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# Lie Symmetries of Inviscid Burgers' Equation

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**Abstract.** The present paper solves completely the problem of the Lie group analysis of nonlinear equation $u_t(x,t) + g(u)u_x(x,t) = 0$, where $g(u)$ is a smooth function of $u$. And apply these results on inviscid Burgers equation.

**Keywords.** Lie group analysis, Burgers equation, Symmetry group.

## 1. Introduction

The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [5]. Such Lie groups are invertible point transformations of both the dependent and independent variables of the differential equations. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and is of great importance to understand and to construct solutions of differential equations. Several applications of Lie groups in the theory of differential equations were discussed in the literature, the most important ones are: reduction of order of ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transformations (for many other applications of Lie symmetries see. [7], [2] and [1]).

Burgers' equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow. It is named for Johannes Martinus Burgers (1895-1981).

For a given velocity $u$ and viscosity coefficient $\nu$, the general form of Burgers' equation is:

$$u_t(x,t) + g(u)u_x(x,t) = \nu\, u_{xx}(x,t), \tag{1}$$

where $g(u)$ is a smooth function of $u$. When $\nu = 0$, Burgers' equation becomes the inviscid Burgers' equation:

$$\text{IBE} \quad : \quad u_t(x,t) + g(u)u_x(x,t) = 0 \tag{2}$$

which is a prototype for equations for which the solution can develop discontinuities (shock waves). Specially, study the geometry of equations

$$u_t(x,t)+u(x,t)u_x(x,t)=0 \tag{3}$$

and

$$u_t(x,t)+\frac{1-u(x,t)}{1+u(x,t)}\,u_x(x,t)=0. \tag{4}$$

This work is a generalization of the paper [6]; i.e. the general form of Lie point symmetries group of the nonlinear equation IBE are presented, and found some special solutions of certain IBE's.

This work is organized as follows. In section 2 we recall some results needed to construct Lie point symmetries of a given system of differential equations. In section 3, we give the general form of an infinitesimal generator admitted by IBE. In section 4, we give the general form of a projectable infinitesimal generator admitted by the equation IBE. In section 5, we determine the group transformation corresponding to every infinitesimal generator obtained by projectable symmetries. In section 6, we show how symmetries may be used to construct some exact solutions for the (3).

## 2. Method of Lie Symmetries

In this section, we recall the general procedure for determining symmetries for any system of partial differential equations (see [7] and [2]). To begin, let us consider the general case of a nonlinear system of partial differential equations of order $n$th in $p$ independent and $q$ dependent variables is given as a system of equations:

$$\Delta_\nu(x,u^{(n)})=0,\qquad \nu=1,\ldots,l, \tag{5}$$

involving $x=(x^1,\ldots,x^p)$, $u=(u^1,\ldots,u^q)$ and the derivatives of $u$ with respect to $x$ up to $n$, where $u^{(n)}$ represents all the derivatives of $u$ of all orders from 0 to $n$. We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system (5):

$$\begin{aligned}\tilde{x}^i &= x^i+s\xi^i(x,u)+O(s^2), && i=1,\ldots,p,\\ \tilde{u}^j &= u^j+s\eta^j(x,u)+O(s^2), && j=1,\ldots,q,\end{aligned} \tag{6}$$

where $s$ is the parameter of the transformation and $\xi^i$, $\eta^j$ are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator $\mathbf{v}$ associated with the above group of transformations can be written as

$$\mathbf{v}=\sum_{i=1}^p\xi^i(x,u)\partial_{x^i}+\sum_{j=1}^q\eta^j(x,u)\partial_{u^j}. \tag{7}$$

A symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions. The invariance of the system (5) under the infinitesimal transformations leads to the invariance conditions (Theorem 2.36 of [7]):

$$\mathrm{Pr}^{(n)}\mathbf{v}\big[\Delta_\nu(x,u^{(n)})\big]=0,\qquad \nu=1,\ldots,l,\qquad \text{whenever}\qquad \Delta_\nu(x,u^{(n)})=0, \tag{8}$$

where $\mathrm{Pr}^{(n)}$ is called the $n^{th}$ order prolongation of the infinitesimal generator given by

$$\mathrm{Pr}^{(n)}\mathbf{v}=\mathbf{v}+\sum_{\alpha=1}^{q}\sum_{J}\phi_J^\alpha(x,u^{(n)})\partial_{u_J^\alpha}, \tag{9}$$

where $J=(j_1,\ldots,j_k)$, $1\le j_k\le p$, $1\le k\le n$ and the sum is over all $J$'s of order $0<\#J\le n$. If $\#J=k$, the coefficient $\phi_J^\alpha$ of $\partial_{u_J^\alpha}$ will only depend on $k$-th and lower order derivatives of $u$, and

$$\phi_\alpha^J(x,u^{(n)})=D_J(\phi_\alpha-\sum_{i=1}^{p}\xi^i u_i^\alpha)+\sum_{i=1}^{p}\xi^i u_{J,i}^\alpha, \tag{10}$$

where $u_i^\alpha:=\partial u^\alpha/\partial x^i$ and $u_{J,i}^\alpha:=\partial u_J^\alpha/\partial x^i$.

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket.

## 3. Symmetries of Equation IBE

We consider the one parameter Lie group of infinitesimal transformations on $(x^1=x, x^2=t, u^1=u)$,

$$\begin{aligned}\tilde{x}&=x+s\xi(x,t,u)+O(s^2),\\ \tilde{t}&=x+s\eta(x,t,u)+O(s^2),\\ \tilde{u}&=x+s\phi(x,t,u)+O(s^2),\end{aligned} \tag{11}$$

where $s$ is the group parameter and $\xi^1=\xi$, $\xi^2=\eta$ and $\phi^1=\phi$ are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form:

$$\mathbf{v}=\xi(x,t,u)\partial_x+\eta(x,t,u)\partial_t+\phi(x,t,u)\partial_u. \tag{12}$$

and, its first prolongation is

$$\mathrm{Pr}^{(1)}\mathbf{v}=\mathbf{v}+\phi^x\,\partial_{u_x}+\phi^t\,\partial_{u_t}, \tag{13}$$

(by (9)), with

$$\begin{aligned}\phi^x&=D_x\phi-u_xD_x\xi-u_tD_x\eta,\\ \phi^t&=D_t\phi-u_xD_t\xi-u_tD_t\eta,\end{aligned} \tag{14}$$

where $D_x$ and $D_t$ are the total derivatives with respect to $x$ and $t$ respectively.

The vector field $\mathbf{v}$ generates a one parameter symmetry group of IBE if and only if

$$\mathrm{Pr}^{(1)}\mathbf{v}[u_t+g(u)u_x]=0,\qquad \text{whenever}\qquad u_t+g(u)u_x=0; \tag{15}$$

by (8); The condition (15) is equivalent to,

$$\phi.g'u_x + g\phi^x + \phi^t = 0, \quad u_t + gu_x = 0; \tag{16}$$

and hence the condition (16) gives the set of defining equations:

$$u_t + g.u_x = 0, \tag{17}$$
$$\phi.g' + g^2.\eta_x + g\eta_t - g.\xi_x - \xi_t = 0, \tag{18}$$
$$g.\phi_x + \phi_t = 0. \tag{19}$$

$x - tg$ and $u$ are characteristics of homogeneous linear first order PDE (19), therefore

$$\phi(t, x, u) = F(u, x - t.g(u)), \tag{20}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary smooth function.

Now, the PDE (18) reduced to

$$\xi_t + g(u)\xi_x = g'(u).F(u, x - t.g(u)) + g(u).\eta_t + g^2(u).\eta_x. \tag{21}$$

$x-tg$ and $u$ are characteristics of homogeneous linear first order PDE $\xi_t+g(u)\xi_x = 0$, and $g(u).\eta(t, x, u) + t.g'(u).F(u, x - t.g(u))$ is a particular solution of (21). Therefore, the solution of (21) is

$$\begin{aligned} \xi(t, x, u) &= g(u).\eta(t, x, u) \\ &\quad +t.g'(u).F(u, x - t.g(u)) + G(u, x - t.g(u)) \end{aligned} \tag{22}$$

where $G : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary smooth function. Therefore, we prove that

