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On Geometric Structures of 2-Dimensional Gas Dynamics Equations

V. Lychagin and V. Yumaguzhin*

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Abstract—In this paper we describe geometric structures on solutions of the 2-dimensional gas dynamics equations. These structures are generated by characteristics of the system. We construct a 1-order tensor differential invariant of the structures. It is shown that there exists a unique linear connection on every solution of the gas dynamics equations such that its torsion coincides with the obtained invariant. We use this invariant to find classes of explicit torsion-free solutions of the gas dynamics equations for polytropic constant volume gas motion.

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1. INTRODUCTION

In this work we investigate geometric structures on solutions of 2-dimensional gas dynamics equations, generated by characteristics. The characteristic variety of the system is given by symmetric 3-form. This form is decomposable and characteristics generate on every cotangent space to a solution a geometric structure consisting of a plane and a cone.

We construct a 1-order tensor differential invariant of this structure. It generates a vector valued 2-forms on solutions.

We show that there exists a unique linear connection on every solution of these equations such that its torsion tensor coincides with the tensor invariant.

Finally for the case of polytropic constant volume gas motion, we find a class of explicit torsion-free solutions.

2. GAS DYNAMICS EQUATIONS

2.1. Adiabatic Gas Motion

The adiabatic 2-dimensional gas motion is described by the following system of differential equations, see for example [2],

$$u_t + uu_x + vu_y + p_x/\rho = 0, \quad v_t + uv_x + vv_y + p_y/\rho = 0, \tag{1}$$
$$\rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) = 0, \quad p_t + up_x + vp_y + A(\rho, p)(u_x + v_y) = 0.$$

Here $(u,\ v)$ is the velocity of gas flow, ρ is its density, p is its pressure and

$$-A(\rho, p)/\rho = (\partial S/\partial\rho)/(\partial S/\partial p),$$

where $S(\rho, p)$ is the entropy, is the square of the speed of sound in the media.

Geometrically the system can be viewed as follows.

Consider the trivial bundle

$$\pi: \quad \mathbb{R}^3 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^3, \qquad \pi: \quad (x^1, x^2, x^3, u^1, \ldots, u^4) \mapsto (x^1, x^2, x^3)$$

*E-mail: yuma@diffiety.botik.ru

and denote by

$$\pi_k : \quad J^k\pi \longrightarrow \mathbb{R}^3, \qquad \pi_k : \quad j_p^k S \mapsto p$$

the bundle of all k-jets of all sections of π. Here we denoted by $j_p^k S$ the k-jet of section S at a point p.

System (1) generates a submanifold $\mathcal{E}$ of the bundle $J^1\pi$ given by equations

$$u_1^1 + u^1 u_2^1 + u^2 u_3^1 + u_2^4/u^3 = 0, \quad u_1^2 + u^1 u_2^2 + u^2 u_3^2 + u_3^4/u^3 = 0, \tag{2}$$
$$u_1^3 + u^1 u_2^3 + u^2 u_3^3 + u^3(u_2^1 + u_3^2) = 0, \quad u_1^4 + u^1 u_2^4 + u^2 u_3^4 + A(u^3, u^4)(u_2^1 + u_3^2) = 0,$$

where u^i, u_j^i are canonical coordinates in $J^1\pi$.

Every section S of the bundle π generates the section $j_k S$ of π_k :

$$j_k S : \quad \mathbb{R}^3 \longrightarrow J^k\pi, \qquad j_k S : \quad p \mapsto j_p^k S.$$

Denote by $L_S^{(1)} \subset J^1\pi$ the image of section $j_1 S$.

Then section S is a solution of system (1) if and only if $L_S^{(1)} \subset \mathcal{E}$.

