Affine Differential Invariants for Planar Curves

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Abstract

In this paper we solve the affine equivalence problem for the graph of functions with real values by finding a complete system of differential invariants for the affine group action.

1 Introduction

The differential invariants associated with a transformation group on a manifold are the fundamental building blocks for understanding the geometry, equivalence, symmetry and other properties of submanifolds. The basic theory of differential invariants dates back to the work of Lie, [5] and Tresse, [7]. However, a complete classification of differential invariants for many of the fundamental transformation groups of physical and geometrical importance remains undeveloped.

Cartan, [1] and [2], proved how his moving frame method could produce the differential invariants for several interesting examples. A new, practical approach to the method of moving frames was recently developed by Fels and Olver, [3] and [4]. The method enables one to algorithmically implement both the practical and theoretical construction of moving frames.

In this paper we solve the following problem, by implementing the Fels and Olver's moving coframe method:

Problem. Given two differentiable ordinary functions $y = f(x)$ and $y = g(x)$ and a number $x_0 \in D_f \cap D_g$, does there exist an affine transformation $T \in A(2)$ and a positive number ε such that

$$T \cdot f(x) = g(T \cdot x) \quad \text{for all} \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon)$$

According to the Theorem 7.7 of Fels and Olver, [3], we only need to find a complete set of differential invariants. And, by implementing the Fels and Olver moving coframe method, we prove that

Theorem. *There exists two differential invariants*

$$I = \frac{sgn(y''')\sqrt{3y''y^{(4)} - 5y'''^2}}{|y''|}$$

$$J = \frac{sgn(y''y''')(9y''^2y^{(5)} - 45y''y'''y^{(4)} + 40y'''^3)}{\left(\sqrt{3y''y^{(4)} - 5y'''^2}\right)^3}$$

of order 4. Furthermore, for any $k \geq 0$, a complete system of functionally independent differential invariants of order $k + 4$ is provided by I, J, $\mathcal{D}J$, $\mathcal{D}^2 J$, ... , and $\mathcal{D}^k J$, where $\mathcal{D} = (D_x)^{-1}D_x$ is the associated differential operator and $D_x = \dfrac{\partial}{\partial x} + y'\dfrac{\partial}{\partial y} + y''\dfrac{\partial}{\partial y'} + \cdots$ is the total derivative operator.

2   Formulation the problem

If $A(2) = GL(2) \propto \mathbf{R}^2$ is the affine transformations group in the plane, then we can identify $A(2) \subseteq GL(3)$ as a subgroup of $GL(3)$ by identifying $(R, a) \in A(2)$ with the 3×3 matrix

$$(1) \qquad \begin{pmatrix} R & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} , \quad ad - bc \neq 0$$

then, the six independent Maurer-Cartan forms are the components of the matrix

$$(2) \qquad \mu := \begin{pmatrix} R & a \\ 0 & 1 \end{pmatrix}^{-1} \cdot d\begin{pmatrix} R & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R^{-1}dR & a^{-1}da \\ 0 & 0 \end{pmatrix}$$

That is,

$$(3) \qquad \begin{aligned} \mu_1 &= \frac{d\,da - b\,dc}{ad - bc} & \mu_2 &= \frac{d\,db - b\,dd}{ad - bc} \\ \mu_3 &= \frac{d\,dx_0 - b\,dy_0}{ad - bc} & \mu_4 &= \frac{a\,dc - c\,da}{ad - bc} \\ \mu_5 &= \frac{a\,dd - c\,db}{ad - bc} & \mu_6 &= \frac{a\,dy_0 - c\,dx_0}{ad - bc} \end{aligned}$$

We define the action of $A(2)$ on $\mathbf{R}^2$ by

$$(4) \qquad \begin{aligned} \begin{pmatrix} \bar{x} \\ \bar{y} \\ 1 \end{pmatrix} &= \begin{pmatrix} R & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} ax + by + x0 \\ cx + dy + y0 \\ 1 \end{pmatrix}, \forall \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \equiv \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 \end{aligned}$$

as usual. This action is effective and transitive.