**Theorem 1.** *Infinitesimal generator of every one parameter Lie group of point symmetries of IBE has the form*

$$\begin{aligned} \mathbf{v} &= \Big(g(u).H(t, x, u) + t.g'(u).F(u, x - t.g(u)) + G(u, x - t.g(u))\Big)\,\partial_x \\ &\quad +H(t, x, u)\,\partial_t + F(u, x - t.g(u))\,\partial_u, \end{aligned} \tag{23}$$

*where $F$, $G : \mathbb{R}^2 \to \mathbb{R}$ and $H : \mathbb{R}^3 \to \mathbb{R}$ are arbitrary smooth functions.*

*If $\mathbf{v}^i = (g.H^i+t.g'.F^i+G^i)\,\partial_x+H^i\,\partial_t+F^i\,\partial_u$ with $i = 1, 2, 3$ and $[\mathbf{v}^1, \mathbf{v}^2] = \mathbf{v}^3$, then*

$$\begin{aligned} F^3 &= (G^1.D_2F^2 - G^2D_2F^1) + (F^1.D_1F^2 - F^2.D_1F^1), \\ G^3 &= (F^1.D_1G^2 - F^2.D_1G^1) + (G^1.D_2G^2 - G^2.D_2G^1), \\ H^3 &= (H^1.D_1H^2 - H^2.D_1H^1) + (G^1.D_3H^2 - G^2.D_3H^1) \\ &\quad +(F^1.D_2H^2 - F^2.D_2H^1), \end{aligned} \tag{24}$$

*where $D_kf$ is derivative with respect to $k^{th}$ variable of function $f$. Therefore, the symmetry Lie algebra of IBE is infinite dimensional.*

## 4. Projectable Symmetries

An one parameter Lie group of point symmetries $g_t$ is called *projectable* or *fiber-preserving*, if the action on the independent variables does not depend on the dependent variables: $g_t.(x,u) = (X(x), U(x,u))$ (see [3] and page 93 of [7]). In this case, $\xi^i$s only depend on $x$; and there are differential invariants in the form $y = I(x)$. Now, may be used to reduce the order of the given system (5) (see page 145 of [7]).

**Lemma 1.** *Let $g(u)$ be a smooth nonconstant function, and $k$ be an integer. Then, the functions $1, g(u), g^2(u), \dots, g^k(u)$ are linearly independent.*

*Proof.* Let there are constants $a_i$ for $i = 0, \dots, k$ such that $a_0 + a_1.g(u) + \cdots + a_k.g^k(u) = 0$. Then, $g(u)$ is a root of polynomial with real coefficients $a_0 + a_1.x + \cdots + a_k.x^k = 0$ for any $u$. Therefore, $g(u)$ is constant, contradicting our assumption. □

By repeating the algorithm of section 3, we find that

**Theorem 2.** *Let $g(u)$ be a smooth nonconstant function. Then, every projectable infinitesimal generator of one parameter Lie group of symmetries of $u_t + g(u)u_x = 0$ has the form*

$$\begin{aligned}&\big(c_4x^2 + c_1tx + c_6x + c_8t + c_7\big)\,\partial_x + \big(c_4xt + c_1t^2 + c_5x + c_2t + c_3\big)\,\partial_t \\ &+ \frac{c_1x + c_8 + \big(2c_4x + c_6 + c_1t - c_4x - 2c_1t - c_2\big).g(u) - (c_4t + c_5).g(u)^2}{g'(u)}\,\partial_u\end{aligned} \tag{25}$$

*where $c_i$'s are arbitrary constants.*

*If $g(u) \equiv C$ be constant, then every projectable infinitesimal generator of one parameter Lie group of symmetries has the form*

$$F(t, x - C.t).\,\partial_x + \big(C.F(t, x - C.t) + H(x - C.t)\big).\,\partial_t + H(x - C.t).\,\partial_u. \tag{26}$$

*Proof.* If we assume $\xi = F(x,t)$ and $\eta = G(x,t)$ in (12), and apply its prolonged on $E := u_t + g(u).u_x = 0$, we conclude that

$$\phi(t,x,u) = \frac{g(u)}{g'(u)}.\Big(g(u).F_x(x,t) + F_t(x,t) - g(u)^2.G_x(x,t) - G_t(x,t)\Big). \tag{27}$$

