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Generalized Collocation Methods

Solutions to Nonlinear Problems

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Preface

Nonlinear Problems in Applied Sciences

This book has been proposed to offer engineers and scientists various mathematical tools, based on generalized collocation methods, to solve nonlinear problems related to partial differential and integro-differential equations. This preface first describes the aims of the book, then outlines its contents, and finally develops a critical analysis focused on the merits (and limits) of the method herein described compared to classical applied mathematics approaches.

Aims and Users

Mathematical problems of interest in technology and in applied sciences are often characterized by nonlinear features which may refer either to the model or to the mathematical problems, and, in some cases, to both. Linearity should be regarded as a special case that is often generated by an artificial simplification of a physical reality.

Mathematical models are already an approximation of a physical reality, which is useful in applied sciences although scientists occasionally even refer to falsification of a physical reality. Linearity assumptions generally increase the gap between the description delivered by a model and the effective behavior of the real system. Therefore, dealing with nonlinear problems can contribute to a fruitful interaction between mathematics and applied sciences.

This reasoning has motivated the search for mathematical methods to deal with nonlinear problems generated by the application of models, stated by partial differential and integro-differential equations, to the analysis of real-world problems. This book specifically deals with the development and application of generalized collocation methods—originally called differential quadrature methods.

The application of these methods is based on the approximation and interpolation of the dependent variables by using suitable polynomials or functions according to their values in the collocation points corresponding to a suitable discretization of the space variable. Lagrange polynomials and sinc functions are usually adopted for the interpolation.

Then the space derivatives, or integral terms, are approximated using interpolation. Replacing them in the evolution equation transforms the initial-boundary value problem into an initial value problem for ordinary differential equations that describe the evolution of the values of the dependent variables in the nodes. Boundary conditions are imposed in the collocation points, corresponding to the boundary of the domain of the independent variables. The solution of the initial-boundary value problem is obtained by solving the initial value problem and then interpolating the solution again.

This book has been written both for scientists and engineers who are interested in modelling real systems by using differential or operator equations, and for university students who have a good knowledge of fundamental mathematics and differential calculus, at a masters course level, and are interested in the application of mathematics to technology and applied sciences.

The aim is to offer an easy-to-use handbook for the implementation of the method and for the development and application of scientific programs. This book deals with mathematical applications and is not addressed to mathematicians interested in conceptual topics and proof theory in general. This book is founded on the idea that ***modelling, mathematical methods, and scientific computation should be dealt with together in a unified presentation***. All the above-mentioned features are part of applied mathematics, and a unified presentation can contribute to a deeper understanding of the subjects.

Scientific programs have been proposed in such a way that the reader can find guidelines for the practical application of the method. The programs make use of the software ***Mathematica***®, which offers a friendly approach to programming, as well as a careful optimization of routines related to the application of algorithms. However, the choice of this software is also due to the authors' personal taste; the reader can use other software depending on his/her own experience.

Contents

The contents of the book are organized in two parts. The first part, consisting of the first two chapters, deals with introductory topics concerning a variety of mathematical models, the related statement of mathematical problems, and general aspects of interpolation techniques.

• Chapter 1 concerns modelling and analytical aspects. The first part of the chapter describes a number of mathematical models, generally nonlinear, which will be used to test the application of the mathematical method at a practical level. This chapter also deals with the statement of the problems obtained by implementing the model with suitable initial and boundary conditions.

• Chapter 2 discusses the technical aspects and the scientific programs concerning the interpolation of functions and surfaces by generalized collocation methods. An analysis is made to show how the selection of the interpolating polynomials and their collocation can be technically organized to obtain the best approximation.

The second part, consisting of Chapters 3, 4, 5, and 6, supported by the Appendix, deals with the development of generalized collocation methods to solve a variety of problems related to nonlinear partial differential equations.

• Chapter 3 deals with the solution of initial-boundary value problems for nonlinear partial differential equations in unbounded domains. Mathematical problems are obtained by assigning the initial conditions, and interpolations are developed using sinc functions.

• Chapter 4 concerns the generalization of the method proposed in Chapter 3 to the solution of initial-boundary value problems in one space dimension. The various problems dealt with in Section 1.3 are technically treated referring to the classical Dirichlet, Neumann, and Robin boundary conditions.

• Chapter 5 discusses the application of generalized collocation methods to the solution of initial-boundary value problems in two space dimensions. The contents are a development of the concepts already proposed in the preceding chapters. The application, however, has to deal with an increased computational difficulty.

• Chapter 6 develops various generalizations of the methods to the solution of some ill-posed problems. Specifically, the chapter addresses problems with nonlinear boundary conditions and problems where the boundary conditions, or source terms, are replaced by additional information on the solution of the mathematical problem. These problems are generated by several interesting engineering applications, where at the boundaries measurements can provide only information on nonlinear functions of the dependent variable rather than a direct measurement of Dirichlet or Neumann boundary conditions. This section also discusses the solution of the problems for integro-differential equations and with a critical analysis focused on the validity of the method with respect to other mathematical tools.

• The Appendix reports the various scientific programs used for the applications proposed in this book. This chapter is not simply a collection of Notebooks, but is also a guide to scientific programming. A useful guide for the application of ***Mathematica*** to the solution of problems of interest in applied sciences can be obtained from the books by Bellomo, Preziosi, and Romano (2000), Romano, Lancellotta, and Marasco (2005), and Lynch

(2007).

All the chapters include several examples and problems that have been solved by computational methods. Full details of the solution technique are given, while the last part of each chapter provides problems and exercises so the reader can practice using the mathematical tools of each chapter.

Critical Analysis

Dealing with nonlinear models and problems in applied sciences is a crucial passage of the application of mathematics to real-world analysis. The mathematical methods developed in this book are also proposed considering their immediate ability to deal with nonlinearities.

The analyses of inverse problems and of initial-boundary value problems with nonlinear boundary conditions are particularly important. This specific feature makes the method interesting for engineering applications as also documented in the valuable book by Chang Shu (1992) *Differential Quadrature and Its Application in Engineering*. Our book is directed towards dealing with mathematical foundations of the method with special attention to computational treatments of nonlinearities. Programming with ***Mathematica*** leads to a rapid and efficient production of scientific computation.

However, it would be naive to try to hide some of the well identified limits of the method. While it efficiently deals with nonlinearity, it suffers from computational complexity problems induced by geometrical features, e.g., problems in more than two space variables, especially when they are defined over complex shapes. Additional difficulties are also generated by problems characterized by oscillating solutions, with high frequency, in the space variables.

Applied scientists will be made aware throughout of the the limits (not only of the advantages) of the effective applicability of generalized collocation methods with respect to other methods of applied mathematics.

Finally, let us stress that this book has also been written for engineering schools or applied sciences faculties, as part of courses of applied mathematics where modelling aspects are followed by simulations suitable for providing a deeper understanding of the mathematical model. Because of these reasons, we hope that the book will help in the research activity of those engineers and applied scientists who are involved in mathematical problems where the interplay between mathematics technology, or applied sciences in general, is not trivial.

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