3 Finding the most general compatible lift

Let the origin of $\mathbf{R}^2$, $z_0 := (0,0)$ be the base of problem. In order to compute the most general compatible lift $\rho : \mathbf{R}^2 \to A(2)$ with the base z_0, we solve the system of 2 equations $\rho(z) \cdot z_0 = z$ for 2 of group parameters in terms of the coordinates z on $\mathbf{R}^2$ and 4 remaining group parameters,

$$(5) \qquad x_0 = x \quad , \quad y_0 = y$$

The most general compatible lift thus has the form

$$(6) \qquad \rho_0(x, y, a, b, c, d) = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c, and d are arbitrary functions of x and y, subject only to the condition $ad - bc \neq 0$, so that the determinant of (6) does not vanish, and hence ρ_0 does take its values in the group $A(2)$.

By substituting the formulae (5) characterizing our compatible lift (6) into the Maurer- Cartan forms (3), we obtain the moving coframe of order zero

$$(7) \qquad \begin{aligned} \mu_1^{(0)} &= \frac{d\,da - b\,dc}{ad - bc} & \mu_2^{(0)} &= \frac{d\,db - b\,dd}{ad - bc} \\[2mm] \mu_3^{(0)} &= \frac{d\,dx - b\,dy}{ad - bc} & \mu_4^{(0)} &= \frac{a\,dc - c\,da}{ad - bc} \\[2mm] \mu_5^{(0)} &= \frac{a\,dd - c\,db}{ad - bc} & \mu_6^{(0)} &= \frac{a\,dy - c\,dx}{ad - bc} \end{aligned}$$

which forms a basis for the space of one-forms on $\mathcal{B}_0 = M \times H$, where $H = A(3)_{z_0} \simeq GL(2)$ is the isotropy group of the base point z_0.

4 First normalization

Let us now consider a curve $N \subset M$. For simplicity, we shall assume that the curve concedes with the graph of a function $y = y(x)$. We restrict the moving coframe forms to the curve, which amounts to replacing the differential dy its "horizontal" component $y'\,dx$. Therefore, the restricted (or horizontal) moving coframe forms are explicitly given by

$$(8) \qquad \begin{aligned} \mu_{1,h}^{(0)} &= \mu_1^{(0)} & \mu_{2,h}^{(0)} &= \mu_2^{(0)} \\[2mm] \mu_{3,h}^{(0)} &= \frac{d - by'}{ad - bc}dx & \mu_{4,h}^{(0)} &= \mu_4^{(0)} \\[2mm] \mu_{5,h}^{(0)} &= \mu_5^{(0)} & \mu_{6,h}^{(0)} &= \frac{ay' - c}{ad - bc}dx \end{aligned}$$

which now depend on first order derivatives, and thus are defined on the first jet space $J^1\mathbf{R}^2 \simeq \mathbf{R}^3$.

The next step in the procedure is to look for invariant combinations of coordinates and group parameters. Among the restricted one-forms (8), there is one linear

dependency, namely $\mu_3 = J\,\mu_6$, where $J = c + d - (a + b)y'$ is a lifted invariant. We can normalize this lifted invariant by setting it equal to one. By solving the equation $J = 1$ for c, we have

(9) $$c = -d + (a + b)y'$$

Substituting (9) into (3) , we find the first order moving coframe

$$
\begin{aligned}
\mu_1^{(1)} &= \frac{da}{a + b} + \frac{by'\,db}{(-d + by')(b + a)} \\[2mm]
&\qquad - \frac{b\,dd}{(-d + by')(b + a)} + \frac{a\,dy'}{-d + by'} \\[2mm]
\mu_2^{(1)} &= -\frac{d\,db}{(-d + by')(b + a)} + \frac{b\,dd}{(-d + by')(b + a)} \\[2mm]
\mu_3^{(1)} &= -\frac{d\,dx}{(-d + by')(b + a)} + \frac{b\,dy}{(-d + by')(b + a)} \\[2mm]
\mu_4^{(1)} &= \frac{da}{a + b} - \frac{ay'\,db}{(-d + by')(b + a)} + \\[2mm]
&\qquad + \frac{a\,dd}{(-d + by')(b + a)} + \frac{b\,dy'}{-d + by'} \\[2mm]
\mu_5^{(1)} &= \frac{(-d + by' + ay')\,db}{(-d + by')(b + a)} - \frac{a\,dd}{(-d + by')(b + a)} \\[2mm]
\mu_6^{(1)} &= \frac{(-d + by' + ay')\,dx}{(-d + by')(b + a)} - \frac{a\,dy}{(-d + by')(b + a)}
\end{aligned}
$$

(10)

which completely characterizes the group transformations on $\mathcal{B}_1$.