By setting this $\phi$ in $\mathbf{v}^{(1)}(E) \equiv 0 \operatorname{mod} E$, we find that

$$\begin{aligned}&F_{tt}(x,t) + (2.F_{xt}(x,t) - G_{tt}(x,t)).g(u) \\ &\quad + (F_{xx}(x,t) - 2G_{xt}(x,t)).g(u)^2 - G_{xx}(x,t).g(u)^3 = 0.\end{aligned} \tag{28}$$

and, by Lemma 1, we obtain a system of PDEs:

$$\begin{aligned}&F_{tt}(x,t) = 0, && 2.F_{xt}(x,t) - G_{tt}(x,t) = 0, \\ &F_{xx}(x,t) - 2G_{xt}(x,t) = 0, && G_{xx}(x,t) = 0.\end{aligned} \tag{29}$$

By solving this system of PDEs, we conclude that

$$\begin{aligned}F(x,t) &= c_4x^2 + (c_6 + c_1t)x + c_8t + c_7, \\ G(x,t) &= c_4xt + c_1t^2 + c_5x + c_2t + c_3.\end{aligned} \tag{30}$$

Now, we put $F$ and $G$ in (27), and find $\eta$. □

**Theorem 3.** *Let $g(u)$ be a smooth nonconstant function. Then, every projectable infinitesimal generator of one parameter Lie group of symmetries of IBE has the form $\mathbf{v} = \sum_{i=1}^{8} a_i \mathbf{v}_i$, where $a_1, \ldots, a_8$ are arbitrary constants and*

$$\begin{aligned}
&\mathbf{v}_1 = \partial_t, && \mathbf{v}_2 = \partial_x,\\
&\mathbf{v}_3 = t\,\partial_x + \frac{1}{g'(u)}\,\partial_u, && \mathbf{v}_4 = t\,\partial_t + x\,\partial_x,\\
&\mathbf{v}_5 = t\,\partial_t - \frac{g(u)}{g'(u)}\,\partial_u, && \mathbf{v}_6 = x\partial_t - \frac{g^2(u)}{g'(u)}\,\partial_u,\\
&\mathbf{v}_7 = t^2\,\partial_t + tx\,\partial_x + \frac{x - t.g(u)}{g'(u)}\,\partial_u, &&\\
&\mathbf{v}_8 = tx\,\partial_t + x^2\,\partial_x + \frac{g(u)(x - t.g(u))}{g'(u)}\,\partial_u. &&
\end{aligned} \tag{31}$$

*These vector fields span a Lie algebra $\mathfrak{g}$ with following commutator table*

| | $\mathbf{v}_1$ | $\mathbf{v}_2$ | $\mathbf{v}_3$ | $\mathbf{v}_4$ | $\mathbf{v}_5$ | $\mathbf{v}_6$ | $\mathbf{v}_7$ | $\mathbf{v}_8$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{v}_1$ | $0$ | $0$ | $\mathbf{v}_2$ | $\mathbf{v}_1$ | $\mathbf{v}_1$ | $0$ | $\mathbf{v}_4+\mathbf{v}_5$ | $\mathbf{v}_6$ |
| $\mathbf{v}_2$ | $0$ | $0$ | $0$ | $\mathbf{v}_2$ | $0$ | $\mathbf{v}_1$ | $\mathbf{v}_3$ | $2\mathbf{v}_4-\mathbf{v}_5$ |
| $\mathbf{v}_3$ | $-\mathbf{v}_2$ | $0$ | $0$ | $0$ | $-\mathbf{v}_3$ | $-\mathbf{v}_4+2\mathbf{v}_5$ | $0$ | $\mathbf{v}_7$ |
| $\mathbf{v}_4$ | $-\mathbf{v}_1$ | $-\mathbf{v}_2$ | $0$ | $0$ | $0$ | $0$ | $\mathbf{v}_7$ | $\mathbf{v}_8$ |
| $\mathbf{v}_5$ | $-\mathbf{v}_1$ | $0$ | $\mathbf{v}_3$ | $0$ | $0$ | $-\mathbf{v}_6$ | $\mathbf{v}_7$ | $0$ |
| $\mathbf{v}_6$ | $0$ | $-\mathbf{v}_1$ | $\mathbf{v}_4-2\mathbf{v}_5$ | $0$ | $\mathbf{v}_6$ | $0$ | $\mathbf{v}_8$ | $0$ |
| $\mathbf{v}_7$ | $-\mathbf{v}_4-\mathbf{v}_5$ | $-\mathbf{v}_3$ | $0$ | $-\mathbf{v}_7$ | $-\mathbf{v}_7$ | $-\mathbf{v}_8$ | $0$ | $0$ |
| $\mathbf{v}_8$ | $-\mathbf{v}_6$ | $-2\mathbf{v}_4+\mathbf{v}_5$ | $-\mathbf{v}_7$ | $-\mathbf{v}_8$ | $0$ | $0$ | $0$ | $0$ |

