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Fuzzy differential invariant (FDI)

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ABSTRACT

In this paper, we have tried to apply the concepts of fuzzy set to Lie groups and fuzzy differential invariant (FDI) in order to provide suitable conditions for applying Lie symmetry method in solving fuzzy differential equations (FDEs). For this, we define a $\mathscr{C}^1$-fuzzy submanifold and fuzzy immersion with some examples. In main section, we defined the fuzzy Lie group and some its relative concepts such as fuzzy transformation group and fuzzy G-invariant. The goal of this paper is to introduce and study new defining for fuzzy Lie group and fuzzy differential invariant (FDI). Also, some illustrative examples are given.

1. Introduction

The fuzzy differential equation (FDE) was first introduced by Kandel and Byatt [13] and then was formulated by Kaleva [12]. Its topics have been rapidly growing in recent years [4,14,16]. The theory of fuzzy differential equations has attracted much attention in recent times because this theory represents a natural way to model dynamical systems under uncertainty [21]. The fuzzy differential equation is a very important topic from the theoretical point of view [8,12,17] as well as the applied point of view [1,4,14,16]; for example, in population models [8], civil engineering [17], and in hydraulics modeling [2]. Fuzzy differential equations were considered by many papers. Numerical techniques were developed in [11,13] and others. Pederson and Sambandham [19,20] study the Euler and Runge–Kutta numerical methods, respectively for hybrid fuzzy differential equations. In some sense, Pederson and Sambandham [12,13] "rewrite the whole literature on numerical solutions of ODEs" in the hybrid fuzzy setting, focusing on the Euler and Runge–Kutta methods, respectively.

Lie symmetry method has long been used to study differential equations. It has been developed into a powerful tool to solve differential equations, to classify them and to establish properties of their solution space. These aspects of Lie group theory have been described in many books and papers [10,18]. Now we want to applicable Lie symmetry method as a analytical method in solving fuzzy differential equations (FDEs). For this, we require to define new fuzzy concepts such as fuzzy Lie group, fuzzy transformation group, fuzzy differential invariant (FDI) and other their relative concepts. The notion of a fuzzy Lie group is depend on the basic concepts in fuzzy topology [3,5,7], $\mathscr{C}^1$-fuzzy manifold and fuzzy differentiable function between two $\mathscr{C}^1$-fuzzy manifolds [9].

2. Preliminaries

A fuzzy topology τ on a group G is said to be *compatible* if the mappings

$$m: \ (G\times G, \tau\times\tau) \to (G,\tau) \quad (x,y)\mapsto xy \tag{1}$$

$$i: \ (G,\tau) \to (G,\tau) \quad x\mapsto x^{-1} \tag{2}$$

are fuzzy continuous [3,15]. A group G equipped with a compatible fuzzy topology τ on G is called *a fuzzy topological group* [6].

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A fuzzy topological vector space, is a vector space E over the field $\mathbb{F}$ of real or complex numbers, if E is equipped with a fuzzy topology τ and $\mathbb{F}$ equipped with the usual topology $\mathscr{F}$, such that following two mappings are fuzzy continuous:

$$(E,\tau)\times(E,\tau)\to(E,\tau)\quad (x,y)\mapsto x+y$$
$$(\mathbb{F},\mathscr{F})\times(E,\tau)\to(E,\tau)\quad (\alpha,x)\mapsto \alpha x.$$

Let E,F be two fuzzy topological vector space, the mapping $\phi:\ E\to F$ is said to be *tangent to 0* if given a neighborhood W of 0_δ, $0<\delta\leqslant 1$, in F there exists a neighborhood V of 0_ϵ, $0<\epsilon<\delta$, in E such that

$$\phi[tV]\subset o(t)W,$$

for some function $o(t)$, and if $f:\ E\to F$ be a fuzzy continuous mapping, the f is said to be *fuzzy differentiable at a point* $x\in E$ if there exists a linear fuzzy continuous mapping $u:\ E\to F$ (It is denoted by $u\in\mathscr{L}(E,F)$.) such that

$$f(x+y)=f(x)+u(y)+\phi(y),\quad y\in E,$$

where ϕ is tangent 0. The mapping u is called *the fuzzy derivative of f in x* that is denoted by $f'(x)$; $f'(x)\in\mathscr{L}(E,F)$. The mapping f is *fuzzy differentiable* if it is fuzzy differentiable at every point of E.