5   Second normalization

As before, we determine new lifted invariants by restricting the first order moving coframe one-forms to the curve $y = y(x)$. This amounts to restricting dy and dy' by their horizontal components $y'dx$ and $y''dx$ respectively, leading to the restricted forms

$$
\begin{aligned}
\mu_{1,h}^{(1)} &= \frac{da}{a + b} + \frac{by'\,db}{(-d + by')(b + a)} \\[2mm]
&\qquad - \frac{b\,dd}{(-d + by')(b + a)} + \frac{by''\,dx}{-d + by'} \\[2mm]
\mu_{2,h}^{(1)} &= -\frac{d\,db}{(-d + by')(b + a)} + \frac{b\,dd}{(-d + by')(b + a)}
\end{aligned}
$$

$$(11) \qquad \mu_{3,h}^{(1)} = \mu_{6,h}^{(1)} = \frac{dx}{a+b}$$

$$\mu_{4,h}^{(1)} = \frac{da}{a+b} - \frac{ay'\ db}{(-d+by')(b+a)}$$

$$+ \frac{a\ dd}{(-d+by')(b+a)} - \frac{ay''\ dx}{-d+by'}$$

$$\mu_{5,h}^{(1)} = \frac{(-d+by'+ay')\ db}{(-d+by')(b+a)} - \frac{a\ dd}{(-d+by')(b+a)}$$

Then we have two linear dependencies, namely

$$\mu_{5,h}^{(1)} = \mu_{1,h}^{(1)} + \mu_{2,h}^{(1)} + K\mu_{3,h}^{(1)} - \mu_{4,h}^{(1)}, \quad \mu_{6,h}^{(1)} = \mu_{3,h}^{(1)},$$

where $K = \dfrac{y''(a+b)^2}{d-by'}$ is a new lifted invariant. Again, we can normalize this lifted invariant by setting it equal to one. By solving the equation $K = 1$ for d, we have

$$(12) \qquad d = (a+b)^2 y'' + by'$$

Note that we can not normalize $K = 0$ since this would require $a = -b$, but then all the one-forms in (10) would have zero denominator. Substituting (12) into (10), we find the second order moving coframe

$$\mu_1^{(2)} = \frac{(3b+a)\ da}{(a+b)^2} + \frac{2b\ db}{(a+b)^2} -$$

$$- \frac{ab\ dy'}{y''(a+b)^3} + \frac{b\ dy''}{y''(a+b)}$$

$$\mu_2^{(2)} = \frac{-2b\ da}{(a+b)^2} + \frac{(a-b)\ db}{(a+b)^2} -$$

$$- \frac{b^2\ dy'}{y''(a+b)^3} - \frac{b\ dy''}{y''(a+b)}$$

$$(13) \qquad \mu_3^{(2)} = \frac{(y'''(a+b)^2+by')\ dx}{y''(a+b)^3} - \frac{b\ dy}{y''(a+b)^3}$$

$$\mu_4^{(2)} = -\frac{(a-b)\ da}{(a+b)^2} - \frac{2a\ db}{(a+b)^2} +$$

$$+ \frac{a^2\ dy'}{y''(a+b)^3} - \frac{a\ dy''}{y''(a+b)}$$

$$\mu_5^{(2)} = \frac{2a\ da}{(a+b)^2} + \frac{(3b+a)\ db}{(a+b)^2} -$$

$$- \frac{ab\ dy'}{y''(a+b)^3} + \frac{a\ dy''}{y''(a+b)}$$

$$\mu_6^{(2)} = \frac{(y'''(a+b)^2-ay')\ dx}{y''(a+b)^3} - \frac{a\ dy}{y''(a+b)^3}$$