**Theorem 4.** 1) *The algebra $\mathfrak{g}$ is semisimple.*

2) *If $A_i$ be the matrix of adjoint transformation*

$$\mathrm{ad}(\mathbf{v}_i) : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}(\mathbf{v}_i)(\mathbf{v}_j) := [\mathbf{v}_i, \mathbf{v}_j], \quad i = 1, \ldots, 8$$

*with respect to base $\{\mathbf{v}_1, \ldots, \mathbf{v}_8\}$, then*

$$\begin{aligned}
A_1 &= E_{14} + E_{15} + E_{23} + E_{47} + E_{57} + E_{68},\\
A_2 &= E_{16} + E_{24} + E_{37} + 2E_{48} - E_{58},\\
A_3 &= -E_{21} - E_{35} - E_{46} + 2E_{56} + E_{78},\\
A_4 &= -E_{11} - E_{22} + E_{77} + E_{88},\\
A_5 &= -E_{11} + E_{33} + E_{87},\\
A_6 &= -E_{12} + E_{43} - 2E_{53},\\
A_7 &= -E_{32} - E_{41} - E_{51} - E_{74} - E_{85},\\
A_8 &= -2E_{42} + E_{52} - E_{61} - E_{73} - E_{84};
\end{aligned} \tag{32}$$

*where, $E_{ij}s$ are $8\times 8-$elementary matrixes, for $i,j=1,\ldots,8$; that is, the $(i,j)$-entry of $E_{i,j}$ is 1, and all other entries are zero.*

## 5. Lie Symmetries of (3)

The (3) is $u_t+u.u_x=0$; i.e., an IBE with $m=1$. In this case, Theorem 2 yields to the following vector fields:

$$\begin{aligned}
&\mathbf{v}_1=\partial_t, && \mathbf{v}_2=\partial_x,\\
&\mathbf{v}_3=t\,\partial_x+\partial_u, && \mathbf{v}_4=t\,\partial_t+x\,\partial_x,\\
&\mathbf{v}_5=t\,\partial_t-u\,\partial_u, && \mathbf{v}_6=x\partial_t-u^2\,\partial_u,\\
&\mathbf{v}_7=t^2\,\partial_t+tx\,\partial_x+(x-t.u)\,\partial_u, && \mathbf{v}_8=tx\,\partial_t+x^2\,\partial_x+u(x-t.u)\,\partial_u.
\end{aligned} \tag{33}$$

To obtain the group transformation which is generated by the infinitesimal generators $\mathbf{v}_i=\xi_i\partial_t+\eta_i\partial_x+\phi_i\partial_u$ for $i=1,\ldots,8$, we need to solve the 8 systems of first order ordinary differential equations,

$$\begin{aligned}
&\frac{d\tilde{x}(s)}{ds}=\xi_i(\tilde{x}(s),\tilde{t}(s),\tilde{u}(s)), && \tilde{x}(0)=x,\\
&\frac{d\tilde{t}(s)}{ds}=\eta_i(\tilde{x}(s),\tilde{t}(s),\tilde{u}(s)), && \tilde{t}(0)=t, \qquad i=1,\ldots,8\\
&\frac{d\tilde{u}(s)}{ds}=\phi_i(\tilde{x}(s),\tilde{t}(s),\tilde{u}(s)), && \tilde{u}(0)=u.
\end{aligned} \tag{34}$$