A bijection $f:\ E\to F$ is said to be a $\mathscr{C}^1$- *fuzzy diffeomorphism* if it and its inverse f^{-1} are fuzzy differentiable, and f' and $(f^{-1})'$ are fuzzy continuous (see to [9]).

Let X be a set. A $\mathscr{C}^1$-*fuzzy atlas on* X is a collection of pairs $\{(A_j,\phi_j)\}_{j\in J}$, which satisfies the following conditions:

(i) Each A_j is a fuzzy set in X and $\sup_j\{\mu_{A_j}(x)\}=1$, for all $x\in X$.
(ii) Each ϕ_j is a bijection, defined on the support of A_j,

$$\{x\in X:\ \mu_{A_j}(x)>0\},$$

which maps A_j onto an open fuzzy set $\phi_j[A_j]$ in some fuzzy topological vector space E_j, and, for each $l\in J$, $\phi_j[A_j\cap A_l]$ is an open fuzzy set in E_j.
(iii) The mapping $\phi_l\circ\phi_j^{-1}$, which maps $\phi_j[A_j\cap A_l]$ is a $\mathscr{C}^1$-fuzzy diffeomorphism for each pair of indices j,l.

Each pair $(A_j,\phi_j)_{j\in J}$ is called *a fuzzy chart* of the fuzzy atlas. If a point $x\in X$ lies in the support of A_j then $(A_j,\phi_j)_{j\in J}$ is said to be a fuzzy chart at x.

Let (X,τ) be a fuzzy topological space. Suppose there exists an open fuzzy set A in X and a fuzzy continuous bijective mapping ϕ defined on the support of A and mapping onto an open fuzzy set V in some fuzzy topological vector space E. Then (A,ϕ) is said to be compatible with the $\mathscr{C}^1$-atlas $\{(A_j,\phi_j)\}_{j\in J}$ if each mapping $\phi_j\circ\phi^{-1}$ of $\phi[A\cap A_j]$ onto $\phi_j[A\cap A_j]$ is a $\mathscr{C}^1$-fuzzy diffeomorphism. Two $\mathscr{C}^1$-fuzzy atlases are compatible if each fuzzy chart of one atlas is compatible with each fuzzy chart of the other atlas. It may be verified immediately that the relation of compatibility between $\mathscr{C}^1$-fuzzy atlases is an equivalence relation. An equivalence class of $\mathscr{C}^1$- fuzzy atlases on X is said to define a $\mathscr{C}^1$-fuzzy manifold on X [9].

If X,Y be the fuzzy manifolds; then the product $X\times Y$ is also a fuzzy manifold.

The function $f:\ X\to Y$ is said to be *fuzzy differentiable* at a point $x\in X$ if there is a fuzzy chart (U,ϕ) at $x\in X$ and a fuzzy chart (V,φ) at $f(x)\in Y$ such that the mapping $\varphi\circ f\circ\phi^{-1}$, which maps $\phi[U\cap f^{-1}[V]]$ into $\varphi[V]$ is fuzzy differentiable at $\phi(x)$. The mapping f is fuzzy differentiable if it is fuzzy differentiable at every point of X; it is a $\mathscr{C}^1$-fuzzy diffeomorphism if $\varphi\circ f\circ\phi^{-1}$ is a $\mathscr{C}^1$-fuzzy diffeomorphism, [9].

