equation) and for the following boundary value problem on an interval $[a, b]$

$$y(a) = y(b) = 0. \tag{11}$$

Then theorem (5) shows that smooth eigenfunctions

$$y(x) = 2\sqrt{\frac{|S_1(x)|}{Q_1(\lambda)}}\sin\left(\frac{\sqrt{Q_1(\lambda)}}{2})\int_a^x \frac{d\tau}{S_1(\tau)}\right)$$

do exist if:

- $S_1 = \lambda + \frac{w}{2} + c_1 \neq 0$ on the interval,
- $Q_1 = (\lambda) > 0$,
- $$\int_a^b \frac{d\tau}{S_1(\tau)} = \frac{2\pi n}{\sqrt{Q_1(\lambda)}}$$

  for $n \in \mathbb{Z}$.

Summarizing we get the following result.

Theorem 6. If a potential w **satisfies the classical KdV** *equation,* $w''' + 6ww' + 4c_1w' = 0$*, then the spectral values* λ *for value boundary problem (11) of Shrödinger Equation (10) are given by the formula*

$$\lambda = \wp(\alpha, g_2, g_3) - 2c_1/3,$$

where α *are solutions of the equations*

$$
\begin{aligned}
&2(b - a)\zeta(\alpha) + \ln\frac{\sigma(b + c - \alpha)\sigma(a + c + \alpha)}{\sigma(a + c - \alpha)\sigma(b + c + \alpha)}\\
&\quad = 2\pi ni, \quad n \in \mathbb{Z},
\end{aligned}
$$

such that $Q_1(\lambda) > 0$ *and*

$$\lambda > -c_1 - \frac{1}{2}\min[w(x), a \le x \le b],$$

or

$$\lambda < -c_1 - \frac{1}{2}\max[w(x), a \le x \le b].$$

Here

$$Q_1(\lambda) = -4\lambda^3 - 8c_1\lambda^2 + q_{11}(w)\lambda + q_{10}(w),$$

and constants $q_{11}(w)$ *and* $q_{10}(w)$ *are values of the first integrals* q_{10} *and* q_{11} *on the solution* w. *The function* $\zeta(\alpha)$ *is the Weierstrass zeta function and* $\sigma(z)$ *is the Weierstrass sigma function with invariants*

$$g_2 = \frac{4\mathbf{c}_1^2 - 6\mathbf{c}_1}{3}, g_3 = -\frac{152\mathbf{c}_1^3}{27} - \frac{5q_{11}(w)\mathbf{c}_1}{3} + 2q_{10}(w).$$

The eigenfunctions corresponding to the eigenvalue λ *have the following form:*

$$y_\lambda(x) = 2\sqrt{\frac{|\lambda + \frac{w(x)}{2} + \mathbf{c}_1|}{Q_1(\lambda)}}\sin\left(\frac{\sqrt{Q_1(\lambda)}}{2}\right.$$
$$\left.\times\left(2(x-a)\zeta(\alpha) + \ln\frac{\sigma(x-\alpha)\sigma(a+\alpha)}{\sigma(x+\alpha)\sigma(a-\alpha)}\right)\right)$$

Proof: Since

$$w = -2\wp(x + c, g_2, g_3) - \frac{2c_1}{3},$$

we have

$$I = \int_a^b \frac{d\tau}{\lambda + \frac{w}{2} + c_1}$$
$$= \int_a^b \frac{d\tau}{\lambda - \wp(\tau + c, g_2, g_3) + 2c_1/3}.$$

Let α be such that

$$\wp(\alpha, g_2, g_3) = \lambda + 2c_1/3.$$

Then (see [7])

$$I = -\int_{a+c}^{b+c} \frac{d\tau}{\wp(\tau, g_2, g_3) - \wp(\alpha, g_2, g_3)}$$
$$= -\frac{1}{\wp'(\alpha, g_2, g_3)}\left(2z\xi(\alpha) + \ln\frac{\sigma(z-\alpha)}{\sigma(z+\alpha)}\right)\Bigg|_{a+c}^{b+c}.$$

Since

$$\wp'^2(x, g_2, g_3) = 4\wp^3(x, g_2, g_3) - g_2\wp(x, g_2, g_3) - g_3$$

and

$$\wp''(x, g_2, g_3) = 6\wp^2(x, g_2, g_3) - g_2/2,$$

we obtain the following values of the first integrals q_{11} and q_{10}:

$$q_{11}(w) = w'' + 3w^2 + 4c_1w - 4c_1^2$$
$$= g_2 - 16/3c_1^2 = -2c_1 - 4c_1^2,$$
$$q_{10}(w) = \frac{2ww'' - w'^2 + 4w^3}{4}$$
$$+ c_1(w'' + 4w^2 + 4c_1w)$$
$$= 2/3c_1g_2 + g_3 - 32/27c_1^3.$$

Note that for

$$\wp(\alpha, g_2, g_3) = \lambda + 2c_1/3$$

we have

$$\wp'^2(\alpha, g_2, g_3)$$
$$= 4\wp^3(\alpha, g_2, g_3) - g_2\wp(\alpha, g_2, g_3) - g_3$$
$$= -Q_1(\lambda).$$

Therefore,

$$\wp'(\alpha, g_2, g_3) = \pm\sqrt{-Q_1(\lambda)}$$

and we get that

$$I = \pm\frac{1}{\sqrt{-Q_1(\lambda)}}\left(2z\xi(\alpha) + \ln\frac{\sigma(z-\alpha)}{\sigma(z+\alpha)}\right)\Bigg|_{a+c}^{b+c}$$
$$= \frac{2\pi n}{\sqrt{Q_1(\lambda)}}.$$

So we have

$$\pm\left(2z\zeta(\alpha) + \ln\frac{\sigma(z-\alpha)}{\sigma(z+\alpha)}\right)\Bigg|_{a+c}^{b+c} = 2\pi ni,$$

where $\zeta(\alpha)$ is the Weierstrass zeta function and $\sigma(z)$ is the Weierstrass sigma function.