6   Third normalization

As before, we determine new lifted invariants by restricting the second order moving coframe one-forms to the curve $y = y(x)$. This amounts to restricting dy , dy' and dy'' by their horizontal components $y'dx$, $y''dx$ and $y'''dx$ respectively, leading to the restricted forms

$$\mu_{1,h}^{(2)} \;=\; \frac{(3b+a)\,da}{(a+b)^2} + \frac{2b\,db}{(a+b)^2} + \frac{b(y'''(a+b)^2 - ay'')\,dx}{y''(a+b)^3}$$

$$\mu_{2,h}^{(2)} \;=\; \frac{-2b\,da}{(a+b)^2} + \frac{(a-b)\,db}{(a+b)^2} - \frac{b(y'''(a+b)^2 + by'')\,dx}{y''(a+b)^3}$$

(14) $$\mu_{3,h}^{(2)} \;=\; \mu_{6,h}^{(2)} = \frac{dx}{b+a}$$

$$\mu_{4,h}^{(2)} \;=\; -\frac{(a-b)\,da}{(a+b)^2} - \frac{2a\,db}{(a+b)^2} - \frac{a(y'''(a+b)^2 - ay'')\,dx}{y''(a+b)^3}$$

$$\mu_{5,h}^{(2)} \;=\; \frac{2a\,da}{(a+b)^2} + \frac{(3b+a)\,db}{(a+b)^2} + \frac{a(y'''(a+b)^2 + bay'')\,dx}{y''(a+b)^3}$$

There exist three linear dependencies, namely

$$\mu_{4,h}^{(2)} \;=\; -2\,\mu_{2,h}^{(2)} + L_1\,\mu_{3,h}^{(2)}$$

$$\mu_{5,h}^{(2)} \;=\; 3\,\mu_{2,h}^{(2)} + L_2\,\mu_{3,h}^{(2)} \quad,\quad \mu_{6,h}^{(2)} = \mu_{3,h}^{(2)}$$

where
$$L_1 = \frac{(a+b)^2 y''' + 3by''}{(a+b)y''} \quad,\quad L_2 = -\frac{(a+b)^2 y''' + (2b-a)y''}{(a+b)y''}$$

are new lifted invariants. Again, we can normalize these lifted invariants by setting $L_1 = 1$. By solving the equation $L_1 = 1$ for a, we have

(15) $$a = -b + \frac{\sqrt{-3by''y'''}}{y'''}$$

and then $L_2 = 0$. Substituting (15) into (13), we find the third order moving coframe

$$\mu_1^{(3)} \;=\; \frac{db}{2b} + \frac{\sqrt{3}\,y'''(by''' - \sqrt{3}\sqrt{-by''y'''})\,dy'}{9y''^2\sqrt{-by''y'''}} +$$

$$+ \frac{\sqrt{3}\,y'''(4by''' + \sqrt{3}\sqrt{-by''y'''})\,dy'}{6y''\sqrt{-by''y'''}} -$$

$$- \frac{\sqrt{3}\,y'''(2by''' + \sqrt{3}\sqrt{-by''y'''})\,dy''}{6y''\sqrt{-by''y'''}}$$

$$\mu_2^{(3)} \;=\; -\frac{\sqrt{3}\,y'''(y'y''' - 3y''^2)\,db}{9y''^2\sqrt{-by''y'''}}$$

$$(16) \qquad \mu_3^{(3)} = -\frac{\sqrt{3}y''' \left(y'y''' - 3y''^2\right) \ dx}{9y''^2\sqrt{-by''y'''}} + \frac{\sqrt{3}y'''^2 \ dy}{9y''^2\sqrt{-by''y'''}}$$

$$\mu_4^{(3)} = \frac{\sqrt{3}ab}{18y''^2\sqrt{-by''y'''}} \left(\left(2by'''(y'y''' - 3y''^2)\right)\right) +$$

$$+ (3y''^2 - 2y'y''')\sqrt{3}\sqrt{-by''y'''}\Big) \ db +$$

$$+ \frac{(by''' - \sqrt{3}\sqrt{-by''y'''}) \ dy'}{3by''^2} + \frac{dy''}{2y''} - \frac{dy'''}{2y'''}$$

$$\mu_5^{(3)} = -\frac{\sqrt{3}y''' \left(3by''^2 - by'y''' + y'\sqrt{3}\sqrt{-by''y'''}\right) \ db}{9y''^2\sqrt{-by''y'''}}$$

$$\mu_6^{(3)} = -\frac{\sqrt{3}y''' \left(3by''^2 - by'y''' + y'\sqrt{3}\sqrt{-by''y'''}\right) \ dx}{9y''^2\sqrt{-by''y'''}}$$