2.2. Symbol

For all natural numbers q and r such that $q > r$, we denote by $\pi_{q,r}$ the natural projections

$$\pi_{q,r} : \quad J^q\pi \longrightarrow J^r\pi, \qquad \pi_{q,r} : \quad j_p^q S \mapsto j_p^r S.$$

Remark that in our case

$$\pi_{1,0}(\mathcal{E}) = J^0\pi.$$

Let $\theta_0 \in J^0\pi$ and let F_{θ_0} be the fiber of $\pi_{1,0}$ over θ_0,

$$F_{\theta_0} = (\pi_{1,0})^{-1}(\theta_0),$$

and let $\mathcal{E}_{\theta_0}$ be the fiber of $\mathcal{E}$ over θ_0,

$$\mathcal{E}_{\theta_0} = \mathcal{E} \cap F_{\theta_0}.$$

Take $\theta_1 \in \mathcal{E}$ and let $\theta_0 = \pi_{1,0}(\theta_1)$. Then, see for example [1], the *symbol of $\mathcal{E}$ at point θ_1* is the tangent space to $\mathcal{E}_{\theta_0}$ at point θ_1,

$$\mathrm{Smbl}_{\theta_1}\mathcal{E} = T_{\theta_1}(\mathcal{E}_{\theta_0}).$$

2.3. Characteristics

Recall the geometrical definition of characteristic covectors, see [1].

Let $\theta_1 = j_p^1 S \in \mathcal{E}$. Denote by $\mathcal{R}_{\theta_1}$ the tangent space to $L_S^{(0)}$ at the point $\theta_0 = \pi_{1,0}(\theta_1)$,

$$\mathcal{R}_{\theta_1} = T_{\theta_0} L_S^{(0)}.$$

Let $\omega = \xi_1 dt + \xi_2 dx + \xi_3 dy$ be a nonzero covector on the tangent space T_p to the base of π at the point p. The pair (θ_1, ω) generates the ray (4-dimensional affine subspace) $l(\theta_1, \omega)$ in the affine space F_{θ_0} by the formula

$$l(\theta_1, \omega) = \{\tilde{\theta}_1 \in F_{\theta_0} | \pi_*(\mathcal{R}_{\theta_1} \cap \mathcal{R}_{\tilde{\theta}_1}) = \mathrm{Ann}\omega\}.$$

Then the covector ω is called *characteristic at point* θ_1 if the the tangent space $T_{\theta_1}\big(l(\theta_1, \omega)\big)$ to $l(\theta_1, \omega)$ at the point θ_1 has a nonzero intersection with the symbol of $\mathcal{E}$ at this point:

$$T_{\theta_1}\big(l(\theta_1, \omega)\big) \cap \mathrm{Smbl}_{\theta_1}\mathcal{E} \neq \{0\}.$$

In the canonical coordinates, the last condition is given by the equation

$$\det\left(\frac{\partial \Phi^i}{\partial u_k^j}\xi_k\right) = 0,$$

where $\Phi^1, \ldots, \Phi^4$ are the left-hand sides of system (2).

Calculating, we get the following condition for characteristic covectors

$$(\xi_1 + u^1\xi_2 + u^2\xi_3)^2\big((\xi_1 + u^1\xi_2 + u^2\xi_3)^2 - \frac{A(u^3, u^4)}{u^3}(\xi_2^2 + \xi_3^2)\big) = 0.$$

This means that the characteristic variety consists of union of the plane P_{θ_1}

$$\xi_1 + u^1\xi_2 + u^2\xi_3 = 0 \tag{3}$$

and the cone C_{θ_1}

$$(\xi_1 + u^1\xi_2 + u^2\xi_3)^2 - \frac{A(u^3, u^4)}{u^3}(\xi_2^2 + \xi_3^2) = 0 \tag{4}$$

in the cotangent space T_p^*.

It is easy to see that

$$\mathrm{P}_{\theta_1} \cap \mathrm{C}_{\theta_1} = \{0\}.$$

Denote by

$$\Omega(\theta_1) = \mathrm{P}_{\theta_1} \cup \mathrm{C}_{\theta_1}$$

the characteristic variety.

This gives us a field of geometric objects, or geometric structure,

$$\Omega : \theta_1 \longmapsto \Omega(\theta_1) \quad \forall \theta_1 \in \mathcal{E}$$

on manifold $\mathcal{E}$.

The restriction $\Omega|_{L_S^{(1)}}$ is a naturally defined geometric structure on a solution $L_S^{(1)}$. We will denote it by Ω_S.

2.4. Bundle of Geometric Structures

We represent the structure Ω_S as a section of the natural structure bundle over $\mathbb{R}^3$.