Finally we get the following equations for α:

$$2(b-a)\zeta(\alpha) + \ln\frac{\sigma(b+c-\alpha)\sigma(a+c+\alpha)}{\sigma(a+c-\alpha)\sigma(b+c+\alpha)} = 2\pi ni.$$

□

In a the similar way one gets the following result.

Theorem 7. *If a potential w satisfies the n-th KdV Equation, then the spectral values λ for value boundary problem (11) of Shrödinger Equation (10) are given by solutions of the Equation*

$$\int_a^b \frac{d\tau}{S_n(\tau)} = \frac{2\pi m}{\sqrt{Q_n(\lambda)}}, \quad m \in \mathbb{Z},$$

such that $Q_n(\lambda) > 0$ *and* $S_n \neq 0$ *on the interval* $[a, b]$.

The eigenfunctions corresponding to the eigenvalue λ *have the following form:*

$$y_\lambda(x) = 2\sqrt{\frac{|S_n(x)|}{Q_n(\lambda)}}\sin\left(\frac{\sqrt{Q_n(\lambda)}}{2}\int_a^x \frac{d\tau}{S_n(\tau)}\right).$$

2.5 Dynamics

It is common to use symmetries to integrate ordinary differential equations. Now we turn it over and we will use ODEs for integrating of evolutionary differential equations.

Let

$$u_t = \varphi\left(x, u, u_x, \ldots, \frac{\partial^k u}{\partial x^k}\right). \qquad (12)$$

We are interested in ordinary differential equations

$$F\left(x, y, y', y'', \ldots, y^{(m-1)}, y^{(m)}\right) = 0$$

for which the Lie equation is satisfied for the given generating function φ. In other words, we find such ODEs for which φ is a generating function of shuffling symmetry.

In this case any solution of the Cauchy problem

$$\left\{ \frac{\partial u}{\partial t} = \varphi\left(x, u, u_x, \ldots, \frac{\partial^{m+k} u}{\partial x^{m+k}}\right), \quad u|_{t=0} = h_0(x) \right.$$

under the condition that $h_0(x)$ satisfies our ordinary differential equation, can be found as a path h_t

$$u(t, x) = h_t(x)$$

in the space of all solutions of the ODE $F(x, y', y'', \ldots, y^{(m-1)}, y^m) = 0$ (see 2.2).

Conditions on F are given by Equation (6):

$$\sum_{i=0}^{m} \frac{\partial F}{\partial p_i}\delta^i(\widetilde{\varphi}) = 0, \quad \text{if } F = 0. \qquad (13)$$

In this equation $\widetilde{\varphi}$ is the restriction of the generating function φ on the equation $F^{-1}(0)$ and all its prolongations:

$$F = 0,$$
$$\mathbf{D}(F) = 0,$$
$$\mathbf{D}^2(F) = 0, \ldots,$$

where

$$\mathbf{D} = \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial p_0} + \cdots + p_m\frac{\partial}{\partial p_{m-1}} + \ldots$$

is the total derivative.

We consider relation (13) as a differential equation on the function F.

The ordinary differential equation $\{F = 0\}$ corresponding to the solution F of this equation we call *dynamics of order m* for evolutionary Equation (12).

Note that in this case in order to find a trajectory of a solution $h(x)$ one should integrate the system

$$\begin{cases} \dot{p_0} = \widetilde{\varphi}, \\ \dot{p_1} = \delta(\widetilde{\varphi}), \\ \cdots\cdots \\ \dot{p_m} = \delta^m(\widetilde{\varphi}) \end{cases}$$

on $\mathcal{E}$ with the initial conditions

$$\begin{cases} p_0|_{t=0} = h(x), \\ p_1|_{t=0} = h'(x), \\ \cdots\cdots \\ p_m|_{t=0} = h^{(m)}(x). \end{cases}$$

Then $p_0(t) = h_t(x)$ gives the unknown function $h_t(x)$, and $u(t,x) = h_t(x)$ is a solution of the evolutionary equation $\frac{\partial u}{\partial t} = \varphi(x, u, u_x, \ldots, \frac{\partial^k u}{\partial x^k})$.

3 Dynamics for the KdV equation

As an illustration of the above method let us describe finite dimensional dynamics for the KdV equation:

$$u_t = u \cdot u_x + u_{xxx}.$$

Substitution $u = 6w$ establishes a relation between KdV equations considered above and this equation. We rewrite functions $z_n(w), s_n(\lambda, w), Q_n(\lambda, w)$ in the canonical coordinates $(p_0, \ldots, p_n, \ldots)$ on the jet spaces where p_0 corresponds to u.