We denote by $\mathscr{K}^n$ the family of all nonempty compact subsets of $\mathbb{R}^n$, the n-dimensional Euclidean space. If $A,B\in\mathscr{K}^n$ and $\lambda\in\mathbb{R}$, then the operations of addition and scalar multiplication are defined as

$$A+B=\{a+b:\ a\in A,\ b\in B\},\quad \lambda A=\{\lambda a:\ a\in A\}.$$

If $A\in\mathscr{K}^n$, we define the *ϵ-neighborhood of A* as the set

$$N(A,\epsilon)=\{x\in\mathbb{R}^n:\ d(x,A)<\epsilon\},$$

where $d(x,A)=\inf_{a\in A}\|x-a\|$ and $\|.\|$ is the usual Euclidean norm on $\mathbb{R}^n$. The Hausdorff $\rho(A,B)$ of $A,B\in\mathscr{K}^n(X)$ is defines by

$$\rho(A,B)=\inf\{\epsilon>0:\ A\subset N(B,\epsilon)\},$$

and the Hausdorff metric on $\mathscr{K}^n$ is defined by

$$h(A,B)=\max\{\rho(A,B),\rho(B,A)\}.$$

Let μ be a fuzzy set of X and $\alpha\in[0,1]$, the set $\{x\in X:\ \mu(x)\geqslant\alpha\}$ is called a *α-level subset of μ* and is symbolized by μ_α. For $\alpha=0$ the support of μ is defined as $\mu_0=\mathrm{supp}(\mu)=\overline{\{x\in X:\ \mu(x)>0\}}$ [4]. There are some of properties for level subsets:

(i) For all $0\leqslant\alpha\leqslant\beta$, we have $\mu_\beta\subseteq\mu_\alpha\subseteq\mu_0$; and
(ii) if $\alpha_n\nearrow\alpha$ then $\mu_\alpha=\bigcap_{n=1}^{\infty}\mu_{\alpha_n}$.

Now, a fuzzy set μ is called *compact* if $\mu_\alpha \in \mathscr{K}^n$ for all $\alpha \in [0,1]$, and also, μ is called *convex* if μ_α is a convex set for all $\alpha \in [0,1]$. Let $X = \mathscr{F}^n$ be the space of all compact and convex fuzzy sets on $\mathbb{R}^n$. If $\mu \in \mathscr{F}$, then μ is called a fuzzy number if the α-level set μ_α is a nonempty compact interval for all $\alpha \in [0,1]$. The sum and the scalar multiplication operations on $\mathscr{F}$ are defined as

$$(\mu+\lambda)(x) = \sup_{y\in\mathbb{R}^n}\{\mu(y)+\lambda(x-y)\} \text{ and } (\lambda\cdot\mu)(x) = \begin{cases} \mu(x) & \text{if } \lambda\neq 0, \\ \chi_0(x) & \text{if } \lambda = 0 \end{cases}$$

where χ_0 is the characteristic function of $\{0\}$. It is well known that the following operations are true for all α-levels:

$$(\mu+\lambda)_\alpha = \mu_\alpha + \lambda_\alpha, \text{ and } (c\cdot\mu)_\alpha = c\mu_\alpha, \quad \forall\alpha\in[0,1],\ c\in\mathbb{R}.$$

We set

$$D(\mu,\lambda) = \sup_{\alpha\in[0,1]} h(\mu_\alpha,\lambda_\alpha), \qquad \forall\alpha\in[0,1].$$

and by it, we can extend the Hausdorff metric h to $\mathscr{F}^n$.

Let $E = (a,b)$ and $F = \mathscr{F}^n$ be the space of all compact and convex fuzzy sets on $\mathbb{R}^n$. The fuzzy mapping $f: (a,b) \to \mathscr{F}^n$ is differentiable in $t_0 \in (0,1)$ if there exists an element $f\prime(t_0) \in \mathscr{F}^n$ such that the limits

$$\lim_{h\to 0^+}\frac{f(t_0+h)-f(t_0)}{h} \text{ and } \lim_{h\to 0^+}\frac{f(t_0)-f(t_0-h)}{h}$$

exist and are equal to $f'(t_0)$ (for more details see to [4]).