Let $\tilde{\Omega}$ be a geometric structure of the form Ω on $\mathbb{R}^3$, that is $\tilde{\Omega}$ consists of hyperplane P_p and cone C_p in cotangent space T_p^* to $\mathbb{R}^3$ at every point $p \in \mathbb{R}^3$ in such a way that

$$\mathrm{P}_p \cap \mathrm{C}_p = \{0\}, \forall p \in \mathbb{R}^3. \tag{5}$$

These fields of hyperplanes and cones can be defined by equations

$$\Omega^1(p)\xi_1 + \Omega^2(p)\xi_2 + \Omega^3(p)\xi_3 = 0,$$

and

$$\Omega^{ij}(p)\xi_i\xi_j = 0$$

respectively. Then condition (5) means

$$\forall \xi \in \mathrm{P}_p \text{ such that } \xi \neq 0 \quad \Omega^{ij}(p)\xi_i\xi_j \neq 0. \tag{6}$$

The structure $\tilde{\Omega}$ can be identified with the section

$$\tilde{\Omega} : \quad p \longmapsto ([\Omega^1(p) : \Omega^2(p) : \Omega^3(p)], [\Omega^{11}(p) : \ldots : \Omega^{23}(p) : \Omega^{33}(p)])$$

of the trivial bundle

$$\tau : \quad \mathbb{R}^3 \times (\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^5) \to \mathbb{R}^3, \quad (p, [q^0 : q^1 : q^2], [r^0 : \ldots : r^5]) \mapsto p.$$

Let E be the open subset of the total space of this bundle defined by the condition (6). Then the bundle

$$\mu = \tau|_E : E \longrightarrow \mathbb{R}^3$$

is a natural bundle of geometric structures of the form Ω.

2.5. Differential Invariants

In this section we construct a first order tensor differential invariant of structures of the form Ω.

To this end we use the approach [3], to construct differential invariants in natural bundles.

2.5.1. Formal symmetries. Let X be a vector field in the base of μ and f_t its flow. The natural lifting $f_t^{(0)}$ of flow f_t to E defines the lifting of X to the vector field $X^{(0)}$ in E.

The value $X^{(0)}_{\theta_0}$ of $X^{(0)}$ at point $\theta_0 \in E = J^0\mu$ is defined by the 1-jet $j^1_p X$ of X at point $p = \mu(\theta_0)$.

Take a point $\theta_1 \in J^1\mu$ such that $\mu_{1,0}(\theta_1) = \theta_0$ and consider the following vector space of 1-jets of vector fields:

$$\mathcal{A}_{\theta_1} = \{j^1_p X | X^{(0)}_{\theta_0} \in \mathcal{R}_{\theta_1}\}.$$

The bracket of vector fields generates the bilinear map

$$[,]: \quad \mathcal{A}_{\theta_1} \times \mathcal{A}_{\theta_1} \to T_p,$$

where

$$[j^1_p X_1, j^1_p X_2] = [X_1, X_2]_p.$$

Then the isotropy algebra $\mathfrak{g}_{\theta_0}$ of θ_0 is defined by

$$\mathfrak{g}_{\theta_0} = \{j^1_p X | X^{(0)}_{\theta_0} = 0\} \subset T_p \otimes T^*_p.$$

It is easy to see, that

$$\mathfrak{g}_{\theta_0} \subset \mathcal{A}_{\theta_1},$$

and the natural projection

$$\rho: \quad \mathcal{A}_{\theta_1} \to T_p, \qquad \rho: \quad j^1_p X \mapsto X_p$$

is a surjection.

A subspace $H \subset \mathcal{A}_{\theta_1}$ is called *horizontal* if

$$\rho|_H: \quad H \to T_p,$$

is an isomorphism.

In the case when $H \subset \mathcal{A}_{\theta_1}$ is horizontal, we have

$$\mathcal{A}_{\theta_1} = \mathfrak{g}_{\theta_0} \oplus H.$$

Any two horizontal subspaces $H, \tilde{H} \subset \mathcal{A}_{\theta_1}$ define a linear map

$$f_{\tilde{H},H} \in \mathfrak{g}_{\theta_0} \otimes T^*_p,$$

where

$$f_{\tilde{H},H}: \quad X_p \mapsto (\rho|_{\tilde{H}})^{-1}(X_p) - (\rho|_H)^{-1}(X_p).$$

On the other hand, if $H \subset \mathcal{A}_{\theta_1}$ is a horizontal subspace and if $f \in \mathfrak{g}_{\theta_0} \otimes T^*_p$, then there exists a unique horizontal subspace $\tilde{H} \subset \mathcal{A}_{\theta_1}$ such that $f = f_{\tilde{H},H}$. Namely, the subspace is spanned by the 1-jets

$$(\rho|_H)^{-1}(X_p) + f(X_p),$$

where $X_p \in T_p$.