7 Fourth normalization

As before, we determine new lifted invariants by restricting the third order moving coframe one-forms to the curve $y = y(x)$. This amounts to restricting dy, dy', dy'' and dy''' by their horizontal components $y'dx$, $y''dx$, $y'''dx$ and $y^{(4)}dx$ respectively, leading to the restricted forms

$$\mu_{1,h}^{(3)} = \frac{db}{2b} + \frac{\sqrt{3}(5y'''^2 - 3y^{(4)}y'')(2by''' + \sqrt{3}\sqrt{-by''y'''}) \ dx}{18y''y'''\sqrt{-by''y'''}}$$

$$\mu_{2,h}^{(3)} = \mu_2^{(3)}, \quad \mu_{3,h}^{(3)} = \mu_{6,h}^{(3)} = \frac{\sqrt{3}y''' \ dx}{\sqrt{-by''y'''}}$$

$$\mu_{4,h}^{(3)} = -2\,\mu_{2,h}^{(2)} + \mu_{3,h}^{(2)}, \quad \mu_{5,h}^{(3)} = 3\,\mu_{2,h}^{(2)}$$

There are four linear dependencies, namely

$$\mu_{3,h}^{(3)} = L_1\mu_{2,h}^{(3)}, \quad \mu_{4,h}^{(3)} = -\mu_{1,h}^{(3)} + L_2\mu_{2,h}^{(3)}$$

$$\mu_{6,h}^{(3)} = L1\mu_{2,h}^{(3)}, \quad \mu_{5,h}^{(3)} = 2\mu_{1,h}^{(3)} + 3\mu_{2,h}^{(3)}$$

$$L_1 = \frac{3y''y'''}{b(3y''y^{(4)} - 5y'''^2)}$$

$$L_2 = \frac{10by'''^2 - 6by''y^{(4)} + 3y''y'''}{b(3y''y^{(4)} - 5y'''^2)}$$

where all of the coefficients are lifted invariants. Again, we can normalize the L_1 setting it equal to one. By solving this equation for b, we have

$$(17) \qquad\qquad b = \frac{y'''}{y^{(4)}} - \frac{3y''}{5y'''}$$

and therefore $L_2 = -1$.

Substituting (5), (9), (12), (15) and (17) in (6) we find the forth order moving frame

$$(18) \qquad \rho_4 \quad : \quad X^{(4)} \longmapsto \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -b + \frac{\sqrt{-3by''y'''}}{y'''} & \frac{y'''}{y^{(4)}} - \frac{3y''}{5y'''} & x \\ -d + (a+b)y' & (a+b)^2 y'' + by' & y \\ 0 & 0 & 1 \end{pmatrix}$$

on the fourth jet space $J^4 = J^4 \mathbf{R}^2 \simeq \mathbf{R}^6$ of curves in $\mathbf{R}^2$ with coordinates $X^{(4)} := (x, y, y', y'', y''', y^{(4)})$.

Substituting (17) into (16), we find the fourth order moving coframe

$$\mu_1^{(4)} = \frac{y'''(3Ry''y^{(4)} - y''y'''^2 - 5Ry'''^2)}{3Ry''^2(3y''y^{(4)} - 5y'''^2)} \, dy' +$$

$$+ \frac{-10Ry'''^2 + 4y''y'''^2 + 3Ry''y^{(4)}}{2Ry''(3y''y^{(4)} - 5y'''^2)} \, dy'' -$$

$$- \frac{y'''(-5R + y'')}{R(3y''y^{(4)} - 5y'''^2)} \, dy'''$$

$$\mu_2^{(4)} = \frac{y'''^3}{3Ry''(3y''y^{(4)} - 5y'''^2)} \, dy' - 2 \frac{y'''^2}{R(3y''y^{(4)} - 5y'''^2)} \, dy'' +$$

$$+ \frac{y''y'''}{R(3y''y^{(4)} - 5y'''^2)} \, dy'''$$

$$\mu_3^{(4)} = \frac{y'''(3y''^2 - y'y''')}{9Ry''^2} \, dx + \frac{y'''^2}{9Ry''^2} \, dy$$

