ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2009, Vol. 30, No. 4, pp. 333–336. © Pleiades Publishing, Ltd., 2009.

Minkowski Metrics on Solutions of the Khokhlov–Zabolotskaya Equation

V. Lychagin and V. Yumaguzhin*

Received June 22, 2009

Abstract—In this paper, we show that every solution of the Khokhlov–Zabolotskaya equation possesses a Minkowski metric. This metric is responsible for transport of singularities of KZ-solutions. We find explicit solutions for which the metrics are either locally-flat or Ricci-flat or conformally-flat.

2000 Mathematics Subject Classification: 35A30, 35Q72, 53B30, 53C50.

DOI: 10.1134/S1995080209040106

Key words and phrases: *Khokhlov–Zabolotskaya equation, jet-bundle, symbol of nonlinear differential operator, metrics, Riemann tensor, Ricci tensor, Weyl tensor, explicit solution.*

1. INTRODUCTION

The geometric singularities of any nonlinear differential equation are moving along Hamiltonian vector fields where the Hamiltonian is defined by the symbol of the linearization of the PDE on a solution, see [3]. In this paper, we consider a 2-order nonlinear differential operator which defines the Khokhlov–Zabolotskaya (KZ) equation. A symbol of the linearization of this operator defines a Minkowski metric on every KZ-solution. The zero-geodesics of the metric (or "light rays") are just trajectories of geometric singularities of KZ-solutions. We investigate various classes of explicit solutions on which the metrics are either locally-flat or Ricci-flat or conformally-flat. It is appeared that only solutions with Ricci-flat metrics are Einstein manifolds and only locally-flat metrics of solutions are projectively-flat.

2. KHOKHLOV–ZABOLOTSKAYA EQUATION

2.1. Nonlinear Differential Operator

The following nonlinear PDE is the Khokhlov–Zabolotskaya equation, see [2],

$$u_{tx} - (uu_x)_x - u_{yy} - u_{zz} = 0. \tag{1}$$

The corresponding nonlinear differential operator may be defined as follows.

Let us consider the trivial bundle

$$\pi : \quad \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R}^4, \qquad \pi : \quad (t, x, y, z, u) \mapsto (t, x, y, z).$$

Denote by $j^k_p S$ the k-jet at a point p of a section S of the bundle π, and by $J^k\pi$ we denote the manifold of all k-jets of all sections of π.

Let

$$\pi_k : \quad J^k\pi \longrightarrow \mathbb{R}^4, \quad \pi_k : \quad j^k_p S \mapsto p$$

be the k-jet bundle of sections of π.

For all natural numbers q and r such that $q > r$, we denote by $\pi_{q,r}$ the natural projections

$$\pi_{q,r} : \quad J^q\pi \longrightarrow J^r\pi, \qquad \pi_{q,r} : \quad j^q_p S \mapsto j^r_p S.$$

*E-mail: yuma@diffiety.botik.ru

Every section S of the bundle π generates the section j_kS of the bundle π_k by the formula

$$j_kS:\quad \mathbb{R}^4 \longrightarrow J^k\pi, \qquad j_kS:\quad p \mapsto j^2_pS.$$

Consider now the nonlinear 2nd order differential operator Δ acting on sections of π by the formula

$$\Delta = \varphi_\Delta \circ j_2,$$

where the function φ_Δ: $J^2\pi \to \mathbb{R}$ is defined by the left hand side of equation (1)

$$\varphi_\Delta(t,x,y,z,u,u_t,\ldots,u_{zz}) = u_{tx} - uu_{xx} - u_{yy} - u_{zz} - u_x^2,$$

where $t,x,y,z,u,u_t,\ldots,u_{zz}$ are the canonical coordinates on the 2-jet bundle $J^2\pi$.

Then the set of all solutions of equation (1) coincides with the set of all sections S of π so that $\Delta(S) = 0$.

2.2. Symbols

Let $\theta_2 \in J^2\pi$, $\theta_1 = \pi_{2,1}(\theta_2)$, $p = \pi_2(\theta_2)$, and let F_{θ_1} be the fiber of projection $\pi_{2,1}$ over θ_1. That is,

$$F_{\theta_1} = (\pi_{2,1})^{-1}(\theta_1).$$

Consider the exact sequence

$$0 \to \mathbb{R} \otimes (T^*_p \odot T^*_p) \xrightarrow{i} J^2_p\pi \xrightarrow{\pi_{2,1}} J^1_p\pi \to 0,$$

where T^*_p is the cotangent space to the base of π at the point p and $(T^*_p \odot T^*_p)$ is the symmetric square of the cotangent space, the map i is defined by the formula

$$i\big(v \otimes (df \odot dg)\big) = j^2_p\left(\frac{1}{2}fgS\right),$$

where f, g and S are smooth functions such that $f(p) = g(p) = 0$, and $S(p) = v$.

