Extended Abstracts of the 41th Iranian International Conference on Mathematics

12-15 September 2010, University of Urmia, Urmia, Iran, pp 00-00

CHARACTERIZATION OF DIFFERENT TYPES OF FOLIATIONS ON THE TANGENT BUNDLE OF A FINSLER MANIFOLD

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Abstract. As the geometric structures that exist in Finsler geometry depend on both point and direction, the tangent bundle of a Finsler manifold is of special importance. In this paper, we have studied the different foliations on the tangent bundle of a Finsler manifold. We have mentioned some results about the vertical foliation. We have also shown that in some particular cases, the horizontal distribution is involutive. As a main result, we have showed that the integrability of the horizontal distribution, leads to a new type of foliation on the tangent bundle. This foliation is created by the set of vector fields which are symmetries of the vertical projector. We have proved that it can be regarded as a Riemannian foliation on the tangent bundle.

1. Introduction and Preliminaries

In sharp contrast to the Riemannian geometry, the geometric structures in Finsler geometry are dependent of both point and direction. Thus, there is a close relationship between the geometry of the tangent

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2000 *Mathematics Subject Classification.* Primary 53C05; Secondary 53C12, 53C15, 53C22.

Key words and phrases. Finsler manifold, foliation, vertical projector, holonomy, symmetry, bundle-like metric.

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bundle of Finsler manifold and the geometry of the Finsler manifold it self. So investigating the foliations on the tangent bundle of a Finsler manifold can be regarded as a powerful device for studying the properties of a Finsler manifold.
Let (M, F) be a m-dimensional Finsler manifold. $\mathcal{T}_M = (TM, M, \pi)$ denotes the tangent bundle, as a base manifold M, a total space TM ($2m$ dimensional) and a projection $\pi : TM \longrightarrow M$. V_zTM is the kernel of $(d\pi)_z : T_zTM \longrightarrow T_{\pi(z)}M$ for $z \in TM_0$. A **non-linear connection** or **horizontal distribution** on TM_0 is a complementary distribution HTM for VTM on TTM^0, which is locally spanned by:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^i}, \quad i \in \{1, ..., m\}.$$

where $N^j_i(x, y)$ are non-linear differentiable functions on TM, called the coefficients of the non-linear connection. So we have the following decomposition : $TTM^0 = HTM^0 \oplus VTM^0$.
By direct calculations, we obtain the following relations:

$$[\frac{\delta}{\delta x^i}, \frac{\delta}{\partial x^j}] = R^k_{ij} \frac{\partial}{\partial y^k} \qquad [\frac{\delta}{\partial x^j}, \frac{\partial}{\partial y^i}] = \frac{\partial N^k_j}{\partial y^i} \frac{\partial}{\partial y^k}.$$

where we have : $\qquad R^k_{ij} = \frac{\delta N^k_i}{\delta x^j} - \frac{\delta N^k_j}{\delta x^i}$.

2. Main results

We denote by $\mathcal{F}_V$ the foliation on TM^0 determined by the fibers of $\pi : TM^0 \longrightarrow M$, and call it the **vertical foliation** on TM^0. In theorems (2.1) and (2.2), we state some results for the vertical foliation:

Theorem 2.1. *If the Sasaki-Finsler metric G is bundle-like for the vertical foliation $\mathcal{F}_V$, then M decomposes into a Riemannian product:*

$$M = M_0 \times M_1 \times ... \times M_p$$

where M_0 is a maximal factor isometric to Euclidean space and each M_i, $i > 0$ is indecomposable. This decomposition is unique up to the order of $M_1, ..., M_p$.

Theorem 2.2. *Let (M, F) be a Finsler manifold, then the following assertions are equivalent:*
(1) The Sasaki-Finsler metric G is bundle-like for the vertical foliation $\mathcal{F}_V$.
(2) $\frac{\delta}{\delta x^i}$ is a H-Killing vector field.
(3) The transversal bundle HTM is holonomy invariant.
(4) The Sasaki Finsler metric on HTM is parallel with respect to the nonlinear connection H.
(5) The parallel translation of the nonlinear connection is an isometry between the fibers as Riemannian spaces for any curve.
(6) The Berwaldian Finsler pair connection (∇^B, H) is h-metrical.

Lemma 2.3. *The horizontal distribution HTM is integrable if and only if $R^k_{ij} = 0$. We denote this foliation by $\mathcal{F}_H$ and call it the horizontal foliation.*

For the horizontal foliation $\mathcal{F}_H$ we can state the following theorem:

Theorem 2.4. *The following assertions are equivalent:*
(1) The Sasaki-Finsler metric is bundle-like for the horizontal foliation $\mathcal{F}_H$.
(2) Along every curve parallel translation is an isometry between the associated Minkowski spaces.
(3) The h-curvature and hv-curvature Finsler tensor fields of the Berwarld connection vanish.
(4)The h-curvature and hv-curvature Finsler tensor fields of the Rund connection vanish.
(5) The holonomy group of the nonlinear connection H is trivial. □

Let J be an almost tangent structure. A vector 1-form $v : \mathcal{X}(TM) \longrightarrow \mathcal{X}(TM)$ satisfying : $Jov = 0, \;\; voJ = J$ is called a **vertical projector**. For a vector 1-form K and $Z \in \mathcal{X}(TM)$ we have the bracket: $[K, Z]_{FN} : \mathcal{X}(TM) \longrightarrow \mathcal{X}(TM)$ given by

$$[K, Z]_{FN}(X) = [K(X), Z] - K[X, Z].$$

We call $X \in \mathcal{X}(TM)$ is a **symmetry** of v if $[v, X]_{FN} = 0$.

Theorem 2.5. *If the horizontal distribution HTM is integrable, then X is the symmetry of v if and only if $X = X^k(x)\frac{\delta}{\delta x^k}$.*

Assume that the horizontal distribution i.e, HTM is integrable. We define: $\Omega = \{X \in \mathcal{X}(TM) : X \text{ is a symmetry } (\text{for } v)\}$

Lemma 2.6. *The distribution Ω is involutive. Hence, Ω creats a foliation on TM denoted by $\mathcal{F}_\Omega$.*

Theorem 2.7. *Condider $(TM, g, \mathcal{F}_\Omega)$, then the following assertions are equivalent:*
(1) $\mathcal{F}_\Omega$ is a Riemannian foliation on TM.
(2) The induced Riemannian metric on $\Omega^\perp$ is parallel with respect to the intrinsic connection $\nabla^\perp$, that is for any $X, Y, Z \in \Gamma(TTM)$:

$$(\nabla^\perp_X g)(\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z) = X(g(\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z) - g(\nabla^\perp_X \widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z) - g(\widetilde{\mathbb{P}}Y, \nabla^\perp_X \widetilde{\mathbb{P}}Z) = 0$$

(3) $\mathbb{P}X(g(\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z)) - g([\mathbb{P}X, \widetilde{\mathbb{P}}Y], \widetilde{\mathbb{P}}Z) - g([\mathbb{P}X, \widetilde{\mathbb{P}}Z], \widetilde{\mathbb{P}}Y) = 0$.
where $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are projection morphisms of TTM on Ω and $\Omega^\perp$ with respect to the decomposition: $TTM = \Omega \oplus \Omega^\perp$, respectively.

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