Exponentiating the infinitesimal symmetries of (3), we get the one parameter groups $g_k(s)$ generated by $\mathbf{v}_k$ for $k=1,\ldots,8$:

$$\begin{aligned}
g_1 &: (t,x,u) \longmapsto (s+t,x,u),\\
g_2 &: (t,x,u) \longmapsto (t,s+x,u),\\
g_3 &: (t,x,u) \longmapsto (t,x+st,s+u),\\
g_4 &: (t,x,u) \longmapsto (e^st,e^sx,u,\\
g_5 &: (t,x,u) \longmapsto (e^st,x,e^{-s}u),\\
g_6 &: (t,x,u) \longmapsto \Big(t+sx,x,\frac{u}{1+su}\Big),\\
g_7 &: (t,x,u) \longmapsto \Big(\frac{t}{1-st},\frac{x}{1-st},u+s(x-tu)\Big),\\
g_8 &: (t,x,u) \longmapsto \Big(\frac{t}{1-sx},\frac{x}{1-sx},\frac{u}{1-s(x-tu)}\Big).
\end{aligned} \tag{35}$$

Consequently,

**Theorem 5.** *If $u = f(x,t)$ is a solution of (3), so are the functions*

$$\begin{array}{rcl} g_1(s)\cdot f(x,t) &=& f(x,s+t),\\ g_2(s)\cdot f(x,t) &=& f(x+s,t),\\ g_3(s)\cdot f(x,t) &=& f(x+st,t)-s,\\ g_4(s)\cdot f(x,t) &=& f(e^s x, e^s t),\\ g_5(s)\cdot f(x,t) &=& e^s\, f(x,e^s t),\\ g_6(s)\cdot f(x,t) &=& \dfrac{f(x,t+sx)}{1-s\,f(x,t+sx)},\\ g_7(s)\cdot f(x,t) &=& \dfrac{1}{1-s\,t}\Big(f\Big(\dfrac{x}{1-st},\dfrac{t}{1-st}\Big)-s\,x\Big),\\ g_8(s)\cdot f(x,t) &=& \big(1-s\,x\big)\,f\Big(\dfrac{x}{1-sx},\dfrac{t}{1-sx}\Big)\div\Big(1-s\,t\,f\Big(\dfrac{x}{1-sx},\dfrac{t}{1-sx}\Big)\Big). \end{array} \tag{36}$$

If we let $u(x,t)=1$ be a constant solution of (3), we conclude that the trivial functions $g_i(s)\cdot 1 = 1$, for $i=1,2,3,4$, $g_5(s)\cdot 1 = e^s$, and $g_6(s)\cdot 1 = 1/(1-s)$ and a nontrivial solution for (3):

$$g_7(s)\cdot 1 = g_8(s)\cdot 1 = \frac{s\,x-1}{s\,t-1}, \tag{37}$$

Now, by applying $g_1$ and $g_2$ on (37), we conclude the solution

$$u(x,t) = \frac{ax+b}{at+c}, \tag{38}$$

where $a$, $b$ and $c$ are arbitrary constants, with $a^2+c^2\neq 0$. By using the other $g_k$'s we can not find any new nontrivial solution for (3).

## 6. Invariant Solutions of (3)

The first advantage of symmetry group methods is to construct new solutions from known solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the (3) to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

The (3) is expressed in the coordinates $(x,t,u)$, so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants $(y,v)$ corresponding to the infinitesimal generator. So using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation.

Now, we find four nontrivial solution of (3).

**1.** First, consider $\mathbf{v}_3 = t\,\partial_x + \partial_u$. To determine independent invariants $I$, we need to solve the first partial differential equations $\mathbf{v}_i$(I)=0, that is

$$\Big(t\,\partial_x + x\partial_x\Big)I = t\,\frac{\partial I}{\partial x} + x\,\frac{\partial I}{\partial x} + 0\,\frac{\partial I}{\partial u}, \tag{39}$$

which is a homogeneous first order PDE. Thus, we solve the associated characteristic ordinary differential equation

$$\frac{dt}{t} = \frac{dx}{x} = \frac{du}{0}. \tag{40}$$

Hence, we obtain two functionally independent invariants $y = x/t$ and $v = u$.