3. C^1-Fuzzy submanifold and fuzzy Lie group

Definition 3.1. A fuzzy differentiable function $\psi: M' \to M$ is called a *fuzzy immersion* if its rank is equal to the dimension of M' at each point of its domain. If its domain is the whole of M', ψ is said to be a fuzzy immersion of M' into M.

Definition 3.2. A $\mathscr{C}^1$-fuzzy manifold M' is said to be a $\mathscr{C}^1$-fuzzy submanifold of a $\mathscr{C}^1$-fuzzy manifold M if

(i) M' is a fuzzy subset of M.
(ii) Natural fuzzy injection $j: M' \to M$ is a fuzzy immersion.

Example 3.3. Clearly $M(n\times n,\mathbb{R})$, the set of real $n\times n$ matrices, is a $\mathscr{C}^1$-fuzzy manifold and $\mathrm{GL}(n,\mathbb{R})$ is a fuzzy subset of it. If $j: \mathrm{GL}(n,\mathbb{R}) \to M(n\times n,\mathbb{R})$ is the natural fuzzy injection and

$$\det\circ j: \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$$

is fuzzy differentiable, then j is a fuzzy immersion so $\mathrm{GL}(n,\mathbb{R})$ is a $\mathscr{C}^1$-fuzzy submanifold of $M(n\times n,\mathbb{R})$.

Definition 3.4. *A fuzzy Lie group G* is a $\mathscr{C}^1$-fuzzy manifold G which is also G is a group, such that the mappings

$$m: (G\times G, \tau\times\tau) \to (G,\tau), \quad i: (G,\tau)\to(G,\tau)$$

defined in (1) and (2), be fuzzy differentiable.

Example 3.5

(i) One of the simplest example of a fuzzy Lie group is $\mathbb{R}^n$ that is commutative fuzzy Lie group. The group operation is given by vector addition. The identity element is the zero vector, and the inverse of a vector x is the vector $-x$. If $\mathbb{R}^n$ equipped with the ordinary fuzzy topology, it is trivial which the mappings

$$m: \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n \quad (x,y)\mapsto x+y,$$
$$i: \mathbb{R}^n\to\mathbb{R}^n \quad x\mapsto x^{-1},$$

are fuzzy differentiable.

(ii) The other example of fuzzy lie group is the general linear group $\mathrm{GL}(n,\mathbb{R})$ consisting of all invertible $n\times n$ real matrices, with matrix multiplication defining the group multiplication, and matrix inversion defining the inverse. In fact, $\mathrm{GL}(n,\mathbb{R})$ is an n^2-dimensional $\mathscr{C}^1$-fuzzy manifold such that

$$m: \mathrm{GL}(n,\mathbb{R}^n)\times\mathrm{GL}(n,\mathbb{R}^n) \to \mathrm{GL}(n,\mathbb{R}^n) \quad (A,B)\mapsto AB,$$
$$i: \mathrm{GL}(n,\mathbb{R}^n)\to\mathrm{GL}(n,\mathbb{R}^n) \quad A\mapsto A^{-1},$$

Definition 3.6. A *fuzzy transformation group* acting on a $\mathscr{C}^1$-fuzzy manifold is determined by a fuzzy Lie group G and fuzzy differentiable map $\Phi: G \times M \to M$, which satisfies

(i) Φ is a fuzzy global surjective,
(ii) $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$, for any $x \in M$ and $g, h \in G$.