In these notations we have the following Lenard's recursion (see [2]):

$$z_0 = 1, \quad z_1 = \frac{p_0}{12} + c_1, \ldots$$

and

$$\mathbf{D}(z_{n+1}) = \frac{1}{4} L(z_n)$$

for $n = 0, 1, \ldots$, and

$$L = \mathbf{D}^3 + \frac{2}{3} p_0 \mathbf{D} + \frac{1}{3} p_1.$$

The functions

$$K_n = \mathbf{D}(z_{n+1})$$

correspond to the n-th stationary KdV equations.

Let

$$S_n = \sum_{i=0}^{n} z_i \lambda^{n-i},$$

and

$$Q_n = 2 S_n \mathbf{D}(K_{n-1}) + \frac{2}{3} p_0 S_n^2 - 4\lambda S_n^2 - K_{n-1}^2 = \sum_{i=0}^{2n+1} q_{ni} \lambda^i$$

Then we have

$$X_\varphi(z_n) = \mathbf{D}^3(z_n) + p_0 z_n,$$

and

$$X_\varphi(K_n) = \mathbf{D}^3(K_n) + p_1 \mathbf{D}(K_n) + p_0 K_n,$$

where $\varphi = p_3 + p_0 p_1$, and

$$X_\varphi = \sum_{i=0} D^i(\varphi) \frac{\partial}{\partial p_i}.$$

This shows that differential equations corresponding to linear combinations of z_n or K_n give finite dimensional dynamics for the KdV equation. Moreover, functions q_{ni} also produce finite dimensional dynamics.

Remark that the order of z_{n+1} is $2n$, and the order of K_n is $2n+1$. Therefore z's give even dimensional and K's odd dimensional dynamics.

Summarizing we arrive to the following result.

Theorem 8. *Let $a_0, \ldots, a_n \in \mathbb{R}$ be constants. Then the differential equations:*

$$\sum_{i=0}^{n} a_i z_i = 0,$$

$$\sum_{i=0}^{n} a_i K_i = 0,$$

$$q_{ni} = 0$$

give finite dimensional dynamics for the KdV equation.

In addition to the above theorem the low dimensional dynamics given by polynomials can be found by direct computations. Below we give and describe some of them in dimensions ≤ 3.

3.1 1st order dynamics

One can check that the functions

$$F = 3p_1^2 + p_0^3 + a_2 p_0^2 + a_1 p_0 + a_0$$

satisfy the equation of the dynamics for $\varphi = p_0 p_1 + p_3$ and arbitrary constants a_0, a_1, a_2.

The solution space **Sol** $(\mathcal{E})$ can be identified with the curve

$$3p_1^2 + p_0^3 + a_2 p_0^2 + a_1 p_0 + a_0 = 0$$

on the (p_0, p_1) plane.

The vector field X_φ on the ODE $\mathcal{E}$ has the following form

$$-\frac{a_2}{3}\left(p_1 \frac{\partial}{\partial p_0} - \frac{a_1 + 2a_2 p_0 + 3p_0^2}{6}\frac{\partial}{\partial p_1}\right).$$

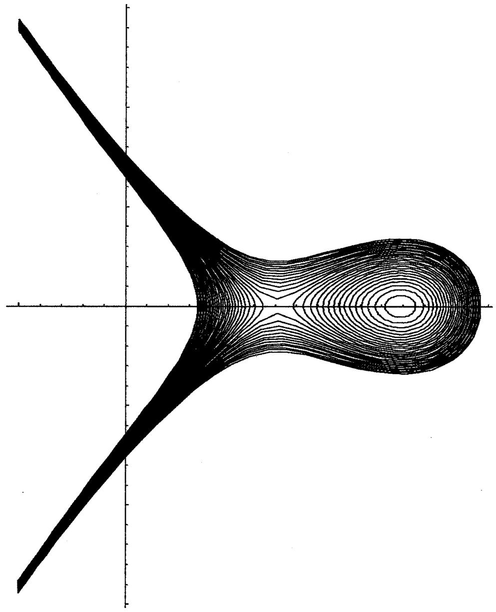
Moreover, the vector field

$$p_1 \frac{\partial}{\partial p_0} - \frac{a_1 + 2a_2 p_0 + 3p_0^2}{6}\frac{\partial}{\partial p_1}$$

is Hamiltonian with respect to the standard symplectic structure $dp_1 \wedge dp_0$ with the Hamiltonian

$$H = 3p_1^2 + p_0^3 + a_2 p_0^2 + a_1 p_0.$$

In other words the curves $H =$ const define the solution spaces and the Hamiltonian flow is the flow generated by KdV.