$$\mu_4^{(4)} = -\frac{3Ry''y^{(4)} - y''y'''^2 - 5Ry'''^2}{3Ry''^2(3y''y^{(4)} - 5y'''^2)} \, dy' -$$

$$- \frac{-20Ry'''^2 - 4y''y'''^2 + 9Ry''y^{(4)}}{2Ry''(3y''y^{(4)} - 5y'''^2)} \, dy'' -$$

$$- \frac{y'''(5R + y'')}{R(3y''y^{(4)} - 5y'''^2)} \, dy'''$$

$$\mu_5^{(4)} = -\frac{y'''(3Ry''y^{(4)} - y''y'''^2 - 5Ry'''^2)}{3Ry''^2(3y''y^{(4)} - 5y'''^2)} \, dy' +$$

$$+ \frac{-15Ry'''^2 - 2y''y'''^2 + 6Ry''y^{(4)}}{Ry''(3y''y^{(4)} - 5y'''^2)} \, dy'' +$$

$$+ \frac{y'''(5R + y'')}{R(3y''y^{(4)} - 5y'''^2)} \, dy'''$$

$$\mu_6^{(4)} = \frac{3y''^3 y''' - y'y''y'''^2 - 5Ry'y'''^2 + 3Ry'y''y^{(4)}}{9Ry''^3} \, dx -$$

$$- \frac{-y''y'''^2 - 5Ry'''^2 + 3Ry''y^{(4)}}{9Ry''^3} \, dy$$

where $R = \sqrt{\dfrac{y'''^2 y''^2}{3y^{(4)}y'' - 5y'''^2}}$.

8   The final step

Now, we determine new lifted invariants by restricting the fourth order moving coframe one-forms to the curve $y = y(x)$. This amounts to restricting dy, dy', dy'', dy''' and $dy^{(4)}$ by their horizontal components $y'dx$, $y''dx$, $y'''dx$, $y^{(4)}dx$ and $y^{(5)}dx$ respectively, leading to the restricted forms

$$\mu_{1,h}^{(4)} = -2\mu_{2,h}^{(3)} + \mu_{4,h}^{(3)}$$

$$\mu_{2,h}^{(4)} = \mu_{3,h}^{(4)} = \mu_{6,h}^{(4)} = \frac{y'''}{3R} \, dx$$

$$\mu_{4,h}^{(4)} = \frac{9y''^2 y^{(5)} - 45y''y'''y^{(4)} + 40y'''^3}{-6y''(3y''y^{(4)} - 5y'''^2)} \, dx$$

$$\mu_{5,h}^{(4)} = -\mu_{2,h}^{(3)} + 2\mu_{4,h}^{(3)}$$

There is one non-constant linear dependency, namely $\mu_{4,h}^{(4)} = L \, \mu_{3,h}^{(4)}$, where

$$L = -\frac{sgn(y''y''')(9y''^2 y^{(5)} - 45y''y'''y^{(4)} + 40y'''^3)}{2\left(\sqrt{3y''y^{(4)} - 5y'''^2}\right)^3}$$

is the fundamental differential invariant of the transformation group. Let $J = -\dfrac{L}{2}$. The remaining one-form

(19) $$\mu_{2,h}^{(4)} = H \, dx, \quad H = \frac{sgn(y''')\sqrt{3y''y^{(4)} - 5y'''^2}}{3|y''|}$$

is the fundamental invariant one form. Let $I = 3H$. All higher order differential invariants can be found by differentiating I with respect ds; for instance, the fundamental fifth order differential invariant is $\mathcal{D}(J)$, where $\mathcal{D} = (D_x)^{-1}D_x$ and $D_x = \dfrac{\partial}{\partial x} + y'\dfrac{\partial}{\partial y} + y''\dfrac{\partial}{\partial y'} + \cdots$ is the total derivative operator.

Now, the Theorem is a conclusion of Theorem 5.16 of Olver, in [6]. Therefore, as

Conclusion. *Every differential invariant for the group action $A(2)$ on $\mathbf{R}^2$ is a function of the J and its derivatives with respect to I.*

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