2.5.2. Spencer cohomologies. Consider the following Spencer δ-complex

$$0 \to \mathfrak{g}^{(1)}_{\theta_0} \hookrightarrow \mathfrak{g}_{\theta_0} \otimes T^*_p \xrightarrow{\partial} T_p \otimes (T^*_p \wedge T^*_p) \to 0, \tag{7}$$

where

$$\mathfrak{g}^{(1)}_{\theta_0} = \left(\mathfrak{g}_{\theta_0} \otimes T^*_p\right) \cap T_p \otimes (T^*_p \odot T^*_p)$$

is the first prolongation of $\mathfrak{g}_{\theta_0}$ and the differential

$$\partial : \quad \mathfrak{g}_{\theta_0} \otimes T_p^* \to T_p \otimes (T_p^* \wedge T_p^*)$$

is defined by the formula

$$\partial(f)(X_p, Y_p) = f(X_p)Y_p - f(Y_p)X_p.$$

Then every horizontal subspace $H \subset \mathcal{A}_{\theta_1}$ generates the 2-form $\omega_H \in T_p \otimes (T_p^* \wedge T_p^*)$ by the formula

$$\omega_H(X_p, Y_p) = [j_p^1 X, j_p^1 Y],$$

where vector fields X and Y are chosen in such a way that $j_p^1 X, j_p^1 Y \in H$, and $X_p \cong j_p^1 X$, and $Y_p \cong j_p^1 Y$ under the isomorphism $\rho|_H : H \to T_p$.

Lemma 2.1. *The cohomology class*

$$[\omega_H] = \omega_H + \partial(\mathfrak{g}_{\theta_0} \otimes T_p^*)$$

is independent under the choice of a horizontal subspace $H \subset \mathcal{A}_{\theta_1}$.

Proof. Suppose $H, \tilde{H}$ are horizontal subspaces of $\mathcal{A}_{\theta_1}$, and suppose that as above $j_p^1 X = (\rho|_H)^{-1}(X_p)$ and $j_p^1 Y = (\rho|_H)^{-1}(Y_p)$. Then

$$(\rho|_{\tilde{H}})^{-1}(X_p) = j_p^1 X + f(X_p)$$

and

$$(\rho|_{\tilde{H}})^{-1}(Y_p) = j_p^1 Y + f(Y_p),$$

where $f = f_{\tilde{H},H}$.

Hence,

$$\begin{aligned}\omega_H(X_p, Y_p) - \omega_{\tilde{H}}(X_p, Y_p) &= [j_p^1 X, j_p^1 Y] - [j_p^1 X + f(X_p), j_p^1 Y + f(Y_p)] \\ &= [j_p^1 Y, f(X_p)] - [j_p^1 X, f(Y_p)] = f(X_p)Y_p - f(Y_p)X_p.\end{aligned}$$

□

This construction gives us in the natural way a function on $J^1\mu$ with values in Spencer cohomologies

$$\omega : \quad \theta_1 \longmapsto \omega_{\theta_1} = [\omega_H].$$

By the construction, this function is a 1st order differential invariant with respect to the action of the pseudogroup of all diffeomorphisms of the base on the bundle μ.

Let Γ be a section of μ and let $L_\Gamma^{(1)}$ be the image of the section $j_1\Gamma$ of μ_1.

Then the restriction of ω on $L_\Gamma^{(1)}$ is a differential invariant of the geometric structure Γ.

2.6. Differential Invariants on Solutions of Gas Dynamics Equations

Let S be a solution of gas dynamics equations (1) and let Ω_S be its geometric structure of the form Ω considered as a section of the bundle μ. It follows from (3) and (4) that this section is defined in standard coordinates of the total space of μ by the formula

$$\Omega_S : \quad p \longmapsto \left(p; u^1, u^2; u^1, u^2, (u^1)^2 - \frac{A(u^3, u^4)}{u^3}, u^1u^2, (u^2)^2 - \frac{A(u^3, u^4)}{u^3}\right),$$

where $u^1, \ldots, u^4$ are components of the solution S.