KdV 1-st order dynamics

Solutions of the equation $F = 0$, have the form

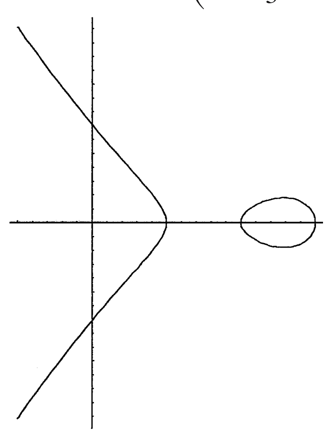
$$u = -12\wp(x + c, g_2, g_3) - \frac{a_2}{3},$$

where $\wp(x, g_2, g_3)$ is the Weierstrass elliptic function with invariants

$$g_2 = \frac{a_2^2 - 3a_1}{108}, \quad g_3 = \frac{27a_0 - 9a_1 a_2 + 2a_2^3}{11664}.$$

The shift of these solutions along X_φ leads us to the following solutions of the KdV equation

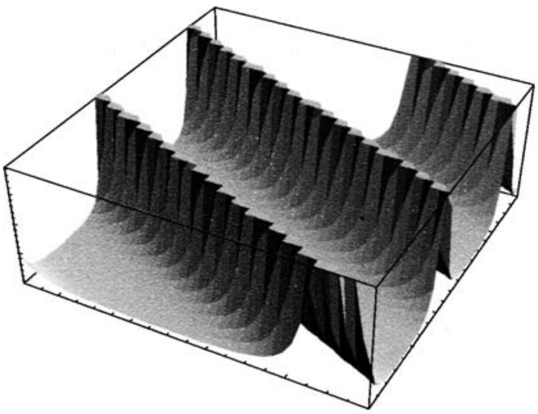
$$u(x,t) = -12\wp\left(x - \frac{a_2 t}{3} + c, g_2, g_3\right) - \frac{a_2}{3}.$$



The solution space

These pictures show the solution space and a trajectory for the 1-st order dynamics with

$$F = p_1^3 + (p_0-1)(p_0-2)(p_0-3).$$



The trajectory

3.2 Second order dynamics

Here we describe the second order dynamics. We shall consider dynamics $F(p_0, p_1, p_2)=0$ which are invariant with respect to the scale symmetry $xp_x + 3tp_t + 2p_0$ for the KdV equation. We assign weight 2 for p_0, weight 3 for p_1 and weight 4 for p_2 and we assume that F is a sum of homogeneous polynomial of degree $\leq n$.

The following list gives non-trivial homogeneous dynamics for small n:
$n=4$

$$F_4 = p_0^2 + 2p_2 + ap_0 + b,$$

$n=6$

$$F_6 = 2(a+3p_0)p_2 - 6p_1^2 + p_0^3 + 3ap_0^3 \\ + 3ap_0^2 + b\, p_0 + c,$$

$n=8$

$$F_8 = p_0^4/4 + p_0^2 p_2 + p_2^2 + a\big(2p_0^3 - 3p_1^2 \\ + 6p_0p_2\big) + b\big(p_0^2 + 2p_2\big) + c$$

where a, b and c are constants, and
$n=10$

$$F_{10} = 8p_2^3 + 9p_1^4.$$

Not that the dynamics F_4 and F_6 coincide with q_{10} and q_{11}.

The previous differential equation $\mathcal{E}_{JT}$ gives us the following dynamics:

$$
\begin{aligned}
&16384c_1^4c_2^2 - 16384c_1^2c_2^3 4096c_2^4 + \frac{8192}{3}c_1^3c_2^2p_0 \\
&- \frac{4096}{3}c_1c_2^3p_0 + \frac{512}{9}c_2^3p_0^2 - \frac{256}{27}c_1^3c_2p_0^3 \\
&- \frac{128}{27}c_1c_2^2p_0^3 - \frac{16}{9}c_1^2c_2p_0^4 + \frac{56}{81}c_2^2p_0^4 - \frac{4}{81}c_1c_2p_0^5 \\
&+ \frac{1}{729}c_1^2p_0^6 + \frac{5c_2p_0^6}{1458} + \frac{5c_1p_0^7}{17496} + \frac{25p_0^8}{1679616} \\
&+ \frac{256}{9}c_1^3c_2p_1^2 - \frac{640}{9}c_1c_2^2p_1^2 - \frac{32}{27}c_2^2p_0p_1^2 \\
&- \frac{8}{27}c_1c_2p_0^2p_1^2 + \frac{2}{243}c_1^2p_0^3p_1^2 + \frac{2}{81}c_2p_0^3p_1^2 \\
&+ \frac{19c_1p_0^4p_1^2}{5832} + \frac{19p_0^5p_1^2}{69984} + \frac{1}{81}c_1^2p_1^4 \\
&- \frac{4}{81}c_2p_1^4 - \frac{1}{486}c_1p_0p_1^4 - \frac{5p_0^2p_1^4}{11664} \\
&- \frac{32}{27}c_1c_2p_1^2p_2 + \frac{8}{81}c_2p_0p_1^2p_2 + \frac{1}{81}c_1p_0^2p_1^2p_2 \\
&+ \frac{5p_0^3p_1^2p_2}{2916} - \frac{1}{972}p_1^4p_2 + \frac{64}{9}c_1^2c_2p_2^2 - \frac{32}{9}c_2^2p_2^2 \\
&+ \frac{16}{27}c_1c_2p_0p_2^2 - \frac{2}{81}c_2p_0^2p_2^2 - \frac{1}{486}c_1p_0^3p_2^2 \\
&- \frac{5p_0^4p_2^2}{23328} + \frac{1}{162}c_1p_1^2p_2^2 + \frac{5p_0p_1^2p_1^2}{1944} + \frac{p_2^4}{1296} \\
&+ \frac{128}{9}p_1^2Q_{20} - 4096c_1^2c_2Q_{21} + 2048c_2^2Q_{21} \\
&- \frac{1024}{3}c_1c_2p_0Q_{21} + \frac{128}{9}c_2p_0^2Q_{21} \\
&+ \frac{32}{27}c_1p_0^3Q_{21} + \frac{10}{81}p_0^4 - \frac{32}{9}c_1p_1^2Q_{21} \\
&- \frac{8}{27}p_0p_1^2Q_{21} - \frac{8}{9}p_2^2Q_{21} + 256Q_{21}^2 \\
&+ 4096c_1^3c_2Q_{22} - 2048c_1c_2^2Q_{22} \\
&+ \frac{2048}{3}c_1^2c_2p_0Q_{22} - \frac{512}{3}c_2^2p_0Q_{22} \\
&+ \frac{128}{9}c_1c_2p_0^2Q_{22} - \frac{32}{27}c_1^2p_0^3Q_{22}
\end{aligned}
$$