If we treat $v$ as a function of $y$, we can compute formulae for the derivatives of $u$ with respect to $x$ and $t$ in terms of $y$, $v$ and the derivatives of $v$ with respect to $y$, along with a single parametric variable, which we designate to be $t$, so that $x$ will be the corresponding principle variable. We find, using the chain rule, that if $u = v = u(y) = u(x/t)$, then

$$\begin{aligned} u_t &= \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} = -\frac{x}{t^2}\,v_y = -\frac{1}{t}\,y\,v_y, \\ u_x &= \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} = \frac{1}{t}\,v_y. \end{aligned} \tag{41}$$

Substituting for $u_t$ and $u_x$ their expressions in the equation (3), we obtain the order ordinary differential equation

$$0 = u_t + uu_x = \frac{1}{t}\,v_y\,\big(-y+v\big). \tag{42}$$

The solutions of this equation are $y = x/t$ and $v = u$. Consequently, we obtain that

$$u(x,t) = \text{cte}, \quad u(x,t) = \frac{x}{t}, \tag{43}$$

are $\mathbf{v}_4$ invariant solutions of equation (3); These solutions belong to set (38).

**2.** The invariants of $\mathbf{v}_1 + \mathbf{v}_3$ are $y = t^2 + 2x$ and $v = u + t$. The reduced equation is $2v(y)\,v'(y) + 1 = 0$; that implies $v(y) = \pm\sqrt{y+a}$. Therefore, $u(x,t) = \pm\sqrt{t^2+2x+a} - t$ is another solution of the equation (3). By applying the one parameter groups $g_i$ on this solution, we conclude a set of three parameter solutions for equation (3):

$$u(x,t) = \pm\sqrt{a^2t^2 + 2ax + bt + c} - at - \frac{b}{2a}. \tag{44}$$

**3.** The invariants of $\mathbf{v}_4+\mathbf{v}_5$ are $y=x/\sqrt{t}$ and $v=u\sqrt{t}$. The reduced equation is $v'(y)\,(y-2\,v(y))+v(y)=0$; that implies $v(y)=\dfrac{1}{2t}\big(x\pm\sqrt{x^2-a^2t}\big)$. Therefore, $u(x,t)=\pm\sqrt{t^2+2x+a}-t$ is another solution of the equation (3). By applying the one parameter groups $g_i$ on this solution, we conclude a set of three parameter solutions for equation (3):

$$u(x,t)=\frac{(2bt-a)x+a\sqrt{x^2+t(bt-a)}}{2t(bt-a)}, \tag{45}$$

where $a$, and $b$ are arbitrary constants.

**4.** The invariants of $\mathbf{v}_8$ are $y=x/t$ and $v=t(x-tu)/(xu)$. The reduced equation is $y\,v'(y)\,v(y)+v^2(y)=0$; that implies $v(y)=a/y$. Therefore

$$u(x,t)=-\frac{1}{t}\mathrm{LW}\big(-te^{a-x}\big) \tag{46}$$

is another solution of (3), where LW is the Lambert W-function; i.e. a function defined by function-equation $f(x)\,.\,e^{f(x)}=x$. By applying the one parameter groups $g_i$ on this solution, we conclude a set of three parameter solutions for (3):

$$u(x,t)=\frac{1}{ct+d}\mathrm{LW}((ct+d)\,e^{a+bt+cx})-\frac{b}{c}, \tag{47}$$

where $a$, $b$, $c$, and $d$ are arbitrary constants.

## 7. Lie Group Analysis of Equation (4)

By Theorem 2, every projectable infinitesimal generator of one parameter Lie group of symmetries of (4) has the form $\mathbf{v}=\sum_{i=1}^{8}a_i\mathbf{v}_i$, where $a_1,\ldots,a_8$ are arbitrary constants and

$$\begin{aligned}
&\mathbf{v}_1=\partial_x, && \mathbf{v}_2=\partial_t,\\
&\mathbf{v}_3=t\,\partial_x+\frac{1}{2}(1+u)^2\,\partial_u, && \mathbf{v}_4=t\,\partial_t+x\partial_x,\\
&\mathbf{v}_5=t\,\partial_t+\frac{1}{2}(1-u^2)\,\partial_u, && \mathbf{v}_6=x\partial_t+(1-u)^2\,\partial_u,\\
&\mathbf{v}_7=t^2\,\partial_t+xt\,\partial_x+\frac{1}{2}(u+1)(x.u+t.u+x-t)\,\partial_u,\\
&\mathbf{v}_8=xt\,\partial_t+x^2\,\partial_x+\frac{1}{2}(u-1)(x.u+t.u+x-t\,\partial_u.
\end{aligned} \tag{48}$$