Example 3.7. $\mathrm{GL}(n, \mathbb{R}^n)$ acts on $\mathbb{R}^n$ as a fuzzy transformation group with the map

$$\Phi: \mathrm{GL}(n, \mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \quad (A, x) \mapsto Ax.$$

Definition 3.8. Let G be a fuzzy Lie group, then $H \subset G$ is called *a fuzzy Lie subgroup* if H is both a subgroup and a $\mathscr{C}^1$-fuzzy submanifold. For instance, $O(n, \mathbb{R})$, that is real orthogonal $n \times n$ matrices, is a fuzzy Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Proposition 3.9. *If a fuzzy Lie group G acts on $\mathscr{C}^1$-fuzzy manifold M as a fuzzy transformation group then so does any fuzzy Lie subgroup H of G.*

Proof. If $j: H \to G$ is the natural fuzzy injection and $id: M \to M$ be a fuzzy identity map. There is a suitable global function $\Phi_H: H \times M \to M$ such that

$$\Phi_H = \Phi \circ (j \times i),$$

where $j \times i: H \times M \to G \times M$. Therefore Φ_H is a fuzzy surjective and fuzzy differentiable function which we have

$$\Phi_H(h, \Phi_H(h', x)) = (\Phi \circ (j \times i))\big(h, (\Phi \circ (j \times i))(h', x)\big) = \Phi(h, \Phi(h', x)) = \Phi(hh', x) = (\Phi \circ (j \times i))(hh', x) = \Phi_H(hh', x)$$

for any $x \in M$ and $h, h' \in H$. □

Example 3.10. In Example 3.5. (ii), the $\mathrm{GL}(n, \mathbb{R})$ is a fuzzy Lie group, and $\mathrm{O}(n, \mathbb{R})$ is one of its fuzzy Lie subgroups, then $\mathrm{O}(n, \mathbb{R})$ also acts on $\mathbb{R}^3$ as a fuzzy transformation group.

Definition 3.11. Let $\Phi: G \times M \to M$ be a fuzzy transformation group. A subset $S \subset M$ is called *fuzzy G-invariant subset of M*, if $\Phi(G \times S) \subseteq S$.

Proposition 3.12. *Consider $\Phi: G \times M \to M$ as a fuzzy transformation group. If a regular $\mathscr{C}^1$-fuzzy submanifold M' of $\mathscr{C}^1$-fuzzy manifold M is G-invariant, then G acts naturally on M' as a fuzzy transformation group.*

Proof. Suppose that $k: G \times M' \to G \times M$ is the natural fuzzy injection, then the function $\Phi': G \times M' \to M'$ induced by $\Phi \circ k$ is fuzzy differentiable that is defines the required action of G on M'. □

Proposition 3.13. *If M/ρ is a quotient $\mathscr{C}^1$-fuzzy manifold of M and equivalence relation ρ is preserved by a fuzzy Lie transformation group G on M then G acts naturally on M/ρ.*

Proof. Let $\alpha: M \to M/\rho$ be natural fuzzy surjection and $\Phi: G \times M \to M$ be fuzzy differentiable map so $\alpha \circ \Phi: G \times M \to M/\rho$ is a fuzzy differentiable function. $G \times (M/\rho)$ is a quotient $\mathscr{C}^1$-fuzzy manifold of $G \times M$ and $\alpha \circ \Phi$ is an invariant of corresponding equivalence relation on $G \times M$. It therefore projects to the fuzzy differentiable function

$$\Phi: G \times M/\rho \to M/\rho \quad (g, \alpha m) \mapsto \alpha(gm).$$

This defines the required action G on M/ρ. □

4. Fuzzy jet spaces and prolongations

Consider the fuzzy initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0, \tag{3}$$

that is studied by Kaleva [12], where $f: [0, a] \times \mathscr{F} \to \mathscr{F}$ is continues and x_0 is a fuzzy number. It is possible to extend the fuzzy differential equation (FDE) in this problem to

$$u_x(x) = f(x, u(x)), \quad u(0) = u_0,$$

where $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$, and u_x is the partial differential equation u respect to x. Also,

$$f: \underbrace{[0, a] \times \ldots \times [0, a]}_{p\ \text{times}} \times \underbrace{\mathscr{F} \times \ldots \times \mathscr{F}}_{q\ \text{times}} \to \underbrace{\mathscr{F} \times \ldots \times \mathscr{F}}_{q\ \text{times}}$$

is continues.