$$-\frac{32}{27}c_2 p_0^3 - \frac{2}{9}c_1 p_0^4 Q_{22} - \frac{5}{486}p_0^5 Q_{22}$$
$$+\frac{32}{9}c_1^2 p_1^2 Q_{22} - \frac{64}{9}c_2 p_1^2 Q_{22}$$
$$-\frac{4}{81}p_0^2 p_1^2 Q_{22}\frac{4}{27}p_1^2 p_2 Q_{22} + \frac{8}{9}c_1 p_2^2 Q_{22}$$
$$+\frac{2}{27}p_0 p_2^2 Q_{22} - 512 c_1 Q_{21} Q_{22} - \frac{128}{3}p_0 Q_{21} Q_{22}$$
$$+256 c_1^2 Q_{22}^2 + \frac{128}{3}c_1 p_0 Q_{22}^2 + \frac{16}{9}p_0^2 Q_{22}^2 = 0$$

Let us look at the dynamics in more details.

3.2.1 $F_4 = p_0^2 + 2p_2 + ap_0 + b$

In this case the vector field $-\frac{2}{a}X_\varphi$ is a restriction of the vector field

$$V_\varphi = p_1\frac{\partial}{\partial p_0} + p_2\frac{\partial}{\partial p_1} - \frac{(a+2p_0)p_1}{2}\frac{\partial}{\partial p_2}$$

on the zero level $F_4 = 0$, and X_φ can be integrated in the same way as for the 1-st order dynamics.

Trajectories of X_φ are given by the formula

$$p_0(t) = -12\wp(t + K, g_2, g_3) - a/2,$$

where

$$g_2 = \frac{a^2 - 12ab}{48}, \quad g_3 = \frac{a^3 - 12ab - 12c^2}{1738}$$

and the constant can be found from the initial data.

These formulas lead us to the following pathes in the solution space

$$u(t,x) = -12\wp\left(t - \frac{ax}{2} + \text{const}, g_2, g_3\right) - a/2$$

with arbitrary invariant g_3 and g_2 given above.

3.2.2 $F_6 = 2(a + 3p_0)p_2 - 6p_1^2 + p_0^3 + ap_0^2 + b\, p_0 + c$

The differential equation $\mathcal{E} = F_6^{-1}(0)$ has singular points at

$$p_0 = -\frac{a}{3},$$

$$p_1^2 = \frac{a^3}{81} - \frac{ab}{18} + \frac{c}{8}.$$

Moreover, in this case symmetries X_{p_1} and X_φ are linearly dependent on the differential equation $\mathcal{E}$ and

$$X_\varphi =$$
$$-H\left(p_1\frac{\partial}{\partial p_0} + \frac{6p_1^2 - p_0^3 - ap_0^2 - bp_0 - c}{2(a+3p_0)}\frac{\partial}{\partial p_1}\right),$$

where

$$H = \frac{ab + 3c + 6bp_0 - 6p_0^3 - 18p_1^2}{2(a+3p_0)^2}$$

is the fist integral for $\mathcal{E}$, and for the vector filed X_φ:

$$X_\varphi(H) = 0$$

on $\mathcal{E}$.

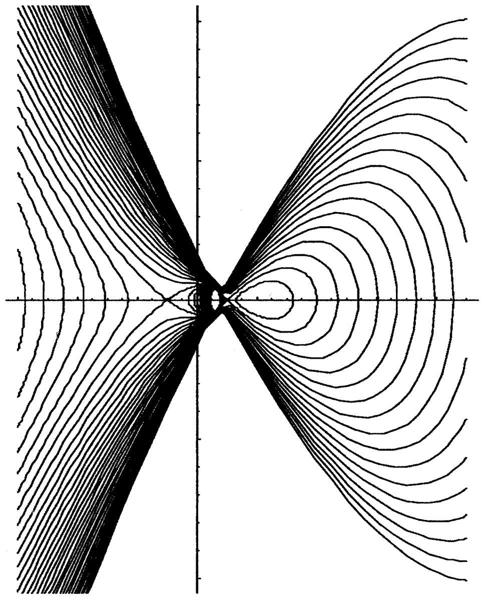
The vector field X_φ has also singularities at the points where $a + 3p_0 \neq 0$, and

$$p_0^3 + ap_0^2 + bp_0 + c = 0, \quad p_1 = 0.$$

Depending on the roots of the polynomial $p_0^3 + ap_0^2 + bp_0 + c$ we have the following three types of phase portraits:

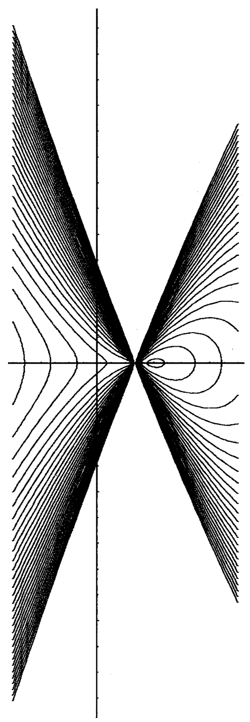
- Three real distinct roots. In the following picture we take roots: $-1, 1, 2$, and

$$F_6 = 2(3p_0 - 2)p_2 - 6p_1^2 + p_0^3 - 2p_0^2 - p_0 + 2$$



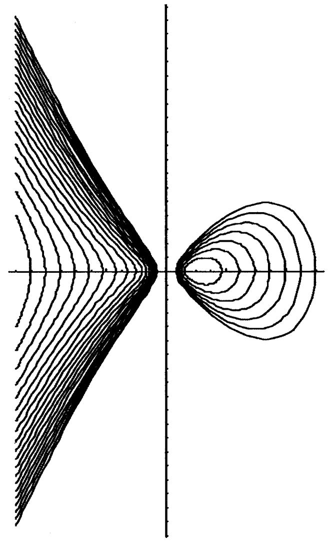
- A real root of multiplicity 2. In the picture we take roots: 1, 1, 2, and

$$F_6 = 2(3p_0 + 4)p_2 - 6p_1^2 + p_0^3 - 4p_0^2 + 5p_0 - 2$$



- Two complex roots. In the picture we take roots: 1, $\frac{-1+\sqrt{-3}}{2}$, $\frac{-1-\sqrt{-3}}{2}$, and

$$F_{61} = 6p_0p_2 - 6p_1^2 + p_0^2 - 1$$



Solutions of the equations $F_6^{-1}(0)$ one can find from the first integral H. So they are solutions of the following 1-st order ODE

$$p_1^2 = -\frac{p_0^3}{3} - kp_0^2 + \frac{b-2ak}{3}p_0 + \frac{ab + 3c - 2a^2k}{18}$$

for some constant k, $H = k$.

Thus solutions of the ODE can be represented in terms of the Weierstrass function as follows

$$u(x) = -12\wp(x + C_0, g_2, g) - k$$

where

$$g_2 = \frac{k(b + k - 2ak)}{12},$$

$$g_3 = \frac{12k^3 - 12ak^2 + 2(a^2 + 3b)k - ab - c}{2592}.$$

Note that along X_φ, function the H is constant and $X_\varphi = -HX_{p_1}$.

Therefore, the corresponding path in the solution space is

$$u(x, t) = -12\wp(x - kt + C_0, g_2, g_3) - k.$$

3.3 $F = 6(p_0 - \lambda)p_2 - 6p_1^2 + (p_0 - \lambda)^3$

This is a special case when the polynomial $p_0^3 + ap_0^2 + bp_0 + c$ has root λ of multiplicity 3. Without loss of generality we can assume that $\lambda = 0$, and we study the ordinary differential equation $\mathcal{E}$, where

$$F = 6p_0p_2 - 6p_1^2 + p_0^3 = 0.$$

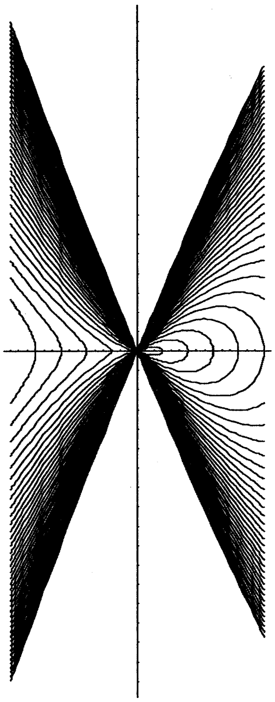
The vector field X_φ on $\mathcal{E}$ is proportional to X_{p_1}

$$X_\varphi = -HX_{p_1}$$

with

$$H = -\frac{p_0^3 + 3p_1^2}{3p_0^2}.$$

Here H is a first integral for ordinary differential equation $\mathcal{E}$ and for the vector field X_φ.