The one parameter groups $g_k(s)$ generated by $\mathbf{v}_k$ for $k = 1, \ldots, 8$ are:

$$\begin{array}{rcccl} g_1 & : & (x,t,u) & \longmapsto & (x+s,t,u), \\ g_2 & : & (x,t,u) & \longmapsto & (x,t+s,u), \\ g_3 & : & (x,t,u) & \longmapsto & \Big(x+ts,t,-\dfrac{s+(s-2).u}{2+s+s.u}\Big), \\ g_4 & : & (x,t,u) & \longmapsto & (e^s x, e^s t, u), \\ g_5 & : & (x,t,u) & \longmapsto & \Big(x, e^s t, \tanh\Big(\dfrac{1}{2}\,s + \dfrac{1}{2}\,\ln\big(\dfrac{1+u}{1-u}\big)\Big)\Big), \\ g_6 & : & (x,t,u) & \longmapsto & \Big(x, t+sx, \dfrac{s+(2-s).u}{2+s-s.u}\Big), \\ g_7 & : & (x,t,u) & \longmapsto & \Big(\dfrac{x}{1-st}, \dfrac{t}{1-st}, \dfrac{2(u+1)}{2+s(x-t)+s(t+x).u} - 1\Big), \\ g_8 & : & (x,t,u) & \longmapsto & \Big(\dfrac{x}{1-sx}, 1 + \dfrac{2(u-1)}{s(x-t)+s(t+x).u-2}\Big). \end{array} \tag{49}$$

Consequently,

**Theorem 6.** *If $u = f(x,t)$ is a solution of* (4)*, so are the functions*

$$\begin{array}{rcl} g_1(s)\cdot f(x,t) & = & f(x+s,t), \\ g_2(s)\cdot f(x,t) & = & f(x,t+s), \\ g_3(s)\cdot f(x,t) & = & \dfrac{s+(s+2).f(x+ts,t)}{2-s-s.f(x+ts,t)}, \\ g_4(s)\cdot f(x,t) & = & f(e^s x, e^s t), \\ g_5(s)\cdot f(x,t) & = & \tanh\Big(-\dfrac{1}{2}\,s + \dfrac{1}{2}\,\ln\Big(\dfrac{1+f(x,e^s t)}{1-f(x,e^s t)}\Big)\Big), \\ g_6(s)\cdot f(x,t) & = & \dfrac{s+(2-s).f(x,t+sx)}{2+s-s.f(x,t+sx)}, \\ g_7(s)\cdot f(x,t) & = & \dfrac{2(f\big(\frac{x}{1-st},\frac{t}{1-st}\big)+1)}{2-s(x-t)-s(t+x).f\big(\frac{x}{1-st},\frac{t}{1-st}\big)} - 1, \\ g_8(s)\cdot f(x,t) & = & 1 - \dfrac{2(f\big(\frac{x}{1-sx},\frac{t}{1-sx}\big)-1)}{s(x-t)+s(t+x).f\big(\frac{x}{1-sx},\frac{t}{1-sx}\big)+2}. \end{array} \tag{50}$$

If we let $u(x,t) = 1$ be a constant solution of (4), we conclude that the trivial functions $g_i(s)\cdot 1 = 1$, for $i = 1, \ldots, 8$, a nontrivial solution for (4):

$$u(x,t) = 1 - \frac{2x}{x+t+c}, \tag{51}$$

where $a$ is an arbitrary constant.

Now using the differential invariants of $\mathbf{v}_3 - \mathbf{v}_2$ for a nontrivial solution of (4). Its differential invariants are $y = x + t^2/2$ and $v = (2 - tu + t)/(u + 1)$. The reduced equation is $v'(y).(v(y)+1)=1$. Therefore,

$$u(x,t)=\frac{t+1\pm\sqrt{1+2x+t^2+2a}}{1\mp\sqrt{1+2x+t^2+2a}}, \tag{52}$$

is a nontrivial solution of the equation (4).

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