Example 4.1. For instance, the fuzzy initial value problem (3) is a FDE with $p = q = 1$, $X \cong \mathbb{R}$ and $U \cong \mathbb{R}$. Let us consider the fuzzy problem

$$\begin{cases} u_x(x,y) + u_y(x,y) = -cu(x,y), \\ u(0,0) = u_0. \end{cases}$$

where $c > 0$ and u_0 is fuzzy number. This is an example of FDE with $p = 2, q = 1$, $X \cong \mathbb{R}^2$ and $U \cong \mathbb{R}$.

A general system of fuzzy differential equation (FDE) involves p independent variables $x = (x^1, \ldots, x^p)$, $x^i \in [0,a];\ 1 \leqslant i \leqslant p$, which we can view as local coordinates on the Euclidean space $X = \underbrace{[0,a] \times \ldots \times [0,a]}_{p\ \text{times}} \cong \mathbb{R}^p$, and q dependent variables $u = (u^1, \ldots, u^q)$, $u^j \in \mathscr{F};\ 1 \leqslant j \leqslant q$, coordinates on $U = \underbrace{\mathscr{F} \times \ldots \times \mathscr{F}}_{q\ \text{times}} \cong \mathbb{R}^q$. The *total space* will be the Euclidean space

$$E = X \times Y = \underbrace{[0,a] \times \ldots \times [0,a]}_{p\ \text{times}} \times \underbrace{\mathscr{F} \times \ldots \times \mathscr{F}}_{q\ \text{times}} \cong \mathbb{R}^{p+q}$$

coordinatized by the independent and dependent variables.

Definition 4.2. A (locally defined) fuzzy diffeomorphism on the total space E that is defined by a fuzzy transformation:

$$(\bar{x}, \bar{u}) = g \cdot (x,u) = (\chi(x,u), \psi(x,u)) \tag{4}$$

is called *fuzzy point transformation*.

Example 4.3. Consider the one-parameter group of rotations

$$g_t \cdot (x,u) = (x\cos t - u\sin t, x\sin t + u\cos t), \tag{5}$$

acting on the space $E = [0,a] \times \mathscr{F} \cong \mathbb{R}^2$ consisting of one independent and one dependent variable. The equation for the transformed function $\bar{f} = g_t \cdot f$ is given in implicit form

$$\bar{x} = x\cos t - u\sin t, \quad \bar{u} = x\sin t + u\cos t,$$

so that $\bar{u} = \bar{f}(\bar{x})$ is found by eliminating x from this two equations. For example, if $u = ax + b$ is affine, then the transformed function is also affine, and given by explicitly by

$$\bar{u} = \frac{\sin t + a\cos t}{\cos t - a\sin t}\bar{x} + \frac{b}{\cos t - a\sin t},$$

which is defined provided $\cot t \neq a$ (see to [18]).

Definition 4.4. For the total space $E = X \times U \cong \mathbb{R}^p \times \mathbb{R}^q$, the nth *fuzzy jet space* $\mathrm{J}^n = \mathrm{J}^n E = X \times U^{(n)}$ is the Euclidean space of dimension

$$p + q^{(n)} \equiv p + q\binom{p+n}{n},$$

whose coordinates consist of the p independent variables x^i of space X, the q dependent variables u^α of $U^{(n)}$, and the derivative coordinates u^α_J; $\alpha = 1, \ldots, q$, $1 \leqslant \#J \leqslant n$. The points in the space $U^{(n)}$ are denoted by $u^{(n)}$, and consist of all the dependent variables and their derivations up to order n; thus the derivative coordinates of a typical point $z \in \mathrm{J}^n$ are denoted by $(x, u^{(n)})$. A fuzzy differentiable function $u = f(x)$ from X to U has nth *prolongation* $u^{(n)} = f^{(n)}(x)$ (also known as the n-jet and denoted $j_n f$.), which is the function from X to $U^{(n)}$ defined by evaluating all the partial derivatives of f up to order n; thus the individual coordinate functions of $f^{(n)}$ are $u^\alpha_J = \partial_J f^\alpha(x)$.