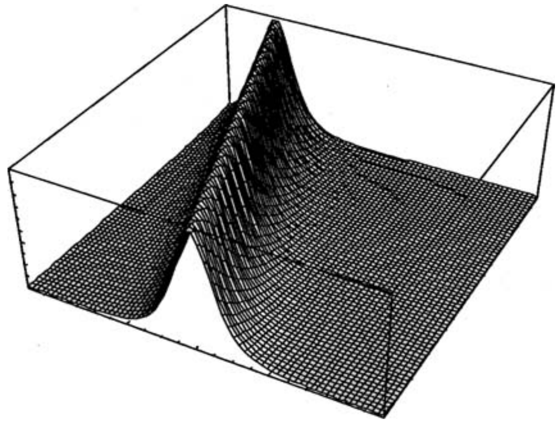


The ODE $\mathcal{E}$ can be solved directly and one gets

$$u(x) = \frac{a^2}{\cosh^2\left(\frac{a(x+b)}{2\sqrt{3}}\right)}.$$

The restriction of H on these solutions has the value $-a^2/3$, and therefore the corresponding path is the solitary wave solution

$$u(x,t) = \frac{a^2}{\cosh^2\left(\frac{a(x+a^2t/3+b)}{2\sqrt{3}}\right)}.$$



3.3.1 $F_{10} = 8p_2^3 + 9p_1^4$

In this case the vector field X_φ on $F_{10}^{-1}(0)$ is proportional to X_{p_1}:

$$X_\varphi = -HX_{p_1}$$

with

$$H = \frac{3p_1^2 - 2p_0p_2}{2p_2}.$$

Solutions of the equation $F_{10} = 0$ have the form

$$u(x) = B - \frac{12}{(x+A)^2},$$

and $H = -B$ for these solutions. Therefore, the corresponding path has the form of a rational solution

$$u(x,t) = B - \frac{12}{(x+A-Bt)^2}.$$

3.3.2 Trivial dynamics

The following ODEs

$$2p_0^2 - 3p_1^2 + 6p_0p_2 + a\big(p_0^2 + 2p_2\big) + b,$$
$$p_0^4/4 + p_0^2p_2 + p_2^2 + a\big(2p_0^3 - 3p_1^2 + 6p_0p_2\big) + b\big(p_0^2 + 2p_2\big) + c,$$
$$p_0^4/4 + p_0^2p_2 + p_2^2 + a\big(2p_0^3 - 3p_1^2 + 6p_0p_2\big) \big(p_0^2 + 2p_2\big) + c$$

give the trivial dynamics: $X_\varphi = 0$ on $\mathcal{E}$-s.

3.4 Third order dynamics

The following dynamics represent nontrivial polynomial dynamics in degree ≤ 10:

$$F_1 = ap_1 + b(p_0 p_1 + p_3) + \frac{1}{2} p_0^2 p_1 - p_1 p_2 + p_0 p_3,$$
$$F_2 = p_3^2 + 2p_1^2 p_2 - a(p_3 + p_0 p_1)^2,$$
$$F_3 = (p_3 + p_1 p_0 + a)(p_1 + bp_0 + c)$$

where a, b, c are constants.

3.4.1 Dynamics for $F_1 = ap_1 + b(p_0 p_1 + p_3) + \frac{1}{2} p_0^2 p_1 - p_1 p_2 + p_0 p_3$

In this case $X_\varphi = HX_{p_1}$, where

$$H = \frac{2p_2 + p_0^2 - 2a}{2(b + p_0)}$$

is a first integral of the ordinary differential equation $F_1^{-1}(0)$.

Solutions of the ordinary differential equations $H = c$, where c is a constant, or

$$p_2 = -\frac{p_0^2}{2} + cp_0 + a + bc$$

can be expressed in terms of the Weierstrass functions:

$$u = -12\wp(x + c_1, g_2, g_3) - c$$

with arbitrary g_3 and

$$g_2 = \frac{(1 - 3b)c^2 - 3ac}{12}.$$

The corresponding pathes in the solution space are

$$u(x, t) = -12\wp(x + ct + c_1, g_2, g_3) - c.$$

3.4.2 Dynamics for $F_2 = p_3^2 + 2p_1^2 p_2 - a(p_3 + p_0 p_1)^2$

Let $a \neq 1$, then $H_\varphi = HX_{p_1}$, where

$$H = \frac{\sqrt{p_0 \pm \sqrt{ap_0^2 + 2(1 - a)p_2}}}{1 - a}$$

is a first integral of ordinary differential equation $F_2^{-1}(0)$.

Solutions of the ordinary differential equations $H = c$, where c is a constant,

$$p_2 = \frac{p_0^2}{2} - cp_0 + c^2(1 - a)$$

can be expressed in terms of the Weierstrass functions:

$$u = 12\wp(x + c_1, g_2, g_3) + c$$

with arbitary g_3 and

$$g_2 = \frac{c^2}{12} + \frac{c^3(1 - a)}{2}.$$

The corresponding pathes in the solution space are

$$u(x, t) = 12\wp(x + ct + c_1, g_2, g_3) + c.$$

In the case $a = 1$, we have

$$F_2 = p_1^2\left(2p_2 - p_0^2\right) - 2p_0 p_1 p_3$$

and $X_\varphi = HX_{p_1}$, where

$$H = \frac{p_0^2 + 2p_2}{2p_0}$$

is a first integral of ODE $F_2^{-1}(0)$.

Solutions of ODEs $H = c$, where c is a constant,

$$p_2 = -\frac{p_0^2}{2} + cp_0$$

can be expressed in terms of the Weierstrass functions:

$$u = -12\wp(x + c_1, g_2, g_3) - c$$

with arbitrary g_3 and

$$g_2 = \frac{c^2}{12}.$$

The corresponding pathes in the solution space are

$$u(x,t) = -12\wp(x + ct + c_1, g_2, g_3) - c.$$

3.6 Fourth order dynamics

The fourth order dynamics are defined by functions

$$F = p_4 + \frac{5p_0p_2}{3} + \frac{5p_1^2}{6} + \frac{5p_0^3}{18} + \left(p_2 + \frac{p_0^2}{2}\right) + bp_0 + c,$$

where a, b, c are constants.