From this exact sequence, we get the natural isomorphism

$$F_{\theta_1} \cong T^*_p \odot T^*_p.$$

Recall, see for example [1], that the *symbol of* Δ *at point* θ_2 *is the restriction of the differential* φ_Δ *on the tangent space* $T_{\theta_2}(F_{\theta_1})$ *of the fiber* F_{θ_1} *at the point* θ_2:

$$\mathrm{Smbl}_{\theta_2}\Delta = (\varphi_\Delta)_*|_{T_{\theta_2}(F_{\theta_1})}.$$

Taking into account the natural identification of $T^*_p \odot T^*_p$ with its tangent space, we can represent $\mathrm{Smbl}_{\theta_2}\Delta$ as an element of $(T^*_p \odot T^*_p)^*$.

This element is the symmetric $(2,0)$ - tensor

$$\partial_t\partial_x - u\partial_x\partial_x - \partial_y\partial_y - \partial_z\partial_z \in T_p \odot T_p.$$

This tensor is non degenerate and therefore it generates an isomorphism $T^*_p \to T_p$. The inverse to this isomorphism is defined a metric of signature $(+,-,-,-)$

$$g(\theta_2) = 4udt^2 + 4dtdx - dy^2 - dz^2 \in T^*_p \odot T^*_p.$$

In other words, the symbol $\mathrm{Smbl}_{\theta_2}\Delta$ of the operator Δ at the point θ_2 defines in a natural way the Minkowski metric $g(\theta_2)$ on the tangent space T_p, where $p = \pi_2(\theta_2)$.

As a result of this construction, we get a field of horizontal Minkowski metrics on the 2-jet bundle $J^2\pi$

$$g:\quad J^2\pi \longrightarrow T^* \odot T^*, \qquad g:\quad \theta_2 \mapsto g(\theta_2).$$

2.3. Metric Structures on Solutions

Let

$$S: \quad (t,x,y,z) \mapsto \big(t,x,y,z,u(t,x,y,z)\big)$$

be a solution of equation (1).

Denote by $L_S^{(2)}$ the image of the section $j_2 S$ of the bundle π_2.

Then, if we identify solutions S and submanifolds $L_S^{(2)} \subset J^2\pi$, the restriction

$$g_S = g|_{L_S^{(2)}} = 4u(t,x,y,z)dt^2 + 4dtdx - dy^2 - dz^2$$

gives a Minkowski metric on the manifold $L_S^{(2)}$, associated with the solution.

2.4. Explicit Solutions

In this section we use the classical differential invariants of metrics to get classes of explicit solutions of the KZ-equation.

2.4.1. Locally-flat solutions. We will find solutions $(L_S^{(2)}, g_S)$ of KZ-equation such that the corresponding metrics g_S are locally-flat. It is known that a metric is locally flat if and only if its curvature tensor is zero.

In our case, nonzero components of the curvature tensor are:

$$\begin{aligned} R_{1212} &= -2S_{xx}, \quad R_{1213} = -2S_{xy}, \quad R_{1214} = -2S_{xz}, \\ R_{1313} &= -2S_{yy}, \quad R_{1314} = -2S_{yz}, \quad R_{1414} = -2S_{zz}. \end{aligned} \tag{2}$$

From (2) we observe that the required solution of equation (1) should also satisfy equations

$$S_{xx}=0, \quad S_{xy}=0, \quad S_{xz}=0, \quad S_{yy}=0, \quad S_{yz}=0, \quad S_{zz}=0.$$

Solving these equations together with KZ-equation, we get the following class of explicit KZ-solutions with locally flat metric g_S:

$$S(t,x,y,z) = h_1(t)yz - \frac{x}{t+C} + h_2(t)y + h_3(t)z + h_4(t),$$

where h_1, h_2, h_3, and h_4 are arbitrary smooth functions and C is an arbitrary constant.

2.4.2. Projectively-flat solutions. We look for solutions $(L_S^{(2)}, g_S)$ of KZ-equation such that metrics g_S are projectively-flat.