If g is a fuzzy point transformation (4), then g acts on functions by transforming their graphs, and hence also naturally acts on the derivatives of the functions. This allows us to define the induced *prolonged fuzzy transformation*

$$(\bar{x}, \bar{u}^{(n)}) = g^{(n)} \cdot (x, u^{(n)}),$$

on the fuzzy jet space $\mathrm{J}^{(n)}$.

Example 4.5. For a rotation in the one-parameter group, considered in Example 4.3. the first prolongation $g^{(1)}_1$ will act on the space coordinatized by (x, u, u_x). The first prolongation of a rotation group is explicitly given by

$$g^{(1)}_t \cdot (x, u, u_x) = \left(x\cos t - u\sin t, x\sin t + u\cos t, \frac{\sin t + u_x\cos t}{\cos t - u_x\sin t}\right), \tag{6}$$

defined for $u_x \neq \cot t$.

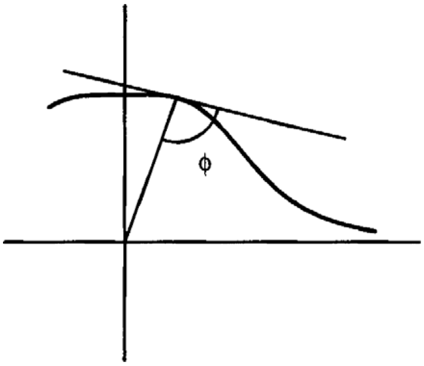

Fig. 1. Rotation group invariant.

Definition 4.6. Let G be a fuzzy point transformation group. A *fuzzy differential invariant* (FDI) for G is a fuzzy differential function μ on J^n which satisfies

$$\mu(g^{(n)}\cdot(x,u^{(n)}))=\mu(x,u^{(n)}),$$

for all $g\in G$ and all $(x,u^{(n)})\in J^n$ where $g^{(n)}\cdot(x,u^{(n)})$ is defined.

Example 4.7. Consider the usual action of the rotation group SO(2) on $E=[0,a]\times\mathscr{F}\cong\mathbb{R}^2$. The radius $r=\sqrt{x^2+u^2}$ is an ordinary fuzzy invariant of SO(2). The first prolongation $SO(2)^{(1)}$ was given in (5), besides the radius r, there is one additional first order fuzzy differential invariant (FDI), which can be taken to be the function

$$w=\frac{xu_x-u}{x+uu_x},$$

provided $x\neq -uu_x$. Geometrically, $\omega=\tan\phi$, where ϕ is the angle between the line from the origin to the point $(x,u)=(x,f(x))$ and the tangent to the graph of $u=f(x)$ at that point; see Fig. 1 ([18]).

5. Applications and new ideas

A fuzzy differential invariant (FDI) is merely an fuzzy invariant, in the standard sense, for a prolonged group of fuzzy transformations acting on the fuzzy jet space J^n. Just as the ordinary fuzzy invariants of a group action serve to characterize fuzzy invariant equations, so fuzzy differential invariants (FDIs) will completely characterize fuzzy invariant systems of fuzzy differential equations (FDEs) for the group. Fuzzy differential invariants (FDI) has any application in solving a fuzzy differential equation (FDE) by Lie symmetry method and we can classify fuzzy differential invariants for many of the groups of physical importance.

There are some fields and new ideas for more works:

- A suitable definition for fuzzy differential operator (FDO)and fuzzy contact invariant (FCI) and investigation of their relative theorems.
- What's the fuzzy differential forms and it's invariance?
- Is there a symmetries fuzzy subgroup for a fuzzy differential equations (FDE)? and how is possible to reach it?

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