It is easy to check that the vector field X_φ and the ODE $F^{-1}(0)$ has the following first integral

$$H_1 = -36bp_0^2 - 12ap_0^2 - 5p_0^4 - 12p_0\big(6c + 5p_1^2\big) + 36\big(6ac - ap_1^2 + p_2^2 - 2p_1p_3\big),$$

and the restriction on $H_1 = k$ admits first integral

$$\begin{aligned} H_2 = {} & 72(2a^2 + 5b)p_0^6 + 120ap_0^7 + 25p_0^8 \\ & + 24p_0^5\big(36ab + 30c + 19p_1^2\big) + 432p_1^4 \\ & \times(3a^2 - 12b - 4p_2) + 2p_0^4(648b^2 - 216ac \\ & + 5k + 684ap_1^2 - 180p_2^2) + \big(-216ac \\ & + k - 36p_2^2\big)^2 - 24_0^p(216(a^2 - b)c - ak \\ & + 36ap_2^2 - 12p_1^2(3a^2 + 9b + 10p_2)) \\ & -216p_1^2(72(a^2 + 2b)c - ak - 12p_2 \\ & \times(4c + ap_2)) - 72p_0^2(72(3ab - c)c \\ & -bk + 10p_1^4 + 36bp_2^2 - 12p_1^2(3ab \\ & + 4c + 6ap_2)) + 24p_0\big(-36ap_1^4 \\ & + 6c\big(-216ac + k - 36p_2^2\big) \\ & + p_1^2(432ac - k + 36p_2(12b + 5p_2))\big). \end{aligned}$$

The last ordinary differential equation $H_2 = k_2$ has two symmetries X_φ and X_{p_1} and they are independent and commute. Therefore, the differential equation can be integrated in quadratures.

Namely, the method discussed above gives us two 1-forms

$$\theta_0 = \frac{p_2dp_0 - p_1dp_1}{G},$$

$$\theta_1 = \frac{Adp_0 + Bdp_1}{G} - dx,$$

where

$$\begin{aligned} G = {} & cp_1 + bp_0p_1 + \frac{1}{2}ap_0^2p_1 + \frac{5}{18}p_0^3p_1 - \frac{p_1^3}{6} \\ & + \frac{3acp_2}{p_1} - \frac{kp_2}{72p_1} - \frac{cp_0p_2}{p_1} - \frac{bp_0^2p_2}{2p_1} \\ & - \frac{ap_0^3p_2}{6p_1} - \frac{5p_0^4p_2}{72p_1} + \frac{1}{2}ap_1p_2 \\ & + \frac{5}{6}p_0p_1p_2 + \frac{p_2^3}{2p_1}, \end{aligned}$$

$$\begin{aligned} A = {} & c + bp_0 + a\left(p_2 + \frac{p_0^2}{2}\right) \\ & + \frac{1}{18}\big(5p_0^3 - 3p_1^2 + 12p_0p_2\big), \end{aligned}$$

$$\begin{aligned} B = {} & \frac{3ac}{p_1} - \frac{k}{72p_1} - \frac{cp_0}{p_1} - \frac{bp_0^2}{2p_1} - \frac{ap_0^3}{6p_1} \\ & - \frac{5p_0^4}{72p_1} - \frac{ap_1}{2} + \frac{p_0p_1}{6} + \frac{p_2^2}{2p_1}, \end{aligned}$$

and integrals $I_0(p_0, p_1), I_1(p_0, p_1)$ such that

$$dI_0 = \frac{p_2}{G}dp_0 - \frac{p_1}{G}dp_1,$$

$$dI_1 = \frac{A}{G}dp_0 + \frac{B}{G}dp_1,$$

and solutions can be found from relations

$$I_0 = c_0, \quad I_1 = x + c_1$$

for some constants c_0, c_1.

References

1. Duzhin, S.V., Lychagin, V.V.: Symmetries of distributions and quadrature of ordinary differential equations. Acta Appl. Math. **24**, no.1, 29–57 (1991)

2. Ibragimov, N.H.: Transformation groups in mathematical physics. M. Nauka (1983)
3. Lax, P.D.: Integrals of nonlinearequations of evolution and solitary waves. Comm. Pure Appl. Math. **21**, 467–490 (1968)
4. Gardner, C.S. Greene, J.M., Kruskal, M.D., Miura, R.M.: Kortweg- de Vries equation and generalizations, VI, Methods for exact solution. Comm. Pure Appl. Math. **27**, 97–133 (1974)
5. Krasilshchik, I.S., Lychagin, V.V., Vinogradov, A.M.: Geometry of Jet Spaces and Non-Linear Differential Equations. Gordon and Breach, New York (1986)
6. Kruglikov Boris, Lychagin Valentin, Mayer brackets and solvability PDEs, 1. Diff. Geometry Appl. **17** 251–272 (2002)
7. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions, Dover, New York (1972)