Recall that a metric space (L, g) is projectively-flat if there exist local coordinates in a neighborhood of every point of L such that geodesic lines of g are represented as straight lines in these coordinates.

It is known, see [4], that (L, g) is projectively-flat if and only if (L, g) is a space of constant curvature. Then the curvature tensor is expressed in terms of the metric in the following way

$$R_{lkij} = K(g_{li}g_{kj} - g_{lj}g_{ki}), \quad K = \text{constant}.$$

Comparing the curvature tensor of g_S with the tensor $(g_S)_{li}(g_S)_{kj} - (g_S)_{lj}(g_S)_{ki}$, we get that $K=0$.

Therefore the only locally-flat solutions of KZ-equation are projectively-flat.

2.4.3. Ricci-flat solutions. We find solutions $(L_S^{(2)}, g_S)$ of KZ-equation such that the Ricci tensor of g_S is zero.

In our case, nonzero components of the Ricci tensor are:

$$\begin{aligned} R_{11} &= -2SS_{xx} - 2S_{yy} - 2S_{zz}, \quad R_{12} = R_{21} = -S_{xx}, \\ R_{13} &= R_{31} = -S_{xy}, \quad R_{14} = R_{41} = -S_{xz}. \end{aligned} \tag{3}$$

From (3) we get that the required solutions of KZ-equation should also satisfy equations

$$S_{xx}=0, \quad S_{xy}=0, \quad S_{xz}=0, \quad S_{yy}+S_{zz}=0.$$

Solving these equations together with KZ-equation, we obtain the following class of explicit solutions of the KZ- equation:

$$S(t,x,y,z) = -\frac{x}{t+C} + h(t,y,z),$$

which are Ricci-flat. Here C is an arbitrary constant and h is a function, satisfying the Laplace equation

$$h_{yy} + h_{zz} = 0.$$

2.4.4. Einstein manifolds. We find solutions $(L_S^{(2)}, g_S)$ of KZ-equation which are Einstein manifolds.

Recall that a Minkowski manifold (M, g) is Einstein if and only if the Ricci tensor R of metric g is proportional to g. That is, $R = \lambda g$, for some constant λ.

Comparing the Ricci tensor of g_S with this metric, we get that the only Ricci-flat solutions of equation (1) are Einstein manifolds.

2.4.5. Conformally-flat solutions. We find solutions $(L_S^{(2)}, g_S)$ of Khokhlov—Zabolotskaya equation such that its metric g_S is conformally-flat.

Recall that a metric is called conformally-flat if, in neighborhood of every point, it can be transformed to the form $e^f g$, where f is a smooth function and g is a flat metric.

It is known that a metric is conformally-flat if and only if its Weyl tensor is zero.

In our case, nonzero components of the Weyl tensor are:

$$W_{1212} = -\frac{1}{2}S_{xx}, \quad W_{1213} = -S_{xy}, \quad W_{1214} = -S_{xz}, \tag{4}$$
$$W_{1313} = -S_{yy} + \frac{1}{3}S_{xx} + S_{zz}, \quad W_{1314} = -2S_{yz},$$

and other nonzero components are linear combinations of these ones.

From (4) we get that the required solutions should satisfy the following equations:

$$S_{xx} = 0, \quad S_{xy} = 0, \quad S_{xz} = 0, \quad S_{yz} = 0, \quad S_{yy} - S_{zz} = 0.$$

Solving these equations together with KZ-equation, we get the following class of explicit solutions of KZ-equation

$$S(t,x,y,z) = \left(\frac{d}{dt}h_1 - h_1^2\right)(y^2 + z^2) + h_1 x + h_2 y + h_3 z + h_4,$$

with conformally-flat metric g_S. Here h_1, h_2, h_3, and h_4 are arbitrary smooth functions in t.

REFERENCES

1. I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and Breach, New York, 1986).
2. A. Kushner, V. Lychagin, and V. Rubtsov, *Contact Gometry and Non-linear Differential Equations* (Cambridge University Press, 2007), p. 496.
3. V. Lychagin, *Singularities of multivalued solutions of nonlinear partial differential equations, and nonlinear phenomena*, Acta Appl. Math. **3** (2), 135 (1985).
4. P. K. Rashevskiy, *Riemann geometry and tensor analysis* (Nauka, Moskow, 1967), p. 664.