Let $\theta_0 = \Omega_S(p)$. Then by direct calculations we get that

$$\mathfrak{g}_{\theta_0} = \left\{ \left(\begin{array}{ccc} a & 0 & 0 \\ -u^2 b & a & b \\ u^1 b & -b & a \end{array} \right) \middle| \, a, b \in \mathbb{R} \right\}.$$

Lemma 2.2. *The tangent space T_p is decomposed to the direct sum of subspaces invariant with respect to the Lie algebra $\mathfrak{g}_{\theta_0}$*

$$T_p = \left\langle \partial_{x^1} + u^1\partial_{x^2} + u^2\partial_{x^3} \right\rangle \oplus \left\langle \partial_{x^2}, \partial_{x^3} \right\rangle .$$

Proof. Take an operator $g \in \mathfrak{g}_{\theta_0}$. Then a, $a+ib$ and $a-ib$ its eigenvalues and the vectors $(1, u^1, u^2)^T$, $(0,1,i)^T$, and $(0,1,-i)^T$ are their eigenvectors respectively. □

This decomposition of T_p generates the following decomposition of $T_p \otimes (T_p^* \wedge T_p^*)$ into direct sum of subspaces invariant with respect to the Lie algebra $\mathfrak{g}_{\theta_0}$

$$T_p \otimes (T_p^* \wedge T_p^*) = \partial(\mathfrak{g}_{\theta_0} \otimes T_p^*) \oplus \left(T_p \otimes (dx^2 \wedge dx^3)\right) .$$

Denote by ω_S the restriction of differential invariant ω to the image of the section $j_1\Omega_S$.

This is a differential invariant and ω_S can be viewed as a vector-valued 2-form.

By strightforward calculations we get that

$$\omega_S = \frac{1}{(u^1)^2 + (u^2)^2}\Big(\big(u^1(u^1_3 + u^2_2) - u^2(u^1_2 - u^2_3)\big)\partial_{x^2} - \big(u^2(u^1_3 + u^2_2) + u^1(u^1_2 - u^2_3)\big)\partial_{x^3}\Big) \otimes (dx^2 \wedge dx^3). \quad (8)$$

Theorem 2.3. *Let $L_S^{(1)}$ be a solution of gas dynamics equations (1). Then there exists a unique linear connection on $L_S^{(1)}$ with a torsion coinciding with differential invariant ω_S.*

Proof. It is easy to check that $\mathfrak{g}_{\theta_0}^{(1)} = \{0\}$. Now it follows from complex (7) that there exist a unique horizontal subspace $H \subset \mathcal{A}_{\theta_1}$, $\theta_1 = j^1_p\Omega_S$, such that $\omega_H \in T_p \otimes (dx^2 \wedge dx^3)$ for every point p of the domain of S. The field of these horizontal subspaces defines the connection. □

2.7. Torsion-Free Solutions

From (8) we get that the condition $\omega_S = 0$ is equivalent to the Cauchy-Riemann equations

$$u^1_2 = u^2_3, \quad u^1_3 = -u^2_2.$$

This means that the velocity $(u,\ v)$ of the gas flow is given by a complex-analytic function in x and y.

Consider now the polytropic constant volume gas motion, that is

$$A(\rho, p) = \gamma p,$$

where γ is a constant, and ρ is a constant in system (1).

Then the condition $\omega_S = 0$ leads to the following class of explicit solutions

$$u = cy + k_1 t + k_{01}, \quad v = -cx + k_2 t + k_{02}, \quad \rho = \text{const},$$

$$p = \rho c^2(x^2 + y^2)/2 + \rho(k_1^2 + k_2^2)t^2 + \rho(k_1k_{01} + k_2k_{02})t - \rho\big(k_1 + c(k_2 t + k_{02})\big)x - \rho\big(k_2 + c(k_1 t + k_{01})\big)y + k_3,$$

where c, k_1, k_2, k_{01}, k_{02}, and k_3 are constants.

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